KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University)
(Established Under Section 3 of UGC Act 1956)
Coimbatore - 641021.
(For the candidates admitted from 2018 onwards)
DEPARTMENT OF PHYSICS

SUBJECT : **MATHEMATICAL PHYSICS - I** **SEMESTER** : **I**
SUBJECT CODE: **18PHU103** **CLASS** : **I B.Sc.Phy**

SCOPE:

To make use of the mathematical methods to solve physics problems and to provide students with basic skills necessary for the application of mathematical methods in physics

Objective:

The main objective of this course is to provide the student with a repertoire of mathematical methods that are essential to the solution of advanced problems encountered in the fields of applied physics and engineering.In addition, this course is intended to prepare the student with mathematical tools and techniques that are required in advanced courses offered in the applied physics

UNIT 1

Basic of C language: Introduction, Data types, Operators and Expressions, Conditional Statements, Input and output Statements (Programs)

UNIT 2

Complex Analysis: Brief revision of Complex numbers & their graphical representation. Roots of Complex Numbers. Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions. Singular functions: poles and branch points, order of singularity. Integration of a function of a complex variable. Cauchy's Integral formula.

UNIT 3

Special Functions Definition – The Beta function – Gamma function – Evaluation of Beta function – Other forms of Beta function – Evaluation of Gamma function – Other forms of Gamma function - Relation between Beta and Gamma functions – Problems.

UNIT4

Matrices Introduction – special types of Matrices – Transpose of a Matrix – The Conjugate of a Matrix – Conjugate Transpose of a Matrix – Symmetric and Anti symmetric – Hermitian and skew Hermitian – Orthogonal and Unitary Matrices – Properties – Characteristics equation – Roots and characteristics vector – Diagonalization of matrices – Cayley – Hamilton theorem – Problems

UNIT 5

Vector Calculus ∇ Operator – Divergence – Second derivative of Vector functions or fields – The Laplacian Operator – Curl of a Vector – Line Integral – Line Integral of a Vector field around an infinitesimal rectangle – Curl of Conservative field – Surface Integral – Volume Integral (without problem) – Gauss's Divergence theorem and it's proof in the simple problems – Stoke's and its proof with simple problems.

SUGGESTED READINGS:

1. Mathematical Physics by Sathya prakash, S.Chand & company, New Delhi.
2. Mathematical Physics by B.D.Gupta, Vikas Publishing house Pvt Ltd, New Delhi.
3. Introduction to Mathematical Physics: Methods & Concepts, By Chun Wa Wong, 2013, Oxford University press, ISBN -978-0-19-964139-0.
4. Mathematical Methods for Physicists: Arfken, Weber, 2005, Harris, Elsevier. Fourier Analysis by M.R. Spiegel, 2004, Tata McGraw-Hill.
5. Essential Mathematical Methods, K.F.Riley and M.P.Hobson, 2011, Cambridge University Press.
6. Mathematical Physics by By Shigeji Fujita, Salvador V. Godoy, 2010, Wiley VCH Verlag GmbH & Co. KGaA, ISBN- 978-3-527-40808-5.
7. Mathematical Physics by Pavan Kumar Chaurasya, Campus Books International publisher, 2013, ISBN - 8180303160, 9788180303166.
8. Mathematical Physics by Harper. C, Prentice Hall of India, 2013.

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DEPARTMENT OF PHYSICS

SUBJECT : **MATHEMATICAL PHYSICS -I** **SEMESTER** : **I**
SUBJECT CODE: **18PHU103** **CLASS** : **I B.Sc.Phy**

LECTURE PLAN
DEPARTMENT OF PHYSICS

S.No	Lecture Duration Hour	Topics to be covered	Support Material/Page.Nos
		UNIT-I	
1	**1**	Introduction	**T9:1**
2	**1**	Data types	**T9:11,12**
3	**1**	Operators and Expressions	**T9:16-27**
4	**1**	Conditional Statements	
5	**1**	Continuation	
6	**1**	Input and output Statements	**T9:27-44**
7	**1**	Continuation	
8	**1**	(Programs)	
9	**1**	Continuation	
10	**1**	Revision	
	Total No Of Hours Planned ForUnit-I=10		
		UNIT-II	
1	**1**	Introduction	**T1:287**
2	**1**	Brief revision of Complex numbers & their graphical representation.	**T1:289**
3	**1**	Roots of Complex Numbers.	
4	**1**	Functions of Complex Variables..	**T1:293**
5	**1**	Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions	**T1:296**
6	**1**	Continuation	
7	**1**	Singular functions: poles and branch points, order of singularity.	**T1:331**

8	**1**	Integration of a function of a complex variable.	
9	**1**	Cauchy's Integral formula	**T1:318**
10	**1**	Continuation	
11	**1**	Revision	
	Total No Of Hours Planned ForUnit-II=11		
		UNIT-III	
1	**1**	Introduction	
2	**1**	Special Functions Definition	**T1:262**
3	**1**	The Beta function	**T1:262**
4	**1**	Gamma function	**T1:262**
5	**1**	Evaluation of Beta function	**T1:263**
6	**1**	Other forms of Beta function	**T1:263**
7	**1**	Evaluation of Gamma function	**T1:264**
8	**1**	Other forms of Gamma function.	**T1:265**
9	**1**	Relation between Beta and Gamma functions	**T1:266**
10	**1**	Problems	
11	**1**	Revision	
	Total No Of Hours Planned ForUnit-III=11		
		UNIT-IV	
1	**1**	Matrices Introduction	**T1:109**
2	**1**	special types of Matrices	**T1:109-113**
3	**1**	Transpose of a Matrix	**T1:112**
4	**1**	The Conjugate of a Matrix	**T1:114**
5	**1**	Conjugate Transpose of a Matrix	
6	**1**	Symmetric and Anti symmetric	**T1:114**
7	**1**	Hermitian and skew Hermitian	**T1:117**
8	**1**	Orthogonal and Unitary Matrices ,Properties	**T1:127,129**
9	**1**	Characteristics equation , Roots and characteristics vector	**T1:159,160**
10	**1**	Diagonalization of matrices	**T1:173**
11	**1**	Cayley – Hamilton theorem	**T1:165-166**
12	**1**	Problems	
13	**1**	Problems	
14	**1**	Problems	
15	**1**	Revision	
	Total No Of Hours Planned ForUnit-IV=15		
		UNIT-V	

1	**1**	Vector Calculus ∇ Operator	**T1:18**
2	**1**	Divergence – Second derivative of Vector functions or fields	**T1:24-28**
3	**1**	The Laplacian Operator	**T1:29**
4	**1**	Curl of a Vector.	
5	**1**	Line Integral – Line Integral of a Vector field around an infinitesimal rectangle	**T1:22**
6	**1**	Curl of Conservative field – Surface Integral – Volume Integral (without problem)	**T1:22,23**
7	**1**	Gauss's Divergence theorem and it's proof in the simple problems	**T1:36**
8	**1**	Stoke's and its proof with simple problems	**T1:47**
9	**1**	Problems	
10	**1**	Revision	
11	**1**	Old Question paper Discussion	
12	**1**	Old Question paper Discussion	
13	**1**	Old Question paper Discussion	
	Total No Of Hours Planned ForUnit-V=13		
Total No Of Hours Planned = 60			

SUGGESTED READINGS:

1. Mathematical Physics by Sathya prakash, S.Chand & company, New Delhi.
2. Mathematical Physics by B.D.Gupta, Vikas Publishing house Pvt Ltd, New Delhi.
3. Introduction to Mathematical Physics: Methods & Concepts, By Chun Wa Wong, 2013, Oxford University press, ISBN -978-0-19-964139-0.
4. Mathematical Methods for Physicists: Arfken, Weber, 2005, Harris, Elsevier. Fourier Analysis by M.R. Spiegel, 2004, Tata McGraw-Hill.
5. Essential Mathematical Methods, K.F.Riley and M.P.Hobson, 2011, Cambridge University Press.
6. Mathematical Physics by By Shigeji Fujita, Salvador V. Godoy, 2010, Wiley VCH Verlag GmbH & Co. KGaA, ISBN- 978-3-527-40808-5.
7. Mathematical Physics by Pavan Kumar Chaurasya, Campus Books International publisher, 2013, ISBN - 8180303160, 9788180303166.

8. Mathematical Physics by Harper. C, Prentice Hall of India, 2013.
9. C Programming and Oops by I.Edwin Dayanand and R.K.Selvakumar, N.V.Publications, pollachi.

UNIT I

Basic of C language: Introduction, Data types, Operators and Expressions, Conditional Statements, Input and output Statements (Program)

BASICS OF C LANGUAGE:

C is a general-purpose high level language that was originally developed by Dennis Ritchie for the Unix operating system. It was first implemented on the Digital Equipment Corporation PDP-11 computer in 1972.

The Unix operating system and virtually all Unix applications are written in the C language. C has now become a widely used professional language for various reasons.

- Easy to learn
- Structured language
- It produces efficient programs.
- It can handle low-level activities.
- It can be compiled on a variety of computers.

Facts about C

- C was invented to write an operating system called UNIX.
- C is a successor of B language which was introduced around 1970
- The language was formalized in 1988 by the American National Standard Institue (ANSI).
- By 1973 UNIX OS almost totally written in C.
- Today C is the most widely used System Programming Language.
- Most of the state of the art software have been implemented using C

Why to use C ?

C was initially used for system development work, in particular the programs that make-up the operating system. C was adopted as a system development language because it produces code that runs nearly as fast as code written in assembly language. Some examples of the use of C might be:

- Operating Systems
- Language Compilers
- Assemblers
- Text Editors
- Print Spoolers
- Network Drivers
- Modern Programs
- Data Bases
- Language Interpreters
- Utilities

DATA TYPES:

C has a concept of 'data types' which are used to define a variable before its use. The definition of a variable will assign storage for the variable and define the type of data that will be held in the location. The value of a variable can be changed any time.

C has the following basic built-in data types.

- int
- float
- double
- char

Please note that there is not a boolean data type. C does not have the traditional view about logical comparison, but thats another story.

int - data type

int is used to define integer numbers.

```
{
    int Count;
    Count = 5;
}
```

float - data type

float is used to define floating point numbers.

```
{
    float Miles;
    Miles = 5.6;
}
```

double - data type

double is used to define BIG floating point numbers. It reserves twice the storage for the number. On PCs this is likely to be 8 bytes.

```
{
    double Atoms;
    Atoms = 2500000;
}
```

char - data type

char defines characters.

```
{
   char Letter;
   Letter = 'x';
}
```

Modifiers

The data types explained above have the following modifiers.

- short
- long
- signed
- unsigned

The modifiers define the amount of storage allocated to the variable. The amount of storage allocated is not cast in stone. ANSI has the following rules:

short int <= int <= long int
float <= double <= long
double

What this means is that a 'short int' should assign less than or the same amount of storage as an 'int' and the 'int' should be less or the same bytes than a 'long int'. What this means in the real world is:

Type	**Bytes**	**Range**	
short int	2	-32,768 -> +32,767	(32kb)
unsigned short int	2	0 -> +65,535	(64Kb)
unsigned int	4	0 -> +4,294,967,295	(4Gb)
int	4	-2,147,483,648 -> +2,147,483,647	(2Gb)
long int	4	-2,147,483,648 -> +2,147,483,647	(2Gb)
signed char	1	-128 -> +127	
unsigned char	1	0 -> +255	
float	4		
double	8		
long double	12		

These figures only apply to todays generation of PCs. Mainframes and midrange machines could use different figures, but would still comply with the rule above.

OPERATORS AND EXPRESSION:

What is Operator? Simple answer can be given using expression *4 + 5 is equal to 9*. Here 4 and 5 are called operands and + is called operator. C language supports following type of operators.

- Arithmetic Operators
- Logical (or Relational) Operators
- Bitwise Operators
- Assignment Operators
- Misc Operators

Arithmetic Operators:

There are following arithmetic operators supported by C language:

Assume variable A holds 10 and variable B holds 20 then:

Operator	Description	Example
+	Adds two operands	A + B will give 30
-	Subtracts second operand from the first	A - B will give -10
*	Multiply both operands	A * B will give 200
/	Divide numerator by denumerator	B / A will give 2
%	Modulus Operator and remainder of after an integer division	B % A will give 0
++	Increment operator, increases integer value by one	A++ will give 11
--	Decrement operator, decreases integer value by one	A-- will give 9

Logical (or Relational) Operators:

The following are the logical operators supported by C language.

Assume variable A holds 10 and variable B holds 20 then:

Operator	**Description**	**Example**
==	Checks if the value of two operands is equal or not, if yes then condition becomes true.	(A == B) is not true.
!=	Checks if the value of two operands is equal or not, if values are not equal then condition becomes true.	(A != B) is true.
>	Checks if the value of left operand is greater than the value of right operand, if yes then condition becomes true.	(A > B) is not true.
<	Checks if the value of left operand is less than the value of right operand, if yes then condition becomes true.	(A < B) is true.
>=	Checks if the value of left operand is greater than or equal to the value of right operand, if yes then condition becomes true.	(A >= B) is not true.
<=	Checks if the value of left operand is less than or equal to the value of right operand, if yes then condition becomes true.	(A <= B) is true.
&&	Called Logical AND operator. If both the operands are non zero then then condition becomes true.	(A && B) is true.
||	Called Logical OR Operator. If any of the two operands is non zero then then	(A || B) is true.

	condition becomes true.	
!	Called Logical NOT Operator. Use to reverses the logical state of its operand. If a condition is true then Logical NOT operator will make false.	!(A && B) is false.

Bitwise Operators:

Bitwise operator works on bits and perform bit by bit operation.

Assume if A = 60; and B = 13; Now in binary format they will be as follows:

A = 0011 1100

B = 0000 1101

A&B = 0000 1100

A|B = 0011 1101

A^B = 0011 0001

~A = 1100 0011

There are following Bitwise operators supported by C language

Operator	Description	Example
&	Binary AND Operator copies a bit to the result if it exists in both operands.	(A & B) will give 12 which is 0000 1100
|	Binary OR Operator copies a bit if it exists in eather operand.	(A | B) will give 61 which is 0011 1101
^	Binary XOR Operator copies the bit if it is set in one operand but not both.	(A ^ B) will give 49 which is 0011 0001
~	Binary Ones Complement Operator is unary and has the efect of 'flipping' bits.	(~A) will give -60 which is 1100 0011
<<	Binary Left Shift Operator. The left operands value is moved left by the number of bits specified by the right operand.	A << 2 will give 240 which is 1111 0000
>>	Binary Right Shift Operator. The left operands value is moved right by the number of bits specified by the right operand.	A >> 2 will give 15 which is 0000 1111

Assignment Operators:

There are following assignment operators supported by C language:

Operator	Description	Example

=	Simple assignment operator, Assigns values from right side operands to left side operand	C = A + B will assigne value of A + B into C
+=	Add AND assignment operator, It adds right operand to the left operand and assign the result to left operand	C += A is equivalent to C = C + A
-=	Subtract AND assignment operator, It subtracts right operand from the left operand and assign the result to left operand	C -= A is equivalent to C = C - A
*=	Multiply AND assignment operator, It multiplies right operand with the left operand and assign the result to left operand	C *= A is equivalent to C = C * A
/=	Divide AND assignment operator, It divides left operand with the right operand and assign the result to left operand	C /= A is equivalent to C = C / A
%=	Modulus AND assignment operator, It takes modulus using two operands and assign the result to left operand	C %= A is equivalent to C = C % A
<<=	Left shift AND assignment operator	C <<= 2 is same as C = C << 2
>>=	Right shift AND assignment	C >>= 2 is same as C = C >> 2

	operator	
&=	Bitwise AND assignment operator	C &= 2 is same as C = C & 2
^=	bitwise exclusive OR and assignment operator	C ^= 2 is same as C = C ^ 2
|=	bitwise inclusive OR and assignment operator	C |= 2 is same as C = C | 2

Short Notes on L-VALUE and R-VALUE:

x = 1; takes the value on the right (e.g. 1) and puts it in the memory referenced by x. Here x and 1 are known as L-VALUES and R-VALUES respectively L-values can be on either side of the assignment operator where as R-values only appear on the right.

So x is an L-value because it can appear on the left as we've just seen, or on the right like this: y = x; However, constants like 1 are R-values because 1 could appear on the right, but 1 = x; is invalid.

Misc Operators

There are few other operators supported by C Language.

Operator	Description	Example
sizeof()	Returns the size of an variable.	sizeof(a), where a is interger, will return 4.
&	Returns the address of an variable.	&a; will give actaul address of the variable.
*	Pointer to a variable.	*a; will pointer to a variable.
? :	Conditional Expression	If Condition is true ? Then value X : Otherwise value Y

Operators Categories:

All the operators we have discussed above can be categorised into following categories:

- Postfix operators, which follow a single operand.
- Unary prefix operators, which precede a single operand.
- Binary operators, which take two operands and perform a variety of arithmetic and logical operations.
- The conditional operator (a ternary operator), which takes three operands and evaluates either the second or third expression, depending on the evaluation of the first expression.
- Assignment operators, which assign a value to a variable.
- The comma operator, which guarantees left-to-right evaluation of comma-separated expressions.

Precedence of C Operators:

Operator precedence determines the grouping of terms in an expression. This affects how an expression is evaluated. Certain operators have higher precedence than others; for example, the multiplication operator has higher precedence than the addition operator:

For example x = 7 + 3 * 2; Here x is assigned 13, not 20 because operator * has higher precedenace than + so it first get multiplied with 3*2 and then adds into 7.

Here operators with the highest precedence appear at the top of the table, those with the lowest appear at the bottom. Within an expression, higher precedenace operators will be evaluated first.

Category	Operator	Associativity
Postfix	() [] -> . ++ - -	Left to right

<table>
<tr><td>Unary</td><td>+ - ! ~ ++ - - (type) * & sizeof</td><td>Right to left</td></tr>
<tr><td>Multiplicative</td><td>* / %</td><td>Left to right</td></tr>
<tr><td>Additive</td><td>+ -</td><td>Left to right</td></tr>
<tr><td>Shift</td><td><< >></td><td>Left to right</td></tr>
<tr><td>Relational</td><td>< <= > >=</td><td>Left to right</td></tr>
<tr><td>Equality</td><td>== !=</td><td>Left to right</td></tr>
<tr><td>Bitwise AND</td><td>&</td><td>Left to right</td></tr>
<tr><td>Bitwise XOR</td><td>^</td><td>Left to right</td></tr>
<tr><td>Bitwise OR</td><td>|</td><td>Left to right</td></tr>
<tr><td>Logical AND</td><td>&&</td><td>Left to right</td></tr>
<tr><td>Logical OR</td><td>||</td><td>Left to right</td></tr>
<tr><td>Conditional</td><td>?:</td><td>Right to left</td></tr>
<tr><td>Assignment</td><td>= += -= *= /= %= >>= <<= &= ^= |=</td><td>Right to left</td></tr>
<tr><td>Comma</td><td>,</td><td>Left to right</td></tr>
</table>

CONDITIONAL STATEMENTS:

C provides two styles of flow control:

- Branching

- Looping

Branching is deciding what actions to take and looping is deciding how many times to take a certain action.

Branching:

Branching is so called because the program chooses to follow one branch or another.

if statement

This is the most simple form of the branching statements.

It takes an expression in parenthesis and an statement or block of statements. if the expression is true then the statement or block of statements gets executed otherwise these statements are skipped.

NOTE: Expression will be assumed to be true if its evaulated values is non-zero.

if statements take the following form:

```
if (expression)
  statement;

or

if (expression)
  {
    Block of statements;
  }

or
```

```
if (expression)
  {
    Block of statements;
  }
else
  {
    Block of statements;
  }
```

or

```
if (expression)
  {
    Block of statements;
  }
else if(expression)
  {
    Block of statements;
  }
else
  {
    Block of statements;
  }
```

? : Operator

The ? : operator is just like an if ... else statement except that because it is an operator you can use it within expressions.

? : is a ternary operator in that it takes three values, this is the only ternary operator C has.

? : takes the following form:

if condition is true ? then X return value : otherwise Y value;

switch statement:

The switch statement is much like a nested if .. else statement. Its mostly a matter of preference which you use, switch statement can be slightly more efficient and easier to read.

```
switch( expression )
   {
     case constant-expression1:      statements1;
     [case constant-expression2:     statements2;]
     [case constant-expression3:     statements3;]
     [default : statements4;]
   }
```

Using break keyword:

If a condition is met in switch case then execution continues on into the next case clause also if it is not explicitly specified that the execution should exit the switch statement. This is achieved by using *break* keyword.

What is default condition:

If none of the listed conditions is met then default condition executed.

Looping

Loops provide a way to repeat commands and control how many times they are repeated. C provides a number of looping way.

while loop

The most basic loop in C is the while loop.A while statement is like a repeating if statement. Like an If statement, if the test condition is true: the statments get executed. The difference is that after the statements have been executed, the test condition is checked again. If it is still true the statements get executed again.This cycle repeats until the test condition evaluates to false.

Basic syntax of while loop is as follows:

```
while ( expression )
{
   Single statement
   or
   Block of statements;
}
```

for loop

for loop is similar to while, it's just written differently. for statements are often used to proccess lists such a range of numbers:

Basic syntax of for loop is as follows:

```
for( expression1; expression2; expression3)
{
```

```
   Single statement
   or
   Block of statements;
 }
```

In the above syntax:

- expression1 - Initialisese variables.
- expression2 - Condtional expression, as long as this condition is true, loop will keep executing.
- expression3 - expression3 is the modifier which may be simple increment of a variable.

do...while loop

do ... while is just like a while loop except that the test condition is checked at the end of the loop rather than the start. This has the effect that the content of the loop are always executed at least once.

Basic syntax of do...while loop is as follows:

```
 do
 {
   Single statement
   or
   Block of statements;
 }while(expression);
```

Break and continue statements

C provides two commands to control how we loop:

- break -- exit form loop or switch.
- continue -- skip 1 iteration of loop.

Examples:

```
#include

main()
{
   int i;
   int j = 10;

   for( i = 0; i <= j; i ++ )
   {
     if( i == 5 )
     {
       continue;
     }
     printf("Hello %d\n", i );
   }
}
```

This will produce following output:

Hello 0

Hello 1

Hello 2

Hello 3

Hello 4

Hello 6

Hello 7

Hello 8

Hello 9

Hello 10

INPUT AND OUTPUT STATEMENTS:

Input : In any programming language input means to feed some data into program. This can be given in the form of file or from command line. C programming language provides a set of built-in functions to read given input and feed it to the program as per requirement.

Output : In any programming language output means to display some data on screen, printer or in any file. C programming language provides a set of built-in functions to output required data.

Here we will discuss only one input function and one output function just to understand the meaning of input and output. Rest of the functions are given into C - Built-in Functions.

printf() function:

This is one of the most frequently used functions in C for output.

Try the following program to understand **printf()** function.

```
#include <stdio.h>

main()
{
  int dec = 5;
```

```
  char str[] = "abc";
  char ch = 's';
  float pi = 3.14;

  printf("%d %s %f %c\n", dec, str, pi,  ch);
}
```

The output of the above would be:

```
5 abc 3.140000 c
```

Here %d is being used to print an integer, %s is being used to print a string, %f is being used to print a float and %c is being used to print a character.

A complete syntax of printf() function is given in C - Built-in Functions

scanf() function:

This is the function which can be used to read an input from the command line.

Try following program to understand **scanf()** function.

```
#include <stdio.h>

main()
{
```

```
  int x;
  int args;

  printf("Enter an integer: ");
  if (( args = scanf("%d", &x)) == 0) {
     printf("Error: not an integer\n");
  } else {
     printf("Read in %d\n", x);
  }
}
```

Here %d is being used to read an integer value and we are passing &x to store the vale read input. Here &indicates the address of variable x.

This program will prompt you to enter a value. Whatever value you will enter at command prompt that will be output at the screen using printf() function. If you enter a non-integer value then it will display an error message.

Enter an integer: 20
Read in 20

A complete set of input output functions is given in C - Built-in Functions

POSSIBLE QUESTIONS

1. Discuss the different types Of Operators?
2. What is Expressions?
3. What is C Language?
4. Explain about the Data types?
5. Explain about the input and output functions?
6. Give an account on conditional statements?
7. Discuss the Nested if ..else statement with an example?
8. What are the features of C language?
9. Define data type.
10. Define decision making statements.
11. Define operator and mention its types.
12. What is the main function()?
13. Write an example for C program
14. Define Expressions.
15. What is conditional statements?
16. what are data types?.Explain it briefly.
17. Write a detailed account on input and output statements
18. Define conditional statements and discuss its types.(Any two)
19. Write a short note on data types.
20. Write a short note on (i)Enumerated data type (ii)Derived data type
21. Write a detailed note on printf and scanf statements in C.

22. Describe the Ladder else if condition with flow chart.
23. What are operators and explain its types?
24. Write a short note on: i)conditional operator (ii)Bitwise operator (iii)Special operator
25. Write a C program for Nested if else statement.

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(For the candidates admitted from 2018 onwards)

DEPARTMENT OF PHYSICS

UNIT I :(Objective Type/Multiple choice Questions each Question carries one Mark)

PART-A (Online Examination)

MATHEMATICAL PHYSICS-I

QUESTIONS	opt1	opt2	opt3	opt4	ANSWER
UNIT-I					
C programming language is developed in	1988	1972	1970	1958	1972
C was designed and written by	Charles Babbage	Dennis Ritchie	Ken Thompson	Peter Norton	Dennis Ritchie
A character variable can at a time store	8	4	1	2	1
How many keywords are in C?	32	30	25	29	32
C programs written for one computer can be run in another without any modifications is called	efficiency	simple	highly portable	fast	highly portable
C language is a	structured programming	object oriented programming	operating system	platform	strucutured programming
Which makes debugging, testing and maintenance of a program easire?	modules	functions	class	loops	modules
C program is a collection of	software	functions	syntax	operations	functions

The execution of the C program begins at	arguments	declaration part	error	main	main
The empty paranthesis immediately following main indicates that the main function has	arguments	no arguments	end of main	syntax error	no arguments
printf is a predefined standard C function for	compliling	reading inout	printing output	terminating	printing output
Every statement in C should end with	colon	comma	semicolon	no symbol	semicolon
The information contained between the paranthesis is called ---------- of the function	arguments	values	address	void	arguments
Which is the newline character?	\t	\n	\o	\\	\n
#define instruction defines values to a	variables	constants	symbolic constants	identifires	symbolic constants
______ Statement tests value of a given variable against list of case values.	switch	ifelse	case	while	case
______ statement causes exit from switch statement.	switch	break	goto	end	break
______ Operator takes 3 operand for making logical decisions.	+	?:	:?	*	?:
_______ Statement branches unconditionally from one point to another in program	switch	break	goto	end	goto
The ___ is powerful decision making statement and is used to control the flow of execution of statements.	if	switch	goto	while	if
_______ is two way decision making statement and is used in conjunction with an expression.	if	switch	goto	while	if
In if condition control is transferred to statements if the value of the expression is______	TRUE	FALSE	one	zero	TRUE
In C multiway decision statement is	while	if else	switch	for	switch

________					
The function _____ is used to check whether the argument is lower case alphabet.	islower	isupper	tolower	toupper	islower
The function ______converts the lower case argument into an upper case alphabet.	islower	isupper	tolower	toupper	toupper
The default statement is executed when	all the case statements are false	one of the case is true	one of the case is false	all the case statements are true	all the case statements are false
The main function section contains	declaration part and defintion part	executable part and initialization part	definition part	declarations part and executable part	declarations part and executable part
The user defined functions are in	subprogram section	main funtion section	link section	documentation section	subprogram section
The sub progarm section contains	main function	user defined funtions	assignment opaerator	defining variables	user defined section
user defined funtions are placed immediately after the	link section	global declaration section	main function section	definition section	main function section
A proper -------------- will make the progarm easily readable	style	numbering	execution	intendation	intendation
Steps in executing a C program remains same irrespective of the	operating system	editor	compiler	language	operating system
A -------------- is a progarm that controls the entire operation of a computer system.	functions	operating system	loops	decision making statements	operaing system
Which is the interface between the hardware and the software?	input	outpt	operating system	compiler	operating system
Syntax errors are checked during	compiling	executing	linking	printing	compiling
Logical errors are checked during	compiling	executing	linking	printing	executing

The file is created with the help of	compiler	system library	text editor	linker	text editor
What is the process of putting together other program files and functions that are required by the program?	compiling	executing	editing	linking	linking
The compiled and the linked program is called the	executable object code	source code	file	data	executable object code
Every C word is classified as either a keyword or a	identifier	data type	constants	variables	identifier
____ refers to finding value that do not change during execution of program.	identifier	data type	constants	variables	constants
____ integer consists of set of digits from 0 to 9 preceded by + or – sign	hexa decimal	decimal	octal	binary	decimal
_____ constant is sequence of character enclosed in double quotes.	string	variable	character	integer	string
Floating point numbers are stored in _____ bits.	8	16	32	24	32
______ is data name that may be used to store a data value.	string	variable	character	integer	variable
_____ lines are not executable statements and are ignored by the complier.	functions	comment	main	argument	comment
Execution of C Program begins from ______ function.	main function	user defined function	declaration	definition section	main function
_____ class provides information about location and visibility of variables.	functions	storage	structure	enum	storage
An arithmetic expression will be evaluated from _______ using rules of precedence of operation.	left to right	righ to left	top to bottom	bottom to top	left to right
____ operator can be used to link related expression together.	comma	dot	AND	OR	comma
_______ function writes character one at a time to terminal.	getchar	putchar	gets	puts	putchar
Reading a single character can be done	getchar	putchar	gets	puts	getchar

by using ______ function.					
______ header file should be included for calling math function in source program	conio.h	stdio.h	math.h	iostream.h	math.h
_____ operator returns number of bytes the operand occupies.	logical	bitwise	arithmetic	size of	size of
______ operator is used for manipulating data at bit level.	logical	bitwise	arithmetic	size of	bitwise
Which is the assignment operator.	=	==	--	?:	=
Characters are usually stored in _______ bits.	8	16	32	24	8
C Language supports ______ logical operators.	2	3	4	5	3
______ may appear anywhere in program but before it is referenced.	commands	#define	#include	function	#define
All C compliers support _________ fundamental data types.	5	4	6	8	4
_____ constant contains single character enclosed within pair of single quote marks.	string	variable	character	integer	character
In a passage of text, individual word, punctuation mark are called ______	keyword	identifier	constants	token	token
C has ______ types of token	10	6	5	9	6
Which refers to the name of the variables, functions and arrays?	Identifiers	constants	keywords	data type	Identifiers
First character in an identifier should be	number	alphabet or underscore	a symbol	space	alphabet or underscore

UNIT-II

Complex Analysis: Brief revision of Complex numbers & their graphical representation. Roots of Complex Numbers. Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions. Singular functions: poles and branch points, order of singularity. Integration of a function of a complex variable. Cauchy's Integral formula.

I. Introduction

The complex number system is merely a logical extension of the real number system. The set of complex numbers includes the real numbers and still more. All complex numbers are of the form

$$x + iy$$

where i = $\sqrt{-1}$. In other words $i^2 = -1$. If y = 0, then the complex number x + iy becomes the real number x. This is why we say that the complex numbers are still "more" than the reals. The real numbers form a proper subset of the reals. We do not mean that the complex numbers are more numerous. We simply mean that they subsume the reals.

Because there are two real numbers (x and y) associated with each complex number, we are able to depict complex numbers using a plane, as opposed to the reals which are depicted on a line. Unlike the real number system, complex numbers are not ordered. This means that it is not meaningful to say $z_1 < z_2$ in the complex number system, even though such a thing is posssible in the reals.

It is possible to define addition and multiplication of complex numbers in the following intuitive ways:

Addition: $(x_1+iy_1) + (x_2 + iy_2) = (x_1+x_2) + i(y_1+y_2)$

Multiplication: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2+x_2y_1)$

The complex number 0 + i0 is the complex counterpart of zero in the reals. It is the complex additive identity. We will at times simply denote it as 0. The multiplicative identity is equal to 1 + i0, which we will at times denote as 1.

A complex number can be written as z, so long as we understand that z = x + iy. It is possible to discuss subtracting and dividing complex numbers. For example

$$z_1 - z_2 = (x_1+iy_1) + (-x_2 + i(-y_2)) = (x_1 - x_2) + i(y_1 - y_2)$$

$$\frac{z_1}{z_2} = (x_1 + iy_1)\left(\frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}\right) = 1$$

In addition to the basic operations of addition, subtraction, multiplication, and division, we can also perform more complicated operations – such as taking the square root.

$$\sqrt{z_1} = a + ib \text{ where } (a+ib)(a+ib) = z_1 = x_1 + iy_1 .$$

Example: Find $\sqrt{3+i4}$

Note that there are two solutions: $\sqrt{3+i4}$ =2+i and $\sqrt{3+i4}$ = -2-i

To check this we note that (2+i)(2+i) = 3 + i4 as is the case with –2-i.

It is not hard to show that there will be *exactly* two complex square roots for any (nonzero) complex number.

The complex conjugate of a complex number z = x + iy is denoted $\bar{z}$ and is defined as (x – iy). The modulus of a complex number is defined as $|z| = \sqrt{z\bar{z}}$.

Representation in the 2-Dimensional Plane

Each complex number can be written as z = x+iy. This means that we can associate an ordered pair (x,y) with each and every complex number z = x+iy. Luckily, this gives us a graphical representation of the complex number system. We can visualize the complex numbers. The 2-dimensional plane that represents the complex numbers is sometimes called the **Argand plane** but was first employed by Gauss. The horizontal axis represents the real numbers which is a 1-dimensional subspace of the plane. The vertical axis represents "pure" complex numbers; or numbers which have no real part, x. A good question is whether or not the complex numbers (field) is isomorphic to R^2 under addition and multiplication.

The unit circle plays an important role in complex numbers since any (non-zero) complex number can be written as a scalar multiple of its corresponding point on the unit circle. For example, the point z = x + iy is a $(x^2 + y^2)$ - multiple of

$$\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

which lies on the unit circle. Seen in this way, the unit circle can generate the entire set of complex numbers through the appropriate multiplication of scalars to points on the unit circle. This is analogous to a point on the real number line (e.g., 1 and –1) being able to generate any other number on the real number line by the appropriate multiplication of a scalar. The unit circle is not unique in this regard, though. Other types of geometric objects containing the origin can do this as well. The value of the circle is that each point on the circle is equidistant from the origin and this distance is equal to 1. Note that the *real* numbers 1 and –1 have distance from zero equal to unity and can generate any real number through multiplication of a scalar. It is interesting, in this regard, that the inverse of a complex number involves the normalization of both the real and (negative) imaginary parts of the number. That is, the denominator is the formula for a circle.

The unit circle separates the plane into two regions. The set of points that are strictly inside the circle (called the open unit disk) and the set of points on and outside the unit circle. The interior of the unit circle (i.e. the unit disk) is particularly important to the stability of certain difference equations. It is similarly involved in determining whether a time series is covariance stationary.

III. Functions of a Complex Variable

We can define complex valued functions of a complex variable. That is, the domain of the function is the complex variable field and the range is also the complex field. We can write this as

$$w = f(z)$$

where both w and z are complex numbers. Such functions have complex numbers as parameters, as well. For example, we can write the following function

$$w = f(z) = \frac{z_o}{z} = \frac{1-i2}{x+iy} = \frac{(x-2y)+i2(x+y)}{x^2+y^2} = u+iv$$

Clearly, w is complex, as is z. The parameter z_o is also complex, but it is a fixed complex number. Note how that u and v have become real multivariate functions of x and y. That is,

$$u = u(x,y) = \frac{x-2y}{x^2+y^2} \quad \text{and} \quad v = v(x,y) = \frac{2(x+y)}{x^2+y^2}$$

As x and y run over all the values in R^2, both u and v are determined, and hence z is determined accordingly. This complex z then determines the value of w.

One of the most useful of all the complex functions is the exponential function. This function has a straightforward relation to the trigonometric functions. We can understand this relation by using a MacLaurin series for the e^x function. To begin with

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Which, if we substitute iθ for x, we get

$$e^{\theta i} = 1 + \frac{(\theta i)}{1!} + \frac{(\theta i)^2}{2!} + \cdots$$

$$= 1 + \frac{\theta i}{1!} - \frac{\theta^2}{2!} - \frac{\theta^3 i}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5 i}{5!} - \frac{\theta^6}{6!} \cdots$$

$$= \{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\} + \{\frac{\theta i}{1!} - \frac{\theta^3 i}{3!} + \frac{\theta^5 i}{5!} - \frac{\theta^7 i}{7!} \cdots\}$$

$$= \cos(\theta) + i\sin(\theta)$$

Note that the point (cos(θ), sin(θ)) is on the unit circle and as θ runs from 0 to 2π, the point moves completely around the circle in a counterclockwise fashion. We can therefore write any complex number as a scalar multiple of $e^{i\theta}$. Usually this is written as

$$z = x + iy = r\cos(\theta) + i\, r\sin(\theta) = re^{i\theta}$$

with θ = arctan(y/x) and $r^2 = x^2 + y^2$.

The complex exponential and its relation to the trigonometric functions is of the greatest imporance in of mathematics. It is incredibly useful and leads to some rather extraordinary and unexpected results.

For example, it allows us to easily compute the following *real* number

$$z = i^i = \frac{1}{\sqrt{e^{\pi}}} \approx 0.207$$

where $i = \sqrt{-1}$. It also allows us to write out the logarithm of a negative number, which was a great controversy during the time of Euler and Leibnitz. That is, we can write

$$z = \ln(-1) = i\pi$$

from which all other negative logarithms can be derived.[1] The logarithm of a complex number can also be derived using this relation. Hence, we have

$$\ln(z) = \ln(x+iy) = \ln(re^{i\theta}) = \ln(r) + i\theta$$

where θ is the angle formed by vectors (x, 0) and (0, y) and where $r^2 = x^2 + y^2$.

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Polynomial equations, even simple ones, have solutions which are surprising to those who look only for real solutions. For example, even the very simple equation

$$z^4 + 1 = 0$$

has FOUR distinct roots (consider $z^2 = i$ and $z^2 = -i$). In general, the Fundamental Theorem of Algebra tells us that there will be exactly n complex roots (possibly repeated) which solve an n^{th} order polynomial equation. Once again, it is important to remember that the coefficients on these polynomial equations can also be complex numbers, as well.

The familiar trigonometric functions of sin(x) and cos(x) can be defined for complex numbers. This is done in the perfectly logical manner as follows:

$$\sin(z) = \sin(x+iy) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos(z) = \cos(x+iy) = \frac{e^{ix} + e^{-ix}}{2},$$

where we remember that $\frac{1}{i} = -i$.

IV. Limits and Derivatives of Complex Functions

Limits in the complex system are complicated by the fact that z = x + iy depends on (x, y) and therefore one can approach $z_o = x_o + iy_o$ along infinitely many paths. For example,

$$z_n = (x_o + \frac{1}{n}) + i(y_o + \frac{1}{n}) \text{ and } z_n^{'} = (x_o - \frac{1}{n}) + i(y_o - \frac{1}{n})$$

both limit to $z_o = x_o + iy_o$, but do so along different paths. Obviously, other more complicated paths are possible. This makes it a little more difficult to define a derivative, which makes use of limits in its defintion.

The derivative of w = f(z), if it exists, is defined by the unique limit

$$\frac{dw}{dz} = \lim_{\delta \to (0+i0)} \frac{f(z+\delta) - f(z)}{\delta}$$

where $\delta \to (0+i0) = 0$, the origin, along *ANY* path.

Example: w = f(z) = z^2 is differentiable. To see this, assume (g(n), h(n)) are functions parametrized such that they limit to the origin as $n \to \infty$. The ordered pair we have assumed maps out any path to the origin and is perfectly general. It is not difficult to show that

$$f'(z) = \lim_{n\to\infty} \frac{\{(x+g(n))+i(y+h(n)\}^2 - \{x+iy\}^2}{g(n)+ih(n)}$$

$$= \lim_{n\to\infty} [2\{x+iy\}+\{g(n)+ih(n)\}]$$

$$=$$

and thus, regardless of the path we take to the origin, the limit remains the same and thus the derivative of f(z) is equal to f '(z) = 2z. ■

V. The Cauchy-Riemann Equations and Complex Differentiation

Suppose that we consider f(z) = z^2 and substitute into this z = x+iy. We can therefore write this function again in the following way:

$$f(z) = z^2 = F(x,y) = (x+iy)^2 = (x^2 - y^2) + i2xy = u(x,y) + iv(x,y)$$

where $u(x,y) = (x^2 - y^2)$ and where v(x,y) = 2xy. Now since z = x+iy, we know that

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

from which it follows that $\frac{\partial x}{\partial z} = \frac{1}{2}$ and $\frac{\partial y}{\partial z} = \frac{1}{2i}$. Now consider the complex derivative f '(z).

$$f'(z) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = (2x-2yi)(\frac{1}{2}) + (-2y+2xi)(\frac{1}{2i})$$

This of course reduces to $f'(z) = 2z$ and the result agrees with the derivative computed in the previous section using limits.

Now suppose that F(x,y) = u(x,y) + iv(x,y) is *any* differentiable complex function. What must be true about the functions u and v ? This is the subject of the Cauchy - Riemann equations.

First, suppose that z changes by x changing alone. Then, assume that z changes by y changing alone. This would give us two expressions for the derivative of

f(z) = F(x,y).

The first (*holding y constant*) can be written as

$$f'(z)\,|_{y\ constant} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + i\frac{\partial v}{\partial x}\frac{\partial x}{\partial z} = \frac{1}{2}\{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\}$$

while the second (*holding x constant*) can be written as

$$f'(z)\,|_{x\ constant} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} + i\frac{\partial v}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\}\,.$$

Now, the derivative of f(z) cannot depend on which way that z is changing (either by x changing alone or alternatively by y changing alone) and so the two expressions for $f'(z)$ must be equal if the derivative exists. This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and that} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

These two equalities are known as the Cauchy-Riemann Equations.

VI. Complex Integration

The first thing to note about complex integration is that it is done in the plane and therefore uses contour or line integration as its basis. A strong understanding of line integration is therefore useful when discussing complex variables. This is not so unexpected since the 2 dimensional plane in complex variables now operates analogously to the real line in elementary calculus. Instead of integrating over some interval, or collection of intevals, we must integrate along some curve or line in 2-space.

The second thing to note about complex integration is that integration no longer implies the measurement of some area. That is, one does not necessarily get a "clean" real number associated with an integral in complex variables. What this implies is that integrals cannot be ordered by size as they can be in real integration. One cannot say that *this area is larger than that area*. This is because there is no ordering of the complex numbers, unlike the reals. Indeed, the integral of a complex function of a complex variable typically yields a complex number.

Here is a simple example to show complex integration:

Example: Let f(z) = z, where z = x+iy. It is obvious that the image of the function f is not a real number; it is complex. Now suppose that we integrate this in the x-y plane along the line y = x from (0,0) to (1,1). We are integrating the function f(z) along the ray from the origin to the point (1,1). This directed line segment is sometimes denoted C for curve, even though it is a straight line segment. We assume that we move from the point (0,0) to the point (1,1), since direction is important. Now let's actually do the integration.

$$\int_C f(z)\,dz = \int_C z\,dz = \int_C (x+iy)(dx+idy)$$

$$=\int_0^1 (x+ix)(1+i)dx$$

$$=\int_0^1 x(1+i)^2\,dx$$

$$= i\int_0^1 2x\,dx$$

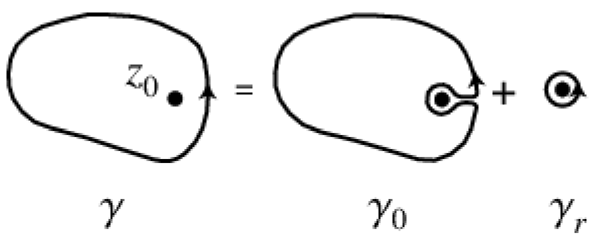
$$= ix^2\Big|_0^1$$

$= i$

This example shows that the complex definite integral of the complex function

f(z) = z is itself a complex number; in fact, it is equal to z = i.

CAUCHY'S INTEGRAL FORMULA:



Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\, dz}{z - z_0}, \tag{1}$$

where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_\gamma \frac{f(z)\, dz}{z - z_0}, \tag{2}$$

defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \tag{3}$$

so

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_0}\frac{f(z)\,dz}{z-z_0}+\oint_{\gamma_r}\frac{f(z)\,dz}{z-z_0}. \qquad (4)$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_r}\frac{f(z)\,dz}{z-z_0}. \qquad (5)$$

Now, let $z\equiv z_0+r\,e^{i\theta}$, so $dz=i\,r\,e^{i\theta}\,d\theta$. Then

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_r}\frac{f\left(z_0+r\,e^{i\theta}\right)}{r\,e^{i\theta}}\,i\,r\,e^{i\theta}\,d\theta \qquad (6)$$

$$=\oint_{\gamma_r}f\left(z_0+r\,e^{i\theta}\right)i\,d\theta. \qquad (7)$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\lim_{r\to 0}\oint_{\gamma_r}f\left(z_0+r\,e^{i\theta}\right)i\,d\theta \qquad (8)$$

$$\oint_{\gamma_r}f(z_0)\,i\,d\theta \qquad (9)$$

$$=i\,f(z_0)\oint_{\gamma_r}d\theta \qquad (10)$$

$$=2\pi\,i\,f(z_0), \qquad (11)$$

giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma,z_0)\,f(z_0)=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}, \qquad (12)$$

where $n(\gamma, z_0)$ is the contour winding number.

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \tag{13}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\oint_\gamma \frac{f(z)\,dz}{z-z_0-h} - \oint_\gamma \frac{f(z)\,dz}{z-z_0}\right] \tag{14}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\oint_\gamma \frac{f(z)\,[(z-z_0)-(z-z_0-h)]\,dz}{(z-z_0-h)(z-z_0)} \tag{15}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\oint_\gamma \frac{h\,f(z)\,dz}{(z-z_0-h)(z-z_0)} \tag{16}$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0)\,f^{(r)}(z_0) = \frac{r!}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^{r+1}}. \tag{19}$$

TAYLOR'S VARIABLE:

The Taylor's series specifies the value of a function at one point in terms of the value of the function and its derivatives at a reference point say . It is an expansion in powers of the change in variable here say (z - a).

Fig.53: Contouri *C* defined in the anticlockwise direction, with parametric equation |z - a| < R

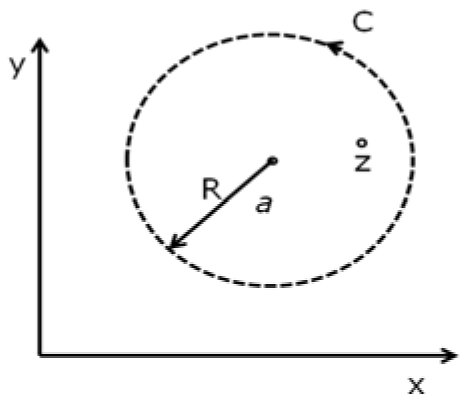
Let *f*(z) be an analytic function inside and on a simple closed curve *C*, with the center as a and radius R. Then at each point inside the contour *C* we define

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots +$$

The power series here converges to $f(z)$ when $|z-a| < R$.

Here R will be the radius of convergence which is defined as the distance from the reference point to the nearest singularity of the function $f(z)$. On $|z-a| = R$ the series may or may not converge while for $|z-a| > R$ the series diverges. If the nearest singularity of $f(z)$ is at ∞, the radius of convergence is at , i.e. the series converges for all values of z.

If a = 0 then the resulting series is often called the **Maclaurin series**.



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$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots +$$

The power series here converges to $f(z)$ when $|z - a| < R$.

Here R will be the radius of convergence which is defined as the distance from the reference point to the nearest singularity of the function $f(z)$. On $|z - a| = R$ the series may or may not converge while for $|z - a| > R$ the series diverges. If the nearest singularity of $f(z)$ is at ∞ , the radius of convergence is at , i.e. the series converges for all values of z.

If a = 0 then the resulting series is often called the **Maclaurin series**.

Proof of the Taylor's series

Let us consider an analytic function $f(z)$ in a neighborhood of a point z = a . Also let C be a circle which lies in this neighborhood and has the center a. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)ds}{(s-z)} \quad ----(1)$$

where,

z: any arbitrary fixed point inside the contour C

s: be the complex variable of integration.

The radius of convergence of the Taylor series is at least equal to the shortest distance from a to the boundary C. It may be larger but then the series may not represent $f(z)$ at all points of Cwhich lie in the interior of the circle of convergence.

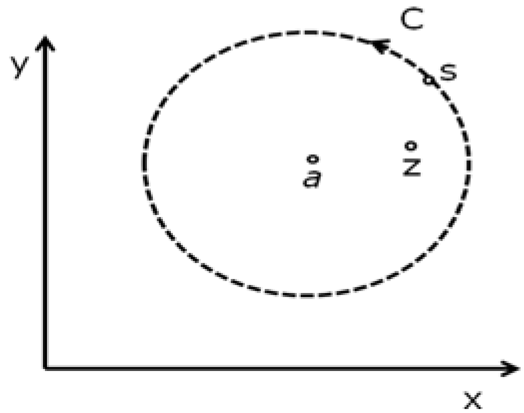

Fig.54: Contour C, with center at a and having the variable of integration as s and any point in the interior as z

We shall first develop $\frac{1}{(s-z)}$ in powers z-a of

$$\frac{1}{(s-z)} = \frac{1}{s-a-(z-a)} = \frac{1}{(s-a)\left(1-\frac{z-a}{s-a}\right)} \quad ----(2)$$

Since s is on C while z is inside C, therefore

$$\left|\frac{z-a}{s-a}\right| < 1 \qquad ----(3)$$

Now, from the geometric progression for |q|<1

$$1+q+q^2+q^n = \frac{1-q^{n+1}}{1-q} \qquad q \neq 1$$

We obtain the relation,

$$\frac{1}{1-q} = 1+q+q^2+q^n+\frac{q^{n+1}}{1-q}$$

Here,

$$q = \frac{z-a}{s-a}$$

Then

$$\frac{1}{1-\left(\frac{z-a}{s-a}\right)} = 1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^n + \frac{\left(\frac{z-a}{s-a}\right)^{n+1}}{1-\left(\frac{z-a}{s-a}\right)}$$

Using 2

$$= 1 + \frac{z-a}{s-a} + \left(\frac{z-a}{s-a}\right)^2 + \cdots + \left(\frac{z-a}{s-a}\right)^n + \frac{\left(\frac{z-a}{s-a}\right)^{n+1}}{\left(\frac{s-z}{s-a}\right)} \quad ----(A)$$

Inserting (A) in (2) which is then put back in (1) to get

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)} + \frac{(z-a)}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^2} + \cdots + \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^{n+1}} + R_n(z) \quad --(4)$$

Since z and a are constants we take the powers of z-a out from under the integral sign and the last term is

$$R_n(z) = \frac{(z-a)^{n+1}}{2\pi i}\oint_C \frac{f(s)ds}{(s-a)^{n+1}(s-z)} \quad ---(5)$$

Using Cauchy’s integral formula

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}} \qquad n = 1,2,3,\ldots$$

We comprehend the series as

$$f(z) = f(a) + \frac{(z-a)}{1\,!} f'(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + R_n(z) \quad ---(6)$$

This representation is called the Taylor's series

Rn (z)is called the Remainder. Since the analytic functions has derivatives to all orders we may take n as large as possible i.e. As $n \to \infty$ we write

$$f(z) = \sum_{m=0}^{\infty} f^m(a) \frac{(z-a)^m}{m!} \quad -----(7)$$

This is the Taylor's series for the function f(z) with center at a. The series in (7) above will clearly converge and represent f(z) if and only if

$$\lim_{n\to\infty} R_n(z) = 0 \qquad ---(8)$$

This we can prove as s is on the contour C while z is inside C thus we will have |s-z| > 0. Since f(z) is analytic inside C and on C it follows that the absolute value of $\frac{f(s)}{(s-z)}$ is bounded thus we can say

$$\left|\frac{f(s)}{(s-z)}\right| < M, \quad \textit{for all values of s on C}$$

Let r be the radius of C then |s-a|=r, for all s on C and C has the length as $2\pi r$

Therefore,

$$|R_n(z)| = \left|\frac{(z-a)^{n+1}}{2\pi i} \oint_C \frac{f(s)ds}{(s-a)^{n+1}(s-z)}\right|$$

$$\leq \frac{|(z-a)|^{n+1}}{2\pi} \left|\frac{M\ 2\pi r}{r^{n+1}}\right| = \frac{M\ r\ |(z-a)|^{n+1}}{r^{n+1}}$$

As $n \to \infty$ then the right hand side is zero.

Example: Express the Taylor series for $f(z) = e^z$

Soln.: The $f(z)$ is an analytic function. We can express the Taylor series about z=0. In general an expansion about z=a is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots +$$

The power series here converges to $f(z)$ when $|z - a| < R$. While about z=0 can be written as

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^n(0)}{n!}z^n + \cdots +$$

$$f(z) = e^z = e^0 = 1$$

$$f'(z) = e^z = e^0 = 1$$

$$f''(z) = e^z = e^0 = 1$$

Thus,

$$f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots + \qquad ; \quad |z| < \infty$$

The first singularity of this function lies at infinity thus the radius of convergence is infinity.

Example: Express the Taylor series for (i) $f(z)$ = sin z (ii) $f(z)$ = cos z (iii) $f(z)$ = sinh z (iv) $f(z)$ = cosh z

Soln.: The given functions in all the cases are analytic functions. The first singularity of these function lies at infinity thus the radius of convergence is infinity. We can express the Taylor series about z=0 as

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 \ldots + \frac{f^n(0)}{n!} z^n + \cdots + \quad ; |z| < \infty$$

(i) $f(z) = \sin z$

$f(z) = \sin\ z = \sin 0 = 0$

$f'(z) = \cos z = \cos 0 = 1$

$f''(z) = -\sin z = -\sin 0 = 0$

$f'''(z) = -\cos z = -\cos 0 = -1$

Thus, we find that $f^{2n}(0) = 0$ while $f^{2n+1}(0) = (-1)^n$ So,

$$f(z) = 0 + \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots + \quad ; |z| < \infty$$

$$\sin z = (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad ; |z| < \infty \quad --- (A)$$

Differentiating each side of (A) with respect to z and interchanging the symbols for differentiation and summation we have we can write the expansion as

(ii) $$\cos z = (-1)^n \frac{z^{2n}}{(2n)!} \quad ; |z| < \infty \quad --- (B)$$

Such a case is possible as term by term differentiation of a power series is allowed. Here the power series is a Maclaurin or Taylor series.

(iii) We can use the identity sin iz = i sinh z, So using (A) by replacing every z by iz

$$\sin iz = (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!}$$

$$\mathrm{i}\, \sinh \mathrm{z} = (\mathrm{i}\,\mathrm{i})^n \frac{\mathrm{i}^{2n+1}\, \mathrm{z}^{2n+1}}{(2n+1)!} = \frac{\mathrm{i}\quad \mathrm{z}^{2n+1}}{(2n+1)!}$$

The first singularity of this function lies at infinity thus the radius of convergence is infinity.

$$sinh\, z = \frac{z^{2n+1}}{(2n+1)!} \;;\quad |z| < \infty \;,\quad ----(C)$$

(iv) A term by term differentiation of equation (C) yields

$$\cosh z = \frac{z^{2n}}{(2n)!} \;;\quad |z| < \infty \;,\quad ----(C)$$

Example: Find the geometric series for the function

(i) $f(z) = \frac{1}{1-z}$ **(ii)** $f(z) = \frac{1}{1+z}$

Soln. : The function $f(z)$ is singular at $z=1$, this point lies on the circle of convergence

Derivative of the function $f(z) = \frac{1}{1-z}$ are of type

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} \quad ; n = 0,1,2,\ldots$$

So the series and in particular the Maclaurin series has

$f^n(0) = n!$

Thus,

$f^n(0) = n!$

Thus,

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 \ldots + \frac{f^n(0)}{n!}z^n + \cdots + \quad ; |z| < 1$$

$$\frac{1}{1-z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^n}{n!} + \cdots + \quad ; |z| < 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad ; |z| < 1$$

Replace every z by (-z) we get

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad ; |z| < 1$$

RESIDUE AND CAUCHY'S RESIDUE THEOREM:

Residue

We recall that if a function *f(z)* is analytic in a simply connected region, then according to Cauchy's integral theorem the value of the integral

$$\oint_C f(z)dz = 0$$

is always zero, where *C* is a closed contour lying wholly in R.

If, on the other hand, the function *f(z)* fails to be analytic at a finite number of points in the interior of the contour C in R, then there is a specific number called the **residue**, which each of these points (points of singularity) contribute to the value of the integral.

We note that a point 'a' is an isolated singularity if the function fails to be analytic at that point and in addition there is some neighborhood throughout which the function is analytic except at the point itself. ***The contribution of the singularity towards the integral is the residue.*** Thus for a non-analytic function the integral

$$\oint_C f(z)dz \neq 0$$

In this case we may represent the function by a Laurent series which converges in the domain 0<|z-a|<R, where R is the distance from 'a' to the nearest singular point of the function f(z). Thus in general we can write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n\,(z-a)^n$$

$$= \cdots + A_n(z-a)^n + \cdots A_2(z-a)^2 + A_1(z-a) + A_0 + \frac{A_{-1}}{z-a} + \frac{A_{-2}}{(z-a)^2} + \frac{A_{-3}}{(z-a)^3} + \cdots + \frac{A_{-n}}{(z-a)^n} + \cdots \qquad ---(i)$$

where we can write

$$A_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-a)^{n+1}} \qquad ; n = 0, \pm 1, \pm 2, \pm 3$$

For the special case n=-1 we have

$$2\pi i\, A_{-1} = \oint_C f(z)dz$$

The coefficient A_{-1} which is the coefficient of $\frac{1}{z-a}$ in the above expansion of the Laurent series is called the RESIDUE of the function at the isolated singular point 'a'. Thus

$$A_{-1} = \underset{z=a}{Res} f(z) = Residue$$

Example: Find the integral of the function $f(z) = \frac{\sin z}{z^4}$ around the unit circle C in the counterclockwise sense.

Soln.

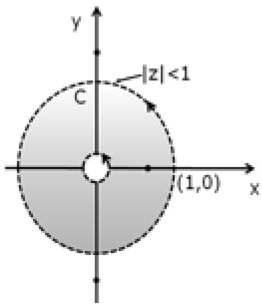

Fig.61: Isolated singular point at z=0 inside a unit circle.

We obtain the Laurent series for an isolated singularity at z=0 of the function as

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Comparing it with (i) we find that the residue is

$$A_{-1} = -\frac{1}{3!}$$

Hence,

$$\oint_C f(z)dz = 2\pi i\, A_{-1} = 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

Example: Show that the integral of the function $f(z) = e^{1/z^2}$ around the circle C, |z|=2 described in the counterclockwise sense is zero.

Soln.

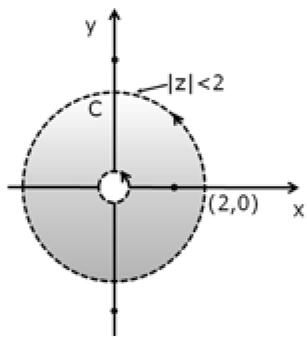

Fig.62: Isolated singular point at z=0 inside a circle |z|<2.

The isolated singularity at point z=0 lies interior to the contour C. By Maclaurin series we know that

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Thus we can write a series for $f(z) = e^{1/z^2}$

$$f(z) = e^{1/z^2} = \sum_{n=0}^{\infty} \frac{1}{z^{2n}\; n!} = 1 + \frac{1}{z^2\;1!} + \frac{1}{z^4\;2!} + \frac{1}{z^6\;3!} + \cdots$$

and comparing it with (i) we find that the residue is $A_{-1} = 0$

Hence,

$$\oint_C f(z)dz = 2\pi i\, A_{-1} = 0$$

Example: Find the residue at simple poles for the function $f(z) = \frac{2-5z}{z^2-z}$

Soln. Using the definition

$$\underset{z=a}{Res} f(z) = A_{-1} = \lim_{z\to a}(z-a)f(z)$$

To evaluate the residue we see that the function $f(z) = \frac{2-5z}{z^2-z}$ has simple poles at z=0 and z=1. Thus, we have at these simple poles

$$\underset{z=0}{Res} f(z) = \lim_{z\to 0}(z-0)\frac{2-5z}{z^2-z} = -2$$

$$\underset{z=1}{Res} f(z) = \lim_{z\to 1}(z-1)\frac{2-5z}{z^2-z} = -3$$

Example: Find the residue for the function $f(z) = \frac{z}{(z-2)(z+1)^2}$

Soln. In this function $f(z) = \frac{z}{(z-2)(z+1)^2}$ we see that we have simple pole at z=2 and a pole of order 2 at z=-1. So, for the simple pole we calculate the residue as

$$\underset{z=1}{Res} f(z) = \lim_{z\to 2}(z-2)\frac{z}{(z-2)(z+1)^2} = \frac{2}{9}$$

For the pole of order k=2 we find the residue at z=-1 by using the general formula

$$A_{-1} = \lim_{z\to a} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\{(z-a)^k f(z)\} = \lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\{(z+1)^2 \frac{z}{(z-2)(z+1)^2}\}$$

$$= \lim_{z\to -1} \frac{d}{dz}\{\frac{z}{z-2}\} = \lim_{z\to -1} \frac{z-2-z}{z-2^2} = -\frac{2}{9}$$

Example: Obtain the Laurent series expansion of the following functions in the neighborhood of the singular points and calculate the residues: (i) (ii)

$f(z) = \frac{\cos z}{z}$ (ii) $f(z) = \frac{e^z}{(z-1)^2}$

Solution

(i) Note that z=0 is the isolated singular point of the function $f(z) = \frac{\cos z}{z}$

We recall the series expansion of cos z as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \qquad -\infty < z < \infty$$

Thus,

$$f(z) = \frac{\cos z}{z} = \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \left(\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots\right)$$

We note that in this series the coefficient of the 1/z term is 1. Thus, $A_{-1} = 1$, therefore is the residue of the given function.

(ii) The function has a double pole at z=1. We expand the exponential series in powers of (z-1)

$$f(z) = \frac{e^z}{(z-1)^2} = \frac{e\ e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

$$= \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \frac{e}{2!} + \frac{e(z-1)}{3!} + \cdots$$

This is the required Laurent series expansion. We observe the coefficient of 1/(z-1) is the residue

What have we achieved so far? We agree that now we can evaluate the residue of a function $f(z)$ with one singular point in a contour using Laurent series expansion. However, how do we proceed when encloses more than one isolated singular points? In such a situation, we have to extend the concept of residue developed so far to more than one singularity. The theorem of residues deals with such a general case and we discuss it in the following section.

Residue Theorem:

We extend the concept of residue developed so far to the case when the integrand has several singularities. Let us consider a positively oriented simple closed contour C, within and on which a function is analytic except for a finite number of singular points $z_1, z_2, z_3, \ldots z_n$ interior to C. If $A_1, A_2, A_3\ldots, A_n$ denote the residues of 'f(z)' at those respective points then

$$\oint_C f(z)dz = 2\pi i\,(A_1 + A_2 + A_3 + \cdots A_n)$$

Proof:

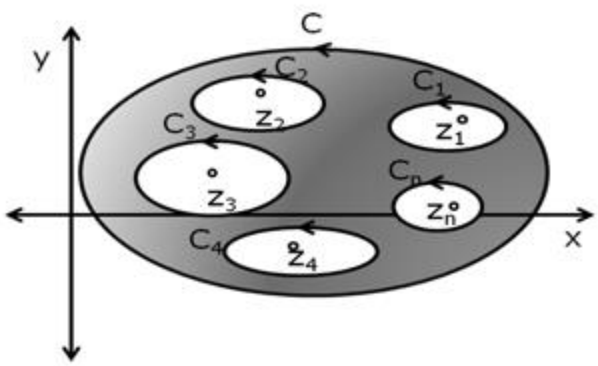

Fig.63: A finite number of singular points $z_1, z_2, z_3 \ldots z_n$ interior to C.

Let the singular points $z_j(j=1,2,\ldots n)$ be the centers of positively oriented circles C_j which are interior to C and are so small that no two of the circles have points in common. That is to say that all the singularities are isolated by small contours.

The circles C_j along with the simple closed contour C form the boundary of a closed region. As we can see in the figure the function is analytic throughout the shaded region, which is a multiply connected domain. Hence, by Cauchy Goursat theorem

$$\oint_C f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \cdots - \oint_{C_n} f(z)dz = 0$$

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$

$$= 2\pi i\, Res[(f(z)_{C_1}) + (f(z)_{C_2}) + \cdots + (f(z)_{C_n}) = 2\pi i\,[A_1 + A_2 + A_3 + \cdots A_n] = 2\pi i \sum_{j=1}^{n} A_j$$

We note that **this theorem is valid only for isolated singularities.** The immense utility of this theorem stems from the fact that it facilitates calculation of a contour integral indirectly through the residues of f (z) at the singularities inside C.

Example: Use the residue theorem to evaluate the integral

$$\oint_C \frac{5z-2}{z(z-1)}dz$$

Where, C is the circle |z|=2 described in counterclockwise sense. Verify your result using (i) Laurent series (ii) partial fractions.

Solution.: The integrand has two singularities at z=0 and z=1 and both of which are interior to the contour C as we see in the figure. We need to find the residues for both the singularities.

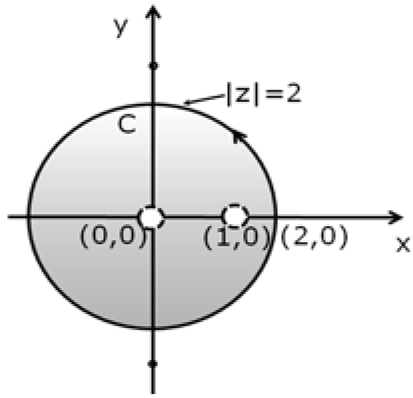

Fig.64: Contour of radius 2, having singularities at z=0 and z=1

(i) Laurent Series approach.

We know that

$$\frac{1}{(1-z)} = 1 + z + z^2 + z^3 + \cdots + \qquad ; |z| < 1$$

So we expand the function in the different domains and observe in 0<|z|<1

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\left(\frac{-1}{(1-z)}\right) = \frac{5z-2}{z}(-1 - z - z^2 - z^3 - \cdots)$$

$$= \frac{2}{z} - 3 - 3z - \cdots$$

The coefficient of 1/z is the residue $A_{-1} = \underset{z=0}{Res} f(z) = 2$.

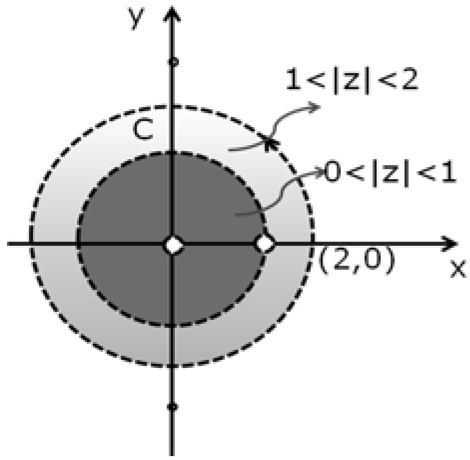

Fig.65: Different domains. Darker circle is for |z|<1 case. The lighter is the annular domain 1<|1|<2

Now we observe in the domain 1<|z|<2

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1}\left(\frac{1}{1+(z-1)}\right)$$

$$\left[5 + \frac{3}{z-1}\right](1-(z-1)+(z-1)^2 - \cdots \quad) \quad ;0 < |z-1| < 1$$

The coefficient of 1/(z-1) is the residue $A_{-1} = \underset{z=1}{Res} f(z) = 3.$

Thus by residue theorem

$$\oint_C \frac{5z-2}{z(z-1)} dz = 2\pi i \left(\underset{z=0}{Res} f(z) + \underset{z=1}{Res} f(z)\right) = 2\pi i(2+3) = 10\pi i$$

(ii) Partial fractions approach

The given function can be expressed as

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{(z-1)}$$

Thus we can write the integral clearly as

$$\oint \frac{5z-2}{z(z-1)}\,dz = \oint \frac{2}{z}dz + \oint \frac{3}{(z-1)}dz = 2\pi i(2+3) = 10\pi i$$

Both the results lead to the same result.

Example: Use the residue theorem to evaluate the integral

$$\oint_C \frac{dz}{z^3(z+3)}$$

where, C is the circle (i)|z|=2 or (ii) |z+2|=3 described in counterclockwise sense

Soln.:The given function is $f(z) = \frac{1}{z^3(z+3)}$. It has singularities at z=0, which is a pole of order 3 and a simple pole at z=-3

(i) We pick the case |z|=2. In this we see that the simple pole z=-3 does not lie in the concerned region. Thus there exists only one singularity at z=0. We evaluate that using the standard formula of the residue for order m

$$A_{-1} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

$$= \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\{(z)^3 \frac{1}{z^3(z+3)}\} = \lim_{z\to 0} \frac{1}{2!}\frac{d}{dz}\{\frac{-1}{(z+3)^2}\} = \lim_{z\to 0} \frac{1}{2!}\frac{2}{(z+3)^3} = \frac{1}{27}$$

Thus using the residue theorem

$$\oint_C \frac{1}{z^3(z+3)}dz = 2\pi i\,(\operatorname*{Res}_{z=0} f(z)) = \frac{2\pi i}{27}$$

(ii) We pick the case |z+2|=3. In this we see that the simple pole z=-3 as well as pole of order 3 at z=0 lie inside the contour. We have already calculated the residue at z=0 in the above part. We need to evaluate not the residue at z=-3 only. It is a simple pole thus we have

$$A_{-1} = \lim_{z\to a} \{(z-a) \;\; f(z)\}$$

$$= \lim_{z\to -3} \{(z+3) \;\; \frac{1}{z^3(z+3)}\} = \lim_{z\to -3} \{\frac{1}{z^3}\} = \frac{-1}{27}$$

Thus using the residue theorem

$$\oint_C \frac{1}{z^3(z+3)} dz = 2\pi i \left(\underset{z=0}{Res} f(z) + \underset{z=-3}{Res} f(z)\right) = \frac{2\pi i}{27} + \frac{-2\pi i}{27} = 0$$

Applications of Residues

The residue theorem is one of the most powerful results of complex variable theory since it finds so many and so varied applications in mathematical analysis and physical sciences. Actually, the usefulness of the *residue theorem* stems from the fact that, even when the integrand is not innocent-looking, it *facilitates the evaluation of the integral by way of a rather straight forward calculation of residues at the singular points of the given function.* Further, the detailed shape of the contour is not relevant, except in so far as it encloses certain singular points. From pragmatic considerations, the residue theorem is of special importance in the evaluation of real integrals. It is not possible to cover all the applications in an elementary course such as this. Therefore, we shall concentrate only on the evaluation of some types of definite integrals using the method of residues. Once you get familiar with the basic principles of this method, you should be able to apply these to more advanced applications.

POSSIBLE QUESTIONS

- State and Prove the Cauchy's integral theorem.
- Find the value of the integral ,(i)along the straight line from z=0 to z=1+I; (ii)along real axis from z=0 to z=1 and then along a line parallel to the imaginary axis from z=1 to z=1+i.
- State and prove Cauchy's integral formula.
- Find the different values of (1+i) 1/3
- Mention the properties of Moduli and Arguments.
- Write a short note on complex conjugates.
- Test the analyticity of the function w=sin z and hence derive that (sin z)=cos z
- Derive the Cauchy's Riemann differential equations.
- Find the value of the integral (i)along y=x 2 ,having (0,0)(3,9)end points (ii)along y=3x between the same points. Do the values depend upon path.

KARPAGAM ACADEMY OF HIGHER EDUCATION

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(For the candidates admitted from 2018 onwards)

DEPARTMENT OF PHYSICS

UNIT I :(Objective Type/Multiple choice Questions each Question carries one Mark)

PART-A (Online Examination)

MATHEMATICAL PHYSICS-I

QUESTIONS	opt1	opt2	opt3	opt4	ANSWER
UNIT-I					
The exponential form of a complex number is	$z = re^{iq}$	$z = e^{iq}$	z = cos q / r	z = r / cos q	$z = re^{iq}$
Any function which satisfies the Laplace equation is known as	harmonic function	conjugate function	single function	analyticc function	harmonic function
A single valued function f(z) which is differentiable at z = zo it is said to be	irregular function	analytic function	periodic function	all the above	analytic function
If a given number is wholly real, it is found in/on	a real axis	imaginary	x-y plane	space	imaginary
A set which entirely consists of interior points is known as	a null set	a bounded set	a closed set	an open set	an open set
The symbol i with the property i^2= -1 was introduced by	Euler	Gauss	Cauchy	Reimann	Euler
In the Argand diagram, the fourth	rectangle	square	cube	none	square

roots of unity forms a -------					
The Conjugate of 1/i is	–i	i	1	-1	–i
The value of $i^2 + i^3 + i^4$ is	irregular function	regular function	infinite	C	C
The sum of n^{th} roots of unity are ------	0	1	2	3	0
In the Argand diagram, the fourth roots of unity forms a -------	square	rectangle	circle	rombus	square
The order of convergence of Newton Raphson method is ----------	1	2	3	4	2
A number of the form a+ib is called a	complex number	real numbers	imaginary numbers	all the above	complex numbers
who represented argand diagram in a complex number	Argand	macmillain	ramanujam	H.K.Dass	Argand
The function of a complex variable is important in solving ____________ in the field of engineering and science	small problems	a large number of problems	simple problems	none of the above	a large number of problems
what is the order of the Euler's method?	h	h/2	h^2	h^3	h^2
The point at which the function is not differeniable is called a ___________ point of a function	poles	branch points	singular points	none of the above	singular points
In which method the accuracy cannot be obtained as the number of the intervals increases	bisection method	newtons method	Euler's method	all the above	Euler's method
A function which satisfies the____________ is known as a harmonic function	laplace equation	wave equation	Bessels equation	all the above	laplace equation
conjucate harmonic function is also a ________ function	harmonic function	analytic function	cosine function	sine function	analytic function
cauchy's intergral theorem is applicable only for the	simply connected	multiple curve	multiconnected	meromorphic	simply connected

_____________ region enclosed by a simple curve					
A simple curve is one which _______________ itself	cross	doesnot cross	deviate	none of the above	doesnot cross
A multi curve is one which ____________ itself	cross	doesnot cross	deviate	none of the above	cross
Multiply connected region is bounded by ________ curves	more than two	one	two	none of the above	more than two
___________ curve encloses more than one separate region	simple curve	multiple curve	semi simple	semi multiple	multiple curve
If the function f(z) is analytic for all finite values of z and is bounded is the ____________	constant	zero	two	three	constant
An important fact in one dimensional case is the singularities are	constant	isolated	non isolated	none of the above	isolated
Analytic point is almost equivalent to	irregular point	regular point	poles	branch point	regular point
ordinary points and analytic points are	equivalent	non-equivalent	zero	constant	equivalent
Two equivalent terms holomorphic points and analytic points often occur with	regular point	irregular points	poles	branch point	regular point
A ___________ is a point, where y is differentiable in a open set around the point	meromorphic points	holonomic point	holomorphic points	none of the above	holomorphic points
___________ points which include regular points and removable singularities	ordinary points	poles	branch points	extraordinary points	ordinary points
The function z^2/(z-1)^3 has	a simple pole at z = 1	a simple pole at z = -1	a pole at z = 1 of order 2	a simple pole at z = 1 of order 3	a simple pole at z = 1 of order 3

The residue of the function 2z+1/z^2-z-2 at z=2 is	1/3	4-Mar	2/3	5/3	5/3
If a given number is wholly real, it is found in/on	a real axis	imaginary axis	x-y plane	space	x-y plane
A set which entirely consists of interior points is known as	an open set	a closed set	a banded set	domain	an open set
If a contour is a unit circle around the origin, then |z| is	1	0	e^{iq}	e^{q}	1
A connected open set is called	an open set	a closed set	a banded set	domain	an open set
Which is the analytic function of complex variable z = x + iy	|Z|	Re Z	Z^{-1}	Log Z	Z^{-1}
Which is the analytic function of complex variable Z=x +iy	Z	sin Z	Log Z	Re Z	Sin Z
Which is the analytic function of complex variable z=X+iY	|Z|	Re Z	Log Z	e^{sinz}	e^{sinz}
Which is not the analytic function of complex variable z=X+iY	|Z|	Re Z	Log Z	Z	Z
Which is not the analytic function of complex variable z=X+iY	|Z|	Re Z	Sin Z	Re Z	Re Z
The function Ze^2/(Z-a)^3 has	a simple pole at Z=a	a simple pole at Z=-a	a pole at Z=a of order 2	a pole at Z=a of order 3	a pole at Z=a of order 3
The symbol i with the property i^2=1 was introduced by	Euler	Gauss	Cauchy	Reimann	Euler
arg (Z_1 / Z_2) is equal to	arg Z_1+ arg Z_2	arg Z_1-arg Z_2	real	imaginary	arg Z_1-arg Z_2
A single valued function f(z) which is differentiable at z = zo it is said to be	irregular function	analytic function	periodic function	all the above	analytic function
The function 1/(Z-1)(Z+1) is analytic	at allpointsy =x	at all points,except Z=1	at all points,exceptZ=-1	both b and c	both b and c

In order that the function $f(z) = \|Z\|^2$ / Z, Z ¹ 0, be continuous at z = 0. we should define f(0) equal to	2	1	0	-1	0
The conjugate of 1/1+i is	1-i	1-i/√ 2	1-i/2	1+i	1-i/√ 2
The conjugate of (1+i) (3+4i) is	1+7i	1-7i	7-i	(-1-7i)	(-1-7i)
The conjugate of 1/i is	(-i)	i	1	(-1)	(-i)
The value of $i^2 + i^3 + i^4$ is	(-i)	i	1	(-1)	(-i)
If Z= a+ib, then real part of Z^{-1} is -----	a/ a^2+b^2	–b/ a^2+b^2	a/ $\sqrt{a^2+b^2}$	–b/$\sqrt{a^2+b^2}$	a/ a^2+b^2
If Z= a+ib, then Im(Z^{-1}) is ----	a/ a^2+b^2	–b/ a^2+b^2	a/ $\sqrt{a^2+b^2}$	–b/$\sqrt{a^2+b^2}$	–b/$\sqrt{a^2+b^2}$
The modulus and argument of √ 3 - i are	2, ∏/6	2, -∏/6	4, ∏/3	4, -∏/3	2, -∏/6
$Z_1 = r_1(\cos \theta_1 + i\sin \theta_1)$ and $Z_2 = r_2(\cos \theta_2 + i\sin \theta_2)$, then arg $Z_1 Z_2$ is ----	$\theta_1 + \theta_2$	$\theta_1 - \theta_2$	$\theta_1 \ \theta_2$	θ_1 / θ_2	$\theta_1 + \theta_2$
The argument of -1 + I is ---	- ∏/4	3∏/4	∏/4	3∏/2	3∏/4
$(1+ e^{-i\theta}) / (1+ e^{i\theta})$ = -------	cos θ + isin θ	cos θ - isin θ	sin θ -icos θ	sin θ +icos θ	cos θ + isin θ
The sum of n^{th} roots of unity are ------	0	1	2	3	0

UNIT 3

Special Functions Definition – The Beta function – Gamma function – Evaluation of Beta function – Other forms of Beta function – Evaluation of Gamma function – Other forms of Gamma function - Relation between Beta and Gamma functions – Problems.

BETA FUNCTION:

The first Eulerian function is generally known as Beta function β (m, n) and is defined by the definite integral

$$\beta\,(m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx \begin{cases} m > 0 \\ n > 0 \end{cases}$$

GAMMA FUNCTION:

The second Eulerian function is generally known as Gamma function Γn and is defined by the definite integral

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx,\ \ n > 0$$

ERROR FUNCTION:

The error function, *erf* (x) or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\ dy$$

This integral occupies a central position in the theory of probability and arises in the solution of certain partial differential equations of physical interest.

SYMMETRY PROPERTY OF BETA FUNCTION:

β (m, n) = β (n, m)

By definition β (m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \begin{cases} m>0 \\ n>0 \end{cases}$ ------------ (1)

Substituting x = 1 – y ∴ dx=-dy in the above equation we get

β (m, n) = $\int_0^1 (1-y)^{m-1}\, y^{n-1}\, dy$

= $\int_0^1 x^{n-1}(1-x)^{m-1}\,dx$ = β (n, m) ------------ (2)

i.e., Beta function β (m, n) is symmetric with respect to m and n. This property is called the symmetry property of Beta function.

EVALUTION OF BETA FUNCTION:

By definition β (m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \begin{cases} m>0 \\ n>0 \end{cases}$ ---------- (1)

Integrating by parts keeping $(1-x)^{n-1}$ as first function, we have

$$\beta\,(m, n) = \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 + \int_0^1 (n-1)(1-x)^{n-2}\frac{x^m}{m}\,dx$$

$$= \frac{n-1}{m}\int_0^1 (1-x)^{n-2}\,x^m\,dx$$

Integrating again by parts, we get

$$\beta\,(m, n) = \frac{(n-1)(n-2)}{m(m+1)}\int_0^1 (1-x)^{n-3}x^{m+1}\,dx$$

Continuing the process of integration by parts and assuming that n is a positive integer, we obtain

$$\beta\,(m, n) = \frac{(n-1)(n-2)\ldots\ldots\ldots\ldots.2\,.1}{m(m+1)\ldots\ldots\ldots(m+n-2)}\int_0^1 x^{m+n-2}\,dx$$

$$= \frac{(n-1)(n-2)\ldots\ldots\ldots\ldots.2\,.1}{m(m+1)\ldots\ldots\ldots(m+n-2)}\left[\frac{x^{m+n-2}}{m+n-1}\right]_0^1$$

$= \frac{n-1\,!}{m(m+1)\ldots\ldots\ldots(m+n-2)(m+n-1)}$ ------------- (2)

Again, if m is also a positive integer, then

β (m, n) = $\frac{(n-1)!(m-1)\,!}{(m+n-1)!}$ ------------ (3)

In case m alone is a positive integer, then in view of the symmetry property β (m, n), we have

β (m, n) = $\frac{(m-1)!}{n(n+1)\ldots\ldots\ldots\ldots(n+m-1)}$ ------------- (4)

OTHER FORMS OF BETA FUNCTION (OR) TRANSFORMATIONS OF BETA FUNCTION:

By definition β (m, n) = $\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$ ------------- (1)

(a) Substituting $x = \frac{y}{1+y}$ $\therefore$ dx = $\frac{dy}{(1+y)^2}$ and $1 - x = \frac{1}{1+y}$, we get

β (m, n) = $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{\mathbf{1}}{(\mathbf{1}+\boldsymbol{y})^{\boldsymbol{n-1}}} \cdot \frac{\boldsymbol{dy}}{(\mathbf{1}+\boldsymbol{y})^{\mathbf{2}}} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$ -------------- (2)

(b) Also since β (m, n) = β (n, m)

$\therefore \beta$ (m, n) = $\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$ ------------- (3)

This equation can be obtained from equation (1) by substituting $x = \frac{1}{1+y}$. Equations (2) and (3), represent other forms of Beta function.

EVALUTION OF GAMMA FUNCTION:

From definition $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

Integrating by parts keeping x^{n-1} as first function, we get

$\Gamma n = \left[-x^{n-1}\,e^{-x}\right]_0^\infty + \int_0^\infty (n-1)x^{n-2}\,e^{-x}\,dx$

$= (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx = (n-1)\,\Gamma\,(\mathrm{n}-1)$

Thus we have $\Gamma n = (n-1)\Gamma(n-1)$ ------------- (1)

Similarly, $\Gamma\,(n-1) = (n-2)\Gamma(n-2)$

Hence it follows that

$\Gamma n = (n-1)\,(\mathrm{n}-2)\Gamma(n-2)$

If n is positive integer, then proceeding as above repeatedly, we get

$\Gamma n = (n-1)(\mathrm{n}-2)(\mathrm{n}-3)\ldots\ldots\ldots.3.2.1\Gamma\,1$

But $\Gamma 1 = \int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty = 1$ ------------- (2)

Hence when n is positive integer

$\Gamma n = (n-1)(\mathrm{n}-2)\ldots\ldots\ldots.3.2.1 = (\mathrm{n}-1)!$ ------------- (3)

Thus, if n is a positive integer, then for all values of n

$\Gamma n = (n-1)\,\Gamma(n-1) = (\mathrm{n}-1)\,!$ ------------- (4)

This is the fundamental property of the Gamma functions,

From this property we may write

$\Gamma(n+1) = n\,\Gamma n \qquad i.e., \Gamma n = \frac{\Gamma(n+1)}{n}$ ------------- (5)

Putting n=0, we get

$\Gamma 0 = \infty$ ------------- (6)

It can be further shown that

$\Gamma(-n) = \infty$ ------------- (7)

OTHER FORMS OF GAMMA FUNCTION(OR) TRANSFORMATIONSOF GAMMA FUNCTION:

By definition $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$ ------------- (1)

(a) Substituting $x = \lambda y \therefore dx = \lambda . dy$ in equation (1), we get

$\Gamma n = \int_0^\infty e^{-\lambda y}\, \lambda^{n-1} . \lambda\, dy = \lambda^n \int_0^\infty e^{-\lambda y} y^{n-1}\, dy$ ------------- (2a)

$\therefore \int_0^\infty e^{-\lambda y}\, y^{n-1}\, dy = \frac{\Gamma n}{\lambda^n}$ -------------- (2b)

(b) Substituting $e^{-x} = y \therefore x = \log_e \frac{1}{y}$ $and\ dx = -\frac{dy}{y}$ in equation (1), we get

$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx = -\int_1^0 (\log_e \frac{1}{y})^{n-1}\, dy = \int_0^1 (\log_e \frac{1}{y})^{n-1}\, dy$ ------------ (3)

(c) Substituting $x^n = y \therefore x = y^{1/n}$ $and\ dx = \frac{1}{n} y^{(1-n)/n}\, dy$ in equation (1) we get,

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx = \int_0^\infty e^{-y^{1/n}}\, y^{(n-1)/n} . \frac{1}{n} y^{(1-n)/n} dy$$

$= \frac{1}{n} \int_0^\infty e^{-y^{1/n}} dy$ -------------(4)

Equations (2a), (3) and (4) represent other forms of Gamma function.

RELATION BETWEEN BETA AND GAMMA FUNCTION:

$\beta\,(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma\,(m+n)}$

The other form of Gamma function Γm is given by

$\Gamma m = \int_0^\infty \lambda^m\, e^{-\lambda x}\, x^{m-1}\, dx$ ------------- (1)

Multiply both sides by $e^{-\lambda}\, x^{n-1}$ and integrating with respect to λ between the limits 0 and ∞, we get

$\Gamma m \int_0^\infty e^{-\lambda}\, \lambda^{n-1}\ \ d\lambda = \int_0^\infty \left[\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1}\, d\lambda\right] x^{m-1}\, dx$ ------------- (2)

But $\int_0^\infty e^{-\lambda}\, \lambda^{n-1}\ \ d\lambda = \Gamma n$

And $\int_0^\infty e^{-\lambda(1+x)}\lambda^{m+n-1}\,d\lambda = \frac{\Gamma(m+n)}{(i+x)^{m+n}}$

Substituting these values in equation (2), we get

$\Gamma m\Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1}\,dx$ ----------- (3)

$= \Gamma m\Gamma n \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$= \Gamma m\Gamma n\beta(m, n)$

$\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$ ----------- (4)

This is an important relation between Beta and Gamma function.

PROBLEMS:

1. Evaluate: $\int_0^\infty e^{-x^2}\,dx$

Solution:

Let $I = \int_0^\infty e^{-x^2}\,dx$ ----------- (1)

Substituting $x = \lambda y \therefore dx = \lambda\,dy$ (where λ is constant) in equation (1), we get

$I = \int_0^\infty e^{-\lambda^2 y^2}.\lambda\,dy$

Multiplying both sides by $e^{-\lambda^2}$ and integrating with respect to λ between the limits 0 and ∞, we get

$I \int_0^\infty e^{-\lambda^2} d\lambda = \int_0^\infty \int_0^\infty e^{-\lambda^2(1+y^2)}\,\lambda.d\lambda.dy$ ----------- (2)

But $\int_0^\infty e^{-\lambda^2(1+y^2)}\,\lambda.d\lambda = \left[\frac{-e^{-\lambda^2(1+y^2)}}{2(1+y^2)}\right]_0^\infty = \frac{1}{2(1+y^2)}$

And $\int_0^\infty e^{-\lambda^2}\,d\lambda = I$

$\therefore I^2 = \int_0^\infty \frac{1}{2(1+y^2)}\,dy = \frac{1}{2}[tan^{-1}y]_0^\infty = \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}$

$\therefore I = \frac{\sqrt{\pi}}{2}$ ------------- (3)

2. To show that $\Gamma m\, \Gamma(1-m) = \frac{\pi}{\sin m\pi}$

Solution:

The relation between Beta and Gamma function is

$\beta\,(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma\,(m+n)}$ ---------- (1)

The transformed form of Beta function β (m, n) is

$\beta\,(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$

$\frac{\Gamma m\, \Gamma n}{\Gamma\,(m+n)} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$ ----------- (2)

Substituting m + n = 1, n = 1 – m in equation (2), we get

$$\frac{\Gamma m\, \Gamma(1-m)}{\Gamma\, 1} = \int_0^\infty \frac{y^{m-1}}{1+y}\, dy$$

Using equation $\int_0^\infty \frac{y^{p-1}}{1+y}\, dy = \pi\, cosec\, p\pi = \frac{\pi}{\sin p\pi}$ and keeping in mind that $\Gamma\, 1 = 1$, we get

$\Gamma m\, \Gamma(1-m) = \frac{\pi}{\sin m\pi}$ ------------ (3)

3. To show that $\Gamma \frac{1}{2} = \sqrt{\pi}$

Solution:

The relation between Beta and Gamma function is

$\beta\,(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma\,(m+n)}$ ---------- (1)

The transformed form of Beta function β (m, n) is

$\beta\,(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$

$\frac{\Gamma m\, \Gamma n}{\Gamma\,(m+n)} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$ ----------- (2)

Substituting m + n = 1, n = 1 – m in equation (2), we get

$$\frac{\Gamma m\,\Gamma(1-m)}{\Gamma\,1} = \int_0^\infty \frac{y^{m-1}}{1+y}\,dy$$

Using equation $\int_0^\infty \frac{y^{p-1}}{1+y}\,dy = \pi\,cosec\,p\pi = \frac{\pi}{\sin p\pi}$ and keeping in mind that $\Gamma\,1 = 1$, we get

$\Gamma m\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ ------------ (3)

Replacing m by $\frac{1}{2}$in equation (3), we get

$$\Gamma\frac{1}{2}\Gamma\frac{1}{2} = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$ ------------ (4)

4. To show that $\Gamma(1-m)\Gamma(1-m) = \frac{m\pi}{\sin m\pi}$

Solution:

The relation between Beta and Gamma function is

$\beta\,(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma\,(m+n)}$ ---------- (1)

The transformed form of Beta function β (m, n) is

$\beta\,(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$

$\frac{\Gamma m\,\Gamma n}{\Gamma\,(m+n)} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$ ----------- (2)

Substituting m + n = 1, n = 1 – m in equation (2), we get

$$\frac{\Gamma m\,\Gamma(1-m)}{\Gamma\,1} = \int_0^\infty \frac{y^{m-1}}{1+y}\,dy$$

Using equation $\int_0^\infty \frac{y^{p-1}}{1+y}\,dy = \pi\,cosec\,p\pi = \frac{\pi}{\sin p\pi}$ and keeping in mind that $\Gamma\,1 = 1$, we get

$\Gamma m\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ ------------ (3)

Multiplying both sides of (3) by m and using

$m\Gamma \mathrm{m} = \Gamma(1-\mathrm{m})$, we get

$\Gamma(1-\mathrm{m})\Gamma(1-\mathrm{m}) = \frac{m\pi}{\sin m\pi}$ ------------ (4)

5. If n is positive integer prove that $2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ \ldots\ldots (2n-1)\sqrt{\pi}$

<u>Solution:</u>

We know that $\Gamma(\mathrm{n}-1) = n\Gamma\mathrm{n}$

$\therefore \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) \ldots\ldots \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}$

$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) \ldots\ldots \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$ (since $\Gamma\frac{1}{2} = \sqrt{\pi}$)

$= \frac{(2n-1)(2n-3)(2n-5)\ldots\ldots 5.3.1.\sqrt{\pi}}{2^n}$

$$2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ \ldots\ldots (2n-1)\sqrt{\pi}$$

POSSIBLE QUESTIONS

1. Define Beta function .
2. Define Gamma function .
3. Define Error function.
4. Evaluation of Beta function
5. Other forms of Beta function
6. Evaluation of Gamma function
7. Other forms of Gamma function
8. Relation between Beta and Gamma functions
9. If n is positive integer prove that $2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ldots\ldots(2n-1)\sqrt{\pi}$
10. To show that $\Gamma(1-\mathrm{m})\Gamma(1-\mathrm{m}) = \frac{m\pi}{\sin m\pi}$
11. To show that $\Gamma\frac{1}{2} = \sqrt{\pi}$
12. To show that $\Gamma\mathrm{m}\,\Gamma(1-\mathrm{m}) = \frac{\pi}{\sin m\pi}$
13. Evaluate: $\int_0^\infty e^{-x^2}\,dx$

KARPAGAM ACADEMY OF HIGHER EDUCATION

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(For the candidates admitted from 2018 onwards)

DEPARTMENT OF PHYSICS

UNIT I :(Objective Type/Multiple choice Questions each Question carries one Mark)

PART-A (Online Examination)

MATHEMATICAL PHYSICS-I

QUESTIONS	opt1	opt2	opt3	opt4	ANSWER
UNIT-III					
The Gamma function is otherwise called	harmonic function	generalised funtion	generalised factorial function	none of these	generalised factorial function
____ are special function of mathematics	Bessels	Legendre	Algebraic	Both a and b	Both a and b
Singular points are of __________ types	1	2	3	4	2
The value of Jo(x) at the origin is	1	0	-1	X	0
From Bessel's functions, the value of $J_{n+1}(x)$ is	$nJ_n(x) + J_n'(x)$	$(n / x) J_n(x) - J_n'(x)$	$nJ_n(x) - J_n'(x)$	$(n / x) J_n(x) + J_n'(x)$	**$(n / x) J_n(x) - J_n'(x)$**
When 'n' is an integer, $J_n(x)$ and $J_{-n}(x)$ are	harmonic function	linearly dependent	linearly independent	orthonormal	linearly independent
Bessel's functions are	indeterminate	simple	oscillatory	critically	oscillatory

		harmonic	function	damped	function
The value of $P_1(x)$ is	x	1	$x^2 / 2$	½ $(x^2 - 1)$	x
The identical roots of the Legendre's functions are	m = ± n	m = ± 1	m = 0 or m = 1	m = 0 or m = -1	**m = ± 1**
If Jo and J_1 are Bessel's functions then $J_1'(x)$ is given by	Jo(x) – 1/x $J_1(x)$	– Jo	Jo(x) + 1/x $J_1(x)$	Jo(x) -1/x $J_1(x)$	**Jo(x) – 1/x** $J_1(x)$
If $J_{n+1}(x) = (2/x) J_n(x) - Jo(x)$ where Jn is the Bessel function of first kind order 'n'. Then 'n' is	0	2	-1	1	1
The value of Po(x) is	1	x	0	-1	1
Let Pn(x) be the Legendre polynomial, then Pn(-x) is equal to	$(-1)^{n+1} P_n'(x)$	$(-1)^n P_n'(x)$	$\mathbf{(-1)^n P_n(x)}$	$P_n''(x)$	$\mathbf{(-1)^n P_n(x)}$
If $P_n(x)$ is the Legendre polynomial of order 'n', then $3x^2 + 3x + 1$ can be expressed as	$3P_2 + 3P_1$	$4P_2+2P_1$ + Po	$3P_2+3P_1$ + Po	$2P_2+3P_1$ + 2Po	$\mathbf{2P_2+3P_1}$ + 2Po
The Legendre polynomial Pn(x) has	n real zeros between 0 and 1	n zeros of which only one is between –1 and +1	2n-1 real zeros between –1 and 1	none of these	**n real** zeros between 0 and 1
The incorrect equation among the following is	$P_o(x) = 1$	$P_1(x) = x$	$P_n(-x) = (-1)^{n+1} P_n(x)$	$P_n(x) = (-1)^{n+1} P_n(x)$	$\mathbf{P_n(-x)} = (-1)^{n+1} P_n(x)$
The value of $P_n(-x)$ is	$- P_n(x)$	$P_n(x)$	$(-1)^n P_n(x)$	$(-1)^n P_n(-x)$	$P_n(x)$
The value of $2J_n'$ is	$J_{n-1} - J_{n+1}$	$J_{n-1} + J_{n+1}$	$J_{n+1} - J_{n+1}$	$2 J_{n+1}$	$\mathbf{J_{n+1}} - J_{n+1}$
The direct method fails if any one of the pivot elements become ----------------.	Zero	one	two	negative	Zero

Γ1/2=	√π	π	0	1	√π
Which of the following is the alter name for the method of false position	Method of chords	method of tangents	method of bisection	regula falsi	regula falsi
Gauss elimination method is a -----------------.	Direct method indirect method	iterative method	convergent	indirect method	Direct method
The rate of convergence in Gauss – Seidel method is roughly --------- times than that of Gauss Jacobi method.	2	3	4	0	2
Example for iterative method -------------.	Gauss elimination	Gauss Seidal	Gauss Jordan	Romberg	Gauss Seidal
In the absence of any better estimates, the initial approximations are taken as---	x = 0, y = 0, z = 0	x = 1, y = 1, z = 1	x= 2, y = 2, z = 2	x= 1, y = 2, z = 3	x = 0, y = 0, z = 0
When Gauss Jordan method is used to solve AX = B, A is transformed into -----	Scalar matrix	diagonal matrix	Upper triangular matrix	lower triangular matrix	diagonal matrix
The modification of Gauss – Elimination method is called ---------------------.	Gauss Jordan	Gauss Seidal	Gauss Jacobbi	Romberg	Gauss Jordan
The ------------- Method produces the exact solution after a finite number of steps.	Gauss Seidal	Gauss Jacobi	iterative	direct.	direct.
In the upper triangular coefficient matrix, all the elements above the diagonal are --------------.	Zero	non – zero	unity	negative.	non – zero
In the upper triangular coefficient matrix, all the elements below the diagonal	Positive	non zero	zero	negative.	zero

are --------------.					
Condition for convergence of Gauss Seidal method is ---------------.	Coefficient matrix is diagonally dominant	pivot element is Zero	Coefficient matrix is not diagonally dominant	none of these.	Coefficient matrix is diagonally dominant
In Gauss elimination method by means of elementary row operations, from which the unknowns are found by ----------------- method.	Forward substitution	back substitution	random	both forward and backward	back substitution
In iterative methods, the solution to a system of linear equations will exist if the absolute value of the largest coefficient is ------------ the sum of the absolute values of all remaining coefficients in each equation.	less than	greater than or equal to	equal to	equal to 0	greater than or equal to
In ------------- iterative method, the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration.	Gauss Seidal	Gauss Jacobi	Gauss Jordon	Gauss elimination.	Gauss Seidal
The direct method fails if any one of the pivot elements become ----------------.	Zero	one	two	negative	Zero
In Gauss elimination method the given matrix is transformed into --------------.	Unit matrix	diagonal matrix	Upper triangular matrix	lower triangular matrix	Upper triangular matrix
If the coefficient matrix is not diagonally dominant, then by ----------- that diagonally	Interchanging rows	interchanging columns	adding zeros	interchanging rows and columns	interchanging rows and columns

dominant coefficient matrix is formed.					
Gauss Jordan method is a -----------------.	Direct method	indirect method	iterative method	convergent	Direct method
Gauss Jocobi method is a -----------------.	Direct method	indirect method	iterative method	convergent	indirect method
The modification of Gauss – Jordan method is called ---------------------.	Gauss Jordan	Gauss Seidal	Gauss Jacobbi	Crouts	Gauss Seidal
Gauss Seidal method always converges for ----------- of systems.	Only the special type	all types	quadratic type	none of these.	Only the special type
In solving the system of linear equations, the system can be written as ---------.	BX = B	AX = A	AX = B	AB = X	AX = B
In solving the system of linear equations, the augment matrix is --------.	(A, A)	(B, B)	(A, X)	(A, B)	(A, B)
In the direct methods of solving a system of linear equations, at first the given system is written as ------------- form.	An augment matrix	a triangular matrix	Constant matrix	coefficient matrix	An augment matrix
All the row operations in the direct methods can be carried out on the basis of --	all elements	pivot element	negative elements	positive elements	pivot element
The direct method fails if -----------.	The first row elements are zero	the first column elements are zero	first row and column elements are zero	None of these	first row and column elements are zero
The elimination of the unknowns is done not only in the equations below, but also in the equations above the leading	Gauss elimination	Gauss Jordan	Gauss Jacobi	Gauss Seidal	Gauss Jordan

diagonal is called ------------.					
In Gauss Jordan method, we get the solution ---------------------.	without using back substitution method	by using back substitution method	by using forward substitution method	without using forward substitution method	without using back substitution method
If the coefficient matrix is diagonally dominant, then ---------- method converges quickly.	Gauss elimination	Gauss Jordan	direct	Gauss Seidal	Gauss Seidal
Which is the condition to apply Jocobi's method to solve a system of equations.	First row is dominant	First column is dominant	Diagonally dominant	upper triangular matrix	Diagonally dominant
Iterative method is a ------------ method.	Direct	indirect	step by step	difficult	indirect

UNIT 4

Matrices Introduction – special types of Matrices – Transpose of a Matrix – The Conjugate of a Matrix – Conjugate Transpose of a Matrix – Symmetric and Anti symmetric – Hermitian and skew Hermitian – Orthogonal and Unitary Matrices – Properties – Characteristics equation – Roots and characteristics vector – Diagonalization of matrices – Cayley – Hamilton theorem – Problems

MATRICES:

A rectangular array of numbers is called a matrix. We shall mostly be concerned with matrices having real numbers as entries. The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS. A matrix having m rows and n columns is said to have the order $m \times n$.

A matrix $\boldsymbol{A}$ of order $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

In a more concise manner, we also denote the matrix $\boldsymbol{A}$ by $[a_{ij}]$ by suppressing its order.

Note: Some books also use

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

to represent a matrix.

A matrix having only one column is called a column vector and a matrix with only one row is called a row vector. Whenever a vector is used, it should be understood from the context whether it is a row vector or a column vector.

Here are a couple of examples of different types of matrices:

Symmetric:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

Diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Lower Triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 7 & 0 \\ 12 & 5 & 3 \end{bmatrix}$$

Zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 1:

Let $A = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 3 & 7 \\ 4 & 6 & 0 \end{bmatrix}$, list out the a_{ij}'s values in ***A***.

Solution:

$a_{11} = 1, a_{12} = 3, a_{13} = 6$

$a_{21} = 2, a_{22} = 3, a_{23} = 7$

$a_{31} = 4, a_{32} = 4, a_{33} = 0$

Example 2:

State the a_{ij}'s values in $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix}$.

Solution:

$a_{11} = 9, a_{12} = 8, a_{13} = 7$

$a_{21} = 6, a_{22} = 5, a_{23} = 4$

$a_{31} = 3, a_{32} = 2, a_{33} = 1$

$a_{41} = 4, a_{42} = 6, a_{43} = 8$

Example 3: Provide two examples of column and row matrices each.

Solution:

We know that, a matrix having only one column is called a column vector or column matrix and a matrix with only one row is called a row vector or row column.

$$1. A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

$$2. A = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 & 9 & 0 \end{bmatrix}$$

Example 4: Is the following matrix classified under the category of matrices?

$$A = \begin{bmatrix} 9 & 7 & 8 \\ 8 & 7 & 9 \\ 6 & 9 & \end{bmatrix}$$

Solution:

A is not a matrix because column three or we can say row three is incomplete.

Example 5: What is the order of the following matrix?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

We know that a matrix having *m* rows and *n* columns is said to have the order $m \times n$, therefore the order of $\boldsymbol{A}$ is 4×3.

Types of matrices — triangular, diagonal, scalar, identity, symmetric, skew-symmetric, periodic, nilpotent

Upper triangular matrix. A square matrix in which all the elements below the diagonal are zero i.e. a matrix of type:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

Lower triangular matrix. A square matrix in which all the elements above the diagonal are zero i.e. a matrix of type

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrix. A square matrix in which all of the elements are zero except for the diagonal elements i.e. a matrix of type

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

It is often written as $D = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$

Scalar matrix. A diagonal matrix in which all of the diagonal elements are equal to some constant "k" i.e. a matrix of type

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

Identity matrix. A diagonal matrix in which all of the diagonal elements are equal to "1" i.e. a matrix of type

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

An identity matrix of order nxn is denoted by $\mathbf{I_n}$.

Transpose of a matrix.

The matrix resulting from interchanging the rows and columns in the given matrix. The transpose of

$$A = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 5 & 4 \end{bmatrix}$$

is

$$A^T = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 8 & 4 \end{bmatrix}$$

The first row of A becomes the first column of A^T, the second row of A becomes the second column of A^T, etc.. It corresponds to a "flip"of the matrix about the diagonal running down from the upper left corner.

Symmetric matrix.

A square matrix in which corresponding elements with respect to the diagonal are equal; a matrix in which $a_{ij} = a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to its transpose; a square matrix in which a flip about the diagonal leaves it unchanged. Example:

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & 5 & 6 \end{bmatrix}$$

Skew-symmetric matrix.

A square matrix in which corresponding elements with respect to the diagonal are negatives of each other; a matrix in which $a_{ij} = -a_{ji}$ where a_{ij} is the element in the i-th row and j-th column; a matrix which is equal to the negative of its transpose. The diagonal elements are always zeros. Example:

$$\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

Direct Sum. Let $A_1, A_2, \ldots, A_s$ be square matrices of respective orders $m_1, m_2, \ldots, m_s$. The generalization

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_s \end{bmatrix} = Diag(A_1, A_2, \ldots, A_s)$$

of the diagonal matrix is called the direct sum of the A_i.

Inverse of a matrix.

If A and B are square matrices such that AB = BA = I where I is the identity matrix , then B is called the inverse of A and we write $B = A^{-1}$. The matrix B also has A as an inverse and we can write $A = B^{-1}$.

Commutative and anti-commutative matrices. If A and B are square matrices such that AB = BA, then A and B are called commutative or are said to commute. If AB = -BA, the matrices are said to **anti-commute.**

Periodic matrix. A matrix A for which $\mathbf{A^{k+1} = A}$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then A is said to be of **period k**. If k = 1, so that

$A^2 = A$, then A is called **idempotent**.

Nilpotent matrix. A matrix A for which $\mathbf{A^p = 0}$, where p is some positive integer. If p is the least positive integer for which $A^p = 0$, then A is said to be**nilpotent of index p**.

DETERMINANT OF A MATRIX:

Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations. As shown by Cramer's rule, a nonhomogeneous system of linear equations has a unique solution iff the determinant of the system's matrix is nonzero (i.e., the matrix is nonsingular). For example, eliminating x, y, and $z$from the equations

$$a_1 x + a_2 y + a_3 z = 0 \tag{1}$$

$$b_1 x + b_2 y + b_3 z = 0 \tag{2}$$

$$c_1 x + c_2 y + c_3 z = 0 \tag{3}$$

gives the expression

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1 = 0, \tag{4}$$

which is called the determinant for this system of equation. Determinants are defined only for square matrices.

If the determinant of a matrix is 0, the matrix is said to be singular, and if the determinant is 1, the matrix is said to be unimodular.

The determinant of a matrix A,

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_n \end{vmatrix} \tag{5}$$

is commonly denoted $\det(\mathsf{A})$, $|\mathsf{A}|$, or in component notation as $\sum(\pm a_1 b_2 c_3 \cdots)$, $D(a_1 b_2 c_3 \cdots)$, or $|a_1 b_2 c_3 \ldots|$(Muir 1960, p. 17). Note that the notation $\det(\mathsf{A})$ may be more convenient when indicating the absolute value of a determinant, i.e., $|\det(\mathsf{A})|$ instead of $||\mathsf{A}||$. The determinant is implemented in the Wolfram Language as Det[*m*].

A 2×2 determinant is defined to be

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv a\,d - b\,c. \tag{6}$$

A $k\times k$ determinant can be expanded "by minors" to obtain

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k2} & a_{k3} & \cdots & a_{kk} \end{vmatrix} \\ &- a_{12}\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k3} & \cdots & a_{kk} \end{vmatrix} + \ldots \pm a_{1k}\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{k(k-1)} \end{vmatrix}.\end{aligned} \tag{7}$$

A general determinant for a matrix A has a value

$$|\mathsf{A}| = \sum_{i=1}^{k} a_{ij}\, C_{ij}, \tag{8}$$

with no implied summation over j and where C_{ij} (also denoted a^{ij}) is the cofactor of a_{ij} defined by

$$C_{ij} \equiv (-1)^{i+j} M_{ij}. \tag{9}$$

and M_{ij} is the minor of matrix A formed by eliminating row i and column j from A. This process is calleddeterminant expansion by minors (or "Laplacian expansion by minors," sometimes further shortened to simply "Laplacian expansion").

A determinant can also be computed by writing down all permutations of $\{1, \ldots, n\}$, taking each permutation as the subscripts of the letters a, b, ..., and summing with signs determined by $\epsilon_p = (-1)^{i(p)}$, where $i(p)$ is the number of permutation inversions in permutation p (Muir 1960, p. 16), and $\epsilon_{n_1 n_2 \cdots}$ is the permutation symbol. For example, with $n = 3$, the permutations and the number of inversions they contain are 123 (0), 132 (1), 213 (1), 231 (2), 312 (2), and 321 (3), so the determinant is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\, b_2\, c_3 - a_1\, b_3\, c_2 - a_2\, b_1\, c_3 + a_2\, b_3\, c_1 + a_3\, b_1\, c_2 - a_3\, b_2\, c_1. \tag{10}$$

If a is a constant and A an $n \times n$ square matrix, then

$$|a\, \mathsf{A}| = a^n\, |\mathsf{A}|. \tag{11}$$

Given an $n \times n$ determinant, the additive inverse is

$$|-\mathsf{A}| = (-1)^n\, |\mathsf{A}|. \tag{12}$$

Determinants are also distributive, so

$$|\mathsf{A}\, \mathsf{B}| = |\mathsf{A}|\, |\mathsf{B}|. \tag{13}$$

This means that the determinant of a matrix inverse can be found as follows:

$$|\mathsf{I}| = \left|\mathsf{A}\, \mathsf{A}^{-1}\right| = |\mathsf{A}|\, \left|\mathsf{A}^{-1}\right| = 1, \tag{14}$$

where I is the identity matrix, so

$$|\mathsf{A}| = \frac{1}{\left|\mathsf{A}^{-1}\right|}. \tag{15}$$

Determinants are multi linear in rows and columns, since

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} \tag{16}$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}. \tag{17}$$

The determinant of the similarity transformation of a matrix is equal to the determinant of the original matrix

$$\left|\mathsf{B}\,\mathsf{A}\,\mathsf{B}^{-1}\right| = |\mathsf{B}|\,|\mathsf{A}|\,\left|\mathsf{B}^{-1}\right| \tag{18}$$

$$= |\mathsf{B}|\,|\mathsf{A}|\,\frac{1}{|\mathsf{B}|} \tag{19}$$

$$= |\mathsf{A}|. \tag{20}$$

The determinant of a similarity transformation minus a multiple of the unit matrix is given by

$$\left|\mathsf{B}^{-1}\,\mathsf{A}\,\mathsf{B} - \lambda\,\mathsf{I}\right| = \left|\mathsf{B}^{-1}\,\mathsf{A}\,\mathsf{B} - \mathsf{B}^{-1}\,\lambda\,\mathsf{I}\,\mathsf{B}\right| \tag{21}$$

$$= \left|\mathsf{B}^{-1}\,(\mathsf{A} - \lambda\,\mathsf{I})\,\mathsf{B}\right| \tag{22}$$

$$= \left|\mathsf{B}^{-1}\right|\,|\mathsf{A} - \lambda\,\mathsf{I}|\,|\mathsf{B}| \tag{23}$$

$$= |\mathsf{A} - \lambda\,\mathsf{I}|. \tag{24}$$

The determinant of a transpose equals the determinant of the original matrix,

$$|\mathsf{A}| = \left|\mathsf{A}^{\mathrm{T}}\right|, \tag{25}$$

and the determinant of a complex conjugate is equal to the complex conjugate of the determinant

$$|\bar{\mathsf{A}}| = \overline{|\mathsf{A}|}. \tag{26}$$

Let ϵ be a small number. Then

$$|I + \epsilon A| = 1 + \epsilon \operatorname{Tr}(A) + O(\epsilon^2), \tag{27}$$

where $\operatorname{Tr}(A)$ is the matrix trace of A. The determinant takes on a particularly simple form for a triangular matrix

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ 0 & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{kk} \end{vmatrix} = \prod_{n=1}^{k} a_{nn}. \tag{28}$$

Important properties of the determinant include the following, which include invariance under elementary row and column operations.

1. Switching two rows or columns changes the sign.

2. Scalars can be factored out from rows and columns.

3. Multiples of rows and columns can be added together without changing the determinant's value.

4. Scalar multiplication of a row by a constant c multiplies the determinant by c.

5. A determinant with a row or column of zeros has value 0.

6. Any determinant with two rows or columns equal has value 0.

Property 1 can be established by induction. For a 2×2 matrix, the determinant is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2 \tag{29}$$

$$= -(b_1 a_2 - a_1 b_2) \tag{30}$$

$$= -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} \tag{31}$$

For a 3×3 matrix, the determinant is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= -\left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right) = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \qquad (32)$$

$$= -\left(-a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}\right) = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

$$= -\left(a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}\right) = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$$

Property 2 follows likewise. For 2×2 and 3×3 matrices,

$$\begin{vmatrix} k\,a_1 & b_1 \\ k\,a_2 & b_2 \end{vmatrix} = k\,(a_1\,b_2) - k\,(b_1\,a_2) \qquad (33)$$

$$= k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad (34)$$

and

$$\begin{vmatrix} k\,a_1 & b_1 & c_1 \\ k\,a_2 & b_2 & c_2 \\ k\,a_3 & b_3 & c_3 \end{vmatrix} = k\,a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} k\,a_2 & c_2 \\ k\,a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} k\,a_2 & b_2 \\ k\,a_3 & b_3 \end{vmatrix} \qquad (35)$$

$$= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad (36)$$

Property 3 follows from the identity

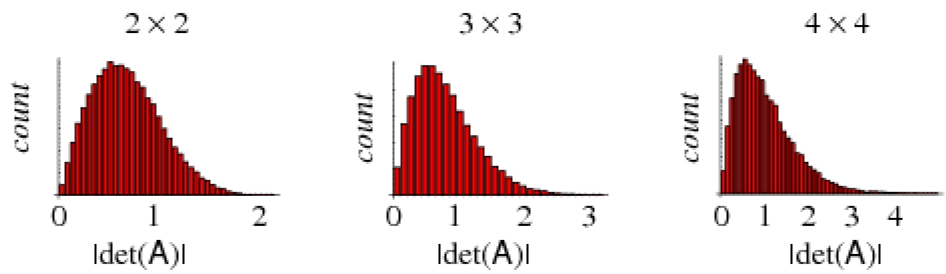
$$\begin{vmatrix} a_1 + k\,b_1 & b_1 & c_1 \\ a_2 + k\,b_2 & b_2 & c_2 \\ a_3 + k\,b_3 & b_3 & c_3 \end{vmatrix} = (a_1 + k\,b_1) \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 + k\,b_2 & c_2 \\ a_3 + k\,b_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 + k\,b_2 & b_2 \\ a_3 + k\,b_3 & b_3 \end{vmatrix}. \qquad (37)$$

If a_{ij} is an $n \times n$ matrix with a_{ij} real numbers, then $\det [a_{ij}]$ has the interpretation as the oriented n-dimensional content of the parallelepiped spanned by the column vectors $[a_{i,1}]$, ..., $[a_{i,n}]$ in $\mathbb{R}^n$. Here, "oriented" means that, up to a change of $+$ or $-$ sign, the number is the n-dimensional content, but the sign depends on the "orientation" of the column vectors involved. If they agree with the standard orientation, there is a $+$ sign; if not, there is a $-$ sign. The parallelepiped spanned by the n-dimensional vectors $\mathbf{v}_1$ through $\mathbf{v}_i$ is the collection of points

$$t_1\ \mathbf{v}_1 + \ldots + t_i\ \mathbf{v}_i, \tag{38}$$

where t_j is a real number in the closed interval $[0, 1]$.

Several accounts state that Lewis Carroll (Charles Dodgson) sent Queen Victoria a copy of one of his mathematical works, in one account, *An Elementary Treatise on Determinants*. Heath (1974) states, "A well-known story tells how Queen Victoria, charmed by *Alice in Wonderland*, expressed a desire to receive the author's next work, and was presented, in due course, with a loyally inscribed copy of *An Elementary Treatise on Determinants*," while Gattegno (1974) asserts "Queen Victoria, having enjoyed *Alice* so much, made known her wish to receive the author's other books, and was sent one of Dodgson's mathematical works." However, in *Symbolic Logic* (1896), Carroll stated, "I take this opportunity of giving what publicity I can to my contradiction of a silly story, which has been going the round of the papers, about my having presented certain books to Her Majesty the Queen. It is so constantly repeated, and is such absolute fiction, that I think it worth while to state, once for all, that it is utterly false in every particular: nothing even resembling it has occurred" (Mikkelson and Mikkelson).



Hadamard (1893) showed that the absolute value of the determinant of a complex $n \times n$ matrix with entries in theunit disk satisfies

$$|\det A| \leq n^{n/2} \quad (39)$$

(Brenner 1972). The plots above show the distribution of determinants for random $n \times n$ complex matrices with entries satisfying $|a_{ij}| < 1$ for $n = 2$, 3, and 4.

RANK OF A MATRIX:

The DETERMINANT of a matrix det *A* or $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$|A| = a_{11} - a_{12}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$

$(a_{22}a_{33}-a_{23}a_{32})$ is called the minor of a_{11} and is usually denoted $|A_{ij}|$ - in this case $|A_{11}|$

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 6 & 9 & 7 \\ 2 & 8 & 1 \end{bmatrix}$$

$$|A| = +3\begin{vmatrix} 9 & 7 \\ 8 & 1 \end{vmatrix} - 5\begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 6 & 9 \\ 2 & 8 \end{vmatrix}$$

= 3(9 – 56) - 5(6 – 14) + 4(48 – 18)

= -141 + 40 + 120

= 19

The COFACTOR of the elements of a_{ij} denoted by c_{ij}
is

$c_{ij} = (-1)^{i+j}|A_{ij}|$

$$|A| = \sum_{j=1}^{n} a_{ij}c_{ij} = \sum_{i}^{n} a_{ij}c_{ij}$$

PROPERTIES OF DETERMINANTS

1. $|A^T| = |A|$

2. $\begin{vmatrix} a_{11} & ka_{12} \\ a_{21} & ka_{22} \end{vmatrix} = ka = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

3. If A is (n x n) then $|kA| = k^n |A|$

4. If a square matrix has two equal rows or columns its determinant is zero

5. If any row (or column) is the multiple of any other row (or column) then its determinant is zero

6. The value of a determinant is unchanged if a multiple of one row (or column) is added to another row (or column)

7. If A is a diagonal matrix of order n then its determinant is $a_{11}a_{22} \ldots a_{nn}$

8. If A is a triangular matrix of order n then its determinant is $a_{11}a_{22} \ldots a_{nn}$

9. If B is the matrix obtained from a square matrix A by interchanging any two rows (or columns) then det B = -det A

10. If A and B are square matrices of the same order then $|AB| = |A|\,|B|$

11. If $A_1, A_2, \ldots, A_s$ are square matrices then $|\text{diag}(A_1, A_2, \ldots, A_s)| = |A_1|\,|A_2| \ldots |A_s|$

12. In general $|A + B|$ does not equal $|A| + |B|$

The RANK of a matrix is equal to the highest order non-zero determinant that can be formed from its sub-matrices

$$A = \begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$$

det A = 0

$$\begin{vmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{vmatrix} = 63$$

Rank of A = 3

The rank of a matrix can also be measured by the maximum number of linearly independent columns of A

This also equals the maximum number of linearly independent rows

$$1\begin{bmatrix}4\\3\\8\\1\end{bmatrix}+2\begin{bmatrix}5\\9\\10\\2\end{bmatrix}+0\begin{bmatrix}2\\6\\7\\9\end{bmatrix}+(-1)\begin{bmatrix}14\\21\\28\\5\end{bmatrix}=0$$

$c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4 = 0$

A FULL COLUMN RANK matrix has the same number of linearily independent columns (rows) equal to the number of columns

A FULL ROW RANK matrix has the same number of linearily independent rows (columns) equal to the number of rows

If A does not have full row and column rank it is SINGULAR
If A does have full row and column rank it is NON-SINGULAR

rank (I_n) = n
rank (kA) = rank (A)
rank (A^T) = rank (A)
If A is (m x n) then rank (A) is $\leq$ min {m, n}
rank $AB \leq$ min{rank (A), rank (B)}

INVERSES

If A and B are matrices of order n such that $AB = BA = I_n$ then B is called the inverse of A

A has an inverse iff it is of full column and row rank – non-singular

$A^{-1} = C^T / |A|$

C^T is the transpose of the matrix of co-factors

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$|A| = 4 - 6 = -2$

$c_{11} = 4$ $c_{12} = -3$ $c_{21} = -2$ $c_{22} = 1$

$$A^{-1} = \frac{-1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$$

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

PROPERTIES OF INVERSES

1. $I^{-1} = I$
2. $(A^{-1})^{-1} = A$
3. $AB = I \quad BA = I$
4. A non-singular A^{-1} non-singular
5. A and B non-singular $(AB)^{-1} = B^{-1}A^{-1}$

LEFT INVERSE of a (m x n) matrix A is the (n x m) matrix B such that $BA = I_n$

RIGHT INVERSE of a (m x n) matrix A is the (n x m) matrix C such that $AC = I_m$

(T x k) design matrix X which has rank k < T
has an infinite number of left inverses including
$(X^T X)^{-1} X^T$

IDEMPOTENT MATRICES

$AA = A$

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.3 & 0.6 \end{bmatrix}$$

Consider $M = [I - X(X^T X)^{-1} X^T]$

KRONECKER PRODUCT

$$A \otimes B$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 1\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 3\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \\ 2\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} & 0\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 & 6 & 6 & 0 \\ 1 & 0 & 3 & 3 & 0 & 9 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 & 0 & 0 \end{bmatrix}$$

EIGEN VALUE AND EIGEN VECTORS:

Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, characteristic values (Hoffman and Kunze 1971), proper values, or latent roots (Marcus and Minc 1988, p. 144).

The determination of the eigenvalues and eigenvectors of a system is extremely important in physics and engineering, where it is equivalent to matrix diagonalization and arises in such common applications as stability analysis, the physics of rotating bodies, and small oscillations of vibrating systems, to name only a few. Each eigenvalue is paired with a corresponding so-called eigenvector (or, in general, a corresponding right eigenvectorand a corresponding left eigenvector; there is no analogous distinction between left and right for eigenvalues).

The decomposition of a square matrix $\mathbf{A}$ into eigenvalues and eigenvectors is known in this work as eigen decomposition, and the fact that this decomposition is always possible as long as the matrix consisting of the eigenvectors of $\mathbf{A}$ is square is known as the eigen decomposition theorem.

The Lanczos algorithm is an algorithm for computing the eigenvalues and eigenvectors for large symmetric sparse matrices.

Let A be a linear transformation represented by a matrix $\mathbf{A}$. If there is a vector $\mathbf{X} \in \mathbb{R}^n \neq \mathbf{0}$ such that

$$\mathsf{A}\,\mathbf{X} = \lambda\,\mathbf{X} \tag{1}$$

for some scalar λ, then λ is called the eigenvalue of A with corresponding (right) eigenvector $\mathbf{X}$.

Letting A be a $k \times k$ square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \tag{2}$$

with eigenvalue λ, then the corresponding eigenvectors satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \tag{3}$$

which is equivalent to the homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4}$$

Equation (4) can be written compactly as

$$(\mathsf{A} - \lambda\,\mathsf{I})\,\mathbf{X} = \mathbf{0}, \tag{5}$$

where I is the identity matrix. As shown in Cramer's rule, a linear system of equations has nontrivial solutions iff thedeterminant vanishes, so the solutions of equation (5) are given by

$$\det(\mathsf{A} - \lambda\,\mathsf{I}) = 0. \tag{6}$$

This equation is known as the characteristic equation of A, and the left-hand side is known as the characteristic polynomial.

For example, for a 2×2 matrix, the eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11}+a_{22}) \pm \sqrt{4\, a_{12}\, a_{21} + (a_{11}-a_{22})^2}\,\right], \tag{7}$$

which arises as the solutions of the characteristic equation

$$x^2 - x\,(a_{11}+a_{22}) + (a_{11}\, a_{22} - a_{12}\, a_{21}) = 0. \tag{8}$$

If all k eigenvalues are different, then plugging these back in gives $k-1$ independent equations for the $k$components of each corresponding eigenvector, and the system is said to be nondegenerate. If the eigenvalues are n-fold degenerate, then the system is said to be degenerate and the eigenvectors are not linearly independent. In such cases, the additional constraint that the eigenvectors be orthogonal,

$$\mathbf{X}_i \cdot \mathbf{X}_j = |\mathbf{X}_i|\,\left|\mathbf{X}_j\right|\,\delta_{i\,j}, \tag{9}$$

where $\delta_{i\,j}$ is the Kronecker delta, can be applied to yield n additional constraints, thus allowing solution for theeigenvectors.

Eigenvalues may be computed in the Wolfram Language using Eigenvalues[*matrix*]. Eigenvectors and eigenvalues can be returned together using the command Eigensystem[*matrix*].

Assume we know the eigenvalue for

$$\mathsf{A}\,\mathbf{X} = \lambda\,\mathbf{X}. \tag{10}$$

Adding a constant times the identity matrix to A,

$$(\mathsf{A} + c\,\mathsf{I})\,\mathbf{X} = (\lambda + c)\,\mathbf{X} \equiv \lambda'\,\mathbf{X}, \tag{11}$$

so the new eigenvalues equal the old plus c. Multiplying A by a constant c

$$(c\,\mathsf{A})\,\mathbf{X} = c\,(\lambda\,\mathbf{X}) \equiv \lambda'\,\mathbf{X}, \tag{12}$$

so the new eigenvalues are the old multiplied by c.

Now consider a similarity transformation of A. Let $|A|$ be the determinant of A, then

$$\left|Z^{-1} A Z - \lambda I\right| = \left|Z^{-1} (A - \lambda I) Z\right| \tag{13}$$

$$= |Z|\, |A - \lambda I|\, \left|Z^{-1}\right| \tag{14}$$

$$= |A - \lambda I|, \tag{15}$$

so the eigenvalues are the same as for A.

POSSIBLE QUESTIONS

8 MARK

- Show that the matrix A = $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is orthogonal.
- Show that the matrix B = $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is unitary.
- Find the eigen values of matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$
- Show that the matrix A = $\begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix}$ is Skew – symmetric but not Skew Hermitian.
- Explain the different types of matrices.(any 5)
- Show that the matrix A = $\begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix}$ is Skew – symmetric but not Skew Hermitian.
- Find the Eigen values of the matrix A = $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
- Find the Rank of following matrices i) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
- Find the eigen values of matrix A = $\begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$
- Find the Rank of following matrices i) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
- If matrix A = $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} -5 & 1 & 3 \\ 4 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ Find A.B
- Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore - 641021.

(For the candidates admitted from 2018 onwards)

DEPARTMENT OF PHYSICS

UNIT I :(Objective Type/Multiple choice Questions each Question carries one Mark)

PART-A (Online Examination)

MATHEMATICAL PHYSICS-I

QUESTIONS	opt1	opt2	opt3	opt4	ANSWER
UNIT-IV					
A diagonal matrix in which all the diagonal elements are equal is called --------.	scalar matrix	diagonal matrix	unit matrix	null matrix	scalar matrix
[3 8 9 -2] is a row matrix of order--------------------	1x 4	4 x 1	1x 1	4 x 4	1x 4
in a square matrix A, $a_{ij} = 0$ for i <j, then it is called a --------------- matrix.	lower triangular	upper triangular	diagonal	triangular	lower triangular
The matrix multiplication of two matrices A and B is possible only if ----------------.	order of A is m x n and order of B is m x n	order of A is m x n and order of B is n x p	order of A is m xm and order of B is n x n	order of A is m x n and order of B is p x n	order of A is m x n and order of B is n x p
If the order of the matrix A is 4 x 5 and the order of the matrix B is 2 x 4 then	2 x 5	2 x 4	4 x 5	4 x 4	2 x 5

the resultant matrix BA has the order-----------.					
A Square matrix such that A' = – A is called --------------	symmetric	skew symmetric	hermit ion	scalar	skew symmetric
The sum of the diagonal elements in a matrix A is the -------------.	Trace of A	unit of A	Transpose of A	inverse of A	Trace of A
If a square matrix A of order n is of the form $A^n = 0$, then it is an ----------- matrix.	Identity	Idempotent	Nilpotent	Orthogonal	Nilpotent
If a square matrix A of order n is of the form $A^n = A$, then it is an ----------- matrix.	Identity	Idempotent	Nilpotent	Orthogonal	Idempotent
$(AB)^{-1}$ = ---------------.	A^{-1}	B^{-1}	$A^{-1}B^{-1}$	$B^{-1}A^{-1}$	$B^{-1}A^{-1}$
$(A^T)^{-1}$ = ---------------.	(A^T)	$(A^{-1})^T$	(A^{-1})	$(A^T)^T$	$(A^{-1})^T$
The ------------ of a square matrix A is the transpose of the matrix formed by replacing the elements of A by their corresponding cofactors.	Transpose	Inverse	Cofactor	Ad joint	Ad joint
The formula for solving the system of simultaneous linear equations by matrix inversion method is ---------------	$X = A B$	$X = A B^{-1}$	$X = A^{-1} B$	$X = B A^{-1}$	$X = A^{-1} B$
The transpose of a matrix A is getting by ----------------.	Interchanging rows into columns only.	Interchanging columns into rows only	Interchanging rows into columns and columns into rows	Taking the same matrix without interchanging the rows and columns.	Interchanging rows into columns only.
The subtraction of any two matrices A and B are possible only if ----.	A and B have same elements	A and B have same order	A and B have different order	A and B have different elements	A and B have same order
Two matrices A and B said to be equal if ---------------.	A and B have same elements	A and B have same order	A and B have different order	A and B have same elements and	A and B have same elements and

				same order	same order
Every matrix is a ----------------------- of it self.	sub matrix	unit matrix	equal matrix	none	sub matrix
The co – factor of an element A_{ij} is defined as ----------------.	$(-1)^{i+j}$ * Determinant obtained by deleting i th row and j th column of A	(-1) * Determinant obtained by deleting i th row and j th column of A	Determinant obtained by deleting i th row and j th column of A	$(-1)^{i+j}$ * Determinant obtained by deleting j th row and i th column of A	$(-1)^{i+j}$ * Determinant obtained by deleting i th row and j th column of A
$A^{-1}A$ = ---------------.	0	1	Identity matrix	Zero matrix	Identity matrix
I*A = ---------------.	A	0	Identity matrix	Zero matrix	A
An identity matrix is also found as a ------------- matrix	Scalar	Diagonal	triangular	scalar and diagonal	scalar and diagonal
In a 3x3 square matrix the minor and the cofactor of the element a_{23} have ---------.	Same sign and same value	same sign and different values	Opposite sign and same value	opposite sign and different values	Opposite sign and same value
If every element of a matrix is multiplied by a constant k ,then the determinant value of the matrix is multiplied by----------------	k	k-1	k^2	k+1	k
The determinant value of the unit matrix of order 2 is ------------------	1	0	-1	2	1
If any one of the row or column of a matrix is zero then the determinant value of the matrix is ------------------	0	positive	negative	one	0
If A is singular ,its inverse is ----------------	null matrix	does not exists	1/Adj A	1/|A|	does not exists
A rectangular matrix will not possesses--------------	inverse	cofactor	determinant	transpose	inverse
For any square matrix (adj A).A = A.	|A| .I	|A |	1 / |A|	A^T	|A| .I

(adj A) is --------------					
For any two square matrices A and B (adj AB) = ----------------	(adj A). (adj B)	(adj B). (adj A)	(adj BA)	(adj B +adj A)	(adj B). (adj A)

UNIT-V

Vector Calculus ∇ Operator – Divergence – Second derivative of Vector functions or fields – The Laplacian Operator – Curl of a Vector – Line Integral – Line Integral of a Vector field around an infinitesimal rectangle – Curl of Conservative field – Surface Integral – Volume Integral (without problem) – Gauss's Divergence theorem and it's proof in the simple problems – Stoke's and its proof with simple problems.

Vector Differential Operator Del i.e.(∇)

the vector differential operation Del is denoted by∇. It is defined as

$$\nabla = \widehat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient Of a Scalar Function

If $\phi(x, y, z)$ be a scalar function then $\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is called the gradient of the scalar function ϕ.

and is denoted by grad ϕ.

$$\text{thus, } grad\ \phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$grad\ \phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$grad\ \phi = \nabla\ \phi$$

Divergence of a vector function

The Divergence of a vector point function $\vec{F}$ is denoted by div$\vec{F}$ and is defined as below.

$$\text{let } \vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

$$div\ F = \vec{\nabla}.\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{\imath}F_1 + \hat{\jmath}F_2 + \hat{k}F_3) = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

it is evident that div F is scalar function.

Physical Interpretation of Divergence

Let us consider the case of a fluid flow. consider a small rectangular parallopiped of dimensions dx, dy, dz parallel to x ,y and z axes respectively.

Let $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$ be the velocity of the fluid at P(x ,y ,z)

∴Mass of fluid flowing in through the face ABCD in unit time

=velocity × Area of the face = V(dy dz)

mass of fluid flowing out across the face PQRS per unit time

$$= \left(V_x + \frac{\partial V}{\partial x}dx\right)(dy\, dz)$$

Net decrease in mass of fluid in the parallelopiped corresponding to the flow along x-axis per unit time

$$V_x\, dy\, dz - \left(V_x + \frac{\partial V}{\partial x}dx\right)dy\, dz$$

$$= -\frac{\partial V_x}{\partial x}dx\, dy\, dz \qquad \text{(minus sign decrease)}$$

similarly, the decrease in mass of fluid to the flow along y-axis $=\frac{\partial V_y}{\partial y}dx\, dy\, dz$

and the decrease in mass of fluid to the flow along z-axis$=\frac{\partial V_z}{\partial z}dx\, dy\, dz$

total decrease of the amount of fluid per unit time $=\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx\, dy\, dz$

Thus the rate of loss of fluid per unit volume $=\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{\imath}V_x + \hat{\jmath}V_y + \hat{k}V_z\right) = \vec{\nabla}.\vec{V} = div\,\vec{V}$$

If the fluid is compressible, there can be gain or less in the volume element. hence $div\,\vec{V} = 0$ and V is called a solenoidal vector function.

Curl

the curl of a vector point function F is defined as below

$$curl\,\vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right)$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$curl\,\vec{F}\ is\ a\ vector\ quantity$$

<u>Physical Meaning Of Curl</u>

We know that $\vec{V} = \vec{\omega} \times \vec{r}$, where ω is the angular velocity, $\vec{V}$ is the linear velocity and $\vec{r}$ is the position vector of a point on the rotating body.

$curl\,\vec{V} = \vec{\nabla} \times \vec{V}$ $\qquad [\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}]$

$[\vec{r} = r_1\hat{\imath} + r_2\hat{\jmath} + r_3\hat{k}]$

$$= \vec{\nabla} \times (\vec{\omega} \times \vec{r}) = \vec{\nabla} \times \left[\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right) \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\right]$$

$$\vec{\nabla} \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \vec{\nabla} \times \left[(\omega_2 z - \omega_3 y)\hat{\imath} - (\omega_1 z - \omega_3 x)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}\right]$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[\left(\omega_2 z - \omega_{3y}\right)\hat{\imath} - (\omega_1 z - \omega_3 x)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{\imath} - (-\omega_2 - \omega_2)j + (\omega_3 + \omega_3)\hat{k} = 2\left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right) = 2\omega$$

curl $\vec{V} = 2\omega$ which shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

If Curl$\vec{F} = 0$, the field F is termed as irrational.

EXAMPLES:

1.Find the divergence and curl of a vector $\vec{V} = (x\,y\,z)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$ at (2,-1,1).

solution,

here, we have

$$\vec{V} = (x\,y\,z)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2)\hat{k}$$

$$Div\,\vec{V} = \nabla.\vec{V}$$

$$Div\,\vec{V} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2)$$

$$= yz - 3x^2 + 2xz - y^2$$

$$= -1 + 12 + 4 - 1 = 14\ at\ (2,-1,1)$$

$$curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= 2yz\hat{\imath} - (z^2 - xy)\hat{\jmath} + (6xy - xz)\hat{k}$$

$$= -2yz\hat{\imath} + (xy - z^2)\hat{\jmath} + (6xy - xz)\hat{k}$$

$$at\ (2,-1,1) = -2(-1)(1)\hat{\imath} + \{(2)(-1) - 1\}\hat{\jmath} + \{6(2)(-1 - 2(1))\}\hat{k}$$

$$= 2\hat{\imath} - 3\hat{\jmath} - 14\hat{k}$$

2.Prove that $(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidial and irrational.

solution,

here $F = (y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$

for solenoidial. we have to prove $\vec{\nabla}.\vec{F} = 0$

now, $\vec{\nabla}.\vec{F} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].\left[(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}\right]$

$$= -2 + 2x - 2x + 2 = 0$$

Thus, $\vec{F}$ is solenoidal. for irrational, we have to prove curl$\vec{F} = 0$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= (3x - 3x)\hat{\imath} - (-2x + 3y - 3y + 2z)\hat{\jmath} + (3z + 2y - 2y - 3z)$$

$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0$$

thus $\vec{F}$ is irrational. hence $\vec{F}$ is solenoidal and irrational.

3.Determine the constants a and b such that curl of vector

$\vec{A} = (2xy + 3yz)\hat{\imath} + (x^2 + axz - 4z^2)\hat{\jmath} - (3xy - byz)\hat{k}$ is zero.

solution,

$$curl\ \vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times ((2xy + 3yz)\hat{\imath} + (x^2 + axz - 4z^2)\hat{\jmath} - (3xy - byz)\hat{k})$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & 3xy - byz \end{vmatrix}$$

$$= [-3x - bz - ax]\hat{\imath} - [-3y - 3y]\hat{\jmath} + [2x + az - 2x - 3z]\hat{k}$$

$$= \big(-x(3 + a) + z(8 - b)\big)\hat{\imath} + 6y\hat{\jmath} + z(-3 + a)\hat{k} = 0$$

3+a=0 and 8-b=0

and a=-3,b=8.

4.Find the constants a, b, c so that

$$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$$

is irrational and hence find function ϕ such that $\vec{F} = \nabla\phi$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= (c + 1)\hat{\imath} - (4 - a)\hat{\jmath} + (b - 2)\hat{k}$$

As $\vec{F}$ is irrational, $\vec{\nabla} \times \vec{F} = 0$

$$= (c + 1)\hat{\imath} - (4 - a)\hat{\jmath} + (b - 2)\hat{k} = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$

$$c + 1 = 0 \qquad 4 - a = 0 \quad b - 2 = 0$$

$$a = 4, b = 2, c = -1$$

now, we have to find ϕ such that $\vec{F} = \nabla\phi$

we, know that

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= \vec{\nabla}\phi.\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$\vec{F}.\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= \left[(x + 2y + 4z)\hat{\imath} + (2x - 3y - z)\hat{\jmath} + (4x - y - 2z)\hat{k}\right].\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= (x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y - 2z)dz$$

$$= x\,dx - 3y\,dy + 2z\,dz + (2y\,dy + 2x\,dx) + (4\text{z dx} + 4\text{x dz}) + (-\text{z d} - \text{y dz})$$

$$\phi = \int x\,dx - 3\int y\,dy + 2\int z\,dz + \int (2y\,dx + 2xdy) + \int (4z\ \ dx + 4x\,dz) - \int (z\,dy + y\,dz)$$

$$\left(\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx + c\right)$$

Line Integral

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the components of $\vec{F}$ along the curve AB.

component of $\vec{F}$ along a tangent PT at P

=Dot product of $\vec{F}$ and unit vector along PT

=$\vec{F}.\frac{\overrightarrow{dr}}{ds}\left(\frac{\overrightarrow{dr}}{ds}\ is\ aunit\ vector\ along\ tangent\ PT\right)$

Line Integral = $\sum \vec{F}.\frac{\overrightarrow{dr}}{ds}\ from\ A\ and\ B\ along\ the\ curve$

$$\therefore Line\ Integral = \int_c \left(\vec{F}.\frac{\overrightarrow{dr}}{ds}\right) ds = \int_c \vec{F}.\overrightarrow{dr}$$

Note (1) work. If $\vec{F}$ represent the variable force acting on a particle along are AB, then the total work done= $\int_A^B \vec{F}.\overrightarrow{dr}$

(2)Circulation. If $\vec{V}$ represent the velocity of a liquid then $\oint_c \vec{v}.\overrightarrow{dr}$ is called the circulation of V round the closed curve c.

If the circulation of V round every closed curve is zero then V is said to be irrational there.

(3)when the path of integration is a closed then notation of integration is $\oint in$ place of $\int$.

If a force $\vec{F} = 2x^2y\hat{\imath} + 3xy\hat{\jmath}$ displace a particle in the x y-plane from (0,0) to (1,4) along a curve $y = 4x^2$. find the work done.

solution,

$$\text{Work done} = \int_c \vec{F}\,.\,\overrightarrow{dr}$$

$$\int_c (2x^2y\hat{\imath} + 3xy\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath})$$

$$= \int_c (2x^2ydx + 3xydy)$$

putting the values of y and dy, we get

$$= \int_0^1 [2x^2(4x^2)dx + 3x(4x^2)8xdx]$$

$$= 104\int_0^1 x^4\,dx = 104\left(\frac{x^5}{5}\right)$$

$$\frac{104}{5}$$

EXAMPLES:

1.Evaluate $\int_c \vec{F}.\overrightarrow{dr}$ when $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$ and C is the boundary of the square in the plane z=0 and bounded by the lines x=0, y=0, x=a, y=a.

solution, $\int_c \vec{F}.\overrightarrow{dr} = \int_{OA} \vec{F}.\overrightarrow{dr} + \int_{AB} \vec{F}.\overrightarrow{dr} + \int_{BC} \vec{F}.\overrightarrow{dr} + \int_{CO} \vec{F}.\overrightarrow{dr}$

here $\vec{r} = x\hat{\imath} + y\hat{\jmath}, \overrightarrow{dr} = dx\hat{\imath} + dy\hat{\jmath}\,, \vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$

$$\vec{F}.\overrightarrow{dr} = x^2\,dx$$

on OA, y=0,

$$\int_{OA} \vec{F}\,.\,\overrightarrow{dr} = \int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\langle 2 \rangle$$

On AB, x=a,

$$dx = 0$$

(1) becomes

$$\therefore \vec{F}.\overrightarrow{dr} = ay\, dy$$

$$\int_{AB} \vec{F}.\overrightarrow{dr} = \int_0^a ay\, dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

⟨3⟩

On BC, y=a,

becomes,

$$dy = 0$$

$$\vec{F}.\overrightarrow{dr} = x^2 dx$$

$$\int_{BC} \vec{F}.\overrightarrow{dr} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{-a^3}{3}$$

⟨4⟩

On, CO, x=0,

(1) becomes,

$$\vec{F}.\overrightarrow{dr} = 0$$

$$\int_{CO} \vec{F}.\overrightarrow{dr} = 0$$

⟨5⟩

adding ⟨2⟩⟨3⟩⟨4⟩⟨4⟩ we get

$$\int_c \vec{F}.\overrightarrow{dr} = \frac{a^3}{2}$$

<u>Surface integral</u>

surface r=f (u, v) is called smooth if f (u, v) posses continuous first order partial derivative.

Let $\vec{F}$ be a vector function and S is the given surface.

surface integral of a vector function$\vec{F}$ over the surface S is defined as the integral of the components $\vec{F}$ of along the normal to the surface.

components of $\vec{F}$ along the normal $= \vec{F}.\hat{n}$, where n is the unit vector to an element ds and

$$\hat{n} = \frac{grad\ f}{|grad\ f|}$$

$$ds = \frac{dx\ dy}{(\hat{n}.\hat{k})}$$

surface integral of F over S is ,

$$= \sum \vec{F}.\hat{n} = \iint_s (\vec{F}.\hat{n}).ds$$

note(1)

flux=$\iint_s (\vec{F}.\hat{n}).ds$ where $\vec{F}$ represents the velocity of the liquid.

If then is said to be a solenoid vector point function.

EXAMPLES:

1.Evaluate $\iint_s (\vec{A}.\hat{n}).ds$ where $\vec{A} = (x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$

and S is the surface of the plane 2x+y+2z=6 in the first octant .

solution,

A vector normal to the surface "S" is given by,

$$\nabla(2x+y+2z) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z) = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

and $\hat{n}$ is the unit vector to surface "s" is given by

$$= \frac{2\hat{\imath} + \hat{\jmath} + 2\hat{k}}{\sqrt{4+1+4}} = \left(\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{2}{3}\hat{k}\right)$$

$$\hat{k}.\hat{n} = \hat{k}\left(\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{2}{3}\hat{k}\right)$$

$$\iint \hat{A}\hat{n}.\overrightarrow{ds} = \iint \vec{A}\hat{n}.\frac{dx\,.\,dy}{\hat{n}.\hat{k}}$$

where R is the projection of S.

$$\text{now, } \vec{A}.\hat{n} = \left[(x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}\right].\left(\tfrac{2}{3}\hat{\imath} + \tfrac{1}{3}\hat{\jmath} + \tfrac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3}(x^2+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$

putting the value of z in (1), we get

$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{2}{3}y(y+6-2x-y) = \frac{4}{3}y(3-x)$$

Hence,

$$\iint_S \vec{A}.\hat{n}.\overrightarrow{ds} = \iint_R \vec{A}.\hat{n}\frac{dx\,dy}{|\hat{k}.\vec{n}|}$$

putting the value of $\vec{A}.\hat{n}.$ from (2) and (3) , we get

$$\iint_S \vec{A}.\hat{n}.\overrightarrow{ds} = \iint_R \frac{4}{3}y(3-x).\frac{3}{2}dx\,dy = \int_0^3\int_0^{6-2x} 2y(3-x)\,dydx$$

$$= \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} = \int_0^3 (3-x)(6-2x)^2\,dx = 4\int_0^3 (3-x)^3\,dx = -(0-81) = 81$$

Volume Integral

let be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_v \vec{F}.dv$

EXAMPLES:

1.If $\vec{F} = 2z\hat{\imath} - x\hat{\jmath} + y\hat{k}$, evaluate $\iiint_v \vec{F}.dv$ where, v is the region bounded by the surfaces. x=0, y=0, x=2, y=4, z=x^2, z=2.

solution,

$$\iiint_v \vec{F}.dv = \iiint_v \left(2z\hat{\imath} - x\hat{\jmath} + y\hat{k}\right)dx\,dy\,dz$$

$$\int_0^2 dx \int_0^4 dy \int_{x^2}^2 \left(2z\hat{\imath} - x\hat{\jmath} + y\hat{k}\right)dz =$$

$$\int_0^2 dx \int_0^4 dy \left[z^2\hat{\imath} - xz\hat{\jmath} + yz\hat{k}\right]_{x^2}^2 =$$

$$\int_0^2 dx \int_0^4 dy \left[4\hat{\imath} - 2x\hat{\jmath} + 2y\hat{k} - x^4\hat{\imath} + x^3\hat{\jmath} - x^2y\hat{k}\right] =$$

$$\int_0^2 dx \left[4\hat{y\imath} - 2xy\hat{\jmath} + y^2\hat{k} - x^4y\hat{\imath} + x^3y\hat{\jmath} - \frac{x^2\,y^2}{2}\hat{k}\right]_0^4$$

$$\int_0^2 \left(16x\hat{\imath} - 8x\hat{\jmath} + 16\hat{k} - 4x^4\hat{\imath} + 4x^3\hat{\jmath} - \frac{8x^3}{3}\hat{k}\right)dx =$$

$$\left[16x\hat{\imath} - 4x^2\hat{\jmath} + 16x\hat{k} - \frac{4x^5}{5}\hat{\imath} + x^4\hat{\jmath} - \frac{8x^3}{3}\hat{k}\right]_0^4 =$$

$$= \frac{32}{5}\hat{\imath} + \frac{32}{3}\hat{k} = \frac{32}{15}\left(3\hat{\imath} + 5\hat{k}\right)$$

<u>Stoke's theorem</u>

statement: surface integral of the component of curl $\vec{F}$ along the normal to the surface S , taken over the surface bounded by curve C is equal to the line integral of the vector point function.

$\vec{F}$ taken along the closed curve C mathematically

$$\oint \overrightarrow{F}.\overrightarrow{dr} = \iint_s curl\,\vec{F}.\hat{n}.ds$$

where $\hat{n} = cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\hat{k}$ is a unit external normal to any surface ds,

proof,

$$\text{let } \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$F = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

on putting the values of $\vec{F}, \overrightarrow{dr}$ in the statement of theorem

$$\oint_c (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$= \iint_s \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).(cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\hat{k})ds$$

$$\oint (F_1\,dx + F_2\,dy + F_3 dz) = \iint_s \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right](cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\hat{k})ds$$

$$\oint (F_1\,dx + F_2\,dy + F_3 dz) = \iint_s \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] ds$$

let us prove,

$$\oint F_1\,dx = \iint_s \left[\left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\right] ds$$

let the equation of the surface S be z=g(x, y). the projection of the surface on x-y plane is region R.

$$\oint F_1(x, y, z) = \oint_c F_1[x, y, g(x, y)]dx$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, g)\,dx\,dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} + \frac{\partial g}{\partial y}\right)$$

the direction cosines of the normal to the surface S be z=g(x, y) are given by $\frac{\cos\alpha}{\frac{-\partial}{\partial x}} = \frac{\cos\beta}{\frac{-\partial}{\partial y}} = \frac{\cos\gamma}{1}$

and dx dy = projection of ds on the x-y plane =ds cos γ

putting the value of ds in R.H.S

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\, ds = \iint_R \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\frac{dx\, dy}{\cos\gamma}$$

$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \iint_R \left(\frac{\partial F_1}{\partial z}\left(\frac{-\partial g}{\partial y}\right) - \frac{\partial F_1}{\partial y}\right)\, dx\, dy$$

$$-\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right)\, dx\, dy$$

from (3) and (4), we get,

$$\oint_c F_1\, dx = \iint_s \left(\frac{\partial F_1}{\partial z}\cos\beta + \frac{\partial F_1}{\partial y}\cos\gamma\right) ds$$

$$\oint_c F_2\, dy = \iint_s \left(\frac{\partial F_2}{\partial}\cos\gamma + \frac{\partial F_2}{\partial}\cos\alpha\right) ds$$

$$\oint_c F_3\, dz = \iint_s \left(\frac{\partial F_3}{\partial}\cos\alpha + \frac{\partial F_3}{\partial}cos\ \beta\beta\right) ds$$

on adding (5), (6), (7)

$\oint_c (F_1\, dx + F_2 dy + F_3 dz) = \iint_S (\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma + \frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha + \frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta)$ ds **proved**

EXAMPLES:

1.Evaluate by strokes theorem $\oint_c (yz\,dx + zx\,dy + xy\,dz)$ where C is the curve $x^2 + y^2 = 1\,, z = y^2$.

solution,

here we have $\oint_c (yz\,dx + zx\,dy + xy\,dz)$

$$= \int (yz\,dx + zx\,dy + xy\,dz).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= curl\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} =$$

$$(x - x)\hat{\imath} + (y - y)\hat{\jmath} + (z - z)\hat{k} = 0$$

$$\oint \vec{F}.\overrightarrow{dr}$$

$$\int curl\,F.\eta\,ds = 0$$

Using strokes theorem or otherwise evaluate ,

$$\int_c (2x - y)dx - (yz^2dy) - (y^2dz)$$

solution,

$$\int_c (2x - y)dx - (yz^2dy) - (y^2dz) = \int_c (2x - y)\hat{\imath} - (yz^2)\hat{\jmath} - (y^2)\hat{k}.\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

by strokes theorem , $\oint \vec{F}.\overrightarrow{dr} = \iint Curl\,\overrightarrow{F}.\hat{n}.\overrightarrow{ds}$

$$Curl\,\overrightarrow{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} =$$

$$(2zy - 2yz)\hat{\imath} + (0 + 1)\hat{k} = \hat{k}$$

$$\oint \vec{F}.\overrightarrow{dr} = \iint \hat{k}\hat{n}.\overrightarrow{ds} = \iint \hat{k}\hat{n}.\frac{dx\,.dy}{\hat{n}.\hat{k}} = \iint dx\,.dy = Area\ of\ the\ circle = \pi$$

$$i.e\ ,\overrightarrow{ds} = \frac{dx\,.dy}{\hat{n}.\hat{k}}$$

2.Find $\iint \vec{F}\hat{n}.\overrightarrow{ds}$ where $\vec{F} = (2x+3y)\hat{\imath} - (xz+y)\hat{\jmath} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre (3, -1, 2) and radius 2.

solution, Let V be the volume enclosed by the surface S.

$$\iint_s \vec{F}\hat{n}.\overrightarrow{ds} = \iiint_v div\,\vec{F}\ dv$$

now, $div\,\vec{F} = \frac{\partial}{\partial x}(2x+3y) + \frac{\partial}{\partial y}[-(xz+y)] + \frac{\partial}{\partial z}(y^2+2z) = 2-1+2 = 3$

$$\iint_s \vec{F}\hat{n}.\overrightarrow{ds} = \iiint_v div\,\vec{F}\ dv = 3\iiint_v dv = 3V$$

again, V is the volume of a sphere of radius 3 .therefore,

$V = \frac{4}{3}\pi r^3 = 36$

$$\iint_s \vec{F}\hat{n}.\overrightarrow{ds} = 3V = 3*36\pi = 108\pi$$

Use Divergence theorem to evaluate $\iint_s \vec{A}.\overrightarrow{ds}$ where $\vec{F} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$

solution,

$$\iint_s \vec{A}.\overrightarrow{ds} = \iiint_v div\,\vec{A}.dV$$

$$= \iiint_v \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}\right)dV$$

$$= \iiint_v (3x^2+3y^2+3z^2)dV = 3\iiint_v (x^2+y^2+z^2)dV$$

on putting $x = r\ \sin\theta\ \cos\phi, y = r\ \sin\theta\ \sin\phi, z = r\cos\theta$, we get

$$= 3\iiint_v r^2(r^2\sin\theta\,dr\,d\theta\,d\phi) = 3\times 8\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta\ \int_0^a r^4\,dr$$

$$= 24(\phi)_0^{\frac{\pi}{2}} - (\cos\theta)_0^{\frac{\pi}{2}} \left(\frac{r^5}{5}\right)_0^a = 24 \times \frac{\pi}{2}(-0+1)\left(\frac{a^5}{5}\right) = \frac{12\,\pi a^5}{5}$$

GAUSS'S DIVERGENCE THEOREM:

Let **B** be a solid region in $\boldsymbol{R^3}$ and let **S** be the surface of **B**, oriented with outwards pointing normal vector. Gauss Divergence theorem states that for a $\boldsymbol{C^1}$ vector field **F,** the following equation holds:

$$\iint_S F \cdot ds = \iiint_B (\nabla \cdot F)dV$$

Note that for the theorem to hold, the orientation of the surface must be pointing outwards from the region **B**, otherwise we'll get the minus sign in the above equation. Note that since **S** is the boundary of**B**, then it is always a closed surface ie: it has no boundary. In other words, the integral of a continuously differentiable vector field across a boundary (flux) is equal to the integral of the divergence of that vector field within the region enclosed by the boundary.

Applications of Gauss Theorem:

- The Aerodynamic Continuity Equation

1. The surface integral of mass flux around a control volume without sources or sinks is equal to the rate of mass storage.
2. If the flow at a particular point is incompressible, then the net velocity flux around the control volume must be zero.
3. As net velocity flux at a point requires taking the limit of an integral, one instead merely calculates the divergence.
4. If the divergence at that point is zero, then it is incompressible. If it is positive, the fluid is expanding, and vice versa.

- Gauss's Theorem can be applied to any vector field which obeys an inverse-square law (except at the origin) such as gravity, electrostatic attraction, and even examples in quantum physics such as probability density.

EXAMPLES:

1. Use the divergence theorem to calculate $\iint_S F \cdot ds$, where **S** is the surface of the box **B**with vertices (±1, ±2, ±3) with outwards pointing normal vector and $F(x, y, z) = (x^2z^3, 2xyz^3, xz^4)$.

Solution: Note that the surface integral will be difficult to compute, since there are six different components to parameterize (corresponding to the six sides of the box) and so one would have to compute six different integrals. Instead, using Gauss Theorem, it is easier to compute the integral (∇·F)of **B.**

First, we compute $(\nabla F) = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$. Now we integrate this function over the region B bounded by S:

$$\iint_S F \cdot ds = \iiint_B (\nabla \cdot F) dV$$
$$= \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} 8xz^3\, dx dy dz$$
$$= 0$$

which is easy to verify.

2. Evaluate $\iint_S (3x\,\mathbf{i} + 2y\,\mathbf{j}) \cdot dA$, where **S** is the sphere given by $x^2 + y^2 + z^2 = 9$.

Solution: We could parametrize the surface and evaluate the surface integral, but it is much faster to use the divergence theorem. Since

$$\mathrm{div}(3x\,\mathbf{i} + 2y\,\mathbf{j}) = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(0) = 3 + 2 = 5$$

The divergence theorem gives:

$$\iint_S (3x\,\mathbf{i} + 2y\,\mathbf{j}) \cdot dA = \iiint_R 5dV = 5 \times (\text{Volume of the sphere}) = 180\pi$$

3. Let **R** be the region in $\boldsymbol{R^3}$ by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$and let **S** be the boundary of the region **R.** Evaluate $\iint_S (y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}) \cdot dA$

Solution:

Since $\operatorname{div}(y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}) = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 2z$

The divergence theorem gives:

$$\iint_S z^2\,\mathbf{k} \cdot dA = \iiint_R 2z dV$$

It is easiest to set up the triple integral in cylindrical coordinates:

$$\iiint_R 2z dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 2z \cdot r \cdot dzdrd\theta$$

$$= 2\pi \int_0^1 \left[z^2 r\right]_{z=r^2}^1 \quad dr$$

$$= 2\pi \int_0^1 \left[r - r^5\right] dr$$

$$= 2\pi \int_0^1 \left(\frac{1}{2} - \frac{1}{6}\right) dr = \frac{2\pi}{3}$$

POSSIBLE QUESTIONS

8 MARK

1. Find $\iint \vec{F}\hat{n}.\overrightarrow{ds}$ where $\vec{F} = (2x+3y)\hat{\imath} - (xz+y)\hat{\jmath} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre (3, -1, 2) and radius 2.
2. Evaluate by strokes theorem $\oint_c (yz\,dx + zx\,dy + xy\,dz)$ where C is the curve $x^2 + y^2 = 1\,, z = y^2$.
3. 1.If $\vec{F} = 2z\hat{\imath} - x\hat{\jmath} + y\hat{k}$, evaluate $\iiint_v \vec{F}.dv$ where, v is the region bounded by the surfaces. x=0, y=0, x=2, y=4, z=x^2, z=2.
4. Evaluate $\iint_s (\vec{A}.\hat{n}).ds$ where $\vec{A} = (x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$and S is the surface of the plane 2x+y+2z=6 in the first octant .
5. Evaluate $\int_c \vec{F}.\overrightarrow{dr}$ when $\vec{F} = x^2\hat{\imath} + xy\hat{\jmath}$ and C is the boundary of the square in the plane z=0 and bounded by the lines x=0, y=0, x=a, y=a.
6. Find the divergence and curl of a vector $\vec{V} = (x\,y\,z)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$ at (2,-1,1).
7. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrational
8. Determine the constants a and b such that curl of vector $\vec{A} = (2xy + 3yz)\hat{\imath} + (x^2 + axz - 4z^2)\hat{\jmath} - (3xy - byz)\hat{k}$ is zero.
9. Find the constants a, b, c so that $\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrational and hence find function ϕ such that $\vec{F} = \nabla\phi$.
10. Verify Stoke's theorem with its proof.
11. State and verify the Gauss's Divergence Theorem.

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore - 641021.

(For the candidates admitted from 2018 onwards)

DEPARTMENT OF PHYSICS

UNIT I :(Objective Type/Multiple choice Questions each Question carries one Mark)

PART-A (Online Examination)

MATHEMATICAL PHYSICS-I

QUESTIONS	opt1	opt2	opt3	opt4	ANSWER
UNIT-V					
Ñ x (Ñ f) =	1	0	2	-1	0
Ñ^2 (r^m) is equal to	mr^{m-1}	$m^2\ r^{m-2}$	$m(m+1)\ r^{m-2}$	$(m+1)\ m\ r^{m-1}$	$m(m+1)\ r^{m-2}$
The divergence theorem enables to convert a surface integral on a closed surface into a ------	line integral	volume integral	surface integral	None	volume integral
If A is solenoidal , then	div A=0	curl A =0	|A| = 0	div (curlA) =0	div A=0
If r is position vector , then Ñx r =	3	2	1	0	0
If f=4xi+yj-2k then Ñ.f=?	1	0	3	2	3
The function f is said to satisfify the laplace equation if	Ñ^2 f	Ñf	Ñ^{4}f	Ñ^{3}f	Ñ^2 f
If r is position vector , then Ñ.r =	0	1	2	3	3
If A is irrotational , then	| A | = 1	ÑxA =0	|A| =0	Ñ.A=0	ÑxA =0

The divergence of the position vector r is	1	2	r	3	3
If r = xi +yj +zk , then Ñ.(ar) is equal to	a	r	0	3a	0
Which of the following is a scalar function ?	Ñ.A	Ñf	Ñ(Ñ.A)	ÑxA	Ñ.A
Given that f = x^2 +y^2 + z^2 , then Ñ^2 f is	1	3	6	0	6
If i, j and k are the unit vectors along the coordinate axes , then (i .i) is	0	1	p	j	1
If i , j and k are the unit vectors along the coordinate axes , then (jxj) is	1	k	0	p	0
If i and j are the unit vectors along x and y projections, then (i.j) is	0	1	k	3	0
If r = xi +yj +zk, then Ñ.r =?	1	2	0	3	3
If i , j and k are the unit vectors along the x, y, z axes , then jxk is equal to	0	1	I	3	I
If r = 2xi -yj +2zk, then Ñ.r =?	0	4	2	3	3
If A = 3i-5j+2k and B = 4i+3j then A.B is equal to	-3	19	-14	11	-3
If A is irrotational , then	Ñ.A =0	ÑxA=0	Ñ.A¹0	ÑxA ¹0	ÑxA=0
A vector A is said to be solenoidal if	ÑxA ¹0	ÑxA=0	Ñ.A¹0	Ñ.A =0	Ñ.A =0
If F is solenoidal ,then	Ñ.F =0	ÑxF=0	Ñ^{2}F=0	Ñ . ÑXF =0	Ñ.F =0
In Stoke's theorem, ò$_c$ A.dr =	òò (Ñ.A) ds	òò (ÑxA) ds	0	òò (ÑA) ds	òò (ÑxA) ds
Ñ (kf) =------	Ñ x k(f)	k (Ñ x f)	k (Ñ f)	Ñ . k(f)	k (Ñ f)
Ñ (f + Y) = -----	Ñ f +Ñ Y	Ñ f -Ñ Y	none	Ñ f * Ñ Y	Ñ f +Ñ Y
Ñ (fY) =	(Ñf)Y -	Ñ(fY) +	Ñ(fY) -	(Ñf)Y + f(ÑY)	(Ñf)Y +

	f(ÑY)	f(ÑY)	f(ÑY)		f(ÑY)
Ñ (f/Y) =	[(Ñf)Y - f(ÑY)]/ Y^2	[(Ñf)Y + f(ÑY)]/ Y^2	[(Ñf)Y + f(ÑY)]/ Y^3	[(Ñf)Y * f(ÑY)]/ Y^2	[(Ñf)Y - f(ÑY)]/ Y^2
Ñ(A + B) = -------	Ñ . A - Ñ . B	Ñ . A + Ñ . B	Ñ . A x Ñ . B	Ñ . B / Ñ . A	Ñ . A + Ñ . B
Ñ. (kA) =------	Ñ x k(A)	k (Ñ x A)	k (Ñ . A)	Ñ . k(A)	k (Ñ . A)
Ñ . (fA) =------	(Ñf) .A + f ((Ñ . A) .A	k (Ñ x f)	k (Ñ f)	(Ñf) .A - f ((Ñ . A) .A	(Ñf) .A + f ((Ñ . A) .A
Ñ x (A + B) = ------	Ñ x A - Ñ x B	Ñ x A + Ñ x B	Ñ x A . Ñ x B	Ñ x A / Ñ x B	Ñ x A + Ñ x B
Ñx(kA) = ------	Ñ . k(A)	k x(Ñ x A)	k x(Ñ . A)	k .(Ñ x A)	k .(Ñ x A)
Ñ x (fA) =------	(Ñf) x A + f ((Ñ x A)	(Ñf) . A + f ((Ñ . A)	(Ñxf) x A + f x(Ñ x A)	(Ñf) x A - f ((Ñ x A)	(Ñf) x A + f ((Ñ x A)
The operator Ñ defined by	¶/¶x -¶/¶y -¶/¶z	¶/¶x +¶/¶y -¶/¶z	¶/¶x -¶/¶y +¶/¶z	¶/¶x +¶/¶y +¶/¶z	¶/¶x +¶/¶y +¶/¶z
The operator Ñ^2 defined by ------	¶^{2}/¶x^2 +¶^{2}/¶y^2 +¶^{2}/¶z^2	¶^{2}/¶x^2 +¶^{2}/¶y^2 -¶^{2}/¶z^2	¶^{2}/¶x^2 -¶^{2}/¶y^2 +¶^{2}/¶z^2	¶^{2}/¶x^2 -¶^{2}/¶y^2 -¶^{2}/¶z^2	¶^{2}/¶x^2 +¶^{2}/¶y^2 +¶^{2}/¶z^2
A single valued function f(x,y,z) is said to be a hormonic function if its second partial derivatives exist and are continuous and if the function satisfies the ------ equation	Integral	Laplace	continuous	Differential	Laplace
If r = xi + yj + zk then, Ñ (1/ r) = -----	–r/r^2	r/r^3	1/r^3	–r/r^3	–r/r^3
The divergence of a curl of a vector is ---	one	three	zero	two	zero
If A =A_1i +A_2j + A_3k, where A1, A2, A3 have continuous second partials, then Ñ . (Ñ x A) =---	2	1	-1	-2	2
If Ñ . V = 0, then the vector V is said to be	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Solenoidal vector

If Ñ x V = 0, then the vector V is said to be	Irrotational vector	Position vector	Zero vector	Solenoidal vector	Irrotational vector
The vector A = x^2z^2i + xyz^2j – xz^3k is	Irrotational vector	Solenoidal vector	Zero vector	Position vector	Solenoidal vector
If f is a hormonic function, then Ñf is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A and B are irrotational, then AxB is	Irrotational vector	Position vector	Solenoidal vector	Zero vector	Zero vector
If A is irrotational, then ÑxA is	1	-1	2	0	0
If A is solenoidal, then Ñ.A is	1	-1	0	-2	0
div (curl A) = ----	0	1	-1	2	0
Curl (grad f) = ---	1	non zero	2	0	1
Curl (A+B) = -----	Curl A – Curl B	curl A + Curl B	curl A * Curl B	curl A / Curl B	curl A + Curl B
d/dt(A.B) = -----	A.dA/dt +dB/dt. B	A.dB/dt +dA/dt .B	A.dA/dt - dB/dt	A.dA/dt + dB/dt	A.dA/dt - dB/dt
If F is constant vector, then curl F = ----	1	2	non zero	0	0
Ñ . (Ñf) =------	Ñ^{2}f	0	Ñf	f	Ñ^{2}f
Grad r^n = -----	nr^{n-1} r	nr^{n-2} r	$(n-1)r^{n-2}$ r	r^{n-1} r	nr^{n-2} r
.------------ is a vector quantity.	Mass	.pressure	volume	force	force
If f is a scalar function and f(t) is a vector function then (ff)	ff ’ - f’f.	.ff ’ + f’f.	ff ’ x f’f	ff ’ / f’f.	ff ’ - f’f.
Gradient of a constant is -----------.	Constant	1	0	gradiant	0
The derivative of the sum of two derivable vector functions f(t) and g(t) of the scalar variable t, is equal to the---------- of their derivatives.	sum	difference	multiple	division	sum
Any scalar function f which satisfies the partial differential equation	harmonic function	Homogeneous function	Nonharmonic function	Nonhomogeneous function	harmonic function
.If f=(x+y)i+xj+zk and s is the	1	2	4	6	6

surface of the cube bounded by the planes x=0,x=1,y=0,y=1,z=0 and z=1 then the surface integral is ----------					
.∭ Δf dv=--------	. ∬ f.n ds	∭ f.n ds	∬ fxn ds	∭ fxn ds	. ∬ f.n ds
.∭ ΔxB dv=--------	∬ nxB ds	. ∭ nxB ds	∬ n.B ds	. ∭ n.B ds	∬ nxB ds
In a Gauss divergence theorem,f is a vector point function --- and ----- in a region v of space	finite and differentiable	infinite and differentiable	finite andNon differentiable	in finite and Nondifferentiable	finite and differentiable

Reg No

(18PHU103)

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE.
DEPARTMENT OF PHYSICS
I B.Sc PHYSICS
First Semester
I-Internal Examination
Mathematical physics-I

Time: 2 hours Maximum: 50 marks

PART-A (20x1=20Marks)

Answer all questions:

1. C was designed and written by ______.
(a) Charles Babbage (b) Dennis Ritchie
(c) Ken Thompson (d) Peter Norton
2. _____ statement causes exit from switch statement.
(a) switch (b) break (c) goto (d) end
3. A proper ------------- will make the program easily readable
(a) style (b) numbering (c) execution
(d) intendation
4. ____ operator returns number of bytes the operand occupies.
(a) logical (b) bitwise (c) arithmetic (d) size of
5. Any function which satisfies the Laplace equation is known as
(a) harmonic function(b) conjugate function
(c) single function (d) analytic function
6. C language is a main function section
(a) structured programming (b) object oriented programming (c) operating system (d) platform
7. The ___ is powerful decision making statement and is used to control the flow of execution of statements.
(a) if (b) switch (c) goto (d) while
8. A ------------- is a program that controls the entire operation of a computer system.
(a) functions (b) operating system
(c) loops (d) decision making statements
9. C Language supports _____ logical operators.
(a) 2 (b) 3 (c) 4 (d) 5
10. Every statement in C should end with
(a) colon (b) comma (c) semicolon (d) no symbol
11. In C multiway decision statement is _______.
(a) while (b) if else (c) switch (d) for
12. Which is the interface between the hardware and the software?
(a) input (b) output (c) operating system
(d) compiler
13. All C compliers support ________fundamental data types.
(a) 5 (b) 4 (c) 6 (d) 8
14. How many keywords are in C?
(a) 32 (b) 30 (c) 25 (d) 29
15. The user defined functions are in
(a) Subprogram section (b) main function section
(c) Link section (d) documentation section
16. Any function which satisfies the Laplace equation is known as
(a) harmonic function (b) conjugate function
(c) single function (d) analytic function
17. A __________ is a point, where y is differentiable in a open set around the point.
(a) meromorphic points (b) holonomic point
(c) holomorphic points (d) none of the above
18. The Conjugate of 1/i is
(a) –i (b) i (c) 1 (d) -1
19. Analytic point is almost equivalent to _______.
(a) regular points (b)irregular points (c)pole points (d)branch points.
20. A set which entirely consists of interior points is known as _____.
(a) an open set (b)a closed set (c)a banded set
(d)domain

PART-B (3x2=6 Marks)

Answer all the questions

21. What are the special features of C language?

22. Define Expression with an example.

23. Define Complex numbers.

PART-C (3x8=24 Marks)

Answer all the questions

24. a. Discuss the various types of Operators.(any 5)

OR

b. Explain the if …..else statement with an example.

25. a. Discuss the types of data in C language..

OR

b. Explain the printf() and scanf() functions in C.

26. a. (i)Write the properties of moduli and Arguments.

(ii)Separate $\log_e Z$ into real and imaginary parts.

OR

b. (i)If $a^2+b^2+c^2=1$ and $b+ic=(i+a)z$ Prove that $(a+ib)/(1+c)=(1+iz)/(1-iz)$.

(ii)Write about the functions of a complex variable.