KARPAGAM ACADEMY OF HIGHER EDUCATION

FACULTY OF ENGINEERING

DEPARTMENT OF SCIENCE AND HUMANITIES

SYLLABUS

14BECS401 PROBABILITY AND QUEUEING THEORY 3 1 0 4 100

OBJECTIVES:

- To understand the fundamental knowledge of probability theory.
- To acquire skills in handling situations involving more than one random variable and functions of random variables.
- To understand and characterize phenomena which evolve with respect to time in a probabilistic manner.

UNIT- I PROBABILITY AND RANDOM VARIABLE

Axioms of probability - Conditional probability - Total probability – Baye's theorem- Random variable - Probability mass function - Probability density function - Properties - Moments - Moment generating functions and their properties.

UNIT- II STANDARD DISTRIBUTIONS

Functions of a random variable - Binomial, Poisson, Geometric, Negative Binomial, Uniform, Exponential, Gamma, Weibull and Normal distributions and their properties.

UNIT- III TWO DIMENSIONAL RANDOM VARIABLES

Joint distributions - Marginal and conditional distributions – Covariance - Correlation and regression - Transformation of random variables - Central limit theorem.

UNIT -IV RANDOM PROCESS AND MARKOV CHAINS

Classification - Stationary process - Markov process - Poisson process - Birth and death process - Markov chains - Transition probabilities - Limiting distributions.

UNIT-V QUEUEING THEORY

Markovian models - M/M/1, M/M/C, finite and infinite capacity - M/M/∞ queues - Finite source model - M/G/1 queue (steady state solutions only) - Pollaczek - Khintchine formula - Special cases.

TEXT BOOKS:

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Ross,S	A first course in probability	Pearson Education, Delhi	2002
2	Medhi,J	Stochastic Process	New Age Publishers ,New Delhi	2009

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1	Veerarajan,T	Statistics and Random Processes	Tata McGraw-Hill, 2nd Edition, New Delhi.	2003
2	Allen,O	Probability, Statistics and Queuing Theory	Academic press, New Delhi.	1999
3	Gross,D. and Harris, C.M	Fundamentals of Queuing theory	John Wiley and Sons, New York.	1998
4	Taha,H.A	Operations Research - An Introduction	Pearson Education Edition Asia, Delhi.	2002

WEBSITES:

1. http://www.cut-theknot.org/probability.shtml
2. http://www.ece.uah.edu/courses/ee420-500
3. http://www.andrewferrier.com/oldpages/queueing_theory
4. http://www.cse.iitb.ac.in/perfnet/cs681/QueuingAnalysis.pdf

STAFF IN-CHARGE **HOD**

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PROBABILITY AND QUEUEING THEORY

Subject Code: 14BECS401

TEXT BOOK, REFERENCES and WEBSITES

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Coimbatore – 641021

FACULTY OF ENGINEERING

DEPARTMENT OF SCIENCE AND HUMANITIES

B.E- AUTOMOBILE ENGINEERING

Subject Code:14BECS401

Subject Name: PROBABILITY AND QUEUEING THEORY

OBJECTIVES OF THE COURSE

- To understand the fundamental knowledge of probability theory.
- To acquire skills in handling situations involving more than one random variable and functions of random variables.
- To understand and characterize phenomena which evolve with respect to time in a probabilistic manner.

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FACULTY OF ENGINEERING

II B.E COMPUTER SCIENCE AND ENGINEERING (2014 Batch)

FOURTH SEMESTER

PROBABILITY AND QUEUEING THEORY

LESSON PLAN

S.NO.	TOPICS TO BE COVERED	HOUR(S)
Unit-I	**UNIT I : PROBABILITY AND RAMDOM VARIABLES**	
	Axioms of Probability	1
	Conditional Probability	1
	Total Probability	1
	Baye's theorem	1
	Tutorial 1 (Baye's theorem)	1
	Random variable	1
	Probability mass function	1
	Probability density function	1
	Properties - Probability	1
	Tutorial 2 (Probability density function)	1
	Moments	1
	Moment generating functions – properties and problems	1
	TOTAL	**12**
Unit-II	**UNIT II : STANDARD DISTRIBUTION**	
	Binomial Distribution(mean, variance, mgf)	1
	Binomial Distribution - problems	1
	Poisson Distribution (mean, variance, mgf)	1
	Geometric Distribution and properties	1
	Tutorial 3(Binomial,Poisson Distribution)	1
	Negative Binomial Distribution	1
	Uniform Distribution and properties	1
	Exponential Distribution and properties	1
	Tutorial 4 (Uniform, Exponential Distribution)	1
	Gamma Distribution and properties	1
	Weibull Distribution and properties	1
	Normal Distribution (mean, variance, mgf)	1
	Normal Distribution- problems and Functions of random variable	1
	TOTAL	**13**
Unit-III	**UNIT III : TWO DIMENSIONAL RANDOM VARIABLES**	
	Joint distribution	1
	Marginal Distributions	1
	Conditional Distributions	1

	Tutorial 5 (Marginal distributions)	1
	Covariance and Correlation	1
	Correlation	1
	Regression	1
	Regression	1
	Tutorial 6 (Correlation & Regression)	1
	Transformation of random variables and Central limit theorem	1
	TOTAL	**10**
Unit-IV	**UNIT IV : RANDOM PROCESSES AND MARKOV CHAINS**	
	Classification	1
	Stationary process	1
	Stationary process	1
	Markov Process	1
	Tutorial 7 (Stationary process)	1
	Poisson Process	1
	Birth and death process	1
	Markov chains	1
	Markov chains - problems	1
	Tutorial 8 (Poisson process)	1
	Transition probabilities	1
	Limiting distributions	1
	TOTAL	**12**
Unit-V	**UNIT V : QUEUEING THEORY**	
	Introduction to queueing theory	1
	M/M/1 infinite capacity	1
	M/M/C infinite capacity	1
	M/M/1 finite capacity	1
	M/M/C finite capacity	1
	Tutorial 9(M/M/1 infinite capacity)	1
	Finite source model	1
	Finite source model M/G/1 queues	1
	M/G/1 queues	1
	Steady state solution	1
	Tutorial 10 (Steady state solution, M/G/1 queues)	1
	Pollaczek - Khinchin formula	1
	Pollaczek - Khinchin formula and Special cases	1
	TOTAL	**13**
	GRAND TOTAL	**50 + 10**

STAFF HOD

UNIT-I

PROBABILITY

Introduction:

The word 'Probability or change' is very frequency used in day-to-day conversation. The Statistician I.J. Good, suggests in his "kinds of Probability" that "the theory of Probability is much older than the human species.

The concept and applications of probability, which is a formal term of the popular word "Change" while the ultimate objective is to facilitate calculation of probabilities in business and managerial, science and technology etc., the specific objectives are to understand the following terminology.

Random Experiment: The term experiment refers to describe, which can be repeated under some given conditions. The experiment whose result (outcomes) depends on change is called Random Experiment.

Example:

1. Tossing of a coin is a random experiment.
2. Throwing a die is a random experiment.
3. Calculation of he mean arterial blood pressure of a person under ideal environmental conditions, by using the formula, Blood pressure = $= \frac{Systoloic\ pressure}{Diastolic\ pressure}\ mm / Hg$ is a random experiment.

Sample Space:

The totality of all possible outcomes of a random experiment is called a sample space and it is denoted by s and a possible outcome are element.

The no. of the coins in a sample space denoted by n(s).

Example:

Tossing a coin n(s)=2={H,T}

Event:

The output or result of a random experiment is called an event or result or outcome.

Example:

1. In tossing of a coin, getting head or tail is an event.
2. In throwing a die getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Events are generally denoted by capital letters A, B, C etc. The events can be of two types. One is simple event and the other is compound event

Favorable event:

The no. of events favorable to an event in a trail is the no.of outcomes which entire the happening of the event.

Mutually Exclusive Events:

Two or more events are said to be mutually exclusive events if the occurrence of one event precludes (excludes or prevents) the occurrence of others, i.e., both cannot happen simultaneously in a single trail.

Example:

1. In tossing of a coin, the events head and tail are mutually exclusive.
2. In throwing a die, all the six faces are mutually exclusive.

Equally Likely Events: Two or more events are said to be equally likely, if there is no reason to expect any one case (or any event) in preference to others. i.e., every outcome of the experiment has equal possibility of occurrence. These are equally likely events.

Exhaustive Number of Cases or Events: The total number of possible outcomes in an experiment is called exhaustive number of cases or events.

Dependent event:

Two events are said to be dependent if the occurance or non occurance of a event in any trail affect the occurance of the other event in other trail.

Classical Definition of Probability: Suppose that an event 'A' can happen in 'm' ways and fails to happen (or non-happen) in 'n' ways, all these 'm+n' ways are supposed equally likely. Then the probability of occurrence (or happening) of the event called its success is denoted by 'P(A)' or simply 'p' and is defined as $P(A) = \frac{m}{m+n} \ldots(1)$ and the probability of non-occurrence (or non-happening) of the event called its failure is denoted by $P(\overline{E})$ or simply 'q' and is defined as. $P(\overline{A}) = \frac{n}{m+n} \ldots(2)$

From (1) and (2) we observe that the probability of an event can be defined as

$$P(event) = \frac{\text{The number of favourable cases for the event}}{\text{Total number of possible cases}}$$

Definition:

Let S be the sample space and A be the event associated with a random experiment. Let n(S) and n(A) be the no .of elements of S & A. Then the probability of the event A occurring denoted as P(A) is defined by

$$P(event) = \frac{\text{The number of favourable cases for the event}}{\text{Total number of possible cases}} = \frac{n(A)}{n(S)}$$

Note:

It follows that, $P(A)+P(\overline{A})=1 \, or \; p+q=1.$

This implies that p=1-q or q=1-p.

Hence $0 \le P(A) \le 1$.

Axiomatic Definition of Probability: Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by P(A), is defined as a real number satisfying he following axioms.

(i) $0 \le P(A) \le 1$

(ii) P(S)=1

(iii) If A and B are mutually exclusive events, $P(A \cup B)=P(A)+P(B)$

(iv) If $A_1, A_2, \ldots, A_n, \ldots$ are a set of mutually exclusive events, $P(A_1 \cup A_2 \cup \ldots \cup A_{n,\ldots})=P(A_1)+P(A_2)+\ldots+P(A_n)+\ldots$

Theorem 1: The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, P(ϕ)=0.

Proof: The certain event S and the impossible event ϕ are mutually exclusive.

Hence $P(S \cup \phi)=P(S)+P(\phi)$ [axiom (iii)]

But $S \cup \phi = S$.

Therefore, $P(S)=P(S)+P(\phi)$

Hence $P(\phi)=0.$

Theorem 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A})=1-P(A) \le 1.$

Proof: A and $\overline{A}$ are mutually exclusive events, such that $A \cup \overline{A} = S$

Therefore, $P(A \cup \overline{A})=P(S)$=1 (Since axiom (ii))

i.e., $P(A)+P(\overline{A})=1.$

Therefore, $P(\overline{A})=1-P(A)$

Since $P(A) \ge 0,$ it follows that $P(\overline{A}) \le 1.$

Theorem 3: If $B \subset A \; then \; P(B) \le P(A).$

Proof: B and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$.

Therefore, $P(B \cup A\overline{B})=P(A)$

i.e., $P(B)+P(A\overline{B})=P(A)$ [axiom (iii)]

Therefore, $P(B) \le P(A).$

Theorem 4: Addition theorem of probability

Statement: For any two events A and B, $P(A\cup B)=P(A)+P(B)-P(A\cap B).$

Proof: Since $(A\cup B)=A\cup(A'\cap B)$ here A and $(A'\cap B)$ are mutually exclusive.

$P(A\cup B)=P[A\cup(A'\cap B)]\ ...(1)$

$=P(A)+P(A'\cap B)$

Again $B=(A\cap B)\cup(A'\cap B)$

Here $(A\cap B)\&(A'\cap B)$ are mutually exclusive events.

$P(B)=P[(A\cap B)\cup(A'\cap B)]\ ...(2)$

$=P(A\cap B)+P(A'\cap B)$

Therefore $P(A'\cap B)=P(B)-P(A\cap B)$

From (1), $P(A\cup B)=P(A)+P(B)-P(A\cap B).$

Conditional Probability: The Conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as, $P(B/A)=\dfrac{P(A\cap B)}{P(A)},$ provided $P(A)\neq 0.$

Rewriting the definition of conditional probability, we get $P(A\cap B)=P(A)\times P(B/A).$ [Product theorem of probability]

Properties:

1. If $A\subset B, P(B/A)=1, Since\ A\cap B=A.$
2. If $B\subset A, P(B/A)\geq P(B), Since\ A\cap B=B,$ and $\dfrac{P(B)}{P(A)}\geq P(B), as\ P(A)\leq P(S)=1.$
3. If A and B are mutually exclusive events, P(B/A)=0, since $P(A\cap B)=0$
4. If P(A)>P(B), P(A/B)>P(B/A).
5. If $A_1\subset A_2, P(A_1/B)\leq P(A_2/B)$.

Independent Events: A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

The product theorem can be extended to any number of independent events: $A_1, A_2,..., A_n$ are n independent events. $P(A_1\cap A_2\cap...\cap A_n)=P(A_1)\times P(A_2)\times...\times P(A_n)$, when this condition is satisfied, the events $A_1, A_2,..., A_n$ are also said to be totally independent. A set of events $A_1, A_2,..., A_n$ is said to be mutually independent if the events are totally independent when considered in sets of 2,3,. . . n events.

Theorem 5: If the events A and B are independent, then so are $\overline{A}\&\overline{B}.$

Proof. $P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})=1-P(A\cup B)$

$=1-[P(A)+P(B)-P(A\cap B)]$ (By addition theorem)

$=1-P(A)-P(B)+P(A)\times P(B)$ {since A and B are independent)

$$= [1 - P(A)] - P(B)[1 - P(A)]$$

$$= P(\overline{A}) \times P(\overline{B})$$

Example 1: In how many different ways can the director of a research laboratory choose two chemists from among seven applicants and three physicists from among nine applicants?

Solution:

The two chemists can be chosen in ${}^{7}C_{2}$ =21 ways

The three physicists can be chosen in ${}^{9}C_{3}$ = 84 ways

Then these two things can be done in 21 x 84 = 1764 ways.

Example 2: What is the probability that a non-leap year contains 53 Sundays?

Solution:

A non-leap year consists of 365 days, of these there are 52 complete weeks and 1 extra day. That day may be any one of the 7 days. So already we have 52 Sundays. For one more Sunday, the probability that getting a one more Sunday is 1/7.

Hence the probability that a non-leap year contains 53 Sundays is 1/7.

Example 3: A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both the white?

Solution:

Given that Balls White(7), Red(6) & Black(5), total 18 balls.

Two balls are drawn at random from 18 balls in ${}^{18}C_{2}$ ways

Two white balls are drawn at random from 7 balls in ${}^{7}C_{2}$ ways.

Hence the required probability = $({}^{7}C_{2})/({}^{18}C_{2}) = 21/153.$

Example 4 : Determine the probability that for a non-defective bolt will be found if out of 600 bolts already examined 12 were defective.

Solution:

Given that out of 600 bolts 12 were defective.

Therefore, probability that a defective bolt will be found = $\frac{12}{600} = \frac{1}{50}$

Therefore, Probability of getting a non-defective bolt = $1 - \frac{1}{50} = \frac{49}{50}$.

Example 5: A fair coin is tossed 4 times. Define the sample space corresponding to this experiment. Also give the subsets corresponding to the following events and find the respective probabilities:

a).More heads than tails are obtained.

b).Tails occur on the even numbered tosses.

Solution:

S= {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

a). Let A be the event is which more heads occur than tails

Then A= {HHHH, HHHT, HHTH. HTHH,THHH}

b).Let B be the event is which tails occur is the second and fourth tosses.

Then B= {HTHT, HTTT,TTHT,TTTT}

$$P(A)=\frac{n(A)}{n(S)}=\frac{5}{16};\ P(B)=\frac{n(B)}{n(S)}=\frac{1}{4}.$$

Example 6: A box contains 4 bad & 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is probability that the other one is also good?

Solution:

Let A = one of the tubes drawn is good and B = the other tube is good .

$P(A\cap B)$ = P(both tubes drawn are good)

$$=\frac{{}^{6}C_{2}}{{}^{10}C_{2}}=\frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., P(B/A) is required.

By definition, $P(B/A)=\dfrac{P(A\cap B)}{P(A)}=\dfrac{1/3}{6/10}=\dfrac{5}{9}.$

Example 7: In a shooting test, the probability of hitting the target is ½ for A , 2/3 for B , 3/4 for C. If all of them five at the target, find the probability that

i). none of them hits the target.

ii). Atleast one of them hits the target.

Solution:

Let A = event of A hitting the target.

$$P(\overline{A})=\frac{1}{2},P(\overline{B})=\frac{1}{3},P(\overline{C})=\frac{1}{4}.$$

$$P(\overline{A}\cap\overline{B}\cap\overline{C})=P(\overline{A})\times P(\overline{B})\times P(\overline{C})$$ (by independence)

i.e., P(none hits the target) = $\dfrac{1}{2}\times\dfrac{1}{3}\times\dfrac{1}{4}=\dfrac{1}{24}$

P(atleast one hits the target) = 1 – P(none hits the target)

$$=1-\frac{1}{24}=\frac{23}{24}.$$

Example:8

Three coins are tossed together find they are exactly 2 head?
Solution:

Total no. of chances by throwing 3 coins are n(S)= 8.

The event A to get exactly 2 heads are A = {HHT, THH,HTH}

n(A)= 3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Example:9

A bag contains 4 red, 5 white and 6 black balls. What is the probability that 2 balls drawn are red and black?
Solution:

Given that Balls White(5), Red(4) & Black(6), total 15 balls.

Two balls are drawn at random from 15 balls in $15C_2$ ways

n(A)= $4C_1$X $6C_1$, Hence the required probability = $\frac{4C_1 X 6C_1}{15C_2} = \frac{8}{35}$

Example :10
A bag contains 3 red and 4 white balls. Two draws are made without replacement.
What is the probability that both balls are red
Solution:
Total no. of balls = 3Red + 4 White = 7 balls

P(Drawing a red ball in the first drawn is red) = $P(A) = \frac{3}{7}$

P(Drawing a red ball in the second drawn is red) = $P(B/A) = \frac{2}{6}$

$$P(A \quad B) = P(A)P(B)$$

$$P(B/A) = \frac{P(A \quad B)}{P(A)}$$

$$P(A \quad B) = P(A)P(B/A)$$

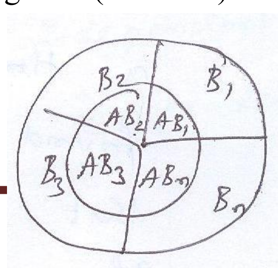
$$= \frac{1}{7}$$

Theorem of Total Probability

Statement: If $B_1, B_{2,\dots,} B_n$ be a set of exhaustive and mutually exclusive events, and A is another event associated with (or caused by) B_i, then $P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$

Proof. The inner circle represents the event A. A can occur along with (or due to) $B_1, B_{2,\dots,} B_n$ that are exhaustive and mutually exclusive.

Therefore, $AB_1, AB_{2,\dots,} AB_n$ are also mutually exclusive.



Therefore, $A = AB_1 + AB_2 + \ldots + AB_n$ (by addition theorem)

Hence $P(A) = P(\sum AB_i)$

$= \sum P(AB_i)$ (since $AB_1, AB_2, \ldots, AB_n$ are mutually exclusive)

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

Baye's theorem on Probability (or) Rule of inverse probability

Statement: If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with (or caused by) B_i, then

$$P(B_i / A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, \; i = 1, 2, \ldots, n$$

Proof. Since by product theorem, $P(A \cap B_i) = P(B_i) \times P(A/B_i)$ . . . (1)

or $P(A \cap B_i) = P(A)\; P(B_i / A)$ . . . (2)

From (1) and (2), $P(A)P(B_i / A) = P(B_i)\; P(A/B_i)$

$$P(B_i / A) = \frac{P(B_i)\; P(A/B_i)}{P(A)} \quad \ldots (3)$$

Therefore from total probability, $P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$ substitute in (3), we get

$$P(B_i / A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, \; i = 1, 2, \ldots, n$$

Example 11: A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag & they are note to be white. What is the chance the all the balls in the bag are white?

Solution:

Since 2 white balls have been drawn out, the bag must have contained 2, 3, 4, or 5 white balls.

Let B_1 = Event of the bag containing 2 white balls.

B_2 = Event of the bag containing 3 white balls.

B_3 = Event of the bag containing 4 white balls.

B_4 = Event of the bag containing 5 white balls.

Let A = Event of drawing 2 white balls.

$$P(A/B_1) = \frac{{}^2C_2}{{}^5C_2} = \frac{1}{10}, \quad P(A/B_2) = \frac{{}^3C_2}{{}^5C_2} = \frac{3}{10}$$

$$P(A/B_3)=\frac{{}^4C_2}{{}^5C_2}=\frac{3}{5},\ \ P(A/B_4)=\frac{{}^5C_2}{{}^5C_2}=1$$

Since the number of white balls in the bag is not known, B_i's are equally likely.

Therefore $P(B_1)=P(B_2)=P(B_3)=P(B_4)=\frac{1}{4}$

By Baye's theorem,

$$P(B_4/A)=\frac{P(B_4)\times P(A/B_4)}{\sum_{i=1}^{4}P(B_i)\times P(A/B_i)}=\frac{\frac{1}{4}\times 1}{\frac{1}{4}\times\left(\frac{1}{10}+\frac{3}{10}+\frac{3}{5}+1\right)}=\frac{1}{2}.$$

Example 12: There are 3 true coins and 1 false coin with 'head' on both sides. A coin is closer at random and tosses 4 times, If 'head' occurs are the 4 times, What is the probability that the false coin has been chosen and used?

Solution:

P(T) = P(the coin is a true coin) = 3/4

P(F) = P(the coin is a false coin) = 1/4

Let A = Event of getting all heads is 4 tosses,

Then, P(A/T) = ½ * ½ * ½ * ½ = 1/16 and P(A/F) = 1

By Baye's theorem,

$$P(F/A)=\frac{P(F)\times P(A/F)}{P(F)\times P(A/F)+P(T)\times P(A/T)}=\frac{\frac{1}{4}\times 1}{\frac{1}{4}\times 1+\frac{3}{4}\times\frac{1}{16}}=\frac{16}{19}.$$

Example 13:

There are three bags , bag one contains 3 white balls , 2 red balls and 4 black balls. Bag two contains 2 white balls, 3 red balls and 5 black balls. Bag three contains 3 white balls, 4 red balls and 2 black balls. One bag is chosen at random and from it 3 balls were drawn out of which 2 balls were white and 1 is red. What is the probability that it is drawn from bag one, two and three?

Solution:

Selection of bags are mutually exclusive events. The selection of the 2 white and 1 red ball is an independent event.

$P(B_1)=P(B_2)=P(B_3)=1/3$

$P(A/B_1)=$ P(Bag 1 selected from 2W&1R ball chosen)

$$=\frac{3C_2X2C_1}{9C_3}$$

=0.07

$P(A/B_2)=$ P(Bag 2 selected from 2W&1R ball chosen)

$$=\frac{2C_2X3C_1}{10C_3}$$

= 0.025

$P(A/B_3)=$ P(Bag 3 selected from 2W&1R ball chosen)

$= \frac{3C_2 X 4C_1}{9C_3}$

$= 0.14$

By using Baye's theorem we have

$P(B_i)$	$P(A/B_i)$	$P(B_i)\ P(A/B_i)$
1/3	0.07	0.0233
1/3	0.025	0.0083
1/3	0.14	0.0466
	$\sum P(B_i)\ P(A/B_i)$	0.0782

$P(B_1/A)$ = P(The balls selected from the first bag)

$= \frac{0.0233}{0.0782}$

$= 0.29$

$P(B_2/A)$ = P(The balls selected from the second bag)

$= \frac{0.008}{0.0782}$

$= 0.102$

$P(B_3/A)$ = P(The balls selected from the third bag)

$= \frac{0.046}{0.0782}$

$= 0.58$

Exercise:

1. In a bolt factory machines A,B,C manufactures 25%,35% and 40% of the total respectively. Out of their output 5%,4% and 2% are defective bolts respectively. A bolt is drawn at random and is found to be defective. What are the probabilities that it was manufactured by the machines A,B and C respectively?

2. A bag contains five balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are found to be white. What is the probability that all the balls in the bag are white?

RANDOM VARIABLES

Definition: A real-valued function defined on the outcome of a probability experiment is called a random variable. A Random variable (RV) is a rule that assigns a numerical value to each possible outcome of an experiment.

1. Discrete Random Variables.
2. Continuous Random Variables

Probability distribution function of X: If X is a random variable, then the function F(x) defined by $F(x) = P\{X \le x\}$ is called the distribution function of X.

1. **Discrete Random Variable:** A random variable whose set of possible values is either finite or countable infinite is called discrete random variable.

Probability Mass Function (pmf): If X is a discrete variable, then the function $p(x) = P[X = x]$ is called the pmf of X. It satisfies two conditions

i) $p(x_i) \geq 0$

ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

Cumulative distribution [discrete R.V] or distribution function of X: The cumulative distribution F(x) of discrete random variable X with probability f(x) is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \; for -\infty < x < \infty$$

Properties of distribution function:

1. $F(-\infty) = 0$
2. $F(\infty) = 1$
3. $0 \leq F(x) \leq 1$
4. $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
5. $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) + P[X = x_1]$
6. $P(x_1 < X < x_2) = F(x_2) - F(x_1) - P[X = x_2]$
7. $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) - P[X = x_2] + P[X = x_1]$

Results:

1. $P(X \leq \infty) = 1$
2. $P(X \leq -\infty) = 0$
3. $P(X > x) = 1 - P[X \leq x]$
4. $P(X \leq x) = 1 - P[X > x]$

Example: 14

A continuous random variable 'X' has a probability density function f(x) = K, $0 \leq x \leq 1$. Find 'K'.

Solution:

Given $f(x) = k, 0 \leq x \leq 1$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{0}^{\infty} kdx = 1$$

k=1

Example 15: A R.V X has the following probability distribution.

x:	-2	-1	0	1	2	3
p(x):	0.1	k	0.2	2k	0.3	3k

Find (1) The value of k, (2) Evaluate P(X<2) and P(-2<X<2) .

Solution:

(1) Since $\sum_{i=1}^{n} p(x_i) = 1$

0.1+k+0.2+2k+0.3+3k = 1

K = 1/15.

(2) P[X<2] = P[x=-2] + P[x=-1] + P[x=0] + P[x=1]

= 0.1 + 1/15 + 0.2 + 2/15

= ½

P[-2 < X < 2] = P[x=-1]+ P[x=0]+ P[x=1]

= 1/15 + 0.2 + 2/15 = 2/5

Example 16:

A random variable X has the following probability function

Values of x	0	1	2	3	4	5	6	7	8
Probability P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

i) Determine the value of 'a'.
ii) Find P(X<3), P(X ≥ 3) and P (0 < X< 5).
iii) Find the distribution function of X.

Solution:

i) To find 'a' value:

Given discrete random variable, $\sum_{i=1}^{\infty} p(x_i) = 1$

a+3a+5a+7a+9a+11a+13a+15a+17a =1

81a=1

a=1/81

ii) To find P(X<3):

P(X<3) = P(X=0)+P(X=1)+P(X=2)

=a+3a+5a

=9a

=1/9

iii) To find $P(X \geq 3)$:

$P(X \geq 3) = 1 - P(X < 3)$

=1-1/9 =8/9

iv) To find $P(0 < X < 5)$:

$P(0 < X < 5) = P(X = 1) + ...P(X = 4)$

= 3a+5a+7a+9a

= 24/81

v) To find the distribution function of X:

Value of x	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a
P(x)	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81
F(x)	1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	1

Example 17: A R.V X has the following function:

X:	0	1	2	3	4	5	6	7
P(X):	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(a) find k (b) Evaluate P[X<6], P[x≥6], (c) Evaluate P[1.5<X<4.5 / X>2] (d) Find P[X<2], P[X>3], P[1<X<5].

Solution:

(a). Since $\sum_{i=1}^{n} p(x_i) = 1$

i.e., $0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$10k^2 + 9k - 1 = 0$

K = -1 or 1/10 (since k=-1 is not permissible, P(X)≥0)

Hence k = 1/10.

(b). P[x≥6] = P[X=6] + P[X=7]

$= 2k^2+7k^2+k$

= 2/100 + 7/100 + 1/10 = 19/100

P[X<6] = 1 - P[x≥6]

= 1 – 19/100

= 81/100

(c). P[1.5<X<4.5 / X>2] $= \dfrac{p[(1.5 < x < 4.5) \cap x > 2]}{p(x > 2)}$ (by conditional probability)

$= \dfrac{p[2 < x < 4.5]}{1 - p(x \le 2)}$

$= \dfrac{p(3) + p(4)}{1 - [p(0) + p(1) + p(2)]}$

$= \dfrac{\frac{2}{10} + \frac{3}{10}}{1 - \left[0 + \frac{1}{10} + \frac{2}{10}\right]} = \dfrac{\frac{5}{10}}{\frac{7}{10}} = \dfrac{5}{7}$

(d). p(X<2) = p[x=0] + p[x=1]

= 0 + k = k = 1/10

$P(X>3) = 1 - p(x \le 3)$

$= 1 - [p(x=0)+p(x=1)+p(x=2)+p(x=3)]$

$= 1 - [0+k+2k+2k]$

$= ½$

$P(1<x<5) = p(x=2) + p(x=3)+ p(x=4)$

$= 2k + 2k + 3k$

$= 7/10$

Example 18: If the R.V. X takes the values 1,2,3 and 4 such that 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4). Find the probability distribution and cumulative distribution function of X.

Solution:

Since X is a discrete random variable.

Let $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k$

$2P(X = 1) = k$ implies that $P(X = 1) = k/2$

$3P(X = 2) = k$ implies that $P(X = 2) = k/3$

$P(X = 3) = k$

$5P(X = 4) = k$ implies that $P(X = 4) = k/5$

Since $\sum_{i=1}^{n} p(x_i) = 1$

i.e., $k/2 + k/3 + k + k/3 = 1$

$k[1/2 + 1/3 + 1 + 1/5] = 1$

Therefore $k = 30/61$

x_i	$p(x_i)$	F(X)
1	P(1) = k/2 = 15/61	F(1) = p(1) = 15/61
2	P(2) = k/3 = 10/61	F(2) = F(1) + p(2) = 15/61 + 10/61 = 25/61
3	P(3) = k = 30/61	F(3) = F(2)+p(3) = 25/61 + 30/61 = 55/61
4	P(4) = k/5 = 6/61	F(4) = F(3)+p(4) = 55/61 + 6/61 = 61/61 = 1

Example 19: A discrete random variable X has the following probability mass function:

X	0	1	2	3	4	5	6	7
P(X)	0	a	2a	2a	3a	a^2	$2a^2$	$7a^2+a$

Find (i) the value of 'a' (ii) $P(X<6)$, $P(X \ge 6)$ (iii) $P(0<X<5)$ (iv) the distribution function of X (v) If $P(X \le x) > 1/2$, find the minimum value of X.

Solution:

(i) Since $\sum_{i=1}^{n} p(x_i) = 1$

i.e., $0+a+2a+2a+3a+a^2+2a^2+7a^2+a = 1$

$10a^2 + 9a - 1 = 0$

$a = -1$ or $1/10$ (since $a=-1$ is not permissible, $P(X) \geq 0$)

Hence $a = 1/10$.

(ii). $P[x \geq 6] = P[X=6] + P[X=7]$

$= 2a^2+7a^2+a$

$= 2/100 + 7/100 + 1/10 = 19/100$

(iii). $P[X<6] = 1 - P[x \geq 6]$

$= 1 - 19/100$

$= 81/100$

(iv). To find P(0<X<5):

$P(0<X<5) = P(X=1)+\ldots..P(X=4)$

$= a+2a+2a+3a$

$= 8a = 8/10$

(v). To find distribution function of X :

x	0	1	2	3	4	5	6	7
P(x)	0	a	2a	2a	3a	a^2	$2a^2$	$7a^2+a$
F(x)	0	1/10	3/10	5/10	8/10	81/100	83/100	1

Minimum value of X:

$P(X \leq x) \quad 1/2$

The minimum value of X for which $P(X \leq x) \quad 0.5$, is the x value is 4.

2. **Continuous Random Variables:** A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Example: The length time during which a vacuum tube installed in a circuit functions is a continuous random variable, number of scratches on a surface, proportion of defective parts among 1000 testes, number of transmitted in error.

Probability density function (pdf): For a continuous R.V X, a probability density function is a function such that (1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx =$ area under f(x) from a to b for any a and b.

Cumulative distribution function: The Cumulative distribution function of a continuous R.V. X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt \quad for -\infty < x < \infty.$$

Mean and variance of the Continuous R.V. X: Suppose X is continuous variable with pdf f(x). The mean or expected value of X, denoted as μ or E(X)

$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$. And the variance of X, denoted as V(X) or σ^2 is $E[X^2] - [E(X)]^2$

Example 20: Given that the pdf of a R.V X is f(x)=kx, 0<x<1. Find k and P(X>0.5)

Solution:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 kx\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 = 1$$

K = 2

P(X > 0.5) = $\int_{0.5}^{\infty} f(x)\,dx$

$= \int_{1/2}^{1} 2x dx$

$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$

= 3/4

Example 21: If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & elsewhere \end{cases}$ is the pdf of a R.V. X. Find k.

Solution:

For a pdf $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here $\int_0^\infty kxe^{-x}\,dx = 1$ [since x>0]

$$k\left[x\left(\frac{e^{-x}}{-1}\right)-1\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 1$$

K =1

Example 22: A continuous R.V. X has he density function $f(x)=\frac{k}{1+x^2}, -\infty < x < \infty.$ find the value of k and the distribution function.

Solution:

Given is a pdf $\int_{-\infty}^{\infty} f(x)dx = 1$, $f(x)=\frac{k}{1+x^2}, -\infty < x < \infty.$

$$k\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = 1$$

$$2k\int_0^{\infty}\frac{1}{1+x^2}dx = 1$$

$$2k\left[\tan^{-1}x\right]_0^\infty = 1$$

$$2k\left[\frac{\pi}{2}-0\right]=1$$

$$\pi k = 1; k=\frac{1}{\pi}$$

$$F(x)=\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x}\frac{1}{\pi}\left(\frac{1}{1+x^2}\right)dx$$

$$=\frac{1}{\pi}\left[\tan^{-1}x\right]_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1}x-\left(\frac{-\pi}{2}\right)\right]$$

$$=\frac{1}{\pi}\left[\tan^{-1}x+\left(\frac{\pi}{2}\right)\right] \quad for -\infty < x < \infty$$

Example:23

A continuous random variable X has a pdf $f(x)=3x^2$, $0 \le x \le 1$. Find a and b such that (i) $P(X \le a) = P(X > a)$ and (ii) $P(X > b) = 0.05.$

Solution:

A continuous random variable X has a pdf $f(x)=3x^2$, $0 \le x \le 1.$

i) To find $P(X \le a) = P(X \quad a)$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 3x^2 dx = 1$$

Since $P(X \le a) = P(X \quad a)$, $P(X \le a) = \frac{1}{2} = 0.5$

$\int_0^a f(x)dx = \frac{1}{2}$, $\int_0^a 3x^2 dx = a^3 = \frac{1}{2}$

a = 0.7937

ii) To find $P(X \quad b) = 0.05$

$\int_b^1 f(x)dx = 0.05$, $\int_b^1 3x^2 dx = 1 - b^3 = 0.05$

$b^3 = 0.95$

$b = (0.95)^{1/3}$

Example 24: If the density function of a continuous R.V. X is given by $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & otherwise \end{cases}$

(1) Find the value of a.

(2) The cumulative distribution function of X.

(3) If x_1, x_2, x_3 are 3 independent observations of X. What is the probability that exactly one of these 3 is greater than1.5?

Solution:

(1) Since f(x) is a pdf, then $\int_{-\infty}^{\infty} f(x)dx = 1$

i.e., $\int_0^3 f(x)dx = 1$

i.e., $\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)dx = 1$

a = ½

(2). (i) If x<0 then F(x) = 0

(ii) If $0 \le x \le 1$ then $F(x) = \int_0^x ax\,dx = \int_0^x \frac{x}{2} dx$

$= \frac{x^2}{4}$

(iii) If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)dx$

$= \int_0^1 ax\,dx + \int_1^x adx$

$= \frac{x}{2} - \frac{1}{4}$

$$(iv)\ If\ 2 \le x \le 3\ then\ F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_0^1 ax\,dx + \int_1^2 a dx + \int_2^x (3a - ax)dx$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

(v) If x>3, then $F(x) = \int_{-\infty}^{x} f(x)dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + \int_3^x f(x)\,dx$$

= 1

(3). $P(X > 1.5) = \int_{1.5}^{3} f(x)dx = \int_{1.5}^{2} \frac{1}{2}dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx$

= ½

Choosing an X and observing its value can be considered as a trail and X>1.5 can be considered as a success.

Therefore, p=1/2, q=1/2.

As we choose 3 independent observation of X, n = 3.

By Bernoulli's theorem, P(exactly one value > 1.5) = P(1 success)

$$= {}^3C_1 \times (p)^1 \times (q)^2 = \frac{3}{8}.$$

Example 25: A RV X has the following distribution

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	3k

(a) find k (b) Evaluate P(X<2) & P(-2<X<2)

Solution:

(a) Σ P(X)=1

6K+0.6=1

K=1/15

Since the distribution is

X	-2	-1	0	1	2	3
P(X)	1/10	1/15	1/5	2/15	3/10	1/5

(b) P(X<2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)

= 1/10 + 1/15 + 1/5 + 2/15 =1/2

& P(-2<X<2) = P(X=-1) + P(X=0) + P(X=1)

= 1/15 + 1/5 + 2/15 = 2/5

Example:26

A continuous random variable X is having the probability density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < \text{x} < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the cumulative distribution function of x.

Solution:

Given $$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < \text{x} < 2 \\ 0, & \text{otherwise} \end{cases}$$

To find cumulative distribution function of x:

i) If $0 < x < 1$ F(x) = $\int_{-\infty}^{x} f(x)dx$

= $\int_{0}^{x} xdx = \frac{x^2}{2}$

ii) If $1 < x < 2$, F(x) = $\int_{-\infty}^{x} f(x)dx$

= $\int_{0}^{1} xdx + \int_{1}^{x} (2 - x)dx$

= $2x - \frac{x^2}{2} - 1$

iii) If $x > 2$, F(x) = $\int_{-\infty}^{x} f(x)dx$

= $\int_{0}^{1} xdx + \int_{1}^{2} (2 - x)dx$

= 1

The cumulative distribution function of x is

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1, & 1 < \text{x} < 2 \\ 1, & \text{x} > 2 \end{cases}$$

UNIT-II

RANDOM VARIABLES

Introduction:

In the last chapter, we introduced the concept of a single random variable. We observed that the various statistical averages or moments of the random variable like mean, variance, standard derivation, skewness give an idea about the characteristics of the random variable.

But in many practical problems several random variables interact with each other and frequently we are interested in the joint behavior of the health conditions of a person, doctors measure many parameters like height, weight, blood pressure, sugar level etc. we should now introduce techniques that help us to determine the joint statistical properties of several random variables.

The concepts like distribution function, density function and moments that we defined for single random variable can be extended to multiple random variables also.

Definition:

Let S be the sample space. Let X=X(S) and Y=Y(S) be two functions each assigning a real no. to each outcome $s \in S$. Then (X,Y) is a two dimensional random variable.

Types of random variables:

1. Discrete random variables

2. Continuous random variables

Two dimensional discrete random variables:

If the possible values of (X,Y) are finite or countably infinite then (X,Y) is called a two dimensional discrete random variables when (X,Y) is a two dimensional discrete random variable the possible values of (X,Y) may be represented as (x_i, y_j) i = 1,2,…..n, j=1,2,….m.

Two dimensional continuous random variables:

If (X,Y) can assume all values in a specified region R in the XY plane (X,Y) is called a two dimensional continuous random variables.

Joint distributions – Marginal and conditional distributions:

(i) Joint Probability Distribution:

The probabilities of two events $A = \{X \le x\}$ and $B = \{Y \le y\}$ have defined as functions of x and y respectively called probability distribution function.

$$F_X(x) = P(X \le x)$$
$$F_Y(y) = P(Y \le y)$$

Discrete random variable important terms:

i) Joint probability function (or) Joint probability mass function:

For two discrete random variables x and y write the probability that X will take the value of x_i, Y will take the value of y_j as, $P(x,y) = P(X = x_i, Y = y_j)$

ie) $P(X = x_i, Y = y_j)$ is the probability of intersection of events $X = x_i \,\&\, Y = y_j$.

$P(X = x_i, Y = y_j) = P(X = x_i \cap Y = y_j)$, The function $P(X = x_i, Y = y_j) = P(x_i, y_j)$ is called a joint probability function for discrete random variables X,Y and it is denoted by P_{ij}.

P_{ij} satisfies the following conditions

(i) $P_{ij} > 0$, for every i,j

(ii) $\sum_j \sum_i P_{ij} = 1$

Continuous random variable (or) Joint Probability Density Function:

Definition:

The joint probability density function if (x,y) be the two dimensional continuous random variable then f(x,y) is called the joint probability density function of (x,y) the following conditions are satisfied.

(i) $f(x,y) \geq 0, \forall x, y \in R$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = 1.$ Where R is a sample space.

Note: $P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d f(x,y)\,dydx$

Joint cumulative distributive function:

If (x,y) is a two dimensional random variable then $F(X,Y) = P(X \leq x, Y \leq y)$ is called a cumulative distributive function of (x,y) the discrete case $F(X,Y) = \sum_j \sum_i P_{ij} = 1$, $y_i \leq y, x_i \leq x.$

In the continuous case $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x,y)\,dxdy$

Properties of Joint Probability Distribution function:

1. $O \le P(x_i, y_j) \le 1$
2. $\sum_i \sum_j P(X_i, Y_j) = 1$
3. $P(X_i) = \sum_j P(X_i, Y_j)$
4. $P(y_i) = \sum_j P(X_i, Y_j)$
5. $P(x_i) \ge P(x_i, y_j)$ for any j
6. $P(y_j) \ge P(x_i, y_j)$ for any i

Properties:

1. The joint probability distribution function F ^{xy} (X, Y) of two random variable X and Y have the following properties. They are very similar to those of the distribution function of a single random variable.
2. $0 \le f_{XY}(x, y) \le 1$
3. $f_{XY}(\infty, \infty) = 1$
4. $f_{XY}(x, y)$ is non decreasing
5. $f_{XY}(-\infty, y) = F_{xy}(x_1, \infty) = 0$
6. For $x_1 < x_2$ and $y_1 < y_2$, $P(x_1 < X \le x_2, Y \le y_1) = F(x_2, y_1) - F(x_1, y_1)$
7. $P(X \le x_1, y_1 < Y \le y_2) = F(x_1, y_2) - F(x_1, y_1)$

   $P(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2, y_2) - F(x_1, y_2) - f(x_2, y_1) + f(x_1, y_1)$
8.
9. $F_Y(y) = F_{XY}(\infty, y) = P(X \le \infty, y \le y) = P(y \le y)$
10. $F_X(x) + F_y(y) - 1 \le F_{XY}(x, y) \le \sqrt{F_X(x) F_Y(y)}$ for all x and y.

These properties can also be easily extended to multi dimensional random variables.

Marginal Probability Distribution function:

(i) Discrete case:

- Let (x,y) be a two dimensional discrete random variable, $P_{ij} = P[X = x_i, Y = y_j]$ then $P(X = x_i) = P_i *$ is called a marginal probability of the function X. Then the collection of the pair $\{x_i, P_i^*\}$ is called a marginal probability of X.
- If $P(Y = y_j) = P_{*j}$ is called a marginal probability of the function Y. Then the collection of the pair $\{y_i, P_{*j}\}$ is called a marginal probability of Y.

(ii) Continuous case:

- The marginal density function of X is defined as $f_x(x) = g(x) = \int_{-\infty}^{\infty} f(x,y)dy$ and
- The marginal density function of Y is defined as $f_y(y) = h(y) = \int_{-\infty}^{\infty} f(x,y)dx$

Conditional distributions:

(i) Discrete case:

- The conditional probability function of X given Y=y $_j$ is given by

$P[X = x_i / Y = Y_j] = P[X = x_i, Y = y_j] / P[Y = y] = P_{ij} / P * j$

The set $\{X = x_i, P_{ij} / P * j\}$, I =1,2,3,…is called the conditional probability distribution of X given $Y = y_j$

- The conditional probability function of Y given X=xi is given by

$P = [Y = y_j \; / \; X = x_i] = \; P[Y = y_j, X = x_i] / P[X = x_i] = P_{ij} / P_i^*$

The set $\{y_{ij} \quad P / P_i^*\}$, j=1,2,3,…is called the conditional probability distribution of Y given $X = x_i$

(ii) Continuous case:

- The conditional probability density function of X is given by $Y = y_j$ is defined as

$f(x/y) = \dfrac{f(x,y)}{h(y)}$, where h(y) is a marginal probability density function of Y.

- The conditional probability density function of Y is given by $X = x_i$ is defined as

$f(y/x) = \dfrac{f(x,y)}{g(x)}$, where g(x) is a marginal probability density function of X.

Independent random variables:

(i) Discrete case:

Two random variable (x,y) are said to be independent if $P(X = x_i \cap Y = y_j) = P(X = x_i)(Y = y_j)$ (ie) $P_{ij} = P_i^* P_{*j}$ for all i ,j.

(ii) Continuous case:

Two random variables (x,y) are said to be independent if $f(x,y) = g(x)h(y),$ where $f(x,y) =$ joint probability density function of x and y,

$g(x) =$ Marginal density function of x,

$h(y) =$ Marginal density function of y.

Marginal Distribution Tables:

Table – I

To calculate marginal distribution when the random variables X takes horizontal values and Y takes vertical values

Y/X	x1	x2	x3	p (y) = p(Y=y)
y1	p11	p21	p31	p(Y=y1)
y2	p12	p22	p32	p(Y=y2)
y3	p13	p23	p33	p(Y=y3)
$P_X(X) = P(x = x)$	$P(x = x1)$	$p(x = x2)$	$p(x = x3)$	

Table – II

To calculate marginal distribution when the random variables X takes vertical values and Y takes horizontal values

Y\X	y1	y2	y3	P $_X$ (x) = P(X=x)
x1	p11	p21	p31	p(X=x1)
x2	p12	p22	p32	p(X=x2)
x3	p13	p23	p33	p(X=x3)
$p(y) = p(y = y)$	$P(y = y1)$	$P(y = y2)$	$P(y = y3)$	

Solved Problems on Marginal Distribution:

Example :1

From the following joint distribution of X and Y find the marginal distribution

X/Y	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

Solution:

The marginal distribution are given in the table below

Y\X	0	1	2	$P_Y(y) =$ P(Y=y)
0	3/28	9/28	3/28	15/28
1	3/14	3/ 14	0	6/14
2	1/28	0	0	1/28
$P_X(x) = P(Y = y)$	$P_X(0) =$ 5/14	$P_X(1) =$ 15/28	$P_X(2) =$ 3/28	1

The marginal Distribution of X

$P_X(0) = P(X = 0) = p(0,0) + p(0,1) + p(0,2)$ = 3/28 + 3/14 + 1/28 = 5/14

$P_X(1) = P(X = 1) = p(1,0) + p(1,1) + p(1,2)$ = 9/28 + 3/14 + 0 = 15/28

$P_X(2) = P(X = 2) = p(2,0) + p(2,1) + p(2,2)$ = 3/28 + 0 + 0 = 3/28

Marginal probability function of X is $P_x(x) = \begin{cases} 5/14, & x = 0 \\ 15/28, & x = 1 \\ 3/28, & x = 2 \end{cases}$

The marginal distributions are

Y/X	1	2	3	$P_Y(y) = p(y = y)$
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21

	5/21	7/21	9/21	1

The marginal distribution of X

$P_x(1) = p(1,1) + (1,2)$ = 2/21 + 3/21

P_x (1) =5/21

$P_x(2) = p(2,1) + (2,2)$ = 3/21 + 4/21

P_x (2) =7/21

$P_x(3) = p(3,1) + p(3,2)$ = 4/21 + 5/21

P_x (3) =9/21

Marginal probability function of X is, $P_x(x) = \begin{cases} 5/21, & x=1 \\ 7/21, & x=2 \\ 9/21, & x=3 \end{cases}$

The marginal distribution of Y

$P_Y(1) = p(1,1) + p(2,1) + p(3,1)$ = 2/21 + 3/21 +4/21

$P_Y(1) = 9/21$

$P_Y(2) = p(1,2) + p(2,2) + p(3,2)$ = 3/21 + 4/21 +5/21

$P_Y(2) = 12/21$

Marginal probability function of Y is $P_Y(y) = \begin{cases} 3/21, & y=1 \\ 4/21, & y=2 \\ & \end{cases}$

Example :2

From the following table for joint distribution of (X, Y) find

i) $P(X \le 1)$ ii) $P(Y \le 3)$ iii) $P(X \le 1, Y \le 3)$ iv) $P(X \le 1/Y \le 3)$

v) $P(Y \le 3/X \le 1)$ vi) $P(X + Y \le 4)$.

X/Y	0	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32

1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Solution:

The marginal distributions are

X / Y	1	2	3	4	5	6	$P_X(x) = P(X = x)$
0	0	0	1/32	2/32	2/32	3/32	8/32 P(x=0)
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16 P(x=1)
2	1/32	1/32	1/64	1/64	0	2/64	8/64 P(x=2)
$P_Y(y) = P(Y = y)$	3/32	3/32	11/64	13/64	6/32	16/64	1
	P(Y=1)	P(Y=2)	P(Y=3)	P(Y=4)	P(Y=5)	P(Y=6)	

i) $P(X \le 1)$

$P(X \le 1) = P(X = 0) + P(X = 1)$

= 8/32 + 10/16

$P(X \le 1)$ = 28/32

ii) $P(Y \le 3)$

$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$

= 3/32 + 3/32 + 11/64

$P(Y \le 3)$ = 23/64

iii) $P(X \le 1, Y \le 3)$

$P(X \le 1, Y \le 3)$ = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)

= 0+0+1/32 + 1/16 + 1/16 + 1/8

$P(X \le 1, Y \le 3)$ = 9/32

iv) $P(X \le 1 / Y \le 3)$

By using definition of conditional probability

$$P[x = x_i / y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]}$$

The marginal distribution of Y

$P_Y(0) = P(Y = 0) = p(0,0) + p(1,0) + p(2,0)$ = 3/28+9/28+3/28 = 15/28

$P_y(1) = P(y = 1) = p(0,1) + p(1,1) + p(2,1)$ = 3/14+3/14+0= 3/7

$P_y(2) = P(y=2) = p(0,2) + p(1,2) + p(2,2)$ = 1/28+0+0 = 1/2

Marginal probability function of Y is $P_y(Y) = \begin{cases} 15/28, & y=0 \\ 3/7, & y=1 \\ 1/28, & y=2 \end{cases}$

Example 3:

The joint distribution of X and Y is given by f(X, Y) =X+Y/21, x=1,2,3 y=1,2.Find the marginal distributions.

Solution:

Given f (X, Y) =X+Y/21, x=1, 2, 3 y=1,2

f (1,1) = 1+1/21 =2/21 =P(1,1)

f (1,2) = 1+2/21 =3/21 =P(1,2)

f (2,1) = 2+1/21 =3/21 =P(2,1)

f (2,2) = 2+2/21 =4/21 =P(2,2)

f (3,1) = 3+1/21 =4/21 =P(3,1)

f (3,2) = 3+2/21 =5/21 =P(3,2)

$$P[X \le 1 / Y \le 3] = \frac{P[X \le 1, Y \le 3]}{P[Y \le 3]} = \frac{9/23}{23/64}$$

$P[X \le 1 / Y \le 3]$ = 18/32

v) $P[Y \le 3 / X \le 1]$

$$P[Y \le 3 / X \le 1] = \frac{P[X \le 3, Y \le 1]}{P[Y \le 1]} = \frac{9/23}{7/8}$$

P[Y≤3/ X≤1]= 9/28

vi) $P(X + Y \le 4)$

P(X + Y≤4)) = P(0,1)+P(0,2)+P(0,3)+ P(0,4)+ P(1,1)+
P(1,2)+P(1,3)+P(2,1)+ P(2,2)
= 0+0+1/32 +2/32 + 1/16 + 1/16 + 1/8+1/32 +1/32

P(X + Y≤4) = 13/32

Example : 4

If the joint P.D.F of (X,Y) is given by p(X,Y)=K(2x+3y),x=0,1,2, y=1,2,3,. Find all the marginal probability distribution .Also find the probability of (X+Y) and P(X+Y >3).

Solution:

Given P(X,Y)= K(2x+3y)

P(0,1)= K(0+3) = 3K

P(0,2)= K(0+6) = 6K

P(0,3)= K(0+9) = 9K

P(1,1)= K(2+3) = 5K

P(1,2)= K(2+6) = 8K

P(1,3)= K(2+9) = 11K

P(2,1)= K(4+3) = 7K

P(2,2)= K(4+6) = 10K

P(2,3)= K(4+9) = 13K

To find K:

The marginal distribution is given in the table.

Y\X	0	1	2	$P_Y(y) = P(Y = y)$
1	3K	5K	7K	15K
2	6K	8K	10K	24K
3	9K	11K	13K	33K
PX(x)=P(X=x)	18K	24K	30K	72K

Total Probability =1

72K = 1

$\boxed{K = 1/72}$

Marginal probability of X & Y:

Substituting K = 1/72 in the above table, we get

Y\X	0	1	2	P_Y (y)=P(Y=y)

1	3/72	5/72	7/72	5/24
2	6/72	8/72	10/72	1/3
3	9/72	11/72	13/72	11/24
P_X (x)=P(X=x)	1/4	11/72	5/12	1

From table, $P_x(0)=1/4$, $p_x(1)=1/3$, $p_x(2)=5/12$

$$P_x(X)=\begin{cases}1/4, x=0\\ 1/3, x=1\\ 5/2, x=2\end{cases}$$

Marginal probability function of x is ,

From table, $p_y(1)=5/24$, $P_y(2)=1/3$, $P_Y(3)=11/24$

$$\mathrm{P_Y(Y)}=\begin{cases}5/24, \mathrm{Y}=1\\ 11/24, \mathrm{y}=2\\ \end{cases}$$

Marginal Probability function of Y is ,

Example :5

From the following table for joint distribution of (X, Y) find

The marginal distributions are

Y/X	1	2	3	$P_Y(y)=P(Y=y)$
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
$P_X(x)=P(X=x)$	5/21	7/21	9/21	1

The marginal distribution of X

P $_X$ (1) = P(1,1)+P(1,2) = 2/21 + 3/21= P ^{X} (3)=9/21

P $_X$ (2) = P(2,1)+P(2,2) = 3/21 + 4/21= Py(2)=7/21

P $_X$ (3) = P(3,1)+P(3,2) = 4/21 + 5/21= P $_X$ (3)=9/21

Marginal probability function of X is $P_x(x) = \begin{cases} 5/21, x=1 \\ 7/21, x=2 \\ 9/21, x=3 \end{cases}$

The marginal distribution of Y

$P_y(1)$ = P (1, 1) + P (2, 1) +P (3, 1)

= 2/21 + 3/21 +4/21= 9/21

$P_y(2)$ = P (1, 2) + P (2, 2) +P (3, 2)

= 3/21 + 4/21 +5/21= 12/21

Marginal probability function of Y is $P_Y(y) = \begin{cases} 3/21, y=1 \\ 4/21, y=2 \end{cases}$

Exercises:

1. Given is the joint distribution of X and Y

Y/X	0	1	2
0	0.02	0.08	0.10
1	0.05	0.20	0.25
2	0.03	0.12	0.15

Obtain 1) Marginal Distribution.

2) The conditional distribution of X given Y =0.

2. The joint probability mass function of X & Y is

X/Y	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the M.D.F of X and Y. Also $(X \le 1, Y \le 1)$ and check if X & Y are independent.

3. Let X and Y have the following joint probability distribution

Y/X	2	4
1	0.10	0.15

3	0.20	0.30
5	0.10	0.15

Show that X and Y are independent.

4. The joint probability distribution of X and Y is given by the following table.

X/Y	1	3	9
2	1/8	1/24	1/12
4	¼	¼	0
6	1/8	1/24	1/12.

i) Find the probability distribution of Y.

ii) Find the conditional distribution of Y given X=2.

ii) Are X and Y are independent.

5. Given the following distribution of X and Y. Find

i) Marginal distribution of X and Y.
ii) The conditional distribution of X given Y=2.

X/Y	-1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

Example : 6

If the joint probability density function of (X, Y) is given by $f(x, y) = 2, \quad 0 \le x \le y \le 1.$. Find marginal density function of X.

Solution:

Given $f(x, y) = 2$, $0 \le x \le y \le 1$

To find marginal density function of x:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{x}^{1} 2dy = 2[1 - x], \quad 0 \le x \le y.$$

Example:7

If the joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0\ , & \text{otherwise}\end{cases}$$

Find (i) $P(X<1\cap Y<3)$ (ii) $P\left(X<1/Y<3\right)$ (iii) $f\left(\frac{y}{x}\right)$.

Solution:

Given $$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0\ , & \text{otherwise}\end{cases}$$

i) **To find** $P(X<1\cap Y<3)$**:**

$$P(X<1\cap Y<3)=\int_0^1\int_2^3 f(x,y)\,dy\,dx$$

$$=\frac{1}{8}\int_0^1\int_2^3(6-x-y)\,dy\,dx$$

$$=\frac{3}{8}$$

ii) **To find** $P\left(X<1/Y<3\right)$

$$P\left(X<1/Y<3\right)=\frac{P(X<1\quad Y<3)}{P(Y<3)}\quad ....(1)$$

To find $P(Y<3)$:

$$P(Y<3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$=\int_0^2\int_2^3\frac{1}{8}(6-x-y)\,dy\,dx$$

$$=\frac{5}{8}$$

Equation (1) becomes $P\left(X<1/Y<3\right)=\frac{3}{5}$

iii) **To find** $f(y/x)$:

We know that $f(y/x) = \dfrac{f(x,y)}{f_x(x)}$

$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy$ $= \dfrac{1}{8}\int_{2}^{4}(6-x-y)dy$

$= \frac{1}{4}(3-x), 0 < x < 2.$

$0 < x < 2, \quad 2 < y < 4.$

$$f(y/x) = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(3-x)} = \frac{6-x-y}{2(3-x)},$$

Example : 8

If the joint distribution of X and Y is given by

$F(x,y) = (1-e^{-x})(1-e^{-y})$, for $x > 0, y > 0$

$= 0$, otherwise

(i) Find the marginal densities of X and Y (ii) Are X and Y independent?

(iii) $P(1 < X < 3, 1 < Y < 2)$

Solution:

Given $F(x,y) = (1-e^{-x})(1-e^{-y})$

$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$

The joint pdf is given by $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y}$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y}(1 - e^{-x} - e^{-y} + e^{-(x+y)})$$

$= e^{-(x+y)}$

$f(x,y) = e^{-(x+y)}, x \ge 0, y \ge 0$

i) The marginal density function of X is $f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$$f(x) = \int_{0}^{\infty} e^{-(x+y)}dy = e^{-x}, x \ge 0$$

The marginal density function of Y is $f(y) = \int_{-\infty}^{\infty} f(x,y)dx$

$$f(y) = \int_0^{\infty} e^{-(x+y)} dx = e^{-y}, y \geq 0$$

ii) Consider $f(x).f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$

ie) X and Y are independent.

iii) $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3).P(1 < Y < 2)$

$$= \int_1^3 f(x)dx.\int_1^2 f(y)dy = \int_1^3 e^{-x}dx \int_1^2 e^{-y}dy$$

$$= \frac{(1-e^2)(1-e)}{e^5}$$

Exercises:

1. The joint p.d.f. of the two dimensional random variable is,

$$f(x,y) = \begin{cases} \frac{8xy}{9}, & 1 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

   (i) Find the marginal density functions of X and Y.

   (ii) Find the conditional density function of Y given X=x.

2. If the joint Probability density function of two dimensional R.V (X,Y) is given by

$$f(x,y) = \begin{cases} \mathrm{x}^2 + \frac{\mathrm{xy}}{3}, & 0 \leq \mathrm{x} \leq 1, 0 \leq \mathrm{y} \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

   Show that X and Y are not independent.

Covariance

It is useful to measure of the relationship between two random variables is called covariance. To define the covariance we need to describe the expected value of a function of two random variables C(x,y).

Covariance:

If X and Y are random variables, than covariance between X and Y is defined as

$$Cov(X,Y) = E\{[X - E(x)][Y - E(y)]\}$$

$$= E\{XY - XE(Y) - E(x)y + E(X)E(Y)\}$$

$$= E(XY) - E(X) - E(Y) - E(X)E(Y) + E(X)E(Y)$$

Covariance (X, Y) = $E(XY) - E(X)E(Y)$ ……………. (A)

If X and y are independent, then $E(XY) = E(X)E(Y)$ ……………… (B)

Substituting (B) in (A), we get Covariance (x, y) = 0

If X and Y are independent, then $Cov(X_, Y) = 0$

Correlation:

If the change in are variable affects a change in the other variable, the variable are said to be correlated In a invariable distribution we may be interested to find out if there is any correlation or co-variance between the two variables under study.

Types of correlation:

1) Positive correlation
2) Negative Correlation

Positive Correlation:

If the two variables deviate in the same direction i.e. If the increase (or decrease) in one results in a corresponding increase (or decrease) in the other, correlation is said to be direct or positive.

Example: The Correlation between

a) The height, and weight of a group of person and
b) Income and expenditure

Negative Correlation:

If the two variable constancy deviate in opposite directions i.e. if (increase 9or decrease) in one result in corresponding decrease (or increase) in the other correlation , is said to be negative.

Example: The Correlation between

a) Price and demand of a commodity and
b) The correlation between volume and pressure of a perfect gas.

Measurement of Correlation:

We can measure the correlation between the two variables by using Karl-Pearson's co -efficient of correction.

Karl-Pearson's Co-Efficient of Correlation:

Correlation co-efficient between two random variable X and Y usually denotes by (X,Y) is a numerical measure of linear. Karl Pearson's co-efficient of correlation between x & y is

r = 1 – 6 $\sum_{i=1}^{n} d_i^{\,2} / n(n^2-1)$, , where $d_i = x_i - y_i$

Relationship between them and detained as

$$r(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y} \quad \text{Where COV}(X,Y) = \frac{1}{n}\sum XY - \overline{XY}$$

$$\sigma_X = \sqrt{\frac{1}{n}\sum X^2 - \overline{X_2}}, \ \overline{X} = \frac{\sum X}{n}$$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum Y^2 - \overline{Y_2}}$$

(n is the number of items in the given data)

Note:

1. Correlation coefficient may also be denoted by r(x,y)

2. If r(x,y) = 0, we say that x & y are uncorrelated.

3. When r = 1 , the correlation is perfect.

Example :9

Calculate the Correlation co-efficient for the following heights (in inches) of father x and their sons y.

Solution:

Method : 1

X	**Y**	**XY**	**X^2**	**Y^2**
67	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184

70 72	69 71	4836 5112	4900 5184	4761 5041
$\sum(x)$ =544	$\sum$ (y) = 552	$\sum XY$ = 37560	$\sum x^2$ = 37028	$\sum y^2$ = 38132

Now

$\overline{X} = 544/8$ =68

$\overline{Y} = 544/8$ =69

$\overline{X}\ \overline{Y} = 68*69 = 4692$

$\sigma_X = \sqrt{1/n \sum x^2 - \overline{x}^2}$

$= \sqrt{37028/8 - 4624}$ =2.121

$= \sqrt{38132/8 - 4761}$ =2.345

r(X,Y)= $\dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

=1/n $\sum xy - \overline{x\ y} / \sigma_x . \sigma_y$

=1/8*37560-4692/2.121*2.345

= 3/4.973

=0.6032

It is positive correlation.

Example:10 Find the co-efficient of Correlation between industrial productions and expose using the following data

Production (x)	55	56	58	59	60	60	62
Export (y)	35	38	37	39	44	43	44

Solution :

X	Y	U =X-58	V =Y-40	UV	U2	V2
55	35	-3	-5	15	9	25
56	38	-2	-2	4	4	4

58	37	0	-3	0	0	9
59	39	1	-1	-1	1	1
60	44	2	4	8	4	16
60	43	2	3	6	4	9
62	44	4	4	16	16	16
		$\sum U=4$	$\sum U=0$	$\sum UV=48$	$\sum U^2 = 38$	$\sum V^2 = 8$

Now $\overline{U} = \sum U / n = 4/7$ =0.5714

$\overline{V} = \sum V / n = 0$ ……… (1)

$\sigma_U = \sqrt{\sum U^2 / n - \overline{U}^{\,2}} = \sqrt{38/7 - (0.5714)^2}$ =2.2588……………….. (2)

$\sigma_V = \sqrt{\sum V^2 - \overline{V} = \sqrt{80/7 - 0 =}}$ 3.38………… (3)

$\therefore r = (X,Y) = r(U,V) = COV(U,V) / \sigma_U * \sigma_V = 6.857/2.258 * 3.38$ = 0.898[using (1), (2) & (3)]

r = 0.79

The value between 0 to 1. So it is positive correlation.

Example :11

Find the Correlation co-efficient for the following data.

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution:

X	**Y**	**U =X-22/4**	**V =Y-24/6**	**UV**	**U^2**	**V^2**
10	18	-3	-1	3	9	1
14	12	-2	-2	4	4	4
18	24	-1	0	0	1	0
22	6	0	-3	0	0	9
26	30	1	1	1	1	1
30	36	2	2	4	4	4

		$\sum U=-3$	$\sum V=-3$	$\sum UV=12$	$\sum U^2 = 19$	$\sum V^2 = 19$

Now $\overline{U} = \sum U / n = -3/6 = -0.5$ ………………(1)

$\overline{V} = \sum V / n = -3/6 = 0.5$ …………………(2)

$COV(U,V) = \dfrac{\sum UV}{n\overline{UV}}$ =1.75

$\sigma_U = \sqrt{\sum U^2 - \overline{U}^2}$

$= \sqrt{19/6 - (0.5)^2} = 1.708$ ……… (3)

$\therefore r(x,y) = 0.6$

The value between 0 to 1. So it is positive correlation

Rank Correlation:

Let us suppose that a group of n individuals are arranged in order of merit or proficiently in possession of two characteristics A & B.

r = 1 – 6 $\sum_{i=1}^{n} d_i^2 / n(n^2 - 1),$, where $d_i = x_i - y_i$

Note:

This formula is called a Spearman's formula .

Solved Problems on Rank Correlation:

Example :12

Find the rank correlation co-efficient from the following data:

Rank in X	1	2	3	4	5	6	7
Rank in Y	4	3	1	2	6	5	7

Solution

X	Y	$d = x_i - y_i$	di^2
1	4	-3	9

2	3	-1	1
3	1	2	4
4	2	2	4
5	6	-1	1
6	5	1	1
7	7	0	0
		$\sum d_i = 0$	$\sum d_i^2 = 20$

Rank Correlation co-efficient

$$r = 1 - 6 \sum_{i=1}^{n} d_i^2 / n(n^2-1), \quad \text{where } d_i = x_i - y_i$$

$$= 1 - 6 \times 20 / 7(49-1) = 0.6429$$

Example : 13 The ranks of some 16 students in mathematics & physics are as follows. Calculate rank correlation co-efficient for proficiency in mathematics & physics.

Rank in Mathematics	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Rank in Physics	1	10	3	4	5	7	2	9	8	11	15	9	14	12	16	13

Solution:

Rank in Mathematics(X)	Rank in Physics(Y)	$d_i = X_i - Y_i$	d_i^2
1	1	0	0
2	10	-8	64
3	3	0	0
4	4	0	0
5	5	0	0

6	7	-1	1
7	2	5	25
8	9	-1	1
9	8	1	1
10	11	-1	1
11	15	-4	16
12	9	3	9
13	14	-1	1
14	12	2	4
15	16	-1	1
16	13	3	9
		$\sum d_i = 0$	$\sum d_i^2 = 136$

Rank correlation co-efficient

$$r = 1 - 6\sum_{i=1}^{n} d_i^2 / n(n^2-1), \quad \text{where } d_i = x_i - y_i$$

r = 0.8

Example : 14

10 competitors in a musical test were ranked by the 3 judges X, Y, Z in the following order

	A	B	C	D	E	F	G	H	I	J
Rank in X	1	6	5	10	3	2	4	9	7	8
Y	3	5	8	4	7	10	2	1	6	9
Z	6	4	9	8	1	2	3	10	5	7

Using Rank correlation method, discuss which panel of Judges has the nearest approach to common likings of music.

X	Y	Z	$D_1 = x_i - y_i$	$D_2 = y_i - z_i$	$D_3 = x_i - z_i$	D_1^2	D_2^2	D_3^2

1	3	6	-2	-3	-5	4	9	25
6	5	4	1	1	2	1	1	4
5	8	9	-3	-1	-4	9	1	16
10	4	8	6	-4	2	36	16	4
3	7	1	-4	6	2	16	36	4
2	10	2	-8	8	0	64	64	0
4	2	3	2	-1	1	4	1	1
9	1	10	8	-9	-1	64	81	1
7	6	5	1	1	2	1	1	4
8	9	7	-1	2	1	1	4	1

$\sum d_1^2 = 200$ $\sum d_2^2 = 214$ $\sum d_3^2 = 60$

The rank correlation between X & Y is

$$r_1 = 1 - 6 \sum_{i=1}^{n} d_i^2 / n(n^2 - 1) = -0.212$$

The rank correlation between Y& Z is

$$r_2 = 1 - 6 \sum_{i=1}^{n} d_i^2 / n(n^2 - 1) = -0.296$$

The rank correlation between X & Z is

$$r_3 = 1 - 6 \sum_{i=1}^{n} d_i^2 / n(n^2 - 1) = 0.636$$

Since the rank correlation between X & Z is maximum and also positive, We conclude that the pair of Judges X & Z has the nearest approach to common likings of music.

Exercises:

1) Calculate the Karl Pearson's co-efficient of correlation from the following data

X	25	26	27	30	32	35
Y	20	22	24	25	26	27

2) Find the co-efficient of correlation of the advertisement cost & sales from the following data

Cost:	39	65	62	90	82	75	98	36	78
Sales:	47	53	58	86	62	68	91	51	84

REGRESSION

Definition:

Regression is a mathematical measure of the average relationship between two or more variables in terms of the original limits of the data.

Lines of regression:

If the variables in a bivariate distribution are related we will cluster around some curve called of regression. If the curve is a straight line, it is and called the line of regression and there is said to be linear regression is said to be curve linear.

The line of regression of y on x is given by $y-\overline{y} = r.\frac{\partial y}{\partial x}(x-\overline{x})$

where r is the correlation coefficient, ∂_y and ∂_X are standard deviation.

The line of regression of X on Y is given by $x-\overline{x} = r.\frac{\partial y}{\partial x}(y-\overline{y})$

Angle between two line of regression:

If the equation of lines of regression of Y on X and X on Y are

$$y-\overline{y} = r.\frac{\partial y}{\partial x}(x-\overline{x}) \quad \text{and} \quad x-\overline{x} = r.\frac{\partial x}{\partial y}(y-\overline{y})$$

The angle $'\theta'$ between the two line of regression is given by

$$\tan\theta = \frac{l-r^2}{r}\left(\frac{\partial y \partial x}{\partial x^2+\partial y^2}\right)$$

Regression coefficients:

Regression coefficient of Y on X , $r\frac{\partial Y}{\partial X} = b_{YX}$ …………. (1)

Regression coefficient of X on Y , $r\frac{\partial X}{\partial Y} = b_{XY}$ ………….. (2)

From (1) and (2) we get

$$r\frac{\partial Y}{\partial X}r\frac{\partial X}{\partial Y} = b_{YX} * b_{YX}$$

Correlation coefficient $r = \pm\sqrt{b_{XY} * b_{YX}}$

The regression coefficients b_{YX} and b_{yx} can be easily obtained by using the following formula.

$$b_{YX} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$$

$$b_{XY} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}$$

Solved Problems on Regression:

Example:15

The equations of two regression lines are 3x+12y=19, 3y+9x=46. Obtain the mean value of X and Y.

Solution:

Given the lines are 3x+12y=19,

3y+9x=46

Since both are passing through $(\bar{x}, \bar{y})$, we get

$3\bar{x} + 12\bar{y} = 19$ …………(1)

$9\bar{x} + 3\bar{y} = 46$ …………..(2)

Solving equation (1) & (2) we get $33\bar{y} = 11$

$\bar{y} = \frac{11}{33} = 0.33$, $\bar{y}$ value sub in equation (1) we get $\bar{x} = 5$

$(\bar{x}, \bar{y})$ = (5, 0.33)

Example:16

From the following data, find

i)The two regression equations.

ii) The co-efficient of correlation between the marks in economics and statistics.

iii)The most likely marks in statistics when marks in economics are 30.

Marks in Economics	25	28	35	32	31	36	29	38	34	32
Marks in Statistics	43	46	49	41	36	32	31	30	33	39

Solution:

X	Y	$X-\overline{X}=X-32$	$Y-\overline{Y}=Y-38$	$(X-\overline{X})^2$	$(Y-\overline{Y})^2$	$(X-\overline{X})(Y-\overline{Y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0
$\sum X=$ 320	$\sum Y=$ 380	$\sum(X-\overline{X})=0$	$\sum(Y-\overline{Y})=0$	$\sum(X-\overline{X})^2=$ 140	$\sum(Y-\overline{Y})^2=$ 398	$\sum(X-\overline{X})(Y-\overline{Y})=$ -93

Here $\overline{X}=\frac{\sum X}{n}$ and $\overline{Y}=\frac{\sum y}{n}$

$=\frac{320}{10}=32$ $\qquad =\frac{380}{10}=38$

Coefficient of regression of Y on X is

$$b_{YX} = \frac{\sum (X-\overline{X})(Y-\overline{Y})}{\sum (X-\overline{X})^2} = \frac{-93}{140}$$

Coefficient of regression of X on Y is

$$b_{XY} = \frac{\sum (X-\overline{X})(Y-\overline{Y})}{\sum (Y-\overline{Y})} = \frac{-93}{398} = -0.2337$$

Equation of the line of regression of X on Y is

$$x-\overline{x} = b_{XY}(y-\overline{y})$$

X – 32 =0.2337(Y-38)

X= -0.2337 y+0.2337 *38 +32

X=-0.23374 +40.8806

Equation of the line of regression of Y on X is

$$y-\overline{y} = b_{yx}(x-\overline{x})$$

Y-38 = -0.6643(x-32)

Y = - 0.6643 x+38+0.6643*32 =-0.6642x+59.2576

Now we have to find the most likely marks in statistics (Y) when marks in economics (X) are 30.we use the line of regression of Y on X.

Y = -0.6643x+59.2575

Put x=30, we get

Y = -0.6643*30+59.2536 =39.3286 =39

Example :17

Height of father and sons are given in centimeters

X:Height of father	150	152	155	157	160	161	164	166
Y:Height of son	154	156	158	159	160	162	161	164

Find the two lines of regression and calculate the expected average height of the son when the height of the father is 154 cm.

Solution:

Let 160 and 159 be assured means of x and y.

x	y	U=X-160	V=Y-159	u^2	v^2	uv
150	154	-10	-5	100	25	50
152	156	-8	-3	64	9	24
155	158	-5	-1	25	1	5
157	159	-3	0	9	0	0
160	160	0	1	0	1	0
161	162	1	3	1	9	3
164	161	4	2	16	4	8
166	164	6	5	36	25	30
		$\sum U = -15$	$\sum V = 2$	$\sum U^2 = -15$	$\sum V^2 = 74$	$\sum UV = 120$

Now $\overline{X} = 158.13$ and $\overline{Y} = 159.25$

Since regression coefficient are independent of change and of origin we have regression coefficient of Y on X

Coefficient of regression of Y on X is

$$b_{YX} = \frac{\sum (X-\overline{X})(Y-\overline{Y})}{\sum (X-\overline{X})^2} = \frac{990}{1783} = 0.555$$

Coefficient of regression of X on Y is

$$b_{XY} = \frac{\sum (X-\overline{X})(Y-\overline{Y})}{\sum (Y-\overline{Y})} = 1.68.$$

Exercise:

1. The two lines of regression are $8x - 10y = 66$ *and* $40x - 18y - 214 = 0$.The variance of X is 9. Find i) the mean values of X and Y ii) Correlation between X and Y.

MA2262 / 80250008 / MA1252 / MA44 / 10177 PQ401
PROBABILITY AND QUEUEING THEORY

NOTES

SEMESTER-IV

QUESTION BANK – III
UNIT– III – MARKOV PROCESSES AND MARKOV CHAINS

PART -A

Problem 1 Distinguish between wide sense stationary (WSS) and strict sense stationary (SSS) random processes.

Solution:

A process is said to be wide sense stationary if it satisfies the following conditions.

i) $E\left[X(t)\right] = \overline{X} =$ Constant

ii) $E\left[X(t)X(t+\tau)\right] = R_{XX}(\tau)$, a function of τ only.

A process is said to be strict sense stationary of it is stationary if all its finite dimensional distributions are invariant under translation of time parameter. i.e.,
$f_X(x_1, x_2, \ldots x_n; t_1, t_2, \ldots t_n) = f_X(x_1, x_2, \ldots x_n; t_1+\Delta, t_2+\Delta, \ldots t_n+\Delta)$ for all $t_1, t_2, \ldots . t_n$ and any real number Δ.

Problem 2 Find the invariant probabilities for the Markov chain $\{X_n : n \geq 1\}$ with state space $\{0,1\}$ and one step tpm $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$.

Solution:

$$\pi P = \pi \;\; is \;\; [\pi_0, \pi_1]\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [\pi_0, \pi_1] \, with \, \pi_0 + \pi_1 = 1$$

$$\Rightarrow \frac{1}{2}\pi_1 = \pi_0 \; and \; \pi_0 + \frac{1}{2}\pi_1 = \pi_1.$$

$$\Rightarrow 2\pi_0 = \pi_1 \;\; and \;\; 2\pi_0 = 1 - \pi_0 \Rightarrow 3\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{3} \therefore \pi_1 = \frac{2}{3}$$

Problem 3 What is a Markov process?

Solution:

If for $t_1 < t_2 < t_3 < \ldots\ldots\ldots t_n < t$, $P\left[X(t) \leq x / X(t_1) = x_1, X(t_2) = x_2 \ldots . X(t_n) = x_n\right]$

$$= P\left[X(t) \leq x / X(t_n) = x_n\right],$$

then the process $\{X(t)\}$ is called a Markov Process.

Problem 4 Define irreducible Markov chain.

Solution:

If in the tprn, $P_{ij}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be irreducible.

Problem 5 State four properties of a Poisson process.

Solution:

i) Poisson process is a Markov process.
ii) Sum of two independent Poisson processes is a Poisson process.
iii) The difference of two independent Poisson processes is not a Poisson process.
iv) The inter-arrival time of a Poisson process with parameter λ has a exponential distribution with mean $\frac{1}{\lambda}$.

Problem 6 Consider the random process $X(t) = \cos(\omega_0 t + \theta)$ where θ is uniformly distributed in the interval $-\pi$ *to* π. Check whether {X (t)} is stationary or not?

Solution:

Since θ is uniformly distributed in $(-\pi, \pi)$ $f(\theta) = \frac{1}{2\pi}$ $-\pi < 0 < \pi$

$$E[X(t)] = E[Cos(\omega_0 t + \theta)] = \int_{-\pi}^{\pi} Cos(\omega_0 t + \theta).\frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi}\left[-Sin(\omega_0 t + \theta)\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{2\pi}\left[Sin(\pi + \omega_0 t) - Sin(\omega_0 t - \pi)\right]$$

$$= \frac{-1}{2\pi}\left[Sin(\omega_0 t) + Sin(\pi - \omega_0 t)\right]$$

$$= \frac{-1}{2\pi}\left[2.Sin(\omega_0 t)\right] = -\frac{Sin\omega_0 t}{\pi}$$

which is a function of t. Therefore $\{X(t)\}$ is not a stationary process.

Problem 7 Prove that a first order stationary process has a constant mean.

Solution:

Consider a random process $\{X(t)\}$ at two different instants t_1 and t_2.

$$E[X(t_1)] = \int_{-\infty}^{\infty} x\, f_x(x, t_1)\, dx$$

$$E[X(t_2)] = \int_{-\infty}^{\infty} x\, f_x(x, t_2)\, dx$$

$$t_2 = t_1 + C$$

$$E[X(t_2)] = \int_{-\infty}^{\infty} x\, f_x(x, t_1 + C)\, dx = \int_{-\infty}^{\infty} x\, f_x(x, t_1)\, dx \quad (\because X(t) \text{ is first order stationary })$$

$$= E\left[X\left(t_1\right)\right]$$

$\therefore E\left[X\left(t_1\right)\right] = E\left[X\left(t_2\right)\right] \quad \therefore E\left[X\left(t\right)\right]$ is a constant.

Problem 8 If patients arrive at a clinic according to poisson process with mean rate of 2 per minutes, find the probabilities that during a 1-minute interval, no patient arrives.

Solution:

Mean arrival rate $= \lambda = 2/$ min.

P[no patient arrives during a 1-minute interval]

$$= P\left[X\left(t\right) = 0\right] = \frac{e^{-\lambda t}\left(\lambda t\right)^0}{0!} = e^{-2} = 0.135$$

Problem 9 Define Poisson random process. Is it a stationary process? Justify the answer.

Solution:

If $X\left(t\right)$ represents the number of occurrences of a certain event in $\left(0,t\right)$, then the discrete random process $\left\{X\left(t\right)\right\}$ is called the Poisson process, provided the following postulates are satisfied.

i) P[1 occurrence in $\left(t, t+\Delta t\right)$] $= \lambda \Delta t + 0\left(\Delta t\right)$

ii) P[0 occurrence in $\left(t, t+\Delta t\right)$] $= 1 - \lambda \Delta t + 0\left(\Delta t\right)$

iii) P[2 or more occurrences in $\left(t, t+\Delta t\right)$] $= 0\left(\Delta t\right)$

iv) $X\left(t\right)$ is independent of the number of occurrences of the event in any interval prior and after the interval $\left(0,t\right)$.

v) The probability that the event occurs a specified number of times in $\left(t_0, t_0 + t\right)$ depends only on t, but not on t_0. Poisson process is not a stationary process, as its satisfied properties are time dependent.

Problem 10 What is meant by steady state distribution of a Markov chain?

Solution:

If a homogenous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called steady state distribution of Markov chain.

Problem 11 What is the super position of n independent Poisson processes with respective average rate $\lambda_1, \lambda_2, ...\lambda_n$?

Solution:

The super position of n independent Poisson processes with average rates $\lambda_1, \lambda_2.......\lambda_n$ is another Poisson process with average rate $\lambda_1 + \lambda_2 + ... + \lambda_n$.

Problem 12 What is a stochastic matrix? When is it said to be regular?

Solution:

If $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ for all i, then the matrix (P_{ij}) is called a stochastic matrix.

A stochastic matrix is said to be a regular matrix, if all the entries of P^m(for some positive integer m) are positive.

PART –A

Problem 13 Two random processes $X(t)$ and $Y(t)$ are defined by $X(t) = A\cos\omega\tau + B\sin\omega\tau$ and $Y(t) = B\cos\omega\tau - A\sin\omega\tau$ show that $X(t)$ and $Y(t)$ are jointly wide sense stationary if A and B are uncorrelated random variables with zero means and the same variances and ω is a constant.

Solution:

Given $E(A) = E(B) = 0$ and $V(A) = V(B)\ 0 \quad \therefore E(A^2) = E(B^2)$

A and B are uncorrelated. $E(AB) = 0$

$R_{YY}(t_1, t_2) = E[X(t_1)Y(t_2)]$

$= E[(A\cos\omega t_1 + B\sin\omega t_1)(B\cos\omega t_2 + A\sin\omega t_2)]$

$= E[AB\cos\omega t_1 \cos\omega t_2 - A^2 \cos\omega t_1 \sin\omega t_2 + B^2 \sin\omega t_1 \cos\omega t_2 - AB\sin\omega t_1 \sin\omega t_2]$

$= \cos\omega t_1 \cos\omega t_2 E(AB) - \cos\omega t_1 \sin\omega t_2 E(A^2) + \sin\omega t_1 \cos\omega t_2 E(B^2) - \sin\omega t_1 \sin\omega t_2 E(AB)$

$= -E(A^2)[\cos\omega t_1 \sin\omega t_2 - \sin\omega t_1 \cos\omega t_2]$

$= E(A^2)[\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2]$

$= \sigma^2 \sin\omega(t_1 - t_2) \qquad \left(E(A^2) = E(B^2) = \sigma^2 say \right)$

= a function of $(t_1 - t_2)$

Now to show that individually they are WSS

$X(t) = A\cos\omega t + B\sin\omega t$

$E[X(t)] = \cos\omega t\ E(A) + \sin\omega t\ E(B) = 0$

$\therefore E[X(t)]$ is a constant.

$R_{XX}(t_1, t_2) = E[X(t_1).X(t_2)]$

$= E[(A\cos\omega t_1 + B\sin\omega t_2) + (A\cos\omega t_2 + B\sin\omega t_2)]$

$= E[A^2 \cos\omega t_1 + \cos\omega t_2 + AB\cos\omega t_1 \sin\omega t_2 + BA\sin\omega t_1 \cos\omega t_2 + B^2 \sin\omega t_1 \sin\omega t_2]$

$= \cos\omega t_1 \cos\omega t_2\ E(A^2) + E(AB)(\cos\omega t_1 \sin\omega t_2 + \sin\omega t_1 \cos\omega t_2) + \sin\omega t_1 \sin\omega t_2\ E(B^2)$

$= E(A^2)[\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2]$

$= \sigma^2 \cos\omega(t_1 - t_2)$

= a function of $(t_1 - t_2)$

Thus $X(t)$ is WSS.

Now $Y(t) = B\cos\omega\tau - A\sin\omega\tau$

$E[Y(t)] = E[B\cos\omega t - A\sin\omega t]$

$= \cos\omega t\, E(B) - \sin\omega t\, E(A) = 0$

$\therefore E[Y(t)]$ is a constant.

$R_{YY}(t_1,t_2) = E[Y(t_1)Y(t_2)]$

$= E[(B\cos\omega t_1 - A\sin\omega t_1)(B\cos\omega t_2 - A\sin\omega t_2)]$

$= E[B^2\cos\omega t_1\cos\omega t_2 + BA\cos\omega t_1\sin\omega t_2 - AB\sin\omega t_1\cos\omega t_2 + A^2\sin\omega t_1\sin\omega t_2]$

$= \cos\omega t_1\cos\omega t_2\, E(B^2) - E(AB)(\cos\omega t_1\sin\omega t_2 - \sin\omega t_1\cos\omega t_2) + (\sin\omega t_1\sin\omega t_2)E(A^2)$

$= E(A^2)[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2]$

$= \sigma^2\cos\omega(t_1 - t_2)$

= a function of $(t_1 - t_2)$

$\therefore \{Y(t)\}$ is a WSS.

$\therefore X(t)$ and $Y(t)$ are jointly WSS.

Problem 14 Find the mean and auto correlation of the Poisson process

<u>Solution:</u>

For a Poisson process, $E[X(t)] = V[X(t)] = \lambda t$ and $E[X^2(t)] = \lambda t + \lambda^2 t^2$

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$

$= E[X(t_1)\{X(t_2) - X(t_1) + X(t_1)\}]$

$= E\{X(t_1)[X(t_2) - X(t_1)]\} + E[X^2(t_1)]$

$= E[X(t_1)]E[X(t_2) - X(t_1)] + E[X^2(t_1)]$

$= \lambda t_1\lambda(t_2 - t_1) + \lambda t_1 + \lambda^2 t_1^2$

$= \lambda^2 t_1 t_2 + \lambda t_1$

$\therefore R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda t_1$

$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E[X(t_1)X(t_2)]$

$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2 = \lambda t_1$

$$r_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{V[X(t_1)]V[X(t_2)]}} = \frac{\lambda t_1}{\sqrt{\lambda t_1.\lambda t_2}} = \sqrt{\frac{t_1}{t_2}}$$

Problem 15 Given a random variable Y with characteristic function $\phi(\omega)=E\left(e^{i\omega y}\right)$ and a random process defined by $X(t)=\cos(\lambda t+y)$; show that $\{X(t)\}$ is stationary in the wide sense if $\varphi(1)=\varphi(2)=0$.

Solution:

$\phi(1)=0 \Rightarrow E\left[e^{iy}\right]=0 \Rightarrow E[\cos y+i\sin y]=0$

$\Rightarrow E(\cos y)=0$ and $E(\sin y)=0$

$\Rightarrow E(\cos 2y)=0$ and $E(\sin 2y)=0$

$\therefore E[X(t)]=E[\cos(\lambda t+y)]$

$=E[\cos\lambda t\cos y-\sin\lambda t\sin y]$

$=\cos\lambda t\,E[\cos y]-\sin\lambda t\,E[\sin y]=0$

$\therefore E[X(t)]=\text{Constant}$

$E[X(t_1)X(t_2)]=E[\cos(\lambda t_1+y)\cos(\lambda t_2+y)]$

$=E[(\cos\lambda t_1\cos y-\sin\lambda t_1\sin y)(\cos\lambda t_2\cos y-\sin\lambda t_2\sin y)]$

$=E\Big[\cos\lambda t_1\cos\lambda t_2\cos^2 y-\cos\lambda t_1\sin\lambda t_2\cos y\sin y-\sin\lambda t_1\cos\lambda t_2\sin y\cos y$

$+\sin\lambda t_1\sin y\lambda t_2\sin^2 y\Big]$

$=\cos\lambda t_1\cos\lambda t_2\,E\left[\cos^2 y\right]-(\cos\lambda t_1\sin\lambda_2+\sin\lambda t_1\cos\lambda t_2)$

$E[\cos y\sin y]+\sin\lambda t_1\sin\lambda t_2\,E\left[\sin^2 y\right]$

$=\cos\lambda t_1\cos\lambda t_2\,E\left[\cos^2 y\right]+\sin\lambda t_1\sin\lambda t_2\,E\left[\sin^2 y\right]-\sin\lambda(t_1+t_2)E[\sin y\cos y]$

$=\cos\lambda t_1\cos\lambda t_2\,E\left[\frac{1+\cos 2y}{2}\right]+\sin\lambda t_1\sin\lambda t_2\,E\left[\frac{1-\cos 2y}{2}\right]-\sin\lambda(t_1+t_2)E\left[\frac{2\sin y\cos y}{2}\right]$

$=\frac{1}{2}[\cos\lambda t_1\cos\lambda t_2+\sin\lambda t_1\sin\lambda t_2]+\frac{1}{2}\cos\lambda t_1\cos\lambda t_2\,E\cos 2y-\frac{1}{2}\sin\lambda t_1\sin\lambda t_2\,E(\cos 2y)$

$-\sin\lambda(t_1+t_2)\frac{1}{2}E[\sin 2y]$

$=\frac{1}{2}\cos\lambda(t_1-t_2)$

= a function of (t_1-t_2) $\therefore \{X(t)\}$ is a WSS.

Problem 16 If $\{X(t)\}$ is a Gaussian Process with $\mu(t)=10$ and $C(t_1,t_2)=16e^{-|t_1-t_2|}$, Find the probability that i) $X(10)\le 8$ ii) $|X(10)-X(6)|\le 4$.

Solution:

$X(10)$ is a normal variant with mean $\mu(10)=10$ and $C(10,10)=16$. Therefore SD = 4.

i) $P[X(10)\le 8]=P\left[\frac{X(10)-10}{4}\le\frac{8-10}{4}\right]$

$$= P[Z \le -0.5] = 0.5 - P[Z \le 0.5]$$
$$= 0.5 - 0.1915 \text{ (from Normal tables)}$$
$$= 0.3085$$

ii) $X(10) - X(6)$ is a normal variant with mean $\mu(10) - \mu(6) = 10 - 10 = 0$

$$V[X(10) - X(6)] = V[X(10)] + V[X(6)] - 2Cov(X(10), X(6))$$
$$= C(10,10) + C(6,6) - 2C(10,6)$$
$$= 16e^0 + 6e^0 - 2e^{-|4|}$$
$$= 31.414$$

$\therefore$ SD of $[X(10) - X(6)] = 5.6048.$

$$\therefore P[|X(10) - X(6)| \le 4] = P\left[\left|\frac{X(10) - X(6) - 0}{5.6045}\right| \le \frac{4}{5.6048}\right]$$
$$= P(|Z| \le 0.2611) = 2 \times 0.2611 = 0.5222$$

Problem 17 A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by trains but if he drives one day, then the next day he is just as likely to drive again he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a '6' appeared. Find i) The probability that he takes a train on the third day. ii) The probability that he drives to work in the long ran.

Solution:

i) Here train (T) and car(C) are the states. The tpm $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Initial state probability distribution

$$P^{(1)} = \left[\frac{5}{6}, \frac{1}{6}\right]$$

P[traveling by car]=P[getting 6]= $\frac{1}{6}$

P[traveling by train]= $\frac{5}{6}$

$$P^{(2)} = P^1 P = \left[\frac{5}{6}, \frac{1}{6}\right] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left[\frac{1}{12}, \frac{11}{12}\right]$$

$$P^{(3)} = P^2 P = \left[\frac{1}{12}, \frac{11}{12}\right] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \left[\frac{11}{24}, \frac{13}{24}\right]$$

Probability that the man travels by train on 3rd day $= \frac{11}{24}$

ii) Let $\pi = (\pi_1, \pi_2)$ be the stationary state distribution of the Markov chain. By property of $\pi P = \pi$

$$(\pi_1,\pi_2)\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}=(\pi_1,\pi_2)$$

$$\frac{1}{2}\pi_2=\pi_1 \text{ and } \pi_1+\frac{1}{2}\pi_2=\pi_2$$

$$2\pi_1=\pi_2$$

Also $\pi_1+\pi_2=1$ (Since π is the probability distribution)

$$\therefore \pi_1+2\pi_1=1 \Rightarrow \pi_1=\frac{1}{3} \therefore \pi_2=\frac{2}{3}$$

$\therefore$ Probability that he travels by car in the long run= $\frac{2}{3}$.

Problem 18 Prove that the difference of two independent Poisson processes is not a Poisson process.

Solution:

Let $X(t)=X_1(t)-X_2(t)$

$$E[X(t)]=E[X_1(t)]-E[X_2(t)]=(\lambda_1-\lambda_2)t$$

$$=E[X^2(t)]=E\{[X_1(t)-X_2(t)]^2\}$$

$$=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)X_2(t)]$$

$$=E[X_1^2(t)]+E[X_2^2(t)]-2E[X_1(t)]E[X_2(t)]$$

$$=(\lambda_1^2t^2+\lambda_1 t)+(\lambda_2^2t^2+\lambda_2 t)-2(\lambda_1 t)(\lambda_2 t)$$

$$=(\lambda_1+\lambda_2)t+(\lambda_1^2+\lambda_2^2)t^2-2\lambda_1\lambda_2t^2$$

$$=(\lambda_1+\lambda_2)t+(\lambda_1-\lambda_2)^2t^2$$

$$\neq(\lambda_1-\lambda_2)t+(\lambda_1-\lambda_2)^2t^2$$

$\therefore X_1(t)-X_2(t)$ is not a Poisson process.

Problem 19 Show that the random process $X(t)=A\cos(\omega t+\theta)$ is wide sense stationary if A and ω are constant and θ is uniformly distributed random variable $(0,2\pi)$

Solution:

Since θ is uniformly distributed in $(0,2\pi)$ p.d.f. $f(\theta)=\frac{1}{2\pi}\ (0\le 0\le 2\pi)$

$$X(t)=A\cos(\omega t+\theta)$$

$$E[X(t)]=\int_0^{2\pi}\frac{1}{2\pi}A\cos(\omega t+\theta)\,d\theta=\frac{A}{2\pi}[\sin(\omega t+\theta)]_0^{\pi}$$

$$=\frac{A}{2\pi}[\sin(\omega t+2\pi)-\sin(\omega t)]$$

$$=\frac{A}{2\pi}[\sin\omega t-\sin\omega t]=0$$

$$R(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$
$$= E\left[A\cos(\omega t_1+\theta).A\cos(\omega t_2+\theta)\right]$$
$$= E\left[A^2\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\right]$$
$$= \frac{A^2}{2}E\left[\cos\omega(t_1+t_2)+2\theta+\cos\omega(t_1-t_2)\right]$$
$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos\omega(t_1+t_2)+2\theta\right]+\cos\left[\omega(t_1-t_2)\right]d\theta$$
$$= \frac{A^2}{2}\left\{\left[\sin(\omega(t_1+t_2)+2\theta)\right]_0^{2\pi}+\cos\omega(t_1-t_2)[0]_0^{v}\right\}$$
$$= \frac{A^2}{4\pi}\left\{\sin(\omega(t_1+t_2)+4\pi)-\sin(\omega(t_1+t_2))+(\cos\omega(t_1+t_2))\right\}2\pi$$
$$= \frac{A^2}{4\pi}\left[\sin\omega(t_1+t_2)-\sin\omega(t_1+t_2)-2\pi\cos\omega(t_1-t_2)\right]$$
$$= \frac{A^2}{4\pi}.2\pi\cos\omega(t_1-t_2) = \frac{A^2}{2}\cos\omega(t_1-t_2)$$

=a function of t_1-t_2

$\therefore$ The process $X(t)$ is W.S.S.

Problem 20 Given a random process $X(t)=10\cos(100t+\theta)$ where θ is uniformly distributed over $(-\pi,\pi)$, prove that the process $\{X(t)\}$ is correlation ergodic.

Solution:

The p.d.f of θ is $f(\theta)=\frac{1}{2\pi}in(-\pi,\pi)$

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right]$$
$$= E\left[10\cos(100t+\theta).10\cos(100(t+\tau)+\theta)\right]$$
$$= \frac{100}{2}E\left[\cos(200t+100\tau+2\theta)+\cos 100\tau\right]$$
$$Now\,E\left[\cos(200t+100\tau+2\theta)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(200t+100\tau+2\theta)d\theta$$
$$= \frac{1}{2\pi}\left[\frac{\sin(200t+100\tau+2\theta)}{2}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{4\pi}\left[(\sin 200t+100\tau+2\pi)-\sin(200t+100\tau-2\pi)\right]$$
$$= 0$$
$$E\left[\cos(100\tau)\right] = \cos 100\tau$$
$$R_{XX}(\tau) = 50\cos 100\tau$$

Ensemble correlation

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt$$

$$= \lim_{T\to\infty} \left\{\frac{1}{2T}\int_{-T}^{T}\left[50\cos 100\tau + 50\cos(200t+100\tau+2\theta)\right]dt\right\}$$

$$= \lim_{T\to\infty} \left\{50\cos 100\tau + \frac{25}{T}\int_{-T}^{T}\cos(200t+100\tau+2\theta)dt\right\}$$

$$= \lim_{T\to\infty} \left\{50\cos 100\tau + \frac{25}{T}\left[\sin\left(\frac{200t+100\tau+2\theta}{200}\right)\right]_{-T}^{T}\right\}$$

$$= \lim_{T\to\infty} \left\{50\cos 100\tau + \frac{25}{T}\times\frac{1}{200}\left[\sin(200T+100\tau+2\theta) - \sin(-200T+100\tau+2\theta)\right]\right\}$$

$$= 50\cos 100\tau + 0$$

$$= R_{XX}(\tau)$$

A process $\{X(t)\}$ is said to be ergodic in correlation if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt = R_{XX}(\tau)$$

Hence the given process is correlation ergodic

Problem 21 Define a random process (stochastic process). Explain the classification of random processes. Give an example to each class.

<u>**Solution:**</u>

A stochastic process is a family of random variable $\{X(s,t)/s\in S, t\in T\}$, where S is called the state space and T is called the index set/space.

<u>Type I</u>

Discrete S and discrete T

Eg: $X(t)$ is the number of defective items found at trials $t = 1,2,3,...$(Discrete random sequence)

<u>Type II</u>

Discrete S and continuous T

Eg : $X(t)$ is the number of telephone calls received during the interval $(0,t), t\geq 0$ (Discrete random process)

<u>Type-III</u>

Continuous S and discrete T

Eg: $X(t)$ is the amount of rainfall measured at time $t = 1,2,3....$(Continuous random sequence)

<u>Type IV</u>

Continuous S and discrete T

Eg : $X(t)$ is the amount of rainfall measured at time $t, t\geq 0$ (Continuous random process)

Problem 22 If the process $\{N(t): t \geq 0\}$ is a Poisson process with parameter λ, obtain $P[N(t)=n]$ and $E[N(t)]$

Solution:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t)$ represent the probability of n occurrences of the event in the interval $(0,t)$.

i.e., $P_n(t) = P\{N(t)=n\}$

$$\therefore P_n(t+\Delta t) = P\{N(t+\Delta t)=n\}$$

$$= P\{n \text{ occurences in the time } (0,t+\Delta t)\}$$

$$= P\begin{cases} n \text{ occurences in the interval } (0,t) \text{ and no occurences in } (t,t+\Delta t) \text{ or} \\ n-1 \text{ occurences in the interval } (0,t) \text{ and } 1 \text{ occurences in } (t,t+\Delta t) \text{ or} \\ n-2 \text{ occurences in the interval } (0,t) \text{ and } 2 \text{ occurences in } (t,t+\Delta t) \text{ or} \ldots \end{cases}$$

$$= P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t + 0 + \ldots$$

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\{P_{n-1}(t) - P_n(t)\}$$

Taking the limits as $\Delta t \to 0$

$$\frac{d}{dt}P_n(t) = \lambda\{P_{n-1}(t) - P_n(t)\} \text{-------- (1)}$$

This is a linear differential equation.

$$\therefore P_n(t)e^{\lambda t} = \int_0^t \lambda P_{n-1}(t)e^{\lambda t} \text{------------- (2)}$$

Now taking $n=1$ we get

$$e^{\lambda t}P_1(t) = \lambda\int_0^t P_0(t)e^{\lambda t}dt \text{--------------- (3)}$$

Now, we have,

$$P_0(t+\Delta t) = P[0 \text{ occurences in } (0,t+\Delta t)]$$

$$= P[0 \text{ occurences in } (0,t) \text{ and } 0 \text{ occurences in } (t,t+\Delta t)]$$

$$= P_0(t)[1-\lambda t]$$

$$P_0(t+\Delta t) - P_0(t) = -P_0(t)(\lambda\Delta t)$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$\therefore$ Taking limit $\Delta t \to 0$

$$\underset{\Delta t \to 0}{Lt} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

$\log P_0(t) = -\lambda t + c$

$P_0(t) = e^{-\lambda t + c}$

$P_0(t) = e^{-\lambda t} e^{c}$

$P_0(t) = e^{-\lambda t} A$ ----------------- (4)

Putting $t = 0$ we get

$P_0(0) = e^0 A = A$

i.e., $A = 1$

$\therefore$(4) we have

$P_0(t) = e^{-\lambda t}$

$\therefore$substituting in (3) we get

$$e^{\lambda t} P_1(t) = \lambda \int_0^t e^{-\lambda t} e^{\lambda t} dt = \lambda \int_0^t dt = \lambda t$$

$P_1(t) = e^{-\lambda t} \lambda t$

Similarly $n = 2$ in (2) we have,

$$P_2(t) e^{\lambda t} = \lambda \int_0^t P_1(t) e^{\lambda t} dt = \lambda \int_0^t e^{-\lambda t} \lambda t e^{\lambda t} dt = \lambda^2 \left(\frac{t^2}{2}\right)$$

$$P_2(t) e^{\lambda t} = \frac{e^{-\lambda t} (\lambda t)^2}{2!}$$

Proceeding similarly we have in general

$$P_n(t) = P\{N(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!},\ n = 0,1,\ldots$$

Thus the probability distribution of $N(t)$ is the Poisson distribution with parameter λt.

$$\text{Mean } = E[N(t)] = \sum n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = (\lambda t) e^{-\lambda t} \sum \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= (\lambda t) e^{-\lambda t} . e^{\lambda t} = \lambda t$$

Problem 23 Let $X(t) = A\cos(\lambda t) + B\sin(\lambda t)$ where A and B are independent normally distributed random variables $N(0,\sigma^2)$. Obtain the covariance function of $\{X(t); -\infty < t < \infty\}$.

Solution:

$E[X(t)] = \cos\lambda t\, E(A) + \sin\lambda t\, E(B) = 0$

$$R_{XX}(t_1,t_2) = E[X(t_1)\ X(t_2)]$$

$$= E\left[A(\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)\right]$$

$$= E\left[A^2 \cos\lambda t_1 \cos\lambda t_2 + B^2 \sin\lambda t_1 \sin\lambda t_2 + \right.$$

$$AB\sin\lambda t_1\cos\lambda t_2 + AB\sin\lambda t_1\cos\lambda t_2]$$
$$= E\left[A^2\cos\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2\right] \qquad E(AB)=E(A)E(B)=0$$
$$= E\left(A^2\right)\cos\lambda\left(t_1-t_2\right)$$
$$= \sigma^2\cos\lambda\left(t_1-t_2\right)$$

Problem 24 The auto correlation function for a stationary process $X(t)$ is given by $R_{XX}(T)=9+2e^{-|T|}$. Find the mean value of the R.V. $y=\int_0^2 X(t)dt$ and variance of $X(t)$.

<u>Solution:</u>

$$E[X(t)] = \lim_{T\to\infty}\sqrt{9+2e^{|T|}} = 3$$
$$E\left[X^2(t)\right] = R_{XX}(0) = 11.$$
$$E[Y(t)] = \int_0^2 E[X(t)]dt = \int_0^2 3dt = 3(t)_0^2 = 6$$
$$V[X(t)] = E\left[X^2(t)\right] - \left\{E[X(t)]\right\}^2 = 11-9 = 2$$

Problem 25 If $\{X(t)\}$ is a WSS process with auto correlation function $R_{XX}(\tau)$ and if $y(t)=X(t+a)-X(t-a)$. Show that $R_{YY}(\tau)=2R_{XX}(\tau)-R_{XX}(\tau+2a)-R_{XX}(\tau-2a)$

<u>Solution</u>:

$$R_{YY}(\tau) = E[y(t)y(t+\tau)]$$
$$= E\left[\left(X(t+a)-X(t-a)\right)\left(X(t+a+\tau)-X(t+\tau-a)\right)\right]$$
$$= E\left[X(t+a)\ X(t+\tau+a)\right] - E\left[X(t+a)\ X(t+\tau-a)\right]$$
$$-E\left[X(t-a)\ X(t+\tau+a)\right] + E\left[X(t-a)\ X(t+\tau-a)\right]$$
$$R_{XX}(\tau) - E\left[X(t+a)X(t+a+\tau-2a)\right] - E\left[X(t-a)X(t-a+\tau+2a)\right] + R_{XX}(\tau)$$
$$= 2R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a)$$

Problem 26 Consider two random processes $X(t)=3\cos(\omega t+\vartheta)$ and $Y(t)=2\cos\left(\omega t+\vartheta-\frac{\pi}{2}\right)$ where θ is a random variable uniformly distribution in $(0,2\pi)$. Prove that $\sqrt{R_{XX}(0)R_{YY}(0)} \geq \left|R_{XY}(\tau)\right|$.

<u>Solution:</u>

$$R_{XX}(\tau) = E\left[X(t)X(t+\tau)\right]$$
$$= E\left[9\cos(\omega t+\theta)\cos\left(\omega(t+\tau)\theta\right)\right]$$
$$= 9E\left[\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)\right]$$

$$= 9E\left[\frac{\cos(2\omega t + \omega t + 2\theta) + \cos\omega\tau}{2}\right]$$

$$= \frac{9}{2}\int_0^{2\pi}\cos\left(\frac{(2\omega t + \omega t + 2\theta) + \cos\omega\tau}{2}\right)\frac{d\theta}{2\pi}$$

$$= \frac{9}{2}\cdot\frac{1}{2\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2}\right]_0^{2\pi} + \cos\omega\tau\,\frac{1}{2\pi}(\theta)_0^{2\pi}$$

$$= \frac{9}{2}\cos\omega\tau$$

$$\therefore R_{XX}(0) = \frac{9}{2}$$

$$R_{YY}(\tau) = E\left[2\cos\left(\omega t + \theta - \frac{\pi}{2}\right).2\cos\left(\omega(t+\tau) + \theta - \frac{\pi}{2}\right)\right]$$

$$= 4E\left[\sin(\omega t + \theta)\sin(\omega t + \omega\tau + \theta)\right]$$

$$= \frac{4}{2}E\left[\cos\left((\omega t + \theta) - (\omega t + \omega\tau + \theta)\right) - \cos(\omega t + \theta) - (\omega t + \omega\tau + \theta)\right]$$

$$= 2\left\{E[\cos\omega\tau] - E\left(\cos(2\omega t + \omega\tau + 2\theta)\right)\right.$$

$$= 2\cos\omega\tau - \int_0^{2\pi}\cos(2\omega t + \omega\tau + 2\theta).\frac{1}{2\pi}d\theta$$

$$= 2\cos\omega\tau$$

$$\therefore R_{YY}(\tau) = 2$$

$$R_{XY}(\tau) = E\left[3\cos(\omega t + \mu)2\cos\left(\omega(t+\tau)\theta - \frac{\pi}{2}\right)\right]$$

$$= E\left[6\cos(\omega\tau + \mu)\sin(\omega t + \omega\tau + \mu)\right]$$

$$= 3E\left[\sin(\omega t + \mu + \omega t + \omega\tau + \mu) - \sin(\omega t + \mu) + (\omega t + \omega\tau + \mu)\right]$$

$$= 3E\left[\sin(2\omega t + 2\mu + \omega\tau) + \sin\omega\tau\right]$$

$$= 3\int_0^{2\pi}\sin(2\omega t + 2\mu + \omega\tau).\frac{1}{2\pi}d\theta + 3\int_0^{2\pi}\sin\omega\tau\,d\theta$$

$$= 0 + 3\sin\omega\tau$$

$$R_{XY}(\tau) = 3\sin\omega\tau$$

$$R_{XX}(0) = \frac{9}{2}$$

$$R_{YY}(0) = 2$$

$$R_{XX}(0)\ \ R_{YY}(\tau) = 9$$

$$\left|R_{XY}(\tau)\right| = |3\sin\omega\tau| \le 3 = \sqrt{9} = \sqrt{R_{XX}(0) = R_{YY}(0)}$$

Problem 27 Define Markov chain and explain how you would classify the states and identify different classes of a Markov chain. Given an example to each class

Solution:

The sequence $\{X_n\}$ is a Markov chain if each X_n is a r.v. and if

$$P\left[X_{n+1}=a_{n+1}/X_n=a_n, X_{n-1}=a_{n-1},\ldots X_0=a_0\right]$$
$$=P\left[X_{n+1}=a_{n+1}/X_n=a_n\right]$$

Let $P=\left(P_{ij}\right)$ be the one-step transition probability matrix.

The state i is recurrent iff $\sum_{n=0}^{\infty} P_{ii}{}^{n}=\infty$

The state i is transient iff $\sum_{n=0}^{\infty} P_{ii}{}^{n}<\infty$

Example:

Consider the p.t.m $P=\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

State '0' is recurrent since $P_{00}=1$ and $\sum_{n=0}^{\infty} 1^n=\infty$.

State 1 and 2 are transcent since $\sum_{n=0}^{\infty} P_{11}{}^{n}=\sum_{n=0}^{\infty}(1/4)^n$ and $\sum_{n=}^{\infty} P_{22}{}^{n}=\sum_{n=0}^{\infty}(1/4)^n$ are $<\infty$

$\therefore \{0\}$ is the recurrent class and $\{1,2\}$ is the transient class.

Problem 28 State the postulates of a Poisson process. State its properties and establish the additive property for the Poisson process.

Solution:

If $X(t)$ represents the number of occurrences of a certain event its (o,t), then the discrete random process $\{X(t)\}$ is called the Poisson process provided the follwing postulates are satisfied.

i) P[1 Occurrence in $(t,t+\Delta t)\big]=\lambda\Delta t+0(\Delta t)$

ii) P[0 Occurrence in $(t,t+\Delta t)\big]=1-\lambda\Delta t+0(\Delta t)$

iii) P[2 Or more Occurrence in $(t,t+\Delta t)\big]=0(\Delta t)$

iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval (0,t).

v) The probability that the event occurs a specified number of limit in (t_0+t_0+t) depends only on t, but not on t_0.

Properties of Poisson process:

1. Poisson process is a Markov Process
2. Sum of two independent Poisson processes is a Poison process.
3. Difference of two independent Poisson processes is not a Poisson process.
4. The inter-arrival time, i.e., the interval between two successive occurrence of a Poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$
5. If the number of occurrences of an event E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λ and if each occurrence of E has a content probability p of being recorded and the recordings are independent of each other, then the number N(t) of the recorded occurrences in t is also a Poisson process with parameter λ P.

Additive property of Poisson process

Let $X(t)=X_1(t)+X_2(t)$ where $\{X_1(t)\}$ and $\{X_2(t)\}$ and independent Poisson processes.

$$P[X(t)=n]=\sum_{r=0}^{n}P[X_1(t)=r]P[X_2(t)=n-r]$$

$$=\sum_{r=0}^{n}\frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!}\frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$=e^{-(\lambda_1+\lambda_2)t}\frac{1}{n!}\sum_{r=0}^{n}nC_r(\lambda_1 t)^r(\lambda_2 t)^{n-r}$$

$$=\frac{1}{n!}e^{-(\lambda_1+\lambda_2)t}(\lambda_1 t+\lambda_2 t)^n=\frac{1}{n!}e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^n$$

$\therefore X_1(t)+X_2(t)$ is a Poisson process with parameter $(\lambda_1+\lambda_2)t$

Problem 29 Let X be the random variable which gives the intervals between two successive occurrences of a Poisson process with parameter λ. Find out the distribution of X.

Solution:

Let E_i and E_{i+1} be two consecutive occurrences. Let E_i takes place at time instant t_i. X is a continuous r.v.

$$P[X>t]=P[E_i+1 \text{ did not occur in } (t_i,t_i+t)]$$

$$=P[\text{No event in an interval of length t}]$$

$$=e^{-\lambda t}\left(\therefore P[X=0]=\frac{e^{-\lambda t}(\lambda t)^0}{0!}\right)$$

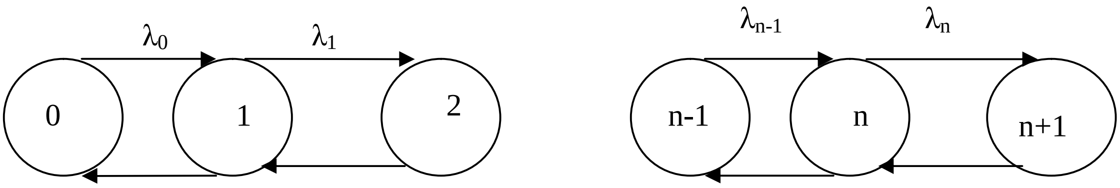
The c.d.f of X is

$$F(t)=P[X\le t]=1-e^{-\lambda t}.$$

$\therefore f(t)=\lambda e^{-\lambda t}; t\ge 0$, which is an exponential distribution with parameter λ.

Problem 30 Draw the state diagram of a birth-death process. Write down the balance equations and obtain expressions for the steady-state probabilities.

Solution:



$$\lambda_\kappa\,\rho_\kappa - \mu_{\kappa+1} = \lambda_{\kappa-1}\,\rho_{\kappa-1} - \lambda_\kappa\,\rho_\kappa, R \geq 1, \text{or } -(\lambda_\kappa + \mu_\kappa)\,\rho_\kappa + \lambda_{\kappa-1}\,\rho_{\kappa-1} + \mu_{\kappa+1}\rho_{\kappa+1} = 0, K \geq 1$$

$$\lambda_0 P_0 = \mu_1 P_1$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$K = 1$$

$$-(\lambda_1 + \mu_1) P_0 + \lambda_0\,\lambda_0 + \mu\,P_2 = 0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

Similarly

$$P_K = \frac{\lambda_0\lambda_1 \dots \lambda_{K-1}}{\mu_1\mu_2 \dots \mu_K} P_0 = \begin{matrix} K-1 \\ 11 \\ i=0 \end{matrix} \left(\frac{\lambda_i}{\mu_{i+1}} \right) P_0$$

Since $\sum_{K=0}^{\infty} P_{K=1}, P_0 = \dfrac{1}{1 + \sum_{K=1}^{\infty} \pi_{i=0}^{K-1} \dfrac{\lambda_i}{\mu_{i+1}}}$

Problem 31 Find the nature of the states of the Markov chain with the TPM $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ and the state space $(1, 2, 3)$.

Solution:

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2.P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^2.P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = P^2$$

$\therefore P^{2n} = P^2 \;\; \& \;\; P^{2n+1} = P$

Also $P_{00}^2 > 0,\ P_{01}^1 > 0,\ P_{02}^2 > 0$

$P_{10}^1 > 0,\ P_{11}^2 > 0,\ P_{12}^1 > 0$

$P_{20}^2 > 0,\ P_{21}^1 > 0,\ P_{22}^2 > 0$

$\Rightarrow$ The Markov chain is irreducible

Also $P_{ii}^2 = P_{ii}^4 = ... > 0$ for all i

$\Rightarrow$ The states of the chain have period 2. Since the chain is finite irreducible, all states are non null persistent. All states are not ergodic.

Problem 32 The one-step T.P.M of a Markov chain $\{X_n;\ n = 0,1,2,...\}$ having state space $S = \{1,2,3\}$ is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $\pi_0 = (0.7,\ 0.2,\ 0.1)$. Find (i) $P(X_2 = 3/X_0 = 1)$ (ii) $P(X_2 = 3)$ (iii) $P(X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 1)$.

Solution:

$$(i)\ P(X_2 = 3/X_0 = 1) = P(X_2 = 3/X_1 = 3)P(X_1 = 3/X_0 = 1) + P(X_3 = 3/X_1 = 2)P(X_1 = 2/X_0 = 1)$$
$$+P(X_2 = 3/X_1 = 1)P(X_1 = 1/X_0 = 1)$$
$$= (0.3)(0.4) + (0.2)(0.5) + (0.4)(0.1) = 0.26$$

$$P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(ii). $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3/X_0 = i)P(X_0 = i)$

$$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2)$$
$$+ P(X_2 = 3/X_0 = 3)P(X_0 = 3)$$
$$= P_{13}^2 P(X_0 = 1) + P_{23}^2 P(X_0 = 2) + P_{33}^2 P(X_0 = 3)$$
$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$$

(iii). $P\left(X_3=2, X_2=3, X_1=3, X_0=1\right)$

$$=P\left[X_0=1, X_1=3, X_2=3\right]P\left[X_3=2/X_0=1, X_1=3, X_2=3\right]$$
$$=P\left[X_0=1, X_1=3, X_2=3\right]P\left[X_3=2/X_2=3\right]$$
$$=P\left[X_0=1, X_1=3\right]P\left[X_2=3/X_0=1, X_1=3\right]P\left[X_3=2/X_2=3\right]$$
$$=P\left[X_0=1, X_1=3\right]P\left[X_2=3/X_1=3\right]P\left[X_3=2/X_2=3\right]$$
$$=P\left[X_0=1\right]P\left[X_1=3/X_0=1\right]P\left[X_2=3/X_1=3\right]P\left[X_3=2/X_2=3\right]$$
$$=(0.4)(0.3)(0.4)(0.7)=0.0336$$

Problem 33 Let $\{X_n;\ n=1,2,3,.....\}$ be a Markov chain with state space $S=\{0,1,2\}$ and 1 – step Transition probability matrix $P=\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}$ (i) Is the chain ergodic? Explain (ii) Find the invariant probabilities.

<u>Solution:</u>

$$P^2=P.P=\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}$$

$$P^3=P^2P=\begin{pmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}\frac{1}{8} & \frac{3}{4} & \frac{1}{8}\\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16}\\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8}\end{pmatrix}$$

$P_{11}^{(3)}>0,\ P_{13}^{(2)}>0,\ P_{21}^{(2)}>0,\ P_{22}^{(2)}>0,\ P_{33}^{(2)}>0$ and all other $P_{ij}^{(1)}>0$

Therefore the chain is irreducible as the states are periodic with period 1
i.e., aperiodic since the chain is finite and irreducible, all are non null persistent
$\therefore$ The states are ergodic.

$$\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}$$

$$\frac{\pi_1}{4}=\pi_0 -------------(1)$$

$$\pi_0+\frac{\pi_1}{2}+\pi_2=\pi_1 -------------(2)$$

$\frac{\pi_1}{4} = \pi_2$ -------------(3)

$\pi_0 + \pi_1 + \pi_2 = 1$-------------(4)

From (2) $\pi_0 + \pi_2 = \pi_1 - \frac{\pi_1}{2} = \frac{\pi_1}{2}$

$\therefore \pi_0 + \pi_1 + \pi_2 = 1$

$\frac{\pi_1}{2} + \pi_1 = 1$

$\frac{3\pi_1}{2} = 1$

$\pi_1 = \frac{2}{3}$

From (3) $\frac{\pi_1}{4} = \pi_2$

$\pi_2 = \frac{1}{6}$

Using (4) $\pi_0 + \frac{2}{3} + \frac{1}{6} = 1$

$\pi_0 + \frac{4+1}{6} = 1$

$\pi_0 + \frac{5}{6} = 1 \Rightarrow \pi_0 = \frac{1}{6}$

$\therefore \pi_0 = \frac{1}{6},\ \pi_1 = \frac{2}{3}\ \&\ \pi_2 = \frac{1}{6}.$

Problem 34 Define autocorrelation function and state its properties. Given that the autocorrelation function for a stationary process $\{X(t)\}$, $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process $\{X(t)\}$.

Solution:

Let $\{X(t)\}$ be a random process. Then the auto correlation function of the process $\{X(t)\}$ is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process and is given by $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$ or $R_{XX}(t,t+\tau) = E[X(t)X(t+\tau)]$

Properties:

(i) $R(\tau)$ is an even function of τ. i.e., $R_{XX}(-\tau) = R_{XX}(\tau)$

(ii) $R(\tau)$ is maximum at $\tau = 0$ i.e., $|R_{XX}(\tau)| \le R_{XX}(0)$

(iii) $R_{XX}(0) = E(X^2(t))$

(iv) $\mu_X^2 = \overline{X}^2 = \underset{|\tau| \to \infty}{Lt} R_{XX}(\tau)$

$$\mu_x^2 = \underset{\tau \to \infty}{Lt} R(\tau) = \underset{\tau \to \infty}{Lt} 25 + \frac{4}{1+6\tau^2} = 25$$

$\therefore \mu_x = 5$

$$E\left(X^2(t)\right) = R_{XX}(0) = 25 + 4 = 29$$

$$Var\left(X(t)\right) = E\left[X^2(t)\right] - E\left[X(t)\right]^2 = 29 - 25 = 4$$

Problem 36 Three are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are inter changed. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?

Solution:

State Space $\{X_n\} = (0,1,2)$ Since the number of ball in the urn A is always 2.

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

$X_n = 0$, $A = 2W$ (Marbles) $B = 3R$ (Marbles)

$X_{n+1} = 0 \qquad P_{00} = 0$

$X_{n+1} = 1 \qquad P_{01} = 1$

$X_{n+1} = 2 \qquad P_{02} = 0$

$X_n = 0$, $A = 1W \,\&\, 1R$ (Marbles) $B = 2R \,\&\, 1W$ (Marbles)

$X_{n+1} = 0 \qquad P_{10} = \frac{1}{6}$

$X_{n+1} = 1 \qquad P_{11} = \frac{1}{2}$

$X_{n+1} = 2 \qquad P_{12} = \frac{1}{3}$

$X_n = 2$, $A = 2R$ (Marbles) $B = 1R \,\&\, 2W$ (Marbles)

$X_{n+1} = 0 \qquad P_{20} = 0$

$X_{n+1} = 1 \qquad P_{21} = \frac{2}{3}$

$X_{n+1} = 2 \qquad P_{22} = \frac{1}{3}$

$P^{(0)} = (1, 0, 0)$ as there is not red marble in A in the beginning.

$P^{(1)} = P^{(0)}P = (0, 1, 0)$

$P^{(2)} = P^{(1)}P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$

$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18}\right)$

$\therefore$P (There are 2 red marbles in A after 3 steps) $= P\{X_3 = 2\} = P_2^{(3)} = \frac{5}{18}$

Let the stationary probability distribution of the chain be $\pi = (\pi_0, \pi_1, \pi_2)$.

By the property of π, $\pi P = \pi$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$$(\pi_0 \ \pi_1 \ \pi_2)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = (\pi_0 \ \pi_1 \ \pi_2)$$

$\frac{1}{6}\pi_1 = \pi_0$

$\pi_0 + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 = \pi_1$

$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_2$

& $\pi_0 + \pi_1 + \pi_2 = 1$

Solving $\pi_0 = \frac{1}{10}, \pi_1 = \frac{6}{10}, \pi_2 = \frac{3}{10}$

P {here are 2 red marbles in A in the long run} = 0.3.

Random Process:-

A random process is a function of time t and the outcomes of a random experiment.

Random variable	Random Process
It is a function on the sample space S.	It is a function on S & T. S - sample space, T - Time set
Random variable $X(s)$ is a real number	$X(s,t)$ is a wave function of t.

Classification of Random Process:-

Random Process can be classified into four types depending on the nature of S and T.

1. Discrete random sequence.

$S \to$ Discrete

$T \to$ Discrete

Ex] The outcome of n^{th} toss of a fair die.

$S = \{1, 2, 3, 4, 5, 6\}$

2. Discrete random Process

$S \rightarrow$ Discrete, $T \rightarrow$ continuous

Ex] No. of telephone calls received in an office during 9.30 AM to 5.30 PM.

$S = \{0, 1, 2, 3 \ldots \}$

3. continuous Ranom sequence.

$S \rightarrow$ continuous, $T \rightarrow$ discrete

Ex] The temperature at the end of n^{th} hour of a day.

here, $T = \{1, 2, 3, 4 \ldots 24\}$

4. continuous Random Process.

$S \rightarrow$ continuous, $T \rightarrow$ continuous

Ex] Amount of rain fall measured at time t.

Mean :-

The mean of random Process $x(t)$ is denoted by $E[x(t)]$

Auto correlation :-

The auto correlation of random Process $x(t)$ is given by

$$R_{xx}(t, t+\tau) = E[x(t) \cdot x(t+\tau)]$$

Stationary Random Process :-

A random Process is stationary if all its statistical Properties (mean, var, Auto correlation.) does not depend

If A random Process is not stationary then it is called evolutionary.

1. First order stationary
2. Second order stationary

First order stationary :-

For First order stationary Process, $E[X(t)]$ is constant.

Second order stationary :-

For second order stationary Process, $R_{XX}(t, t+\tau)$ is a function of τ.

Wide sense stationary : (WSS)

A Random Process is call wide sense stationary if $E[X(t)]$ is constant and $R_{XX}(t, t+\tau)$ is a function of τ.

1. The Process $x(t)$ whose Probability distribution under certain condition is given by

$$P[x(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3 \cdots \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$$

show that $x(t)$ is Not stationary

Soln :-

Given that,

$$P[x(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, 3 \cdots \\ \dfrac{at}{1+at}, & n = 0 \end{cases}$$

$x(t)=n$	0	1	2	3	$\cdots$
$P[x(t)]$	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	$+\cdots$

$$E[x(t)] = \sum x(t) \cdot P[x(t)]$$

$$= \sum (n) \cdot p(n)$$

$$= \frac{1}{(1+at)^2} + \frac{2\,at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{1+at}\left\{1 + \frac{2\,at}{1+at} + \frac{3(at)^2}{1+at} + \cdots\right\}$$

WKT $1+2x+3x^2+\cdots=(1-x)^{-2}$

$$=\frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}(1+at)^2$$

$$\boxed{E[x(t)]=1}$$

$$E[x^2(t)]=\sum x^2(t)\cdot P[x(t)]$$

$$=\sum n^2\,P(n)$$

$$=\sum(n^2+n-n)\,P(n)$$

$$=\sum\left[(n^2+n)\cdot P(n)-n\cdot P(n)\right]$$

$$=\sum(n^2+n)\,P(n)-\sum n\cdot P(n)$$

$$=\left\{\frac{2}{(1+at)^2}+\frac{6(at)}{(1+at)^3}+\frac{12(at)^2}{(1+at)^4}+\cdots\right.$$

$$=\frac{2}{(1+at)^2}\left\{1+\frac{3at}{1+at}+\frac{6(at)^2}{(1+at)^2}+\cdots\right.$$

WKT, $1+3x+6x^2+\dots = (1-x)^{-3}$

$$= \frac{2}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}(1+at)^3 - 1 = 2(1+at) - 1.$$

$$= 2 + 2at - 1 = 1 + 2at$$

$$\text{var} = E[x^2(t)] - \text{Mean}^2$$

$$= 1 + 2at - 1$$

$$\boxed{\text{var} = 2at}.$$

$\therefore$ ~~This Process is not stationary.~~

Variance of $x(t)$ depends t.

$\therefore x(t)$ is not stationary.

---

2. Show that the random Process $x(t) = A\cos(\lambda t) + B\sin$ is WSS if $E[A] = E[B] = 0$; $E[A^2] = E[$ & $E[AB] = 0$ where A and B are random variables.

Soln:-

Here, A and B are random variables.

with $E[A] = E[B] = 0$; $E[A^2] = E[B^2]$ & $E[AB] = 0$

Given, $X(t) = A\cos(\lambda t) + B\sin(\lambda t)$

$E[X(t)] = \cos(\lambda t)\cdot E[A] + \sin(\lambda t)\, E[B]$

$= 0$, constant

$R_{XX}(t, t+\tau) = E[X(t)\cdot \cancel{X}(t+\tau)]$

$X(t) = A\cos(\lambda t) + B\sin(\lambda t)$

$X(t+\tau) = A\cos(\lambda t + \lambda\tau) + B\sin(\lambda t + \lambda\tau)$

$X(t)\cdot X(t+\tau) = A^2\cos(\lambda t)\cdot\cos(\lambda t + t\tau)$

$+ AB\cos(\lambda t)\sin(\lambda t + \lambda\tau) + AB$

$\sin(\lambda t)\cos(\lambda t + \lambda\tau) + B^2\sin(\lambda t)\cdot$

$\sin(\lambda t + \lambda\tau)$

$E[X(t)\cdot X(t+\tau)] = \cos(\lambda t)\cdot\cos(\lambda t + \lambda\tau) E[A^2] +$

$\cos(\lambda t)\cdot\sin(\lambda t + \lambda\tau) E[AB] +$

$\sin(\lambda t)\cdot\cos(\lambda t + \lambda\tau) E[AB] +$

$\sin(\lambda t)\cdot\sin(\lambda t + \lambda\tau) E[B^2]$ $\boxed{E[A^2]}$

$= \cos(\lambda t)\cdot\cos(\lambda t + \lambda\tau) E[A^2] + \sin(\lambda t)$

$\sin(\lambda t + \lambda\tau) E[A^2]$

$= E[A^2] \{ \cos(\lambda t) \cos(\lambda t + \lambda \tau) + \sin(\lambda t)$

$\sin(\lambda t + \lambda \tau) \}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= E[A^2] \cos(\lambda t - \lambda t - \lambda \tau)$

$= E[A^2] \cdot \cos(-\lambda \tau)$

---

2. Show that $x(t) = A \cos(\omega t + \theta)$ is wide sense stationary if A and ω are constants and θ is uniformly distributed random variable in $(0, 2\pi)$

soln:-

Given $x(t) = A \cos(\omega t + \theta)$

here, A & ω are constants,

θ is uniformly distributed random variable in $(0, 2\pi)$.

then ~~The pdf is~~

$$f(\theta) = \frac{1}{b-a} = \frac{1}{2\pi}$$

$$E[x(t)] = A \cdot E[\cos(\omega t + \theta)]$$

$$= A \int \cos(\omega t + \theta)\, f(\theta)\, d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} \cos(\omega t + \theta)\, d\theta$$

$$= \frac{A}{2\pi}\left[\sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\omega t+2\pi) - \sin(\omega t)\right]$$

$$= \frac{A}{2\pi}\left[\sin(\omega t) - \sin(\omega t)\right]$$

$$\boxed{E[X(t)] = 0 \text{, constant}}$$

$$R_{XX}(t, t+\tau) = E[x(t)\cdot x(t+\tau)]$$

$$x(t) = A\cos(\omega t+\theta)$$

$$x(t+\tau) = A\cos(\omega t+\omega\tau+\theta)$$

$$x(t)\cdot x(t+\tau) = A^2\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta)$$

WKT,

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

$$x(t)\cdot x(t+\tau) = \frac{A^2}{2}\left[\cos(-\omega\tau)+\cos(2\omega t+2\theta+\omega\tau)\right]$$

$$x(t)\cdot x(t+\tau) = \frac{A^2}{2}\cos(-\omega\tau)+\frac{A^2}{2}\cos(2\omega t+2\theta+\omega\tau)$$

$$E[x(t)\cdot x(t+\tau)] = \frac{A^2}{2}\cos(-\omega\tau)+\frac{A^2}{2}E[\cos(2\omega t+2\theta+\omega\tau)]$$

$\rightarrow$ ①

$$E[\cos(2\omega t + 2\theta + \omega\tau)] = \int \cos(2\omega t + 2\theta + \omega\tau)\, f(\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\omega t + 2\theta + \omega\tau)\, d\theta .$$

$$= \frac{1}{2\pi} \left[\frac{\sin(2\omega t + 2\theta + \omega\tau)}{2} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[\sin(2\omega t + 4\pi + \omega\tau) - \sin(2\omega t + \omega\tau) \right] = 0$$

$\rightarrow (2)$

By using (2) in (1).

$$R_{xx}(t, t+\tau) = \frac{A^2}{2} \cos(-\omega\tau), \text{ not a function of } t.$$

$\therefore$ x (t) is WSS.

The transition Probability matrix of a Markov chain have 3 states 1, 2, 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and

the initial distribution is $p^{(0)} = [0.7 \;\; 0.2 \;\; 0.1]$

Find probability of $P(X_2 = 3)$ and ~~Probability of~~

Probability of $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$.

Soln:-

Given that,

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$p^{(0)} = [0.7 \;\; 0.2 \;\; 0.1]$$

$$p^{(1)} = p^{(0)}.P = [0.7 \;\; 0.2 \;\; 0.1] \begin{bmatrix} 0.1 & 0.5 \\ 0.6 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$$

$$= \begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$
$$= [0.22 \;\; 0.43 \;\; 0.35]$$

$$p^{(2)} = p^{(1)}.P = [0.22 \;\; 0.43 \;\; 0.35] \begin{bmatrix} 0.1 & 0.5 \\ 0.6 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$$

$$= \begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$
$$= [0.385 \;\; 0.336 \;\; 0.279]$$

i) $P(x_2 = 3) = 0.279$

ii) $P(x_3 = 2,\ x_2 = 3,\ x_1 = 3,\ x_0 = 2)$

$= P_{32} \cdot P_{33} \cdot P_{23} \cdot P(x_0 = 2)$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= 0.0048$

The transition Probability matrix of a Markov chain have three states 1, 2, 3 is

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$ and the initial distribution $P^{(0)} = [0.5 \quad 0.3 \quad 0.2]$.

i) Find $P(x_2 = 2)$ and $P(x_3 = 3, x_2 = 2, x_1 = 1, x_0 =$

Soln:-

Given $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$

$$P^{(0)} = [0.5 \quad 0.3 \quad 0.2]$$

$$P^{(1)} = P^{(0)} \cdot P = [0.5 \quad 0.3 \quad 0.2] \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

$= [0.21 \quad 0.39 \quad 0.40]$

$$P^{(2)} = P^{(1)} \cdot P = [0.21 \quad 0.39 \quad 0.40] \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

$= [0.24 \quad 0.417 \quad 3.42)$

i) $P(X_2 = 2] = 0.417$.

ii) $P(X_3 = 3, X_2 = 2, X_1 = 1, X_0 = 3)$

$= P_{23} \; P_{12} \; P_{31} \cdot P(X_0 = 3)$

$= 0.3 \times 0.3 \times 0.4 \times 0.2$

$= 0.0072$.

Note :-

The limiting distribution of a Markov chain is given by $\pi(I - P) = 0$

1. Find the limiting state Probabilites associated with the following TPM (transmit… Probabity matrix $P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$

suppose that the Probability of a dry day following a rainy day is $1/3$ and the Probability of rainy day following a dryday is $1/2$ given that May 1 is dry day.

i) Find the Probability that May 3 is dry day.

ii) Find the Probability that May 5 is also a dry day.

Soln:-

Here, states are Dry (or) Rainy

$$\therefore \text{TPM is } P = \begin{array}{c c} & \begin{array}{cc} D & R \end{array} \\ \begin{array}{c} D \\ R \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$$\text{May 1}: P^{(0)} = \begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array}$$

$$\text{May 2}: P^{(1)} = P^{(0)} \cdot P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \end{array}$$

$$\text{May 3}: P^{(2)} = P^{(1)} \cdot P = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} 0.4166 & 0.5833 \end{bmatrix} \end{array}$$

$$\text{May 4}: P^{(3)} = P^{(2)} \cdot P = \begin{bmatrix} 0.4166 & 0.5833 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} D & R \end{array} \\ \begin{bmatrix} 0.4027 & 0.5971 \end{bmatrix} \end{array}$$

May 4 : $P^{(3)} = P^{(2)} \cdot P = [0.4166 \quad 0.5833] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

$$\begin{array}{cc} D & R \end{array}$$
$$= [0.4027 \quad 0.5971]$$

May 5 : $P_0^{(4)} = P_1^{(3)} \cdot P = [0.4027 \quad 0.5971] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

$$\begin{array}{cc} D & R \end{array}$$
$$= [0.4003 \quad 0.5994]$$

P(May 3 is dry day) = 0.4166

P(May 5 is dry day) = 0.4003.

3) A man either drives a car or catches a train to go to office each day. He never goes two days in a row by train but if he drives one day, the next day he is just as likely to drive again as he is to travel by train. Suppose that on the first day of the week, the man tosses a fair die and drove to work if and only if a 6 appeared.

i) Find the Probablity that he takes on the train on the third day.

ii) Find the Probability that he drives to

Soln:-

Here the states are car, train.

$$\text{TPM is } P = \begin{array}{c|cc} & C & T \\ \hline C & 1/2 & 1/2 \\ T & 1 & 0 \end{array}$$

$$\text{Day 1} : P^{(0)} = \begin{array}{cc} C & T \\ [1/6 & 5/6] \end{array}$$

$$\text{Day 2} : P^{(1)} = P^{(0)} \cdot P = \begin{bmatrix} 1/6 & 5/6 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{array}{cc} C & T \\ [0.9166 & 0.08333] \end{array}$$

$$\text{Day 3} : P^{(2)} = P^{(1)} \cdot P = [0.91668 \quad 0.08333] \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{array}{cc} C & T \\ [0.5416 & 0.4583] \end{array}$$

P(He takes train on day 3) = 0.4583.

The limiting dist is $\pi(I - P) = 0$

$$[\pi_1 \ \pi_2] \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \right\} = [0 \quad 0]$$

$$[\pi_1, \ \pi_2] \begin{bmatrix} 1/2 & -1/2 \\ -1 & 1 \end{bmatrix} = [0 \quad 0]$$

$$\frac{\pi_1}{2} - \pi_2 = 0 \longrightarrow (1)$$

$$\frac{-\pi_1}{2} + \pi_2 = 0 \longrightarrow (2)$$

$$\pi_1 + \pi_2 = 1 \longrightarrow (3)$$

$\pi_1 - \frac{3}{4}\pi_2 - \frac{3}{4}\pi_3 = 0 \longrightarrow (1)$

$-\pi_1 + \pi_2 - \frac{1}{4}\pi_3 = 0 \longrightarrow (2)$

$0 + -\frac{1}{4}\pi_2 + \pi_3 = 0 \longrightarrow (3)$

$\pi_1 + \pi_2 + \pi_3 = 1 \longrightarrow (4)$

solve (1), (2) & (4)

$\pi_1 = 0.4285$

$\pi_2 = 0.4571$

$\pi_3 = 0.1142$

$\pi = [0.4285 \quad 0.4571 \quad 0.1142]$

Poisson Process

$x(t) \rightarrow$ No. of occurrences of an event in $(0, t)$.

2MP $x(t)$ follows Poisson Process
VI

if,

(i) P(1 occurrence in $(t, t+\Delta t)) = \lambda\Delta t + O$

(ii) P(0 occurrence in $(t, t+\Delta t)) = 1 - \lambda\Delta t + O$

(iii) P(2 (or) more occurrence in $(t, t+\Delta t)) = O(\Delta t)$

(iv) $X(t)$ is independent of no of occurrences of the event in any interval before and after

(v) $P(\text{event in } (t_0, t_0+\tau))$ depends τ and not on t_0).

Probability Law of Poisson Process:-

If $x(t)$ follows Poisson Process,

then Probability of $P[x(t) = n] = \dfrac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}$,

$n = 0, 1, 2, \ldots$

Proof:-

$P_n(t) = P[x(t) = n]$

→ Probability of n occurences in $(0, t)$

$P_n(t+\Delta t) = P[n$ occurrence in $(0,t)$ & 0 occurence in $(t, t+\Delta t)]$.

$P[n-1]$ occurences in $(0, t)$ & 1 occurence in $(t, t+\Delta t)]$

$P_n(t+\Delta t) = P_n(t)[1 - \lambda \Delta t] + P_{n-1}(t) \cdot \lambda \Delta t$

$= P_n(t) - \lambda P_n(t) \Delta t + P_{n-1}(t) \lambda \Delta t$

$P_n(t+\Delta t) - P_n(t) = P_{n-1}(t) \cdot \lambda \Delta t - \lambda P_n(t) \cdot \Delta t$

$\div \Delta t$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda P_{n-1}(t) - \lambda P_n(t)$$

Take lim
$\Delta t \to 0$

$$P_n'(t) = \lambda P_{n-1}(t) - \lambda P_n(t) \to (1)$$

Let the soln be $P_n(t) = \frac{(\lambda t)^n}{n!} f(t)$

$$P_n'(t) = \frac{(\lambda t)^n}{n!} f'(t) + f(t) \frac{n\lambda(\lambda t)^{n-1}}{n!}$$

$$\frac{(\lambda t)^n}{n!} f'(t) + f(t) \frac{\lambda(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!} f(t)$$

$$- \frac{\lambda(\lambda t)^n}{n!} f(t)$$

$$\frac{(\lambda t)^n}{n!} f'(t) = -\frac{\lambda(\lambda t)^n}{n!} f(t)$$

$$f'(t) = -\lambda f(t)$$

$$\Rightarrow \frac{f'(t)}{f(t)} = -\lambda$$

$$\Rightarrow \int \frac{f'(t)}{f(t)} dt = \int -\lambda dt + \log c$$

$$\Rightarrow \log [f(t)] = -\lambda t + \log c$$

$$\Rightarrow \log [f(t) - \log c] = -\lambda t$$

$$\Rightarrow \log \left[\frac{f(t)}{c}\right] = -\lambda t$$

$$\frac{f(t)}{c} = e^{-\lambda t}$$

$$\Rightarrow \boxed{f(t) = c \cdot e^{-\lambda t}}$$

$f(0) = c$

$P_n(t) = \frac{(\lambda t)^n}{n!} f(t)$

$P_0(t) = f(t)$

$P_0(0) = f(0)$

$\boxed{1 = c}$

$\Rightarrow f(t) = e^{-\lambda t} \Rightarrow P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

Mean of Poisson Process :-

In poisson, $P[X(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, n = 0, 1, 2 \ldots$

$$\text{Mean} = E[X(t)] = \sum X(t) \cdot P[X(t)]$$

$$= \sum_{n=0}^{\infty} n \cdot p(n)$$

$$= \sum_{n=0}^{\infty} n \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t} \left\{ \frac{(\lambda t)^1}{0!} + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots \right\}$$

$$= \lambda t \, e^{-\lambda t} \left\{ 1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots \right\}$$

WKT, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^{+x}$

$$= \lambda t \, e^{-\lambda t} \cdot e^{\lambda t}$$

Variance of poisson Process:-

In poisson Process

$$P[x(t)=n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n=0,1,2,\ldots$$

$$E[x^2(t)] = \Sigma x^2(t)\, P[x(t)]$$

$$= \sum_{n=0}^{\infty} n^2 P(n)$$

$$= \sum_{n=0}^{\infty} (n^2+n-n)\, P(n)$$

$$= \sum_{n=0}^{\infty} [(n^2-n)\, P(n) + n\Sigma P(n)]$$

$$= \sum_{n=0}^{\infty} n(n-1)P(n) + \sum_{n=0}^{\infty} n\Sigma P(n)$$

$$= \sum_{n=0}^{\infty} n(n-1)\frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$$

$\to n(n-1)!$
$n(n-1)(n-2)!$

$$= e^{-\lambda t}\sum_{n=2}^{\infty}\frac{(\lambda t)^n}{(n-2)!} + \lambda t$$

$$= e^{-\lambda t}\left\{\frac{(\lambda t)^2}{0!} + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + \cdots\right\} + \lambda t$$

$$= e^{-\lambda t}.(\lambda t)^2\left\{1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots\right\} + \lambda t$$

$$= (\lambda t)^2 e^{-\lambda t}.e^{\lambda t} + \lambda t$$

$$E[x^2(t)] = (\lambda t)^2 + \lambda t$$

$$\mathrm{var}[x(t)] = E[x^2(t)] - \mathrm{mean}^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$$

Note :-

In poisson Process mean $= \lambda t$, not a constant poisson Process is not stationary.

Prove that the difference of two independent poisson Process is not stationary poisson.

<u>Proof</u> :-

Let $x(t), y(t)$ are two independent poisson process with parameter λ_1 & λ_2.

then,

$E[x(t)] = \lambda_1 t.$	$E[y(t)] = \lambda_2 t$
$\mathrm{var}[x(t)] = \lambda_1 t$	$\mathrm{var}[y(t)] = \lambda_2 t.$

$$z(t) = x(t) \cdot y(t)$$

$$E[z(t)] = E[x(t)] \cdot E[y(t)]$$

$$= \lambda_1 t \sim \lambda_2 t$$

$$= (\lambda_1 - \lambda_2) t$$

$$\mathrm{var}[z(t)] = \mathrm{var}[x(t) - y(t)]$$

WKT $var[ax+by] = a^2 var(x) + b^2 var(y)$

$= var[x(t)] + var[y(t)]$

$= \lambda_1 t + \lambda_2 t$

$= (\lambda_1 + \lambda_2)t.$

$\therefore z(t)$ is not poisson.

Note :-

The inter arrival time of a poisson process follows exponential distribution.

In exponential $P(T > t) = e^{-\lambda t}$ → Interarrival time

Queue is formed if the service is not immediately available. Queue may be decreased in size by giving additional service ~~of~~ facilities. Long Queues may result in lost sales and lost customers. Queue refers to customer waiting for service in a line.

Characteristics of Queuing System :-

1. Input (or) Arrival pattern
2. service pattern
3. No. of service channels
4. system capacity
5. Queue discipline.

Input (or) arrival pattern

Since the customers arrive in a random order, the arrival customer follows Poisson process.

i) Arrival Rate :-

Arrival Rate = λ customer per unit time.

UNIT 5

ii) Inter arrival time :-

$$\text{Inter arrival time} = \frac{1}{\lambda}\ \text{time}$$

. service pattern :-

It is specified when it is known how many customers can be served at a time

i) service Rate :-

$$\text{service Rate} = \mu \text{ customer per unit time}$$

ii) service time :-

$$\text{service time} = \frac{1}{\mu}\ \text{time}.$$

3. No. of service channel :-

Two Provide service we may have one counter or many counter in series or parallel.

No. of service channels is denoted by C.

4. system capacity :-

It any no. of customers or allowed to Joint in a queue. Then, the capacity is infinite sometime there is a limited waiting space, ~~show~~ so that the queue becomes large further customer cannot be allowed to Joint in the queue.

Queue discipline :-

It is a manner by which the customers or selected for service

1. FIFO → First In First Out
(or)
2. FCFS → First come First served.

2. LIFO → Last In First out
(or)
3. SIRO → Selection In Random order

4. SIR → service in Priority

Kendall's Notation -

Kendall's Notation of a Queuing system

(a|b|c) : (d|e)

a → arrival distribution

b → service distribution

c → No. of servers

d → system capacity

e → Queue discipline.

Modell :- Basic term

$(M/M/1) : (\infty / FCFS) \rightarrow$

Single server with infinite capacity

1. $\rho = \dfrac{\lambda}{\mu} \rightarrow$ utilisation factor
Traffic intensity
P(server is busy)

2. $P_0 = 1 - \rho \rightarrow$ P(server is free)

3. $P_n = \rho^n \cdot P_0$

4. $L_s = \dfrac{\rho}{1-\rho}$

5. $L_q = L_s - \rho$

6. $W_s = \dfrac{L_s}{\lambda}$

7. $W_q = \dfrac{L_q}{\lambda}$

8. $P(N > k) = \rho^{k+1}$

$N \rightarrow$ No. of customers in system.

9. $P(M > k) = \rho^k$

$M \rightarrow$ No. of customers in Queue.

10. $P(W_s > t) = e^{(\lambda - \mu)t}$

11. $P(W_q > t) = \rho \cdot e^{(\lambda - \mu)t}$.

A Super Market as a single carier. During the peek hours, customers arrive at a rate of 20 per hour. The average no. of customers that can be prolessed by the carrie is 24 perhour.

i) Find the avg no. of customers in the system and in queue.

ii) The average time a customer spend in the system and in the queue.

iii) Probability that the carier is idle.

iv) Probablity that the carier is busy.

Soln:-

customer → peoples

server → carhier

No. of server → 1

Capacity → ∞

Given model is $(M|M|1):(\infty/FIFS)$

Arrival rate = 20 cus/hour

$\lambda = 20$ cus/hour

Service rate = 24 cus/hour

$\mu = 24$ cus/hour

$$\rho = \frac{\lambda}{\mu} = \frac{20}{24}$$

i) $L_s = \frac{\rho}{1-\rho} = \frac{20/24}{1-20/24} = \frac{20/24}{4/24} = \frac{20}{24} \times \frac{24}{4}$

$= 5$

$L_q = L_s - \rho = 5 - \frac{20}{24} = 4.1666$.

ii) $W_s = \frac{L_s}{\lambda} = \frac{5}{20} = \frac{1}{4}$ hour

$W_q = \frac{L_q}{\lambda} = \frac{4.1666}{20} = 0.2083$ hour

iii) P (cashier is idle) = P(No customers)

$= P_0 = 1 - \rho = 1 - \frac{20}{24} = \frac{4}{24} = \frac{1}{6}$

iv) P (cashier is busy) = $\rho = \frac{20}{24}$

$= 5/6$.

. customers arrive at a watch repair shop in a poisson process at a rate of one per every 10 min and the service time is exponential with mean 8 mins.

i) Find the average no. of customer in the shop and in the queue.

ii) Average waiting timing, A customer spend in the shop and in the queue.

iii) What is the probability that the server is idle?

Soln :-

customer → people

server → Repairman

No. of server → 1

capacity → ∞

Given Model $(M/M/1) : (\infty / FCFS)$

Arrival rate = 1 cus / 10 min

$\lambda = \frac{1}{10}$ cus / min

service time = 8 min / cus

$\frac{1}{\mu} = 8$ min / cus

$\mu = \frac{1}{8}$ cus / min

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{10}}{\frac{1}{8}} = \frac{8}{10} = \frac{4}{5}$$

i) $L_s = \frac{\rho}{1-\rho} = \frac{4/5}{1-4/5} = \frac{4}{5} \times \frac{5}{1} = 4.$

$L_q = L_s - \rho = 4 - \frac{4}{5} = \frac{16}{5} = 3.2$

ii) $W_s = \frac{L_s}{\lambda} = \frac{4}{1/10} = 40$ min

$W_q = \frac{L_q}{\lambda} = \frac{3.2}{1/10} = 32$ min

iii) P (server is idle) $= P_0 = 1 - \rho$

$= 1 - \frac{4}{5}$

$= \frac{1}{5}$

---

3 Customers arrive at a sales counter, managed by a single person according to a poisson process with mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the avg waiting time of a customer.

Soln:-

customer → peoples

server → sales man

No. of server → 1

capacity → ∞.

Given model is $(M|M|1):(\infty/FCFS)$

Arrival rate = 20 cus/hour

$\lambda = 20$ cus/hour

service time = 100 sec/cus

$\frac{1}{\mu} = 100$ sec/cus

$$\mu = \frac{1}{100} \text{ cus/sec}$$

$$= \frac{1}{100} \times \frac{3600}{3600} \text{ cus/sec} \qquad \text{hour} = \frac{\text{sec}}{3600}$$

$$\mu = 36 \text{ cus/hour}$$

$$\rho = \frac{\lambda}{\mu} = \frac{20}{36} = \frac{5}{9}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{5/9}{1-5/9} = \frac{5}{9} \times \frac{9}{4} = 5/4$$

$$W_s = \frac{L_s}{\lambda} = \frac{5/4}{20} = \frac{5}{4 \times 20} = \frac{1}{16} \text{ hour}$$

$$\boxed{W_s = \frac{1}{16} \text{ hour}}$$

Model 2 (basic terms) $= (\rho, P_0, \lambda')$

$(M/M/1) : (N/FCFS)$ - single server with finite capacity.

1. $\rho = \frac{\lambda}{\mu}$ → utilisation factor

Traffic intensity

p(server is busy)

2. $P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{N+1}} & \rho \neq 1 \\ \frac{1}{N+1} & \rho = 1 \end{cases}$

$P_n = \rho^n P_0$

3. Effective arrival rate $\lambda' = \mu(1 - P_0)$

4. $L_s = \begin{cases} \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}, & \rho \neq 1 \\ \frac{N}{2}, & \rho = 1 \end{cases}$

5. $L_q = L_s - \frac{\lambda'}{\mu}$

6. $W_s = \frac{L_s}{\lambda'}$

7. $W_q = \frac{L_q}{\lambda'}$.

8. P(customers turned away) = P_N

Trains arrive at a yard every 15 mins and service time is 33 minutes. If the line capacity of the yard is limited to 5 trains, Find i) Probability that the yard is empty.

ii) Effective arrival rate.

iii) Average No. of trains in the system.

iv) Average No. of trains in the queue.

v) Average time a train is Expected to spend in the yard and in the queue.

Soln:-

Customer → Train

server → yard

No. of server → 1

Capacity → 5

Given Model is (M/M/1) : (N/FIFS)

Here $N = 5$

Inter arrival time = 15 min/Train

$$\frac{1}{\lambda} = 15 \text{ min/Train}$$

$$\lambda = \frac{1}{15} \text{ Train/min}$$

Service time = 33 min / Train

$\frac{1}{\mu} = 33$ min / Train.

$\mu = \frac{1}{33}$ Train / min

$\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/33} = \frac{33}{15} = 2.2$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-2.2}{1-2.2^6} = \frac{-1.2}{-112.3799} = 0.0106$

$\lambda' = \mu(1-P_0) = \frac{1}{33}(1-0.0106) = 0.0299$

i) P (yard is empty) = P (No train) = $P_0 = 0.0106$

ii) Effective arrival rate = $\lambda' = 0.02999$.

iii) $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

$= \frac{2.2}{1-2.2} - \frac{6(2.2)^6}{1-(2.2)^6}$

$= \frac{2.2}{-1.2} - \frac{680.2794}{-112.3799}$

$= -1.8333 + 6.0533 = 4.22$

iv) $L_q = L_s - \frac{\lambda'}{\mu} = 4.22 - \frac{0.0299}{1/33}$

$= 4.22 - 0.9867$

Soln:-

customer → people

server → Barber.

No. of server → 1

capacity → 7

Given Model is $(M/M/1):(N/FCFS)$

here $N = 7$.

Interarrival time = 20 min/cus

$$\frac{1}{\lambda} = 20 \text{ min/cus}$$

$$\lambda = \frac{1}{2} \text{ cus/min}$$

Service time = 15 min/cus

$$\frac{1}{\mu} = 15 \text{ min/cus}$$

$$\mu = \frac{1}{15} \text{ cus/min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1/20}{1/15} = \frac{15}{20} = \frac{3}{4} = 0.75$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-0.75^{8}} = 0.2778$$

$$\lambda' = \mu(1-P_0) = \frac{1}{15}(1-0.2778)$$

$$= 0.0481$$

i) P(customer directly into barber chair) = P(No customer) = $P_0 = 0.2778$.

ii) $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

$= \frac{0.75}{1-0.75} - \frac{8(0.75)^8}{1-(0.75)^8} = \frac{0.75}{0.25} - \frac{0.8009}{0.8999}$

$= 3 - 0.8900$

$L_s = 2.11$

iii) $L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{0.0481}{1/15}$

$= 2.11 - 0.72145$

$L_q = 1.3385$

iv) $W_s = \frac{L_s}{\lambda'} = \frac{2.11}{0.0481} = 43.8669$ min

$W_q = \frac{L_q}{\lambda'} = \frac{1.3885}{0.0481} = 28.8669$ min

v) Effective arrival rate $= \lambda' = 0.0481$

vi) P(atleast 3 cus in the shop) $= P(N \geq 3)$

$= 1 - P(N < 3)$

$= 1 - P(0, 1, 2)$

$= 1 - \{P_0 + \rho P_0 + \rho^2 P_0\}$

$$= 1 - \{0.2778 + 0.75(0.2778) + (0.75)^2 (0.2778)\}$$

$$= 1 - \{0.6424\}$$

$$= 0.3576.$$

vii) $P(\text{customer turnaway}) = P_7$

$$= \rho^7 P_0$$

$$= (0.75)^7 \times (0.2778)$$

$$= 0.0370$$

patient

Model 3 :-

(M | M | C) : (∞ | FCFS) - Multiple server with infinite capacity.

Basic term:-

1. $\rho = \frac{\lambda}{C\mu}$ → utilisation factor

P(A server is busy)

2. $P_0 = \left\{ 1 + \frac{C\rho}{1!} + \frac{(C\rho)^2}{2!} + \dots + \frac{(C\rho)^C}{C!} \right\}^{-1}$

↓

Probability of no customer.

P(All servers are idle)

~~3. $P(N \le S)$ $\frac{(C\rho)^C}{C(1-\rho)}$~~

~~↓~~

~~P(All servers was busy)~~

~~Basic~~

4 P(N. of system)

3. $P(N \geq C) = \frac{(C\rho)^C}{C!\,(1-\rho)} P_0$

↓

P(All servers are busy)

4. $P_n = \begin{cases} \frac{(C\rho)^n}{n!} \rho^n P_0, & n \le C \\ \frac{C^C}{C!} \rho^n P_0, & n \geq C \end{cases}$

5. $L_q = \frac{\rho}{1-\rho} P(N \geq C)$

vi) $L_s = L_q + C\rho$

vii) $W_s = \frac{L_s}{\lambda}$

ix) $W_q = \frac{L_q}{\lambda}$

A supermarket has two girls running up sales at the counters. If the service time for each customer with mean 4 min and If people arrive in poisson fashion at the rate of 6 per hour; Find the following.

i) what is the probability that a customer Has to wait for service

) Average system and queue length

) Average time spent by the customer in the Super Market & in the queue

) what is the expected percentage of idle time for each girl?

Soln:

customer → people

Server → Sale girls

No. of server → 2

Capacity → ∞

Given model is $(M/M/C):(\infty/FCFS)$

Service time = 4 min/cus

$$\frac{1}{\mu} = 4 \text{ min/cus}$$

$$\boxed{\mu = \frac{1}{4}} \text{ cus/min}$$

Arrival rate = 10 cus/hour

$$\lambda = 10 \text{ cus/hour}$$

$$\boxed{\lambda = \frac{1}{6}} \text{ cus/}\cancel{\text{hour}}\text{ min}$$

$$\rho = \frac{\lambda}{c\mu} = \frac{\frac{1}{6}}{2 \times \frac{1}{4}} = \frac{1}{\cancel{6}_3} \times \frac{\cancel{4}^2}{2} = 0.3333$$

$$\rho = 0.3333$$

$$c\rho = 0.6666$$

$$P_0 = \left\{ 1 + \frac{c\rho}{1!} + \frac{(c\rho)^2}{2!\,(1-\rho)} \right\}^{-1}$$

$$= \left\{ 1 + \frac{0.6666}{1} + \frac{(0.6666)^2}{2\,(1-0.3333)} \right\}^{-1}$$

$$= \left\{ 1 + 0.6666 + 0.3332 \right\}^{-1}$$

$$P_0 = 0.5$$

$$P(N \geq c) = \frac{(c\rho)^2}{2!\,(1-\rho)} P_0 = \frac{(0.6666)^2 \times 0.5}{2\,(1-0.3333)}$$

$$P(N \geq c) = 0.1666$$

P(Customer has to wait) = P(All servers are busy)

= 0.1666

$$L_q = \frac{\rho}{1-\rho} \; P(N \geq C)$$

$$= \frac{0.3333}{1-0.3333} \times 0.1666$$

$$L_q = 0.0832$$

$$L_s = L_q + C\rho$$

$$= 0.0832 + 0.6666$$

$$L_q = \cancel{0.05}\; 0.7498$$

1) $W_s = \frac{L_s}{\lambda}$

$$= \frac{0.7498}{1/6} = 4.4988 \text{ min}$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.0832}{1/6} = 0.4992 \text{ min}$$

) $P(\text{A girl is idle}) = 1 - \rho = 1 - 0.3333$

$$= 0.6667.$$

% value = 66.67%.

---

2. There are three typists in an office. Each typists can type an average of 6 letters Per hour. If letters arrive for being type at the rate of 15 letters Per hour.

i) what is the Probability of all the typist will be busy?

ii) what is the average no. of letters waiting to be typed?

iii) what is the average time a letter has to spend on for waiting and for being type.

Soln :-

Customer → Letters

Server → Typist

No. of server → 3

Capacity → ∞

Given $(M/M/C):(\infty/FCFS)$

Here, $C = 3$

service rate = 6 letters / hour

$\mu = 6$ letters / hour.

Arrival rate = 15 letters / hour

$\lambda = 15$ letters / hour

$$\rho = \frac{\lambda}{C\mu} = \frac{15}{3 \times 6} = 0.8333$$

$$C\rho = 2.4999$$

$$P_0 = \left\{ 1 + \frac{C\rho}{1!} + \frac{(C\rho)^2}{2!} + \frac{(C\rho)^3}{3!(1-\rho)} \right\}^{-1}$$

$$= \left\{ 1 + \frac{2.4999}{1} + \frac{2.4999^2}{2} + \frac{2.4999^3}{6(1-0.8333)} \right\}^{-1}$$

$$= \left\{ 1 + 2.4999 + 3.1247 + 15.6200 \right\}^{-1}$$

$$P_0 = 0.0449$$

$$P(N \geq c) = \frac{(c\rho)^3}{3!(1-\rho)} P_0 = 15.6200 \times 0.0449$$

$$= 0.7013$$

i) P(All typists is busy) $= P(N \geq c) = 0.7013$

ii) $$L_q = \frac{\rho}{1-\rho} P(N \geq c) = \frac{0.8333}{1-0.8333} \times 0.7013$$

$$L_q = 3.5056$$

$$L_s = L_q + c\rho = 6.0055$$

iii) $W_s = L_s / \lambda = 0.4003$ hour.

Model A : (M/M/C) : (N/FCFS) → Multiple server with Finite capacity

1. $\rho = \frac{\lambda}{C\mu}$ → Traffic intensity
P(A server is busy)

2. $P_0 = \left[1 + \frac{C\rho}{1!} + \frac{(C\rho)^2}{2!} + \dots + \frac{(C\rho)^C [1+\rho+\rho^2+\dots+\rho^{N-C}]}{C!} \right]$

3. $P_n = \begin{cases} \frac{C^n}{n!} \rho^n P_0 & , n \le C \\ \frac{C^C}{C!} \rho^n P_0 & , n > C \end{cases}$

4. $\lambda' = \lambda(1-P_N)$ → The system is full

5. $L_q = \frac{(C\rho)^C}{C!} \cdot \frac{\rho P_0}{(1-\rho)^2} \cdot \left[1-\rho^{N-C} - (N-C)(1-\rho)\rho^{N-C}\right]$

6. $L_s = L_q + \frac{\lambda'}{\mu}$

7. $W_s = \frac{L_s}{\lambda'}$

8. $W_q = \frac{L_q}{\lambda'}$

A Barber shop has 2 barbers and 3 chairs for waiting customers. Assume that customers arrive in Poisson fashion at a rate of 5 per hour and that each barber services customers according to Exponential with a mean of 15 mins. Further If a customer arrives and there are No empty chairs in the shop, he will leave.

i) What is the Probability that the system is empty?

ii) what is the expected no. of customer in the shop?

iii) what is the expected no. of customer in the queue?

iv) what is the expected waiting time and in the queue?

v) Find the steady state Probabilities of various no. of Customers in the shop?

Soln:-

Customer → People

Server → Barber

No. of server → 2

Capacity → 5

Given Model is $(M/M/C):(N/FCFS)$

Here, $c=2$; $N=5$; $N-C=3$

Arrival rate = 5 cus/hour

$\lambda = 5$ cus/hour

$\lambda = \frac{1}{12}$ cus/min

service time = 15 min/cus

$\frac{1}{\mu} = 15$ min/cus

$\mu = 1/15$ cus/min

$$\rho = \frac{\lambda}{C\mu} = \frac{1/12}{2 \times 1/15} = \frac{1}{12} \times \frac{15}{2} = \frac{5}{8} = 0.625$$

$\rho = 0.625$

$C\rho = 1.25$

$$P_0 = \left\{ 1 + \frac{C\rho}{1!} + \frac{(C\rho)^2}{2!}\left[1 + \rho + \rho^2 + \rho^3\right] \right\}^{-1}$$

$$= \left\{ 1 + \frac{1.25}{1} + \frac{(1.25)^2}{2}\left[1 + 0.625 + 0.625^2 + 0.625^3\right] \right\}^{-1}$$

$$= \{1 + 1.25 + 1.7654\}^{-1}$$

$P_0 = 0.2490$

$$P_N = \frac{c^c}{c!} \rho^N P_0 = \frac{2^2}{2!} (0.625)^5 \times 0.2490 = 0.0474$$

$$\lambda' = \lambda(1 - P_N) = \frac{1}{12} (1-0.0474) = 0.0793$$

i) P(system is empty) = P(No customer)

$$P_0 = 0.2490.$$

$$\text{iii) } L_q = \frac{(c\rho)^c}{c!} \cdot \frac{\rho P_0}{(1-\rho)^2} \cdot \left[1-\rho^{N-c} - (N-c)(1-\rho)\rho^{N-c}\right]$$

$$= \frac{(1.25)^2}{2!} \times \frac{0.625 \times 0.2490}{(1-0.625)^2} \times \left[1-0.625^3 \times 3 \times (1-0.625)(0.625)^3\right]$$

$$= 0.7812 \times 1.1066 \times 0.4812$$

$$L_q = 0.4159$$

$$\text{ii) } L_s = L_q + \frac{\lambda'}{\mu}$$

$$= 0.4159 + \frac{0.0793}{1/15}$$

$$= 0.4159 + 0.0793 \times 15$$

$$L_s = 1.6054$$

$$\text{iv) } W_s = \frac{L_s}{\lambda'} = \frac{1.6054}{0.0793} = 20.2446 \text{ min}$$

$$W_q = \frac{L_q}{\lambda'} = \frac{0.4159}{0.0793} = 5.2446 \text{ min}$$

v) WKT $P_n = \begin{cases} \frac{c^n}{n!} \rho^n P_0 & n \le c \\ \frac{c^c}{c!} \rho^n P_0 & n > c \end{cases}$

$P_0 = 0.2490$

$P_1 = \frac{c^1}{1!} \rho^1 P_0 = 2 \times 0.625 \times 0.2490 = 0.3112$

$P_2 = \frac{c^2}{2!} \times \rho^2 P_0 = \frac{2^2}{2} \times 0.625^2 \times 0.2490 = 0.1945$

$P_3 = \frac{c^c}{c!} \times \rho^3 P_0 = \frac{2^2}{2} \times 0.625^3 \times 0.2490 = 0.1215$

$P_4 = \frac{c^c}{c!} \times \rho^4 P_0 = \frac{2^2}{2} \times 0.625^4 \times 0.2490 = 0.0759$

$P_5 = 0.0474.$

At a port, there are 6 unloading berths and 4 unloading crews. When all the berths are full arriving ships are diverted to an overflow facility 20kms down the river. Tankers arrive according to a poisson process with a mean of 1 every 2 hours. It takes for an unloading crew, on the average 10 hours to unload a tanker, the unloading time follows an exponential distribution. Find.

i) How many tankers are at the port on the average

ii) How long does a tanker spend at the port on the average?

iii) What is the average arrival rate at the overflow facility?

Soln :-

customer → ship

server → crew

No. of server → 4

Capacity → 6

Given model is $(M | M | C) : (N | FCFS)$

$c = 4 ; N = 6 \; ; N - c = 2$

Arrival rate = 1 ship / 2 hour.

$\lambda = \frac{1}{2}$ ship/hour

service time = 10 hrs / ship

$\frac{1}{\mu} = 10$ hrs / ship

$\mu = \frac{1}{10}$ ship/hours

$$\rho = \frac{\lambda}{c\mu} = \frac{\frac{1}{2}}{4 \times \frac{1}{10}} = \frac{1}{2} \times \frac{10}{4} = 1.25$$

$\rho = 1.25 \qquad c\rho = 5$

$$P_0 = \left\{ 1 + \frac{c\rho}{1!} + \frac{(c\rho)^2}{2!} + \frac{(c\rho)^3}{3!} + \frac{(c\rho)^4}{4!} \left[1 + \rho + \rho^2 \right] \right\}^{-1}$$

$$= \left\{ 1 + \frac{5}{1} + \frac{5^2}{2} + \frac{5^3}{6} + \frac{5^4}{24} \left[1 + 1.25 + 1.25^2 \right] \right\}$$

$= \{1+5+12.5+20.8333+99.2838\}^{-1}$

$P_0 = 0.0072$

$P_N = \frac{C^C}{C!} \rho^N P_0 = \frac{4^4}{4!} \times 1.25^6 \times 0.0072 = 0.2929$

$\lambda' = \lambda(1-P_N) = \frac{1}{2}(1-0.2929)$

$\lambda' = 0.35351$

i) $L_q = \frac{(C\rho)^C}{C!} \frac{\rho P_0}{(1-\rho)^2}\left[1-\rho^{N-C} - (N-C)(1-\rho)\rho^{N-C}\right]$

$= \frac{5^4}{4!} \times \frac{1.25 \times 0.0072}{(1-1.25)^2}\left[1 - 1.25^2 - 2 \times (1-1.25) \times 1.25^2\right]$

$= 26.0416 \times 0.144 \times 0.2187$

$L_q = 0.8201$

$L_s = L_q + \frac{\lambda'}{\mu}$

$= 0.8201 + \frac{0.3535}{1/10}$

$= 0.8201 + 0.3535 \times 10$

$L_s = 4.3551$

ii) $W_s = \frac{L_s}{\lambda'} = \frac{4.3551}{0.355} = 12.3199$ hr

iii) Average arrival rate at overflow facility $= \lambda \cdot P_N$

$= \frac{1}{2} \times 0.2929$

$= 0.1464.$

$(M/G/1) : (\infty / GD)$ - PK Formula

$N \rightarrow$ No. of customers in the system at t.

$N' \rightarrow$ No. of customers in the system at $t+T$, when two consecutive customers have just left after getting service.

At steady state, $E[N] = E[N']$
$E[N^2] = E[N'^2]$ $\rightarrow$ (1)

Here, T is service time, continuous R.V.

$f(t) \rightarrow$ Pdf of T.

$E[T] \Rightarrow$ Mean of T

$Var(T) = E[T^2] - Mean^2$

$Var(T) = E[T^2] - \{E[T]\}^2$

$$\boxed{E[T^2] = Var(T) + \{E[T]\}^2} \longrightarrow (4)$$

Let M denote the no. of arrivals during T

$$N' = \begin{cases} M, & N = 0 \\ N-1+M, & N > 0 \end{cases}$$

$$\delta = \begin{cases} 1, & N = 0 \\ 0, & N > 0 \end{cases}$$

$$N\delta = \begin{cases} N, & N = 0 \\ 0, & N > 0 \end{cases} = 0$$

$$\delta^2 = \begin{cases} 1, & N = 0 \\ 0, & N > 0 \end{cases} = \delta$$

impartan

$N\delta = 0$; $\delta^2 = \delta$

$$(1) \leftarrow \boxed{N' = N - 1 + M + \delta} \rightarrow (3)$$

Taking E on both sides

$E[N'] = E[N] - 1 + E[M] + E[\delta]$

At steady state,

$$E[N] = E[N] - 1 + E[M] + E[\delta]$$

$$0 = -1 + E[M] + E[\delta]$$

$$E[\delta] = 1 - E[M] \longrightarrow (4)$$

$$(3)^2 \Rightarrow N'^2 = (N - 1 + M + \delta)^2 \qquad (a+b+c+d)^2.$$

$$N'^2 = N^2 + 1 + M^2 + \delta^2 - 2N + 2NM + 2N\delta - 2M - 2\delta + 2M\delta$$

$$N'^2 = N^2 + 1 + M^2 - \delta - 2N + 2NM - 2M + 2M\delta$$

Taking E

$$E[N'^2] = E[N^2] + 1 + E[M^2] - E[\delta] - 2E[N] + 2E[N]E[M] - 2E[M] + 2E[M]E[\delta]$$

At steady state $\qquad E[\delta] = 1 - E[M]$

$$E[N^2] = E[N^2] + 1 + E[M^2] - 1 + E[M] - 2E[N] + 2E[N]E[M] - 2E[M] + 2E[M] - 2\{E[M]\}^2$$

$$0 = E[M^2] + E[M] - 2E[N] + 2E[N]E[M] - 2\{E[M]\}^2$$

$$2E[N] - 2E[N]E[M] = E[M^2] + E[M] - 2\{E[M]\}^2$$

(5)

$$2E[N]\{1-E[M]\} = E[M^2] + E[M] - 2\{E[M]\}^2$$

$$E[N] = \frac{E[M^2] + E[M] - 2\{E[M]\}^2}{2\{1-E[M]\}} \rightarrow (5)$$

M follows poisson process with parameter λ

$E[M|T] = \lambda T$

$Var(M|T) = \lambda T$

$E[M^2|T] - Mean^2 = \lambda T$

$E[M^2|T] - \lambda^2 T^2 = \lambda T$

$E[M^2|T] = \lambda T + \lambda^2 T^2$

WKT $E[E[Y|X]] = E[Y]$

$E[M] = E[\lambda T] = \lambda E[T]$

$E[M^2] = \lambda E[T] + \lambda^2 E[T^2]$

$(5) \Rightarrow$

$$E[N] = \frac{\lambda E[T] + \lambda^2 E[T^2] + \lambda E[T] - 2\lambda^2\{E[T]\}^2}{2\{1-\lambda E[T]\}}$$

$$= \frac{\lambda^2 E[T^2] + 2\lambda E[T]\{1-\lambda E[T]\}}{2\{1-\lambda E[T]\}}$$

$$= \frac{\lambda^2 E[T^2]}{2\{1-\lambda E[T]\}} + \lambda E[T]$$

$$L_s = \lambda E[T] + \frac{\lambda^2 var(T) + \lambda^2\{E[T]\}^2}{2\{1-\lambda E[T]\}}$$

Take,

$$E[T] = \frac{1}{\mu}$$

$$Ls = \frac{\lambda}{\mu} + \frac{\lambda^2 var(T) + \frac{\lambda^2}{\mu^2}}{2\{1-\frac{\lambda}{\mu}\}}$$

$$\therefore \rho = \frac{\lambda}{\mu}$$

$$\boxed{Ls = \rho + \frac{\lambda^2 var(T) + \rho^2}{2(1-\rho)}}$$

Note 1:-

when service time is constant, $var(T) = 0$

the PK formula becomes as

$$Ls = \rho + \frac{\rho^2}{2(1-\rho)}$$

Note 2:-

(M/G/1) model becomes

(M/M/1) model when service time

follows exponential distribution.

Explanation:-

$$\left(Var(T) = \frac{1}{\mu^2} \right.$$

$$\frac{1}{\mu} \Rightarrow L_s = \rho + \frac{\lambda^2 Var(T) + \rho^2}{2(1-\rho)}$$

$$L_S = \rho + \frac{\rho^2 + \rho^2}{2(1-\rho)}$$

$$\left\{ \frac{\lambda}{\mu} = \rho + \frac{2\rho^2}{2(1-\rho)} \right.$$

$$= \frac{\rho - \rho^2 + \rho^2}{1-\rho}$$

$$\left. L_S = \frac{\rho}{1-\rho} \right)$$

A one barber shop takes exactly 25 mins to complete one haircut. If customers arrivals at the barber shop in a Poisson fashion at an average rate of one every 40 mins.

i) How long the average a customer spend in the shop.

ii) Find the average time a customer must ~~weigt~~ wait for service.

Soln :-

Customer → People

Server → Barber

No. of server → 1

Capacity → ∞

Given Model is $(M/G/1):(\infty/FCFS)$

Service time = 25 min / cus

$\frac{1}{\mu} = 25$ min / cus

$\mu = \frac{1}{25}$ cus / min.

Arrival rate = 1 cus / 40 min

$\lambda = \frac{1}{40}$ cus / min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/40}{1/25} = \frac{1}{40} \times 25$$

$\rho = 0.625$

service time is constant

$\Rightarrow var(T) = 0$

i) $L_s = \rho + \frac{\lambda^2 var(T) + \rho^2}{2(1-\rho)}$

$$= 0.625 + \frac{0.625^2}{2(1-0.625)}$$

Soln :-

Customer → People

Server → Barber

No. of server → 1

Capacity → ∞

Given Model is $(M/G/1):(\infty/FCFS)$

Service time = 25 min / cus

$\frac{1}{\mu} = 25$ min / cus

$\mu = \frac{1}{25}$ cus / min.

Arrival rate = 1 cus / 40 min

$\lambda = \frac{1}{40}$ cus / min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/40}{1/25} = \frac{1}{40} \times \frac{25}{1}$$

$\rho = 0.625$

Service time is constant

$\Rightarrow var(T) = 0$

$$\text{i) } L_S = \rho + \frac{\lambda^2 var(T) + \rho^2}{2(1-\rho)}$$

$$= 0.625 + \frac{0.625^2}{2(1-0.625)}$$

$$Ws = \frac{Ls}{\lambda} = \frac{1.1458}{1/40} = 1.1458 \times 40 = 45.832\ min$$

ii) $Lq = Ls - \rho = 1.1458 - 0.625 = 0.5208$

$$Wq = \frac{Lq}{\lambda} = 0.5208 \times 40 = 20.832\ min$$

An atomatic car box faility operates with only one bay. cars arrive acconding to poisson distribution with a mean of 4 cars per hour and may weight in the failitties parking lot. If the bay is busy. The parking lot is large enough to accomidate any no. of cars. Determine

i) Mean No. of customers in the system

ii) Mean no. of customers in the queue.

iii) Mean waiting of a customer in the system

iv) Mean waiting time of custome in the queue,

if the service time for all the cars

a) is constants and equal to 10 mins

b) Follows Normal distribution with mean 12 mins and standard deviation 3 mins.

c) Follows uniform distribution between 8 mins and 12 mins.

ii) $L_q = L_s - \rho = 1.3330 - 0.6666 = 0.6664$.

iii) $W_s = \frac{L_s}{\lambda} = 1.3330 \times 15 = 19.995$ min

iv) $W_q = \frac{L_q}{\lambda} = 0.6664 \times 15 = 9.996$ min.

b) service time follows normal with mean 12 min & standard deviation 3 min

service time $= 12$ min / car

$$\frac{1}{\mu} = 12 \text{ min / car}$$

$$\mu = \frac{1}{12} \text{ car / min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/12} = \frac{1}{15} \times \frac{12}{1} = 0.8$$

S.D $= 3$ min

var(t) $= 9$ min

i) $$L_s = \rho + \frac{\lambda^2 \operatorname{var}(t) + \rho^2}{2(1-\rho)}$$

$$= 0.8 + \frac{9/15^2 + 0.8^2}{2(1-0.8)}$$

$$= 0.8 + \frac{0.68}{0.4} = 2.5$$

$L_s = 2.5$

ii) $L_q = L_s - \rho = 2.5 - 0.8 = 1.7$

iii) $W_s = \frac{L_s}{\lambda} = 2.5 \times 15 = 37.5$ min

iv) $W_q = \frac{L_q}{\lambda} = 1.7 \times 15 = 25.5$ min,

c) service time follows uniform between 8 min

$$\text{service time} = \frac{b+a}{2} = 10\text{min}$$

$$\frac{1}{\mu} = 10\text{min} / \text{car}$$

$$\mu = \frac{1}{10} \text{ car/min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{15} \div \frac{10}{\cdot 1} = 0.6666$$

$$Var(T) = \frac{(b-a)^2}{12} = \frac{16}{12} = 1.3333$$

i) $$L_s = \rho + \frac{\lambda^2 Var(T) + \rho^2}{2(1-\rho)}$$

$$= 0.6666 + \frac{\frac{1.3333}{15^2} + 0.6666^2}{2(1-0.6666)}$$

$$= 0.6666 + \frac{0.4502}{0.6668}$$

$L_s = 1.3417$.

UNIT 5

PROBABILITY AND QUEUEING THEORY
UNIT V

S.NO	Questions	opt 1	opt 2	opt 3	opt 4	opt 5	opt 6	Answer
1	Non markovian queueing model, the inter arrival and inter service times were assumed to follow _______ distribution	exponential	possion	binomial	normal			exponential
2	For the non Mrkovian queues _________ formula is used to calculate the averange number of customers in the system	Pollaczek - Khinchine	exponential	possion	binomial			Pollaczek - Khinchine
3	In PK formula the _______ follow a poission process with rate of arrival λ	service	arrival	queue	closed			arrival
4	In PK formula the arrival follow a __________process with rate of arrival λ	exponential	possion	binomial	normal			poission
5	In PK formula the arrival follow a poission process with rate of arrival _____	μ	α	λ	β			λ
6	_________ of queues can be described as a group of nodes	network	service	arrival	infinte			network
7	Network of queues can be described as a group of ____________	exponential	possion	nodes	normal			nodes
8	In an _____ network, customers enter the sysyem from outside and after service at one or more queues, eventually leave the system	closed queueing	open queueing	arrival queueing	service queueing			open queueing
9	In an open queueing network, customers enter the system from _________ and after service at one or more queues, eventually leave the system	between	inside	outside	none of this			outside
10	In an open queueing network, customers enter the sysyem from outside and after service at one or more queses, eventually ________ the system	enter	leave	inside	none of this			leave
11	A ________ network doesnot have any external arrival or departures	closed queueing	open queueing	arrival queueing	service queueing			closed queueing
12	A closed queueing network doesnot have any ___________or departures	external service	internal service	external arrival	internal arrival			external arrival
13	Arrival rate is denoted by ____	μ	α	λ	β			λ
14	Service rate is denoted by ____	μ	α	λ	β			μ
15	In multi server queues, there are many channels which provides the _______ facilities	same serivice	different serivice	same arrival	different arrival			same service
16	In_________queues, there are many channels which provides the same serivice facilities	single server	two server	three server	multi server			multi server
17	If any number of customers are allowed to join the queues then the capacity of the system is ______	one	five	finite	infinite			infinite

18	No of customers in the system ______	queue size	service size	arrival pattern	service pattern			queue size
19	No of customers in the system is denoted by______	n	a	c	d			n
20	The state in which there are n customers in the system _____	S_n	P_n	μ_n	L_q			S_n
21	Transient state probability that exactly n customers are in the system at the time t ______	S_n	P_n(t)	P_n	λ_n			P_n(t)
22	Steady state probability of having n customers in the system	S_n	P_n(t)	P_n	λ_n			P_n
23	Mean arrival rate ______	μ	α	λ	β			λ
24	Mean arrival rate when there are n customers in the system ______	λ_n	μ_n	L_s	L_q			λ_n
25	Mean service rate ______	μ	α	λ	β			μ
26	Mean service rate when there are n customers in the system ______	λ_n	μ_n	L_s	L_q			μ_n
27	In (M/M/1):(infinity /FIFO), the expected number of customers in the system L_s = ______	λ/(μ-λ)	μ/(μ-λ)	μ/(μ+λ)	λμ/(μ-λ)			λ/(μ-λ)
28	In (M/M/1):(infinity /FIFO), the expected queue size L_s = __________	λ/(μ-λ)	μ/(μ-λ)	μ/(μ-λ)	λμ/(μ-λ)			λ/(μ-λ)
29	In (M/M/1):(infinity /FIFO), the expected number of customers in the queue L_q = ______	W_s/(λ/μ)	W_q/(λ/μ)	L_s-(λ/μ)	L_q/(λ/μ)			L_s-(λ/μ)
30	In (M/M/1):(infinity /FIFO), the expected waiting time of a customers in the system W_s =______	W_s/λ	W_q/(λ/μ)	L_s/λ	L_q/λ			L_s/λ
31	In (M/M/1):(infinity /FIFO), the expected waiting time of a customers in the queue W_q= ______	W_s/λ	W_q/(λ/μ)	L_s/λ	L_q/λ			L_q/λ
32	In (M/M/1):(infinity /FIFO), the avereage number of customers in non-empty queues L_w = ______	λ/(μ-λ)	μ/(μ-λ)	μ/(μ+λ)	λμ/(μ-λ)			μ/(μ-λ)
33	Avereage waiting time of a customer in the system _______	W_s	W_q	L_s	L_q			W_s
34	Avereage waiting time of a customer in the queue _______	W_s	W_q	L_s	L_q			W_q
35	Average number of customers in the system is also denoted by ____	N_s	E(N_s)	L_s	E(L_s)			E(N_s)
36	By ________ in the system we mean the number of customer in the queue + the person who is getting serviced	W_s	W_q	L_s	L_q			L_s
37	Let N be the number of customers in the system, then the number of customer in the queue is _________	N	N-1	n	n-1			N-1
38	The average number of customers in the queues can also denoted as	N_q	E(N_q)	E(L_q)	E(L_s)			E(N_q)
39	If λ is the parameter of the exponetial distribution then the mean of the distribution is ________	λ	1-λ	1/λ	1+λ			1/λ
40	------ distribution follows Markovian process.	arrival	both interarrival and inter service	inter service	inter arrival			arrival
41	FCFS stands for------------.	first come first serve	first come finish service	first come fast serve	finish come first serve			first come first serve

42	The probability of the number of customers in the system exceeds k is given by ----------.	p[N > k]	p[N < k]	p[N ≥ k]	p[N ≤ k]			p[N > k]
43	Kendall’s notation for representing queueing model is ----------.	(a/b/c) : (d/e)	(a/b) : (c/d)	(a/b) : (c/d/e)	(a) : (b/c/d/e)			(a/b/c) : (d/e)
43	(a/b/c/):(d/e) is called ---------- notation for representing queueing models.	Kendall’s	cauchy’s	lagrrange’s	Newton’s			Kendall’s
44	r = λ/μ stands for --------	traffic intensity	mean arrival rate	expectation	mean service rate			traffic intensity
45	Arrival distribution follows -----------process.	Poisson	exponential	Uniform	Normal			Poisson
46	Probability of the system being empty is given by----------.	1- (λ/μ)	(λ/μ)-2	2- (λ/μ)	1+ (λ/μ)			1- (λ/μ)
47	Inter arrival distribution follows---------process.	Poisson	exponential	Uniform	Normal			exponential
48	If the characteristics of a ququqing system are independent of time, then the system is said to be in ----------- state.	steady	unsteady	transient	idle			steady
49	In M/M/1 queueing system mean arrival rate is 12 per hour and mean service rate is 24 per hour.The average number of customers in the system is ----------.	1	2	3	4			1
50	If the traffic intensity is 0.7 and the average arrival rate is 4 per minute then mean service rate is-------.	5.7143	6.7143	7.7143	8.7143			5.7143
51	In M/M/1 queueing system if the mean arrival rate is 6 per minute and mean service rate is 8 per minute then the average number of customers in the system is ----------.	½ minute	1 minute	3 minutes	1/3 minute			3 minutes
52	In M/M/1 queueing system mean arrival rate is 10 per hour and mean service rate is 12 per hour then the traffic intensity is ----------.	3/4	1 1/3	5/6	1 1/5			5/6
53	A super market has two girls attending to sales at the counters. If the service rate is ¼ and the mean arrival rate is 1/6 then the traffic intensity in given by ----.	½ minute	1 minute	3 minutes	1/3 minute			1/3 minute
54	The inter arrival rate is 6 per minute and mean service rate is 8 per minute then the average length of the queue is …………	3	4	5	2			4
55	If people arrive at one man barbershop and if the arrival rate is 1/12 per minute and mean service rate is 1/10 per minute then the probability of the customer go straight to the barbers chair is ………..	1/6	1/5	2/5	1			1/6
56	If people arrive at one man barbershop and if the arrival rate is 1/12 per minute and mean service rate is 1/10 per minute then the expected number of customers in the barber sh5op is…………	5	6	4	3			5
57	In M/M/1 queueing system mean arrival rate is 3 per hour and mean service rate is 4 per hour then the traffic intensity is ----------.	3/4	1 1/3	1 1/2	2/3			3/4
58	In M/M/1 queueing system mean arrival rate is 12 per hour and mean service rate is 30 per hour then the probability that there is no customers in the system is------.	0.6	0.5	0.2	0.25			0.6
59	Inter service distribution follows---------process.	Poisson	Exponential	Uniform	Normal			Exponential

1/λ

PROBABILITY AND QUEUEING THEORY
UNIT IV

S.NO	Questions	opt 1	opt 2	opt 3	opt 4	Answer
1	____________ is a function of the possible outcomes of an experiment and also time.	Random variable	Random process	Random line	Random Number	Random process
2	If t is fixed, then {X(s,t)} is a _____________	Random variable	Number	Single time function	Double time function	Random variable
3	If s and t is fixed, then {X(s,t)} is a _____________	Random variable	Number	Single time function	Double time function	Number
4	If s is fixed, then {X(s,t)} is a _____________	Random variable	Number	Single time function	Double time function	Single time function
5	If the index set 'T' is direct ,then the random process denoted by ______________	{x(n)}	X (n)	x	N	{x(n)}
6	A function of the possible outcomes of an experiment is _____________	Random variable	Random process	Stochastic process	Number	Random variable
7	The probabilistic model used for characterizing a random signal is called _________	Stochastic process				
8	In random process outcomes are mapped into ____________form which is function at time t	line	wave	perpendicular	parellel	wave
9	If X is continuous and t can have any of a continuum of values, then X(t) is called as_________	continuous random process	discrete random process	continuous random sequences	discrete random sequences	continuous random process
10	If X assumes only discrete and t is continuous, then X(t) is called as ___________	continuous random process	discrete random process	continuous random sequences	discrete random sequences	discrete random process
11	If X is continuous but time t takes only discrete values, then it is called as ________________	continuous random variable	discrete random variable	continuous random sequences	discrete random sequences	continuous random sequences
12	A random process X(s,t) in which both s and t are discrete is called	continuous random process	discrete random process	continuous random sequences	discrete random sequences	discrete random sequences
13	A process is called ___________ process if the future values of any sample function cannot be predicted exactly from observed values	deterministic	non deterministic	strong stationary	jointly stationary	non deterministic
14	A process is called non deterministic process if the future values of any sample function cannot be predicted exactly from ______________	past values	observed values	future values	joint values	observed values
15	A process is called ___________ process if the future values of any sample function cannot be predicted exactly from past values	deterministic	non deterministic	strong stationary	jointly stationary	deterministic
16	A process is called deterministic process if the future values of any sample function cannot be predicted exactly from ______________	past values	observed values	future values	joint values	past values
17	A random process is called a ------- , if all its finite dimensional distributions are invariant under translation of time parameter.	wide sence stationary process	Strict sense stationary process	evolutionary process	covariance stationary	Strict sense stationary process
18	A random process is said to be stationary if its _______________ etc are constants	mean	variance	moments	all the above	all the above
19	A random process is said to be stationary if its mean, variable, moments, etc are ______	past values	observed values	constant	variable	constant
20	A continuous random process satisfying markov property is known as _______________________	continuous paramerter markov process	discrete paramerter markov process	continuous paramerter markov chain	discrete paramerter markov chain	continuous paramerter markov process

21	A continuous random sequence satisfying markov property is known as ________________________	continuous paramerter markov process	discrete paramerter markov process	continuous paramerter markov chain	discrete paramerter markov chain	discrete paramerter markov process
22	A discrete random sequence satisfying markov property is known as	continuous paramerter markov process	discrete paramerter markov process	continuous paramerter markov chain	discrete paramerter markov chain	discrete paramerter markov chain
23	A discrete random process satisfying markov property is known as	continuous paramerter markov process	discrete paramerter markov process	continuous paramerter markov chain	discrete paramerter markov chain	continuous paramerter markov chain
24	A markov chain is said to be __________ , if every state can be reached from every other state, where P_(ii)(n)>0 for some n.	irreducible	non - irreducible	recurrent state	return state	irreducible
25	P_(ii)(n)>0, for some n>1, then we call the state I of the markov chain as ________	irreducible	non - irreducible	recurrent state	return state	return state
26	A sum of two independent poisson process is ____________	normal process	stochastic process	poisson process	binomial process	poisson process
27	A ___________ order stationary process has a constant mean	first	second	third	fourth	first
28	A first order ___________ process has a constant mean	stochastic	non deterministic	stationary	nonstationary	stationary
29	A first order stationary process has a constant ____________	mean	variance	moments	all the above	mean
30	A first order stationary process has a __________ mean	constant	variable 'x'	variable 't'	none of this	constant
31	A random process that is not stationary in any sense is called as _______	normal process	stochastic process	poisson process	evolutionary process	evolutionary process
32	A state I is said to be _______ if and only if there is positive probability that the process will not return to this state	transient	irregular	periodic	aperiodic	transient
32	A random process is called a --- if its mean is constant and the autocorrelation depends only on the time difference.	weakly stationary proces	strict sense stationary process	strong sense stationary process	evolutionary	weakly stationary proces
33	The tpm of a chain is --- matrix, since all Pij ≥ 0 and the sum of all elements of any row of the transition matrix is equal to 1.	stochastic	singular	non singular	row	stochastic
34	A stochastic matrix is said to be a regular matrix, if all the entries of P ^ m are---	positive	negative	zero	fractional	positive
35	The state i is said to be --- if the return to state i is uncertain.	transient	persistent	recurrent	non null persistent	transient
36	Let X denote the number of telephone calls received in the interval (0,t). Then {X(t)} is a	continuous random process	discrete random process	continuous random sequences	discrete random sequences	discrete random process
37	The Poisson process is a ___	Markov process	Strict sense stationary process	weak sense process	weak sense process	Markov process
38	The tpm of an irreducible chain is an ---matrix.	Irreducible	reducible	singular	non singular	Irreducible
39	Binomial process is a ___	strict sense stationary process	wide sense proces	weak sense process	ergodic process	strict sense stationary process
40	The maximum temperature of a particular place is (0, t). The set of possible values of X is continuous in continuous time is an example of ------	continuous random process	discrete random process	continuous random sequences	discrete random sequences	continuous random process
41	If mean ≠ constant then the process is called --	first-order stationary process.	not a first-order stationary process.	second-order stationary process.	stationary process of order three.	not a first-order stationary process.
42	Markov process is classified into ---- types.	4	2	3	5	4
43	The transition probability matrix of a finite state Markov chain is a ___	Row matrix	column matrix	square matrix	identity matrix	square matrix
44	The family of all functions X(s,t) is called ----	random process	random signal	discrete process	discrete signal	random process
45	There exists processes where the statistical value like mean, variance are constants. Such processes are called ----	deterministic process	non-deterministic process	stationary process	wide sense process	stationary process
46	SSS process is----	strict sense stationary process	wide sense proces	weak sense process	ergodic process	strict sense stationary process
47	A random process in which the future value depends only on the present value and not on the past values is called a -----	Markov process	Markov chain	Irreducible chain	Periodic state	Markov process
48	The derivative of mean of a stationary process is---	1	3	4	0	0
49	RXY(t) =0 if the processes are---	independent	orthogonal	dependent	same	orthogonal
50	Two random processes are said to be uncorrelated if	$C_{XY}(t_1,t_2) = 0$	$C_{XY}(t_1,t_2) = Rxy(t_1,t_2) + E[X(t_1)]\ E[Y(t_2)]$	$C_{XY}(t_1,t_2) = Rxy(t_1,t_2)\ E[X(t_1)]\ E[Y(t_2)]$	$C_{XY}(t_1,t_2) = Rxy(t_1,t_2) /\ E[X(t_1)]\ E[Y(t_2)]$	$C_{XY}(t_1,t_2) = 0$
51	The state i is said to be ----- if its mean recurrence time is finite.	non null persistent	null persistent	transient	recurrent	non null persistent
52	The state i is said to be ----- if its mean recurrence time is infinite.	non null persistent	null persistent	transient	recurrent	null persistent
53	If the transition probability matrix is regular, then the homogeneous Markov chain is	regular	irregular	periodic	aperiodic	regular
54	If the period di =1, then state i is said to be -----	regular	irregular	periodic	aperiodic	aperiodic
56	A non null persistent and aperiodic state is called ---	stochastic	ergodic	WSS	SSS	ergodic

57	The set of possible values of any individual members of the random process is called ___ space.	vector	state	random	universal	state
58	The discrete parameter Markov process is called a ____	weakly stationary process	covariance stationary process	wide-sense stationary process	Markov chain	Markov chain
59	Two states i and j which are accessible to each other are said to ___	Irreducible	reducible	communicate	absorbing	communicate
60	A state is said to be an ___ state if no other state is accessible from it.	Irreducible	reducible	communicate	absorbing	absorbing
61	All states of a finite irreducible markov chain are ___	recurrent	reducible	communicate	absorbing	recurrent
62	All regular Markov chain are ___	Markov process	ergodic process	WSS	SSS	ergodic process

TWO DIMENSIONAL RANDOM VARIABLES
UNIT III

S.NO	Questions	opt 1	opt 2	opt 3	opt 4	Answer
1	Let S be the sample space. Let X &Y be two functions each assigning a real number to each outcome. Then (X,Y) is a ------------------	two dimensional random variable	one dimensional random variable	Marginal distribution	Conditional distribution	two dimensional random variable
2	If the possible values of (X,Y) are finite or countably infinite, then (X,Y) is called a ------.	two dimensional contiouous random variable	one dimensional discrete random variable	two dimensional contiouous random variable	one dimensional contiouous random variable	two dimensional contiouous random variable
3	The function f (x,y) = P(X=xi, Y=yj) = P (xi,yj) is called the -----for continuous random variable X & Y.	joint probability density function	joint probability mass function	joint probability function	cumulative distribution function	joint probability density function
4	The correlation between the heights and weights in increasing order of a group of persons is	positive	negative	zero	partial	positive
5	The value of the correlation coefficient lies ---- .	between -1 and 1	between 1 and 2	between 2 and 3	between 0 and 1	between -1 and 1
6	If the curve is a straight line, then it is called the -----	the line of correlation	the line of regression	the line of covariance	the line of variance	the line of regression
7	The function f (x,y) = P(X=xi, Y=yj) = P (xi,yj) is called the -----for continuous random variable X & Y.	joint probability density function	joint probability mass function	joint probability function	cumulative distribution function	joint probability mass function
8	The set {yj. pj.) is called -------.	distribution function	marginal distribution function of Y	marginal distribution function of X.	cumulative distribution function	marginal distribution function of Y
9	The value of F(- infinity , y) =	1	2	3	0	0
10	If X & Y are independent then the correlation coefficient r =	1	2	0	3	0
11	_________is a mathematical measure of the average relationship between two or more variables interms of the original limits of the data	Correlation	Regression	covariance	mean	Regression
12	If the correlation coefficient r =0, we get------	tanq=π	tanq=π/3	tanq=π/4	tanq=π/2	tanq=π/2
13	The coefficient of correlation is independent of change of_____ and ________	scale, origin	vector, origin	variable, constant	interer, origin	scale, origin
14	The regression analysis confined to their study of only two variable at a time is called __________regression.	Simple	Multiple	Linear	two	Simple
15	Joint probability is the probability of the _________occurrence of two or more events.	Simultaneous (or) joint	Conditional	Marginal probability	density function	Simultaneous (or) joint
16	If X=Y , then correlation coefficient between them is _______	1	zero	less than one	greater than one	1
17	If the possible values of (X,Y) are finite, then (X,Y) is called a ------.	continuous	Two dimensional random variable	one dimensional random variable	both one dimensional and two dimensional	Two dimensional random variable
18	If X & Y are -----random variable , then f(x,y) is called joint probability density function.	Discrete	continuous	both discrete and contiouous	one dimensional random variable	continuous
19	The regression analysis confined to their study of only two variable at a time is called __________regression.	two	Simple	Multiple	Linear	Simple
20	In Rank correlation the correction factor is added for each ________ value.	Repeated	Non-repeated	single	fractional	Repeated
21	When the correlation coefficient is equal to___________ the correlation is perfect and positive.	1	2	0	3	1
22	The set {xi. pj.) is called -------.	distribution function	marginal distribution function of Y	marginal distribution function of X.	cumulative distribution function	marginal distribution function of X.
23	The correlation between volume and pressure of a perfect gas is	positive	negative	partial	multiple	negative
24	If the value of y decreases as the value of x increases then there is ----- correlation between two variables.	positive	negative	partial	multiple	negative
25	The correlation between the income and expenditure is_____.	positive	negative	partial	multiple	positive
26	Two random variables X and Y with joint probability density function of f(x,y) is said toindependent if ____.	f(x,y)=f(x)+f(y)	f(x,y)=f(x)-f(y)	f(x,y)=f(x)*f(y)	f(x,y)=f(x)/f(y)	f(x,y)=f(x)*f(y)
27	The value of F(- infinity , infinity) =	1	2	3	0	1
28	The ordinal number indicating the position of a given attributes in the ranking is called ____	Deviation	Karl Pearson's	Rank	Regression	Rank
29	If the variation of one variable has no relation with the variation on the other is called ___ correlation.	positive	negative	zero	multiple	zero
30	___ method gives us exact measure of degree of correlation between two variables.	Spearman's	Karl Pearson's	Rank	Cauchy's	Spearman's
31	If fx(x)>0, then f(y/x) = ______.	f(x,y)*f(x).	f(x,y)-f(x)	f(x,y)+f(x).	f(x,y)/f(x)	f(x,y)/f(x)
32	Cov(aX, bY) =______.	b Cov(X,Y)	a Cov(X,Y)	ab Cov(X,Y)	Cov(aX, bY)	ab Cov(X,Y)
33	If X and Y are statically independent then Cov (X,Y) = __	1	0	2	3	0
34	If f_Y(Y)>0, then f(X/Y) =	f(x,y)*f(x).	f(x,y)-f(x)	f(x,y)+f(x).	f(x,y)/f(x)	f(x,y)/f(y)
35	The points in the scatter diagram will cluster around some curve called---	the curve of correlation	the curve of regression	correlation	regression	the curve of regression
36	The ___ theorem holds for random variables with finite mean and variance.	Baye's	Central limit	Einstein-Wiener	Binomial	Central limit
37	If two random variables are independent, then the density function of their ___ is given by the convolution of their density functions.	quotient	multiply	subtract	sum	sum
38	The coefficient of variation is defined as ___	(standard deviation / mean) + 100	(standard deviation / mean) * 100	(standard deviation / mean) - 100	(mean / standard deviation) * 100	(standard deviation / mean) * 100
39	Relation between corellation and regression is given by----.	r = Square root [b_(yx) * b_(xy)]	r = Square root [b_(yx) / b_(xy)]	r = Square root [b_(yx) + b_(xy)]	r = Square root [b_(yx) - b_(xy)]	r = Square root [b_(yx) * b_(xy)]
40	In ___ correlation relationship between three or more variables is studied.	simple	partial	multiple	linear	multiple
41	Perfect positive correlation is also called ___ correlation.	direct	indirect	inverse	partial	direct
42	Two random variables are said to be orthogonal if _______	correlation is zero	rank is zero	covariance is zero	one	correlation is zero
43	Two random variables are said to be uncorrelated if correlation coefficient is____	zero	one	two or more	orthogonal	zero
44	The coefficient of correlation___	cannot be positive	cannot be negative	can be either positive or negative	zero	can be either positive or negative
45	If X=Y , then correlation cofficient between them is _______	1	zero	less than one	gerater than one	1
46	Correlation means relationship between ______ variables	two	one	two or more	three	two or more

STANDARD DISTRIBUTIONS
UNIT II

S.NO	Questions	opt 1	opt 2	opt 3	opt 4	Answer
1	Binomial distribution is symmetrical if	p = q = ½	p = q = ¾	p = q = 4/5	p = q = 2/3	p = q = ½
2	A normal curve has an ___	Elliptic	parabolic	hyperbolic	asymptote	asymptote
3	Variance of binomial distribution is_______.	npq	np	nq	square root of (npq)	npq
4	"The number of printing errors at each page of a book" is a example of ___ distribution.	Normal	Uniform	Binomial	Poisson	Poisson
5	Moment generating function of Uniform density function is	e^bt- e^at/ t (b-a)	e^bt+ e^at/ t (b+a)	e^bt+ e^at/ t (b-a)	e^bt- e^at/ t (b+a)	e^bt- e^at/ t (b-a)
6	For binomial distribution ______	variance -1= mean	variance = mean	variance > mean	variance < mean	variance < mean
7	------------- is a non negative continuous random variable.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Gamma distribution
8	Standard deviation of binomial distribution is ____	√npq	np(q-p)	npq	npq(q-p)	√npq
9	The density function of the Uniform distribution is _________.	1/ ba a<x<b	1/ (b+a) a<x<b	1/ ba a>x>b	1/ (b-a) a<x<b	1/ (b-a) a<x<b
10	Moment generating function of Binomial distribution	Mx (t) = (1 – p) ^n	Mx (t) = (pe^t + q)^ n	Mx (t) = (pe^t – p) ^n	Mx (t) = (e^t + q)^ n	Mx (t) = (pe^t +q)^ n
11	The mean and variance of a standard normal distribution is ___	N(1,2)	N(0,1)	N(0,2)	N(0,40)	N(0,1)
12	Variance of Uniform density function is	b /2	[(b-a)^2]/12	ba /2	[(b+a)^2] /12	[(b-a)^2]/12
13	A continuous random variable X has a probability density function f(x)= K, 0 <= x <= 1. Find K.	1	2	3	4	1
14	If X is normally distributed with mean 1 and S.D. ½ , find the probability that X > 2.	0.0288	0.0544	0.0228	0.0882	0.0228
15	Mean of Uniform density function is___________	b /2	(b-a) /2	ba /2	(b+a) /2	(b+a) /2
16	The formula of variance is ___________	Var[X] = E(X^2) – [E(X)]^ 2	**Var[X] = E(X^2) – [E(X)]**	Var[X] = E(X) – [E(X)]^ 2	Var[X] = E(X) – [E(X)].	Var[X] = E(X^2) – [E(X)]^ 2
17	Binomial distribution is	nCx(p^x)q^(n-x)	nCx(p^x)q^(n+x)	nCx(p^-x)q^(n-x)	nCx(p^-x)q^(n+x)	nCx(p^x)q^(n-x)
18	Mean of Poisson distribution is ____________	λ	λ/t	λ-t	λ+t	λ
19	Gamma of n =	(n-1) !	(n +1) !	n!	0!	(n-1) !
20	Moment generating function of Exponential distribution is	λ/(λ-t)	2λ/(λ-t)	λ/(λ+t)	2λ/(λ-+t)	λ/(λ-t)
21	Variance of Poisson distribution is	λ	2λ	3λ	4λ	λ
22	Exponential distribution is	λe^(λx)	λe^(-λx)	λe^(-2λx)	λe^(2λx)	λe^(-λx)
23	Gamma of (1/2) =	(π^2)/2	square root of π/2	square root of π/3	square root of π	square root of π
24	Moment generating function of Poisson distribution	e^ [λ(e^t-1)]	e^ [λ(e^t+1)]	e^ [λ(e^t-2)]	e^ [-λ(e^t-1)]	e^ [λ(e^t-1)]
25	Mean of Exponential distribution is	1/λ	λ	λ/2	2/λ	1/λ
26	Geometric distribution is given byP{X=x}=_____where x=1,2,…	pq^(x-1)	pq^(x-2)	pq^(x+1)	pq^(2x-1)	pq^(x-1)
27	If the mean and variance of a binomial variate are 8 and 6, then the probability of failure is given by______.	q=3/4	q=4/3	q=1/4	q=1/3	q=3/4
28	If the mean and variance of a binomial variate are 20 and 16, then the probability of success is given by______.	p=1/5	p=2/5	p=3/5	p=3/4	p=1/5
29	If n=5 and p=1/2 then the mean of a binomial variate is ______.	0.50	2.50	3.5	4.5	2.5
30	The mean of a poisson variate is 2 . Find its variance.	2	3	1	2.5	2
31	Poisson distribution is the limiting case of __________distribution.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Binomial distribution

32	The other name of uniform distribution is__________.	Binomial distribution	Gamma distribution	Poisson distribution	rectangular distribution	rectangular distribution
33	In a Uniform distribution if X is distributed uniformly on (0,30) then its density function is given by______.	F(x)= 1/13	F(x)= 2/13	F(x)= 1/3	F(x)= 1/30	F(x)= 1/30
34	______ is larger than the mean for a negative binomial distribution	Variance	Standard deviation	Mean deviation	Quartile deviation	Variance
35	Mean of binomial distribution is ____	np	npq	n+1	n	np
36	Third moment of Binomial distribution is ____	n(q-p)	np(q-p)	npq	npq(q-p)	npq(q-p)
37	For a negative binomial distribution______	Var(X) > E(X)	Var(X) < E(X)	Var(X) = E(X)	Var(X) / E(X)	Var(X) > E(X)
38	If X follows a Poisson distribution such that P(X=1) = 1/4 and P(X=2) = 3/8, find P(X=3).	0.123	1.234	2.34	0.375	0.375
39	The height of persons in a country is a random variable of the type____	continuous random variable	neither discrete nor continuous random vaiables	Continuous as well as discrete random variable	discrete random variable	continuous random variable
40	A family of parametric distrbution in which mean is equal to variance is _______	Binomial distribution	Gamma distribution	normal Distribution	Poissson distribution	Poissson distribution
41	A family of parametric distrbution in which mean is always greater than its variance is _______	Binomial distribution	Gamma distribution	Geometric Distribution	Poissson distribution	Geometric Distribution
42	The _________ distribution has the memory less property.	Gamma distribution	Geometric distribution	Geometric distribution	Poissson distribution	Geometric distribution
43	The mean of the binomial distribution is _____ than its variance	greater than	Less than	more	normal	greater than
44	Mean and variance of geometric distribution are__________	related	correlated	rectangle	range	related
45	A distribution where the mean and median have different values is not a _________ distribution	normal	binomial	poisson	gamma	normal
46	Normal distribution was invented by _____	Laplace	De-Moivre	Gauss	all the above	all the above

PROBABILITY AND RAMDOM VARIABLES

UNIT I

S.NO	Questions	opt 1	opt 2	opt 3	opt 4	opt 5	opt 6
1	"It is possible to live without water" There is ------	Certainty	uncertainty	possibility	impossibility		
2	In drawing a card from the pack of cards, the numbers of cases favorable to the event of getting a diamond card is	7	12	10	3		
3	Probability of the possible events is	1	0	2	3		
4	"Everyday the sun rises in the east" Look the statement and there is-------	certainty	uncertainty	possibility	impossibility		
5	" Probably Arun gets that job" There is -----	certainty	uncertainty	possibility	impossibility		
6	In the theory of probability we represent 'certainty' by ---	1	2	10	0		
7	In the theory of probability we represent 'uncertainty' by ---	1	2	a positive fraction	0		
8	In the theory of probability we represent 'impossibility' by ---	1	2	a positive fraction	0		
9	The value of uncertainty lies between----	1 to 10	10 to 15	0 and 1	-1 to 1		
10	An action or an operation which can produce any result or outcome is called a -----	Random experiment	Probability	Statistics	mathematics		
11	The outcomes of an action is known as -----	event	Trial	Random experiment	Theory		
12	The another name of Random experiment is ----	event	Trial	Random experiment	Theory		
13	The another name of events is ----	event	Trial	Random experiment	cases		
14	Rolling of die is a -----	Trial	event	Random experiment	cases		
15	In a rolling of die, getting 6 is ----	Trial	event	Random experiment	cases		
16	An event whose occurrence is inevitable when an experiment is performed is called as -------	cases	Trial	Certain Event	Random experiment		

17	Another name of certain event is----	event	Uncertain event	Compound event	Sure event
18	An event which can never occur when an experiment is performed is called an -----	Impossible event	Compound event	Sure event	Certain Event
19	There are ---- types of events.	7	9	10	2
20	An event is called ---- if it corresponds to a single possible out come of the experiment.	simple	Compound event	Certain event	Composite event
21	An event is called ---- if it does not correspond to a single possible out come of the experiment.	simple	Compound event	Certain event	Uncertain event
22	Another name of compound event is----	simple	Composite event	Certain event	Uncertain event
23	In rolling of single die, occurrence of 8 is an ------	simple	Compound event	Certain event	Impossible event
24	In rolling of single die, the chance of getting 5 is a -------	simple	Compound event	Certain event	Impossible event
25	In rolling of single die, the chance of getting 2,4,6(even numbers) are -------	simple	Compound event	Certain event	Impossible event
26	The totality of all possible outcomes of a random experiment is called a	Sample space.	Certain event	Impossible event	Compound event
27	A possible outcome or element in a sample space is called a ------	Sample event	Certain event	Sample point	Compound event
28	In throwing a die, all the outcomes 1,2,3,4,5 and 6 are together constitute a sample space. Getting any on eof the face upwards is called -----	Sample event	Certain event	Sample point	Compound event
29	The number of cases favorable to an event in a trial are the number of outcomes which entail the happening of the event is called---	Favourable event	Certain event	Sample point	Compound event
30	In a tossing of two coins, the number if cases favorable to the event of getting a head are ---	7	9	10	3
31	In drawing a card from the pack of cards, the numbers of cases favorable to the event of getting a court card are	7	12	10	3
32	The out comes are said to be -------------if none of them is expected to occur in preference to other.	Equally likely	Exhaustive events	Dependent events.	Complementary events.

33	Two events are said to be -----------when both cannot happen simultaneously in a single trial .	Mutually exclusive	Exhaustive events	Dependent events.	Complementary events.
34	Out comes are said to be ---- when they include all possible outcomes.	Mutually exclusive	Exhaustive events	Dependent events.	Complementary events.
35	Two or more events are considered to be ------if the occurrence of an event does not affect the occurrence of the other.	Mutually exclusive	Exhaustive events	Dependent events.	Independent events.
36	Two events are said to be-------, if the occurrence or nonoccurrence of an event in any trial affects the occurrence of the other event in other trials.	Mutually exclusive	Exhaustive events	Dependent events.	Independent events.
37	Let A and B are two events. Then A is called ----------- of B if A and B are mutually exclusive and exhaustive.	Complementary event	Exhaustive events	Dependent events.	Independent events.
38	p + q = -----	7	9	1	3
39	If P(A) is 1, the event A is called a ----	Cases	Trial	Certain Event	Random experiment
40	If P(A) is 0, the event A is called a ----	Cases	Trial	Certain Event	Impossible event
41	Probability of the impossible events is	0	9	8	3
42	The rule from the original sample space to a numerical sample space, subjected to constraints is called a_____.	Random variable	Random constant	constant	variable
43	Random variable is a ---- which maps the numerical or non-numerical sample space of the random experiment to real values.	Imaginary valued function	Real valued function	Complex valued function	Rational valued function
44	Random variable is a real valued function which is also -----	Multi-valued function	double-valued function	Single valued function	triple-valued function
45	The set of values which the random variable X takes is called --- of the random variable.	Random variable	One-dimensional random variable	Two dimensional random variable	Spectrum
46	------ is one which takes on real values.	Real random variable	Complex random variable	Multi dimensional random variable	Continuous random variable

47	------- is defined in terms of real random variable.	Real random variable	Complex random variable	Multi dimensional random variable	Continuous random variable
48	A random variable X is --- if it assumes only discrete values.	spectrum	complex	continuous	discrete
49	The vector random variable is also known as ------	random vector	scalar vector	one-dimensional random variable	two dimensional random variable
50	A function which assigns a vector to each point of sample space is called a----	vector random variable	scalar vector	one-dimensional random variable	two dimensional random variable
51	One-dimensional random variable is also called as -----	scalar-valued random variable	scalar vector	one-dimensional random variable	two dimensional random variable
52	The spectrum of random variable can have -----	positive value	negative value	positive integer value	negative integer value
53	Probability of a single real value in a continuous random variable is---	two	three	four	zero
54	Probability distribution function and probability density function of a continuous random variable are -----	continuous	discrete	single	partial
55	For a discrete random variable, probability distribution function is right continuous and probability density function is -----	continuous	discrete	single	partial
56	For a mixed random variable, probability distribution function and probability density function are -----	continuous	discrete	single	mixed
57	For a discrete random variable, the probability density function represents the----.	probability mass function	probability distribution function	probability density function	none of these

58	If P(X< or =k) = P(X>k), then each probability is equal to ----	1	2	3	½
59	A density function may correspond to different ------	probability mass function	probability distribution function	probability density function	random variable
60	A discrete random variable can be considered as a limiting case of --- with impulse distribution.	continuous random variable	discrete random variable	single random variable	mixed random variable

Answer

impossibility

12

1

certainty

uncertainty

1

a positive fraction

0

0 and 1

Random experiment

event

Trial

cases

Trial

event

Certain Event

Sure event

Impossible event

2

simple

Compound event

Composite event

Impossible event

simple

Compound event

Sample space.

Sample point

Sample point

Favourable event

3

12

Equally likely

Mutually
exclusive
Exhaustive
events

Independent
events.

Dependent
events.

Complementary
event
1

Certain Event

Impossible
event
0
Random
variable

Real valued
function

Single valued
function

Spectrum

Real random
variable

Complex
random variable

discrete

random vector

vector random
variable

scalar-valued
random variable

negative value

zero

continuous

discrete

mixed

probability
mass function

½

random variable

continuous
random variable