14BEEC401 PROBABILITY AND RANDOM PROCESS 3 1 0 4 100

OBJECTIVES:

- To gain knowledge in measures of central tendency and probability.
- Acquire skills in handling situations involving more than one random variable and functions of random variables.
- To understand the knowledge of random process.

UNIT- I MEASURES OF CENTRAL TENDENCY AND PROBABILITY

Measures of central tendency – Mean, Median, Mode and Standard Deviation - SPSS Software Demonstration.
Probability - Random variable - Axioms of probability - Conditional probability - Total probability – Baye's theorem - Probability mass function - Probability density functions.

UNIT- II STANDARD DISTRIBUTIONS

Functions of a random variable - Binomial, Poisson, Uniform, Exponential, Gamma, and Normal distributions - Moment generating functions, Characteristic function and their properties.

UNIT -III TWO DIMENSIONAL RANDOM VARIABLES

Joint distributions - Marginal and conditional distributions – Covariance - Correlation and regression - Transformation of random variables - Central limit theorem.

UNIT- IV CLASSIFICATION OF RANDOM PROCESS

Definition and examples - first order, second order, strictly stationary, wide – sense stationary and Ergodic processes - Markov process - Binomial, Poisson and Normal processes - Sine wave process.

UNIT -V CORRELATION AND SPECTRAL DENSITIES

Auto correlation - Cross correlation - Properties – Power spectral density – Cross spectral density - Properties – Wiener-Khintchine relation – Relationship between cross power spectrum and cross correlation function - Linear time invariant system - System transfer function –Linear systems with random inputs – Auto correlation and cross correlation functions of input and output.

TEXT BOOKS:

S. No.	**Author(s) Name**	**Title of the book**	**Publisher**	**Year of Publication**
1	Gupta, S.C. and Kapur, V.K	Fundamentals of Mathematical Statistics	Sultan Chand and Sons, New Delhi.	2007
2	Veerarajan,T.	Probabilitiy, Statistics and Random process	Tata McGraw-Hill Publications, Second Edition, New Delhi	2002

REFERENCES:

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Henry Stark and John W. Woods	Probability and Random Processes with Applications to Signal Processing	Pearson Education, Third edition, Delhi	2002
2	Ochi, M.K	Applied Probability and Stochastic Process	John Wiley & Sons, New York	1990
3	Peebles Jr, P.Z	Probability Random Variables and Random Signal Principles	Tata McGraw-Hill Pubishers, New Delhi.	2002
4	Ross, S	A first Course in Probability	Pearson Education, New Delhi (Chap 2 to 8)	2002

WEBSITES:

1. www.cut-theknot.org/probability.shtml
2. www.ece.uah.edu/courses/ee420-500
3. http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT %20Guwahati/probabilityrp/index.htm
4. www.mhhe.com/engcs/electrical/popoulis
5. 5. http://hmdc.harvard.edu/projects/SPSS_Tutorial/spsstut.shtml

KARPAGAM ACADEMY OF HIGHER EDUCATION
Faculty of Engineering
Department of Science and Humanities
LECTURE PLAN

Name : Mr. B. Arun   Class : II- ECE
Subject: PROBABILITY AND RANDOM PROCESS   Subject Code : 14BEEC401

S.NO.	TOPICS TO BE COVERED	HOUR(S)
UNIT – I - MEASURES OF CENTRAL TENDENCY AND PROBABILITY		
1.	Introduction - Basic Definitions - Measures of central tendency and Dispersion	1
2.	Discrete data - Mean, Median, Mode and Standard Deviation - Problems	1
3.	Ungrouped data - Mean, Median, Mode and Standard Deviation – Problems	1
4.	Grouped data - Mean, Median, Mode and Standard Deviation – Problems	1
5.	Tutorial – 1 - Mean, Median, Mode and Standard Deviation	1
6.	Probability – Basic Definitions – Problems	1
7.	Axioms of Probability, Conditional Probability – Problems	1
8.	Total Probability theorem and Baye's theorem	1
9.	Baye's theorem – Problems	1
10.	Concepts of Random variables - Probability Mass Function (PMF) – Problems	1
11.	Probability Density Function (PDF) – Problems	1
12.	Tutorial – 2 (Baye's theorem, Probability mass function, Probability density function)	1
13.	SPSS soft ware demonstration	1
	Total	**13**
UNIT – II - STANDARD DISTRIBUTIONS		
14.	Concepts of Discrete and Continuous distributions - Functions of random variable	1
15.	Binomial Distribution - Moment generating functions (MGF) and Characteristic function (CF) – Problems	1
16.	Binomial Distribution – Problems	1
17.	Poisson Distribution – MGF, CF and Problems	1
18.	Tutorial - 3 – Binomial and Poisson distribution	1
19.	Uniform Distribution - MGF, CF and Problems	1
20.	Exponential Distribution - MGF, CF and Problems	1
21.	Gamma Distribution - MGF, CF and Problems	1
22.	Normal Distribution - MGF, CF and Problems	1
23.	Distribution Properties – Discrete and Continuous cases	1
24.	Tutorial – 4 – Uniform, Exponential and Normal distribution	1
25.	Chebyshev's inequality	1
	Total	**12**
UNIT – III - TWO DIMENSIONAL RANDOM VARIABLES		
26.	Concepts of Joint, Marginal and Conditional distribution	1
27.	Joint Distributions – Problems	1
28.	Marginal Distributions – Problems	
29.	Conditional Distributions – Problems	1

30.	Tutorial – 5 - Marginal and Conditional Distribution	1
31.	Abstract of Covariance, Correlation and Regression	1
32.	Correlation – Problems	1
33.	Regression – Problems	1
34.	Problems based on Correlation and Regression	1
35.	Transformation of random variables and Concept of Central limit theorem	1
36.	Tutorial – 6 - Correlation and Regression	1
	Total	**11**
UNIT IV - CLASSIFICATION OF RANDOM PROCESS		
37.	Concepts of Random Process - Definition and Problems	1
38.	First and Second order Classification	1
39.	Concepts of Strictly Sence Stationary process (SSS)	1
40.	Problems based on SSS	
41.	Tutorial – 7 - SSS	1
42.	Ergodic process – Definition and Problems	1
43.	Idea of Markov Process	1
44.	Binomial Processes – Definition	1
45.	Poisson Process and Normal Processes – Problems	1
46.	Concepts of Wide Sence Stationary process (WSS)	1
47.	Problems based on WSS	
48.	Tutorial – 8 - WSS	1
49.	Concepts of Sine wave process	1
	Total	**13**
UNIT V - CORRELATION AND SPECTRAL DENSITIES		
50.	Introduction - Auto correlation Properties	1
51.	Cross Correlation Properties – Problems	1
52.	Power spectral density – Problems	1
53.	Tutorial – 9 - Cross Correlation and Power spectral density	1
54.	Cross spectral density – Definition and Problems	1
55.	Wiener – Khintchine relation – Definition	1
56.	Relationship between cross power spectrum and cross correlation function	1
57.	Tutorial -10 - Cross power spectrum and cross correlation function	1
58.	Linear time invariant system and System transfer function	1
59.	Concepts of Linear systems with random inputs	1
60.	Auto correlation and cross correlation functions of inputs and output	1
	Total	**11**
	Theory hours	**50**
	Tutorial hours	**10**
	Grand total	**50+10 = 60**

Staff In charge **HOD**

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Faculty of Engineering
Department of Science and Humanities
PROBABILITY AND RANDOM PROCESS

TEXT BOOK:

S. No.	Author(s) Name	Title of the book	Publisher	Year of Publication
1	Lavvy C. Andrews, Ronald L. Phillips	Mathematical Techniques for Engineers and scientists	Prentice-Hall of India Private limited, New Delhi.	2005

REFERENCES:

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1	Henry Stark and John W. Woods	Probability and Random Processes with Applications to Signal Processing	Pearson Education, Third edition, Delhi	2002
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3. http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT%20Guwahati/probabilityrp/index.htm
4. www.mhhe.com/engcs/electrical/popoulis
5. http://hmdc.harvard.edu/projects/SPSS_Tutorial/spsstut.shtml

UNIT - I

BASIC PROBABILITY

Introduction:

The word 'Probability or change' is very frequency used in day-to-day conversation. The Statistician I.J. Good, suggests in his "kinds of Probability" that "the theory of Probability is much older than the human species.

The concept and applications of probability, which is a formal term of the popular word "Change" while the ultimate objective is to facilitate calculation of probabilities in business and managerial, science and technology etc., the specific objectives are to understand the following terminology.

Random Experiment: The term experiment refers to describe, which can be repeated under some given conditions. The experiment whose result (outcomes) depends on change is called Random Experiment.

Example:

1. Tossing of a coin is a random experiment.
2. Throwing a die is a random experiment.
3. Calculation of he mean arterial blood pressure of a person under ideal environmental conditions, by using the formula, Blood pressure = $= \frac{Systoloic\ pressure}{Diastolic\ pressure}\ mm / Hg$ is a random experiment.

Sample Space:

The totality of all possible outcomes of a random experiment is called a sample space and it is denoted by s and a possible outcome are element.

The no. of the coins in a sample space denoted by n(s).

Example:

Tossing a coin n(s)=2={H,T}

Event:

The output or result of a random experiment is called an event or result or outcome.

Example:

1. In tossing of a coin, getting head or tail is an event.
2. In throwing a die getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Events are generally denoted by capital letters A, B, C etc. The events can be of two types. One is simple event and the other is compound event

Favorable event:

The no. of events favorable to an event in a trail is the no.of outcomes which entire the happening of the event.

Mutually Exclusive Events:

Two or more events are said to be mutually exclusive events if the occurrence of one event precludes (excludes or prevents) the occurrence of others, i.e., both cannot happen simultaneously in a single trail.

Example:

1. In tossing of a coin, the events head and tail are mutually exclusive.
2. In throwing a die, all the six faces are mutually exclusive.

Equally Likely Events: Two or more events are said to be equally likely, if there is no reason to expect any one case (or any event) in preference to others. i.e., every outcome of the experiment has equal possibility of occurrence. These are equally likely events.

Exhaustive Number of Cases or Events: The total number of possible outcomes in an experiment is called exhaustive number of cases or events.

Dependent event:

Two events are said to be dependent if the occurance or non occurance of a event in any trail affect the occurance of the other event in other trail.

Classical Definition of Probability: Suppose that an event 'A' can happen in 'm' ways and fails to happen (or non-happen) in 'n' ways, all these 'm+n' ways are supposed equally likely. Then the probability of occurrence (or happening) of the event called its success is denoted by 'P(A)' or simply 'p' and is defined as $P(A)=\frac{m}{m+n}$ ...(1) and the probability of non-occurrence (or non-happening) of the event called its failure is denoted by $P(\overline{E})$ or simply 'q' and is defined as. $P(\overline{A})=\frac{n}{m+n}$...(2)

From (1) and (2) we observe that the probability of an event can be defined as

$$P(event)=\frac{The\ number\ of\ favourable\ cases\ for\ the\ event}{Total\ number\ of\ possible\ cases}$$

Definition:

Let S be the sample space and A be the event associated with a random experiment. Let n(S) and n(A) be the no .of elements of S & A. Then the probability of the event A occurring denoted as P(A) is defined by

$$P(event)=\frac{The\ number\ of\ favourable\ cases\ for\ the\ event}{Total\ number\ of\ possible\ cases} = \frac{n(A)}{n(S)}$$

Note:

It follows that, $P(A)+P(\overline{A})=1$ or $p+q=1$.

This implies that p=1-q or q=1-p.

Hence $0 \le P(A) \le 1$.

Axiomatic Definition of Probability: Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by P(A), is defined as a real number satisfying he following axioms.

(i) $0 \le P(A) \le 1$

(ii) P(S)=1

(iii) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$

(iv) If $A_1, A_{2,...,} A_{n,...}$ are a set of mutually exclusive events, $P(A_1 \cup A_2 \cup ... \cup A_{n,...}) = P(A_1) + P(A_2) + ... + P(A_n) + ...$

Theorem 1: The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, P(ϕ)=0.

Proof: The certain event S and the impossible event ϕ are mutually exclusive.

Hence $P(S \cup \phi) = P(S) + P(\phi)$ [axiom (iii)]

But $S \cup \phi = S$.

Therefore, $P(S) = P(S) + P(\phi)$

Hence $P(\phi) = 0.$

Theorem 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A}) = 1 - P(A) \le 1.$

Proof: A and $\overline{A}$ are mutually exclusive events, such that $A \cup \overline{A} = S$

Therefore, $P(A \cup \overline{A}) = P(S)$ =1 (Since axiom (ii))

i.e., $P(A) + P(\overline{A}) = 1.$

Therefore, $P(\overline{A}) = 1 - P(A)$

Since $P(A) \ge 0,$ it follows that $P(\overline{A}) \le 1.$

Theorem 3: If $B \subset A\, then\, P(B) \le P(A).$

Proof: B and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$.

Therefore, $P(B \cup A\overline{B}) = P(A)$

i.e., $P(B) + P(A\overline{B}) = P(A)$ [axiom (iii)]

Therefore, $P(B) \le P(A).$

Theorem 4: Addition theorem of probability

Statement: For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

Proof: Since $(A\cup B)=A\cup(A'\cap B)$ here A and $(A'\cap B)$ are mutually exclusive.

$P(A\cup B)=P[A\cup(A'\cap B)]\ ...(1)$

$=P(A)+P(A'\cap B)$

Again $B=(A\cap B)\cup(A'\cap B)$

Here $(A\cap B)\&(A'\cap B)$ are mutually exclusive events.

$P(B)=P[(A\cap B)\cup(A'\cap B)]\ ...(2)$

$=P(A\cap B)+P(A'\cap B)$

Therefore $P(A'\cap B)=P(B)-P(A\cap B)$

From (1), $P(A\cup B)=P(A)+P(B)-P(A\cap B).$

Conditional Probability: The Conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as, $P(B/A)=\dfrac{P(A\cap B)}{P(A)}$, provided $P(A)\neq 0.$

Rewriting the definition of conditional probability, we get $P(A\cap B)=P(A)\times P(B/A).$ [Product theorem of probability]

Properties:

1. If $A\subset B, P(B/A)=1, Since\ A\cap B=A.$
2. If $B\subset A, P(B/A)\geq P(B), Since\ A\cap B=B,$ and $\dfrac{P(B)}{P(A)}\geq P(B), as\ P(A)\leq P(S)=1.$
3. If A and B are mutually exclusive events, P(B/A)=0, since $P(A\cap B)=0$
4. If P(A)>P(B), P(A/B)>P(B/A).
5. If $A_1\subset A_2, P(A_1/B)\leq P(A_2/B).$

Independent Events: A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

The product theorem can be extended to any number of independent events: $A_1, A_{2,...,} A_n$ are n independent events. $P(A_1\cap A_2\cap ...\cap A_n)=P(A_1)\times P(A_2)\times ...\times P(A_n)$, when this condition is satisfied, the events $A_1, A_{2,...,} A_n$ are also said to be totally independent. A set of events $A_1, A_{2,...,} A_n$ is said to be mutually independent if the events are totally independent when considered in sets of 2,3,. . . n events.

Theorem 5: If the events A and B are independent, then so are $\overline{A}\&\overline{B}.$

Proof. $P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})=1-P(A\cup B)$

$=1-[P(A)+P(B)-P(A\cap B)]$ (By addition theorem)

$=1-P(A)-P(B)+P(A)\times P(B)$ {since A and B are independent)

$=[1-P(A)]-P(B)[1-P(A)]$

$$= P(\overline{A}) \times P(\overline{B})$$

Example 1: In how many different ways can the director of a research laboratory choose two chemists from among seven applicants and three physicists from among nine applicants?

Solution:

The two chemists can be chosen in ${}^{7}C_{2}$ =21 ways

The three physicists can be chosen in ${}^{9}C_{3}$ = 84 ways

Then these two things can be done in 21 x 84 = 1764 ways.

Example 2: What is the probability that a non-leap year contains 53 Sundays?

Solution:

A non-leap year consists of 365 days, of these there are 52 complete weeks and 1 extra day. That day may be any one of the 7 days. So already we have 52 Sundays. For one more Sunday, the probability that getting a one more Sunday is 1/7.

Hence the probability that a non-leap year contains 53 Sundays is 1/7.

Example 3: A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both the white?

Solution:

Given that Balls White(7), Red(6) & Black(5), total 18 balls.

Two balls are drawn at random from 18 balls in ${}^{18}C_{2}$ ways

Two white balls are drawn at random from 7 balls in ${}^{7}C_{2}$ ways.

Hence the required probability = $({}^{7}C_{2})/({}^{18}C_{2}) = 21/153.$

Example 4 : Determine the probability that for a non-defective bolt will be found if out of 600 bolts already examined 12 were defective.

Solution:

Given that out of 600 bolts 12 were defective.

Therefore, probability that a defective bolt will be found = $\frac{12}{600} = \frac{1}{50}$

Therefore, Probability of getting a non-defective bolt = $1 - \frac{1}{50} = \frac{49}{50}$.

Example 5: A fair coin is tossed 4 times. Define the sample space corresponding to this experiment. Also give the subsets corresponding to the following events and find the respective probabilities:

a).More heads than tails are obtained.

b).Tails occur on the even numbered tosses.

Solution:

S= {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

a). Let A be the event is which more heads occur than tails

Then A= {HHHH, HHHT, HHTH. HTHH,THHH}

b).Let B be the event is which tails occur is the second and fourth tosses.

Then B= {HTHT, HTTT,TTHT,TTTT}

$$P(A)=\frac{n(A)}{n(S)}=\frac{5}{16};\ P(B)=\frac{n(B)}{n(S)}=\frac{1}{4}.$$

Example 6: A box contains 4 bad & 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is probability that the other one is also good?

Solution:

Let A = one of the tubes drawn is good and B = the other tube is good .

$P(A\cap B)$ = P(both tubes drawn are good)

$$=\frac{{}^{6}C_{2}}{{}^{10}C_{2}}=\frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., P(B/A) is required.

By definition, $P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1/3}{6/10}=\frac{5}{9}.$

Example 7: In a shooting test, the probability of hitting the target is ½ for A , 2/3 for B , 3/4 for C. If all of them five at the target, find the probability that

i). none of them hits the target.

ii). Atleast one of them hits the target.

Solution:

Let A = event of A hitting the target.

$$P(\overline{A})=\frac{1}{2},P(\overline{B})=\frac{1}{3},P(\overline{C})=\frac{1}{4}.$$

$$P(\overline{A}\cap\overline{B}\cap\overline{C})=P(\overline{A})\times P(\overline{B})\times P(\overline{C})\text{ (by independence)}$$

i.e., P(none hits the target) = $\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}=\frac{1}{24}$

P(atleast one hits the target) = 1 – P(none hits the target)

$$=1-\frac{1}{24}=\frac{23}{24}.$$

Example:8

Three coins are tossed together find they are exactly 2 head?

Solution:

Total no. of chances by throwing 3 coins are n(S)= 8.

The event A to get exactly 2 heads are A = {HHT, THH,HTH}

n(A)= 3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Example:9

A bag contains 4 red, 5 white and 6 black balls. What is the probability that 2 balls drawn are red and black?

Solution:

Given that Balls White(5), Red(4) & Black(6), total 15 balls.
Two balls are drawn at random from 15 balls in $15C_2$ ways

n(A)= $4C_1$X $6C_1$, Hence the required probability = $\frac{4C_1 X\, 6C_1}{15C_2} = \frac{8}{35}$

Example :10

A bag contains 3 red and 4 white balls. Two draws are made without replacement.
What is the probability that both balls are red

Solution:

Total no. of balls = 3Red + 4 White = 7 balls

P(Drawing a red ball in the first drawn is red) = $P(A) = \frac{3}{7}$

P(Drawing a red ball in the second drawn is red) = $P(B/A) = \frac{2}{6}$

$$P(A \cap B) = P(A)P(B)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A)P(B/A)$$

$$= \frac{1}{7}$$

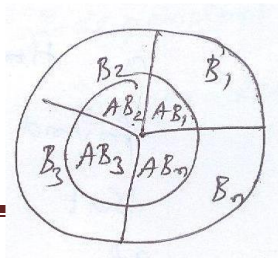
Theorem of Total Probability

Statement: If $B_1, B_{2,...,} B_n$ be a set of exhaustive and mutually exclusive events, and A is another event associated with (or caused by) B_i, then $P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$

Proof. The inner circle represents the event A. A can occur along with (or due to) $B_1, B_{2,...,} B_n$ that are exhaustive and mutually exclusive.

Therefore, $AB_1, AB_{2,...,} AB_n$ are also mutually exclusive.

Therefore, $A = AB_1 + AB_2 + \ldots + AB_n$ (by addition theorem)



Hence $P(A) = P(\sum AB_i)$

$= \sum P(AB_i)$ (since $AB_1, AB_2, \ldots, AB_n$ are mutually exclusive)

$$P(A) = \sum_{i=1}^{n} P(B_i) P(A/B_i)$$

Baye's theorem on Probability (or) Rule of inverse probability

Statement: If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with (or caused by) B_i, then

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, i = 1,2,\ldots,n$$

Proof. Since by product theorem, $P(A \cap B_i) = P(B_i) \times P(A/B_i)$ . . . (1)

or $P(A \cap B_i) = P(A)\ P(B_i/A)$ . . . (2)

From (1) and (2), $P(A) P(B_i/A) = P(B_i)\ P(A/B_i)$

$$P(B_i/A) = \frac{P(B_i)\ P(A/B_i)}{P(A)} \quad . . . (3)$$

Therefore from total probability, $P(A) = \sum_{i=1}^{n} P(B_i) P(A/B_i)$ substitute in (3), we get

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, i = 1,2,\ldots,n$$

Example 11: A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag & they are note to be white. What is the chance the all the balls in the bag are white?

Solution:

Since 2 white balls have been drawn out, the bag must have contained 2, 3, 4, or 5 white balls.

Let B_1 = Event of the bag containing 2 white balls.

B_2 = Event of the bag containing 3 white balls.

B_3 = Event of the bag containing 4 white balls.

B_4 = Event of the bag containing 5 white balls.

Let A = Event of drawing 2 white balls.

$$P(A/B_1) = \frac{^2C_2}{^5C_2} = \frac{1}{10},\ P(A/B_2) = \frac{^3C_2}{^5C_2} = \frac{3}{10}$$

$$P(A/B_3) = \frac{^4C_2}{^5C_2} = \frac{3}{5},\ P(A/B_4) = \frac{^5C_2}{^5C_2} = 1$$

Since the number of white balls in the bag is not known, B_i's are equally likely.

Therefore $P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$

By Baye's theorem,

$$P(B_4 / A) = \frac{P(B_4) \times P(A/B_4)}{\sum_{i=1}^{4} P(B_i) \times P(A/B_i)} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times \left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1 \right)} = \frac{1}{2}.$$

Example 12: There are 3 true coins and 1 false coin with 'head' on both sides. A coin is closer at random and tosses 4 times, If 'head' occurs are the 4 times, What is the probability that the false coin has been chosen and used?

Solution:

P(T) = P(the coin is a true coin) = 3/4

P(F) = P(the coin is a false coin) = 1/4

Let A = Event of getting all heads is 4 tosses,

Then, P(A/T) = ½ * ½ * ½ * ½ = 1/16 and P(A/F) = 1

By Baye's theorem, $P(F / A) = \frac{P(F) \times P(A/F)}{P(F) \times P(A/F) + P(T) \times P(A/T)} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{16}{19}.$

Example 13:

There are three bags , bag one contains 3 white balls , 2 red balls and 4 black balls. Bag two contains 2 white balls, 3 red balls and 5 black balls. Bag three contains 3 white balls, 4 red balls and 2 black balls. One bag is chosen at random and from it 3 balls were drawn out of which 2 balls were white and 1 is red. What is the probability that it is drawn from bag one, two and three?

Solution:

Selection of bags are mutually exclusive events. The selection of the 2 white and 1 red ball is an independent event.

$P(B_1)=P(B_2)=P(B_3)=1/3$

$P(A/B_1) =$ P(Bag 1 selected from 2W&1R ball chosen)

$= \frac{3C_2 X 2C_1}{9C_3}$

=0.07

$P(A/B_2) =$ P(Bag 2 selected from 2W&1R ball chosen)

$= \frac{2C_2 X 3C_1}{10C_3}$

= 0.025

$P(A/B_3) =$ P(Bag 3 selected from 2W&1R ball chosen)

$= \frac{3C_2 X 4C_1}{9C_3}$

= 0.14

By using Baye's theorem we have

$P(B_i)$	$P(A/B_i)$	$P(B_i)\ P(A/B_i)$
1/3	0.07	0.0233
1/3	0.025	0.0083
1/3	0.14	0.0466
	$\sum P(B_i)\ P(A/B_i)$	0.0782

$P(B_1/A)$ = P(The balls selected from the first bag)

$= \dfrac{0.0233}{0.0782}$

= 0.29

$P(B_2/A)$ = P(The balls selected from the second bag)

$= \dfrac{0.008}{0.0782}$

= 0.102

$P(B_3/A)$ = P(The balls selected from the third bag)

$= \dfrac{0.046}{0.0782}$

= 0.58

Exercise:

1. In a bolt factory machines A,B,C manufactures 25%,35% and 40% of the total respectively. Out of their output 5%,4% and 2% are defective bolts respectively. A bolt is drawn at random and is found to be defective. What are the probabilities that it was manufactured by the machines A,B and C respectively?

2. A bag contains five balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are found to be white. What is the probability that all the balls in the bag are white?

RANDOM VARIABLES

Definition: A real-valued function defined on the outcome of a probability experiment is called a random variable. A Random variable (RV) is a rule that assigns a numerical value to each possible outcome of an experiment.

1. Discrete Random Variables.
2. Continuous Random Variables

Probability distribution function of X: If X is a random variable, then the function F(x) defined by $F(x) = P\{X \le x\}$ is called the distribution function of X.

1. **Discrete Random Variable:** A random variable whose set of possible values is either finite or countable infinite is called discrete random variable.

Probability Mass Function (pmf): If X is a discrete variable, then the function $p(x) = P[X = x]$ is called the pmf of X. It satisfies two conditions

i) $p(x_i) \geq 0$

ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

Cumulative distribution [discrete R.V] or distribution function of X: The cumulative distribution F(x) of discrete random variable X with probability f(x) is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \text{ for } -\infty < x < \infty$$

Properties of distribution function:

1. $F(-\infty) = 0$
2. $F(\infty) = 1$
3. $0 \leq F(x) \leq 1$
4. $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
5. $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) + P[X = x_1]$
6. $P(x_1 < X < x_2) = F(x_2) - F(x_1) - P[X = x_2]$
7. $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) - P[X = x_2] + P[X = x_1]$

Results:

1. $P(X \leq \infty) = 1$
2. $P(X \leq -\infty) = 0$
3. $P(X > x) = 1 - P[X \leq x]$
4. $P(X \leq x) = 1 - P[X > x]$

Example 14: A R.V X has the following probability distribution.

x:	-2	-1	0	1	2	3
p(x):	0.1	k	0.2	2k	0.3	3k

Find (1) The value of k, (2) Evaluate P(X<2) and P(-2<X<2) .

Solution:

(1) Since $\sum_{i=1}^{n} p(x_i) = 1$

0.1+k+0.2+2k+0.3+3k = 1

K = 1/15.

(2) P[X<2] = P[x=-2] + P[x=-1] + P[x=0] + P[x=1]

= 0.1 + 1/15 + 0.2 + 2/15

= ½

P[-2 < X < 2] = P[x=-1]+ P[x=0]+ P[x=1]

= 1/15 + 0.2 + 2/15 = 2/5

Example 15:

A random variable X has the following probability function

Values of x	0	1	2	3	4	5	6	7	8
Probability P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

i) Determine the value of 'a'.
ii) Find P(X<3), P(X ≥ 3) and P (0 < X< 5).
iii) Find the distribution function of X.

Solution:

i) To find 'a' value:

Given discrete random variable, $\sum_{i=1}^{\infty} p(x_i) = 1$

a+3a+5a+7a+9a+11a+13a+15a+17a =1

81a=1

a=1/81

ii) To find P(X<3):

P(X<3) = P(X=0)+P(X=1)+P(X=2)

=a+3a+5a

=9a

=1/9

iii) To find $P(X \geq 3)$:

$P(X \geq 3) = 1 - P(X < 3)$

=1-1/9 =8/9

iv) To find $P(0 < X < 5)$:

$P(0 < X < 5) = P(X = 1) + ...P(X = 4)$

= 3a+5a+7a+9a

= 24/81

v) To find the distribution function of X:

Value of x	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a
P(x)	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81
F(x)	1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	1

Example 16: A R.V X has the following function:

X:	0	1	2	3	4	5	6	7
P(X):	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(a) find k (b) Evaluate P[X<6], P[x≥6], (c) Evaluate P[1.5<X<4.5 / X>2] (d) Find P[X<2], P[X>3], P[1<X<5].

Solution:

(a). Since $\sum_{i=1}^{n} p(x_i)=1$

i.e., $0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$10k^2+ 9k – 1 = 0$

K = -1 or 1/10 (since k=-1 is not permissible, $P(X)\geq 0$)

Hence k = 1/10.

(b). $P[x\geq 6] = P[X=6] + P[X=7]$

$= 2k^2+7k^2+k$

= 2/100 + 7/100 + 1/10 = 19/100

$P[X<6] = 1 - P[x\geq 6]$

= 1 – 19/100

= 81/100

(c). P[1.5<X<4.5 / X>2] $=\frac{p[(1.5<x<4.5)\cap x>2]}{p(x>2)}$ (by conditional probability)

$$=\frac{p[2<x<4.5]}{1-p(x\leq 2)}$$

$$=\frac{p(3)+p(4)}{1-[p(0)+p(1)+p(2)]}$$

$$=\frac{\frac{2}{10}+\frac{3}{10}}{1-\left[0+\frac{1}{10}+\frac{2}{10}\right]}=\frac{\frac{5}{10}}{\frac{7}{10}}=\frac{5}{7}$$

(d). p(X<2) = p[x=0] + p[x=1]

= 0 + k = k = 1/10

$P(X>3) = 1 – p(x\leq 3)$

= 1 – [p(x=0)+p(x=1)+p(x=2)+p(x=3)]

= 1 – [0+k+2k+2k]

= ½

P(1<x<5) = p(x=2) + p(x=3)+ p(x=4)

= 2k + 2k + 3k

= 7/10

Example 17: If the R.V. X takes the values 1,2,3 and 4 such that 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4). Find the probability distribution and cumulative distribution function of X.

Solution:

Since X is a discrete random variable.

Let 2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k

2P(X = 1) = k implies that P(X = 1) = k/2

3P(X = 2) = k implies that P(X = 2) = k/3

P(X = 3) = k

5P(X = 4) = k implies that P(X = 4) = k/5

Since $\sum_{i=1}^{n} p(x_i)=1$

i.e., k/2 + k/3 + k + k/3 = 1

k[1/2 + 1/3 + 1 + 1/5] = 1

Therefore k = 30/61

x_i	$p(x_i)$	F(X)
1	P(1) = k/2 = 15/61	F(1) = p(1) = 15/61
2	P(2) = k/3 = 10/61	F(2) = F(1) + p(2) = 15/61 + 10/61 = 25/61
3	P(3) = k = 30/61	F(3) = F(2)+p(3) = 25/61 + 30/61 = 55/61
4	P(4) = k/5 = 6/61	F(4) = F(3)+p(4) = 55/61 + 6/61 = 61/61 = 1

Example 18: A discrete random variable X has the following probability mass function:

X	0	1	2	3	4	5	6	7
P(X)	0	a	2a	2a	3a	a^2	$2a^2$	$7a^2+a$

Find (i) the value of 'a' (ii) $P(X < 6)$, $P(X \geq 6)$ (iii) $P(0 < X < 5)$ (iv) the distribution function of X (v) If $P(X \leq x) > 1/2$, find the minimum value of X.

Solution:

(i) Since $\sum_{i=1}^{n} p(x_i)=1$

i.e., $0+a+2a+2a+3a+a^2+2a^2+7a^2+a = 1$

$10a^2 + 9a - 1 = 0$

a = -1 or 1/10 (since a=-1 is not permissible, P(X)≥0)

Hence a = 1/10.

(ii). P[x≥6] = P[X=6] + P[X=7]

$= 2a^2+7a^2+a$

= 2/100 + 7/100 + 1/10 = 19/100

(iii). P[X<6] = 1 - P[x≥6]

= 1 – 19/100

= 81/100

(iv). To find $P(0<X<5)$:

$P(0<X<5) = P(X=1)+\ldots..P(X=4)$

$= a+2a+2a+3a$

$= 8a = 8/10$

(v). To find distribution function of X :

x	0	1	2	3	4	5	6	7
P(x)	0	a	2a	2a	3a	a^2	$2\,a^2$	$7\,a^2+a$
F(x)	0	1/10	3/10	5/10	8/10	81/100	83/100	1

Minimum value of X:

$P(X \le x) > 1/2$

The minimum value of X for which $P(X \le x) > 0.5$, is the x value is 4.

Example 19: A RV X has the following distribution

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	3k

(a) find k (b) Evaluate $P(X<2)$ & $P(-2<X<2)$

Solution:

(a) $\sum P(X)=1$

$6K+0.6=1$

$K=1/15$

Since the distribution is

X	-2	-1	0	1	2	3
P(X)	1/10	1/15	1/5	2/15	3/10	1/5

(b) $P(X<2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 1/10 + 1/15 + 1/5 + 2/15 = 1/2$

& $P(-2<X<2) = P(X=-1) + P(X=0) + P(X=1)$

$= 1/15 + 1/5 + 2/15 = 2/5.$

Moments

The moment generating function (MGF) of a random variable X (about origin) whose probability function f(x) is given by

$$M_x(t) = E(e^{tx}) \quad = \sum_{x=\infty}^{\infty} e^{tx} P(x), \text{ for a discrete probability distribution}$$

where t is real parameter and the integration or summation being extended to the entire range of x.

Example 20

The probability function of an infinite discrete distribution is given by $P(X = x) = \frac{1}{2^x}, x = 1, 2, ...\infty$. Find the mean and variance of the distribution. Also find P(X is even).

Solution

We know that

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + ...$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + ...\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} \qquad \text{[Using } (1-x)^{-1} = 1 + x + x^2 + ...]$$

$$= \frac{e^t}{2}\left[\frac{(2-e^t)^{-1}}{2^{-1}}\right]$$

$$M_x(t) = \frac{e^t}{2-e^t} = (2-e^t)^{-1} e^t$$

$$M_x'(t) = -e^t(2-e^t)^{-2}(-e^t) + (2-e^t)^{-1}e^t$$

$$= e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t$$

$$M_x''(t) = 2(2-e^t)^{-2}e^{2t} + e^{2t}(-2)(2-e^t)^{-3}(-e^t) + (2-e^t)^{-1}e^t + e^t(-1) + (2-e^t)^{-2}(-e^t)$$

Now E(X) = Mean = $M_x'(0) = 1 + 1 = 2$

$E(X^2) = M_x''(0) = 6$

Mean $\mu_1' = 2$

Variance = $E(X^2) - [E(X)]^2$

= 6-4 =2

Now p(X = even) = p(x = 2) + p(x = 4)+……

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + ...$$

$$= \frac{\left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1/4}{1-1/4} = \frac{1}{4} \text{ x } \frac{4}{4-1} = \frac{1}{3}$$

MGF	**Mean**	**Variance**	**p(x=even)**
$e^t(2-e^t)^{-1}$	2	2	$1/3$

UNIT - II

RANDOM VARIABLES

Introduction:

In the last chapter, we introduced the concept of a single random variable. We observed that the various statistical averages or moments of the random variable like mean, variance, standard derivation, skewness give an idea about the characteristics of the random variable.

But in many practical problems several random variables interact with each other and frequently we are interested in the joint behavior of the health conditions of a person, doctors measure many parameters like height, weight, blood pressure, sugar level etc. we should now introduce techniques that help us to determine the joint statistical properties of several random variables.

The concepts like distribution function, density function and moments that we defined for single random variable can be extended to multiple random variables also.

Continuous Random Variables: A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Example: The length time during which a vacuum tube installed in a circuit functions is a continuous random variable, number of scratches on a surface, proportion of defective parts among 1000 testes, number of transmitted in error.

Probability density function (pdf): For a continuous R.V X, a probability density function is a function such that (1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)dx = 1$ (3)

$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx =$ area under f(x) from a to b for any a and b.

Cumulative distribution function: The Cumulative distribution function of a continuous R.V. X is

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \ \ for -\infty < x < \infty.$

Mean and variance of the Continuous R.V. X: Suppose X is continuous variable with pdf f(x). The mean or expected value of X, denoted as μ or E(X)

$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$. And the variance of X, denoted as V(X) or σ^2 is $E[X^2] - [E(X)]^2$

Example: 1

A continuous random variable 'X' has a probability density function f(x) = K, $0 \leq x \leq 1$. Find 'K'.

Solution:

Given $f(x) = k, 0 \leq x \leq 1$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^\infty k\,dx = 1$$

k=1

Example 2: Given that the pdf of a R.V X is f(x)=kx, 0<x<1. Find k and P(X>0.5)

Solution:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 kx\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 = 1$$

K = 2

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{1/2}^{1} 2x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

= 3/4

Example 3: If $f(x)=\begin{cases} kxe^{-x}, & x>0 \\ 0, & elsewhere \end{cases}$ is the pdf of a R.V. X. Find k.

Solution:

For a pdf $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here $\int_0^\infty kxe^{-x}\,dx = 1$ [since x>0]

$$k\left[x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 1$$

K =1

Example 4: A continuous R.V. X has he density function $f(x) = \frac{k}{1+x^2}, -\infty < x < \infty.$ find the value of k and the distribution function.

Solution:

Given is a pdf $\int_{-\infty}^{\infty} f(x)dx = 1$, $f(x) = \frac{k}{1+x^2}, -\infty < x < \infty.$

$$k\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = 1$$

$$2k\int_{0}^{\infty} \frac{1}{1+x^2}dx = 1$$

$$2k\left[\tan^{-1} x\right]_0^{\infty} = 1$$

$$2k\left[\frac{\pi}{2} - 0\right] = 1$$

$$\pi k = 1; k = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1} x - \left(\frac{-\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x + \left(\frac{\pi}{2}\right)\right] \quad for -\infty < x < \infty$$

Example:5

A continuous random variable X has a pdf $f(x) = 3x^2,\ 0 \le x \le 1.$ Find a and b such that (i) $P(X \le a) = P(X > a)$ and (ii) $P(X > b) = 0.05.$

Solution:

A continuous random variable X has a pdf $f(x) = 3x^2,\ 0 \le x \le 1.$

i) To find $P(X \le a) = P(X \succ a)$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{0}^{1} 3x^2 dx = 1$$

Since $P(X \le a) = P(X \succ a)$, $P(X \le a) = \frac{1}{2} = 0.5$

$$\int_{0}^{a} f(x)dx = \frac{1}{2}\ ,\ \int_{0}^{a} 3x^2 dx = a^3 = \frac{1}{2}$$

a = 0.7937

ii) To find $P(X \succ b) = 0.05$

$$\int_{b}^{1} f(x)dx = 0.05,\quad \int_{b}^{1} 3x^2 dx = 1 - b^3 = 0.05$$

$b^3 = 0.95$

$b = (0.95)^{1/3}$

Example 6: If the density function of a continuous R.V. X is given by $f(x)=\begin{cases} ax, & 0\le x\le 1 \\ a, & 1\le x\le 2 \\ 3a-ax, & 2\le x\le 3 \\ 0, & otherwise \end{cases}$

(1) Find the value of a.

(2) The cumulative distribution function of X.

(3) If x_1, x_2, x_3 are 3 independent observations of X. What is the probability that exactly one of these 3 is greater than1.5?

Solution:

(1) Since f(x) is a pdf, then $\int_{-\infty}^{\infty} f(x)dx=1$

i.e., $\int_0^3 f(x)dx=1$

i.e., $\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a-ax)dx = 1$

a = ½

(2). (i) If x<0 then F(x) = 0

(ii) If $0\le x\le 1\, then\, F(x)=\int_0^x ax\,dx=\int_0^x \frac{x}{2}dx$

$= \frac{x^2}{4}$

$(iii)\ If\ 1\le x\le 2\, then\, F(x)=\int_{-\infty}^{x} f(x)dx$

$=\int_0^1 ax\,dx+\int_1^x adx$

$=\frac{x}{2}-\frac{1}{4}$

$(iv)\ If\ 2\le x\le 3\, then\, F(x)=\int_{-\infty}^{x} f(x)dx$

$=\int_0^1 ax\,dx+\int_1^2 adx+\int_2^x (3a-ax)dx$

$=\frac{3x}{2}-\frac{x^2}{4}-\frac{5}{4}$

(i) If x>3, then $F(x)=\int_{-\infty}^{x} f(x)dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + \int_3^x f(x)\,dx$$

$$= 1$$

(3). $P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx = \int_{1.5}^{2} \frac{1}{2}dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx$

$$= ½$$

Choosing an X and observing its value can be considered as a trail and X>1.5 can be considered as a success.

Therefore, p=1/2, q=1/2.

As we choose 3 independent observation of X, n = 3.

By Bernoulli's theorem, P(exactly one value > 1.5) = P(1 success)

$$= {}^3C_1 \times (p)^1 \times (q)^2 = \frac{3}{8}.$$

Example:7

A continuous random variable X is having the probability density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the cumulative distribution function of x.

Solution:

Given $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

To find cumulative distribution function of x:

i) If $0 < x < 1$ F(x) = $\int_{-\infty}^{x} f(x)dx$

$$= \int_0^x x dx = \frac{x^2}{2}$$

ii) If $1 < x < 2$, F(x) = $\int_{-\infty}^{x} f(x)dx$

$$= \int_0^1 x dx + \int_1^x (2 - x)dx$$

$$= 2x - \frac{x^2}{2} - 1$$

iii) If $x > 2$, F(x) = $\int_{-\infty}^{x} f(x)dx$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$
$$= 1$$

The cumulative distribution function of x is $F(x) = \begin{cases} \dfrac{x^2}{2}, & 0 < x < 1 \\ 2x - \dfrac{x^2}{2} - 1, & 1 < x < 2 \\ 1, & x > 2 \end{cases}$

CONTINUOUS RANDOM VARIABLE DISTRIBUTIONS

Normal distribution:

Definition:

A continuous random variable X is said to follow a normal distribution with mean μ and variance σ^2, if its density function is given by the probability law

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}.e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,\ \sigma > 0,\ -\infty < \mu < \infty.$$

If X follows normal distribution with mean μ and standard deviation σ, then it is denoted by $N \sim (\mu, \sigma)$ sometimes $N(\mu, \sigma^2)$ can also be used.

Solution:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx}.\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$. \qquad = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

put $z = \dfrac{x-\mu}{\sigma}$ If $x = -\infty$, $z = -\infty$

$\sigma dz = dx$ If $x = \infty$, $z = \infty$

$$. \qquad = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)}.e^{\frac{-z^2}{2}} \sigma dz$$

. $$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z^2-2t\sigma z)}{2}} dz \qquad \because \frac{-(z^2-2t\sigma z)}{2} = \frac{-1}{2}\left[(z-\sigma t)^2 - \sigma^2 t^2 - \frac{(z-\sigma t)^2}{2} + \frac{\sigma^2 t^2}{2}\right]$$

$$= e^{\mu t} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-\sigma t)^2 + \frac{\sigma^2 t^2}{2}} dz$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-\sigma t)^2} dz$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du$$

$u = z - \sigma t$ $\qquad z = \infty,\ u = \infty$

$du = dz$ $\qquad z = -\infty, u = -\infty$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} . \frac{1}{\sqrt{2\pi}} . \sqrt{2\pi} \qquad \left[\int_{0}^{\infty} e^{\frac{-u^2}{2}} du = \sqrt{2\pi}\right]$$

$$\therefore M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} .$$

Example: 8

A normal distribution has mean $\mu = 20$ and S.D $\sigma = 10$. Find $P(15 \le X \le 40)$.

Solution:

Given $\mu = 20$, $\sigma = 10$

The normal variate $Z = \frac{X - \mu}{\sigma} = \frac{X - 20}{10}$

When $X = 15$, $Z = \frac{X-20}{10} = \frac{15-20}{10} = -0.5$

$X = 40$, $Z = \frac{40-20}{10} = 2$

$$\therefore P(15 \le X \le 40) = P(-0.5 \le Z \le 2)$$
$$= P(-0.5 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= P(0 \le Z \le 0.5) + P(0 \le Z \le 2)$$
$$= 0.1915 + 0.4772 \qquad [\text{Using normal table}]$$
$$= 0.6687$$

Example 9

If X is a normal variate with mean 1 and variance 4. Y is another normal variate independent of X with mean 2 and variance 3. What is the distribution of X+2Y.

Solution:

Given X and Y are independent normal variates.

X+2Y is also a normal variate by additive property.

$\therefore$ Mean of (X+2Y) = E(X+2y)

$= E(X) + E(2Y)$
$= E(X) + 2E(Y)$
$= 1 + 2 \times 2 \quad [E(X)=1, E(Y)=2]$
$= 5$

$Var(X+2Y) = Var(X) + Var(2Y)$
$= 1^2 Var(X) + 2^2 Var(Y)$
$= 1 \times 4 + 4 \times 3 = 16$

$\therefore$ X+2Y follows normal distribution with mean 5 and variance 16.

Gamma Distribution:

The continuous random variable X is said to follow a Gamma distribution with parameter λ if its probability function is given by,

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, 0 < x < \infty \\ 0, & otherwise \end{cases}$$

Note: 1

A continuous random variable X whose probability density function is

$f(x) = \dfrac{a^{\lambda} e^{-ax} x^{\lambda-1}}{\Gamma(\lambda)}$, $a > 0,\ \lambda > 0,\ 0 < x < \infty$ is called a Gamma distribution with two parameters a and λ.

Note: 2

When $a = 1$

$f(x) = \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}$, which is called the sample Gamma distribution or standard Gamma distribution.

Note: 3

Sometimes the definition of Gamma distribution is given by taking

$$a = \frac{1}{\beta},\quad f(x) = \frac{1}{\beta^{\lambda}} \cdot \frac{e^{\frac{-x}{\beta}} x^{\lambda-1}}{\Gamma(\lambda)}, x \geq 0$$

Find the moment generating function of Gamma distribution:

Solution:

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x)dx = \int_0^{\infty} e^{tx} \cdot \frac{e^{-x} \cdot x^{\lambda-1}}{\Gamma(\lambda)} dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx}.e^{-x}.x^{\lambda-1}dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-t)}.x^{\lambda-1}dx$$

put $(1-t)x = u$ *If* $x = 0, \quad u = 0$

$(1-t)dx = du$ *If* $x = \infty, \quad u = \infty$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-u}.\left(\frac{u}{1-t}\right)^{\lambda-1}\left(\frac{du}{1-t}\right) = \frac{1}{\Gamma(\lambda)} \int_0^\infty \frac{u^{\lambda-1}e^{-u}}{(1-t)^\lambda} du$$

$$= \frac{1}{\Gamma(\lambda).(1-t)^\lambda}.\Gamma(\lambda) = \frac{1}{(1-t)^\lambda} \qquad \left[\Gamma(n) \int_0^\infty x^{n-1}e^{-x}dx\right]$$

$$M_X(t) = (1-t)^{-\lambda}, \quad |t| < 1.$$

Find the mean and variance of Gamma distribution:

Solution:

$M_X(t) = (1-t)^{-\lambda}$

$M'_X(t) = -\lambda(1-t)^{-\lambda-1}(-1)$ $\qquad \mu'_1 = M'_X(0) = \lambda$ 1

$M''_X(t) = \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)$ $\qquad \mu'_1 = M''_X(0) = \lambda(\lambda+1)$ 2

Variance $\mu_2 = \mu'_2 - \mu'^2_1 = \lambda(\lambda+1) - \lambda^2 = \lambda^2 + \lambda - \lambda^2$

$\therefore$ Variance $= \lambda$.

Hence mean and variance of Gamma distribution $= \lambda$

Gamma			
p.d.f	MGF	Mean	Variance
$\frac{e^{-x}x^{\lambda-1}}{\Gamma(\lambda)}, \lambda > 0, \quad 0 < x < \infty$	$(1-t)^{-\lambda}$	λ	λ

Exponential distribution:

A continuous random variable X is said to follow an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$$

Find the moment generating function of exponential distribution:

Solution:

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx \qquad \left[Here \quad f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases} \right]$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^\infty \qquad \left[\because \int e^{ax} dx = \frac{e^{ax}}{a} \right]$$

$$= \frac{\lambda}{-(\lambda - t)} \left[e^{-\infty} - e^{-0} \right]$$

$$= \frac{\lambda}{(\lambda - t)} \qquad \left[\therefore e^{-\infty} = 0, \ e^0 = 1 \right]$$

$\therefore$ The MGF $= \dfrac{\lambda}{\lambda - t}, \ \lambda > t$

Find the mean and variance of exponential distribution:

Solution:

We know that MGF is,

$$M_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{t}{\lambda}}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-1} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \ldots\ldots\ldots\ldots + \frac{t^r}{\lambda^r} + \ldots\ldots.$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{2!}\left(\frac{2!}{\lambda^2}\right) + \ldots\ldots\ldots\ldots + \frac{t^r}{\lambda^r}\left(\frac{r!}{\lambda^2}\right)$$

$$M_X(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r$$

$\therefore$ Mean $\mu_1' = \ coefficient \ of \ \dfrac{t'}{1!} = \dfrac{1}{\lambda}$

$$\mu_2' = \ coefficient \ of \ \frac{t^2}{2!} = \frac{2}{\lambda^2}$$

Now , variance $\mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Variance $= \frac{1}{\lambda^2} = 1/\lambda^2$

Exponential			
p.d.f	MGF	Mean	Variance
$\lambda e^{-\lambda x}, x > 0$	$\frac{\lambda}{\lambda - t}, \lambda > t$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Memoryless property of the Exponential distribution:

If X is exponentially distributed , then $P\left(X > s+t \big/ X > s\right) = P(X > t)$ for any $s, t > 0$

Proof:

$$P(X > k) = \int_k^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{\lambda} \right]_k^\infty = -e^{-\infty} + e^{-\lambda k} = e^{-\lambda k}$$

Also, $P\left(X > s+t \big/ X > s\right) = \frac{P(X > s+t \; and \; X > s)}{P(X > s)}$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda s} . e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

$\therefore P\left(X > s+t \big/ X > s\right) = P(X > t)$

Thus $P(X > t) = e^{-\lambda t}$.

Example: 10

The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a) What is the probability that the repair time exceeds $2h$?

(b) What is the conditional probability that a repair takes atleast $11h$ given that its duration exceeds $8h$?

Solution:

Let X be the random variable which represents the time to repair the machine then the density function of X is given by,

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{1}{2}x}, \quad x > 0$$

(a) $P(X > k) = e^{-\lambda k}$

$$P(X > 2) = e^{-\frac{1}{2}\times 2} = e^{-1}$$

(b) $P\left(X \geq 11 / X > 8\right) = P\left(X \geq 8+3 / X > 8\right) = P(X > 3)$ $\left[\because P\left(X > s+t / X > s\right) = P(X > s)\right]$

by memoryless property

$$P(X > t) = e^{-\lambda t} = e^{-\frac{1}{2}\times 3} = e^{-1.5}$$

$\therefore P(X > 3) = e^{-1.5}$.

BIVARITE RANDOM VARIABLES

Definition:

Let S be the sample space. Let X=X(S) and Y=Y(S) be two functions each assigning a real no. to each outcome $s \in S$. Then (X,Y) is a two dimensional random variable.

Types of random variables:

1. Discrete random variables
2. Continuous random variables

Two dimensional discrete random variables:

If the possible values of (X,Y) are finite or countably infinite then (X,Y) is called a two dimensional discrete random variables when (X,Y) is a two dimensional discrete random variable the possible values of (X,Y) may be represented as (x_i, y_j) i = 1,2,…..n, j=1,2,….m.

Two dimensional continuous random variables:

If (X,Y) can assume all values in a specified region R in the XY plane (X,Y) is called a two dimensional continuous random variables.

Joint distributions – Marginal and conditional distributions:

(i) Joint Probability Distribution:

The probabilities of two events $A = \{X \leq x\}$ and $B = \{Y \leq y\}$ have defined as functions of x and y respectively called probability distribution function.

$F_X(x) = P(X \leq x)$

$F_Y(y) = P(Y \leq y)$

Discrete random variable important terms:

i) Joint probability function (or) Joint probability mass function:

For two discrete random variables x and y write the probability that X will take the value of x_i , Y will take the value of y_j as, $P(x,y) = P(X = x_i, Y = y_j)$

ie) $P(X = x_i, Y = y_j)$ is the probability of intersection of events $X = x_i$ & $Y = y_j$.

$P(X = x_i, Y = y_j) = P(X = x_i \cap Y = y_j)$, The function $P(X = x_i, Y = y_j) = P(x_i, y_j)$ is called a joint probability function for discrete random variables X,Y and it is denoted by P_{ij} .

P_{ij} satisfies the following conditions

(i) $P_{ij} > 0$, for every i,j

(ii) $\sum_j \sum_i P_{ij} = 1$

Continuous random variable (or) Joint Probability Density Function:

Definition:

The joint probability density function if (x,y) be the two dimensional continuous random variable then f(x,y) is called the joint probability density function of (x,y) the following conditions are satisfied.

(i) $f(x,y) \geq 0, \forall x, y \in R$

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$. Where R is a sample space.

Note: $P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d f(x,y)dydx$

Joint cumulative distributive function:

If (x,y) is a two dimensional random variable then $F(X,Y) = P(X \leq x, Y \leq y)$ is called a cumulative distributive function of (x,y) the discrete case $F(X,Y) = \sum_j \sum_i P_{ij} = 1$, $y_i \leq y, x_i \leq x$.

In the continuous case $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x,y)dxdy$

Properties of Joint Probability Distribution function:

1. $O \leq P(x_i, y_j) \leq 1$
2. $\sum_i \sum_j P(X_i, Y_j) = 1$
3. $P(X_i) = \sum_j P(X_i, Y_j)$
4. $P(y_i) = \sum_j P(X_i, Y_j)$
5. $P(x_i) \geq P(x_i, y_j)$ for any j
6. $P(y_j) \geq P(x_i, y_j)$ for any i

Properties:

1. The joint probability distribution function F xy (X, Y) of two random variable X and Y have the following properties. They are very similar to those of the distribution function of a single random variable.
2. $0 \leq f_{XY}(x,y) \leq 1$

3. $f_{XY}(\infty,\infty)=1$

4. $f_{XY}(x,y)$ is non decreasing

5. $f_{XY}(-\infty,y)=F_{xy}(x_1,\infty)=0$

6. For $x_1<x_2$ and $y_1<y_2$, $P(x_1<X\le x_2, Y\le y_1)=F(x_2,y_1)-F(x_1,y_1)$

7. $P(X\le x_1, y_1<Y\le y_2)=F(x_1,y_2)-F(x_1,y_1)$

8. $P(x_1<X\le x_2, y_1<Y\le y_2)=F(x_2,y_2)-F(x_1,y_2)-f(x_2,y_1)+f(x_1,y_1)$

9. $F_Y(y)=F_{XY}(\infty,y)=P(X\le\infty, y\le y)=P(y\le y)$

10. $F_X(x)+F_y(y)-1\le F_{XY}(x,y)\le\sqrt{F_X(x)F_Y(y)}$ for all x and y.

These properties can also be easily extended to multi dimensional random variables.

Marginal Probability Distribution function:

(i) Discrete case:

- Let (x,y) be a two dimensional discrete random variable, $P_{ij}=P[X=x_i, Y=y_j]$ then $P(X=x_i)=P_i*$ is called a marginal probability of the function X. Then the collection of the pair $\{x_i, P_i^*\}$ is called a marginal probability of X.
- If $P(Y=y_j)=P_{*j}$ is called a marginal probability of the function Y. Then the collection of the pair $\{y_i, P_{*j}\}$ is called a marginal probability of Y.

(ii) Continuous case:

- The marginal density function of X is defined as $f_x(x)=g(x)=\int_{-\infty}^{\infty} f(x,y)dy$ and
- The marginal density function of Y is defined as $f_y(y)=h(y)=\int_{-\infty}^{\infty} f(x,y)dx$

Conditional distributions:

(i) Discrete case:

- The conditional probability function of X given Y=y $_j$ is given by

$P[X=x_i/Y=Y_j]=P[X=x_i, Y=y_j]/P[Y=y]=P_{ij}/P*j$

The set $\{X=x_i, P_{ij}/P*j\}$, I =1,2,3,…is called the conditional probability distribution of X given $Y=y_j$

- The conditional probability function of Y given X=xi is given by

$P=[Y=y_j\ /\ X=x_i]=\ P[Y=y_j, X=x_i]/P[X=x_i]\ =P_{ij}/P_i^*$

The set { $y_{ij}\ \ P/P_i^*$ }, j=1,2,3,…is called the conditional probability distribution of Y given $X=x_i$

(ii) Continuous case:

- The conditional probability density function of X is given by $Y=y_j$ is defined as

$f(x/y) = \frac{f(x,y)}{h(y)}$, where h(y) is a marginal probability density function of Y.

- The conditional probability density function of Y is given by $X = x_i$ is defined as

$f(y/x) = \frac{f(x,y)}{g(x)}$, where g(x) is a marginal probability density function of X.

Independent random variables:

(i) Discrete case:

Two random variable (x,y) are said to be independent if $P(X = x_i \cap Y = y_j) = P(X = x_i)(Y = y_j)$ (ie) $P_{ij} = P_i^* P_{*j}$ for all i ,j.

(ii) Continuous case:

Two random variables (x,y) are said to be independent if $f(x,y) = g(x)h(y)$, where $f(x,y) =$ joint probability density function of x and y,

$g(x) =$ Marginal density function of x,

$h(y) =$ Marginal density function of y.

Marginal Distribution Tables:

Table – I

To calculate marginal distribution when the random variables X takes horizontal values and Y takes vertical values

Y/X	x1	x2	x3	p (y) = p(Y=y)
y1	p11	p21	p31	p(Y=y1)
y2	p12	p22	p32	p(Y=y2)
y3	p13	p23	p33	p(Y=y3)
$P_X(X) = P(x = x)$	$P(x = x1)$	$p(x = x2)$	$p(x = x3)$	

Table – II

To calculate marginal distribution when the random variables X takes vertical values and Y takes horizontal values

Y\X	y1	y2	y3	P $_X$ (x) = P(X=x)
x1	p11	p21	p31	p(X=x1)
x2	p12	p22	p32	p(X=x2)
x3	p13	p23	p33	p(X=x3)
$p(y) = p(y = y)$	$P(y = y1)$	$P(y = y2)$	$P(y = y3)$	

Solved Problems on Marginal Distribution:

Example :11

From the following joint distribution of X and Y find the marginal distribution

X/Y	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

Solution:

The marginal distribution are given in the table below

Y\X	0	1	2	$P_Y(y)$ = P(Y=y)
0	3/28	9/28	3/28	15/28

1	3/14	3/ 14	0	6/14
2	1/28	0	0	1/28
$P_X(x) = P(Y = y)$	$P_X(0) =$ 5/14	$P_X(1) =$ 15/28	$P_X(2) =$ 3/28	1

The marginal Distribution of X

$P_X(0) = P(X = 0) = p(0,0) + p(0,1) + p(0,2)$ = 3/28 + 3/14 + 1/28 = 5/14

$P_X(1) = P(X = 1) = p(1,0) + p(1,1) + p(1,2)$ = 9/28 + 3/14 + 0 = 15/28

$P_X(2) = P(X = 2) = p(2,0) + p(2,1) + p(2,2)$ = 3/28 + 0 + 0 = 3/28

Marginal probability function of X is $P_x(x) = \begin{cases} 5/14, & x = 0 \\ 15/28, & x = 1 \\ 3/28, & x = 2 \end{cases}$

The marginal distributions are

Y/X	1	2	3	$P_Y(y) = p(y = y)$
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
$P_x(x) = P(x = x)$	5/21	7/21	9/21	1

The marginal distribution of X

$P_x(1) = p(1,1) + (1,2)$ = 2/21 + 3/21

$\boxed{P_x(1) = 5/21}$

$P_x(2) = p(2,1) + (2,2)$ = 3/21 + 4/21

$\boxed{P_x(2) = 7/21}$

$P_x(3) = p(3,1) + p(3,2)$ = 4/21 + 5/21

$\boxed{P_x(3) = 9/21}$

Marginal probability function of X is, $P_x(x) = \begin{cases} 5/21, & x = 1 \\ 7/21, & x = 2 \\ 9/21, & x = 3 \end{cases}$

The marginal distribution of Y

$P_Y(1) = p(1,1) + p(2,1) + p(3,1)$ = 2/21 + 3/21 +4/21

$P_Y(1) = 9/21$

$P_Y(2) = p(1,2) + p(2,2) + p(3,2)$ = 3/21 + 4/21 +5/21

$P_Y(2) = 12/21$

Marginal probability function of Y is $P_Y(y) = \begin{cases} 3/21, & y = 1 \\ 4/21, & y = 2 \\ & \end{cases}$

Example :12

From the following table for joint distribution of (X, Y) find

i) $P(X \leq 1)$ ii) $P(Y \leq 3)$ iii) $P(X \leq 1, Y \leq 3)$ iv) $P(X \leq 1 / Y \leq 3)$

v) $P(Y \leq 3 / X \leq 1)$ vi) $P(X + Y \leq 4)$.

X/Y	0	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32

1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Solution:

The marginal distributions are

X / Y	1	2	3	4	5	6	$P_X(x) = P(X = x)$
0	0	0	1/32	2/32	2/32	3/32	8/32 P(x=0)
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16 P(x=1)
2	1/32	1/32	1/64	1/64	0	2/64	8/64 P(x=2)
$P_Y(y) = P(Y = y)$	3/32	3/32	11/64	13/64	6/32	16/64	1
	P(Y=1)	P(Y=2)	P(Y=3)	P(Y=4)	P(Y=5)	P(Y=6)	

i) $P(X \le 1)$

$P(X \le 1) = P(X = 0) + P(X = 1)$

= 8/32 + 10/16

$P(X \le 1)$ = 28/32

ii) $P(Y \le 3)$

$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$

= 3/32 + 3/32 + 11/64

$P(Y \le 3)$ = 23/64

iii) $P(X \le 1, Y \le 3)$

$P(X \le 1, Y \le 3)$ = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)

= 0+0+1/32 + 1/16 + 1/16 + 1/8

$P(X \le 1, Y \le 3)$ = 9/32

iv) $P(X \le 1 / Y \le 3)$

By using definition of conditional probability

$$P[x = x_i / y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]}$$

The marginal distribution of Y

$P_Y(0) = P(Y = 0) = p(0,0) + p(1,0) + p(2,0)$ = 3/28+9/28+3/28 = 15/28

$P_y(1) = P(y = 1) = p(0,1) + p(1,1) + p(2,1)$ = 3/14+3/14+0= 3/7

$P_y(2) = P(y = 2) = p(0,2) + p(1,2) + p(2,2)$ = 1/28+0+0 = 1/2

Marginal probability function of Y is $P_y(Y) = \begin{cases} 15/28, & y = 0 \\ 3/7, & y = 1 \\ 1/28, & y = 2 \end{cases}$

Example 13:

The joint distribution of X and Y is given by f(X, Y) =X+Y/21, x=1,2,3 y=1,2.Find the marginal distributions.

Solution:

Given f (X, Y) =X+Y/21, x=1, 2, 3 y=1,2

f (1,1) = 1+1/21 =2/21 =P(1,1)
f (1,2) = 1+2/21 =3/21 =P(1,2)
f (2,1) = 2+1/21 =3/21 =P(2,1)
f (2,2) = 2+2/21 =4/21 =P(2,2)
f (3,1) = 3+1/21 =4/21 =P(3,1)
f (3,2) = 3+2/21 =5/21 =P(3,2)

$$P[X \le 1 / Y \le 3] = \frac{P[X \le 1, Y \le 3]}{P[Y \le 3]} = \frac{9/23}{23/64}$$

$P[X \le 1 / Y \le 3] =$ 18/32

v) $P[Y \le 3 / X \le 1]$

$$P[Y \le 3 / X \le 1] = \frac{P[X \le 3, Y \le 1]}{P[Y \le 1]} = \frac{9/23}{7/8}$$

P[Y≤3/ X≤1]= 9/28

vi) $P(X + Y \le 4)$

P(X + Y≤4)) = P(0,1)+P(0,2)+P(0,3)+ P(0,4)+ P(1,1)+ P(1,2)+P(1,3)+P(2,1)+ P(2,2)
= 0+0+1/32 +2/32 + 1/16 + 1/16 + 1/8+1/32 +1/32

P(X + Y≤4) = 13/32

Example : 14

If the joint P.D.F of (X,Y) is given by p(X,Y)=K(2x+3y),x=0,1,2, y=1,2,3,. Find all the marginal probability distribution .Also find the probability of (X+Y) and P(X+Y >3).

Solution:

Given P(X,Y)= K(2x+3y)

P(0,1)= K(0+3) = 3K
P(0,2)= K(0+6) = 6K
P(0,3)= K(0+9) = 9K
P(1,1)= K(2+3) = 5K
P(1,2)= K(2+6) = 8K
P(1,3)= K(2+9) = 11K
P(2,1)= K(4+3) = 7K
P(2,2)= K(4+6) = 10K
P(2,3)= K(4+9) = 13K

To find K:

The marginal distribution is given in the table.

Y\X	0	1	2	$P_Y(y) = P(Y = y)$
1	3K	5K	7K	15K
2	6K	8K	10K	24K
3	9K	11K	13K	33K
PX(x)=P(X=x)	18K	24K	30K	72K

Total Probability =1

72K = 1

K = 1/72

Marginal probability of X & Y:

Substituting K = 1/72 in the above table, we get

Y\X	0	1	2	P_Y(y)=P(Y=y)
1	3/72	5/72	7/72	5/24
2	6/72	8/72	10/72	1/3
3	9/72	11/72	13/72	11/24
P_X (x)=P(X=x)	1/4	11/72	5/12	1

From table, $P_x(0)=1/4$, $p_x(1)=1/3$, $p_x(2)=5/12$

Marginal probability function of x is , $P_x(X)=\begin{cases}1/4, x=0\\ 1/3, x=1\\ 5/2, x=2\end{cases}$

From table, $p_y(1)=5/24$, $P_y(2)=1/3$, $P_Y(3)=11/24$

Marginal Probability function of Y is , $P_Y(Y)=\begin{cases}5/24, Y=1\\ 11/24, y=2\\ \end{cases}$

Example :15

From the following table for joint distribution of (X, Y) find
The marginal distributions are

Y/X	1	2	3	$P_Y(y)=P(Y=y)$
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
$P_X(x)=P(X=x)$	5/21	7/21	9/21	1

The marginal distribution of X

P $_X$ (1) = P(1,1)+P(1,2) = 2/21 + 3/21= P x (3)=9/21

P $_X$ (2) = P(2,1)+P(2,2) = 3/21 + 4/21= Py(2)=7/21

P $_X$ (3) = P(3,1)+P(3,2) = 4/21 + 5/21= P $_X$ (3)=9/21

Marginal probability function of X is P $_x$ (x) = $\begin{cases}5/21, x=1\\ 7/21, x=2\\ 9/21, x=3\end{cases}$

The marginal distribution of Y

P $_y$ (1) = P (1, 1) + P (2, 1) +P (3, 1)

= 2/21 + 3/21 +4/21= 9/21

$P_y(2)$ = P (1, 2) + P (2, 2) +P (3, 2)

= 3/21 + 4/21 +5/21= 12/21

Marginal probability function of Y is $P_Y(y)=\begin{cases}3/21, y=1\\ 4/21, y=2\end{cases}$

Exercises:

1. Given is the joint distribution of X and Y

Y/X	0	1	2

0	0.02	0.08	0.10
1	0.05	0.20	0.25
2	0.03	0.12	0.15

Obtain 1) Marginal Distribution.
2) The conditional distribution of X given Y =0.

2. The joint probability mass function of X & Y is

X/Y	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the M.D.F of X and Y. Also $(X \leq 1, Y \leq 1)$ and check if X & Y are independent.

3. Let X and Y have the following joint probability distribution

Y/X	2	4
1	0.10	0.15
3	0.20	0.30
5	0.10	0.15

Show that X and Y are independent.

4. The joint probability distribution of X and Y is given by the following table.

X/Y	1	3	9
2	1/8	1/24	1/12
4	¼	¼	0
6	1/8	1/24	1/12.

i) Find the probability distribution of Y.
ii) Find the conditional distribution of Y given X=2.
ii) Are X and Y are independent.

5. Given the following distribution of X and Y. Find
i) Marginal distribution of X and Y.
ii) The conditional distribution of X given Y=2.

X/Y	-1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

Example : 16

If the joint probability density function of (X, Y) is given by $f(x,y)=2,\quad 0\le x\le y\le 1.$ Find marginal density function of X.

Solution:

Given $f(x,y)=2,\ 0\le x\le y\le 1$

To find marginal density function of x:

$$g(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{x}^{1} 2dy=2[1-x],\ 0\le x\le y.$$

Example:17

If the joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0\ , & \text{otherwise}\end{cases}$$

Find (i) $P(X<1\cap Y<3)$ (ii) $P\left(X<1/Y<3\right)$ (iii) $f\left(\frac{y}{x}\right)$.

Solution:

Given $f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0\ , & \text{otherwise}\end{cases}$

i) **To find** $P(X<1\cap Y<3)$:

$$P(X<1\cap Y<3)=\int_0^1\int_2^3 f(x,y)dydx$$

$$=\frac{1}{8}\int_0^1\int_2^3 (6-x-y)dydx$$

$$=\frac{3}{8}$$

ii) **To find** $P\left(X<1/Y<3\right)$

$$P\left(X<1/Y<3\right)=\frac{P(X<1\cap Y<3)}{P(Y<3)}\quad \text{..................(1)}$$

To find $P(Y<3)$:

$$P(Y<3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx$$

$$=\int_0^2\int_2^3 \frac{1}{8}(6-x-y)dydx$$

$$=\frac{5}{8}$$

Equation (1) becomes $P\left(X<1/Y<3\right)=\frac{3}{5}$

iii) **To find** $f(y/x)$:

We know that $f(y/x)=\dfrac{f(x,y)}{f_x(x)}$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{8}\int_2^4 (6-x-y)dy$$

$$= \tfrac{1}{4}(3-x), 0 < x < 2.$$

$$f(y/x) = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(3-x)} = \frac{6-x-y}{2(3-x)}, \quad 0 < x < 2, \quad 2 < y < 4.$$

Example : 18

If the joint distribution of X and Y is given by

$F(x,y) = (1-e^{-x})(1-e^{-y})$, for $x > 0, y > 0$

= 0 , otherwise

(i) Find the marginal densities of X and Y (ii) Are X and Y independent?

(iii) $P(1 < X < 3, 1 < Y < 2)$

Solution:

Given $F(x,y) = (1-e^{-x})(1-e^{-y})$

$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$

The joint pdf is given by $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y}$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y}(1 - e^{-x} - e^{-y} + e^{-(x+y)})$$

$$= e^{-(x+y)}$$

$$f(x,y) = e^{-(x+y)}, x \geq 0, y \geq 0$$

i) The marginal density function of X is $f(x) = \displaystyle\int_{-\infty}^{\infty} f(x,y)dy$

$$f(x) = \int_0^{\infty} e^{-(x+y)}dy = e^{-x}, x \geq 0$$

The marginal density function of Y is $f(y) = \displaystyle\int_{-\infty}^{\infty} f(x,y)dx$

$$f(y) = \int_0^{\infty} e^{-(x+y)}dx = e^{-y}, y \geq 0$$

ii) Consider $f(x).f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$

ie) X and Y are independent.

iii) $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3).P(1 < Y < 2)$

$$= \int_1^3 f(x)dx . \int_1^2 f(y)dy = \int_1^3 e^{-x}dx \int_1^2 e^{-y}dy$$

$$= \frac{(1-e^2)(1-e)}{e^5}$$

Exercises:

1. The joint p.d.f. of the two dimensional random variable is,

$$f(x,y)=\begin{cases}\dfrac{8xy}{9}, & 1<x<y<2\\ 0, & \text{otherwise}\end{cases}$$

(i) Find the marginal density functions of X and Y.
(ii) Find the conditional density function of Y given X=x.

2. If the joint Probability density function of two dimensional R.V (X,Y) is given by

$$f(x,y)=\begin{cases}x^2+\dfrac{xy}{3}, & 0\le x\le 1,\, 0\le y\le 2\\ 0\ , & \text{otherwise}\end{cases}.$$

Show that X and Y are not independent.

UNIT-3.

CLASSIFICATION OF RANDOM PROCESSES

Definition and examples - first order, second order, strictly stationary, wide-sense stationary and Ergodic processes - Markov processes - Binomial, Poisson and Normal processes - Sine wave process - Random telegraph process.

3.0 Introduction

A random process is conceptually an extension of a random variable.

A random variable is a function of time is called a random process.

New problems in various branches of Engineering and Science, do not fit into the frame work of the classical probability theory. Such problems arouses us to study the processes, that is, phenomena that takes place in time. It is necessary to develop random processes which is a family of random variables that is indexed by a parameter such as time. Many problems that arise in Physics, Chemistry and other fields can be solved by using random processes. In this unit, we give a simple solution to the mathematical problems which use random processes technique.

A comparison of Random variable and random process.

	Random variable	Random process
1.	A function of the possible outcomes of an experiment. i.e., $X(s)$	A function of the possible outcomes of an experiment and also time. i.e., $X(s, t)$.
2.	Outcome is mapped into a number 'x'	Outcomes are mapped into wave form which is a function of time 't'.

3.1 DEFINITION AND EXAMPLES

3.1. (a) Random process

A random process is a collection (or ensemble) of random variables $\{X(s,t)\}$ that are functions of a real variable, namely time 't' where $s \in S$ (Sample space) and $t \in T$ (parameter set or index set)

Examples :

1. The daily stock price.
2. The wireless signal received by a cell phone over time.
3. The image intensity over 1 $c.m^2$ regions.

State space

The set of possible values of any individual member of the random process is called state space.

Any individual member itself is called a **sample function** or **a realisation of the processes.**

Note (i) If 's' and 't' are fixed, $\{X(s,t)\}$ is a number,

(ii) If 't' is fixed, $\{X(s,t)\}$ is a random variable.

(iii) If 's' is fixed. $\{X(s,t)\}$ is a single time function.

(iv) If 's' and 't' are variables, $\{X(s,t)\}$ is a collection of random variables that are time functions.

Notation : As the dependence of a random process on 's' is obvious, 's' will be omitted in the notation of a random process. If the parameter set 'T' is discrete, the random process will be noted by $\{X(n)\}$ or $\{Xn\}$. If the parameter set 'T' is continuous, the process will be denoted by $\{X(t)\}$.

3.1 (b) Classification of process [A.U CBT Dec. 2009]

It is convenient to classify random processes according to the characteristics of t and the random variable $X = X(t)$ at time t. We shall consider only four cases based on t and X having values in the ranges $-\infty < t < \infty$ and $-\infty < x < \infty$

1. Continuous random process
2. Continuous random sequence
3. Discrete random process
4. Discrete random sequence

X(t) \ t	Continuous t	Discrete t
Continuous X(t)	**1. Continous random process** If both X and t are continuous, the random process is called as continuous random process. **Example :** X(t) represents the maximum temperature at a place in the interval (0, t)	**2. Continuous random sequence** If X is continuous and t is discrete, the random process is called as continuous random sequence. **Example :** X_n represents the temperature at the end of the nth hour of a day, in the interval (1, 24).
Discrete X(t)	**3. Discrete random process.** If X is discrete and t is continuous, the random process is called as discrete random process. **Example :** X(t) represents the number of telephone calls received in the interval (0, t) S = {0, 1, 2, 3, ...}	**4. Discrete random sequence.** If both X and t are discrete, then the random process is called as discrete random sequence. **Example :** If X_n represents the outcome of the nth toss of a fair die, then $\{X_n : n \geq 1\}$ is a discrete random sequence. Since T = {1, 2, 3, ...} and S = {1, 2, 3, 4, 5, 6}

We can classify random process in another way also. It can be classified as

1. Deterministic random process
2. Non-deterministic random process

1. Deterministic random process	2. Non-deterministic random process.
A random process is called a deterministic random process, if all the future values can be predicted from past observations.	A random process is called a non-deterministic random process, if future values of any sample function cannot be predicted from past observations.
Example : Consider a random process $X(t) = A \cos(\omega t + \theta)$. This consists of a family of pure sine waves and it is completely specified in terms of the random variables A and θ. Hence, it is a deterministic random process.	**Example :** In the case of dissolving of sugar crystals in coffee, it consists of a family of functions that cannot be described in terms of finite number of parameters. The future sample function cannot be determined from the past sample functions and so it is a non-deterministic random process.

3.1. (c) Statistical (Ensemble) Averages :

(i) Mean $= E[X(t)] = \int_{-\infty}^{\infty} x f(x,t)\, dx$

(ii) Auto correlation function of $[X(t)]$

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$$

(or) $R_{XX}(t, t+\tau) = E[X(t)\, X(t+\tau)]$

where τ = time difference $= t_2 - t_1$

(iii) Auto covariance of $[X(t)]$

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)]\, E[X(t_2)]$$

$$C_{XX}(t, t) = E[X^2(t)] - \left[E[X(t)]\right]^2 \quad [\because t_1 = t_2 = t]$$

$$= \text{Var}\, [X(t)]$$

(iv) Correlation coefficient of $[X(t)]$

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1)\, C_{XX}(t_2, t_2)}}$$

Note : $\rho_{XX}(t, t) = 1$

(v) Cross correlation

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

(or) $R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)]$

(vi) Cross covariance

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - E[X(t_1)]\, E[Y(t_2)]$$

(or) $C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[X(t)]\, E[Y(t+\tau)]$

(vii) Cross correlation coefficient

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sqrt{C_{XY}(t_1, t_1)\, C_{XY}(t_2, t_2)}}$$

Example 3.1.1.

Define a random process. Explain the classification of random process. Give an example to each case. **[A.U. N/D 2003]**

Solution :

See Page No. 3.2 and 3.3.

EXERCISE 3.1

1. What is the difference between an RV and a random process ?
2. What is the difference between random sequence and random processes ?
3. What is a discrete random sequence ? Give an example.
4. What is a continuous random sequence ? Give an example.
5. What is a continuous random process ? Give an example.
6. What do you mean by the mean and variance of a random process?

3.2. FIRST ORDER STRICTLY STATIONARY PROCESSES

3.2. (a) Stationary process (or) Strictly stationary process (or) Strict sense stationary process [SSS processes]

[A.U. N/D. 2004]

A random process $X(t)$ is said to be stationary in the strict sense, if its statistical characteristics do not change with time.

i.e., the random processes $X(t_1)$ and $X(t_2)$ where $t_2 = t_1 + \Delta$ will have all statistical properties the same.

STATIONARY PROCESS

Formula : $E[X(t)] = \text{Constant}$ and $V[X(t)] = \text{Constant}$

3.2. (b) Jointly stationary in the strict sense.

Two real-valued random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the strict sense, if the joint distribution of $X(t)$ and $Y(t)$ are invariant under translation of time.

3.2. (c) First order stationary process

A random process is called stationary to order one, if its first-order density function does not change with a shift in time origin.

In otherwords,

$f_x(x_1 : t_1) = f_x(x_1 : t_1 + \Delta)$ must be true for any t_1 and any real number Δ if $X(t)$ is to be a first order stationary process.

Theorem 1 : A first order stationary random process has a constant mean. (OR) The first order stationary random process X(t) has independent of t.

Let $X(t)$ be a first order stationary random process

$$\Rightarrow f(x, t+\varepsilon) = f(x, t) \quad \dots (1) \text{ where } t, \varepsilon \text{ are arbitrary.}$$

To prove : $E[X(t)] = \text{constant}$
i.e., **To Prove :** $E[X(t+\varepsilon)] = E[X(t)]$

Proof :

$$E[X(t+\varepsilon)] = \int_{-\infty}^{\infty} x \; f(x, t+\varepsilon)\, dx$$

$$= \int_{-\infty}^{\infty} x \; f(x, t)\, dx \quad \text{by (1)}$$

$$= E[X(t)]$$

Hence, $E[X(t)] = \text{constant.}$

Theorem 2 : A first order stationary random process has a constant variance.

Let $X(t)$ be a first order stationary random process.

$$\Rightarrow f(x, t+\varepsilon) = f(x, t) \quad \dots (1) \text{ where } t, \varepsilon \text{ are arbitrary.}$$

> **To prove :** $Var[X(t)] = \text{constant}$
>
> i.e., **To prove :** $Var[X(t+\varepsilon)] = Var[X(t)]$

Proof : $Var[X(t+\varepsilon)] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x, t+\varepsilon)\, dx$ where $\mu = E[X(t)]$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x, t)\, dx \quad \text{by (1)}$$

$$= Var[X(t)]$$

Hence, $Var[X(t)] = \text{constant.}$

Note 1 : First-order densities of a SSS process are independent of time. i.e., $E[X(t)] = $ a constant.

Note 2 : A random process that is not stationary in any sense is called an evolutionary process.

Note 3 : The mean and variance of a first-order stationary process are constants.

Note 4 : If the process is first order stationary, then Mean $= E[X(t)] = $ constant

Note 5 : A second order stationary process is also a first order stationary process.

Note 6 : If $E[X(t)] = $ constant and $R(t_1, t_2) = $ a function of $(t_1 - t_2)$, the random process $X(t)$ need not be a SSS process.

I. Example for First Order Stationary Process

Example 3.2.1

Show that the random process $X(t) = A\sin(\omega t + \phi)$ where A and ω are constants, ϕ is a random variable uniformly distributed in $(0, 2\pi)$ is first order stationary.

Solution : Given : $X(t) = A\sin(\omega t + \phi)$

where 'ϕ' is uniformly distributed in $(0, 2\pi)$

$$\Rightarrow f(\phi) = \frac{1}{2\pi - 0} = \frac{1}{2\pi}, \quad 0 < \phi < 2\pi$$

[$\because$ From the definition of uniform distribution]

> **To prove :** $X(t)$ is first order stationary.
>
> i.e, **To prove :** $E[X(t)] = $ constant

Proof : $E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\phi)\, d\phi$

$$= \int_{0}^{2\pi} A\sin(\omega t + \phi) \frac{1}{2\pi}\, d\phi$$

$$= \frac{A}{2\pi} \int_{0}^{2\pi} \sin(\omega t + \phi)\, d\phi$$

$$= \frac{A}{2\pi} \left[-\cos(\omega t + \phi)\right]_0^{2\pi}$$

$$= \frac{-A}{2\pi} \left[\cos(\omega t + \phi)\right]_0^{2\pi}$$

$$= \frac{-A}{2\pi} \left[\cos(\omega t + 2\pi) - \cos\omega t\right]$$

$$= \frac{-A}{2\pi} \left[\cos(2\pi + \omega t) - \cos\omega t\right]$$

$$= \frac{-A}{2\pi} \left[\cos\omega t - \cos\omega t\right] \quad [\because \cos(2\pi + \theta) = \cos\theta]$$

$$= 0 = \text{a constant}$$

Hence, $X(t)$ is a first order stationary process.

II. Example for SSS Process

Example 3.2.2

Consider the random process X (t) = cos (ω_o t + θ), where θ is uniformly distributed in the interval $-\pi$ to π. Check whether X (t) is stationary or not ? Find the first and second moments of the process.

[A.U. A/M. 2004] [A.U N/D 2010]

Solution: Given : $X(t) = \cos(\omega_0 t + \theta)$, where

'θ' is uniformly distributed in $(-\pi, \pi)$

$$\Rightarrow \quad f(\theta) = \frac{1}{\pi - (-\pi)} = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

[∵ From the definition of uniform distribution]

To prove :	$X(t)$ is a SSS process.
i.e., To prove :	(i) $E[X(t)]$ = constant,
	(ii) Var $[X(t)]$ = constant.

Proof : (i) $E[X(t)] = \int_{-\infty}^{\infty} X(t) f(\theta)\, d\theta$

$$= \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta)\, d\theta$$

$$= \frac{1}{2\pi} \Big[\sin(\omega_0 t + \theta)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} [\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)]$$

$$= \frac{1}{2\pi} [\sin(\pi + \omega_0 t) + \sin(\pi - \omega_0 t)]$$

[∵ $\sin(-\theta) = -\sin\theta$]

$$= \frac{1}{2\pi} [-\sin \omega_0 t + \sin \omega_0 t]$$

[∵ $\sin(\pi + \theta) = -\sin\theta$; $\sin(\pi - \theta) = \sin\theta$]

$$= \frac{1}{2\pi} [0]$$

$$= 0 \quad \text{[First moment]}$$

i.e., $E[X(t)]$ = constant

(ii) $E[X^2(t)] = E[\cos^2(\omega_0 t + \theta)]$

$$= E\left[\frac{1 + \cos[2(\omega_0 t + \theta)]}{2}\right] \quad \left[\text{Formula : } \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right]$$

$$= \frac{1}{2} E[1 + \cos(2\omega_0 t + 2\theta)]$$

$$E[X^2(t)] = \frac{1}{2} E[1] + \frac{1}{2} E[\cos(2\omega_0 t + 2\theta)] \quad \text{... (1)}$$

$$E[1] = \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta$$

$$= \frac{1}{2\pi} \Big[\theta\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} [\pi - (-\pi)]$$

$$= \frac{1}{2\pi} (2\pi)$$

$$= 1$$

$$E[\cos(2\omega_0 t + 2\theta)] = \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta)\frac{1}{2\pi}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\sin(2\omega_0 t + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

[Formula : $\int \cos\theta\, d\theta = \sin\theta$]

$$= \frac{1}{4\pi}\left[\sin(2\omega_0 t + 2\theta)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\sin(2\omega_0 t + 2\pi) - \sin(2\omega_0 t - 2\pi)\right]$$

$$= \frac{1}{4\pi}\left[\sin(2\pi + 2\omega_0 t) + \sin(2\pi - 2\omega_0 t)\right]$$

[$\because \sin(-\theta) = -\sin\theta$]

$$= \frac{1}{4\pi}\left[\sin 2\omega_0 t - \sin 2\omega_0 t\right]$$

[$\because \sin(2\pi + \theta) = \sin\theta$; $\sin(2\pi - \theta) = -\sin\theta$]

$$= \frac{1}{4\pi}[0] = 0$$

$\therefore$ (1) $\Rightarrow E[X^2(t)] = \frac{1}{2}(1) + 0 = \frac{1}{2}$ [Second moment]

$$\boxed{\text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2}$$

$$= \frac{1}{2} - (0)^2$$

$$= \frac{1}{2} = \text{constant}$$

Hence, $X(t)$ is a SSS process.

Example 3.2.3

If the random process X(t) takes the value −1 with probability $\frac{1}{3}$ and takes the value 1 with probability $\frac{2}{3}$, find whether X(t) is a stationary process or not.

Solution : Given :

$X(t) = n$	−1	1
P_n	$\frac{1}{3}$	$\frac{2}{3}$

To prove : $X(t)$ is a SSS process

i.e., To prove : (i) $E[X(t)]$ = Constant
(ii) $\text{Var}[X(t)]$ = Constant

Proof :

(i) $E[X(t)] = \sum_{n=-1}^{1} n P_n$

$$= (-1)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right) = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$$

= a constant.

(ii) $E[X^2(t)] = \sum_{n=-1}^{1} n^2 P_n$

$$= (-1)^2\left(\frac{1}{3}\right) + (1)^2\left(\frac{2}{3}\right) = \frac{1}{3} + \frac{2}{3} = 1$$

$$\text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$$

$$= 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

= a constant

Hence, $X(t)$ is a SSS process.

Example 3.2.4

Show that, if the process $X(t) = a\cos\omega t + b\sin\omega t$ is SSS, where 'a' and 'b' are independent random variables, then they are normal.

Solution : Given : $X(t) = a\cos\omega t + b\sin\omega t$

$E[a] = E[b] = 0$ ... (1) and $E[ab] = E[a]E[b] = 0$ ... (2)

$E[a^2] = E[b^2] = \sigma^2$ ... (3)

To prove : $X(t)$ is a SSS process

i.e., To prove : (i) $E[X(t)] = \text{constant}$

(ii) $Var[X(t)] = \text{constant}$

Proof :

(i) $E[X(t)] = E[a\cos\omega t + b\sin\omega t]$

$= \cos\omega t\; E[a] + \sin\omega t\; E[b]$

$= \cos\omega t\,(0) + \sin\omega t\,(0)$ by (1)

$= 0 = \text{constant}$

(ii) $E[X^2(t)] = E\left[(a\cos\omega t + b\sin\omega t)^2\right]$

$= E\left[a^2\cos^2\omega t + b^2\sin^2\omega t + 2ab\cos\omega t\sin\omega t\right]$

$= E[a^2]\cos^2\omega t + E[b^2]\sin^2\omega t + 2E[ab]\cos\omega t\sin\omega t$

$= \sigma^2\cos^2\omega t + \sigma^2\sin^2\omega t + 0$ by (2) & (3)

$= \sigma^2[\cos^2\omega t + \sin^2\omega t]$

$= \sigma^2(1)$

$= \sigma^2$

$\therefore Var[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$

$= \sigma^2 - 0$

$= \sigma^2 = \text{constant}$

$\therefore$ Hence, $X(t)$ is a SSS process.

III. Example for not SSS process

Example 3.2.5

Consider the random process $X(t) = \cos(t+\phi)$, where ϕ is a random variable with density function $f(\phi) = 1/\pi$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, check whether the process is stationary or not.

[A.U. May 2000]

[A.U CBT M/J 2010, A.U Tvli. A/M 2009] [A.U N/D 2010]

Solution : Given : $X(t) = \cos(t+\phi)$, $f(\phi) = \frac{1}{\pi}$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f(\phi)\, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\frac{1}{\pi}\,d\phi$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(t+\phi)\,d\phi$$

$$= \frac{1}{\pi}\left[\sin(t+\phi)\right]_{\phi=-\pi/2}^{\phi=\pi/2}$$

$$= \frac{1}{\pi}\left[\sin\left(t+\frac{\pi}{2}\right) - \sin\left(t-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) + \sin\left(\frac{\pi}{2}-t\right)\right] \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{\pi}[\cos t + \cos t]$$

$[\because \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta\ ;\ \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta]$

$$= \frac{2}{\pi}\cos t \neq \text{constant.}$$

Hence, $X(t)$ is not a SSS process.

Example 3.2.6

The process $\{X(t)\}$ whose probability distribution under certain conditions is given by,

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, \quad n = 0$$

Show that it is "not stationary" (or evolutionary).

[A.U. M/J 2006] [A.U. N/D 2007] [A.U. A/M 2008]
[A.U Tvli M/J 2010, Trichy A/M 2010, N/D 2010, N/D 2011]
[A.U M/J 2012, N/D 2012]

Solution : The probability distribution of $X(t)$ is

$X(t) = n$	0	1	2	3	...
P_n	$\frac{at}{1+at}$ (P_0)	$\frac{1}{(1+at)^2}$ (P_1)	$\frac{at}{(1+at)^3}$ (P_2)	$\frac{(at)^2}{(1+at)^4}$ (P_3)	... .

(i) $E[X(t)] = \sum_{n=0}^{\infty} n P_n$

$$= (0)(P_0) + (1)(P_1) + (2)(P_2) + (3)(P_3) + \ldots$$

$$= 0 + (1)\frac{1}{(1+at)^2} + (2)\frac{at}{(1+at)^3} + (3)\frac{(at)^2}{(1+at)^4} + \ldots$$

$$= \frac{1}{(1+at)^2}\left[1 + (2)\frac{at}{1+at} + (3)\frac{(at)^2}{(1+at)^2} + \ldots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \ldots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \left(\frac{at}{1+at}\right)\right]^{-2} \quad [\because (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots]$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\frac{1}{(1+at)^{-2}} = \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$$E[X(t)] = \sum_{n=0}^{\infty} n P_n = 1 \quad \ldots (1)$$

i.e., $E[X(t)] = \text{constant}$

(ii) $E[X^2(t)] = \sum_{n=0}^{\infty} n^2 P_n$

$$= \sum_{n=0}^{\infty} [n(n+1) - n] P_n \quad [\because n^2 = n(n+1) - n]$$

$$= \sum_{n=0}^{\infty} n(n+1) P_n - \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n(n+1) P_n - 1 \quad \ldots \text{by (1)}$$

$$= \left[(0)(1)P_0 + (1)(2)P_1 + (2)(3)P_2 + (3)(4)P_3 + \ldots\right] - 1$$

$$= \left[0 + 1.2\frac{1}{(1+at)^2} + 2.3\frac{at}{(1+at)^3} + 3.4\frac{(at)^2}{(1+at)^4} + \ldots\right] - 1$$

$$= \frac{1}{(1+at)^2}\left[1.2 + 2.3\left(\frac{at}{1+at}\right) + 3.4\left(\frac{at}{1+at}\right)^2 + \ldots\right] - 1$$

$$= \frac{1}{(1+at)^2}\left[(2)\left(1 - \frac{at}{1+at}\right)^{-3}\right] - 1$$

$$\left[\because (1-x)^{-3} = \frac{1}{2}[1.2 + 2.3x + 3.4x^2 + \ldots]\right]$$

$$= \frac{1}{(1+at)^2}(2)\left(\frac{1+at-at}{1+at}\right)^{-3} - 1$$

$$= \frac{1}{(1+at)^2}(2)\left(\frac{1}{1+at}\right)^{-3} - 1$$

$$= \frac{1}{(1+at)^2}(2)\left(1+at\right)^{3} - 1$$

$$= 2(1+at) - 1$$

$$= 2 + 2at - 1$$

i.e., $E[X^2(t)] = 1 + 2at$ ... (2)

(iii) $\text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$

$= 1 + 2at - (1)^2$ by (1) & (2)

$= 1 + 2at - 1$

$= 2at$

$\neq$ constant

Here $E[X(t)]$ = constant but Var $[X(t)] \neq$ constant.

$\therefore$ The given process is not a stationary process.

Example 3.2.7

Show that the random process $X(t) = A\cos(\omega_0 t + \theta)$ is not stationary, if A and ω_0 are constants and θ is uniformly distributed random variable in $(0, \pi)$ [AU Dec. 2005, April 2007]

Solution : Given : $X(t) = A\cos(\omega_0 t + \theta)$,

where 'θ' is uniformly distributed in $(0, \pi)$.

$$\Rightarrow f(\theta) = \frac{1}{\pi - 0}, \quad 0 < \theta < \pi$$

$$= \frac{1}{\pi} \qquad [\because \text{From the definition of uniform distribution}]$$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f(\theta)\, d\theta$$

$$= \int_0^{\pi} A\cos(\omega_0 t + \theta)\frac{1}{\pi}\, d\theta$$

$$= \frac{A}{\pi}\int_0^{\pi} \cos(\omega_0 t + \theta)\, d\theta$$

$$= \frac{A}{\pi}\left[\sin(\omega_0 t + \theta)\right]_0^{\pi} \qquad \left[\because \int \cos\theta\, d\theta = \sin\theta\right]$$

$$= \frac{A}{\pi}\left[\sin(\omega_0 t + \pi) - \sin\omega_0 t\right]$$

$$= \frac{A}{\pi}\left[-\sin\omega_0 t - \sin\omega_0 t\right] \qquad [\because \sin(\pi + \theta) = -\sin\theta]$$

$$= \frac{A}{\pi}\left[-2\sin\omega_0 t\right]$$

$$= -\frac{2A}{\pi}\sin\omega_0 t \neq \text{constant.}$$

$\therefore X(t)$ is not a stationary process.

Example 3.2.8

Verify whether the sine wave process X(t), where X (t) = Y cos ω t and Y is uniformly distributed in (0, 1) is a strict sense stationary process.

Solution :

Given : $X(t) = Y \cos \omega t$,

where 'Y' is uniformly distributed in (0, 1).

$$\Rightarrow Y = \frac{1}{1-0} = 1 , \quad 0 < Y < 1$$

[∵ From the definition of uniform distribution]

$$E[X(t)] = \int_{-\infty}^{\infty} X(t)\, Y\, dY$$

$$= \int_0^1 \cos \omega t \, dY$$

$$= \cos \omega t \int_0^1 dY$$

$$= \cos \omega t \, \left[Y\right]_0^1$$

$$= \cos \omega t \, [1-0]$$

$$= \cos \omega t$$

= a function of t

≠ constant

∴ $X(t)$ is not a SSS process.

Example 3.2.9

A random process has sample functions of the form $X(t) = A \cos(\omega t + \theta)$ in which A and ω are constants and θ is a random variable. Prove this process is not stationary, if it is not uniformly distributed over a range of 2π.

Solution : It is given that the random variable θ is not uniformly distributed.

Let the distribution be $f(\theta)$, it is not a constant.

$$E[X(t)] = \int_0^{2\pi} A \cos(\omega t + \theta) f(\theta)\, d\theta \neq \text{constant}$$

This involves a time component and is not constant which indicates that the process is not a stationary process.

EXERCISE 3.2

1. Define a strict-sense stationary process and give an example.
2. Define a k^{th} order stationary process. When will it become a SSS process ?
3. What is the first order stationary process ?
4. Show that the random process $X(t) = 100 \sin(\omega t + \theta)$ is first order stationary, if it is assumed that ω is constant and θ is uniformly distributed in (0, 2 π).
5. Consider the random process $X(t) = A \cos(\omega t + \phi)$ where ω is a random variable with density functions $f(\omega)$ and φ, a random variable uniform in the interval $(-\pi, \pi)$ and independent of ω, prove that $X(t)$ is a first order stationary with zero means.
6. Consider the process $X(t) = 10 \sin(200t + \phi)$ where φ is uniformly distributed in the interval $(-\pi, \pi)$. Check whether the process is stationary or not.
7. Give an example of stationary random process and justify your claim. **[A.U N/D 2005]**

3.3 SECOND-ORDER AND WIDE-SENSE STATIONARY PROCESS

3.3 (a) Second-order stationary process

A process is said to be second order stationary, if the 2nd order density function satisfies.

$$f(x_1, x_2 : t_1, t_2) = f(x_1, x_2 : t_1 + \delta, t_2 + \delta), \forall x_1, x_2 \text{ and } \delta$$

3.3 (b)

SSS	WSS
Strict Sense Stationary Process (or) Strictly stationary process (or) Stationary process	**Wide-Sense Stationary Process** Weak-Sense Stationary process (or) Covariance stationary process
Def. : A random process $X(t)$ is said to be SSS, if its statistical characteristics do not change with time i.e. (i) $E[X(t)]$ = constant (ii) $\text{Var}[X(t)]$ = constant	**Def. :** A random process $X(t)$ is said to be WSS, if it satisfies. (i) $E[X(t)]$ = constant (ii) $R(t_1, t_2)$ = function of time difference. i.e. a function of $(t_1 - t_2)$ Note : $\tau = t_1 - t_2$
Note : Every WSS process need not be a SSS process of order 2.	**Note :** A SSS process of order two is a WSS process but the converse is not true.
Example for SSS : 1. Bernoulli's process is a SSS. 2. Strong sense white noise. 3. Weak sense white noise	**Example for WSS :** 1. A random telegraph signal process is a WSS. 2. Random binary transmission process is a WSS which is not mean-ergodic. 3. Sinusoid with random phase.
Example for not SSS 1. Semi random telegraph signal process 2. Poisson process is not a stationary process.	**Example for not WSS** 1. Poisson process is not a WSS. 2. Random walk is not a WSS.

3.3. (c) N-th order stationary.

The stationary concept can be defined by considering any number of random variables of the process.

Def. Stationary to order N.

In general, a process is stationary to order N, if for N random variables of the process considered at times $t_1, t_2, \ldots t_N$, their N-th order joint density function is invariant with time origin shift.

$f_x(x_1, x_2, \ldots x_N : t_1, t_2, \ldots t_N) = f_x(x_1, x_2, \ldots, x_N : t_1 + \delta, t_2 + \delta, \ldots, t_N + \delta)$

for all $t_1, t_2, \ldots t_N$ and δ.

Theorem 1.

If X(t) is a second-order process, then its second-order probability density function is a function only of time differences.

[OR] If X(t) is a second-order stationary process, the autocorrelation function is a function of time difference.

[OR] Prove that the autocorrelation of a SSS process X(t) is a function of $(t_1 - t_2)$

Proof : Let $X(t)$ be a second-order stationary process.

$\Rightarrow f(x_1, x_2 ; t_1, t_2) = f(x_1, x_2 ; t_1 + \delta, t_2 + \delta)$ for any δ.

Put $\delta = -t_2$, then $f(x_1, x_2 ; t_1, t_2) = f(x_1, x_2 ; t_1 - t_2, 0)$

i.e., $f(x_1, x_2 ; t_1, t_2) = f(x_1, x_2 ; t_1 - t_2)$

We know that the autocorrelation function

$$R(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2 ; t_1, t_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2, t_1 - t_2)\, dx_1\, dx_2$$

$$= R(t_1 - t_2)$$

$\therefore$ Autocorrelation function is a function of time difference.

Note :

1. A second-order stationary process is also a first order stationary.
2. The second-order densities of a SSS process are functions of $\tau = t_1 - t_2$

Theorem 2.

If a random process X(t) is WSS then it must also be covariance stationary. [AU 2003]

Proof : Given : $X(t)$ is WSS

$\Rightarrow$ (i) $E[X(t)] = \mu$ = a constant.

(ii) $R(t_1, t_2)$ = a function of $(t_1 - t_2)$

The autocovariance function is given by

$$C(t_1, t_2) = R(t_1, t_2) - E[X(t_1) X(t_2)]$$

$$= R(t_1 - t_2) - E[X(t_1)]\, E[X(t_2)]$$

$$= R(t_1 - t_2) - (\mu)(\mu)$$

$$= R(t_1 - t_2) - \mu^2$$

which depends only on the time difference.

Hence, $X(t)$ is covariance stationary.

Def: Jointly Wide-Sense stationary processes :

Two random processes $X(t)$ and $Y(t)$ are called jointly wide-sense stationary if

(i) $X(t)$ is a WSS process

(ii) $Y(t)$ is a WSS process

(iii) $R(t_1, t_2) = E[X(t_1) Y(t_2)] = R(t_1 - t_2)$

Theorem 3.

If $\{X(t)\}$ is a WSS process with auto correlation function $R_{xx}(\tau)$ and if $Y(t) = X(t+a) - X(t-a)$. Show that $R_{YY}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau + 2a) - R_{xx}(\tau - 2a)$ [A.U. A/M 2003, 2006]

Solution :

By definition,

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)] \quad \text{-- (1)}$$

Given :

$$Y(t) = X(t+a) - X(t-a) \quad \text{-- (2)}$$

Consider,

$$\text{LHS} = R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)] \quad \text{-- (3)}$$

Substituting (2) in (3), we get

$$R_{YY}(\tau) = E\{[X(t+a) - X(t-a)]\,[X(t+a+\tau) - X(t-a+\tau)]\}$$

$$= E[X(t+a) X(t+a+\tau) - X(t-a) X(t+a+\tau) - X(t+a) X(t+\tau-a) + X(t-a) X(t-a+\tau)]$$

$$= E[X(t+a) X(t+a+\tau)] - E[X(t-a) X(t-a+(2a+\tau))] - E[X(t+a) X(t+a+(\tau-2a))] + E[X(t-a) X(t-a+\tau)]$$

$[\because t+a+\tau = t-a+2a+\tau$

$t-a+\tau = t+a-2a+\tau]$

Let $z = t+a$; $u = t-a$

Then, $R_{YY}(\tau) = E[X(z) X(z+\tau)] - E[X(u) X(u+2a+\tau)]$

$= -E[X(z) X(z+\tau-2a)] + E[X(u) X(u+\tau)]$

$= R_{XX}(\tau) - R_{XX}(2a+\tau) - R_{XX}(\tau-2a) + R_{XX}(\tau)$ (By (1))

$= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$

= R.H.S.

I. Example for Stationary of second order

Example 3.3.1

A random process is described by X (t) = A sin t + B cos t where A and B are independent random variables with zero means and equal variances (or equal S.D). Show that the process is stationary of second order.

Solution : Given : $X(t) = A \sin t + B \cos t$ ... (1)

$E[A] = 0, \quad E[B] = 0$ ... (2)

$E[AB] = E[A]\,E[B]$ [∵ A and B are independent random variables]

$= (0)\,(0)$

i.e., $E[AB] = 0$ ... (3)

$E[A^2] = \sigma^2, \; E[B^2] = \sigma^2$ ... (4)

To prove : $X(t)$ is a stationary of second order

i.e., To prove : (i) $E[X(t)] = \text{constant}$

(ii) $E[X^2(t)] = \text{constant}$.

Proof: (i) $E[X(t)] = E[A \sin t + B \cos t]$

$= \sin t\, E[A] + \cos t\, E[B]$

$= \sin t\,(0) + \cos t\,(0)$ ... by (2)

$= 0 = \text{a constant}$.

(ii) $E[X^2(t)] = E\left[(A \sin t + B \cos t)^2\right]$

$= E[A^2 \sin^2 t + B^2 \cos^2 t + 2AB \sin t \cos t]$

$= \sin^2 t\; E[A^2] + \cos^2 t\, E[B^2] + 2 \sin t \cos t\, E[AB]$

$= \sin^2 t\,(\sigma^2) + \cos^2 t\,(\sigma^2) + 0$ ... by (3) & (4)

$= \sigma^2 [\sin^2 t + \cos^2 t]$

$= \sigma^2 (1) = \sigma^2 = \text{a constant}$

Hence, the process $X(t)$ is stationary of second order.

Example 3.3.2

Consider the random process V (t) = cos (ω t + θ), where θ is a random variable with probability density $V(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta < \pi \\ 0, & \text{otherwise} \end{cases}$

(i) Show that first and second moments of V (t) are independent of time.

(ii) If θ is a constant, will be ensemble mean of V (t) be time-independent?

Solution : Given : $V(t) = \cos(\omega t + \theta), \; V(\theta) = \frac{1}{2\pi}, \; -\pi \le \theta < \pi$

(i) $E[V(t)] = E[\cos(\omega t + \theta)]$

$= \int_{-\pi}^{\pi} \cos(\omega t + \theta)\, \frac{1}{2\pi}\, d\theta$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta$

$= \frac{1}{2\pi} \left[\sin(\omega t + \theta)\right]_{-\pi}^{\pi}$

$= \frac{1}{2\pi} \left[\sin(\omega t + \pi) - \sin(\omega t - \pi)\right]$

$= \frac{1}{2\pi} \left[\sin(\pi + \omega t) + \sin(\pi - \omega t)\right]$ [∵ $\sin(-\theta) = -\sin\theta$]

$= \frac{1}{2\pi} \left[-\sin \omega t + \sin \omega t\right]$ [∵ $\sin(\pi + \theta) = -\sin\theta$, $\sin(\pi - \theta) = \sin\theta$]

$= \frac{1}{2\pi}\,[0] = 0 = \text{a constant}$.

$=$ independent of time.

UNIT - 4

...RRELATION AND SPECTRAL DENSITIES

...to correlation - Cross correlation - Properties - Power ...pectral density - Cross spectral density - Properties, ...iener-Khintchine relation - Relationship between cross ...ower spectrum and cross correlation function

...UTO CORRELATION - PROPERTIES

... Auto correlation

...f the process $\{X(t)\}$ is either wide sense stationary or strict ...stationary then $E\{X(t)\,X(t+\tau)\}$ is a function of τ, denoted ...$_{XX}(\tau)$ or $R(\tau)$ or $R_X(\tau)$. This function $R_{XX}(\tau)$ is called the ...correlation function of the process $\{X(t)\}$.

i.e., $R_{XX}(\tau) = E\{X(t)\,X(t+\tau)\}$

... Properties

...PERTY 1 : The mean square value of the Random process may ...btained from the Auto correlation function. $R_{XX}(\tau)$, by putting ...

...: We know that

$$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$$

$$R_{XX}(0) = E[X(t)\,X(t)]$$

$$= E[X^2(t)]$$

i.e., $R_{XX}(0)$ is the mean square value.

i.e., 2nd moment of the random process.

...PERTY 2 : $R_{XX}(\tau)$ is an even function of τ **[A.U A/M 2011]**

i.e., $R_{XX}(\tau) = R_{XX}(-\tau)$. **[A.U CBT M/J 2010]**

...: We know that $R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$

|||ly $R_{XX}(-\tau) = E[X(t)\,X(t-\tau)]$

Put $t - \tau = P$

$$t = P + \tau$$

$$R_{XX}(-\tau) = E[X(P+\tau)\, X(P)]$$

$$= E[X(P)\, X(P+\tau)]$$

$$= R_{XX}(\tau)$$

PROPERTY 3 : The maximum value of $R_{XX}(\tau)$ is attained at the point $\tau = 0$ i.e., $|R_{XX}(\tau)| \le R_{XX}(0)$

[A.U Trichy N/D 2011]

Proof : Consider $E\{[X(t_1) \pm X(t_2)]^2\} \ge 0$

$$\Rightarrow E[X^2(t_1) + X^2(t_2) \pm 2X(t_1)X(t_2)] \ge 0$$

$$\Rightarrow E[X^2(t_1)] + E[X^2(t_2)] \pm 2E[X(t_1)X(t_2)] \ge 0$$

Since $R_{XX}(0) = E[X^2(t)]$

$$\Rightarrow R_{XX}(0) + R_{XX}(0) \pm 2R_{XX}(t_1, t_2) \ge 0$$

$$\Rightarrow 2R_{XX}(0) \ge 2|R_{XX}(\tau)|$$

$$\Rightarrow R_{XX}(0) \ge |R_{XX}(\tau)|$$

PROPERTY 4 : If a random process X (t) has no periodic components and if X (t) is of non zero mean, then

$$\lim_{|\tau| \to \infty} R_{XX}(\tau) = [E(X)]^2$$

Proof : We have $R_{XX}(\tau) = E[X(t_1) . X(t_2)]$

$$= E(X_1 X_2)$$

where $t_2 = t_1 + \tau$

If there are no periodic components, as $|\tau| \to \infty$, X_1 and X_2 can be considered as independent.

$$\therefore R_{XX}(\tau) = E(X_1) . E(X_2)$$

Since X_1 and X_2 are from the same random process at two different time instants,

$$E(X_1) = E(X_2) = E(X)$$

$$\therefore \lim_{\tau \to \infty} R_{XX}(\tau) = [E(X)]^2$$

If the random variables have zero mean, then the auto correlation function $R_{XX}(t_1, t_2)$ is zero

$$\lim_{|\tau| \to \infty} R_{XX}(t_1, t_2) = \lim_{|\tau| \to \infty} E[X(t_1) . X(t_1+\tau)]$$

$$= E[X(t_1)] . E[X(t_1+\tau)] = 0$$

PROPERTY 5 : If X (t) is periodic, then its auto correlation function is also periodic.

Proof : Consider $R(\tau \pm T_o) = E[X(t) . X(t+\tau \pm T_o)]$

Since X (t) is periodic

$$X(t+\tau \pm T_o) = X(t+\tau)$$

$$\therefore R(\tau \pm T_o) = E[X(t) . X(t+\tau)]$$

$$= R(\tau)$$

Hence, R (τ) is periodic.

PROPERTY 6 : If the random process $Z(t) = X(t) + Y(t)$ where X(t) and Y (t) are random process, then

$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$

Proof : Consider $R_{ZZ}(\tau) = E[Z(t) . Z(t+\tau)]$

$$= E[\{X(t) + Y(t)\}\{X(t+\tau) + Y(t+\tau)\}]$$

$$= E[X(t) . X(t+\tau) + X(t) . Y(t+\tau) + Y(t) . X(t+\tau) + Y(t) . Y(t+\tau)]$$

$$= E[X(t) . X(t+\tau)] + E[X(t) . Y(t+\tau)] + E[Y(t) . X(t+\tau)] + E[Y(t) . Y(t+\tau)]$$

$$= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

Example 4.1.1

State any two properties of an auto correlation function.

[A.U. A/M 2004]

Solution : See property 1, 2 & 3.

Example 4.1.2

Given that the autocorrelation function for a stationary ergodic process with no periodic components is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and variance of the process {X(t)}. [A.U. N/D 2004, N/D 2005]

Solution : Given $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$ **[A.U N/D 2011]**

We know that
$$[\bar{X}]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$$
$$= \lim_{\tau \to \infty} \left[25 + \frac{4}{1+6\tau^2}\right]$$
$$= 25 + \frac{4}{\infty} \qquad \left[\because \frac{1}{\infty} = 0\right]$$
$$= 25 + 0$$
$$= 25$$
$$\bar{X} = 5$$

$\therefore$ Mean of the process $X(t) = E[X(t)] = 5$

By the property of the auto correlation, we have

$$E[X^2(t)] = R_{XX}(0) = 25 + \frac{4}{1+0} = 25 + 4 = 29$$

$$\therefore \text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$$
$$= 29 - (5)^2 = 29 - 25 = 4$$

Example 4.1.3

A Stationary Random process has an Auto correlation function and is given by $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$. Find the mean and variance of the process.

[AU. Dec. 2003]

Solution : Given : $R_{XX}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$

Since the random process is stationary, by Property 4 of the Auto correlation function,

We have
$$\left[E[X]\right]^2 = [\bar{X}]^2 = \lim_{|\tau| \to \infty} R_{XX}(\tau)$$
$$= \lim_{|\tau| \to \infty} \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$$
$$= \lim_{|\tau| \to \infty} \frac{\tau^2\left[25 + \frac{36}{\tau^2}\right]}{\tau^2\left[6.25 + \frac{4}{\tau^2}\right]}$$
$$= \lim_{|\tau| \to \infty} \frac{\left[25 + \frac{36}{\tau^2}\right]}{\left[6.25 + \frac{4}{\tau^2}\right]}$$
$$= \frac{25}{6.25} = 4 \qquad \left[\because \frac{1}{\infty} = 0\right] \qquad \text{... (1)}$$

$$\therefore \text{Mean} = \bar{X} = 2$$

i.e., $E[X(t)] = 2$ ... (2)

$$\text{Variance } \sigma^2 = E\left[X^2(t)\right] - \left[E[X(t)]\right]^2$$

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$$E[X^2(t)] = R_{XX}(0) = \frac{25(0)+36}{6.25(0)+4} = 9 \text{ by property 1} \quad \dots (3)$$

$$(2) \Rightarrow \text{Variance } \sigma^2 = E[X^2] - [E[X]]^2$$

$$= 9 - 4 = 5$$

Example 4.1.4

Find the mean and variance of a stationary process whose auto correlation function is given by $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$. **[AU May 2006]**

Solution : Given : $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}$

We know that $[\overline{X}]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$

$$= \lim_{\tau \to \infty} \left[18 + \frac{2}{6+\tau^2}\right]$$

$$= 18 + \frac{2}{\infty}$$

$$= 18 + 0$$

$$= 18$$

$$\overline{X} = \sqrt{18} = 3\sqrt{2}$$

$\therefore$ Mean of the process $X(t) = E[X(t)] = 3\sqrt{2}$

By the property of the auto correlation, we have

$$E[X^2(t)] = R_{XX}(0) = 18 + \frac{2}{6+0} = 18 + \frac{2}{6} = 18 + \frac{1}{3} = \frac{54+1}{3}$$

$$= \frac{55}{3}$$

$$\therefore \text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$$

$$= \frac{55}{3} - 18 = \frac{55-54}{3} = \frac{1}{3}$$

Example 4.1.5

Find the mean and variance of the stationary process {X(t)}, whose ACF is given by $R(\tau) = 2 + 4e^{-2|\tau|}$ **[A.U N/D 2010]**

Solution : Given $R_{XX}(\tau) = 2 + 4e^{-2|\tau|}$

We know that $\left[\overline{X}\right]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$

$$= \lim_{\tau \to \infty} 2 + 4e^{-2|\tau|} = 2 + 4e^{-\infty}$$

$$= 2 + 4(0) = 2$$

$$\overline{X} = \sqrt{2}$$

$\therefore$ Mean of the process $X(t) = E[X(t)] = \sqrt{2}$

By the property of the auto correlation, we have

$$E[X^2(t)] = R_{XX}(0) = 2 + 4e^{-0} = 2 + 4 = 6$$

$$\therefore \text{Var}[X(t)] = E[X^2(t)] - \left[E[X(t)]\right]^2$$

$$= 6 - (\sqrt{2})^2$$

$$= 6 - 2$$

$$= 4$$

Example 4.1.6

Find the variance of the stationary process {X(t)} whose ACF is given by $R(\tau) = 16 + \frac{9}{1+6\tau^2}$ **[A.U M/J 2006] [A.U A/M 2010] [A.U A/M 2011] [A.U. M/J 2012]**

Solution : Given : $R_{XX}(\tau) = 16 + \frac{9}{1+6\tau^2}$

We know that $\left[\overline{X}\right]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$

$$= \lim_{\tau \to \infty} \left[16 + \frac{9}{1+6\tau^2}\right]$$

$$= 16 + \frac{9}{\infty} = 16 + 0 = 16$$

$$\overline{X} = 4$$

$\therefore$ Mean of the process $X(t) = E[X(t)] = 4$

By the property of the auto correlation, we have

$$E[X^2(t)] = R_{XX}(0) = 16 + \frac{9}{1+0} = 16 + 9 = 25$$

$$\therefore \text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]]^2$$

$$= 25 - 16 = 9$$

Example 4.1.7

The auto correlation function for a stationary process X (t) is given by $R_{XX}(\tau) = 9 + 2e^{-|\tau|}$. Find the mean of the random variable $Y = \int_0^2 X(t)\,dt$ and variance of X (t). [A.U. A/M 2003]

Solution : Given : $R_{XX}(\tau) = 9 + 2e^{-|\tau|}$

By the property of auto correlation function

$$(\overline{X})^2 = \mu_X^2 = [E[X(t)]]^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$$

$$= \lim_{\tau \to \infty} 9 + 2e^{-|\tau|}$$

$$= 9 + 2e^{-\infty}$$

$$= 9 + 0$$

$$= 9 \qquad [\because e^{-\infty} = 0]$$

$$\therefore \mu_X = E[X(t)] = 3$$

$$E[X^2(t)] = R_{XX}(0) = 9 + 2e^{-|0|} = 9 + 2 = 11$$

Hence $\text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]]^2 = 11 - (3)^2$

$$= 11 - 9 = 2$$

Given $Y(t) = \int_0^2 X(t)\,dt$

Mean of $Y(t) = E[Y(t)] = E\left[\int_0^2 X(t)\,dt\right] \quad [\because Y(t) = \int_0^2 X(t)\,dt]$

$$= \int_0^2 E[X(t)]\,dt = \int_0^2 3dt = 3\int_0^2 dt$$

$$= 3[t]_0^2 = 3(2-0) = 6$$

Example 4.1.8

The random process X (t) is stationary with E [X (t)] = 1 and $R(\tau) = 1 + e^{-2|\tau|}$. Find the mean and variance of $S = \int_0^1 X(t).dt$

Solution :

Consider $E[S] = E\left[\int_0^1 X(t).dt\right]$

$$= \int_0^1 E[X(t)].dt = 1 \qquad [\because E[X(t)] = 1]$$

We have $E[S^2] = \int_{-1}^{1} (1 - |\tau|)(1 + e^{-2|\tau|})\,d\tau$

$$= \int_{-1}^{0} (1+\tau)(1+e^{2\tau})\,d\tau + \int_0^1 (1-\tau)(1+e^{-2\tau})\,d\tau$$

$$= 1.5 + \frac{e^{-2}}{2}$$

$$\text{Var}(S) = E(S^2) - [E(S)]^2$$

$$= 1.5 + \frac{e^{-2}}{2} - 1$$

$$= \frac{1}{2}(1 + e^{-2})$$

Example 4.1.9

A stationary random process X (*t*) with mean 2 has the auto correlation function $R_{XX}(\tau) = 4 + e^{-\frac{|\tau|}{10}}$. Find the mean and variance of $Y = \int_0^1 X(t)\,.\,dt$.

[A.U M/J 2012]

Solution :

Let Z (*t*) be a WSS random process. Consider a random variable

$$\omega = \int_K^{K+T} Z(t)\,.\,dt \qquad \text{where } T > 0$$

Then $\quad E(\omega^2) = \int_{-T}^{T} (T - |\tau|)\, R_{ZZ}(\tau)\,.\,d\tau$

Consider $\quad E[Y] = E\left[\int_0^1 X(t)\,.\,dt\right]$

$$= \int_0^1 E[X(t)]\,.\,dt$$

$$= 2\,.\int_0^1 dt = 2$$

Comparing ω and Y, we have K = 0 and T = 1

$$\therefore E(Y^2) = \int_{-1}^{1} (1 - |\tau|)\, R_{XX}(\tau)\,.\,d\tau$$

$$= \int_{-1}^{1} (1 - |\tau|)\left(4 + e^{-\frac{|\tau|}{10}}\right) d\tau$$

$$= \int_{-1}^{1} (1 + \tau)\left(4 + e^{\frac{\tau}{10}}\right).\,d\tau + \int_0^1 (1 - \tau)\left(4 + e^{-\frac{\tau}{10}}\right) d\tau$$

$$= 200\, e^{-0.1} - 176$$

Consider $\quad Var(Y) = E(Y^2) - [E(Y)]^2$

$$= 200\,.\,e^{-0.1} - 176 - 4$$

$$= 200\,.\,e^{-0.1} - 180$$

$$= 20\,(10\,.\,e^{-0.1} - 9)$$

Example 4.1.10

If X (t) is a random process with mean '3' and auto correlation of $9 + 4\,.\,e^{-0.2\,\tau}$, find the mean and variance of the random variable Z = X (5).

Solution : $\quad E(Z) = E[X(5)]$

Since the mean of X(*t*) = 3,

$$E[X(5)] = 3$$

$$Var(Z) = E(Z^2) - [E(Z)]^2$$

Consider $\quad E(Z^2) = E[X^2(5)]$

$$= E[X(5)\,.\,X(5)]$$

This is same as Auto correlation of X(*t*) at $t_1 = 5$ and $t_2 = 5$.

$$\therefore E[X(5)\,.\,X(5)] = R_{XX}(\tau)$$

$$= R_{XX}(t_2 - t_1)$$

$$= R_{XX}(0)$$

$$\therefore E(Z^2) = [9 + 4\,.\,e^{-0.2\tau}]_{\tau=0} = 13$$

$$\therefore Var[Z] = 13 - 9 = 4.$$

Example 4.1.11

Let X(t) be a WSS random process with auto correlation $R_{XX}(\tau) = A \cdot e^{-\alpha|\tau|}$. Find the second moment of the random variable Y = X(5) − X(2).

Solution : The second moment of Y is $E(Y^2)$ i.e.,

$$E[|X(5) - X(2)|^2] = E[X^2(5) + X^2(2) - 2X(5)X(2)]$$

$$= E[X^2(5)] + E[X^2(2)] - 2 \cdot E[X(2)X(5)]$$

Since $E[X(t_1) \cdot X(t_2)] = R_{XX}(t_2 - t_1)$

$$\Rightarrow \quad E[X(2) \cdot X(5)] = R_{XX}(3) = A \cdot e^{-3\alpha}$$

$$\therefore E[|X(5) - X(2)|^2] = E[X^2(5)] + E(X^2(2)] - 2A \cdot e^{-3\alpha} \quad \dots (1)$$

We have the mean square value

$$E[X^2(t)] = R_{XX}(0)$$

$$\therefore \quad E[X^2(t)] = A \cdot e^{-\alpha|\tau|}\Big]_{\tau=0} = A$$

$$\Rightarrow \quad E[X^2(5)] = A$$

$$\Rightarrow \quad E[X^2(2)] = A$$

$$\therefore E[|X(5) - X(2)|^2] = E[X^2(5)] + E(X^2(2)] - 2A \cdot e^{-3\alpha}$$

$$= A + A - 2A \cdot e^{-3\alpha}$$

$$= 2A - 2A\, e^{-3\alpha} = 2A(1 - e^{-3\alpha}).$$

Example 4.1.12

If $S = \int_0^{10} X(t)\,dt$. Find also the mean and variance of S, if $E[X(t)] = 8$ and $R_{XX}(\tau) = 64 + 10\,e^{-2|\tau|}$

Solution : Given : $E[X(t)] = 8$

$$E[S] = E\int_0^{10} X(t)\,dt$$

$$= \int_0^{10} E[X(t)]\,dt$$

$$= \int_0^{10} 8\ dt$$

$$= 8\left[t\right]_0^{10}$$

$$= 8[10 - 0]$$

$$= 80$$

$$S = \int_0^{10} X(t)\,dt = 10\,\bar{X}_T$$

where $X(t)$ is defined in (0, 10) so that $T = 10$

$$V(S) = V[10\bar{X}_T] = 100\,V(\bar{X}_T)$$

Formula : If $X(t)$ is defined in $(0, T)$

$$V(\bar{X}_T) = \frac{1}{T}\int_{-T}^{T}\left[1 - \frac{|\tau|}{T}\right] C_{XX}(\tau)\,d\tau$$

$$\therefore V(S) = 100\,\frac{1}{10}\int_{-10}^{10}\left[1 - \frac{|\tau|}{10}\right][C_{XX}(\tau)]\,d\tau$$

$$= 10\int_{-10}^{10}\left[1 - \frac{|\tau|}{10}\right]\left[R_{XX}(\tau) - E[X(t)]^2\right]d\tau$$

$$= 10\int_{-10}^{10}\left[1 - \frac{|\tau|}{10}\right]\left[64 + 10e^{-2|\tau|} - 64\right]d\tau$$

$$= 10\int_{-10}^{10}\left[1 - \frac{|\tau|}{10}\right]10\,e^{-2|\tau|}\,d\tau$$

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$$= (100)(2)\int_0^{10}\left[1-\frac{|\tau|}{10}\right]e^{-2|\tau|}\,d\tau$$

$$= 200\int_0^{10}\left(1-\frac{\tau}{10}\right)e^{-2\tau}\,d\tau$$

$$= 20\int_0^{10}(10-\tau)\,e^{-2\tau}\,d\tau$$

$$= 20\left[(10-\tau)\left[\frac{e^{-2\tau}}{(-2)}\right]-(-1)\left[\frac{e^{-2\tau}}{(-2)^2}\right]\right]_0^{10}$$

$$= 20\left[-\left(\frac{10-\tau}{2}\right)e^{-2\tau}+\frac{e^{-2\tau}}{4}\right]_0^{10}$$

$$= 20\left[\left(-0+\frac{e^{-20}}{4}\right)-\left(-5+\frac{1}{4}\right)\right]$$

$$= 20\left[\frac{e^{-20}}{4}-\left(\frac{-19}{4}\right)\right]$$

$$= \frac{20}{4}\left[e^{-20}+19\right]$$

$$= 5\left[19+e^{-20}\right]$$

Example 4.1.13

Check whether the following are valid autocorrelation function.

(a) $2\sin\pi\tau$ **(b)** $\frac{1}{1+4\tau^2}$

Solution : (a) Given : $R_{XX}(\tau) = 2\sin\pi\tau$

$$R_{XX}(-\tau) = 2\sin\pi(-\tau) = -2\sin\pi\tau$$

As $R_{XX}(\tau) \neq R_{XX}(-\tau)$ this function is invalid.

(b) Given : $R_{XX}(\tau) = \frac{1}{1+4\tau^2}$

$$R_{XX}(-\tau) = \frac{1}{1+4(-\tau)^2} = \frac{1}{1+4\tau^2} = R_{XX}(\tau)$$

$$R_{XX}(0) = 1$$

For any other τ, $R_{XX}(\tau) < R_{XX}(0)$, so the given function is valid.

Example 4.1.14

Check whether the following functions are valid Autocorrelation functions.

(a) $R_{XX}(\tau) = \frac{25\tau^2}{4+5\tau^2}$ **(b)** $R_{XX}(\tau) = \tau^3+\tau^2$

(c) $R_{XX}(\tau) = \cos(\tau) + \frac{|\tau|}{T}$ **[AU Dec. 2005]**

Solution : If $R_{XX}(\tau)$ is an auto correlation function then

$$R_{XX}(\tau) = R_{XX}(-\tau)$$

(a) $R_{XX}(\tau) = \frac{25\tau^2}{4+5\tau^2}$

$$R_{XX}(-\tau) = \frac{25\tau^2}{4+5\tau^2} = R_{XX}(\tau)$$

∴ Given is a valid Autocorrelation function.

(b) $R_{XX}(\tau) = \tau^3+\tau^2$

$$R_{XX}(-\tau) = -\tau^3+\tau^2 \neq R_{XX}(\tau)$$

∴ Given is not a valid Autocorrelation function.

(c) $R_{XX}(\tau) = \cos(\tau) + \frac{|\tau|}{T}$

$$R_{XX}(-\tau) = \cos(-\tau) + \frac{|-\tau|}{T}$$

$$= \cos\tau + \frac{|\tau|}{T} = R_{XX}(\tau)$$

∴ Given is a valid Autocorrelation function.

Example 4.1.15

Find the mean Square value of the random process whose Auto Correlation is $\frac{A^2}{2}\cos\omega\tau$

[AU April 07]

Solution : Given : $R_{XX}(\tau) = \frac{A^2}{2}\cos\omega\tau$

by Property 1 of Auto correlation,

$$\text{Mean Square value} = R_{XX}(0) = E[X^2(t)]$$

$$= \frac{A^2}{2}\cos 0$$

$$= \frac{A^2}{2} \qquad [\because \cos 0 = 1]$$

$$\therefore \text{Mean Square Value} = \frac{A^2}{2}$$

Example 4.1.16

Consider a random process X (t) = A cos ω t, where 'ω' is a constant and A is a random variable, uniformly distributed over (0, 1). Find its auto correlation.

Solution :

Given : A is a r.v. uniformly distributed over (0, 1) ∴ we get

$$f(A) = \begin{cases} \frac{1}{1-0} & in\ (0,1) \\ 0 & otherwise \end{cases} = \begin{cases} 1 & in\ (0,1) \\ 0 & otherwise \end{cases} \quad \text{... (1)}$$

Given : $X(t) = A\cos\omega t$

$X(t+\tau) = A\cos\omega(t+\tau)$

We know that

$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$ by definition

$$= E[(A\cos\omega t)(A\cos\omega(t+\tau)]$$

$$= \cos\omega t + \cos\omega(t+\tau)\,E[A^2]$$

$$= \cos\omega t\cos\omega(t+\tau)\int_0^1 A^2 f(A)\,dA$$

$$= \cos\omega t\cos\omega(t+\tau)\int_0^1 A^2 A \text{ by (1)}$$

$$= \cos\omega t\cos\omega(t+\tau)\left[\frac{A^3}{3}\right]_0^1$$

$$= \cos\omega t\cos\omega(t+\tau)\left[\frac{1}{3}-0\right]$$

$$= \frac{1}{3}\cos\omega t\cos\text{-}(t+\tau)$$

Example 4.1.17

A random process X (t) is defined as

X (t)	= A	for $0 \le t \le 1$
	= 0	otherwise

where A is a random variable that is uniformly distributed from −θ to θ. Prove that the auto correlation function of X (t) is $\frac{\theta^2}{3}$.

Solution : Given : A is a r.v. that is uniformly distributed from −θ to θ, ∴ we get

$$f(A) = \frac{1}{\theta-(-\theta)},\ \text{in } (-\theta,\theta)$$

$$= \frac{1}{2\theta},\ \text{in } (-\theta,\theta)$$

$R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$ by definition

Given $X(t) = A \Rightarrow X(t+\tau) = A$

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$$\therefore \; R_{XX}(\tau) = E[(A)(A)] = E[A^2]$$

$$= \int_{-\theta}^{\theta} A^2 f(A)\, dA$$

$$= \int_{-\theta}^{\theta} A^2 \frac{1}{2\theta}\, dA$$

$$= \frac{1}{2\theta} \int_{-\theta}^{\theta} A^2\, dA$$

$$= \frac{1}{2\theta}\, 2 \int_{0}^{\theta} A^2\, dA \qquad [\because A^2 \text{ is an even function}]$$

$$= \frac{1}{\theta}\left[\frac{A^3}{3}\right]_0^{\theta}$$

$$= \frac{1}{3\theta}\left[A^3\right]_0^{\theta}$$

$$= \frac{1}{3\theta}[\theta^3 - 0]$$

$$= \frac{\theta^2}{3}$$

Example 4.1.18

Find the mean and autocorrelation and auto covariance of the Poisson process. **[A.U. A/M 2003]**

Sol. The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

$$\therefore \; P_n(t) = P\{X(t) = n\} = \frac{-e^{\lambda t}(\lambda t)^n}{\lfloor n}, \; n = 0, 1, 2, \ldots \infty$$

The mean of the Poisson process is given by

$$E[X(t)] = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x}$$

$$= e^{-\lambda t} \cdot \lambda t \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{\lfloor x-1} = e^{-\lambda t} \cdot \lambda t \cdot e^{\lambda t} = \lambda t$$

$$E[X^2(t)] = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} [x(x-1) + x]\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1)\, p(x) + \sum_{x=0}^{\infty} x\, P(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda t}(\lambda t)^x}{\lfloor x} + E(x)$$

$$= e^{-\lambda t} (\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{\lfloor x-2} + \lambda t$$

$$= e^{-\lambda t} (\lambda t)^2 \cdot e^{\lambda t} + \lambda t = (\lambda t)^2 + \lambda t$$

$$\therefore \; \text{Var}(X) = E(X^2) - [E(X)]^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$$

The auto correlation of the Poisson process is given by

$$R_{xx}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$

$$= E[X(t_1)\left[X(t_2) - X(t_1) + X(t_1)\right]]$$

$$= E[X(t_1)(X(t_2) - X(t_1))] + E[X^2(t_1)]$$

$$= E[X(t_1)]\, E[X(t_2 - t_1)] + E[X^2(t_1)],$$

$\because X(t)$ is a process of independent increments.

$$R_{xx}(t_1, t_2) = \lambda t_1 \cdot \lambda(t_2 - t_1) + \lambda^2 t_1^2 + \lambda t_1, \text{ if } t_2 \geq t_1$$

$$= \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^2 t_1^2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 + \lambda \min\{t_1, t_2\}$$

The auto covariance of the Poisson process is given by

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - E[X(t_1)] \cdot E[X(t_2)]$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - (\lambda t_1 . \lambda t_2)$$

$$= \lambda t_1 , \text{ if } t_2 \geq t_1$$

$$= \lambda \min \{t_1 , t_2\}$$

$$r_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{Var\{X(t_1)\}} . \sqrt{Var\{X(t_2)\}}}$$

$$= \frac{\lambda t_1}{\sqrt{\lambda t_1} . \sqrt{\lambda t_2}} = \sqrt{\frac{t_1}{t_2}} ; \text{ if } t_2 \geq t_1$$

Example 4.1.19

If {X (t)} is a WSS process with auto correlation function $R_{XX}(\tau)$ and if Y (t) = X (t + a) − X (t − a), show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau + 2a) - R_{XX}(\tau - 2a)$. [A.U. A/M 2003]

Solution : We know that,

$$R_{XX}(\tau) = E[X(t) X(t+\tau)] \quad \dots (1)$$

Given : $Y(t) = X(t+a) - X(t-a) \quad \dots (2)$

L.H.S $R_{YY}(\tau) = E[Y(t) Y(t+\tau)] \quad \dots (3)$

$$R_{YY}(\tau) = E\{[X(t+a) - X(t-a)][X(t+a+\tau) - X(t-a+\tau)]\}$$

$$= E[X(t+a) X(t+a+\tau) - X(t-a) X(t+a+\tau) - X(t+a) X(t+\tau-a) + X(t-a) X(t-a+\tau)]$$

$$= E[X(t+a) X(t+a+\tau)] - E[X(t-a) X(t-a+(2a+\tau)) - E[X(t+a) X(t+a+(\tau-2a))] + E[X(t-a) X(t-a+\tau)] \quad \dots (4)$$

$[\because t+a+\tau = t-a+2a+\tau , \quad t-a+\tau = t+a-2a+\tau]$

Let $z = t+a , \quad u = t-a$

Then, $R_{YY}(\tau) = E[X(z) X(z+\tau)] - E[X(u) X(u+2a+\tau)]$

$$= -E[X(z) X(z+\tau-2a)] + E[X(u) X(u+\tau)]$$

$$= R_{XX}(\tau) - R_{XX}(2a+\tau) - R_{XX}(\tau-2a) + R_{XX}(\tau) \quad \text{(By (1))}$$

$$= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$$

= R.H.S.

Example 4.1.20

Define auto correlation function and prove that for a WSS process {X (t)}, $R_{XX}(-Z) = R_{XX}(\tau)$. [A.U. A/M 2003]

Solution : See property 2.

Example 4.1.21

If X (t) and Y (t) are independent WSS processes with zero means, find the auto correlation of Z (t) where Z (t) = a . X (t) . Y (t).

Solution :

$$R_{ZZ}(\tau) = E[a . X(t) . Y(t) . a X(t+\tau) Y(t+\tau)]$$

$$= a^2 . E[X(t) . X(t+\tau) . Y(t) . Y(t+\tau)]$$

$$= a^2 . E[X(t) . X(t+\tau)] E.[Y(t) . Y(t+\tau)]$$

$$= a^2 . R_{XX}(\tau) . R_{YY}(\tau).$$

Example 4.1.22

Consider a random process X (t) = B cos (50t + ϕ) where B and ϕ are independent RVS. B is a random variable with mean 0 and variance 1. ϕ is uniformly distributed in the interval $[-\pi, \pi]$. Find mean and auto correlation of the process. [A.U. A/M 2004]

Solution : Given that 'B' is a RV with mean 0 and variance 1.

(i.e.,) $E(B) = 0$ and $\sigma^2(B) = 1$.

Since ϕ is uniformly distributed in $[-\pi, \pi]$ is pdf is given by

$$f(\phi) = \frac{1}{2\pi}, \quad -\pi \leq \phi \leq \pi.$$

Consider $E[X(t)] = E[B\cos(50t+\phi)]$

$= E(B).E[\cos(50t+\phi)]$, $\because$ B and ϕ are independent.

$= 0 \qquad \because E(B) = 0$

$\therefore$ the mean, $E[X(t)] = 0$

The autocorrelation is defined as,

$$R(t_1, t_2) = E[X(t_1).X(t_2)]$$

$$= E[B\cos(50t_1+\phi) . B\cos(50t_2+\phi)]$$

$$= E[B^2\cos(50t_1+\phi) . \cos(50t_2+\phi)]$$

$$= \frac{E(B^2)}{2} E[\cos(50(t_1+t_2)+2\phi) + \cos(50(t_1-t_2)]$$

$$= \frac{1}{2}\{E[\cos(50(t_1+t_2)+2\phi)] + \cos(50(t_1-t_2))\},$$

$$\because \sigma^2(B) = 1$$

$$\Rightarrow E(B^2) - E(B)]^2 = 1$$

$$\Rightarrow E.(B^2) = 1 \because E(B) = 0$$

$$= \frac{1}{2}\left\{\int_{-\pi}^{\pi}\cos(50(t_1+t_2)+2\phi)\frac{1}{2\pi}d\phi + \cos(50(t_1-t_2))\right\}$$

$$= \frac{1}{2}\left[\left(\frac{\sin(50(t_1+t_2)+2\phi)}{2}\right)_{-\pi}^{\pi}\frac{1}{2\pi} + \cos(50(t_1-t_2))\right]$$

$$= \frac{1}{2}\left[\frac{1}{4\pi}\left(\sin(50(t_1+t_2)) - \sin(50(t_1+t_2))\right) + \cos(50(t_1-t_2))\right]$$

$$= \frac{1}{2}\cos(50(t_1-t_2))$$

$\therefore R(t_1, t_2) = \frac{1}{2}\cos(50(t_1-t_2))$ which is a function of $t_1 - t_2$.

...ample 4.1.23.

...d the auto correlation function of the periodic time function ...$\{(t)\} = A\sin\omega t$ [A.U A/M 2010]

...ution : Given $X(t) = A\sin\omega t$

$$X(t+\tau) = A\sin\omega(t+\tau)$$

$$X(t)X(t+\tau) = A\sin\omega t\, A\sin\omega(t+\tau)$$

$$= A^2\sin\omega t\,\sin(\omega t+\omega\tau)$$

$$= A^2\sin(\omega t+\omega\tau)\sin\omega t$$

$$= A^2\frac{1}{2}[\cos(\omega t+\omega\tau-\omega t) - \cos(\omega t+\omega\tau+\omega t)]$$

$$= \frac{A^2}{2}[\cos\omega\tau - \cos(2\omega t+\omega\tau)]$$

...e time autocorrelation function

$$...(t, t+\tau) = \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt$$

$$= \frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}[\cos\omega\tau - \cos(2\omega t+\omega\tau)]\,dt$$

$$= \frac{1}{2T}\frac{A^2}{2}\int_{-T}^{T}\cos\omega\tau\,dt - \frac{A^2}{4T}\int_{-T}^{T}\cos(2\omega t+\omega\tau)\,dt$$

$$= \frac{A^2}{(2T)(2)}\cos\omega\tau\,\left[t\right]_{-T}^{T} - \frac{A^2}{4T}\left[\frac{\sin(2\omega t+\omega\tau)}{2\omega}\right]_{-T}^{T}$$

$$= \frac{A^2}{(2T)(2)}(2T)\cos\omega\tau - \frac{A^2}{8\omega T}\left[\sin(2\omega t+\omega\tau)\right]_{-T}^{T}$$

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$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{8\omega T}\left[\sin(2\omega T+\omega\tau) - \sin(-2\omega T+\omega\tau)\right]$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{8\omega T}\left[2\cos\left[\frac{2\omega T+\omega\tau-2\omega T+\omega\tau}{2}\right]\sin\left[\frac{2\omega T+\omega\tau+2\omega T-\omega\tau}{2}\right]\right]$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{4\omega T}\cos\omega\tau\,\sin 2\omega T$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{4\omega T}\cos\omega\tau\,\sin\left[2\frac{2\pi}{T}T\right] \quad \left[\because \omega = 2\pi f = \frac{2\pi}{T}\right]$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{4\omega T}\cos\omega\tau\,\sin 4\pi$$

$$= \frac{A^2}{2}\cos\omega\tau - 0 \quad [\because \sin 4\pi = 0]$$

$$= \frac{A^2}{2}\cos\omega\tau$$

EXERCISE 4.1

1. Find the mean, variance of the stationary process $\{X(t)\}$, whose autocorrelation function is given by $R(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$

2. Find the variance of the stationary process $\{X(t)\}$, whose autocorrelation is given by $R(\tau) = 2 + 4e^{-2|\tau|}$

3. Find the variance of the stationary process $\{X(t)\}$ whose autocorrelation is given by $R(\tau) = 16 + \frac{9}{1+6\tau^2}$

   [Ans. $V[X(t)] = 9$]

4. If the autocorrelation of a process $\{X(t)\}$ is $R(t_1, t_2)$ and if $Y(t) = X(t+a) - X(a)$, where a is a constant, express $R_{YY}(t_1, t_2)$ in terms of R's.

If $\{X(t)\}$ is a WSS process with $E\{X(t)\} = 2$ and $R_{YY}(z) = 4 + e^{-|z|/10}$, find the mean and variance of $S = \int_0^1 X(t)\,dt$.

[Ans. mean = 2; variance = $20\,(10e^{-0.1} - 9)$]

Check whether the following are valid Autocorrelation functions.
(a) $\cos(\omega_o\tau)$; (b) $\sin(\omega_0\tau)$; (c) $\frac{\tau}{\tau+1}$; (d) $1 - \frac{|\tau|}{T}$; (e) $1 + 3\tau^2$.

[Ans. (a) Yes, (b) No, (c) No, (d) Yes, (e) Yes]

Find the mean and variance of the process $X(t)$ for which $R_{XX}(\tau) = 50e^{-10|\tau|} + 25\cos(5\tau) + 10$

[Ans. ± 3.162, 85.75]

A random process $\{X(t)\}$ has Autocorrelation function $R_{XX}(\tau) = e^{-3|\tau|}\cos\omega\tau + 36$. Another random process $Y(t)$ has Autocorrelation function $R_{YY}(\tau) = 68(\tau) + 2e^{-|a\tau|}$, $X(t)$ and $Y(t)$ are statistically independent of each other. A third random process is formed as $3X(t)\,Y(t)$. Find the mean and variance of each process.

[Ans. 6, 1, 0, 6, 0, 888]

9. If the Autocorrelation function of a stationary process $R_{XX}(\tau) = 36 + \frac{4}{1+3\tau^2}$, find the mean and variance of the process.

   [Ans. 6, 4]

10. Given the Autocorrelation function of a stationary process of $49 + \frac{6}{1+3\tau^2}$. Find the mean and variance.

   [Ans. 7, 6]

4.2 CROSS CORRELATION - PROPERTIES

4.2.(a) Cross correlation

Let $\{X(t)\}$ and $\{Y(t)\}$ be two random processes. Then the cross correlation between them is defined as

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)] = R_{XY}(\tau)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y: t, t+\tau)\, dx\, dy$$

4.2.(b) Properties

PROPERTY 1 : $R_{XY}(\tau) = R_{YX}(-\tau)$

Consider $R_{XY}(t_1, t_2) = E[X(t_1)\,.\,Y(t_2)]$

i.e., $R_{XY}(\tau) = E[X(t)\,.\,Y(t+\tau)]$

Consider $R_{XY}(-\tau) = E[X(t-\tau)\,.\,Y(t)]$

Put $t - \tau = a$

$\Rightarrow \quad t = a + \tau$

$\therefore\ R_{YX}(-\tau) = E[X(a)\,.\,Y(a+\tau)] = R_{XY}(\tau)$

PROPERTY 2 : If X (t) and Y (t) are two random processes and $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are their respective auto correlation functions, then $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,.\,R_{YY}(0)}$ **[A.U N/D 2010]**

[A.U CBT A/M 2011] [A.U N/D 2012]

Consider $E\left\{\left[\dfrac{X(t)}{\sqrt{R_{XX}(0)}} - \dfrac{Y(t)}{\sqrt{R_{YY}(0)}}\right]^2\right\} \ge 0$

$$E\left[\frac{X^2(t)}{R_{XX}(0)} + \frac{Y^2(t)}{R_{YY}(0)} - 2\,.\,\frac{X(t)\,.\,Y(t)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}\right] \ge 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)}\right] + E\left[\frac{Y^2(t)}{R_{YY}(0)}\right] - 2\,.\,E\left[\frac{X(t)\,.\,Y(t)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}\right] \ge 0$$

$$= \frac{1}{R_{XX}(0)}\,.\,E[X^2(t)] + \frac{1}{R_{YY}(0)}\,.\,E[Y^2(t)]$$

$$- 2\,.\,E\left[\frac{X(t)\,.\,Y(t)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}\right] \ge 0$$

Since $R_{XX}(0) = E[X^2(t)]$

and $R_{YY}(0) = E[Y^2(t)]$

$$= 2 - 2\,.\,E\left[\frac{X(t)\,.\,Y(t)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}\right] \ge 0$$

$$\Rightarrow \quad 1 \ge E\left[\frac{X(t)\,.\,Y(t)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}\right]$$

$$\Rightarrow \quad 1 \ge \frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)\,.\,R_{YY}(0)}}$$

$$\Rightarrow \quad |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,.\,R_{YY}(0)}$$

PROPERTY 3 : If X (t) and Y (t) are two random process, then

$$|R_{XY}(\tau)| \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

[A.U Trichy M/J 2011]

Consider that the geometric mean of two positive quantities is less than their arithmetic mean.

i.e., $\sqrt{R_{XX}(0)\,.\,R_{YY}(0)} \le \dfrac{R_{XX}(0) + R_{YY}(0)}{2}$

Since $\quad R_{XY}(\tau) \le \sqrt{R_{XX}(0) + R_{YY}(0)}$

It implies that

$$R_{XY}(\tau) \le \sqrt{R_{XX}(0)\,.\,R_{YY}(0)}$$

PROPERTY 4 : If the random process X (t) and Y (t) are independent then

$$R_{XY}(\tau) = E(X)\,.\,E(Y)$$

$$R_{XY}(\tau) = E[X(t)\,.\,Y(t+\tau)]$$

$$= E[X(t)]\,.\,E[Y(t+\tau)]$$

$$= E(X)\,.\,E(Y)$$

PROPERTY 5 : If the random process X (t) and Y (t) are of zero mean,

$$\lim_{\tau \to \infty} R_{XY}(\tau) = \lim_{\tau \to \infty} R_{YX}(\tau) = 0$$

Consider $\lim_{\tau \to \infty} R_{XY}(\tau) = \lim_{\tau \to \infty} E[X(t) . Y(t+\tau)]$

As $\tau \to \infty$, X (t) and Y (t) can be considered as independent.

$$\lim_{\tau \to \infty} R_{XY}(\tau) = E[X(t)] . E[Y(t+\tau)] = 0$$

PROPERTY 6 : The auto correlation and cross correlation of two random processes X (t) and Y (t) can be expressed as a matrix called correlation matrix as

$$R(t_1, t_2) = \begin{bmatrix} R_{XX}(t_1, t_2) & R_{XY}(t_1 t_2) \\ R_{YX}(t_1, t_2) & R_{YY}(t_1, t_2) \end{bmatrix}$$

Let $(t_2 - t_1) = \tau$

If the above matrix is written as

$$R(\tau) = \begin{bmatrix} R_{XX}(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_{YY}(\tau) \end{bmatrix}$$

then the random process X (t) and Y (t) are jointly widesense stationary processes.

PROPERTY 7 : Two random processes X (t) and Y (t) are said to be uncorrelated if their cross correlation function is equal to the product of their means.

$$R_{XY}(\tau) = E[X(t) . Y(t+\tau)]$$

$$= E[X(t)] . E[Y(t+\tau)]$$

i.e., X (t) and Y (t) are independent.

X(t) and Y(t) are said to be incoherent or orthogonal if $R_{XY}(\tau) = 0$.

Incoherent or orthogonal processes are uncorrelated processes with E(X) and / or E(Y) = 0.

4.2.(c) Cross covariance and correlation coefficient

The covariance between the random variables X (t_1) and X (t_2) where these are from the same random process X (t)] is called auto covariance, $C_{XX}(t_1, t_2)$.

The corresponding correlation coefficient is

$$\rho_{XX}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1) . C_{XX}(t_2, t_2)}} \quad (1)$$

We have $C_{XX}(t_1, t_2) = E[\{X(t_1) - E[X(t_1)]\}\{X(t_2) - E[X(t_2)]\}]$

$$= R_{XX}(t_1, t_2) - E[X(t_1)] . E[X(t_2)] \quad (2)$$

Let $t_1 = t_2 = t$.

$$C_{XX}(t, t) = R_{XX}(t, t) - [E(X)]^2$$

$$= R_{XX}(0) - [E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

$$= Var[X(t)]$$

So, the auto covariance of two random variables observed at the same time instant represents the variance of X (t).

If $t_1 = t_2$, then

$$\rho = \frac{C_{XX}(t_1, t_1)}{\sqrt{C_{XX}(t_1, t_1) . C_{XX}(t_1, t_1)}} = 1$$

The covariance between two random variables X(t_1) and Y(t_2) [which are from different random processes X(t) and Y(t)] is called cross covariance of X(t) and Y(t).

$$\therefore C_{XY}(t_1, t_2) = E[\{X(t_1) - E[X(t_1)]\}\{Y(t_2) - E[Y(t_2)]\}]$$

$$= E[X(t_1) . Y(t_2) - X(t_1) . E[Y(t_2)] - E[X(t_1)] . Y(t_2) + E[X(t_1)] . E[Y(t_2)]]$$

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$$= E[X(t_1) \cdot Y(t_2)] - E[X(t_1)] \cdot E[Y(t_2)] - E[X(t_1)] \cdot E[Y(t_2)] + E[X(t_1)] \cdot E[Y(t_2)]$$

$$= E[X(t_1) \cdot Y(t_2)] - E[X(t_1)] \cdot E[Y(t_2)]$$

$$= R_{XY}(t_1, t_2) - E[X(t_1)] \cdot E[Y(t_2)]$$

The cross correlation coefficient of the random processes X(t) and Y(t) is $\rho_{XY}(t_1, t_2) = \dfrac{C_{XY}(t_1, t_2)}{\sqrt{C_{XX}(t_1, t_1) \cdot C_{YY}(t_2, t_2)}}$

Example 4.2.1

Consider two random process $X(t) = 3\cos(\omega t + \theta)$ and $Y(t) = 2\cos\left(\omega t + \theta - \dfrac{\pi}{2}\right)$ where θ is a random variable uniformly distributed in $(0, 2\pi)$ prove that $\sqrt{R_{XX}(0)\, R_{YY}(0)} \geq |R_{XY}(\tau)|$.

[A.U A/M 2003]

Solution : Given θ is uniformly distributed in $(0, 2\pi)$

$$\therefore f(\theta) = \frac{1}{2\pi}, \quad 0 < x < 2\pi$$

Given $X(t) = 3\cos(\omega t + \theta)$ and $Y(t) = 2\cos(\omega t + \theta - \pi/2)$

$$X(t+\tau) = 3\cos[\omega(t+\tau) + \theta] = 3\cos(\omega t + \omega\tau + \theta)$$

$$Y(t+\tau) = 2\cos[\omega(t+\tau) + \theta - \pi/2]$$

$$= 2\cos(\omega t + \omega\tau + \theta - \pi/2)$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \quad \ldots (1)$$

$$X(t)X(t+\tau) = 3\cos(\omega t + \theta)\, 3\cos(\omega t + \omega\tau + \theta)$$

$$= 9\left[\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\right]$$

$$= \frac{9}{2}\left[2\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta)\right]$$

$$= \frac{9}{2}\left[\cos(\omega t + \theta + \omega t + \omega\tau + \theta) + \cos(\omega t + \theta - \omega t - \omega\tau - \theta)\right]$$

$$[2\cos A\cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{9}{2}[\cos(2\omega t + 2\theta + \omega\tau) + \cos(-\omega\tau)]$$

$$= \frac{9}{2}\left[\cos(2\omega t + 2\theta + \omega\tau) + \cos\omega\tau\right] [\because \cos(-\theta) = \cos\theta]$$

$$(1) \Rightarrow R_{XX}(t, t+\tau) = E\left[\frac{9}{2}[\cos(2\omega t + 2\theta + \omega\tau) + \cos\omega\tau]\right]$$

$$= \frac{9}{2}E[\cos(2\omega t + 2\theta + \omega\tau)] + \frac{9}{2}\cos\omega\tau \quad (2)$$

$$= \frac{9}{2}\int_0^{2\pi}\cos(2\omega t + \omega\tau + 2\theta) \cdot \frac{1}{2\pi}\,d\theta + \frac{9}{2}\cos\omega\tau.$$

$$= \frac{9}{4\pi}\left[\frac{1}{2}\sin(2\omega t + 2\theta + \omega\tau)\right]_0^{2\pi} + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{8\pi}[\sin(2\omega t + \omega\tau) - \sin(2\omega t + \omega\tau)] + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{2}\cos\omega\tau$$

$$\therefore R_{XX}(\tau) = \frac{9}{2}\cos\omega\tau$$

$$\therefore R_{XX}(0) = \frac{9}{2}$$

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)] \quad \ldots (2)$$

$$Y(t)Y(t+\tau) = 2\cos\left(\omega t + \theta - \frac{\pi}{2}\right) 2\cos\left(\omega t + \omega\tau + \theta - \frac{\pi}{2}\right)$$

$$= 2\left[2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)\cos\left(\omega t + \omega\tau + \theta - \frac{\pi}{2}\right)\right]$$

$$= 2\left[2\cos\left(\omega t+\theta-\frac{\pi}{2}+\omega t+\omega\tau+\theta-\frac{\pi}{2}\right)\right.$$
$$\left.+\cos\left(\omega t+\theta-\frac{\pi}{2}-\omega t-\omega\tau-\theta+\frac{\pi}{2}\right)\right]$$
$$= 2\left[\cos(2\omega t+2\theta+\omega\tau-\pi)+\cos(-\omega\tau)\right]$$
$$= 2\cos\left[\pi-(2\omega t+2\theta+\omega\tau)\right]+2\cos(\omega\tau)$$
$$[\because \cos(-\theta)=\cos\theta]$$
$$= -2\cos(2\omega t+2\theta+\omega\tau)+2\cos(\omega\tau)$$

$$(2) \Rightarrow R_{YY}(t,t+\tau) = -2E\left[\cos(2\omega t+2\theta+\omega\tau)\right]+2E(\cos\omega\tau)$$
$$= -2\int_0^{2\pi}\cos(2\omega t+2\theta+\omega\tau)\frac{1}{2\pi}\,d\theta+2\cos\omega\tau$$
$$= -\frac{2}{2\pi}\left[\frac{1}{2}\sin(2\omega t+2\theta+\omega\tau)\right]_0^{2\pi}+2\cos\omega\tau$$
$$= -\frac{1}{2\pi}\left[\sin(2\omega t+4\pi+\omega\tau)-\sin(2\omega t+\omega\tau)\right]+2\cos\omega\tau$$
$$= -\frac{1}{2\pi}\left[\sin(2\omega t+\omega\tau)-\sin(2\omega t+\omega\tau)\right]+2\cos\omega\tau$$
$$= 2\cos\omega\tau$$

$$R_{YY}(0) = 2$$

$$R_{XY}(t,t+\tau) = E\left[X(t)\,Y(t+\tau)\right] \quad \text{... (3)}$$

$$X(t)\,Y(t+\tau) = 3\cos(\omega t+\theta)\;2\cos(\omega t+\omega\tau+\theta-\pi/2)$$
$$= 3\left[2\cos(\omega t+\theta)\cos\left(\omega t+\omega\tau+\theta-\frac{\pi}{2}\right)\right]$$
$$= 3\left[2\sin(\omega t+\omega\tau+\theta)\cos(\omega t+\theta)\right]$$
$$\left[\because \cos\left(\theta-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta\right]$$
$$= 3\left[\sin(\omega t+\omega\tau+\theta+\omega t+\theta)+\sin(\omega t+\omega\tau+\theta-\omega t-\theta)\right]$$
$$= 3\left[\sin(2\omega t+2\theta+\omega\tau)+\sin\omega\tau\right]$$

$$(3) \Rightarrow R_{XY}(t,t+\tau) = E\left[3\left[\sin(2\omega t+2\theta+\omega\tau)+\sin\omega\tau\right]\right]$$
$$= 3E\left[\sin(2\omega t+2\theta+\omega\tau)+3E(\sin\omega\tau)\right]$$
$$= 3\int_0^{2\pi}\sin(2\omega t+2\theta+\omega\tau)\frac{1}{2\pi}\,d\theta+3\sin\omega\tau$$
$$= \frac{3}{2\pi}\int_0^{2\pi}\sin(2\omega t+2\theta+\omega\tau)\,d\theta+3\sin\omega\tau$$
$$= \frac{3}{2\pi}\left[\frac{-\cos(2\omega t+2\theta+\omega\tau)}{2}\right]_0^{2\pi}+3\sin\omega\tau$$
$$= \frac{-3}{4\pi}\left[\cos(2\omega t+2\theta+\omega\tau)\right]_0^{2\pi}+3\sin\omega\tau$$
$$= \frac{-3}{4\pi}\left[\cos(2\omega t+4\pi+\omega\tau)-\cos(2\omega t+\omega\tau)\right]+3\sin\omega\tau$$
$$= \frac{-3}{4\pi}\left[\cos(2\omega t+\omega\tau)-\cos(2\omega t+\omega\tau)\right]+3\sin\omega\tau$$
$$= \frac{-3}{4\pi}[0]+3\sin\omega\tau$$
$$= 3\sin\omega\tau$$
$$= R_{XY}(\tau) \quad \because X(t) \text{ and } Y(t) \text{ are jointly WSS.}$$

$$|R_{XX}(\tau)| = |3\sin\omega\tau| \le 3$$

$$R_{XX}(0)\,R_{YY}(0) = \left(\frac{9}{2}\right)(2) = 9$$

$$\sqrt{R_{XX}(0)\,R_{YY}(0)} = 3$$

$$\therefore |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

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4.3 POWER SPECTRAL DENSITY - PROPERTIES

4.3.(a) Spectrum of $x(t)$

Let $x(t)$ be a deterministic signal. The Fourier Transform of $x(t)$ is defined as

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

Here $X(\omega)$ is called "Spectrum of $x(t)$".

4.3.(b) Power spectral density (or) Power density spectrum.

The power spectral density $S_X(\omega)$ of a continuous time random process $X(t)$ is defined as the Fourier transform of $R_{XX}(\tau)$:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau \quad \text{... (1)}$$

Thus, taking the inverse Fourier transform of $S_{XX}(\omega)$, we obtain

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega \quad \text{... (2)}$$

Equations (1) and (2) are known as the Wiener-Khinchin relations.

4.3.(c) Properties

PROPERTY 1 : For a wide sense stationary random process, power spectral density at zero frequency gives the area under the graph of Auto correlation.

Proof : We have $S_{XX}(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$

$$\Rightarrow \quad S_{XX}(0) = \int_{-\infty}^{\infty} R(\tau)\, d\tau$$

Which is the total area under the graph of the auto correlation function.

***PROPERTY 2* : The mean square value of a wide-sense stationary process is equal to the total area under the graph of the spectral density.**

Proof : We have $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i2\pi\omega\tau}\, d\omega$

Taking $\tau = 0$ we get

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

By the property of the autocorrelation function, we have

$$E[X^2(t)] = R_{XX}(0)$$

$$\text{Hence } E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

PROPERTY 3 : The PSD of a real valued random process is an even function of frequency. (or) The spectral density function of a real random process is an even function. **[A.U Tvli A/M 2011]**

Proof : We have $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$

|||ly $S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{i\omega\tau}\, d\tau$

Let $\tau = -u$ $\quad \tau \to -\infty \Rightarrow u \to \infty$

$d\tau = -du$ $\quad \tau \to \infty \Rightarrow u \to -\infty$

$$\therefore S_{XX}(-\omega) = \int_{\infty}^{-\infty} R_{XX}(-u)\, e^{i\omega(-u)}\, (-du)$$

$$= -\int_{\infty}^{-\infty} R_{XX}(-u)\, e^{i\omega(-u)}\, du$$

$$= \int_{-\infty}^{\infty} R_{XX}(-u)\, e^{i\omega(-u)}\, du$$

$$= \int_{-\infty}^{\infty} R_{XX}(-\tau)\, e^{i\omega(-\tau)}\, d\tau$$

Since τ is a dummy variable.

$$= R_{XX}(-\tau) \qquad \dots (1)$$

Since $\quad R_{XX}(\tau) = R_{XX}(-\tau)$

$$(1) \Rightarrow \quad S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= S_{XX}(\omega)$$

PEOPERTY 4 : A wide sense stationary random process has a non-negative power density spectrum.

Proof : Since the mean square value is always positive, PSD is also positive.

PROPERTY 5 : The spectral density and the autocorrelation function of a real WSS process form a Fourier conine transform pair.

Proof : $\quad S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, \{\cos\omega\tau - i\sin\omega\tau\}\, d\tau$

$$= 2\int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\, d\tau \qquad \text{[Since } R(\tau) \text{ is even]}$$

= Fourier cosine transform of $|2R(\tau)|$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,(\cos\tau\omega + i\sin\tau\omega)\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\tau\omega\, d\omega \qquad \text{[Since } S(\omega) \text{ is even]}$$

= Fourier inverse cosine transform of $\left[\frac{1}{2}S(\omega)\right]$

Example 4.3.1

If $R(\tau) = e^{-2\lambda|\tau|}$ is the autocorrelation function of a random process X (t), obtain the spectral density of X (t). [A.U. Dec. 2004]

Solution : The spectral density of X (t) is given by,

$$S_{XX}(\omega) = S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\,(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= 2\int_{0}^{\infty} e^{-2\lambda|\tau|}\cos\omega\tau\, d\tau \qquad \left[\because -i\int_{-\infty}^{\infty} e^{-2\lambda(\tau)}\sin\omega\tau\, d\tau = 0\right]$$

$$= 2\int_{0}^{\infty} e^{-2\lambda\tau}\cos\omega\tau\, d\tau \qquad \because |\tau| = \tau \text{ in } (0,\infty)$$

$$= \left[\frac{2e^{-2\lambda\tau}}{4\lambda^2+\omega^2}(-2\lambda\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty} = \frac{4\lambda}{4\lambda^2+\omega^2}$$

Example 4.3.2

Given that a process X(t) has the autocorrelation function $R_{XX}(\tau) = Ae^{-\alpha|\tau|}\cos(\omega_0\tau)$ where $A > 0$, $\alpha > 0$ and ω_0 are real constants, find the power spectrum of X(t). [A.U. Dec. 2003]

Solution : Given : $R_{XX}(\tau) = Ae^{-a|\tau|}\cos(\omega_0\tau)$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A e^{-a|\tau|}\cos(\omega_0\tau)\, e^{-i\omega_0\tau}\, d\tau$$

$$= \int_{-\infty}^{0} A e^{+a\tau}\cos(\omega_0\tau)\, e^{-i\omega\cdot\tau}\, d\tau + \int_{0}^{\infty} A e^{-a\tau}\cos(\omega_0\tau)\, e^{-i\omega_0\tau}\, d\tau$$

$$= A\int_{-\infty}^{0} e^{(a-i\omega_0)\tau}\cos(\omega_0\tau)\,d\tau + \int_{0}^{\infty} A\,e^{-(a+i\omega_0)\tau}\cos(\omega_0\tau)\,d\tau$$

$$= A\left[\frac{e^{(a-i\omega_0)\tau}}{(a-i\omega_0)^2+\omega_0^2}\left[(a-i\omega_0)\cos\omega_0\tau+\omega_0\sin\omega_0\tau\right]\right]_{-\infty}^{0}$$

$$+ A\left[\frac{e^{-(a+i\omega_0)\tau}}{(a+i\omega_0)^2+\omega_0^2}\left[-(a+i\omega_0)\cos\omega_0\tau+\omega_0\sin\omega_0\tau\right]\right]_{0}^{\infty}$$

$$= A\left[\frac{a-i\omega_0}{(a-i\omega_0)^2+\omega_0^2}-0\right]+A\left[\frac{0-[-(a+i\omega_0)]}{(a+i\omega_0)^2+\omega_0^2}\right]$$

$$= A\left[\frac{a-i\omega_0}{(a-i\omega_0)^2+\omega_0^2}+\frac{a+i\omega_0}{(a+i\omega_0)^2+\omega_0^2}\right]$$

$$= A\left\{\frac{(\alpha-i\omega)[\alpha^2+\omega_0^2-\omega^2+2i\alpha\omega]+(\alpha+i\omega)[(\alpha^2+\omega_0^2-\omega^2)-2i\alpha\omega]}{[(\alpha^2+\omega_0^2-\omega^2)-2i\alpha\omega][(\alpha^2+\omega_0^2-\omega^2)+2i\alpha\omega]}\right\}$$

$$= A\left\{\frac{\alpha[\alpha^2+\omega_0^2-\omega^2+2i\alpha\omega+\alpha^2+\omega_0^2-\omega^2-2i\alpha\omega] + i\omega[\alpha^2+\omega_0^2-\omega^2-2i\alpha\omega-\alpha^2-\omega_0^2+\omega^2-2i\alpha\omega]}{(\alpha^2+\omega_0^2-\omega^2)^2+4\alpha^2\omega^2}\right\}$$

$$= \frac{A\{2\alpha(\alpha^2+\omega_0^2-\omega^2)+i\omega(-4i\alpha\omega)\}}{(\alpha^2+\omega_0^2-\omega^2)^2+4\alpha^2\omega^2}$$

$$= \frac{A\{2\alpha(\alpha^2+\omega_0^2-\omega^2)+4\alpha\omega^2\}}{(\alpha^2+\omega_0^2-\omega^2)^2+4\alpha^2\omega^2} = \frac{A\{2\alpha(\alpha^2+\omega_0^2+\omega^2)\}}{(\alpha^2+\omega_0^2-\omega^2)^2+4\alpha^2\omega^2}$$

Example 4.3.3

The power spectral density of a random process $\{X(t)\}$ is given by

$S_{XX}(\omega) = \begin{cases}\pi, & |\omega|<1\\ 0, & \text{elsewhere}\end{cases}$ Find its autocorrelation function. **[A.U. A/M. 2003]**

Sol. $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1}\pi\,e^{i\omega\tau}\,d\omega$

$$= \frac{1}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{e^{i\tau}-e^{-i\tau}}{i\tau}\right]$$

$$= \frac{1}{\tau}\left[\frac{e^{i\tau}-e^{-i\tau}}{2i}\right] = \frac{1}{\tau}\sin\tau$$

Example 4.3.4

The power spectral density of a WSS process is given by

$S(w) = \begin{cases}\frac{b}{a}(a-|w|); & |w|\le a\\ 0; & |w|>a\end{cases}$ **[A.U. Model, A.U Trichy A/M 2010]**

Find the auto-correlation fn. of the process. **[AU N/D 2007]**

Sol. The autocorrelation function R (τ) is given by

$$R_{XX}(\tau) = R(\tau) = F^{-1}(S(w)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(w).e^{i\tau w}\,dw$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|w|)\,e^{i\tau w}\,dw$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}[a-|w|][\cos\tau w+i\sin\tau w]\,dw$$

$$= \frac{1}{2\pi}\frac{b}{a}\left[\int_{-a}^{a}[a-|w|](\cos\tau w)\,dw + i\int_{-a}^{a}[a-|w|](\sin\tau w)\,dw\right.$$

$(a-|w|)\cos\tau w$ is even [If $f(x)$ is even then $f(-x)=f(x)$]

$(a-|w|)\sin\tau w$ is odd [If $f(x)$ is odd then $f(-x)=-f(x)$]

$$R_{XX}(\tau) = \frac{1}{2\pi}\frac{b}{a}\left[2\int_{0}^{a}[a-|w|]\cos\tau w\,dw+0\right]$$

$$= \frac{b}{a}\frac{1}{\pi}\int_{0}^{a}[a-|w|]\cos\tau w\,dw \qquad \left[\because \int_{-a}^{a} g(x)dx=0 \text{ if } g(x) \text{ is odd}\right]$$

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$$= \frac{b}{a\pi}\int_0^a [a - |w|] \cos \tau w \, dw$$

$$= \frac{b}{a\pi}\int_0^a (a - w) \cos \tau w \, dw \qquad [\because |w| = w \text{ in } (0, a)]$$

$$= \frac{b}{a\pi}\left[(a - w)\frac{\sin \tau w}{\tau} - \frac{\cos \tau w}{\tau^2}\right]_0^a$$

$$= \frac{b}{a\pi}\left[\frac{1 - \cos a\tau}{\tau^2}\right] = \frac{b}{a\pi\tau^2} 2\sin^2\frac{a\tau}{2} = \frac{2b}{a\pi\tau^2}\sin^2\left(\frac{a\tau}{2}\right)$$

Example 4.3.5

The power spectrum of a WSS process X = {X (t)} is given by $S(w) = \frac{1}{(1+\omega^2)^2}$. Find the auto-correlation function and average power of the process. **[A.U. Model]**

Solution : Given : $S(\omega) = S_{XX}(\omega) = \frac{1}{(1+\omega^2)^2}$

[$S_{XX}(\omega)$ is the Fourier transform of $R_{XX}(\tau)$]

$$S_{XX}(\omega) = \left[\frac{1}{(1+i\omega)(1-i\omega)}\right]^2 \quad \text{... (1)}$$

$$\frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} = \frac{1-i\omega+1+i\omega}{(1+i\omega)(1-i\omega)} = \frac{2}{(1+i\omega)(1-i\omega)}$$

$$\frac{1}{2}\left[\frac{1}{1+i\omega} + \frac{1}{1-i\omega}\right] = \frac{1}{(1+i\omega)(1-i\omega)}$$

$$\therefore (1) \Rightarrow S_{XX}(\omega) = \left[\frac{1}{2}\left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega}\right)\right]^2$$

$$= \frac{1}{4}\left[\frac{1}{(1+i\omega)^2} + \frac{1}{(1-i\omega)^2} + \frac{2}{1+\omega^2}\right]$$

$R_{XX}(\tau)$ is the inverse Fourier Transform of $S_{XX}(\omega)$

$R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$

$$= \frac{1}{4}\left[F^{-1}\left[\frac{1}{(1+i\omega)^2} + \frac{1}{(1-i\omega)^2} + \frac{2}{1+\omega^2}\right]\right]$$

$$= \frac{1}{4}\left[F^{-1}\left[\frac{1}{(1+i\omega)^2}\right] + F^{-1}\left[\frac{1}{(1-i\omega)^2}\right] + F^{-1}\left[\frac{2}{1+\omega^2}\right]\right]$$

$$= \frac{1}{4}\left[u(\tau)\,\tau e^{-\tau} + u(\tau)\,\tau e^{\tau} + e^{-|\tau|}\right] \quad \text{... (2)}$$

Since $F^{-1}\left[\frac{1}{(\alpha+i\omega)^2}\right] = u(\tau)\,\tau e^{-\alpha\tau}$,

where $u(\tau)$ unit step function, $\alpha > 0$

$$F^{-1}\left[\frac{1}{(\alpha-i\omega)^2}\right] = u(\tau)\,\tau e^{-\alpha\tau}$$

$$F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}$$

Average power of $X(t) = R_{XX}(0)$ ... (3)

$$(2) \Rightarrow R_{XX}(0) = \frac{1}{4}\left[0 + 0 + e^{-0}\right] = \frac{1}{4}[1] = \frac{1}{4} = 0.25$$

$\therefore$ (3) $\Rightarrow$ Average power of $X(t) = 0.25$

Example 4.3.6

The power spectral density of a zeromean wide stationary process X(t) is given by $S(w) = \begin{cases} k; & |w| < w_0 \\ 0; & \text{otherwise} \end{cases}$ where k is a constant. Show that X (t) and X $\left(t + \frac{\pi}{100}\right)$ are uncorrelated. **[A.U. Model]**

Solution. $R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(w)\, e^{i\tau w}\, dw$

$$= \frac{k}{2\pi}\int_{-w_0}^{w_0} e^{i\tau w}\, dw = \frac{k}{2\pi}\left(\frac{e^{i\tau w}}{i\tau}\right)_{-w_0}^{w_0}$$

$$= \frac{k}{2\pi} \cdot \frac{e^{i\tau w_o} - e^{-i\tau w_o}}{i\tau}$$

$$= \frac{k}{2\tau} \cdot \sin w_o \tau \left(\because \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin \theta\right)$$

$$\therefore \ E\left(X\left(t + \frac{\pi}{w_o}\right) X(t)\right) = R_X\left(\frac{\pi}{w_o}\right) = \frac{k}{\pi \tau} \sin w_o \left(\frac{\pi}{w_o}\right) = 0$$

Since the mean of the process is zero,

$$c\left\{X\left(t + \frac{\pi}{w_o}\right) . X(t)\right\} = E\left\{X\left(t + \frac{\pi}{w_o}\right) X(t)\right\} = 0$$

$\therefore$ X (t) and X $\left(t + \frac{\pi}{w_o}\right)$ are uncorrelated.

Example 4.3.7

The power spectral density function of a wide sense stationary process is given by

$$S(w) = \begin{cases} 1; & |w| < w_0 \\ 0; & \text{otherwise} \end{cases}$$ **[A.U. Model] [A.U A/M 2011]**

Find the autocorrelation function of the process. **[A.U M/J 2007]**

Soln. The autocorrelation function R (τ) is given by

$$R(\tau) = F^{-1}[S(w)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(w)\, e^{i\tau w}\, dw$$

$$= \frac{1}{2\pi} \int_{-w_o}^{w_o} e^{i\tau w}\, dw$$

$$= \frac{1}{2\pi} \int_{-w_o}^{w_o} (\cos \tau w + i \sin \tau w)\, dw$$

$$= \frac{1}{2\pi} \int_{-w_o}^{w_o} \cos \tau w\, dw \quad \left(\because \text{sine is an odd fn. } \int_{-w_o}^{w_o} \sin \tau w = 0\right)$$

$$= \frac{2}{2\pi} \int_{0}^{w_o} \cos \tau w\, dw$$

$$= \frac{1}{\pi} \left[\frac{\sin \tau w}{\tau}\right]_0^{w_o} = \frac{1}{\pi} \frac{\sin \tau w_o}{\tau}$$

Example 4.3.8

The autocorrelation function of a wide sense stationary random process is given by R $(\tau) = \alpha^2 e^{-2\lambda|\tau|}$. Determine the power spectral density of the process. **[A.U. Model, Trichy A/M 2010]**

Solution. Power spectral density **[A.U A/M 2011, Trichy M/J 2011]**

$$S(w) = \int_{-\infty}^{\infty} \alpha^2 e^{-2\lambda|\tau|} e^{-iw\tau} d\tau$$

$$= \alpha^2 \int_{-\infty}^{0} e^{2\lambda} e^{-iw\tau} d\tau + \alpha^2 \int_{0}^{\infty} e^{-2\lambda\tau} . e^{-iw\tau} . d\tau$$

$$= \alpha^2 \int_{-\infty}^{0} e^{(2\lambda - iw)\tau} d\tau + \alpha^2 \int_{0}^{\infty} e^{-(2\lambda + iw)\tau} d\tau$$

$$= \alpha^2 \left[\frac{e^{(2\lambda - iw)\tau}}{2\lambda - iw}\right]_{-\infty}^{0} + \alpha^2 \left[\frac{e^{-(2\lambda + iw)\tau}}{-(2\lambda + iw)}\right]_0^{\infty}$$

$$= \alpha^2 \left[\frac{1}{2\lambda - iw}\right] + \alpha^2 \left[\frac{1}{2\lambda + iw}\right]$$

$$= \alpha^2 \frac{(2\lambda + iw + 2\lambda - iw)}{(2\lambda - iw)(2\lambda + iw)} = \frac{4\lambda \alpha^2}{4\lambda^2 + w^2}$$

4.4 CROSS SPECTRAL DENSITY - PROPERTIES

4.4.(a) Cross spectral density (or) Cross power density spectrum

Consider two jointly wide sense stationary random processes X(t) and Y(t).

Let $R_{XY}(\tau)$ and $R_{YX}(\tau)$ be their cross correlation functions. Then, the cross spectral densities are

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{Xy}(\tau)\, e^{-i\omega\tau}\, d\tau$$

and

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

Note 1 : The above relations are collectively referred to as the cross-spectral densities of the random processes $\{x(t)\}$ and $\{y(t)\}$.

Note 2 : The power of the two processes is defined as

$$P_{XY} = \frac{1}{2T}\int_{-T}^{T} x_T(t)\, y_T(t)\, dt$$

$$P_{YX} = \frac{1}{2T}\int_{-T}^{T} y_T(t)\, x_T(t)\, dt$$

The average power is

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega$$

$$P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\, d\omega$$

4.4(b) Properties of Cross Power Density Spectrum

[A.U Tvli M/J 2011]

PROPERTY 1 : $S_{YX}(\omega) = S_{XY}(-\omega)$

We have $\quad S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) \,.\, e^{-i\omega\tau} \,.\, d\tau$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) \cdot e^{-i\omega\tau} \cdot d\tau$$

Consider $S_{XY}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{i\omega\tau} \cdot d\tau$

$$= \int_{-\infty}^{\infty} R_{XY}(-\tau) \cdot e^{i\omega\tau} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau) \cdot e^{i\omega\tau} \cdot d\tau$$

$$= S_{YX}(\omega)$$

PROPERTY 2 : Real part of $S_{XY}(\omega)$ is an even function of ω.

Consider $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-i\omega\tau} \cdot d\tau$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$\therefore$ R.P of $S_{XY}(\omega) = \int_{-\infty}^{\infty} XY(\tau) \cdot \cos\omega\tau \cdot d\tau$

Consider R.P. of $S_{XY}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos[(-\omega)\tau]\, d\tau$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \cos\omega\tau \cdot d\tau$$

Hence proved.

Similarly, real part of $S_{YX}(\omega)$ is an even function of ω.

PROPERTY 3 : Imaginary part of $S_{XY}(\omega)$ is an odd function of ω.

Imaginary part of $S_{XY}(\omega)$ is $-\int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin\omega\tau \cdot d\tau$

Consider imaginary part of $S_{XY}(-\omega) = -\int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin(-\omega\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot \sin\omega\tau \cdot d\tau$$

Hence proved.

Similarly, imaginary part of $S_{XY}(\omega)$ is an odd function of ω.

PROPERTY 4 : $S_{XY}(\omega) = 0$ if X (t) are orthogonal.

[A.U Tvli M/J 2011]

We have $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-i\omega\tau} \cdot d\tau$

Two processes $X(t)$ and $Y(t)$ are said to be orthogonal if

$$R_{XY}(\tau) = 0$$

$$\therefore S_{XY} = 0$$

Similarly, $S_{YX} = 0$ if X (t) and Y (t) are orthogonal.

Property 5 : If X (t) and Y (t) are uncorrelated,

$$S_{XY}(\omega) = E(X) \cdot E(Y) \cdot \delta(\omega)$$

We have $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-i\omega\tau} \cdot d\tau$

$$= \int_{-\infty}^{\infty} E[X(t) \cdot Y(t+\tau)] \cdot e^{-i\omega\tau} \cdot d\tau$$

X (t) and Y (t) are said to be uncorrelated.

If $R_{XY}(\tau) = E[X(t)] \cdot E[Y(t)]$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} E[X(t)] \cdot E[Y(t)] \cdot e^{-i\omega\tau} \cdot d\tau$$

$$= E[X(t)] \cdot E[Y(t)] \int_{-\infty}^{\infty} e^{-i\omega\tau} \cdot d\tau$$

$$= E[X(t)] \cdot E[Y(t)] \cdot \delta(\omega)$$

Example 4.4.1.

The cross-power spectrum of real random process X(t) and Y(t) is given by

$$S_{XY}(\omega) = \begin{cases} a + \dfrac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha,\ \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the cross-correlation function. [AU A/M 2003, Tvli M/J 2011]

Solution : Given $S_{XY}(\omega) = a + \dfrac{ib\omega}{\alpha}, \ -\alpha < \omega < \alpha$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left[a + \frac{ib\omega}{\alpha}\right] e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} a\, e^{i\omega\tau}\, d\omega + \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{ib\omega}{\alpha}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\int_{-\alpha}^{\alpha} e^{i\omega\tau}\, d\omega + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha}\omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\left[\omega\left[\frac{e^{i\omega\tau}}{i\tau}\right] - (1)\left[\frac{e^{i\omega\tau}}{(i\tau)^2}\right]\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi}\frac{1}{i\tau}\left[e^{i\omega\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\left[\frac{\omega}{i\tau}e^{i\omega\tau} + \frac{e^{i\omega\tau}}{\tau^2}\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi}\frac{1}{i\tau}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right]$$

$$+ \frac{ib}{2\pi\alpha}\left[\left(\frac{\alpha}{i\tau}e^{i\alpha\tau} + \frac{e^{i\alpha\tau}}{\tau^2}\right) - \left(\frac{-\alpha}{i\tau}e^{-i\alpha\tau} + \frac{e^{-i\alpha\tau}}{\tau^2}\right)\right]$$

$$= \frac{a}{2\pi}\frac{1}{i\tau}\left[2i\sin\alpha\tau\right] + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}e^{i\alpha\tau} + \frac{e^{i\alpha\tau}}{\tau^2} + \frac{\alpha}{i\tau}e^{-i\alpha\tau} - \frac{e^{-i\alpha\tau}}{\tau^2}\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\left(e^{i\alpha\tau} + e^{-i\alpha\tau}\right) + \frac{1}{\tau^2}\left(e^{i\alpha\tau} - e^{-i\alpha\tau}\right)\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}(2\cos\alpha\tau) + \frac{1}{\tau^2}(2i\sin\alpha\tau)\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{b}{\pi\tau}\cos\alpha\tau - \frac{b}{\pi\alpha\tau^2}\sin\alpha\tau$$

$$= \frac{a\tau\alpha\sin\alpha\tau + b\alpha\tau\cos\alpha\tau - b\sin\tau}{\pi\alpha\tau^2}$$

$$= \frac{1}{\pi\alpha\tau^2}\left[(a\alpha\tau - b)\sin\alpha\tau + b\alpha\tau\cos\alpha\tau\right]$$

Example 4.4.2

The cross-power spectrum of real random process X (t) and Y (t) is given by

$$S_{XY}(\omega) = \begin{cases} a + jb\omega, & |\omega| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the cross correlation function. [A.U N/D 2010] [A.U A/M 2011, N/D 2011]

Solution : Given : $S_{XY}(\omega) = a + jb\omega, \ |\omega| < 1$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(a + jb\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\left[a\, e^{j\omega\tau} + jb\omega\, e^{j\omega\tau}\right] d\omega$$

$$= \frac{a}{2\pi}\int_{-1}^{1} e^{j\omega\tau}\, d\omega + \frac{jb}{2\pi}\int_{-1}^{1}\omega\, e^{j\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1} + \frac{jb}{2\pi}\left[\omega\left[\frac{e^{j\omega\tau}}{j\tau}\right] - (1)\left[\frac{e^{j\omega\tau}}{j^2\tau^2}\right]\right]_{-1}^{1}$$

5.23
of
ction
(1)
2)

$$= \frac{a}{2\pi}\frac{1}{j\tau}\left[e^{j\omega\tau}\right]_{-1}^{1} + \frac{jb}{2\pi}\left[\frac{\omega}{j\tau}e^{j\omega\tau} + \frac{1}{\tau^2}e^{j\omega\tau}\right]_{-1}^{1}$$

$[\because j^2 = -1]$

$$= \frac{a}{2\pi}\frac{1}{j\tau}\left[e^{j\tau} - e^{-j\tau}\right] + \frac{jb}{2\pi}\left[\left(\frac{1}{j\tau}e^{j\tau} + \frac{1}{\tau^2}e^{j\tau}\right) - \left(-\frac{1}{j\tau}e^{-j\tau} + \frac{1}{\tau^2}e^{-j\tau}\right)\right]$$

$$= \frac{a}{2\pi}\frac{1}{j\tau}2j\sin\tau + \frac{jb}{2\pi}\left[\frac{1}{j\tau}e^{j\tau} + \frac{1}{\tau^2}e^{j\tau} + \frac{1}{j\tau}e^{-j\tau} - \frac{1}{\tau^2}e^{-j\tau}\right]$$

$$= \frac{a}{\pi\tau}\sin\tau + \frac{jb}{2\pi}\left[\frac{1}{j\tau}(e^{j\tau} + e^{-j\tau}) + \frac{1}{\tau^2}(e^{j\tau} - e^{-j\tau})\right]$$

$$= \frac{a}{\pi\tau}\sin\tau + \frac{jb}{2\pi}\left[\frac{1}{j\tau}2\cos\tau + \frac{1}{\tau^2}2j\sin\tau\right]$$

$$= \frac{a}{\pi\tau}\sin\tau + \frac{b}{\pi\tau}\cos\tau - \frac{b}{\pi\tau^2}\sin\tau$$

$$= \frac{1}{\pi\tau^2}\left[(a\tau - b)\sin\tau + b\tau\cos\tau\right]$$

Example 4.4.3

If the cross-correlation of two processes {X(t)} and {Y(t)} is $R_{XY}(t, t+\tau) = \frac{AB}{2}[\sin(\omega_0\tau) + \cos(\omega_0(2t+\tau))]$ where A,B are ω_0 and constants. Find the cross power spectrum.

[AU May 2004] [A.U. M/J 2012]

Solution : The time average is given by

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\,d\tau$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{AB}{2}[\sin(\omega_0\tau) + \cos(\omega_0(2t+\tau))\,dt$$

$$= \frac{AB}{2}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin(\omega_0\tau)\,d\tau + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(\omega_0(2t+\tau)\,dt)\right]$$

$$= \frac{AB}{2}\left[\sin(\omega_0\tau) + \lim_{T\to\infty}\frac{1}{4T}\left[\sin(\omega_0(2t+\tau))\right]_{-T}^{T}\right]$$

$$= \frac{AB}{2}\sin(\omega_0\tau) + 0$$

$[\because$ Max. value of $\sin\theta$ is 1 and $\frac{-1}{T}\to 0$, as $T\to\infty]$

The cross power spectrum is

$$S_{XY}(\omega) = \text{Fourier Transform of } \frac{AB}{2}\sin(\omega_0\tau)$$

$$= \int_{-\infty}^{\infty}\frac{AB}{2}\sin(\omega_0\tau)e^{-i\omega\tau}d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty}\sin(\omega_0\tau)[\cos\omega\tau - i\sin\omega\tau]\,d\tau$$

$[\because e^{-i\theta} = \cos\theta - i\sin\theta]$

$$= \frac{AB}{2}\left[\int_{-\infty}^{\infty}\sin(\omega_0\tau)\cos(\omega\tau)\,d\tau - i\int_{-\infty}^{\infty}\sin(\omega_0\tau)\sin(\omega\tau)\,d\tau\right]$$

$$= \frac{AB}{2}\left[\frac{1}{2}\int_{-\infty}^{\infty}(\sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau)\,d\tau - \frac{i}{2}\int_{-\infty}^{\infty}(\cos(\omega-\omega_0\tau - \cos(\omega+\omega_0)\tau)\,d\tau\right]$$

$[\because \cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B]$

$$= \frac{AB}{4}\Big[\int_{-\infty}^{\infty} [\sin(\omega_0+\omega)\tau + \sin(\omega_0-\omega)\tau$$
$$-i\cos(\omega-\omega_0)\tau + i\cos(\omega+\omega_0)\tau]\, d\tau\Big]$$

$$= \frac{AB}{4}\Big[\int_{-\infty}^{\infty} [i(\cos(\omega+\omega_0)\tau - i\sin(\omega+\omega_0)\tau)$$
$$-i[\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau]\, d\tau\Big]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{-iAB}{4}\int_{-\infty}^{\infty}\left[e^{-i(\omega-\omega_0)\tau} - e^{-i(\omega+\omega_0)\tau}\right] d\tau$$

$$= \frac{-iAB}{4}\left[\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau}\, d\tau - \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau}\, d\tau\right]$$

$$= \frac{-iAB}{4}\,[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)]$$

$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau$ is the dirac delta function such that $\int_{-\infty}^{\infty}\delta(\omega)\, d\omega = 1$

so we get

$$S_{XY}(\omega) = \frac{-i\pi AB}{2}\,[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

Example 4.4.4

If X(t) and Y(t) are uncorrelated random processes, then find the power spectral density of Z if Z(t) = X(t) + Y(t). Also find the cross-spectral density $S_{XZ}(\omega)$ and $S_{YZ}(\omega)$. **[A.U. M/J 2012]**

Solution : If $X(t)$ and $Y(t)$ are uncorrelated random processes, then, their cross covariance

$$C_{XY}(t, t+\tau) = 0$$

$$R_{XY}(t, t+\tau) - E[X(t)].E[Y(t)] = 0$$

i.e., $R_{XY}(t, t+\tau) = E[X(t)].E[Y(t)]$

$$R_{XY}(\tau) = E[X(t)].E[Y(t)] \quad \text{... (1)}$$

|||rly $R_{XY}(\tau) = E[Y(t)].E[X(t)] = R_{XY}(\tau)$ ... (2)

By definition we have the autocorrelation function of

$$Z(t) = X(t) + Y(t)$$

$$R_{ZZ}(\tau) = E[Z(t).Z(t+\tau)]$$

$$= E\{[X(t)+Y(t)].[X(t+\tau)+Y(t+\tau)]\} \quad [\because \text{by (3)}]$$

$$= E\{X(t).X(t+\tau) + X(t).Y(t+\tau) + Y(t).X(t+\tau) + Y(t).Y(t+\tau)\}$$

$$= E[X(t).X(t+\tau)] + E[X(t).Y(t+\tau)] + E[Y(t).X(t+\tau)] + E[Y(t).Y(t+\tau)]$$

$$= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) \quad \text{[By definition}$$

i.e., $R_{ZZ}(\tau) = R_{XX}(\tau) + 2R_{XY}(\tau) + R_{YY}(\tau)$ ... (4) $[\because$ by (2)]

$\therefore$ The power spectral density of $\{Z(t)\}$ is

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} [R_{XX}(\tau) + 2R_{XY}(\tau) + R_{YY}(\tau)]\, e^{-i\omega\tau}\, d\tau \quad \text{by (4)}$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau + 2\int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

i.e., $S_{ZZ}(\omega) = S_{XX}(\omega) + 2S_{XY}(\omega) + S_{YY}(\omega)$ ... (5)

[$\because$ by definition of power spectral density and cross spectral density]

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Cross-correlation function is

$$R_{XZ}(\tau) = E[X(t).Z(t+\tau)]$$
$$= E\{X(t).[X(t+\tau)+Y(t+\tau)]\}$$
$$= E[X(t).X(t+\tau)+X(t).Y(t+\tau)]$$
$$= E[X(t).X(t+\tau)+E[X(t).Y(t+\tau)]$$
$$\therefore\ R_{XZ}(\tau) = R_{XX}(\tau)+R_{XY}(\tau) \quad \text{... (6)}$$

$$S_{XZ}(\omega) = \int_{-\infty}^{\infty} R_{XZ}(\omega)\, e^{-i\omega\tau}\, d\tau$$
$$= \int_{-\infty}^{\infty} [R_{XX}(\tau)+R_{XY}(\tau)]\, e^{-i\omega\tau}\, d\tau \quad \text{by (6)}$$
$$= \int_{-\infty}^{\infty} R_{XX}(\tau).e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} R_{XY}(\tau).e^{-i\omega\tau}\, d\tau$$

i.e., $S_{XZ}(\omega) = S_{XX}(\omega) + S_{XY}(\omega)$

|||rly we get

$$S_{YZ}(\omega) = S_{YZ}(\omega) + S_{YY}(\omega)$$

EXERCISE 4.4

1. If the cross-correlation of two processes $\{x(t)\}$ and $\{y(t)\}$ is $R_{XY}(t, t+\tau) = \frac{AB}{2}[\sin(\omega_o\tau) + \cos(\omega_o(2t+\tau))]$ where A, B are ω_o and constants find the cross power spectrum.

2. Show that if $S_{XY}(\omega)$ is the cross power spectrum, then $R_e[S_{XY}(\omega)]$ is an even function.

4.5 WIENER - KHINTCHINE RELATION

4.5.(a) Wiener-Khintchine Theorem

[A.U N/D 2010]
[A.U A/M 2011, N/D 2011, CBT N/D 2011, CBT A/M 2011]
[A.U Trichy N/D 2011] [A.U N/D 2012]

If $X_T(\omega)$ is the Fourier transform of the truncated random process defined as

$$X_T(t) = \begin{cases} X(t) & \text{for } |t| \le T \\ 0 & \text{for } |t| > T \end{cases}$$

where $[X(t)]$ is a real WSS process with power spectral density function $S(\omega)$, then

$$S(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T}E\{|X_T(\omega)|^2\}\right]$$

Proof : Given : $X_T(\omega)$ is the Fourier Transform of $X_T(t)$

$$\Rightarrow X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t}\, dt$$
$$= \int_{-T}^{T} X(t)\, e^{-i\omega t}\, dt \qquad [\because X_T(t) = X(t) \text{ for } |t| \le T]$$

$$|X_T(\omega)|^2 = [X_T(\omega)]\,[X_T(-\omega)]$$
$$= \left[\int_{-T}^{T} X(t_2)\, e^{-i\omega t_2}\, dt_2\right]\left[\int_{-T}^{T} X(t_1)\, e^{i\omega t_1}\, dt_1\right]$$
$$= \int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)\, e^{-i\omega(t_2-t_1)}\, dt_2\, dt_1$$

$$E\left[|X_T(\omega)|^2\right] = \int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]\, e^{-i\omega(t_2-t_1)}\, dt_2\, dt_1$$
$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\, e^{-i\omega(t_2-t_1)}\, dt_2\, dt_1 \quad \text{... (1)}$$
$$[\because E[X(t_1)X(t_2)] = R_{XX}(t_1,t_2)]$$

Let $t = t_1$, $dt = dt_1$

$\tau = t_2 - t_1$
$= t_2 - t$
$d\tau = dt_2$
$t_2 = \tau + t$

$$J = \begin{vmatrix} \frac{\partial t_1}{\partial t} & \frac{\partial t_1}{\partial \tau} \\ \frac{\partial t_2}{\partial t} & \frac{\partial t_2}{\partial \tau} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$dt_2\, dt_1 = |J|\, d\tau\, dt = d\tau\, dt$$

The limits of t are	The limits of τ are
when $t_1 = -T \Rightarrow t = -T$	when $t_2 = -T \Rightarrow \tau = -T - \tau$
$t_1 = T \Rightarrow t = T$	$t_2 = T \Rightarrow \tau = T - t$

$$\therefore (1) \Rightarrow E\left[|X_T(\omega)|^2\right] = \int_{-T}^{T} \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau\, dt$$

[$\because X(t)$ is WSS]

$$E\left[|X_T(\omega)|^2\right] = \left[\int_{-T}^{T} dt\right] \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \left[t\right]_{-T}^{T} \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= 2T \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$\lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right] = \lim_{T\to\infty} \frac{1}{2T}\, 2T \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \lim_{T\to\infty} \int_{-T-t}^{T-t} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= S_{XX}(\omega), \text{ by definition}$$

$$\therefore S_{XX}(\omega) = \lim_{T\to\infty} \left[\frac{1}{2T} E\left[|X_T(\omega)|^2\right]\right]$$

Aliter : Given : $X_T(\omega) = \int_{-\infty}^{\infty} X_T(t)\, e^{-i\omega t}\, dt = \int_{-T}^{T} X(t)\, e^{-i\omega t}\, dt$

Since $\{X(t)\}$ is real

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T(-\omega)$$

$$= \left[\int_{-T}^{T} X(t_1)\, e^{-i\omega t_1}\, dt_1\right] \left[\int_{-T}^{T} X(t_2)\, e^{i\omega t_2}\, dt_2\right]$$

$$= \int_{-T}^{T} \int_{-T}^{T} X(t_1)\, X(t_2)\, e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

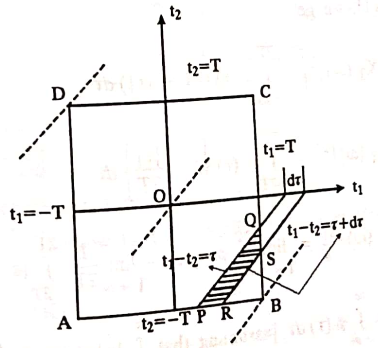
$$\therefore E\,|X_T(\omega)|^2 = \int_{-T}^{T} \int_{-T}^{T} E\left\{X(t_1)\, X(t_2)\right\} e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

$$= \int_{-T}^{T} \int_{-T}^{T} R(t_1 - t_2)\, e^{-i\omega(t_1 - t_2)}\, dt_1\, dt_2$$

[Since $X(t)$ is WSS]

$$= \int_{-T}^{T} \int_{-T}^{T} \phi(t_1 - t_2)\, dt_1\, dt_2, \text{ say} \qquad \dots (1)$$

The double integral (1) is evaluated over the area of the square ABCD bounded by $t_1 = -T, T$ and $t_2 = -T, T$ as shown in the figure.



5.5

1)

We divide the area of the square into a number of strips like **PQRS**, where PQ is given by $t_1 - t_2 = \tau$ and RS is given by $t_1 - t_2 = \tau + d\tau$.

When PQRS is at the initial position D, $t_1 - t_2 = -2T$. When it is at the final position B, $t_1 - t_2 = 2T$. Hence when τ varies from $-2T$ to $2T$, the area ABCD is covered.

Now $dt_1\, dt_2$ = elemental area of the $t_1 t_2$-plane

= area of PQRS ... (2)

$(t_1)_P = \tau - T$ and PB (=BQ) $= T-(\tau - T) = 2T - \tau$, if $\tau > 0$

$= 2T + \tau$, if $\tau < 0$

When $\tau > 0$,

Area of PQRS $= \Delta PBQ - \Delta RBS$

$$= \frac{1}{2}(2T-\tau)^2 - \frac{1}{2}(2T-\tau-d\tau)^2$$

$$= (2T-\tau)\, d\tau, \text{ omitting } (d\tau)^2 \quad \text{... (3)}$$

From (2) and (3),

$$dt_1\, dt_2 = (2T - |\tau|)\, d\tau \quad \text{... (4)}$$

Using (4) in (1), we get,

$$E\{|X_T(\omega)|^2\} = \int_{-2T}^{2T} \phi(\tau)\,(2T - |\tau|)\, d\tau$$

$$\therefore \quad \frac{1}{2T} E\{|X_T(\omega)|^2\} = \int_{-2T}^{2T} \phi(\tau) \left\{1 - \frac{|\tau|}{2T}\right\} d\tau$$

$$\lim_{T\to\infty} \frac{1}{2T} E\{|X_T(\omega)|^2\} = \lim_{T\to\infty} \int_{-2T}^{2T} \phi(\tau)\, d\tau - \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} |\tau|\, \phi(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \phi(\tau)\, d\tau \text{ [assuming that } \int_{-\infty}^{\infty} |\tau|\, \phi(\tau)\, d\tau \text{ is bounded]}$$

$$= \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau \quad \text{[provided } \int_{-\infty}^{\infty} |\tau|\, R(\tau)\, e^{-i\omega\tau}\, d\tau \text{ is bounded]}$$

$= S(\omega)$, by definition of $S(\omega)$.

Example 4.5.1

Calculate the power spectral density of a stationary random process whose Auto correlation is $R_{XX}(\tau) = e^{-\alpha|\tau|}$

Solution : According to Wiener - Khinchine Relation,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,.\,e^{-i\omega\tau}\,.\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|}\,.\,e^{-i\omega\tau}\,.\,d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau}\,.\,e^{-i\omega\tau}\,.\,d\tau + \int_{0}^{\infty} e^{-\alpha\tau}\,.\,e^{-i\omega\tau}\,.\,d\tau$$

$$= \int_{-\infty}^{0} e^{(\alpha - i\omega)\tau}\,.\,d\tau + \int_{0}^{\infty} e^{-(\alpha+i\omega)\tau}\, d\tau$$

$$= \left[\frac{e^{(\alpha - i\omega)\tau}}{(\alpha - i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+i\omega)\tau}}{-(\alpha+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

Example 4.5.2

If the auto correlation function of a ωSS process is $R(\tau) = K\,e^{-K|\tau|}$, show that its spectral density is given by

$$S(\omega) = \frac{2}{1 + (\omega/K)^2}$$

Solution :

5.5

(1)

1

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} K.e^{-K|\tau|}.e^{-i\omega\tau}.d\tau$$

$$= \int_{-\infty}^{0} K.e^{K\tau}.e^{-i\omega\tau}.d\tau + \int_{0}^{\infty} K.e^{-K\tau}.e^{-i\omega\tau}.d\tau$$

$$= \int_{-\infty}^{0} K.e^{(K-i\omega)\tau}.d\tau + \int_{0}^{\infty} K.e^{-(K+i\omega)\tau}.d\tau$$

$$= K\left[\frac{e^{(K-i\omega)\tau}}{(K-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(K+i\omega)\tau}}{-(K+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{K}{K-i\omega} + \frac{K}{K+i\omega}$$

$$= \frac{2K^2}{K^2+\omega^2} = \frac{2}{1+\left(\frac{\omega}{K}\right)^2}$$

Example 4.5.3

Find the PSD of a random process X (t) if E [X (t)] = 1 and $R_{XX}(\tau) = 1 + e^{-\alpha|\tau|}$.

Solution : $S(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}.d\tau$

$$= \int_{-\infty}^{\infty} [1+e^{-\alpha|\tau|}]\, e^{-i\omega\tau}.d\tau$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau}.d\tau + \int_{-\infty}^{0} e^{\alpha\tau}.e^{-i\omega\tau}.d\tau + \int_{0}^{\infty} e^{-\alpha\tau}.e^{-i\omega\tau}.d\tau$$

$$= \delta(\omega) + \frac{1}{\alpha-i\omega} + \frac{1}{\alpha+i\omega}$$

$$= \delta(\omega) + \frac{2\alpha}{\alpha^2+\omega^2}$$

4.6 RELATIONSHIP BETWEEN CROSS POWER SPECTRUM AND CROSS CORRELATION FUNCTION

Cross correlation function and cross spectral density of the random processes X (t) and Y (t) form a Fourier Transform pair.

If X (t) and Y (t) are jointly wide sense stationary.

$$R_{XY}(t, t+\tau) = R_{XY}(\tau)$$

$$\therefore\ S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$\therefore$ $S_{XY}(\omega)$ Fourier Transform $R_{XY}(\tau)$

Example 4.6.1

Determine the cross-correlation function corresponding to the cross-power density spectrum $S_{XY}(w) = \frac{8}{(\alpha+jw)^2}$

Solution : $R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(w)\, e^{jw\tau}\, dw$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{2}{(\alpha+jw)^3}\, e^{jw\tau}\, dw$$

The integral cannot be evaluated to consider $S_{XY}(w) = 4G(w)$

where $G(w) = \frac{2}{(\alpha+jw)^2}$

By Fourier transform pairs

$$g(\tau) = u(\tau)\,\tau^2 e^{-\alpha\tau} \Leftrightarrow G(w) \quad ...(1)$$

From the linearity property of Fourier transforms and by (1) we get

$$R_{XY}(\tau) = 4u(\tau)\,\tau^2 e^{-\alpha\tau}$$

Note under power spectrum

5.5

1)

$$S_{XX}(w) = \int_{-\infty}^{\infty} A\,[P_{XX}(t, t+\tau)]\, e^{-jw\tau}\, d\tau$$

where $A\,[R_{XX}(t, t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t, t+\tau)\, dt$

where A is the time average.

Note under cross spectrum

$$S_{XY}(w) = \int_{-\infty}^{\infty} A\,[R_{XY}(t, t+\tau)]\, e^{-jw\tau}\, d\tau$$

where $A\,[R_{XY}(t, t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$

where A is the time average.

Example 4.6.2

If the cross-correlation of two processes {X(t)} and {Y(t)} is $R_{XY}(t, t+\tau) = \frac{AB}{2}[\sin(\omega_o \tau) + \cos(\omega_o(2t+\tau))]$ where A, B are ω_o and constants. Find the cross power spectrum.

Solution : We know that the Time average is given by

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{AB}{2} [\sin(\omega_o \tau) + \cos\omega_o(2t+\tau)]\, dt$$

$$= \frac{AB}{2} \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sin(\omega_o \tau)\, dt + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos(\omega_o(2t+\tau))\, dt \right]$$

$$= \frac{AB}{2} \left[\sin(\omega_o \tau) + \lim_{T \to \infty} \frac{1}{4T} [\sin(\omega_o(2t+\tau))]_{-T}^{T} \right]$$

$$= \frac{AB}{2} \sin(\omega_o \tau) + 0 \qquad (1)$$

[∵ Max. value of $\sin\theta$ is 1 and $\frac{1}{T} \to 0$, as $T \to \infty$]

The cross power spectrum is given by

$$S_{XY}(\omega) = \text{Fourier Transform of } \frac{AB}{2} \sin(\omega_o \tau)$$

$$= \int_{-\infty}^{\infty} \frac{AB}{2} \sin(\omega_o \tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{AB}{2} \int_{-\infty}^{\infty} \sin(\omega_o \tau) [\cos\omega\tau - i \sin\omega\tau]\, d\tau$$

[∵ $e^{-i\theta} = \cos\theta - i\sin\theta$]

$$= \frac{AB}{2} \left[\int_{-\infty}^{\infty} \sin(\omega_o \tau)\cos(\omega\tau)\, d\tau - i \int_{-\infty}^{\infty} \sin(\omega_o \tau)\sin(\omega\tau)\, d\tau \right]$$

$$= \frac{AB}{2} \left[\frac{1}{2} \int_{-\infty}^{\infty} (\sin(\omega_o + \omega)\tau + \sin(\omega_o - \omega)\tau)\, d\tau - \frac{i}{2} \int_{-\infty}^{\infty} (\cos(\omega - \omega_o)\tau - \cos(\omega + \omega_o)\tau)\, d\tau \right]$$

[∵ $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$$= \frac{AB}{4} \left[\int_{-\infty}^{\infty} [\sin(\omega_o + \omega)\tau + \sin(\omega_o - \omega)\tau - i\cos(\omega - \omega_o)\tau + i\cos(\omega + \omega_o)\tau]\, d\tau \right]$$

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$$= \frac{AB}{4} \left[\int_{-\infty}^{\infty} [i (\cos (\omega + \omega_o) \tau - i \sin (\omega + \omega_o) \tau) \right.$$

$$\left. -i [\cos (\omega - \omega_o) \tau - i \sin (\omega - \omega_o) \tau]] \, d\tau \right]$$

$$[\because \sin (-\theta) = -\sin \theta]$$

$$= \frac{-i\,AB}{4} \int_{-\infty}^{\infty} [e^{-i(\omega - \omega_o)\tau} - e^{-i(\omega + \omega_o)\tau}] \, d\tau$$

$$= \frac{-iAB}{4} \left[\int_{-\infty}^{\infty} e^{-i(\omega - \omega_o)\tau} \, d\tau - \int_{-\infty}^{\infty} e^{-i(\omega + \omega_o)\tau} \, d\tau \right]$$

$$= \frac{-i\,AB}{4} [2\pi\,\delta (\omega - \omega_o) - 2\pi\,\delta (\omega + \omega_o)]$$

where $\delta (\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \, d\tau$ is the dirac delta function such that $\int_{-\infty}^{\infty} \delta (\omega) \, d\omega = 1.$

Hence, $S_{XY} (\omega) = \frac{-i\,\pi\,AB}{2} [\delta (\omega - \omega_o) + \delta (\omega + \omega_o)]$

UNIT - 5
LINEAR SYSTEMS WITH RANDOM INPUTS

Linear time invariant system - System transfer function - Linear systems with random inputs - Auto correlation and cross correlation functions of input and output - white noise.

5.1 Linear Time Invariant System - System transffer function - Linear system with random inputs

5.1(a) System

A system is defined by a functional relationship between the input $x(t)$ and the output $y(t)$ as

$$y(t) = f\{x(t)\}, \quad -\infty < t < \infty$$

5.1.(b) Linear system

A system with functional relationship $f\{x(t)\}$ is linear, if, for any two inputs $x_1(t)$ and $x_2(t)$, the output of the system can be defined as $f\{(a_1 x_1(t) + a_2 x_2(t)\} = a_1 f\{x_1(t)\} + a_2 f\{x_2(t)\}$

where a_1 and a_2 are constants

5.1.(c) Time invariance

Time invariance is defined as a property of linear systems that if the input is time shifted by an amount τ, the corresponding output will also be time shifted by the same amount.

i.e., if $\quad f\{x(t)\} = y(t)$ then

$$f\{x(t-\tau)\} = y(t-\tau), \quad -\infty < \tau < \infty$$

A system that does not meet the condition is called time varying system.

5.1.(d) Causality

Causality is defined as a property of linear systems that the system response at time t depends only on the past values of the input.

i.e., If $y(t)$ is the output of the system for the input $x(t)$, then $y(t_1) = f\{x(t_1)\},\ t \le t_1$.

Note : All physically realisable systems are causal systems.

5.1.(e) Stable

A system is stable if for every bounded input, the system given bounded output.

If $\int_{-\infty}^{\infty} h(t)\, dt < \infty$ i.e., $h(t)$ is absolutely integrable, the system is said to be stable.

5.1.(f) Unit impulse response to the system

Let us consider input of the system as the unit impulse function $\delta(t-a)$ at $t = a$. We know that

$$\delta(t-a) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } a - \frac{\varepsilon}{2} \le t \le a + \frac{\varepsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

where $\varepsilon \to 0$

$$\textbf{Then } \int_{-\infty}^{\infty} \phi(t)\,\delta(t-a)\, dt = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} \phi(t)\frac{1}{\varepsilon}\, dt = \phi(a)$$

If $a = 0$, then we have

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\, dt = \phi(0) \quad \ldots (1)$$

Put $X(t) = \delta(t)$. Then by the convention relation

$$Y(t) = \int_{-\infty}^{\infty} h(u)\,\delta(t-u)\, du$$

$$= \int_{-\infty}^{\infty} h(t-u)\,\delta(u)\, du \quad \text{(by the convolution property)}$$

$$= h(t-0) \quad \text{[by (1)]}$$

$$= h(t)$$

which is the system weighting function.

Linear time invariant system

5.1(g) Time - invariant system - transfer function

By Fourier transformation of $y(t)$ we may derive an equivalent characterization in the frequency domain. If $x(\omega)$, $y(\omega)$ and $H(\omega)$ are the respective fourier transforms of $x(t)\, y(t)$ and $n(t)$, then

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\xi)\, h(t-\xi)\, d\xi\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\xi) \left[\int_{-\infty}^{\infty} h(t-\xi)\, e^{-i\omega(t-\xi)}\right] e^{-i\omega\xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} x(\xi)\, H(\omega)\, e^{-i\omega\xi}\, d\xi$$

$$= X(\omega)\, H(\omega)$$

where $H(\omega)$ is called the transfer function of the system.

The above relation show that the response of any linear time - invariant system is equal to the product of the transform of the input signal and the transform of the network impulse response.

PROPERTY 1 : Show that $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ **where** $S_{XX}(\omega)$ **and** $S_{YY}(\omega)$ **are the power spectral density functions of the input X(t) and the output Y(t) and H(ω) is the system transfer function.** **[A.U Trichy M/J 2011, CBT N/D 2011, N/D 2011]**

Sol. Let $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$

$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha)\, d\alpha$$

$$\therefore X(t+\tau)\, Y(t) = \int_{-\infty}^{\infty} X(t+\tau)\, X(t-\alpha)\, h(\alpha)\, d\alpha$$

$$\therefore E\left\{X(t+\tau)\,Y(t)\right\} = \int_{-\infty}^{\infty} E\left\{X(t+\tau)\,X(t-\alpha)\right\} h(\alpha)\,d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau+\alpha)\,h(\alpha)\,d\alpha \qquad \text{[since } \{X(t)\} \text{ is WSS]}$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-\beta)\,h(-\beta)\,d\beta \qquad \text{[putting } \beta = -\alpha]$$

i.e., $R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)$ ... (1)

Similarly, $R_{yx}(\tau) = R_{xx}(\tau)^* h(\tau)$ ... (1a)

Now $Y(t)\;Y(t-\tau) = \int_{-\infty}^{\infty} X(t-\alpha)\,Y(t-\tau)\,h(\alpha)\,d\alpha$

$$\therefore E\left\{Y(t)\,Y(t-\tau)\right\} = \int_{-\infty}^{\infty} R_{xy}\;(\tau-\alpha)\,h(\alpha)\,d\alpha$$

assuming that $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS

i.e., $R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$ ... (2)

Taking Fourier transforms of (1) and (2), we get

$S_{xy}(\omega) = S_{xx}(\omega)\,H*(\omega)$ ... (3)

where $H*(\omega)$ is the conjugate of $H(\omega)$ and

$S_{yy}(\omega) = S_{xy}(\omega)\,H(\omega)$ ... (4)

Inserting (3) in (4), $S_{yy}(\omega) = |H(\omega)|^2\,S_{xx}(\omega)$

PROPERTY 2 : If the input X (t) and its output Y (t) are related by Y (t) = $\int_{-\infty}^{\infty}$ h(u) X (t – u) du, then the system is a linear time-invariant system. **[A.U. M/J 2012]**

Proof. First, we prove the linearily,

Consider, $X(t) = a_1 X_1(t) + a_2 X_2(t)$ ... (1)

Then $\quad Y(t) = \int_{-\infty}^{\infty} h(u)\,X(t-u)\,du$

$$= \int_{-\infty}^{\infty} h(u)\,[a_1 X_1(t-u) + a_2 X_2(t-u)]\,du \qquad \text{By (1)}$$

$$= a_1 \int_{-\infty}^{\infty} h(u)\,X_1(t-u)\,du + a_2 \int_{-\infty}^{\infty} h(u)\,X_2(t-u)\,du$$

$$= a_1 Y_1(t) + a_2 Y_2(t)$$

Hence the system is linear.

Now, we prove that the system is a time invariant system.

Replacing t by $t+k$, we get

$$Y(t) = \int_{-\infty}^{\infty} h(u)\,X[(t+k)-u]\,du$$

$$= Y(t+k)$$

$\therefore$ The system is time invariant.

Hence the system is linear time invariant system.

PROPERTY 3 : If {X (t)} is a WSS process and if Y (t) = $\int_{-\infty}^{\infty}$ h(u) X (t – u) du, then $R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

[A.U N/D 2011][A.U. M/J 2012]

Proof : Given, $Y(t) = \int_{-\infty}^{\infty} h(u)\,X(t-u)\,du$ ... (1)

We know that

$$R_{XY}(\tau) = E\,[X(t)\;Y(t+\tau)]$$

$$= E\left[X(t)\int_{-\infty}^{\infty} h(u)\,X(t+\tau-u)\,du\right] \qquad \text{By (1)}$$

$$= \int_{-\infty}^{\infty} E\,[X(t)\,X(t+\tau-u)]\,h(u)\,du$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau-u)\,h(u)\,du \qquad \text{[since } \{X(t)\} \text{ is a WSS]}$$

$\therefore R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$ (By convolution) ... (2)

PROPERTY 4. **If $\{X(t)\}$ is a WSS process and if**

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du, \text{ then}$$

$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$ **where * denotes the convolution.** **[A.U N/D 2011]**

Proof : Given : $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$ ... (1)

$$R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, Y(t+\tau)\, du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t-u)\, Y(t+\tau)]\, h(u)\, du$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau+u)\, h(u)\, du$$

Put $u = -\alpha \Rightarrow du = -d\alpha$

$$= \int_{\infty}^{-\infty} R_{XY}(\tau-\alpha)\, h(-\alpha)\, (-d\alpha)$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau-\alpha)\, h(-\alpha)\, d\alpha$$

$$= R_{XY}(\tau) * h(-\tau)$$

PROPERTY 5. If $\{X(t)\}$ is a WSS process and if

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du \text{ then}$$

$$S_{XY}(\omega) = S_{XX}(\omega) * H(\omega)$$ **[A.U. M/J 2012]**

Proof : Given $Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)$

$$R_{XY}(\tau) = E[X(t)\, Y(t+\tau)]$$

$$= E\left[[X(t)] \int_{-\infty}^{\infty} h(u)\, X(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)\, X(t+\tau-u)]\, h(u)\, du$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau-u)\, h(u)\, du$$

$$= R_{XX}(\tau) * h(\tau)$$

Taking Fourier Transform on both sides we get

$$F[R_{XY}(\tau)] = F[R_{XX}(\tau) * h(\tau)]$$

$$= F[R_{XX}(\tau)]\, F[h(\tau)]$$

$\Rightarrow \quad S_{XY}(\omega) = S_{XX}(\omega)\, H(\omega)$ by the definition of spectrum.

Example 5.1.1

A WSS process X(t) with $R_{xx}(\tau) = A\,.\,e^{-a|\tau|}$ where 'A' and 'a' are real positive constants is applied to the i/P of an LTI systems with $h(t) = e^{-bt}\,.\,u(t)$ where b is a real positive constant. Find the PSD of the O/P of the system. **[A.U A/M 2010]**

Solution : The transfer functions is given by

$$H(\omega) = \frac{1}{J\omega + b}$$

$$\text{So, } |H(\omega)|^2 = \frac{1}{\omega^2 + b^2}$$

PSD of the i/P $X(t) = \text{F.T}\,[R_{XX}(\tau)]$

$$= F.T\left[A\,.\,e^{-a|\tau|}\right]$$

$$= A\,.\,\frac{2a}{\omega^2 + a^2}$$

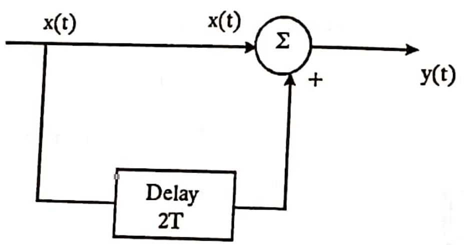
PSD of the O/P of the system be $S_{YY}(\omega)$.

$$S_{YY}(\omega) = \left[|H(\omega)|^2\right] \times [\text{PSD of the } i/P]$$

$$= \left(\frac{1}{\omega^2+b^2}\right)\left(\frac{2Aa}{a^2+\omega^2}\right)$$

Example 5.1.2

A WSS random process X(t) with PSD $S_{XX}(\omega)$ is applied as the *i/p* to the system expressed as



Find the PSD of Y(t).

The O/P $\quad Y(t) = X(t) + X(t-2T)$

$$\therefore Y(\omega) = X(\omega) + e^{-J2\omega T}.X(\omega)$$

$$\therefore Y(\omega) = X(\omega)\left[1+e^{-J2\omega T}\right]$$

The system transfer function

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + e^{-J2\omega T}$$

O/P $\quad$ PSD $= \left(|H(\omega)|^2\right) \times (i/P\ PSD)$

$$= \left[(1+\cos 2\omega T)^2 + \sin^2 2\omega T\right] S_{XX}(\omega)$$

$$= (2 + 2\cos 2\omega T)\, S_{XX}(\omega)$$

$$S_{YY}(\omega) = 4\cos^2\omega T . S_{XX}(\omega)$$

Example 5.1.3

X(t) is the i/P voltage to a circuit and Y(t) is the O/P voltage. X(t) is a stationary random process with zero mean and auto correlation $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean of Y(t) and its PSD if the system function $H(\omega) = \frac{1}{J\omega+2}$

Solution : $\quad H(\omega) = \frac{1}{\omega^2+4}$

$$\Rightarrow H(0) = \frac{1}{4}$$

$$E[Y(t)] = E[X(t)].H(0) = 0$$

$$|H(\omega)|^2 = \frac{1}{\omega^2+4}$$

i/P PSD $\quad S_{XX}(\omega) = F[R_{XX}(\tau)]$

$$= F\left[e^{-2|\tau|}\right] = \frac{4}{\omega^2+4}$$

$\therefore$ O/P PSD $\quad S_{YY}(\omega) = |H(\omega)|^2 . S_{XX}(\omega)$

$$= \frac{4}{(\omega^2+4)^2}$$

$$\therefore E(Y^2) = \int_{-2}^{0} (2+\tau)(9+2e^{\tau})\,d\tau + \int_{0}^{2} (2-\tau)(9+2e^{-\tau})\,d\tau$$

$$= \left[18\tau + 4e^{\tau} + \frac{9\tau^2}{2} + 2e^{\tau}(\tau-1)\right]_{-2}^{0} + \left[18\tau - 4e^{-\tau} - \frac{9\tau^2}{2} 2e^{-\tau}(\tau+1)\right]_{0}^{2}$$

$$\therefore E(Y^2) = 40.542$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= 40.542 - 36 = 4.542$$

Example 5.1.4

A random process n(t) has a PSD G (f) = $\frac{\eta}{2}$ for $-\infty \le f \le \infty$. The random process is passed through a low pass filter which has a transfer function H(f) = 2 for $-f_M \le f \le f_M$ and H(f) = 0 otherwise. Find the PSD of the waveform at the O/P of the filter.

Solution : $H(f) = 2$ for $-f_M \le f \le f_M$

$= 0$ otherwise.

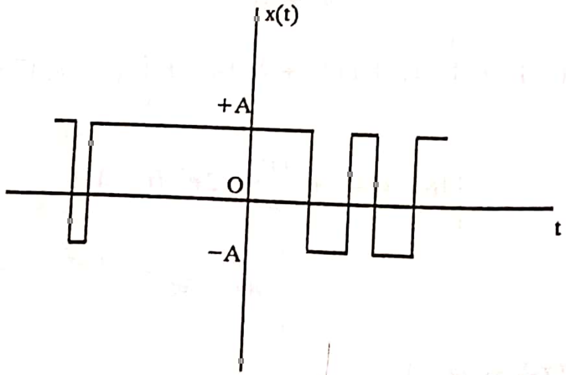
PSD of the O/P of the filter $= |H(f)|^2 \times i/P\ PSD$

$= 4 \times \frac{\eta}{2} = 2\eta$

Example 5.1.5

Determine the power spectral density of a random signal shown in the figure where the signal assumes either +A or –A with equal probability and the probability of 'n' such changes occur in the time interval τ by the poisson distribution

$P(n, \tau) = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$ where λ is the average number of changes per unit time.



Solution : First, let us find the auto correlation function of the given signal $x(t)$.

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Since $X(t)$ and $X(t+\tau)$ are two discrete random variables having the two possible values +A and –A, the product $X(t)X(t+\tau)$ is equal to $+A^2$ or $-A^2$ depending on whether $X(t) = X(t+\tau)$ or $X(t) = -X(t+\tau)$. This is turn depends on whether the number of changes during the time interval τ is even or odd.

Therefore, using Poisson distribution,

$$P[X(t) = X(t+\tau)] = P[n \text{ even}]$$

$$= e^{-\lambda t} \sum_{\substack{n=0 \\ n=\text{even}}}^{\infty} \frac{(\lambda\tau)^n}{n!} = e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^{2n}}{(2n)!}$$

$$= e^{-\lambda\tau} \cosh(\lambda\tau)$$

and $P[X(t) = -X(t+\tau)] = P[n \text{ odd}]$

$$= e^{-\lambda\tau} \sum_{\substack{n=0 \\ n=\text{odd}}} \frac{(\lambda\tau)^n}{n!} = e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^{2n+1}}{(2n+1)!}$$

$$= e^{-\lambda\tau} \sinh(\lambda\tau)$$

Thus, auto correlation function is given by

$$R_{XX}(\tau) = (+A^2)\, P[X(t) = X(t+\tau)] + (-A^2)\, P[X(t) = -X(t+\tau)]$$

$$= A^2 e^{-\lambda\tau}[\cosh(\lambda\tau) - \sinh(\lambda\tau)]$$

$$= A^2 e^{-2\lambda\tau}$$

Since $R_{XX}(\tau)$ is an even function of τ, we may write,

$$R_{XX}(\tau) = A^2 e^{-2\lambda|\tau|}$$

The power spectral density, $\delta_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} A^2 e^{-2\lambda|\tau|} e^{-j\omega\tau} d\tau$$

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Using the results we get,

$$\delta_{XX}(\omega) = \frac{4\lambda A^2}{4\lambda^2 + \omega^2}$$

Example 5.1.6

Prove that the system $Y(b) = \int_{-\infty}^{\infty} h(\xi)\, X(t-\xi)\, d\xi$ is a linear time-invariant system.

Solution : Let $X(t) = a_1 X_1(t) + a_2 X_2(t)$

Then $Y(t) = \int_{-\infty}^{\infty} h(\xi)\,[a_1 X_1(t-\xi) + a_2 (t-\xi)]\, d\xi$

$$= a_1 Y_1(t) + a_2 Y_2(t)$$

∴ the system is linear. If $X(t)$ is replaced by $X(t+h)$ then

$$Y(t+h) = \int_{-\infty}^{\infty} h(\xi)\, X(\overline{t+h} - \xi)\, d\xi$$

∴ the system is time invariant.

Example 5.1.7

Show that {X (t)} is a WSS process then the output {Y (t)} is a WSS process. [A.U N/D 2010, N/D 2011, N/D 2012, Trichy N/D 2011]

Solution : If the input to a time-invariant, stable linear system is a WSS process, then the output will also be a WSS process. (i.e.,) To show that if $\{X, (t)\}$ is a WSS process then the output $\{Y(t)\}$ is a WSS process.

Proof : We know that the input and output are related by

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du \quad \dots (1)$$

$$\Rightarrow \quad E\{Y(t)\} = \int_{-\infty}^{\infty} h(u)\, E[X(t-u)]\, du$$

Since $\{X(t)\}$ is a WSS process, Mean is a constant

(i.e.,) $E[X(t-u)]$ is a constant.

Hence, $E[Y(t)] = E[X'(t-u)] \int_{-\infty}^{\infty} h(u)\, du$

$$= \overline{X}_u \int_{-\infty}^{\infty} h(u)\, du$$

= a finite constant, independent of t.

[∵ system is a stable system]

∴ $E[Y(t)]$ = a constant.

Next, we show that the autocorrelation $R_{YY}(t, t+\tau)$ depends on τ.

Now, by definition,

$$R_{YY}(t, t+\tau) = E[Y(t) \cdot Y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\, X(t-u_1)\, h(u_2)\, X(t+\tau-u_2)\, du_1\, du_2\right]$$

[By using (1)]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, E[X(t-u_1)\, X(t+\tau-u_2)]\, du_1\, du_2 \quad \dots (1)$$

Since $\{X(t)\}$ is a WSS procss.

$E[X(t-u_1)\, X(t+\tau-u_2)]$ is a function of τ, say $g(\tau)$.

Hence, (1) becomes

$$R_{YY}(t, t+\tau) = g(\tau) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, du_1\, du_2$$

$$= \text{a function of } \tau.$$

Hence the output $\{Y(t)\}$ is also WSS process.

Example 5.1.8

Find the mean square value of the processes whose power spectral density is as given below :

(a) $\dfrac{1}{\omega^4 + 10\omega^2 + 9}$ **; (b)** $\dfrac{\omega^2 + 2}{\omega^4 + 13\omega^2 + 36}$

To find mean-square value of the process, we can find its auto correlation function and substitute $\tau = 0$.

(a) $\dfrac{1}{\omega^4 + 10\omega^2 + 9}$

$$S_X(\omega) = \frac{1}{\omega^4 + 10\omega^2 + 9}$$

$$= \frac{1}{(\omega^2+9)(\omega^2+1)} = \frac{1}{8}\left(\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right)$$

$R_{XX}(\tau)$ is Fourier inverse transform of

$$\frac{1}{8}\left(\frac{1}{\omega^2+1}\right) - \frac{1}{8}\left(\frac{1}{\omega^2+9}\right)$$

$$R_{XX}(\tau) = \frac{1}{8}\cdot\frac{1}{2}e^{-|\tau|} - \frac{1}{8}\cdot\frac{1}{6}e^{-3|\tau|}$$

$$= \frac{1}{16}e^{-|\tau|} - \frac{1}{48}e^{-3|\tau|}$$

The mean square value is $R_{XX}(\tau)$ at $\tau = 0$

$$R_{XX}(0) = \frac{1}{16} - \frac{1}{48} = \frac{2}{48} = \frac{1}{24}$$

(b) $S_X(\omega) = \dfrac{\omega^2 + 2}{\omega^4 + 13\omega^2 + 36}$

$$S_X(\omega) = \frac{\omega^2+2}{\omega^4 + 13\omega^2 + 36} = \frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)}$$

$$= \frac{(\omega^2+4) - 2}{(\omega^2+9)(\omega^2+4)}$$

$$= \frac{1}{\omega^2+9} - \frac{2}{(\omega^2+9)(\omega^2+4)}$$

$$= \frac{1}{\omega^2+9} - \frac{2}{5}\left(\frac{1}{\omega^2+4} - \frac{1}{\omega^2+9}\right)$$

$$= \frac{7}{5}\cdot\frac{1}{\omega^2+9} - \frac{2}{5}\cdot\frac{1}{\omega^2+4}$$

Taking Fourier inverse transform of $S_X(\omega)$, we get $R_{XX}(\tau)$,

$$\text{So, } R_{XX}(\tau) = \frac{7}{5}\cdot\frac{1}{6}\cdot e^{-3|\tau|} - \frac{2}{5}\cdot\frac{1}{4}\cdot e^{-2|\tau|}$$

$$= \frac{7}{30}e^{-3|\tau|} - \frac{2}{20}e^{-2|\tau|}$$

$$R_{XX}(0) = \frac{7}{30} - \frac{2}{20} = \frac{14-6}{60} = \frac{8}{60}$$

Example 5.1.9

The auto correlation function of the Poisson increment process is given by $R(\tau) = \begin{cases} \lambda^2 & ; \text{for } |\tau| > \epsilon \\ \lambda^2 + \dfrac{\lambda}{\epsilon}\left(1 - \dfrac{|\tau|}{\epsilon}\right) & ; \text{for } |\tau| \le \epsilon \end{cases}$.

Prove that its spectral density is given by

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda \sin^2(\omega\epsilon/2)}{\epsilon^2\omega^2}.$$

[AU M/J 2007, N/D 2011]

Solution : $S(\omega) = \int_{-\in}^{\in} \left\{\lambda^2 + \frac{\lambda}{\in}\left(1 - \frac{|\tau|}{\in}\right)\right\} e^{-i\omega\tau}\, d\tau$

$$+ \int_{-\infty}^{\in} \lambda^2 e^{-i\omega\tau} d\tau + \int_{\in}^{\infty} \lambda^2 e^{-i\omega\tau} d\tau$$

$$= \frac{\lambda}{\in}\int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right) e^{-i\omega\tau} d\tau + \int_{-\infty}^{\infty} \lambda^2 e^{-i\omega\tau} d\tau$$

$$= \frac{2\lambda}{\in}\int_{0}^{\in}\left(1 - \frac{\tau}{\in}\right)\cos\omega\tau\, d\tau + F\{\lambda^2\}$$

where F (λ^2) is the Fourier transform of λ^2.

$$= \frac{2\lambda}{\in}\left[\left(1 - \frac{\tau}{\in}\right)\frac{\sin\omega\tau}{\omega} + \frac{1}{\in}\left(\frac{-\cos\omega\tau}{\omega^2}\right)\right]_0^{\in} + F(\lambda^2)$$

$$= \frac{2\lambda}{\in^2\omega^2}(1 - \cos\omega \in) + F(\lambda^2)$$

$$S(\omega) = \frac{4\lambda\sin^2(\omega \in/2)}{\in^2\omega^2} + F(\lambda^2) \quad \text{... (1)}$$

The Fourier inverse transform of S (ω) is given by

$$R(\tau) = F^{-1}\{S(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\tau\omega}\, d\omega$$

Let us now find R (τ) corresponding to S (ω) = $2\pi\lambda^2\delta(\omega)$, where $\delta(\omega)$ is the unit impulse function.

$$\text{(i.e.,) } R(\tau) = F^{-1}\{2\pi\lambda^2\delta(\omega)\}$$

$$= \frac{2\pi\lambda^2}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)\, e^{i\tan\omega}\, d\omega$$

$$= \lambda^2 \qquad \left[\text{since } \int_{-\infty}^{\infty}\phi(t)\,\delta(t) = \phi(0)\right]$$

$$\therefore F(\lambda^2) = 2\pi\lambda^2\delta(\omega) \quad \text{... (2)}$$

Inserting (2) in (1) we get

$$(1) \Rightarrow S(\omega) = \frac{4\lambda\sin^2(\omega \in/2)}{\in^2\omega^2} + 2\pi\lambda^2\,\delta(\omega)$$

Example 5.1.10

The short-time moving average of a process {X(t)} is defined as $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)\, ds$. Prove that X(t) and Y(t) are related by means of a convolution type integral. Find the unit impulse response of the system also.

Solution : Given : $Y(t) = Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)\, ds$ ... (1)

Putting $s = t - u$ and treating t as a parameter, (1) becomes

$$Y(t) = \frac{1}{T}\int_{0}^{T} X(t-u)\, du \quad \text{... (2)}$$

Now we define the unit impulse response of the system as follows

$$h(t) = \begin{cases}\frac{1}{T}\,; & \text{for } 0 \le t \le T\\ 0\,, & \text{otherwise}\end{cases}$$

$$(2) \Rightarrow Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

Which is a convolution type integral.

Example 5.1.11

If the input x(t) and the output y(t) are connected by the differential equation $T\frac{dy(t)}{dt} + y(t) = x(t)$, prove that they can be related by means of a convolution type integral. Assume that x(t) and y(t) are zero for $t \le 0$.

Solution : Given $y'(t) + \frac{1}{T}y(t) = \frac{1}{T}x(t)$ is a linear equation.

$$y(t) = \frac{1}{T}\int_0^t x(t-y)\,e^{-u/T}\,du \quad \text{... (1)}$$

Given : $x(t) = 0,$ for $t < 0$

$\therefore \quad x(t-u) = 0,$ for $t < u$

$$(1) \Rightarrow y(t) = \frac{1}{T}\int_0^{\infty} x(t-u)\,e^{-u/T}\,du \quad \text{... (2)}$$

Now if we define

$$h(t) = \begin{cases} \frac{1}{T}\,e^{-t/T}, & \text{for } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$(2) \Rightarrow y(t) = \int_{-\infty}^{\infty} h(t)\,x(t-u)\,du$$

Hence the result.

Example 5.1.12

A system has an impulse response $h(t) = e^{-\beta t}U(t)$, find the power spectral density of the output Y(t) corresponding to the input X(t).

[A.U N/D 2010, N/D 2012]

Solution : Given : $X(t) \to$ input process

$Y(t) \to$ output process

We know that $S_{YY}(\omega) = |H(\omega)|^2\, S_{XX}(\omega)$ ... (1)

$H(\omega)$ is the Fourier transform of the function $h(t)$

The unit step function $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$

$$\therefore h(t) = \begin{cases} 0, & t < 0 \\ e^{-\beta t}, & t \ge 0 \end{cases}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt$$

$$= \int_0^{\infty} e^{-\beta t}\,e^{-i\omega t}\,dt$$

$$= \int_0^{\infty} e^{-(\beta + i\omega)t}\,dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)}\right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega}\left[e^{-(\beta+i\omega)t}\right]_0^{\infty}$$

$$= \frac{-1}{\beta+i\omega}[0-1]$$

$$= \frac{1}{\beta+i\omega}$$

$$|H(\omega)| = \frac{1}{|\beta+i\omega|} = \frac{1}{\sqrt{\beta^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}$$

$$(1) \Rightarrow S_{YY}(\omega) = \frac{1}{\beta^2+\omega^2}\,S_{XX}(\omega)$$

Example 5.1.13

Examine whether the following systems are linear :

(a) $y(t) = \alpha\, x(t)$, where α is a scalar.

(b) $y(t) = t\, x(t)$

(c) $y(t) = x^2(t)$ **[A.U CBT M/J 2010, Tvli A/M 2011]**

Solution : (a) Let $y_1(t)$ and $y_2(t)$ be the output signals corresponding to the input signals $x_1(t)$ and $x_2(t)$, respectively, i.e.,

$$y_1(t) = \alpha x_1(t) \text{ and } y_2(t) = \alpha x_2(t)$$

For any scalars c_1 and c_2, the output signal for the input signal $x(t) = c_1 x_1(t) + c_2 x_2(t)$ is given by

$$y(t) = \alpha x(t) = \alpha [c_1 x_1(t) + c_2 x_2(t)]$$

$$= c_1 [\alpha x_1(t)] + c_2 [\alpha x_2(t)]$$

$$= c_1 y_1(t) + c_2 y_2(t) \text{ from this we get}$$

the given system is linear.

(b) Let $y_1(t)$ and $y_2(t)$ be the output signals corresponding to the input signals $x_1(t)$ and $x_2(t)$, respectively, i.e.,

$$y_1(t) = t x_1(t) \text{ and } y_2(t) = t x_2(t)$$

For any scalars c_1 and c_2, the output signal for the input signal $x(t) = c_1 x_1(t) + c_2 x_2(t)$ is given by

$$y(t) = tx(t) = t[c_1 x_1(t) + c_2 x_2(t)]$$

$$= c_1 [tx_1(t)] + c_2 [tx_2(t)]$$

$$= c_1 y_1(t) + c_2 y_2(t) \text{ from this we get}$$

the given system is linear.

(c) Let $y_1(t)$ and $y_2(t)$ be the output signals corresponding to the input signals $x_1(t)$ and $x_2(t)$, respectively, i.e.,

$$y_1(t) = x_1^2(t) \text{ and } y_2(t) = x_2^2(t)$$

For any scalars c_1 and c_2, the output signal for the input signal $x(t) = c_1 x_1(t) + c_2 x_2(t)$ is given by

$$y(t) = x^2(t) = [c_1 x_1(t) + c_2 x_2(t)]^2$$

$$= c_1^2 x_1^2(t) + c_2^2 x_2^2(t) + 2 c_1 c_2(t) x_2(t)$$

But $c_1 y_1(t) + c_2 y_2(t) = c_1 x_1^2(t) + c_2 x_2^2(t)$

Since $y(t) \neq c_1 y_1(t) + c_2 y_2(t)$, we conclude that the given system is non-linear.

Example 5.1.14

Examine whether the following systems are time-invariant.

(a) $y(t) = \alpha x(t)$ (b) $y(t) = tx(t)$

(c) $y(t) = x(t) - x(t-a)$

[A.U CBT A/M 2011]

Solution : (a) Given : $y(t) = f[x(t)] = \alpha x(t)$ ... (1)

Let the input $x(t)$ alone be shifted by h time units so that

(1) $\Rightarrow$ $y(t) = f[x(t+h)] = \alpha x(t+h)$ ... (2)

Now, if the output is shifted by h time units

(1) $\Rightarrow y(t+h) = \alpha x(t+h)$ ... (3)

From (2) & (3) we get $y(t+h) = y(t)$

$\therefore$ The given system is time-invariant.

(b) Given : $y(t) = f[x(t)] = tx(t)$ ... (1)

Let the input $x(t)$ alone be shifted by h time units so that

(1) $\Rightarrow y(t) = f[x(t+h)] = t\ x(t+h)$ ... (2)

Now, if the output is shifted by h time units

(1) $\Rightarrow$ $y(t+h) = (t+h)\ x(t+h)$

$= t\ x(t+h) + h\ x(t+h)$ ... (3)

From (2) & (3) we get $y(t+h) \neq y(t)$

$\therefore$ The given system is not time-invariant.

(c) Given : $y(t) = f[x(t)] = x(t) - x(t-a)$ ... (1)

Let the input $x(t)$ alone be shifted by h time units so that

(1) $\Rightarrow y(t) = f[x(t+h)] = x(t+h) - x(t+h-a)$ ... (2)

Now, if the output is shifted by h time units

(1) $\Rightarrow y(t+h) = x(t+h) - x(t+h-a)$ ... (3)

From (2) & (3) we get $y(t+h) = y(t)$

$\therefore$ The given system is time-invariant.

Example 5.1.15

Examine whether the following systems are causal :

(a) $y(t) = x(t) - x(t-a)$
(b) $y(t) = tx(t)$
(c) $y(t) = x(t+2)$

Solution : (a) and (b) The given systems are causal because the present value of $y(t)$ depends only on the present or previous values of the input $x(t)$.

(c) The given system is not causal because the present value of $y(t)$ depends on the future values of the input $x(t)$.

Example 5.1.16

The power spectral density of a signal X(t) is $S_{XX}(\omega)$ and its power is P. Find the power of the signal aX(t). **[A.U CBT M/J 2010]**

Solution : The system described is linear.

Let $Y(t)$ be $aX(t)$

So, $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$= a^2 S_{XX}(\omega)$

Power of $Y(t)$ $= 2\int_0^{\infty} S_{YY}(\omega)\, df$

$= 2\int_0^{\infty} a^2 S_{XX}(\omega)\, df$

$= 2a^2 \int_0^{\infty} S_{XX}(\omega)\, df$

$= 2a^2 P$

5.2 Auto correlation and cross correlation functions of input and output.

5.2.(a) Auto - correlation function of response

Let X (t) be wise-sense stationary. The auto correlation function of Y (t) is

$$R_{YY}(t, t+\tau) = E\{Y(t)\, Y(t+\tau)\} \quad \text{... (1)}$$

WKT $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\varepsilon)\, X(t-\varepsilon)\, d\varepsilon$

$$Y(t+\tau) = \int_{-\infty}^{\infty} h(\varepsilon_2)\, X(t+\tau-\varepsilon_2)\, d\varepsilon_2 \quad \text{... (2)}$$

Sub (2) in (1)

$$\therefore R_{YY}(t, t+\tau) = E\left\{\int_{-\infty}^{\infty} h(\varepsilon_1)\, X(t-\varepsilon_1)\, d\varepsilon_1 \int_{-\infty}^{\infty} h(\varepsilon_2)\, X(t+\tau-\varepsilon_2)\, d\varepsilon_2\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\,\{X(t-\varepsilon_1)\, X(t+\tau-\varepsilon_2)\}\, h(\varepsilon_1)\, h(\varepsilon_2)\, d\varepsilon_1\, d\varepsilon_2$$

which reduces to

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\varepsilon_1-\varepsilon_2)\, h(\varepsilon_1)\, h(\varepsilon_2)\, d\varepsilon_1 \varepsilon_2$$

because X (t) is assumed wide-sense stationary.

Y (t) is a wide-sense stationary if X (t) is wide-sense stationary because $R_{yy}(\tau)$ does not depend on t and E {y (t)} is a constant.

$R_{YY}(\tau)$ shown the two fold convalution of the input auto correlation function with the network's impulse response.

i.e., $R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$

5.2.(b) Cross - correlation functions of input and output

The cross-correlation function between the input X (*t*) and the output Y (*t*) is given by

(i) $R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$

(ii) $R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$ [A.U CBT M/J 2010]

Proof : [A.U Tvli. M/J 2010] [A.U N/D 2011]

The cross-correlation function of X (*t*) and Y (*t*) is

$$R_{XY}(t, (t+\tau) = E\{X(t)\, Y(t+\tau)\} \quad \dots (1)$$

Now

$$Y(t+\tau) = h(t) * X(t+\tau)$$

$$= \int_{-\infty}^{\infty} h(\varepsilon)\, X(t+\tau-\varepsilon)\, d\varepsilon \quad \dots (2)$$

Sub (2) in (1)

$$R_{XY}(t, t+\tau) = E\left\{X(t) \int_{-\infty}^{\infty} h(\varepsilon)\, X(t+\tau-\varepsilon)\, d\varepsilon\right\}$$

$$= \int_{-\infty}^{\infty} E\{X(t)\, X(t+\tau-\varepsilon)\}\, h(\varepsilon)\, d\varepsilon \quad \dots (3)$$

If X (*t*) is wide-sense stationary, equation (3) reduces to

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\varepsilon)\, h(\varepsilon)\, d\varepsilon$$

which is the convolution $R_{XX}(\tau)$ with $h(\tau)$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$

A similar development shown that

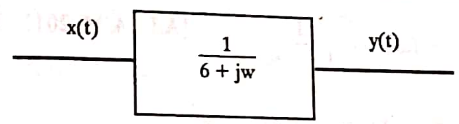
$$R_{YX}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\varepsilon)\, h(-\varepsilon)\, d\varepsilon$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$$

From the above it is very clear that cross-correlation functions depend on τ and not on absolute time '*t*'

Example 5.2.1

Consider a linear system as shown below.



X(t) is the input and Y(t) is the output of the system. The auto correlation of x(t) is $R_{xx}(\tau) = 3.\,\delta(\tau)$. Find the PSD, auto correlation function and mean square value of the output Y(t).

Solution : We have the relation

$$\text{Output PSD} = |H(\omega)|^2 \times \text{input PSD}$$

$$\text{Input PSD} = F[R_{xx}(\tau)]$$

$$= F[3.\delta(\tau)] = 3$$

$$\therefore \quad \text{Output PSD} = \frac{3}{36+\omega^2}$$

$$R_{YY}(\omega) = F^{-1}(\text{input PSD})$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3}{36+\omega^2} . e^{j\omega\tau.d\omega}$$

We have $e^{-a|\tau|} \overset{F}{\longleftrightarrow} \dfrac{2a}{a^2\omega^2}$

$$\therefore F^{-1}(\text{output PSD}) = F^{-1}\left[\frac{3 \times 26}{12(6^2+\omega^2)}\right]$$

$$= \frac{3}{12} \exp(-6|\tau|)$$

Mean square value $E[Y^2(t)] = R_{YY}(0)$

$$= \frac{3}{12} = \frac{1}{4}$$

Example 5.2.2

Consider a system with transfer function $\frac{1}{1+i\omega}$. An input signal with autocorrelation function $m\,\delta(\tau) + m^2$ is fed as input to the system. Find the mean and mean-square value of the output. **[A.U A/M 2011, M/J 2012]**

Solution : Given, $H(\omega) = \frac{1}{1+i\omega}$

and $R_{XX}(\tau) = m\,\delta(\tau) + m^2$

$$S_X(\omega) = m + 2\pi m^2 \delta(\omega)$$

We know that, $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$

$$= \left|\frac{1}{1+i\omega}\right|^2 \cdot \left[m + 2\pi m^2 \delta(\omega)\right]$$

$$= \frac{1}{1+\omega^2} \cdot \left[m + 2\pi m^2 \delta(\omega)\right]$$

$R_{YY}(\tau)$ is the Fourier inverse transform of $S_Y(\omega)$.

So, $R_{YY}(\tau) = \frac{m}{2} e^{-|\tau|} + m^2$

We know that $\lim_{\tau \to \infty} R_{XX}(\tau) = \overline{X}^2$

So $\overline{X}^2 = m^2$

$\overline{X} = m$

Also $H(0) = 1$

We know that $\overline{Y} = H(0)\,\overline{Y}$

So, mean of the output $= \overline{Y} = 1, m = m$

Mean-square value of the output $= \overline{Y}^2 = R_{YY}(0) = \frac{m}{2} + m^2$

Example 5.2.3

A signal $x(t) = \text{sinc } 2Bt$ is applied to an integrator with transfer function $|H(f)| = \frac{1}{1+\left(\frac{f}{W}\right)^2}$. Find the output power density spectrum and output autocorrelation function when $B < W$.

Note : The function $\text{sinc } x = \frac{\sin \pi x}{\pi x}$

Given $x(t) = \text{sinc } 2Bt$

$$X(f) = \frac{1}{2B}, \quad -B \le f \le B$$

$$S_X(f) = \left|\frac{1}{2B}\right|^2 = \frac{1}{4B^2}$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= \left[\frac{1}{1+\left(\frac{f}{W}\right)^2}\right]^2 \cdot \frac{1}{4B^2} \qquad -B \le f \le B$$

When $B < W$,

$$S_Y(f) = 1 . \frac{1}{4B^2} = S_X(f)$$

$$R_{YY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4B^2} e^{i\omega\tau} d\omega = \frac{1}{2B} \sin 2B\tau$$

Example 5.2.4

Find the output power density spectrum and output autocorrelation function for a system with $h(t) = e^{-t}$, $t \ge 0$, for an input with power density spectrum $\frac{\eta_0}{2}$, $-\infty < f < \infty$.

Solution : Given $h(t) = e^{-t}, t \ge 0$

So, $H(f) = \frac{1}{1 + i(2\pi f)}$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= \frac{1}{1+(2\pi f)^2} \cdot \frac{\eta_0}{2}, \qquad -\infty < f < \infty$$

The autocorrelation of the output $R_{YY}(\tau)$ is Fourier inverse of $S_Y(f)$

$$R_{YY}(\tau) = \frac{\eta_0}{4} e^{-|\tau|}, \qquad -\infty < \tau < \infty$$

Example 5.2.5

A linear system is described by the impulse response

$$h(t) = \frac{1}{RC} e^{-\left(\frac{t}{RC}\right)}$$

Assume an input signal whose autocorrelation function is $B\,\delta(\tau)$. Find the autocorrelation, mean and power of the output. [A.U A/M 2011]

Given : $h(t) = \frac{1}{RC} e^{-\left(\frac{t}{RC}\right)}$

Let $y(t)$ be the output.

Mean value of the output $= \overline{y(t)} = \overline{x(t)} \int_{-\infty}^{\infty} h(t)\, dt = 0$

The autocorrelation of the output $y(t)$ is

$$R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_{XX}(\tau + \alpha - \beta)\, d\alpha\, d\beta$$

Given $R_{XX}(\tau) = B\,\delta(\tau)$

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) . h(\beta) . B\,\delta(\tau + \alpha - \beta)\, d\alpha\, d\beta$$

$$= B \int_{-\infty}^{\infty} h(\alpha) \int_{-\infty}^{\infty} h(\beta)\, \delta(\tau + \alpha - \beta)\, d\beta\, d\alpha$$

$$= B \int_{0}^{\infty} h(\alpha) h(\tau + \alpha)\, d\alpha$$

As $h(t) = \frac{1}{RC} e^{-\left(\frac{1}{RC}\right)}$,

$$R_{YY}(\tau) = B \int_0^{\infty} \frac{1}{RC} e^{-\left(\frac{\alpha}{RC}\right)} \frac{1}{RC} e^{-\left(\frac{\tau+\alpha}{RC}\right)}\, d\alpha$$

$$= \frac{B}{(RC)^2} \int_0^{\infty} e^{-\frac{\tau}{RC}} e^{-\frac{2\alpha}{RC}}\, d\alpha$$

$$= \frac{B}{(RC)^2} . e^{-\frac{\tau}{RC}} \int_0^{\infty} e^{-\frac{2\alpha}{RC}}\, d\alpha$$

$$= \frac{B}{(RC)^2} e^{-\frac{\tau}{RC}} . \left[\frac{e^{-\frac{2\alpha}{RC}}}{\frac{-2}{RC}}\right]_0^{\infty}$$

$$= \frac{B}{(RC)^2} \cdot \frac{RC}{-2} . e^{-\frac{\tau}{RC}} (-1)$$

$$= \frac{B}{2RC} e^{-\frac{\tau}{RC}}, \ \tau \geq 0$$

As $R_{YY}(\tau)$ is an even function of τ,

$$R_{YY}(\tau) = \frac{B}{2RC} e^{\frac{\tau}{RC}}, \ \tau \leq 0$$

So, $$R_{YY}(\tau) = \frac{B}{2RC} e^{\frac{-|\tau|}{RC}}, \ -\infty < \tau < \infty$$

The power spectral density of the output $y(t)$ is the Fourier transform of $R_{YY}(\tau)$

$$S_Y(f) = \frac{B}{2RC} \cdot \frac{\left(\frac{2}{RC}\right)}{\left(\frac{1}{RC}\right)^2 + (2\pi f)^2}$$

$$= \frac{B}{1 + (RC)^2 + (2\pi f)^2}$$

$$\text{Power} = \int_{-\infty}^{\infty} S_Y(f)\, df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B}{1 + (RC)^2 (2\pi f)^2} d\omega$$

$$= \frac{B}{2RC}$$

Example 5.2.6

X (t) is the input voltage to a circuit (system) and Y (t) is the output voltage. {X (t)} is a stationary random process with $\mu_x = 0$ and $R_{xx}(\tau) = e^{-\alpha|\tau|}$. Find μ_y, $Y_{yy}(w)$, if the power transfer function is $H(w) = \frac{R}{R + i Lw}$ **[A.U N/D 2008]**

Solution : $Y(t) = \int_{-\infty}^{\infty} h(\varepsilon)\, X(t-\varepsilon)\, d\varepsilon$

$$\therefore E[Y(t)] = \int_{-\infty}^{\infty} h(\varepsilon)\, E\{X(t-\varepsilon)\}\, d\varepsilon$$

$$= 0 \qquad [\because E\{X(t-\varepsilon)\} = \mu_x = 0]$$

$$Y_{xx}(w) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-iw\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-iw\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-iw\tau} d\tau$$

$$= \left\{\frac{e^{(\alpha - iw)\tau}}{\alpha - iw}\right\}_{-\infty}^{0} + \left\{\frac{e^{-(\alpha + iw)\tau}}{-(\alpha + iw)}\right\}_{0}^{\infty}$$

$$= \frac{1}{\alpha - iw} + \frac{1}{\alpha + iw} = \frac{2\alpha}{\alpha^2 + w^2}$$

Now

$$Y_{YY}(w) = Y_{xx}(w)\, |H(w)|^2$$

$$= \frac{2\alpha}{\alpha^2 + w^2} \times \frac{R^2}{R^2 + L^2 wx}$$

$$= \frac{\{(2\alpha R^2/(R^2 - L^2\alpha^2))\}}{\alpha^2 + w^2} + \frac{\{2\alpha R^2/(\alpha^2 - R^2/L^2)\}}{R^2 + L^2 w^2}$$

[Using partial fraction]

$$= \frac{2\alpha\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2 - \alpha^2} \times \frac{1}{\alpha^2 + wz} + \frac{2\alpha R^2/L^2}{\alpha^2 - \left(\frac{R}{L}\right)^2} \times \frac{1}{\left(\frac{R}{L}\right)^2 + w^2}$$

$$X_{YY}(w) = \lambda \frac{1}{\alpha^2 w^2} + \mu \frac{1}{\left(\frac{R}{L}\right)^2 + w^2} \text{ (say)}$$

Example 5.2.7.

Assume a random process X(t) is given as input to a system with transfer function $H(\omega) = 1$ for $-\omega_0 < \omega < \omega_0$. If the autocorrelation function of the input process is $\frac{N_0}{2}\delta(t)$, find the autocorrelation function of the output process. **[AU A/M 2010, Trichy M/J 2011]**

Solution : Given : $R_{XX}(\tau) = \frac{N_0}{2}\delta(t)$

$$H(\omega) = 1, \ -\omega_0 < \omega < \omega_0$$

$$S_{XX}(\omega) = \frac{N_0}{2}$$

Power spectral density of $Y(t) = S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$= (1)^2 \frac{N_0}{2}$$

$$= \frac{N_0}{2}$$

The output of ACF $= R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{N_0}{2}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau}\, d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi i\tau}\left[e^{i\omega\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi i\tau}\left[e^{i\omega_0\tau} - e^{-i\omega_0\tau}\right]$$

$$= \frac{N_0}{2\pi\tau}\left[e^{i\omega_0\tau} - \frac{e^{-i\omega_0\tau}}{2i}\right]$$

$$= \frac{N_0}{2\pi\tau}\sin\omega_0\tau$$

Example 5.2.8.

If X(t) is the input voltage to a circuit and Y(t) is the output voltage. {X(t)} is a stationary random process with $\mu_x = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find the mean μ_Y and power spectrum $S_{YY}(\omega)$ of the output if the system transfer function is given by $H(\omega) = \frac{1}{\omega + 2i}$

[A.U N/D 2010, N/D 2012]

Solution : Given : $\mu_x = 0$, $R_{XX}(\tau) = e^{-2|\tau|}$

We know that for a linear time invariant system

$$Y(t) = h(t) * X(t)$$

$$= \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

$$\therefore \quad E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u)\, X(t-u)\, du\right]$$

$$\mu_Y = \int_{-\infty}^{\infty} h(u)\; E[X(t-u)]\, du$$

$$= \int_{-\infty}^{\infty} h(u)\; \mu_X\, du \qquad [\because X(t) \text{ is stationary}$$

$$\Rightarrow E[X(t)] = \mu_X = E[X(t-u)]]$$

$$= 0 \qquad [\because \mu_X = 0]$$

We know that $S_{YY}(\omega) = S_{XX}(\omega)\,|H(\omega)|^2$ ... (1)

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|}\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau}\, e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-2\tau}\, e^{-i\omega\tau}\, d\tau$$

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$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau}\, d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \left[\left(\frac{1}{2-i\omega}\right) - 0\right] + \left[(0) - \left(\frac{1}{-(2+i\omega)}\right)\right]$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega} = \frac{2+i\omega+2-i\omega}{4+\omega^2}$$

$$S_{XX}(\omega) = \frac{4}{2^2+\omega^2}$$

$$H(\omega) = \frac{1}{\omega+2i} = \frac{1}{\omega+2i}\,\frac{\omega-2i}{\omega-2i} = \frac{\omega-2i}{\omega^2+2^2}$$

$$|H(\omega)| = \sqrt{\left(\frac{\omega}{\omega^2+4}\right)^2 + \left(\frac{2}{\omega^2+4}\right)^2} = \sqrt{\frac{\omega^2+4}{(\omega^2+4)^2}} = \frac{1}{\sqrt{\omega^2+4}}$$

$$\therefore (1) \Rightarrow S_{YY}(\omega) = \left(\frac{4}{4+\omega^2}\right)\left(\frac{1}{\omega^2+4}\right) = \frac{4}{(\omega^2+4)^2}$$

EXERCISE 5.1 & 5.2

1. Check whether the following systems are time in invariance or time-variant.
   (a) $y(t) = x(-t)$; (b) $y(t) = x(t) \cos \omega_0 t$
   [Ans. Time-Variant, time-variant]

2. Check whether the following systems are linear or non-linear
   (a) $y(t) = x(t^2)$; (b) $y(t) = Ax(t) + B$
   [Ans. Linear, non-linear]

3. Check whether the following systems are causal or non-causal.
   (a) $y(t) = x(t) - 3x(t)$; (b) $y(t) = x(2t)$
   [Ans. Causal, Non-causal]

4. A random signal with power spectral density $\frac{\eta_0}{2}$ is given as input to a system with transfer function $\frac{1}{1+i2\pi fRC}$. Find the power and the autocorrelation function of the output.
   [Ans. $\frac{1}{4RC}\,\eta_0$, $\frac{\eta_0}{4RC}\, e^{\frac{-|\tau|}{RC}}$]

5. A circuit has an impulse response $h(t) = \frac{1}{T}$, for $0 \le t \le T$
   $= 0$, otherwise
   Find an expression for the power density spectrum of the output in terms of that of input.
   [Ans. $\frac{\sin^2(\pi f T)}{(\pi f T)^2}\, S_{XX}(f)$]

6. X(t), a stationary random process with zero mean is given as input to a system with transfer function $H(f) = \frac{R}{R+i(2\pi f)L}$. The autocorrelation of the input is $e^{-\beta|\tau|}$. Find the mean and power autocorrelation of the input is of output process. Check whether output process is stationary.
   [Ans. 0, $\frac{2\beta}{\beta^2-\omega^2}$, Yes]

7. A zero mean random sequence with autocorrelation function $R_{XX}(k) = \begin{cases} 1, & k=0 \\ 0, & k \ne 0 \end{cases}$ is fed to a system with impulse response

$h(k) = \begin{cases} 2, & k = 0.1 \\ 0, & k > 1 \end{cases}$. Find the mean and power density spectrum of the output sequence.

[Ans. 0, 4 + 4 cos $2\pi f$]

8. A system has a transfer function as $\dfrac{1}{1 + j\left(\dfrac{f}{1000}\right)}$. If the input to the system is a zero-mean stationary random process with power density spectrum as 10^{-9}, find the power of the output.

[Ans. $10^{-9}(1000\pi)$]

9. A random process has a power density spectrum $S_X(\omega) = 98\pi\,\delta(\omega) + 49$. It is applied to a system with transfer function $H(s) = \dfrac{4}{s+49}$. Find the mean value of the output process.

[Ans. $\frac{4}{7}$]

10. A signal has a power spectrum as $S_X(\omega) = \begin{cases} 10 & |\omega| \le 10\pi \\ 0 & |\omega| > 10\pi \end{cases}$.
It is passed through a system with transfer function

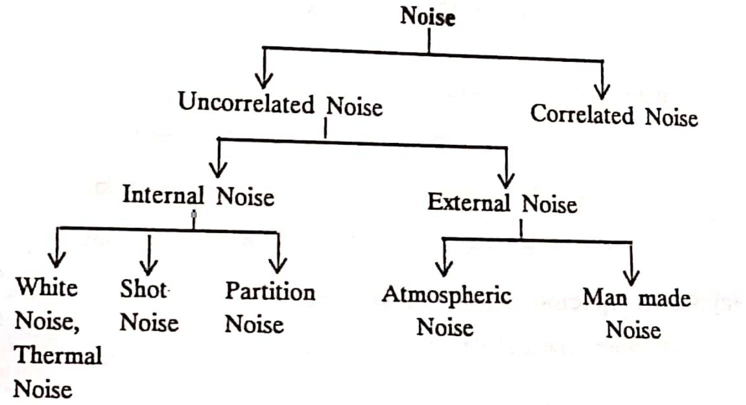
$$|H(\omega)|^2 = \begin{cases} 1 - \dfrac{|\omega|}{20\pi} & |\omega| \le 20\pi \\ 0 & |\omega| > 20\pi \end{cases}$$

. Find the output power of the system.

[Ans. 75 watts]

5.3 White NOISE

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signals in communication systems and over which we have incomplete control.



(a) Shot Noise

The discrete nature of electrons causes a signal disturbance called Shot Noise.

Note : Shot noise arises in electronic devices such as diodes and transistors because of the discrete nature of current flow in these devices.

(b) Thermal Noise

This noise is due to the random motion of free electrons in a conducting medium such as a resistor.

(or)

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor.

(c) White Noise (or) Gaussian Noise [A.U Tvli. M/J 2010]

The noise analysis of communication systems is based on an idealized form of noise called White Noise. [A.U A/M 2011]

(d) Power spectral density of thermal noise

The power spectral density of the noise current due to the free electrons is given by

$$S_i(w) = \left[\frac{2KT\,G_\alpha^2}{\alpha^2 + w^2}\right] = \frac{2KTG}{1 + \left(\frac{w}{\alpha}\right)^2}$$

where K is the Boltzmann's constant

α is the average number of collision/second.

T is the ambient temperature in degrees kelvin.

G is the conductance of the conducting medium.

(e) Power spectral density of shot noise.

The power spectral density of $i_n(t)$ is given by

$$S_i(w) = \bar{n}\,|I_e(w)|^2$$

The shot noise current consists of two components, a constant current component I_o and the time varying component $i_n(t)$

The component $i_n(t)$ as it is random, cannot be specified as a function of time. However $i_n(t)$ represents a stationary random signal and can be specified by its power density spectrum. Since there are $\hat{n}$ pulses per second it is reasonable to expect that the power density spectrum at $i_n(t)$ will be $\bar{n}$ times the energy density spectrum of $i_e(t)$. Thus if $i_e(t) \longrightarrow I_e(w)$.

(f) Band-Limited White Noise [A.U Tvli. M/J 2010]

Noise having a non-zero and constant spectral density over a finite frequency band and zero elsewhere is called band-limited white noise (i.e.,) if $\{N(t)\}$ is a band-limited white noise then

$$S_{NN}(w) = \begin{cases} \frac{N_o}{2}, & |w| \le w_B \\ 0, & \text{elsewhere} \end{cases}$$

Properties

1. $E[N^2(t)] = \frac{N_o w_B}{2\pi}$

2. $R_{NN}(\tau) = \frac{N_o w_B}{2\pi}\left(\frac{\sin w_B T}{w_B T}\right)$

3. $N(t)$ and $N\left(t + \frac{K\pi}{w_B}\right)$ are independent,

   where K is a non-zero integer.

(g) Filters

Filtering is commonly used in electrical systems to reject undesirable signals and niose and to select the desired signal.

The commonly used filters are narrow-band filters (i.e.,) band pass and low pass filters.

(i) If the system function $H(w)$ is defined as

$$H(w) \neq 0, \text{ for } w_o - \frac{\varepsilon}{2} < w < w_o + \frac{\varepsilon}{2}$$
$$= 0, \text{ otherwise}$$

then the filter is called a band pass filter.

(ii) If the system function $H(w)$ is defined as

$$H(w) \neq 0, \text{ for } w - \frac{\varepsilon}{2} < w < w + \frac{\varepsilon}{2}$$
$$= 0, \text{ otherwise}$$

then the filter is called a low pass filter.

Note : The equation $S_{yy}(w) = |H(w)|^2 S_{xx}(w)$ shows that the spectral properties of a signal can be modified by passing it through a linear time-invariant system with the appropriate transfer function.

Example 5.3.1

Calculate the rms noise voltage generated in a bandwidth of 15 kHz, by a resistor of 2 kΩ operating at 20° C. Find the available noise power over this bandwidth. Find the noise PSD.

Solution : The mean square value of thermal noise voltage is

$$v_n^2 = 4Rk\ TB$$

where $k = 1.38 \times 10^{-23}\ J/^{o}k$

$T = 20 + 273 = 293^{o}\ k$

$B = 15$ kHZ ; and $R = 2000\ \Omega$

$$\therefore v_n^2 = 4 \times 2000 \times 1.38 \times 10^{-23} \times 293 \times 15 \times 10^{3}$$

$$= 48.52 \times 10^{-14}\ (\text{volts})^2$$

$\therefore$ rms noise voltage $= 6.96 \times 10^{-7}$ volts

Available noise power $=$ kTB watts

$$= 1.38 \times 10^{-23} \times 293 \times 15 \times 10^{3}$$

$$= 6.06 \times 10^{-17} \text{ watts}$$

$$\text{Noise PSD} = \frac{\text{Noise power}}{\text{Bandwidth}} = kT \text{ w/Hz}$$

$$= (1.38 \times 10^{-23} \times 293) \text{ w/Hz} = 404.34 \times 10^{-23} \text{ w/Hz.}$$

Example 5.3.2

Find the PSD of the thermal noise voltage across the terminals 1 and 2 for the following circuit.

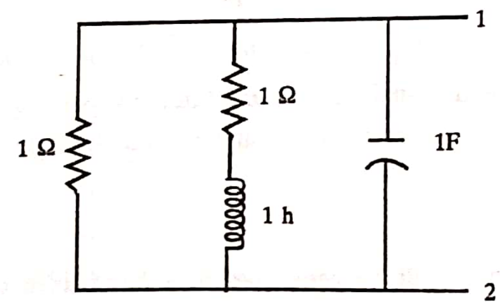


Fig.

Solution : The PSD of thermal noise voltage is

$$G(f) = 2\ kT\ R\ (f)$$

Consider
$$Y_{12}(f) = 1 + \frac{1}{1+Jw} + Jw$$

$$= \frac{1+Jw+1+(1+Jw)Jw}{1+Jw}$$

$$= \frac{(2-w^2)+2Jw}{1+Jw}$$

$$Z_{12}(f) = \frac{1+Jw}{(2-w^2)+2Jw}$$

$$= \frac{1+Jw}{(2-w^2)+2Jw} \times \frac{(2-w^2)-2Jw}{(2-w^2)-2Jw}$$

The real part of above Z (f) is

$$R(f) = \frac{2-w^2+2w^2}{(2-w^2)+4w^2} = \frac{2+w^2}{4+w^4}$$

The thermal noise PSD is $G(f) = 2\,kT.R(f)$

$$= \frac{2kT(2+w^2)}{4+w^4}$$

Example 5.3.3

The O/P thermal noise voltage of a parallel RC circuit is passed through an LPF with cutoff frequency 'f_c' and then through an amplifier of gain 9. Find the O/P noise power of the amplifier.

Solution : We have the *i/p* PSD of the above cascade section is

$$G_i(f) = 2kTR$$

The overall transfer function is the product of the transfer functions of RC section,

LPF section and the amplifier section.

$$H(f) = 1 \text{ for } f \le |f_c|$$
$$= 0 \text{ elsewhere}$$

$\therefore$ The overall H (f) $= \frac{1}{1+JwRC} \cdot 1.9$

$= \frac{9}{1+JWRC}$ for $|f| \leq f_c$

$= 0$ elsewhere

$\therefore$ *O/p* PSD G_o (f) $= |H(f)|^2 . G_i(f)$

$$= \left|\frac{9}{1+JwRC}\right|^2 . 2kTR$$

$$= \left(\frac{81}{1+4\pi^2 f^2 R^2 C^2}\right) 2kTR$$

$$= \frac{162.kTR}{1+4\pi^2 f^2 R^2 C^2}$$

o/p Noise power $= \int_{-f_c}^{f_c} \frac{162\,kTR}{1+4\pi^2 f^2 R^2 c^2} \cdot df$

$$= \frac{162\,kTR}{4\pi^2 R^2 C^2} \int_{-f_c}^{f_c} \frac{1}{f^2 + \left(\frac{1}{2\pi RC}\right)^2} \cdot df$$

$$= \frac{162\,kTR}{4\pi^2 R^2 C^2} . 2\pi RC . \tan^{-1}[2\pi f RC]\Big]_{-f_c}^{f_c}$$

$$= \frac{162kT}{\pi C} . \tan^{-1}(2\pi f_c RC) \text{ watts.}$$

Example 5.3.4

A white noise signal of zero mean and PSD $\frac{\eta}{2}$ is applied to an ideal LPF whose bandwidth is B. Find the auto correlation of the O/P noise signal.

Solution : We have *i/p* PSD $G_i(f) = \frac{\eta}{2}$

and LPF transfer function is H (f) $= 1$ for $|f| \leq B$

$= 0$ elsewhere

$\therefore$ *O/p* PSD $G_0(f) = |H(f)|^2 . \frac{\eta}{2}$

$= \frac{\eta}{2}$ for $|f| \leq B$

Since auto correlation R $(\tau) \xleftrightarrow{F.T}$ PSD

$$R(\tau) = \int_{-B}^{B} G_o(f)\, e^{J2\pi f\tau} . df$$

$$= \int_{-B}^{B} \frac{\eta}{2} . e^{J2\pi f\tau} . df$$

$$= \frac{\eta}{2}\left[\frac{e^{J2\pi f\tau}}{J2\pi\tau}\right]_{-B}^{B}$$

$$= \frac{\eta}{2} . \left[\frac{e^{J2\pi B\tau} - e^{-J2\pi B\tau}}{J2\pi\tau}\right]$$

$$= \eta . B . \frac{\sin 2\pi B\tau}{2\pi B\tau}$$

Example 5.3.5

A white noise signal with PSD $\frac{\eta}{2}$ is applied to an RC LPF. Find the auto correlation of the O/P signal of the filter.

Solution : The transfer function of RC LPF is

$$H(f) = \frac{1}{1+J2\pi fRC}$$

The *o/p* PSD $= |H(f)|^2 . \frac{\eta}{2}$

$$= \frac{1}{1+4\pi^2 f^2 R^2 C^2} \cdot \frac{\eta}{2}$$

Auto correlation $= F^{-1}$ [O/P PSD]

$$= F^{-1}\left[\frac{\eta}{2} \cdot \frac{1}{1+4\pi^2 f^2 R^2 C^2}\right]$$

We have $e^{-a|t|} \cdot a(t) \xleftrightarrow{F.T} \frac{2a}{a^2 + 4\pi^2 f^2}$

using the above result

$$\therefore R(\tau) = \frac{\eta}{4RC} \cdot e^{-\frac{|\tau|}{RC}}$$

Example 5.3.6

If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and $C(t_1, t_2) = 16 e^{-|t_1 - t_2|}$ find the probability that (i) $X(10) \le 8$ and (ii) $|X(10) - X(6)| \le 4$. **[A.U CBT Dec. 2009]**

Sol. If $\{X(t)\}$ is a Gaussian process, then any member of the process is a normal RV.

Therefore, X (10) is a normal RV with mean $\mu(10) = 10$ and variance $C(10, 10) = 16$.

$$P\{X(10) \le 8\} = P\left\{\frac{X(10) - 10}{4} \le -0.5\right\}$$

$= P\{Z \le -0.5\}$ (where Z is the standard normal RV)

$= 0.5 - P\{0 \le Z \le 0.5\}$

$= 0.5 - 0.1915$ (from normal tables)

$= 0.3085$

$X(10) - X(6)$ is also a normal RV with mean $\mu(10) - \mu(6) = 10 - 10 = 0$.

$$\text{Var}\{X(10) - X(6)\} = \text{Var}\{X(10)\} + \text{Var}\{X(6)\} - 2\,\text{Co var}\{X(10), X(6)\}$$

$= C(10, 10) + C(6, 6) - 2C(10, 6)$

$= 16 + 16 - 2 \times 16e^{-4}$

$= 31.4139$

$$\text{Now } P\{|X(10) - X(6)| \le 4\} = P\left\{\frac{|X(10) - X(6)|}{5.6048} \le \frac{4}{5.6048}\right\}$$

$= P\{|Z| \le 0.7137\}$

$= 2 \times 0.2611$

$= 0.5222$

where $\mu_1 = E\{X(t_i)\}$ and Λ is the third order square matrix (λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$.

$$E\{X(t)\} = \lim_{\tau \to \infty} R_x(\tau) = \lim_{\tau \to \infty} e^{-|\tau|} = 0$$

$$\therefore \lambda_{ij} = C\{X(t_i)\, X(t_j)\} = R(t_i - t_j)$$

$$\therefore \lambda_{11} = R(t_1, t_1) = R(t, t) = R(0) = 1$$

$$\lambda_{12} = R(t, t+1) = R(1) = e^{-1} \text{ etc.}$$

$$\therefore \Lambda = \begin{pmatrix} 1 & \frac{1}{e} & \frac{1}{e^2} \\ \frac{1}{e} & 1 & \frac{1}{e} \\ \frac{1}{e^2} & \frac{1}{e} & 1 \end{pmatrix} \text{ and } |\Lambda| = \left(1 - \frac{1}{e^2}\right)^2$$

$$|\Lambda|_{11} = 1 - \frac{1}{e^2} \quad |\Lambda|_{12} = -\frac{1}{e} + \frac{1}{e^3} \quad |\Lambda|_{13} = 0 \text{ etc.}$$

Therefore, the required joint pdf is given by

$$f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}\left(1-\frac{1}{e^2}\right)} \exp$$

$$\left[-\frac{1}{2\left(1-\frac{1}{e^2}\right)^2}\left\{\left(1-\frac{1}{e^2}\right)x_1^2 - \frac{2}{e}\left(1-\frac{1}{e^2}\right)x_1x_2 + \left(1-\frac{1}{e^4}\right)x_2^2 - \frac{2}{e}\left(1-\frac{1}{e^2}\right)x_2x_3 + \left(1-\frac{1}{e^2}\right)x_3^2\right\}\right]$$

$$\text{(i.e.,) } f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}\left(1-\frac{1}{e^2}\right)} \exp$$

$$\left[-\frac{1}{2\left(1-\frac{1}{e^2}\right)}\left\{x_1^2 - \frac{2}{e}x_1x_2 + \left(1+\frac{1}{e^2}\right)x_2^2 - \frac{2}{e}x_2x_3 + x_3^2\right\}\right]$$

Example 5.3.7

If $\{Y(t)\}$ is the square law detector process and if $Z(t) = Y(t) - E\{Y(t)\}$ show that the spectral density of $\{Z(t)\}$ is given by $S_{zz}(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} S_{xx}(\alpha)\, S_{xx}(w-\alpha)\, d\alpha$, where $S_{xx}(w)$ is the input spectral density.

Solution :

$$E\{Z(t)\,Z(t-\tau)\} = E[\{Y(t) - E[Y(t)]\}\{Y(t-\tau) - E[Y(t-\tau)]\}]$$

$$= E\{Y(t)\,Y(t-\tau)\} - E\{Y(t)\}\,E\{Y(t-\tau)\}$$

$$\text{(i.e.,) } R_{zz}(\tau) = R_{yy}(\tau) - E\{Y(t)\}\,E\{Y(t-\tau)\}$$

$$= R_{xx}^2(0) + 2R_{xx}^2(\tau) - R_{xx}^2(0)$$

[see square law detector process]

$$= 2R_{xx}^2(\tau)$$

Taking Fourier transforms

$$S_{zz}(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} S_{xx}(\alpha)\, S_{xx}(w-\alpha)\, d\alpha$$

Example 5.3.8

Obtain the autocorrelation for an ideal low pass stochastic process.

Solution : Let the spectral density function of the low pass process $\{X(t)\}$ be $S_{xx}(w)$, in $|w| < w_B$.

Let the complex form of Fourier series of $S_{xx}(w)$ in $(-w_B, w_B)$ be

$$S_{xx}(w) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi w/w_B} \quad \dots (1)$$

where c_n is given by

$$c_n = \frac{1}{2w_B}\int_{-w_B}^{w_B} S_{xx}(w)\, e^{-in\pi w/w_B}\, dw \quad \dots (2)$$

Taking the inverse Fourier inverse tansforms of (1)

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum c_n\, e^{in\pi w'/w_B}\, e^{i\tau w'}\, dw'$$

$$= \sum \frac{1}{2\pi}\int_{-w_B}^{w_B}\frac{1}{2w_B}\int_{-w_B}^{w_B} S_{xx}(w)\, e^{-in\pi w/w_B}\, dw\; e^{i\left(\frac{n\pi}{w_B}+\tau\right)w'}\, dw' \quad \text{[since } \{X(t)\} \text{ is low pass]}$$

$$= \sum \frac{1}{2w_B}\int_{-w_B}^{w_B} R_{xx}\left(-\frac{n\pi}{w_B}\right) e^{i\left(\frac{n\pi}{w_B}\right)w'}\, w'\, dw'$$

$$= \sum R_{XX}\left(-\frac{n\pi}{w_B}\right)\frac{1}{w_B}\left\{\frac{\sin\left(\frac{n\pi}{w_B}+\tau\right)w'}{\left(\frac{n\pi}{w_B}+\tau\right)}\right\}_0^{w_B}$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}\left(-\frac{n\pi}{w_B}\right)\frac{\sin\left(\frac{n\pi}{w_B}+\tau\right)w_B}{\left(\frac{n\pi}{w_B}+\tau\right)w_B}$$

$$= \sum_{n=-\infty}^{\infty} R_{XX}\left(\frac{n\pi}{w_B}\right)\frac{\sin\left(\tan-\frac{n\pi}{w_B}\right)w_B}{\left(\tau-\frac{n\pi}{w_B}\right)w_B} \quad \text{[Changing } n \text{ to } -n\text{]}$$

Let us assume that the values of X (t) at $t = n$ T, n = ... –3, –2, –1, 0, 1, 2, 3, ... are given using which we can construct X (t), where $T = \frac{\pi}{w_B}$.

$$\therefore R_{XX}(\tau) = \sum_{n=-\infty}^{\infty} R_{XX}(nT)\frac{\sin w_B(\tau - nT)}{w_B(\tau - nT)}$$

Thus, when {X (t)} is a low process, its autocorrelation is found out by summation.

Example 5.3.9

Consider a white Gaussian noise of zero mean and power spectral density $N_0/2$ applied to a low pass RC filter whose transfer function is H (f) = $\frac{1}{1+i2\pi fRC}$. Find the autocorrelation function of the output random process.

Solution : Given : $H(f) = \frac{1}{1+i2\pi fRC}$

$$|H(f)| = \frac{1}{|1+i2\pi fRC|} = \frac{1}{\sqrt{1+4\pi^2 f^2 R^2 C^2}}$$

$$|H(f)|^2 = \frac{1}{1+4\pi^2 f^2 R^2 C^2} \quad \text{... (1)}$$

Given : $S_{XX}(f) = \frac{N_0}{2}$ ... (2) [∵ the input is a white noise]

The simple RC circuit for which the transfer function is given in a linear time invariant system.

The power spectral densities of the input $\{X(t)\}$ and the output $\{Y(t)\}$ of a linear system are connected by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \quad \text{... (3)}$$

In the given problem the transfer function is expresed interms of the frequency f.

$$(3) \Rightarrow S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$

$$= \frac{1}{1+4\pi^2 f^2 R^2 C^2}\frac{N_0}{2} \quad \text{by (1) \& (2)}$$

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(f)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+4\pi^2 f^2 R^2 C^2}\frac{N_0}{2}\, e^{i2\pi f\tau}\, d(2\pi f)$$

Since $\omega = 2\pi f$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+4\pi^2 f^2 R^2 C^2}\frac{N_0}{2}\, e^{i2\pi\tau f}\, 2\pi\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{1}{1+4\pi^2 f^2 R^2 C^2}\, e^{i2\pi\tau f}\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}\left[\frac{1}{4\pi^2 R^2 C^2}\right]\left[\frac{1}{\frac{1}{4\pi^2 R^2 C^2}+f^2}\right] e^{i2\pi\tau f}\, df$$

$$R_{YY}(\tau) = \frac{N_0}{8\pi^2 R^2 C^2} \int_{-\infty}^{\infty} \frac{e^{i(2\pi\tau)f}}{\left(\frac{1}{2\pi RC}\right)^2 + f^2} df \quad \ldots (4)$$

Compare the integral in (4) with $\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2} dx$,

which can be evaluated by contour integration technique

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2} dx = \frac{\pi}{a} e^{-|m|a}$$

$$\therefore (4) \Rightarrow R_{YY}(\tau) = \frac{N_0}{8\pi^2 R^2 C^2} \frac{\pi}{\left(\frac{1}{2\pi RC}\right)} e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)}$$

$$= \frac{N_0}{8\pi^2 R^2 C^2} 2\pi^2 RC \quad e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)}$$

$$R_{YY}(\tau) = \frac{N_0}{4RC} e^{-|\tau|/RC}$$

The mean square value of $\{Y(t)\}$ is given by

$$E[Y^2(t)] = R_{YY}(0) = \frac{N_0}{4RC} e^{-\frac{0}{RC}} = \frac{N_0}{4RC} e^{-0} = \frac{N_0}{4RC}(1) = \frac{N_0}{4RC}$$

Example 5.3.10

If Y (t) = A cos $(\omega_0 t + \theta)$ + N (t), where A is a constant, θ is a random variable with a uniform distribution in $(-\pi, \pi)$ and {N (t)} is a band-limited Gaussian white noise with a power spectral density.

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

[A.U N/D 2010]

find the power spectral density of {Y (t)}. Assume that N (t) and θ are in independent.

[AU M/J 2007, N/D 2012]

Solution : Given : $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$$Y(t+\tau) = A\cos\left[\omega_0(t+\tau) + \theta\right] + N(t+\tau)$$

$$= A\cos[\omega_0 t + \omega_0\tau + \theta] + N(t+\tau)$$

$$Y(t)\,Y(t+\tau) = \left[A\cos(\omega_0 t + \theta) + N(t)\right]\left[A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)\right]$$

$$= A^2 \cos(\omega_0 + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)$$
$$+ A\cos(\omega_0 t + \theta) N(t+\tau) + A\cos(\omega_0 t + \omega_0\tau + \theta) N(t)$$
$$+ N(t) N(t+\tau)$$

$$R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E\left[A^2\cos(\omega_0 + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)\right.$$
$$+ A\cos(\omega_0 t + \theta) N(t+\tau) + A\cos(\omega_0 t + \omega_0\tau + \theta) N(t)$$
$$\left.+ N(t) N(t+\tau)\right]$$

$$= A^2 E[\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)]$$
$$+ A E[\cos(\omega_0 t + \theta) N(t+\tau)] + A E[\cos(\omega_0 t + \omega_0\tau + \theta) N(t)]$$
$$+ E[N(t) N(t+\tau)]$$

$$= \frac{A^2}{2} E[2\cos(\omega_0 t + \omega_0\tau + \theta)\cos(\omega_0 t + \theta)]$$
$$+ A E[\cos(\omega_0 t + \theta) N(t+\tau)]$$
$$+ A E[\cos(\omega_0 t + \omega_0\tau + \theta) N(t)]$$
$$+ E[N(t) N(t+\tau)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 t + \omega_0\tau + \theta + \omega_0 t + \theta) + \cos(\omega_0 t + \omega_0\tau + \theta - \omega_0 t - \theta)]$$
$$+ A\, E[\cos(\omega_0 t + \theta) N(t+\tau)]$$
$$+ A\, E[\cos(\omega_0 t + \omega_0\tau + \theta) N(t)]$$
$$+ E[N(t) N(t+\tau)]$$

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$$= \frac{A^2}{2} E[\cos(2\omega_0 t + 2\theta + \omega_0\tau) + \cos(\omega_0\tau)]$$
$$+ A\ E[\cos(\omega_0 t + \theta)]\, E\left(N(t+\tau)\right)$$
$$+ A\ E[\cos(\omega_0 t + \omega_0\tau + \theta)]\ E[N(t)]$$
$$+ R_{NN}(\tau) \qquad [\because N(t) \text{ is stationary}] \qquad \dots (1)$$

Given θ is uniformly distributed in $(-\pi, \pi)$

$$\therefore f(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

$$E[\cos(\omega_0 t + \theta)] = \int_{-\infty}^{\infty} \cos(\omega_0 t + \theta) f(\theta)\, d\theta$$
$$= \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi}\, d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos\omega_0 t \cos\theta - \sin\omega_0 t \sin\theta]\, d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\omega_0 t \cos\theta\, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\omega_0 t \sin\theta\, d\theta$$
$$= \frac{1}{2\pi} \cos\omega_0 t \int_{-\pi}^{\pi} \cos\theta\, d\theta - \frac{1}{2\pi} \sin\omega_0 t \int_{-\pi}^{\pi} \sin\theta\, d\theta$$
$$= \frac{1}{2\pi} (\cos\omega_0 t)\ (2) \int_0^{\pi} \cos\theta\, d\theta - \frac{1}{2\pi} \sin\omega_0 t\ (0)$$

[since $\cos\theta$ is even, $\sin\theta$ is odd]

$$= \frac{1}{\pi} \cos\omega_0 t \left[\sin\theta\right]_0^{\pi} - 0$$
$$= \frac{1}{\pi} \cos\omega_0 t\ [0 - 0] = 0 \qquad \dots (2)$$

$$E[\cos(\omega_0 t + \omega_0\tau + \theta)] = 0 \qquad \dots (3)$$

$$E[\cos(2\omega_0 t + 2\theta + \omega_0\tau)] = \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta + \omega_0\tau) \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta + \omega_0\tau)\, d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\cos(2\omega_0 t + \omega_0\tau)\cos 2\theta - \sin(2\omega_0 t + \omega_0\tau)\sin 2\theta\right] d\theta$$
$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0\tau) \int_{-\pi}^{\pi} \cos 2\theta\, d\theta - \frac{1}{2\pi} \sin(2\omega_0 t + \omega_0\tau) \int_{-\pi}^{\pi} \sin 2\theta\, d\theta$$
$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0\tau)\ 2 \int_0^{\pi} \cos 2\theta\, d\theta - \frac{1}{2\pi} \sin(2\omega_0 t + \omega_0\tau)\ (0)$$

[since $\cos 2\theta$ is even $\sin 2\theta$ is odd]

$$= \frac{1}{\pi} \cos(2\omega_0 t + \omega_0\tau) \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0\tau) \left[\sin 2\theta\right]_0^{\pi}$$
$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0\tau) [0 - 0]$$
$$= 0 \qquad \dots (4)$$

$$(1) \Rightarrow R_{YY}(t, t+\tau) = \frac{A^2}{2} \cos\omega_0\tau + 0 + 0 + 0 + R_{NN}(\tau)$$

by using (2), (3) & (4)

$$R_{YY}(t, t+\tau) = \frac{A^2}{2} \cos(\omega_0\tau) + R_{NN}(\tau)$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} \left[\frac{A^2}{2} \cos\omega_0\tau + R_{NN}(\tau)\right] e^{-i\omega\tau}\, d\tau$$
$$= \frac{A^2}{2} \int_{-\infty}^{\infty} \cos\omega_0\tau\, e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} R_{NN}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \pi \frac{A^2}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{N_0}{2}$$

$$[\because \text{ Given } S_{NN}(\omega) = \frac{N_0}{2}]$$

Example 5.3.11

White noise with two sided PSD $\frac{\eta}{2}$ is passed through a low pass RC network with time constant τ = RC and thereafter through ideal amplifier with a voltage gain 10.

(a) Write the expression for auto correlation function $R_n(\tau)$ of the white noise (b) Write the expression for the PSD of the noise at the O/P of the amplifier.

Solution : (a) $R_n(\tau) = \int_{-\infty}^{\infty} G_n(f) . e^{J 2\pi f \tau} . df$

$$= \int_{-\infty}^{\infty} \left(\frac{\eta}{2}\right) . e^{J 2\pi f \tau} . df$$

$$= \left(\frac{\eta}{2}\right) . \delta(f)$$

(b) O/P noise PSD $G_{no}(f) = |H(f)|^2 \times G_{ni}(f)$

$$H(f) = \frac{10}{1 + J 2\pi f \tau} \Rightarrow |H(f)|^2 = \frac{100}{1 + 4\pi^2 f^2 \tau^2}$$

$$\therefore G_{no}(f) = \frac{100}{1 + (2\pi f \tau)^2} . \frac{\eta}{2}$$

$$= \frac{50\eta}{1 + (2\pi f \tau)^2}$$

Example 5.3.12

Find the auto correlation function of Gaussian White Noise. **[A.U A/M 2010]**

Solution : The power spectral density of Gaussian white noise is

$S_{XX}(\omega) = \frac{N_0}{2}$, when N_0 is a positive real valued constant.

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_0}{2} e^{i\omega\tau} d\omega$$

$$= \frac{N_0}{4\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} (\cos\omega\tau + i\sin\omega\tau) d\omega$$

$$= \frac{N_0}{2\pi} \int_{0}^{\infty} \cos\omega\tau \, d\omega, \text{ since } \sin\omega\tau \text{ is an odd function}$$

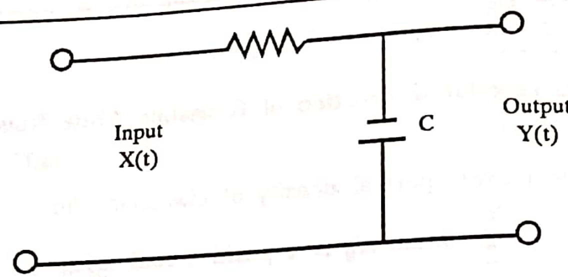
$$= \frac{N_0}{2} \delta(\tau)$$

Where $\delta(\tau) = \int_{0}^{\infty} \cos\omega\tau \, d\omega$

Example 5.3.13

If a white Gaussian noise X(t) with zero-mean and spectral density $\frac{N_0}{2}$ is applied to a low-pass R_C filter shown in the figure, determine the auto correlation of the output Y(t). **[A.U N/D 2011]**

Solution : Given $S_{XX}(\omega) = \frac{N_0}{2}$ and we know R_C filter has filter transfer function as



$$H(i\omega) = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{1 + i\omega RC}$$

Hence, $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$$= \left(\frac{1}{1+i\omega RC}\right)\left(\frac{1}{1-i\omega RC}\right)\frac{N_0}{2}$$

$$= \frac{N_0}{2}\left[\frac{1}{2}\left[\frac{1}{1+i\omega RC} + \frac{1}{1-i\omega RC}\right]\right] \text{ by partial fraction}$$

$$= \frac{N_0}{4}\left[\frac{1}{1+i\omega RC} + \frac{1}{1-i\omega RC}\right]$$

To find the auto correlation, take the inverse Fourier transform of both the terms.

Hence $R_{YY}(\tau) = F^{-1}\left[\frac{N_0}{4}\left[\frac{1}{1+i\omega RC} + \frac{1}{1-i\omega RC}\right]\right]$

$$= \frac{N_0}{4}\left[F^{-1}\left[\frac{1}{1+i\omega RC}\right] + F^{-1}\left[\frac{1}{1-i\omega RC}\right]\right]$$

$$= \frac{N_0}{4}\left[F^{-1}\left[\frac{1}{RC}\cdot\frac{RC}{1+i\omega RC}\right] + F^{-1}\left[\frac{1}{RC}\cdot\frac{RC}{1-i\omega RC}\right]\right]$$

$$= \frac{N_0}{4}\left[F^{-1}\left[\frac{1}{RC}\cdot\frac{1}{\left(\frac{1}{RC}+i\omega\right)}\right] + F^{-1}\left[\frac{1}{RC}\cdot\frac{1}{\left(\frac{1}{RC}-i\omega\right)}\right]\right]$$

$$= \frac{N_0}{4}\left[\frac{1}{RC}e^{-\tau/RC}.u(\tau) + \frac{1}{RC}e^{\tau/Rc}u(\tau)\right]$$

where $u(\tau)$ is the unit step function. Hence

$$R_{YY}(\tau) = \frac{N_0}{4RC}e^{-|\tau|/RC}$$

Example 5.3.14

Define thermal noise and white noise.

Solution : It is noise occurring because of the random motion of free electrons in some conducting medium. Thermal noise generated in resistors, semiconductors is assumed to have zero mean, stationary Gaussian random process $[N(t)]$ with power spectral density that is flat over a wide range frequencies. Such noise is said to be **white noise.**

Usually the spectral density of white noise is denoted by $\frac{N_0}{2}$.

Example 5.3.15

Define band-limited white noise. **[A.U N/D 2010]**

Solution : Noise with non-zero and constant spectral density over a finite frequency band is called band-limite white noise i.e.,

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega| \le \omega_B \\ 0, & \textit{otherwise} \end{cases}$$

Example 5.3.16

Find the autocorrelation, and average power of white noise.

(OR) If the PSD of white noise is $\frac{N_0}{2}$, find its ACF.

Solution : We know that $S_{NN}(\omega) = \frac{N_0}{2}$ Hence, as

$$\int_{-\infty}^{\infty} \frac{N_0}{2}\,\delta(\tau)\,e^{-i\omega\tau}\,d\tau = \frac{N_0}{2}$$

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the autocorrelation $R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$

$$\text{Average power} = \int_{-\infty}^{\infty} S_{NN}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{N_0}{2}\, d\omega \to \infty$$

Example 5.3.17

If $\{X(t)\}$ is a band limited process such that $S_{XX}(\omega) = 0$, when $|\omega| > \sigma$, prove that $2[R_{XX}(0) - R_{XX}(\tau)] \le \sigma^2\tau^2 R_{XX}(0)$.

[A.U A/M 2010]

Solution :

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\cos\tau\omega\, d\omega \quad \text{[since } S_{XX}(\omega) \text{ is even]}$$

$$R_{XX}(0) - R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1 - \cos\tau\omega)\, d\omega$$

[$\because$ $\{X(t)\}$ is band limited]

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega) \times 2\sin^2\left(\frac{\tau\omega}{2}\right) d\omega \quad \text{... (1)}$$

We know that, $|\sin\theta| \le \theta$

$$\therefore \sin^2\theta \le \theta^2$$

$$\therefore 2\sin^2\left(\frac{\tau\omega}{2}\right) \le \frac{\tau^2\omega^2}{2} \quad \text{... (2)}$$

$$R_{XX}(0) - R_{XX}(\tau) \le \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\frac{\tau^2\omega^2}{2}\, d\omega \quad \text{by (1)}$$

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\, d\omega$$

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

i.e., $\le \frac{\sigma^2\tau^2}{2} R_{XX}(0)$

Example 5.3.18

It is given that $R_x(\tau) = e^{-|\tau|}$ for a certain stationary Gaussian random process $\{X(t)\}$. Find the joint pdf of the Random variables $X(t), X(t+1), X(t+2)$.

Solution : Let the RVs by $X(t_1), X(t_2), X(t_3)$

The joint pdf of $\{X(t_1), X(t_2), X(t_3)]$ is given by

$F(x_1, x_2, x_3, t_1, t_2, t_3) =$

$$\frac{1}{(2\pi)^{3/2}|\Lambda|^{1/2}} \exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{3}\sum_{j=1}^{3} |\Lambda|_{ij}(x_i - \mu_i)(x_j - \mu_j)\right]$$

where $\mu_i = E[X(t_i)]$ and Λ is the third order square matrix (λ_{ij}),

where $\lambda_{ij} = C\{X(t_1), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$.

$$E\{X(t)\} = \lim_{\tau\to\infty} R_x(\tau) = \lim_{\tau\to\infty} e^{-|\tau|} = 0$$

$$\therefore \lambda_{ij} = C\{X(t_i)\, X(t_j)\} = R(t_i - t_j)$$

$$\therefore \lambda_{11} = R(t_1, t_1) = R(t, t) = R(0) = 1$$

$$\lambda_{12} = R(t, t+1) = R(1) = e^{-1} \text{ etc.}$$

$$\therefore \Lambda = \begin{pmatrix} 1 & \frac{1}{e} & \frac{1}{e^2} \\ \frac{1}{e} & 1 & \frac{1}{e} \\ \frac{1}{e^2} & \frac{1}{e} & 1 \end{pmatrix} \text{ and } |\Lambda| = \left(1 - \frac{1}{e^2}\right)^2$$

$$|\Lambda|_{11} = 1-\frac{1}{e^2} \qquad |\Lambda|_{12} = -\frac{1}{e}+\frac{1}{e^3} \qquad |\Lambda|_{13} = 0 \text{ etc.}$$

$\therefore$ the required joint pdf is given by

$$f(x_1,x_2,x_3) = \frac{1}{(2\pi)^{3/2}\sqrt{\left(1-\frac{1}{e^2}\right)}}$$

$$\exp\left[-\frac{1}{2\left(1-\frac{1}{e^2}\right)^2}\left\{\left(1-\frac{1}{e^2}\right)x_1^2-\frac{2}{e}\left(1-\frac{1}{e^2}\right)x_1x_2 + \left(1-\frac{1}{e^4}\right)x_2^2-\frac{2}{e}\left(1-\frac{1}{e^2}\right)x_2x_3+\left(1-\frac{1}{e^2}\right)x_3^2\right\}\right]$$

i.e.,
$$f(x_1,x_2,x_3) = \frac{1}{(2\pi)^{3/2}\left(1-\frac{1}{e^2}\right)}$$

$$\exp\left[-\frac{1}{2\left(1-\frac{1}{e^2}\right)}\left\{x_1^2-\frac{2}{e}x_1x_2+\left(1+\frac{1}{e^2}\right)x_2^2-\frac{2}{e}x_2x_3+x_3^2\right\}\right]$$

Example 5.3.19

Let Z and θ be independent random variables such that Z has a density function

$$f(z) = \begin{cases} 0, & \text{in } z<0 \\ ze^{-z^2/2}, & \text{in } z>0 \end{cases}$$

and θ is uniformly distributed in $(0, 2\pi)$. Show that $\{X_t;\ -\infty<t<\infty\}$ is a Gaussian process, if $X_t = Z\cos(2\pi t+\theta)$

Solution :

Now first we find the density function of

$$W = \cos(2\pi t+\theta), \text{ where } f_\theta(\theta) = \frac{1}{2\pi}$$

Since $w = \cos(2\pi t+\theta)$, $\theta = \cos^{-1}(w) - 2\pi t$

There are only two values of θ in $(0, 2\pi)$ for a given value of w. Let tham be θ_1 and θ_2.

By the transformation rule

$$f_w(w) = f_\theta(\theta_1)\left|\frac{d\theta_1}{dw}\right| + f_\theta(\theta_2)\left|\frac{d\theta_2}{dw}\right|$$

Let us now find the first order density of $X = ZW$, where X_t has been taken as X.

Now we introduce the auxiliary variable $Y = W$, so that we may find the joint pdf of (X, Y)

$$x = zw \text{ and } y = w$$

i.e.,
$$z = \frac{x}{y} \text{ and } w = y$$

$$\therefore f_{XY}(x,y) = |J|\, f_{ZW}(z,w)$$

where
$$J = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{vmatrix} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1\end{vmatrix} = \frac{1}{|y|}$$

$\therefore f_X(x)$ is the marginal density function of X.

i.e.,
$$f_X(x) = \int_{-1}^{1}\frac{1}{|y|}\, f_Z(z) f_W(w)\, dy, \text{ where } z = \frac{x}{y} \text{ and } w = y$$

[by independence of Z and W]

$$= \int_{-1}^{1}\frac{1}{|y|}\,\frac{x}{y}\, e^{-x^2/2y^2}\,\frac{1}{\pi\sqrt{1-y^2}}\, dy,\ \frac{x}{y}>0$$

$$= \begin{cases} \frac{1}{\pi}\int_{-1}^{0} -\frac{x}{y^2}\, e^{-x^2/2y^2}\,\frac{1}{\sqrt{1-y^2}}\, dy, & \text{where } x<0 \text{ and } y<0 \quad (1) \\ \frac{1}{\pi}\int_{0}^{1} \frac{x}{y^2}\, e^{-x^2/2y^2}\,\frac{1}{\sqrt{1-y^2}}\, dy, & \text{where } x>0 \text{ and } y>0 \quad (2)\end{cases}$$

Changing y to $-y'$, we note that the integral in (1) becomes

$$\frac{1}{\pi}\int_0^1 \frac{x}{y'^2} e^{-x^2/2y'^2} \frac{1}{\sqrt{1-y'^2}}\, dy'$$, which is the same as integral (2).

$$\therefore f_X(x) = \frac{1}{\pi}\int_0^1 \frac{x}{y^2} e^{-x^2/2y^2} \frac{1}{\sqrt{1-y^2}}\, dy, \quad -\infty < x < \infty \qquad \dots (3)$$

Put $\frac{x^2}{2y^2} = t$ in (3), treating x as a parameter,

$$\text{Then } f_X(x) = \frac{1}{\pi}\int_{x^2/2}^{\infty} \frac{1}{\sqrt{2t - x^2}} e^{-t}\, dt \qquad \dots (4)$$

Put $t - \frac{x^2}{2} = u$, treating x as a parameter,

$$\text{Then } f_X(x) = \frac{1}{\pi\sqrt{2}}\int_0^{\infty} e^{-x^2/2} u^{-1/2} e^{-u}\, du$$

$$= \frac{1}{\pi\sqrt{2}} e^{-x^2/2}\, \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \infty < x < \infty$$

Thus each member of the process $\{X_t\}$ follows a normal distribution with mean zero and variance 1.

$\therefore a_1 X t_1 + a_2 X t_2 + \dots + a_n X t_n$ also follows a normal distribution, for any set of $a_1, a_2, \dots a_n$.

$\therefore \{X_1, X_2, \dots, X_{t_n}\}$ are jointly normal for any n

$\therefore$ the process $\{X_t\}$ is Gaussian.

Example 5.3.20

If $\{N(t)\}$ is a band limited white noise centered at a carrier frequency ω_0 such that $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & |\omega - \omega_0| < \omega_B \\ 0, & \text{otherwise} \end{cases}$

Find the autocorrelation of $\{N(t)\}$. **[AU M/J 2012]**

Solution : Given : $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & -\omega_B + \omega_0 < \omega < \omega_B + \omega_0 \\ 0, & \text{otherwise} \end{cases}$

$$R_{NN}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\frac{N_0}{2}\int_{-\omega_B+\omega_0}^{\omega_B+\omega_0} e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\frac{N_0}{2}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_B+\omega_0}^{\omega_B+\omega_0} = \frac{1}{2\pi}\frac{N_0}{2i\tau}\left[e^{i\omega\tau}\right]_{-\omega_B+\omega_0}^{\omega_B+\omega_0}$$

$$= \frac{1}{2\pi}\frac{N_0}{2i\tau}\left[e^{i(\omega_B+\omega_0)\tau} - e^{i(-\omega_B+\omega_0)\tau}\right]$$

$$= \frac{1}{2\pi}\frac{N_0}{\tau}\left[\frac{e^{(\omega_B+\omega_0)i\tau} - e^{(-\omega_B+\omega_0)i\tau}}{2i}\right]$$

$$= \frac{1}{2\pi}\frac{N_0}{\tau}\left[\frac{e^{\omega_B i\tau} - e^{-\omega_B i\tau}}{2i}\right] e^{\omega_0 i\tau}$$

$$= \frac{1}{2\pi}\frac{N_0}{\tau} \sin\omega_B\tau\, [\cos\omega_0\tau + i\sin\omega_0\tau]$$

$$= \frac{N_0}{\pi}\frac{\omega_B}{2}\left[\frac{\sin\omega_B\tau}{\omega_B\tau}\right][\cos\omega_0\tau + i\sin\omega_0\tau]$$

$$R_{NN}(\tau) = \frac{N_0}{\pi}\frac{\omega_B}{2}\left[\frac{\sin\omega_B\tau}{\omega_0\tau}\right]\cos\omega_0\tau \qquad [\because R_{NN}(\tau) \text{ is a real function}]$$

EXERCISE 5.3

1. Calculate the rms noise voltage generated in a bandwidth of 30 kHz, by a resistor of 4 kΩ operating at 40° C. Find the available noise power over this bandwidth. Find the noise PSD.

2. If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and $C(t_1, t_2) = 16 e^{-|t_1 - t_2|}$ find the probability that
   (i) $X(10) \le 8$ and (ii) $|X(10) - X(6)| \le 4$.
   (iii) $X(8) \le 6$

PRP - UNIT - I - ONLINE						
S.No	**Questions**	**OPT 1**	**OPT 2**	**OPT3**	**OPT 4**	**ANSWERS**
1	This single expression in statistics is known as ____________	measures	average	skew	group	average
2	Which average is affected most by extreme observations______	mode	median	geometric mean	arithmetic mean	geometric mean
3	Which of the following is the most unstable average_________	mode	median	geometric mean	harmonic mean	mode
4	The sum of deviations taken from arithmetic mean is________	minimum	zero	maximum	one	minimum
5	The sum of square deviations taken from arithmetic mean is_____	zero	maximum	minimum	one	minimum
6	When calculating the average growth of economy,the correct mean to use is______________	weighted mean	Geometric mean	arithmetic mean	median	geometric mean
7	When observation in the data is zero, then its geometric mean is________________	Negative	zero	positive	normal	zero
8	The best measure of central tendency is______________	arithmetic mean	Geometric mean	Harmonic mean	median	arithmetic mean
9	The point of inersection of the less than and more than gives correspponds to_____________	mean	median	geometric mean	mode	median
10	Median is same as_________quartile	first	second	third	four	second
11	Median is a _______average	first	second	positional	normal	positional
12	Median is dividing the series when arranged as an array into______ parts	two	three	four	normal	two
13	Median and mode are called __________average	first	second	positional	normal	positional
14	The geometric mean of a set of values lies between arithmetic mean and_________	harmonic mean	Geometric mean	mean	median	harmonic mean
15	In a symmetrical distrbution mean_________median________mode	is equal to,is equal to	is equal to,less than	less tnan or equal to	greater than or equal to	is equal to,is equal to
16	Harmonic mean is the _______ of the arithmetic mean of the values	positional	proposional	reciprocal	equal	reciprocal
17	The ___and ________ mark off the limits with in which the middle 50 % of the items lie	quartile one and three	deviation and one	median and the three	deviation	quartile one and three
18	__________ can be calculated from a frequeny distribution with open end classes	median or mode	mode	mean or median	deviation	median or mode
19	In the calculation of ______ all the observations are taken into cosideraion	mean	mode	median	divation	mean
20	Median is the average suited for _______classes	open -end	middle	center	sub	open-end

21	When calculating the average rate of debt expansion for a company, the correct mean to use is the_____	arithmetic mean	weighted mean	geometric mean	either a (or) c	geometric mean
22	The mode has all the following disadvantages except________	a data set may have no modal value	every value in a data set may be a mode	a multimodal data set is difficult to analyze	the mode is unduly affected by extream values	the mode is unduly affected by extreme value
23	If one event is unaffected by the outcome of another event, the two events are said to be _____	dependent	independent	mutually exclusive	event	independent
24	If P(A or B)=P(A), then_____________	A and B are mutually exclusive	venn diagram	P(A)+P(B)	deviation	A and B are mutually exclusive
25	The simple probability of an occurrence of an event is called the ____________	bayesian probability	joint probability	mariginal probabiity	conditional probability	marginal probability
26	Why are the events of a coin toss mutually exclusive_______	the out come of any toss is not affected by the out come of those preceding	both a head and a tail cannot turn up on any one toss	the probabiity of getting a head and the probability of getting a tail	all of these	both a head and tail cannot turn up on any one toss
27	What is the probabiity that a ball drawn at random from the urn is blue____________	0.1	0.4	0.6	1	0.6
28	The set of all possible outomes of an activity is the __________	sample space	event	independent	mode	sample space
29	Events that cannot happen together are called_______________	mutually exculsive	event	exclusive	mode	mutually exculsive
30	What is the median of the numbers 4,12, 11, 6, 2?	2	4	5	11	4
31	What is the median of the numbers 3, 11, 6, 5, 4, 7, 12, 3 and 10?	4	5	6	7	6
32	What is the mean of the squares of the first ten natural numbers?	30.25	31.67	38.5	50.5	38.5
33	What is the mean of these numbers: 12, -1, 8, 2, -10, 0, -5, 3, 20, -2?	6.3	5.3	3.7	2.7	2.7
34	What is the mean of the numbers 8, 9, 13 and 18?	10	11	12	16	12
35	A booklet has 12 pages with the following numbers of words: 271, 354, 296, 301, 333, 326, 285, 298, 327, 316, 287 and 314. What is the mean number of words per page?	307	309	311	313	309

36	The classical school of thought on probability assumes that all possible outcomes of an experiment are____	Equally likely	Mutually exclusive	Mutually exclusive and equally likely	Independent	Mutually exclusive and equally likely
37	What is the probability of getting an even number when a die is tossed___	1/3	1/2	1/6	1/9	1/2
38	What is the probability of getting more than 2 when a die is tossed___	1/3	1/2	2/3	1/9	2/3
39	The probability of drawing a spade from a pack of cards is_____	1/52	1/13	4/13	1/4	1/4
40	If the outcome of one event does not influence another event, then the two events are_____	mutually exculsive	Dependent	independent	Equally likely	independent
41	A density function may correspond to different ______	probability mass function	probability distribution function	probability density function	random variable	random variable
42	For a discrete random variable, the probability density function represents the _____	probability mass function	probability distribution function	probability density function	none of these	probability mass function
43	Probability of a single real value in a continuous random variable is _____	two	three	four	zero	zero
44	A random variable X is_____ if it assumes only discrete values.	spectrum	complex	continuous	discrete	discrete
45	If P(A) is 1, the event A is called a _____	Cases	Trial	Certain Event	Random experiment	Certain Event
46	p + q = _____, here p is success and q is failure events	7	9	1	3	1
47	In rolling of single die, the chance of getting 2,4,6 (even numbers) are ______	simple	Compound event	Certain event	impossible event	Compound event
48	A numerical measure of uncertainty is practiced by the important branch of statistics called _______	Theory of mathematics	Theory of physics	Theory of statistics	Theory of probability	Theory of probability
49	What is the probability of getting a sum 9 from two throws of a dice?	1/6	1/9	8/9	2/9	1/9
50	Three unbiased coins are tossed. What is the probability of getting at most two heads?	3/4	1/4	3/8	7/8	7/8
51	A bag contains 6 black and 8 white balls. One ball is drawn at random. What is the probability that the ball drawn is white?	3/4	4/7	1/8	3/7	4/7

PRP - UNIT - II - ONLINE

S.No	Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
1	Binomial distribution is symmetrical if__________	p = q = ½	p = q = ¾	p = q = 4/5	p = q = 2/3	p = q = ½
2	A normal curve has an ___	Elliptic	parabolic	hyperbolic	asymptote	asymptote
3	Variance of binomial distribution is_______.	npq	np	nq	square root of (npq)	npq
4	"The number of printing errors at each page of a book" is a example of ___ distribution.	Normal	Uniform	Binomial	Poisson	Poisson
5	Moment generating function of Uniform density function is	e^bt- e^at/ t (b-a)	e^bt+ e^at/ t (b+a)	e^bt+ e^at/ t (b-a)	e^bt- e^at/ t (b+a)	e^bt- e^at/ t (b-a)
6	For binomial distribution ______	variance -1= mean	variance = mean	variance > mean	variance < mean	variance < mean
7	_________ is a non negative continuous random variable.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Gamma distribution
8	Standard deviation of binomial distribution is ____	√npq	np(q-p)	npq	npq(q-p)	√npq
9	The density function of the Uniform distribution is _________.	1/ ba a<x<b	1/ (b+a) a<x<b	1/ ba a>x>b	1/ (b-a) a<x<b	1/ (b-a) a<x<b
10	Moment generating function of Binomial distribution	Mx (t) = (1 – p) ^n	Mx (t) = (pe^t + q)^ n	Mx (t) = (pe^t – p) ^n	Mx (t) = (e^t + q)^ n	Mx (t) = (pe^t +q)^ n
11	The mean and variance of a standard normal distribution is ___	N(1,2)	N(0,1)	N(0,2)	N(0,40)	N(0,1)
12	Variance of Uniform density function is	b /2	[(b-a)^2]/12	ba /2	[(b+a)^2] /12	[(b-a)^2]/12
13	A continuous random variable X has a probability density function f(x)= K, 0 <= x <= 1. Find K.	1	2	3	4	1
14	If X is normally distributed with mean 1 and S.D. ½ , find the probability that X > 2.	0.0288	0.0544	0.0228	0.0882	0.0228
15	Mean of Uniform density function is___________	b /2	(b-a) /2	ba /2	(b+a) /2	(b+a) /2
16	The formula of variance is ___________	Var[X] = E(X^2) – [E(X)]^ 2	Var[X] = E(X^2) – [E(X)]	Var[X] = E(X) – [E(X)]^ 2	Var[X] = E(X) – [E(X)].	Var[X] = E(X^2) – [E(X)]^ 2
17	Binomial distribution is	nCx(p^x)q^(n -x)	nCx(p^x)q^(n +x)	nCx(p^- x)q^(n-x)	nCx(p^- x)q^(n+x)	nCx(p^x)q^(n -x)
18	Mean of Poisson distribution is ____________	λ	λ/t	λ-t	λ+t	λ
19	Gamma of n =	(n-1) !	(n +1) !	n!	0!	(n-1) !
20	Moment generating function of Exponential distribution is	λ/(λ-t)	2λ/(λ-t)	λ/(λ+t)	2λ/(λ-+t)	λ/(λ-t)
21	Variance of Poisson distribution is	λ	2λ	3λ	4λ	λ
22	Exponential distribution is	λe^(λx)	λe^(-λx)	λe^(-2λx)	λe^(2λx)	λe^(-λx)

23	Gamma of (1/2) =	(π^2)/2	square root of π/2	square root of π/3	square root of π	square root of π
24	Moment generating function of Poisson distribution	e^ [λ(e^t-1)]	e^ [λ(e^t+1)]	e^ [λ(e^t-2)]	e^ [-λ(e^t-1)]	e^ [λ(e^t-1)]
25	Mean of Exponential distribution is	1/λ	λ	λ/2	2/λ	1/λ
26	Geometric distribution is given byP{X=x}=_____where x=1,2,…	pq^(x-1)	pq^(x-2)	pq^(x+1)	pq^(2x-1)	pq^(x-1)
27	If the mean and variance of a binomial variate are 8 and 6, then the probability of failure is given by______.	q=3/4	q=4/3	q=1/4	q=1/3	q=3/4
28	If the mean and variance of a binomial variate are 20 and 16, then the probability of success is given by______.	p=1/5	p=2/5	p=3/5	p=3/4	p=1/5
29	If n=5 and p=1/2 then the mean of a binomial variate is ______.	0.50	2.50	3.5	4.5	2.5
30	The mean of a poisson variate is 2 . Find its variance.	2	3	1	2.5	2
31	Poisson distribution is the limiting case of __________distribution.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Binomial distribution
32	The other name of uniform distribution is__________.	Binomial distribution	Gamma distribution	Poisson distribution	rectangular distribution	rectangular distribution
33	In a Uniform distribution if X is distributed uniformly on (0,30) then its density function is given by______.	F(x)= 1/13	F(x)= 2/13	F(x)= 1/3	F(x)= 1/30	F(x)= 1/30
34	______ is larger than the mean for a negative binomial distribution	Variance	Standard deviation	Mean deviation	Quartile deviation	Variance
35	Mean of binomial distribution is ____	np	npq	n+1	n	np
36	Third moment of Binomial distribution is ____	n(q-p)	np(q-p)	npq	npq(q-p)	npq(q-p)
37	For a negative binomial distribution______	Var(X) > E(X)	Var(X) < E(X)	Var(X) = E(X)	Var(X) / E(X)	Var(X) > E(X)
38	If X follows a Poisson distribution such that P(X=1) = 1/4 and P(X=2) = 3/8, find P(X=3).	0.123	1.234	2.34	0.375	0.375
39	The height of persons in a country is a random variable of the type____	continuous random variable	neither discrete nor continuous random vaiables	Continuous as well as discrete random variable	discrete random variable	continuous random variable
40	A family of parametric distrbution in which mean is equal to variance is _______	Binomial distribution	Gamma distribution	normal Distribution	Poissson distribution	Poissson distribution
41	A family of parametric distrbution in which mean is always greater than its variance is _______	Binomial distribution	Gamma distribution	Geometric Distribution	Poissson distribution	Geometric Distribution
42	The _________ distribution has the memory less property.	Gamma distribution	Geometric distribution	Geometric distribution	Poissson distribution	Geometric distribution

43	The mean of the binomial distribution is _____ than its variance	greater than	Less than	more	normal	greater than
44	Mean and variance of geometric distribution are___________	related	correlated	rectangle	range	related
45	A distribution where the mean and median have different values is not a _________ distribution	normal	binomial	poisson	gamma	normal
46	Normal distribution was invented by _____	Laplace	De-Moivre	Gauss	all the above	all the above

PRP - UNIT - III - ONLINE

S.No	Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
1	The coefficient of correlation is independent of change of_____ and ________	scale,origin	vector,origin	variable, constant	interer, origin	scale,origin
2	When r = 0 the line of regression are ______to each other.	parallel	perpendicular	straight line	circular	perpendicular
3	The relationship between three or more variables is studied with the help of _________ correlation.	multiple	rank	perferct	spearman's rank	multiple
4	The coefficient of correlation is under-root of two _____	regression coefficients	rank coefficient	Regression equation	regression line	regression coefficient
5	The coefficient of correlation___	has no limits	can be less than 1	can be more than 1	varies between + or - one	varies between + or - one
6	which of the following is the highest range of r__________	0 and 1	minus one and 0	minus one and one	zero	minus one and one
7	The coefficient of correlation is independent of ________	change of scale only	change of origin only	both change of scale and origin	change of variables	both change of scale and origin
8	The coefficient of correlation___	cannot be positive	cannot be negative	can be either positive or negative	zero	can be either positive or negative
9	COV(X,Y)= __________	E(XY)-E(X)E(Y)	E(XY)+E(X) E(Y)	E(XY)	Var(X,Y)	E(XY)-E(X)E(Y)
10	Two random variables with non zero correlation are said to be___	correlation	regression	rank	variables	regression
11	Correlation means relationship between ______ variables	two	one	two or more	three	two or more
12	A Mathematical measure of the average relationship between two variables is called______	correlation	regression	rank	variables	correlation

13	The covariance of two independent random variable is ________	Zero	two	three	two or more	Zero
14	Two random variables are said to be orthogonal if _______	correlation is zero	rank is zero	covariance is zero	one	correlation is zero
15	Two random variables are said to be uncorrelated if correlation coefficient is____	zero	one	two or more	orthogonal	zero
16	Regression analysis is a mathematical measures of the average relationship between ________variable	two or more	one	Two variables	three	two or more
17	The regresision analysis confined to ther study of only two variable at a time is called__________regression	Simple	Multiple	Linear	two	Simple
18	If r=0, then the regression coefficient are________	zero	one	threee	constant	zero
19	The equation of the fitted stright line is _____	y=ax+b	y=a+bx	y=mx+c	y=mx	y=ax+b
20	If X=Y , then correlation cofficient between them is _______	1	zero	less than one	gerater than one	1
21	The greater the value of r__________obtained through regression analysis	the better are estimates	the worst are the estimates	really makes no difference	good estimates	the better are estimates
22	Where r is zero the regression lines cut each other making an angle of______	30 degree	60 degree	90 degree	neither of the above	neither of the above
23	The father the two regression lines cut each other_______	Greater will be degree of correlation	The less will be the degree of correlation	does not matter	the worst are the estimates	the less will be the degree of correlation
24	The regression lines cut each other at the point of :______	Average of X and Y	Average of X only	Average of Y only	average of both(a) and (b)	average of X and Y
25	When the two regression lines coincide, then r is :_________	0	-1	1	0.5	1
26	The variable , we are trying to predict is called the _________	depentent variable	indepent variable	constant	normal	Dependent variable
27	Both the regression coefficients cannot ______one	exceed	exact	plus or minus	negative	exceed
28	The regression analysis measures ______between variables	dependence	independence	constant	normal	Dependence
29	If the possible values of (X,Y) are finite, then (X,Y) is called a______	two dimensional random variable	onedimensional random variable	both a and b	infinte	two dimensional random variable

30	If X & Y are continuous random variable , then f(x,y) is _____	joint probability function	joint probability density function	both a and b	infinte	both a and b
31	Joint probability is the probability of the _________occurrence of two or more events.	Simultaneous (or) joint	Conditional	Mariginal probability	density function	Simultaneous (or) joint
32	The order of arrangement is important in _______	permutation	Gambling	joint	density	Permutation
33	If X & Y are____random variable , then f(x,y) is called joint probability function.	discrete	continuous	both a and b	infinte	continuous
34	If the value of y decreases as the value of x increases then there is ______correlation between two variables.	negative	perfect positive	both a and b	infinte	negative
35	The correlation between the income and expenditure is _____	positive	negative	finite	both a and b	positive
36	correlation between price and demand of commodity is ____	positive	finite	negative	both a and b	negative
37	If X and Y are independent , then _____	E(XY) = E(X) + E(Y)	E(XY) = E(X) - E(Y)	E(XY) = E(X) E(Y)	E(XY) = E(X)/E(Y)	E(XY) = E(X) E(Y)
38	correlation coefficient does not exceed _____	unity	5	0	2	unity
39	Two independent variables are _____	correlated	uncorrelated	both a and b	positive	uncorrelated
40	In Rank correlation the correction factor is added for each _____value.	repeated	Non-repeated	indefinite	both a and b	repeated
41	When r = 1 or -1 the the line of regression are ____ to each other.	parallel	perpendicular	straight line	circular	parallel
42	If the curve is a straight line, then it is called the ____	the line of correlation	the line of regression	covariance	both a and b	the line of regression
43	If the curve is not a straight line, then it is called the ____	covariance	the line of correlation	the curvilinear	the line of regression	the curvilinear
44	when r is ____ the correlation is perfect and positive.	1	2	3	0	1
45	If X and Y are independent , then	E(XY)=0	E(X) E(Y)=0	Cov(X,Y) =0	E(XY)=1	Cov(X,Y) =0
46	Two random variables X and Y with joint pdf f(x,y) is said to independent if ____	f(x,y) = f(x) +f(y)	f(x,y) = f(x) / f(y)	f(x,y) = f(x) * f(y)	f(x,y) = f(x) - f(y)	f(x,y) = f(x) * f(y)
47	Cov(X,Y)=_____	E[{ X- E(X) }* { Y – E(Y) }]	E[{ X- E(X) }+ { Y – E(Y) }]	E[{ X- E(X) }- { Y – E(Y) }]	E[{ X- E(X) } { Y – E(Y) }]	E[{ X- E(X) } { Y – E(Y) }]

48	The correlation coefficient is used to determine______	A specific value of the y-variable given a specific value of the x-variable	A specific value of the x-variable given a specific value of the y-variable	The strength of the relationship between the x and y variables	is the same as r-square	The strength of the relationship between the x and y variables
49	The coefficient of correlation_____	is the square of the coefficient of determination	is the square root of the coefficient of determination	is the same as r-square	can never be negative	is the square root of the coefficient of determination
50	The correlation between two variables is of order_____	2	1	0	3	0

PRP - UNIT - IV - ONLINE						
S.No	**Questions**	**OPT 1**	**OPT 2**	**OPT3**	**OPT 4**	**ANSWERS**
1	The probabilistic model used for characterizing a -------is called a random process	random process	random signal	random model	random variable	random process
2	The Random process is also called as__	Markov process	WSS	SSS	stochastic process	stochastic process
3	The family of all functions X(s,t) is called ----	random process	random signal	random variables	random model	random process
4	A---- is a collection of Random variables that are functions of t and s.	Random process	random function	random signal	random model	Random process
5	A non null persistent and aperiodic state is called ---	stochastic	ergodic	WSS	SSS	ergodic
6	If X is continuous and t can have any of a continous of values, then X(t) is called as	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
7	If X assumes only discrete and t is continuous, then X(t) is called as	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Discrete random process

8	Let X denote the number of telephone calls received in the interval (0,t). Then {X(t)} is a	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Discrete random process
9	Thermal agitation noise in conductors is an example of ------	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
10	Let X denote the maximum temperature at a place in the interval (0,t). Then {X(t)} is a	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
11	The outcome of the n-th toss of a fair dice is an example of ------	Continuous Random process	Discrete random process	Continuous random sequence	Discrete random sequence	Discrete random sequence
12	A random process for which X is continuous but time takes only discrete values is called a ---	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous random sequence
13	A random process for which X is discrete and time takes only discrete values is called a ---	Continuous Random process	Discrete random process	Continuous random sequence	Discrete random sequence	Discrete random sequence
14	The set of possible values of any individual members of the random process is called ___ space.	vector	state	random	universal	state
15	If the process is first order stationary, then mean is	negative	positive	constant	unique	constant
16	A stochastic matrix is said to be a regular matrix, if all the entries of Pm are---	positive	negative	zero	square matrix	positive
17	The discrete parameter Markov process is called a ____	weakly stationary process	covariance stationary process	wide-sense stationary process	Markov chain	Markov chain

18	A random process is called a --- if its mean is constant and the autocorrelation depends only on the time difference.	weakly stationary process	covariance stationary process	wide-sense stationary process	All the above	All the above
19	If the transition probability matrix is regular, then the homogeneous Markov chain is	regular	irregular	square matrix	unique	regular
20	The n-th order stationary process is stationary to order ___	n	n+1	n*n	n-1	n-1
21	A random process is called a ------- , if all its finite dimensional distributions are invariant under translation of time parameter.	Wide-sense stationary process	Strict sense stationary process	Markov process	Covariance stationary process	Strict sense stationary process
22	All regular Markov chain are ___	Markov process	ergodic process	WSS	SSS	ergodic process
23	The transition probability matrix of a finite state Markov chain is a __ matrix.	row	column	square	identity	square
24	A random process is ____ if it is ergodic in the mean and the auto correlation function.	first-order stationary proces	Wide-sense ergodic	WSS	SSS	Wide-sense ergodic
25	A random process that is not stationary in any sense is called as ------	Evolutionary process	Strict sense stationary process	WSS	Markov process	Evolutionary process
26	A continuous random sequence satisfying Markov property is known as ---- as t is discrete &{Xi} is continuous.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	discrete parameter Markov process
27	The Markov chain is ___ if there is only one class.	Irreducible	reducible	Poisson	Binomial	Irreducible
28	The Binomial process is ___.	strongly stationary process	Markov process	wide-sense stationary process	covariance stationary process	Markov process
29	A state is said to be ___ if its period is 1.	Markov process	ergodic	aperiodic	periodic	aperiodic
30	A state is said to be aperiodic if its period is ____	0	1	2	3	1

31	The state ‘i’ is called an ___ state if it communications with every state it leads to.	essential	ergodic	aperiodic	identity	essential
32	The Poisson process is a ____ process	Markov	WSS	WSE	SSS	Markov
33	A Random process in which all type of ensemble averages are interchangeable with the corresponding time averages is called an ___ process	Markov	WSS	WSE	ergodic	ergodic
34	A Random process is ___, if it is ergodic in the mean and the auto correlation funtion	Markov	Wide ergodic process	WSS	SSS	Wide ergodic process
35	___ process has limited historical dependency	Markov	Wide ergodic process	WSS	SSS	Markov
36	A first order linear differential equation is a ___	WSS	Wide ergodic process	Markovian	SSS	Markovian
37	Two states i and j which are accessible to each other are said to ___	Irreducible	reducible	communicate	absorbing	communicate
38	A state is said to be an ___ state if no other state is accessible from it.	Irreducible	reducible	communicate	absorbing	absorbing
39	A state i is ___, if starting in i, the expected time until the process returns to state i is finite.	negative recurrent	positive recurrent	recurrent	Irreducible	positive recurrent
40	In a fintie ___, all recurrent states are positive recurrent	negative recurrent	markov chain	recurrent	Irreducible	markov chain
41	In a fintie markov chain, all recurrent states are ___.	negative recurrent	positive recurrent	recurrent	Irreducible	positive recurrent
42	All states of a finite irreducible markov chain are ___	recurrent	reducible	communicate	absorbing	recurrent
43	All states of a finite ____ markov chain are recurrent.	Irreducible	reducible	communicate	absorbing	Irreducible
44	A state i is called an --- state if ti communicates with every state it leads to.	Irreducible	reducible	essential	absorbing	essential
45	A special case of ergodic markov chain is ___ markov chain	reducible	essential	aperiodic	regular	regular
46	A special case of ___ markov chain is regular markov chain	reducible	essential	aperiodic	ergodic	ergodic
47	Positive recurrent, aperiodic states are called ___	reducible	essential	aperiodic	ergodic	ergodic
48	A random process is called a ___ random process if all the future values can be predicted from past observations.	non-deterministic	deterministic	stationary	markov	deterministic

49	A random process is called a deterministic random process if all the future values ____ be predicted from past observations.	can	cannot	should	may	can
50	A random process is called a ___ random process if all the future values of any sample function cannot be predicted from past observations.	non-deterministic	deterministic	stationary	markov	non-deterministic
51	A random process is called a non-deterministic random process if all the future values ____ be predicted from past observations.	can	cannot	should	may	cannot
52	___ explains the time invariance of certain properties of the random process	reducible	stationarity	aperiodic	ergodic	stationarity
53	A continuous random process satisfying Markov property is known as ---- as t is continuous &{Xi} is also continuous.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	Continuous parameter Markov process
54	A discrete random sequence satisfying Markov property is known as ---- as t is discrete &{Xi} is also discrete.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	discrete parameter Markov chain
55	A discrete random process satisfying Markov property is known as ---- as t is continuous & {Xi} is discrete.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	Continuous parameter Markov chain

PRP - UNIT - V - ONLINE

S.NO	QUESTIONS	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
1	The Cross covariance, C_{XX} (t_1,t_2) is	Rxx(t1,t2) E[X(t1)] E[X(t2)]	Rxx(t1,t2) / E[X(t1)] E[X(t2)]	Rxx(t1,t2) + E[X(t1)] E[X(t2)]	Rxx(t1,t2) – E[X(t1)] E[X(t2)]	Rxx(t1,t2) – E[X(t1)] E[X(t2)]
2	Ergodicity is a weaker condition than----.	stationary	cross correlation	auto correlation	SSS	stationary
3	If X(t) & Y(t) are orthogonal, then S_{YX}(f) =	0	1	between 0 to 1	4	0
4	A ------ is defined by a functional relationship between the input and the output as y(t) = f{x(t)}	system	linear	non-linear	unique	system

5	The mean of the derivative of a stationary process is---	1	3	4	0	0
6	The cross correlation of the two random processes is	Rxy(t1,t2) = E[X(t1) +Y(t2)]	Rxy(t1,t2) = E[X(t1) /Y(t2)]	Rxx(t1,t2) = E[X(t1) Y(t2)]	Rxy(t1,t2) = E[X(t1) Y(t2)]	Rxy(t1,t2) = E[X(t1) Y(t2)]
7	A random process {X(t)} is called --- if all its ensemble averages equals appropriate time averages.	stochastic	ergodic	WSS	SSS	ergodic
8	If the auto correlation function of a random process exists over a finite time range, the power density spectrum exists over--- frequency range.	infinite	finite	unique	zero	infinite
9	A system is --- if the principle of superposition does not hold good.	system	linear	non-linear	unique	non-linear
10	The power density spectrum of a linear system is a ---function.	imaginary	real valued	constant	identity	real valued
11	When the correlation is defined between two random variables each from two different processes or two sample functions each from different processes, the correlation function are called as ----- function.	Cross correlation	Auto correlation	SSS	WSS	Cross correlation
12	Cross – correlation does not necessarily have a maximum at ------	point	origin	constant	unique	origin
13	The auto correlation function of E(sinwt) and E sin (wt + q) are---	same	odd	even	not defined	same
14	The auto covariance of the random process is the --- of the random variables obtained by observing the process at time t_1 and t2 respectively.	mean	covariance	time	auto correlation	covariance
15	The important time and frequency parameters relationship of random process is called as	Fourier series	Einstein – Wienern-Khinchine relationship	Markov process	Binomial process	Einstein – Wiener-Khinchin relationship

16	___ theorem provides an alternative method for finding the power spectral density function.	Einstein	Wienern-Khinchine	Poisson	Binomial	Wienern-Khinchine
17	The cross spectral density of two orthogonal processes is----	0	1	2	3	0
18	The imaginary part of $S_{XY}(f)$ is an ---- function of f.	odd	even	constant	unique	odd
19	If the auto correlation function of a stationary random process exists over an infinite time range, its power density spectrum exists over--- frequency range.	infinite	finite	unique	zero	finite
20	$R_{XY}(t)$ =0 if the processes are---	independen t	orthogonal	random	random signal	orthogonal
21	___ is defined as a property of linear systems that if the input is time shifted by an amount, the corresponding output will also be time shifted by the same amount.	Time invariance	Causality	Causal	stable	Time invariance
22	The auto correlation function is a __ order moment.	first	second	higher	nth	second
23	The _____ function is a second order moment.	correlation	cross correlation	auto correlation	time cross correlation	auto correlation
24	The unit of power density spectrum is ___	km/hour	sq.units	cu.units	watts per hertz	watts per hertz
25	The ___ spectral density of two orthogonal processes is 0	auto	cross	correlation	time cross	cross
26	The ___ relationship relates time and frequency characteristics of a random process.	Einstein-Wiener-Khinchin	Euler - Einstein	RMS	cross-power density and cross-correlation function	Einstein-Wiener-Khinchin

27	If the ___ function has periodic components, then the corresponding process also will have periodic components.	autocorrelat ion	crosscorrel ation	correlation	time cross	autocorrelat ion
28	If the autocorrelation function has periodic components, then the corresponding process also will have ____components.	aperiodic	WSS	periodic	ergotic	periodic
29	S(f) gives the distribution of power of {X(t)} as a funtion of frequency and hence is called the ___ function.	autocorrelat ion	crosscorrel ation	power spectral density	ergotic	power spectral density
30	The mean square value of a ___ process is equal to the total area under the graph of the spectral denstiy.	WSS	SSS	WSE	ergotic	WSS
31	The mean square value of a wise-sense stationary process is equal to the total____under the graph of the spectral denstiy.	volume	amount	density	area	area
32	The value of the ___ funtion at zero frequency is equal to the total area under the graph of the autocorrelation funtion.	autocorrelat ion	crosscorrel ation	spectral density	ergotic	spectral density
33	The value of the spectral density funtion at ____ frequency is equal to the total area under the graph of the autocorrelation funtion.	0	1	2	3	0
34	The spectral density function of a real random process is an __ funtion	odd	even	constant	unique	even
35	The spectral density function of a ___ random process is an even funtion	complex	real	imaginary	constant	real
36	The spectral density and the autocorrelation function of a real WSS process form a ___ pair.	Fourier transform	Fourier cosine transform	Fourier sine transform	Fourier series	Fourier cosine transform
37	If the system operates only on the varibale t treating s as a parameter, it is called a ___.	linear	deterministi c	stochastic	system	deterministi c

38	If the system operates on both t (time) and s (parameter), it is called ___	linear	deterministic	stochastic	system	stochastic
39	If Y(t+h)=f[X(t+h)], then f is called a ___ system	time-invariant	invariant	cross-invariant	auto-invariant	time-invariant
40	If the value of the output Y(t) depends only on the past values on the input X(t), then the system is called a __ system	linear	deterministic	stochastic	causal	causal
41	If the output Y(t) at a given time depends only on X(t) and not an any other past or future values X(t), then the system f is called a ___ system	power density	power transfer	memoryless	causal	memoryless
42	If the input of the system is the unit impulse function, then the output is the system ___ funtion.	unit impulse	unit impulse response	weighting	unit impulse response or weighting	unit impulse response or weighting
43	h(t) is denoted as ___ function	unit impulse response	unit impulse	time-invariant	causal	unit impulse response
44	If a system is such that its input X(t) and its output Y(t) are related by a ___, then the system is a linear time-invariant system.	Einstein-Wiener-Khinchin	Euler - Einstein	RMS	convolution integral	convolution integral
45	If a system is such that its input X(t) and its output Y(t) are related by a convolution integral then the system is a linear ____ system.	time-invariant	invariant	cross-invariant	auto-invariant	time-invariant
46	If the input to a time-invariant, stable linear system is a WSS process, the output will a __ process	SSS	WSS	WSE	ergotic	WSS
47	If the input to a ___ linear system is a WSS process, the output will also a WSS process	time-invariant	invariant	unit impulse	time-invariant, stable	time-invariant, stable

48	___ is the Fourier transform of the unit impulse response funtion of the system.	power density function	power transfer function	power density spectrum	causal	power transfer function
49	The spectral denstiy of any WSS process is ___	positive	negative	very small	non-negative	non-negative
50	H(omega) is called as ___ function	system	power transfer	time-invariant	system or power transfer	system or power transfer
51	The another name of the system weighting function is ___ function	unit impulse response	unit impulse	time-invariant	causal	unit impulse response
52	R (tau) is called the ____ function	autocorrelation	crosscorrelation	time-invariant	ergotic	autocorrelation
53	R (tau) is an ____ function	odd	even	unique	constant	even
54	R(tau) is maximum at (tau) =	1	-1	0	infinity	0
55	If the processes {X(t)} and {Y(t)} are orthogonal, then R_{XY}(tau) =	1	-1	0	infinity	0
56	The concepts of ergodicity deals with equality of ___ averages and __ averages.	continuous, ensemble	time, ensemble	time, stationary	discrete, ensemble	time, ensemble
57	___ theorem provides a sufficient condition for the mean-ergodicity of a random process.	Wiener-Khinchin	Euler	Einstein	Mean-Ergodic	Mean-Ergodic

PRP - UNIT - II - ONLINE

S.No	Questions	OPT 1	OPT 2	OPT3	OPT 4	ANSWERS
1	Binomial distribution is symmetrical if__________	p = q = ½	p = q = ¾	p = q = 4/5	p = q = 2/3	p = q = ½
2	A normal curve has an ___	Elliptic	parabolic	hyperbolic	asymptote	asymptote
3	Variance of binomial distribution is_______.	npq	np	nq	square root of (npq)	npq
4	“The number of printing errors at each page of a book” is a example of ___ distribution.	Normal	Uniform	Binomial	Poisson	Poisson
5	Moment generating function of Uniform density function is	e^bt- e^at/ t (b-a)	e^bt+ e^at/ t (b+a)	e^bt+ e^at/ t (b-a)	e^bt- e^at/ t (b+a)	e^bt- e^at/ t (b-a)
6	For binomial distribution ______	variance -1= mean	variance = mean	variance > mean	variance < mean	variance < mean
7	_________ is a non negative continuous random variable.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Gamma distribution
8	Standard deviation of binomial distribution is ____	√npq	np(q-p)	npq	npq(q-p)	√npq
9	The density function of the Uniform distribution is _________.	1/ ba a<x<b	1/ (b+a) a<x<b	1/ ba a>x>b	1/ (b-a) a<x<b	1/ (b-a) a<x<b
10	Moment generating function of Binomial distribution	Mx (t) = (1 – p) ^n	Mx (t) = (pe^t + q)^ n	Mx (t) = (pe^t – p) ^n	Mx (t) = (e^t + q)^ n	Mx (t) = (pe^t +q)^ n
11	The mean and variance of a standard normal distribution is ___	N(1,2)	N(0,1)	N(0,2)	N(0,40)	N(0,1)
12	Variance of Uniform density function is	b /2	[(b-a)^2]/12	ba /2	[(b+a)^2] /12	[(b-a)^2]/12
13	A continuous random variable X has a probability density function f(x)= K, 0 <= x <= 1. Find K.	1	2	3	4	1
14	If X is normally distributed with mean 1 and S.D. ½ , find the probability that X > 2.	0.0288	0.0544	0.0228	0.0882	0.0228
15	Mean of Uniform density function is___________	b /2	(b-a) /2	ba /2	(b+a) /2	(b+a) /2
16	The formula of variance is ___________	Var[X] = E(X^2) – [E(X)]^ 2	Var[X] = E(X^2) – [E(X)]	Var[X] = E(X) – [E(X)]^ 2	Var[X] = E(X) – [E(X)].	Var[X] = E(X^2) – [E(X)]^ 2
17	Binomial distribution is	nCx(p^x)q^(n-x)	nCx(p^x)q^(n+x)	nCx(p^-x)q^(n-x)	nCx(p^-x)q^(n+x)	nCx(p^x)q^(n-x)
18	Mean of Poisson distribution is ____________	λ	λ/t	λ-t	λ+t	λ
19	Gamma of n =	(n-1) !	(n +1) !	n!	0!	(n-1) !
20	Moment generating function of Exponential distribution is	λ/(λ-t)	2λ/(λ-t)	λ/(λ+t)	2λ/(λ-+t)	λ/(λ-t)
21	Variance of Poisson distribution is	λ	2λ	3λ	4λ	λ
22	Exponential distribution is	λe^(λx)	λe^(-λx)	λe^(-2λx)	λe^(2λx)	λe^(-λx)
23	Gamma of (1/2) =	(π^2)/2	square root of π/2	square root of π/3	square root of π	square root of π
24	Moment generating function of Poisson distribution	e^ [λ(e^t-1)]	e^ [λ(e^t+1)]	e^ [λ(e^t-2)]	e^ [-λ(e^t-1)]	e^ [λ(e^t-1)]

25	Mean of Exponential distribution is	1/λ	λ	λ/2	2/λ	1/λ
26	Geometric distribution is given byP{X=x}=_____where x=1,2,…	pq^(x-1)	pq^(x-2)	pq^(x+1)	pq^(2x-1)	pq^(x-1)
27	If the mean and variance of a binomial variate are 8 and 6, then the probability of failure is given by______.	q=3/4	q=4/3	q=1/4	q=1/3	q=3/4
28	If the mean and variance of a binomial variate are 20 and 16, then the probability of success is given by______.	p=1/5	p=2/5	p=3/5	p=3/4	p=1/5
29	If n=5 and p=1/2 then the mean of a binomial variate is ______.	0.50	2.50	3.5	4.5	2.5
30	The mean of a poisson variate is 2 . Find its variance.	2	3	1	2.5	2
31	Poisson distribution is the limiting case of __________distribution.	Binomial distribution	Gamma distribution	Poisson distribution	negative binomial distribution	Binomial distribution
32	The other name of uniform distribution is__________.	Binomial distribution	Gamma distribution	Poisson distribution	rectangular distribution	rectangular distribution
33	In a Uniform distribution if X is distributed uniformly on (0,30) then its density function is given by______.	F(x)= 1/13	F(x)= 2/13	F(x)= 1/3	F(x)= 1/30	F(x)= 1/30
34	______ is larger than the mean for a negative binomial distribution	Variance	Standard deviation	Mean deviation	Quartile deviation	Variance
35	Mean of binomial distribution is ____	np	npq	n+1	n	np
36	Third moment of Binomial distribution is ____	n(q-p)	np(q-p)	npq	npq(q-p)	npq(q-p)
37	For a negative binomial distribution______	Var(X) > E(X)	Var(X) < E(X)	Var(X) = E(X)	Var(X) / E(X)	Var(X) > E(X)
38	If X follows a Poisson distribution such that P(X=1) = 1/4 and P(X=2) = 3/8, find P(X=3).	0.123	1.234	2.34	0.375	0.375
39	The height of persons in a country is a random variable of the type____	continuous random variable	neither discrete nor continuous random vaiables	Continuous as well as discrete random variable	discrete random variable	continuous random variable
40	A family of parametric distrbution in which mean is equal to variance is _______	Binomial distribution	Gamma distribution	normal Distribution	Poissson distribution	Poissson distribution
41	A family of parametric distrbution in which mean is always greater than its variance is _______	Binomial distribution	Gamma distribution	Geometric Distribution	Poissson distribution	Geometric Distribution
42	The _________ distribution has the memory less property.	Gamma distribution	Geometric distribution	Geometric distribution	Poissson distribution	Geometric distribution
43	The mean of the binomial distribution is _____ than its variance	greater than	Less than	more	normal	greater than
44	Mean and variance of geometric distribution are__________	related	correlated	rectangle	range	related
45	A distribution where the mean and median have different values is not a _________ distribution	normal	binomial	poisson	gamma	normal
46	Normal distribution was invented by _____	Laplace	De-Moivre	Gauss	all the above	all the above

PRP - UNIT - III - ONLINE						
S.No	**Questions**	**OPT 1**	**OPT 2**	**OPT3**	**OPT 4**	**ANSWERS**
1	The coefficient of correlation is independent of change of_____ and ________	scale,origin	vector,origin	variable, constant	interer, origin	scale,origin
2	When r = 0 the line of regression are ______to each other.	parallel	perpendicular	straight line	circular	perpendicular
3	The relationship between three or more variables is studied with the help of _________ correlation.	multiple	rank	perferct	spearman's rank	multiple
4	The coefficient of correlation is under-root of two _____	regression coefficients	rank coefficient	Regression equation	regression line	regression coefficient
5	The coefficient of correlation___	has no limits	can be less than 1	can be more than 1	varies between + or - one	varies between + or - one
6	which of the following is the highest range of r__________	0 and 1	minus one and 0	minus one and one	zero	minus one and one
7	The coefficient of correlation is independent of ________	change of scale only	change of origin only	both change of scale and origin	change of variables	both change of scale and origin
8	The coefficient of correlation___	cannot be positive	cannot be negative	can be either positive or negative	zero	can be either positive or negative
9	COV(X,Y)= __________	E(XY)-E(X)E(Y)	E(XY)+E(X)E(Y)	E(XY)	Var(X,Y)	E(XY)-E(X)E(Y)
10	Two random variables with non zero correlation are said to be___	correlation	regression	rank	variables	regression
11	Correlation means relationship between ______ variables	two	one	two or more	three	two or more
12	A Mathematical measure of the average relationship between two variables is called______	correlation	regression	rank	variables	correlation
13	The covariance of two independent random variable is ________	Zero	two	three	two or more	Zero
14	Two random variables are said to be orthogonal if _______	correlation is zero	rank is zero	covariance is zero	one	correlation is zero
15	Two random variables are said to be uncorrelated if correlation coefficient is____	zero	one	two or more	orthogonal	zero
16	Regression analysis is a mathematical measures of the average relationship between ________variable	two or more	one	Two variables	three	two or more
17	The regresision analysis confined to ther study of only two variable at a time is called__________regression	Simple	Multiple	Linear	two	Simple
18	If r=0, then the regression coefficient are________	zero	one	threee	constant	zero
19	The equation of the fitted stright line is _____	y=ax+b	y=a+bx	y=mx+c	y=mx	y=ax+b

20	If X=Y , then correlation cofficient between them is _______	1	zero	less than one	gerater than one	1
21	The greater the value of r__________obtained through regression analysis	the better are estimates	the worst are the estimates	really makes no difference	good estimates	the better are estimates
22	Where r is zero the regression lines cut each other making an angle of______	30 degree	60 degree	90 degree	neither of the above	neither of the above
23	The father the two regression lines cut each other_______	Greater will be degree of correlation	The less will be the degree of correlation	does not matter	the worst are the estimates	the less will be the degree of correlation
24	The regression lines cut each other at the point of :______	Average of X and Y	Average of X only	Average of Y only	average of both(a) and (b)	average of X and Y
25	When the two regression lines coincide, then r is :_________	0	-1	1	0.5	1
26	The variable , we are trying to predict is called the _________	depentent variable	indepent variable	constant	normal	Dependent variable
27	Both the regression coefficients cannot ______one	exceed	exact	plus or minus	negative	exceed
28	The regression analysis measures ______between variables	dependence	independence	constant	normal	Dependence
29	If the possible values of (X,Y) are finite, then (X,Y) is called a______	two dimensional random variable	onedimensional random variable	both a and b	infinte	two dimensional random variable
30	If X & Y are continuous random variable , then f(x,y) is _____	joint probability function	joint probability density function	both a and b	infinte	both a and b
31	Joint probability is the probability of the _________occurrence of two or more events.	Simultaneous (or) joint	Conditional	Mariginal probability	density function	Simultaneous (or) joint
32	The order of arrangement is important in _______	permutation	Gambling	joint	density	Permutation
33	If X & Y are____random variable , then f(x,y) is called joint probability function.	discrete	continuous	both a and b	infinte	continuous
34	If the value of y decreases as the value of x increases then there is ______correlation between two variables.	negative	perfect positive	both a and b	infinte	negative
35	The correlation between the income and expenditure is _____	positive	negative	finite	both a and b	positive
36	correlation between price and demand of commodity is ____	positive	finite	negative	both a and b	negative
37	If X and Y are independent , then _____	E(XY) = E(X) + E(Y)	E(XY) = E(X) - E(Y)	E(XY) = E(X) E(Y)	E(XY) = E(X)/E(Y)	E(XY) = E(X) E(Y)

38	correlation coefficient does not exceed _____	unity	5	0	2	unity
39	Two independent variables are _____	correlated	uncorrelated	both a and b	positive	uncorrelated
40	In Rank correlation the correction factor is added for each _____value.	repeated	Non-repeated	indefinite	both a and b	repeated
41	When r = 1 or -1 the the line of regression are ____ to each other.	parallel	perpendicular	straight line	circular	parallel
42	If the curve is a straight line, then it is called the ____	the line of correlation	the line of regression	covariance	both a and b	the line of regression
43	If the curve is not a straight line, then it is called the ____	covariance	the line of correlation	the curvilinear	the line of regression	the curvilinear
44	when r is ____ the correlation is perfect and positive.	1	2	3	0	1
45	If X and Y are independent , then	E(XY)=0	E(X) E(Y)=0	Cov(X,Y) =0	E(XY)=1	Cov(X,Y) =0
46	Two random variables X and Y with joint pdf f(x,y) is said to independent if ____	f(x,y) = f(x) +f(y)	f(x,y) = f(x) / f(y)	f(x,y) = f(x) * f(y)	f(x,y) = f(x) -	f(x,y) = f(x) * f(y)
47	Cov(X,Y)=_____	E[{ X- E(X) }* { Y – E(Y) }]	E[{ X- E(X) }+ { Y – E(Y) }]	E[{ X- E(X) }- { Y – E(Y) }]	E[{ X- E(X) } { Y – E(Y) }]	E[{ X- E(X) } { Y – E(Y) }]
48	The correlation coefficient is used to determine______	A specific value of the y-variable given a specific value of the x-variable	A specific value of the x-variable given a specific value of the y-variable	The strength of the relationship between the x and y variables	is the same as r-square	The strength of the relationship between the x and y variables
49	The coefficient of correlation_____	is the square of the coefficient of determination	is the square root of the coefficient of determination	is the same as r-square	can never be negative	is the square root of the coefficient of determination
50	The correlation between two variables is of order_____	2	1	0	3	0

PRP - UNIT - IV - ONLINE						
S.No	**Questions**	**OPT 1**	**OPT 2**	**OPT3**	**OPT 4**	**ANSWERS**
1	The probabilistic model used for characterizing a -------is called a random process	random process	random signal	random model	random variable	random process
2	The Random process is also called as__	Markov process	WSS	SSS	stochastic process	stochastic process
3	The family of all functions X(s,t) is called ----	random process	random signal	random variables	random model	random process
4	A---- is a collection of Random variables that are functions of t and s.	Random process	random function	random signal	random model	Random process
5	A non null persistent and aperiodic state is called ---	stochastic	ergodic	WSS	SSS	ergodic
6	If X is continuous and t can have any of a continous of values, then X(t) is called as	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
7	If X assumes only discrete and t is continuous, then X(t) is called as	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Discrete random process
8	Let X denote the number of telephone calls received in the interval (0,t). Then {X(t)} is a	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Discrete random process
9	Thermal agitation noise in conductors is an example of ------	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
10	Let X denote the maximum temperature at a place in the interval (0,t). Then {X(t)} is a	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous Random process
11	The outcome of the n-th toss of a fair dice is an example of ------	Continuous Random process	Discrete random process	Continuous random sequence	Discrete random sequence	Discrete random sequence
12	A random process for which X is continuous but time takes only discrete values is called a ---	Continuous Random process	Discrete random process	Continuous random sequence	Discrete sequence	Continuous random sequence

13	A random process for which X is discrete and time takes only discrete values is called a ---	Continuous Random process	Discrete random process	Continuous random sequence	Discrete random sequence	Discrete random sequence
14	The set of possible values of any individual members of the random process is called ___ space.	vector	state	random	universal	state
15	If the process is first order stationary, then mean is	negative	positive	constant	unique	constant
16	A stochastic matrix is said to be a regular matrix, if all the entries of Pm are---	positive	negative	zero	square matrix	positive
17	The discrete parameter Markov process is called a ____	weakly stationary process	covariance stationary process	wide-sense stationary process	Markov chain	Markov chain
18	A random process is called a --- if its mean is constant and the autocorrelation depends only on the time difference.	weakly stationary process	covariance stationary process	wide-sense stationary process	All the above	All the above
19	If the transition probability matrix is regular, then the homogeneous Markov chain is	regular	irregular	square matrix	unique	regular
20	The n-th order stationary process is stationary to order ___	n	n+1	n*n	n-1	n-1
21	A random process is called a ------- , if all its finite dimensional distributions are invariant under translation of time parameter.	Wide-sense stationary process	Strict sense stationary process	Markov process	Covariance stationary process	Strict sense stationary process
22	All regular Markov chain are ___	Markov process	ergodic process	WSS	SSS	ergodic process
23	The transition probability matrix of a finite state Markov chain is a __ matrix.	row	column	square	identity	square
24	A random process is ____ if it is ergodic in the mean and the auto correlation function.	first-order stationary proces	Wide-sense ergodic	WSS	SSS	Wide-sense ergodic
25	A random process that is not stationary in any sense is called as ------	Evolutionary process	Strict sense stationary process	WSS	Markov process	Evolutionary process

26	A continuous random sequence satisfying Markov property is known as ---- as t is discrete &{Xi} is continuous.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	discrete parameter Markov process
27	The Markov chain is ___ if there is only one class.	Irreducible	reducible	Poisson	Binomial	Irreducible
28	The Binomial process is ___.	strongly stationary process	Markov process	wide-sense stationary process	covariance stationary process	Markov process
29	A state is said to be ___ if its period is 1.	Markov process	ergodic	aperiodic	periodic	aperiodic
30	A state is said to be aperiodic if its period is ____	0	1	2	3	1
31	The state 'i' is called an ___ state if it communications with every state it leads to.	essential	ergodic	aperiodic	identity	essential
32	The Poisson process is a ____ process	Markov	WSS	WSE	SSS	Markov
33	A Random process in which all type of ensemble averages are interchangeable with the corresponding time averages is called an ___ process	Markov	WSS	WSE	ergodic	ergodic
34	A Random process is ___, if it is ergodic in the mean and the auto correlation funtion	Markov	Wide ergodic process	WSS	SSS	Wide ergodic process
35	___ process has limited historical dependency	Markov	Wide ergodic process	WSS	SSS	Markov
36	A first order linear differential equation is a ___	WSS	Wide ergodic process	Markovian	SSS	Markovian
37	Two states i and j which are accessible to each other are said to ___	Irreducible	reducible	communicate	absorbing	communicate
38	A state is said to be an ___ state if no other state is accessible from it.	Irreducible	reducible	communicate	absorbing	absorbing
39	A state i is ___, if starting in i, the expected time until the process returns to state i is finite.	negative recurrent	positive recurrent	recurrent	Irreducible	positive recurrent
40	In a fintie ___, all recurrent states are positive recurrent	negative recurrent	markov chain	recurrent	Irreducible	markov chain
41	In a fintie markov chain, all recurrent states are ___.	negative recurrent	positive recurrent	recurrent	Irreducible	positive recurrent
42	All states of a finite irreducible markov chain are ___	recurrent	reducible	communicate	absorbing	recurrent
43	All states of a finite ____ markov chain are recurrent.	Irreducible	reducible	communicate	absorbing	Irreducible
44	A state i is called an --- state if ti communicates with every state it leads to.	Irreducible	reducible	essential	absorbing	essential
45	A special case of ergodic markov chain is ___ markov chain	reducible	essential	aperiodic	regular	regular
46	A special case of ___ markov chain is regular markov chain	reducible	essential	aperiodic	ergodic	ergodic
47	Positive recurrent, aperiodic states are called ___	reducible	essential	aperiodic	ergodic	ergodic

48	A random process is called a ___ random process if all the future values can be predicted from past observations.	non-deterministic	deterministic	stationary	markov	deterministic
49	A random process is called a deterministic random process if all the future values ____ be predicted from past observations.	can	cannot	should	may	can
50	A random process is called a ___ random process if all the future values of any sample function cannot be predicted from past observations.	non-deterministic	deterministic	stationary	markov	non-deterministic
51	A random process is called a non-deterministic random process if all the future values ____ be predicted from past observations.	can	cannot	should	may	cannot
52	___ explains the time invariance of certain properties of the random process	reducible	stationarity	aperiodic	ergodic	stationarity
53	A continuous random process satisfying Markov property is known as ---- as t is continuous &{Xi} is also continuous.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	Continuous parameter Markov process
54	A discrete random sequence satisfying Markov property is known as ---- as t is discrete &{Xi} is also discrete.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	discrete parameter Markov chain
55	A discrete random process satisfying Markov property is known as ---- as t is continuous & {Xi} is discrete.	Continuous parameter Markov process	discrete parameter Markov process	discrete parameter Markov chain	Continuous parameter Markov chain	Continuous parameter Markov chain

PRP - UNIT - V - ONLINE						
S.NO	**QUESTIONS**	**OPTION 1**	**OPTION 2**	**OPTION 3**	**OPTION 4**	**ANSWER**
1	The Cross covariance, C_{XX} (t_1,t_2) is	Rxx(t1,t2) E[X(t1)] E[X(t2)]	Rxx(t1,t2) / E[X(t1)] E[X(t2)]	Rxx(t1,t2) + E[X(t1)] E[X(t2)]	Rxx(t1,t2) – E[X(t1)] E[X(t2)]	Rxx(t1,t2) – E[X(t1)] E[X(t2)]
2	Ergodicity is a weaker condition than----.	stationary	cross correlation	auto correlation	SSS	stationary
3	If X(t) & Y(t) are orthogonal, then $S_{YX}(f)$ =	0	1	between 0 to 1	4	0
4	A ------ is defined by a functional relationship between the input and the output as y(t) = f{x(t)}	system	linear	non-linear	unique	system
5	The mean of the derivative of a stationary process is---	1	3	4	0	0
6	The cross correlation of the two random processes is	Rxy(t1,t2) = E[X(t1) +Y(t2)]	Rxy(t1,t2) = E[X(t1) /Y(t2)]	Rxx(t1,t2) = E[X(t1) Y(t2)]	Rxy(t1,t2) = E[X(t1) Y(t2)]	Rxy(t1,t2) = E[X(t1) Y(t2)]
7	A random process {X(t)} is called --- if all its ensemble averages equals appropriate time averages.	stochastic	ergodic	WSS	SSS	ergodic
8	If the auto correlation function of a random process exists over a finite time range, the power density spectrum exists over--- frequency range.	infinite	finite	unique	zero	infinite
9	A system is --- if the principle of superposition does not hold good.	system	linear	non-linear	unique	non-linear
10	The power density spectrum of a linear system is a ---function.	imaginary	real valued	constant	identity	real valued
11	When the correlation is defined between two random variables each from two different processes or two sample functions each from different processes, the correlation function are called as ----- function.	Cross correlation	Auto correlation	SSS	WSS	Cross correlation
12	Cross – correlation does not necessarily have a maximum at ------	point	origin	constant	unique	origin
13	The auto correlation function of E(sinwt) and E sin (wt + q) are---	same	odd	even	not defined	same

14	The auto covariance of the random process is the --- of the random variables obtained by observing the process at time t_1 and t2 respectively.	mean	covariance	time	auto correlation	covariance
15	The important time and frequency parameters relationship of random process is called as	Fourier series	Einstein – Wienern-Khinchine relationship	Markov process	Binomial process	Einstein – Wiener-Khinchin relationship
16	___ theorem provides an alternative method for finding the power spectral density function.	Einstein	Wienern-Khinchine	Poisson	Binomial	Wienern-Khinchine
17	The cross spectral density of two orthogonal processes is----	0	1	2	3	0
18	The imaginary part of $S_{XY}(f)$ is an ---- function of f.	odd	even	constant	unique	odd
19	If the auto correlation function of a stationary random process exists over an infinite time range, its power density spectrum exists over--- frequency range.	infinite	finite	unique	zero	finite
20	$R_{XY}(t)$ =0 if the processes are---	independent	orthogonal	random	random signal	orthogonal
21	___ is defined as a property of linear systems that if the input is time shifted by an amount, the corresponding output will also be time shifted by the same amount.	Time invariance	Causality	Causal	stable	Time invariance
22	The auto correlation function is a __ order moment.	first	second	higher	nth	second
23	The _____ function is a second order moment.	correlation	cross correlatio	auto correlatio	time cross correlation	auto correlation
24	The unit of power density spectrum is ___	km/hour	sq.units	cu.units	watts per hertz	watts per hertz
25	The ___ spectral density of two orthogonal processes is 0	auto	cross	correlation	time cross	cross
26	The ___ relationship relates time and frequency characteristics of a random process.	Einstein-Wiener-Khinchin	Euler - Einstein	RMS	cross-power density and cross-correlation function	Einstein-Wiener-Khinchin
27	If the ___ function has periodic components, then the corresponding process also will have periodic components.	autocorrelation	crosscorrelation	correlation	time cross	autocorrelation

28	If the autocorrelation function has periodic components, then the corresponding process also will have ____components.	aperiodic	WSS	periodic	ergotic	periodic
29	S(f) gives the distribution of power of {X(t)} as a funtion of frequency and hence is called the ___ function.	autocorrelation	crosscorrelation	power spectral density	ergotic	power spectral density
30	The mean square value of a ___ process is equal to the total area under the graph of the spectral denstiy.	WSS	SSS	WSE	ergotic	WSS
31	The mean square value of a wise-sense stationary process is equal to the total____under the graph of the spectral denstiy.	volume	amount	density	area	area
32	The value of the ___ funtion at zero frequency is equal to the total area under the graph of the autocorrelation funtion.	autocorrelation	crosscorrelation	spectral density	ergotic	spectral density
33	The value of the spectral density funtion at ____ frequency is equal to the total area under the graph of the autocorrelation funtion.	0	1	2	3	0
34	The spectral density function of a real random process is an __ funtion	odd	even	constant	unique	even
35	The spectral density function of a ___ random process is an even funtion	complex	real	imaginary	constant	real
36	The spectral density and the autocorrelation function of a real WSS process form a ___ pair.	Fourier transform	Fourier cosine transform	Fourier sine transform	Fourier series	Fourier cosine transform
37	If the system operates only on the varibale t treating s as a parameter, it is called a ___.	linear	deterministic	stochastic	system	deterministic
38	If the system operates on both t (time) and s (parameter), it is called ___	linear	deterministic	stochastic	system	stochastic
39	If Y(t+h)=f[X(t+h)], then f is called a ___ system	time-invariant	invariant	cross-invariant	auto-invariant	time-invariant
40	If the value of the output Y(t) depends only on the past values on the input X(t), then the system is called a __ system	linear	deterministic	stochastic	causal	causal

41	If the output Y(t) at a given time depends only on X(t) and not an any other past or future values X(t), then the system f is called a ___ system	power density	power transfer	memoryless	causal	memoryless
42	If the input of the system is the unit impulse function, then the output is the system ___ funtion.	unit impulse	unit impulse response	weighting	unit impulse response or weighting	unit impulse response or weighting
43	h(t) is denoted as ___ function	unit impulse response	unit impulse	time-invariant	causal	unit impulse response
44	If a system is such that its input X(t) and its output Y(t) are related by a ___, then the system is a linear time-invariant system.	Einstein-Wiener-Khinchin	Euler - Einstein	RMS	convolution integral	convolution integral
45	If a system is such that its input X(t) and its output Y(t) are related by a convolution integral then the system is a linear ____ system.	time-invariant	invariant	cross-invariant	auto-invariant	time-invariant
46	If the input to a time-invariant, stable linear system is a WSS process, the output will a __ process	SSS	WSS	WSE	ergotic	WSS
47	If the input to a ___ linear system is a WSS process, the output will also a WSS process	time-invariant	invariant	unit impulse	time-invariant, stable	time-invariant, stable
48	___ is the Fourier transform of the unit impulse response funtion of the system.	power density	power transfer	power density s	causal	power transfer function
49	The spectral denstiy of any WSS process is ___	positive	negative	very small	non-negative	non-negative
50	H(omega) is called as ___ function	system	power transfer	time-invariant	system or power transfer	system or power transfer
51	The another name of the system weighting function is ___ function	unit impulse response	unit impulse	time-invariant	causal	unit impulse response
52	R (tau) is called the ____ function	autocorrelation	crosscorrelation	time-invariant	ergotic	autocorrelation
53	R (tau) is an ____ function	odd	even	unique	constant	even
54	R(tau) is maximum at (tau) =	1	-1	0	infinity	0

55	If the processes {X(t)} and {Y(t)} are orthogonal, then R_{XY}(tau) =	1	-1	0	infinity	0
56	The concepts of ergodicity deals with equality of ___ averages and __ averages.	continuous, ensemble	time, ensemble	time, stationary	discrete, ensemble	time, ensemble
57	___ theorem provides a sufficient condition for the mean-ergodicity of a random process.	Wiener-Khinchin	Euler	Einstein	Mean-Ergodic	Mean-Ergodic