KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)
(Established Under Section 3 of UGC Act 1956)
Coimbatore - 641021.
(For the candidates admitted from 2016 onwards)

SUBJECT : **Classical Mechanics**
SEMESTER : **V** **SUBJECT CODE**: **16PHU504A**
CLASS : **III B.Sc.PHYSICS**

Scope: Classical mechanics was the first branch of Physics to be discovered, and is the foundation upon which all other branches of Physics are built. Moreover, classical mechanics has many important applications in other areas of science, such as Astronomy (*e.g.*, celestial mechanics), Chemistry (*e.g.*, the dynamics of molecular collisions), Geology (*e.g.*, the propagation of seismic waves, generated by earthquakes, through the Earth's crust), and Engineering (*e.g.*, the equilibrium and stability of structures). Classical mechanics is also of great significance outside the realm of science.

Objective: The emphasis of the course is on applications in solving problems of interest to physicists. Students are to be examined on the basis of problems, seen and unseen.

UNIT I

Classical Mechanics of Point Particles: Review of Newtonian Mechanics; Application to the motion of a charge particle in external electric and magnetic fields- motion in uniform electric field, magnetic field- gyroradius and gyrofrequency, motion in crossed electric and magnetic fields. Generalized coordinates and velocities, Hamilton's principle, Lagrangian and the Euler-Lagrange equations, one-dimensional examples of the Euler-Lagrange equations- one-dimensional Simple Harmonic Oscillations and falling body in uniform gravity; applications to simple systems such as coupled oscillators Canonical momenta & Hamiltonian. Hamilton's equations of motion.

UNIT II

Applications: Hamiltonian for a harmonic oscillator, solution of Hamilton's equation for Simple Harmonic Oscillations; particle in a central force field- conservation of angular momentum and energy.

UNIT III

Small Amplitude Oscillations: Minima of potential energy and points of stable equilibrium, expansion of the potential energy around a minimum, small amplitude oscillations about the minimum, normal modes of oscillations example of N identical masses connected in a linear fashion to (N -1) - identical springs.

UNIT IV

Special Theory of Relativity: Postulates of Special Theory of Relativity. Lorentz Transformations. Minkowski space. The invariant interval, light cone and world lines. Space-time diagrams. Time -

dilation, length contraction and twin paradox. Four-vectors: space-like, time-like and light-like. Four-velocity and acceleration. Metric and alternating tensors. Four-momentum and energy-momentum relation. Doppler effect from a four-vector perspective. Concept of four-force. Conservation of four-momentum. Relativistic kinematics. Application to two-body decay of an unstable particle.

UNIT V

Fluid Dynamics: Density □ and pressure P in a fluid, an element of fluid and its velocity, continuity equation and mass conservation, stream-lined motion, laminar flow, Poiseuille's equation for flow of a liquid through a pipe, Navier-Stokes equation, qualitative description of turbulence, Reynolds number.

Suggested Readings

1. Classical Mechanics, H.Goldstein, C.P. Poole, J.L. Safko, 3rd Edn. 2002,Pearson Education.
2. Mechanics, L. D. Landau and E. M. Lifshitz, 1976, Pergamon.
3. Classical Electrodynamics, J.D. Jackson, 3rd Edn., 1998, Wiley.
4. The Classical Theory of Fields, L.D Landau, E.M Lifshitz, 4th Edn., 2003, Elsevier.
5. Introduction to Electrodynamics, D.J. Griffiths, 2012, Pearson Education.
6. Classical Mechanics, P.S. Joag, N.C. Rana, 1st Edn., McGraw Hall.
7. Classical Mechanics, R. Douglas Gregory, 2015, Cambridge University Press.
8. Classical Mechanics: An introduction, Dieter Strauch, 2009, Springer.
9. Solved Problems in classical Mechanics, O.L. Delange and J. Pierrus, 2010, Oxford Press

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DEPARTMENT OF PHYSICS

SUBJECT: Classical Mechanics
SEMESTER: III
SUBJET CODE: 16PHP504A **CLASS: III B.Sc Physics**

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
		Unit - 1	
1.	1 Hr	Review of Newtonian Mechanics	T1(26-27)
2.	1 Hr	Application to the motion of charged partical in electric and Magnetic field, Charged particle in uniform electric field, magnetic field	T1(47-48), T1(49-50), T1(50-51)
3.	1 Hr	Gyroradius and gyrofrequency, motion of charged particle in crossed electric and magnetic field	T1(51), T1(53-54)
4.	1 Hr	Generalized coordinates and velocities, Hamilton's principle	T1(17), T1(18), T1(21)
5.	1 Hr	Lagrangiana and Euler-lagrangian equation	T1(39-40)
6.	1 Hr	One dimensional simple harmonic oscillator, falling body in uniform gravity	T1(125-126), T1(137-138)
7.	1 Hr	Application to coupled oscillator, canonical momenta and Hamiltonian	T1(377-378), T1(379)
8.	1 Hr	Hamilton's equation of motion	T1(334-335)
9.	1 Hr	Revision	
		Total number of Hours planned for unit-1	9 Hrs
		Unit - 2	
1.	1 Hr	Hamiltonian for a Harmonic oscillator	T1(123-126)
2.	1Hr	Continuation of Hamiltonain for a harmonic oscillator	T1(123-126)
3.	1 Hr	Solution of Hamilton's equation for simple harmonic oscillator	T1(126-128)
4.	1 Hr	Contination of Hamilton's equation for simple harmonid oscillator	T1(126-128)
5.	1 Hr	Particle in a central force field	T1(133-134)
6.	1 Hr	Conservation of angular momentum	T1(3-4)
7.	1 Hr	Conservation of Energy	T1(4-5)
8.	1 Hr	Revision	

Total number of Hours planned for unit - 2			8 Hrs
Unit – 3			
1.	1 Hr	Minima of potential energy and and points of stable equilibrium	T1(243-244)
2.	1 Hr	Expansion of the potential energy around a minimum	T1(244-245)
3.	1 Hr	Small amplitude oscillations about the minimum	T1(246-248)
4.	1 Hr	Small amplitude oscillations about the minimum - Continuation	T1(246-248)
5.	1 Hr	Normal modes of oscillations	T1(249-250)
6.	1 Hr	Examples ofr N identical masses connected in a linear fashion to (N-1) identical springs	T1(251-254)
7.	1 Hr	Contination of normal modes of oscillation	T1(251-254)
8.	1 Hr	N-identical masses to (N-1) identical springs - Derivation	T1(261-263)
9.	1Hr	Revision	
Total number of hours planned for unit - 3			9 Hrs
Unit - 4			
1.	1 Hr	Postulates of special theory of relativity, Lorentz transformation	T1(277), T1(279-281)
2.	1 Hr	Minkowski space, The invariant interval, light cone and world lines	T1(281-282), T1(283)
3.	1 Hr	Space time diagram, Time dilation	T1(284-285)
4.	1 Hr	Length contraction, Twin paradox	T1(286-287)
5.	1 Hr	Four vectors-space-like, time-like and light-like	T1(288-290)
6.	1 Hr	Four velocity and acceleration, Metric and alternating tensors	T1(290-291), T1(291-293)
7.	1 Hr	Four momentum and energy-momentum relation	T1(298-299)
8.	1 Hr	Doppler Effect from a four-vector prospective, concept of four force, Conservation of four momentum	T1(301-302), T1(303), T1(304)
9.	1 Hr	Relativistic kinematics, two body decay	T1(306-307)
10.	1 Hr	Revision	
Total number of hours planned for unit - 4			10 Hrs
Unit – 5			
1.	1 Hr	Density and pressure P in a fluid	T1(323)
2.	1 Hr	An element of fluid and its velolcity	T1(324-325)
3.	1 Hr	Contuinuty equation and mass conservation	T1(326-327)
4.	1 Hr	Stream-lined motion	T1(327-328)
5.	1 Hr	Laminar flow	T1(329)
6.	1 Hr	Poiseuille’s equation for flow of a liquid through a pipe	T1(329-330)
7.	1 Hr	Navier’s – Stokes equation	T1(330-331)
8.	1 Hr	Qualitativ description of turbulence	T1(331-332)
9.	1 Hr	Reynolds number, Revision	T1(333)
10.	1 Hr	Previuous year question paper discussion	
11.	1 Hr	Previous year question paper discussion	

12.	1 Hr	Previous year question paper discussion	
Total number of hours planned for Unit - 5			12 Hrs

Textbooks

T1- Classical Mechanics, H. Goldstein, C.P. Poole and J. L. Safko, 3rd Edition, 2002. Pearson Education

Reference Book

R1 – Classical Mechanics, R. S. Joag, N. C. Rana, Ist Edition, McGraw Hill

UNIT I

Classical Mechanics of Point Particles: Review of Newtonian Mechanics; Application to the motion of a charge particle in external electric and magnetic fields- motion in uniform electric field, magnetic field-gyroradius and gyrofrequency, motion in crossed electric and magnetic fields. Generalized coordinates and velocities, Hamilton's principle, Lagrangian and the Euler-Lagrange equations, one-dimensional examples of the Euler-Lagrange equations- one-dimensional Simple Harmonic Oscillations and falling body in uniform gravity; applications to simple systems such as coupled oscillators Canonical momenta & Hamiltonian. Hamilton's equations of motion.

Review of Newtonian Mechanics

Newton's first law of motion

Newton's first law was actually discovered by Galileo and perfected by Descartes (who added the crucial proviso ``in a straight line"). This law states that if the motion of a given body is not disturbed by external influences then that body moves with *constant* velocity. In other words, the displacement $\mathbf{r}$ of the body as a function of time t can be written

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}\,t,$$

where $\mathbf{r}_0$ and $\mathbf{v}$ are *constant* vectors. As illustrated in Figure, the body's trajectory is a *straight-line* which passes through point $\mathbf{r}_0$ at time $t = 0$ and runs parallel to $\mathbf{v}$. In the special case in which $\mathbf{v} = 0$ the body simply remains at rest.

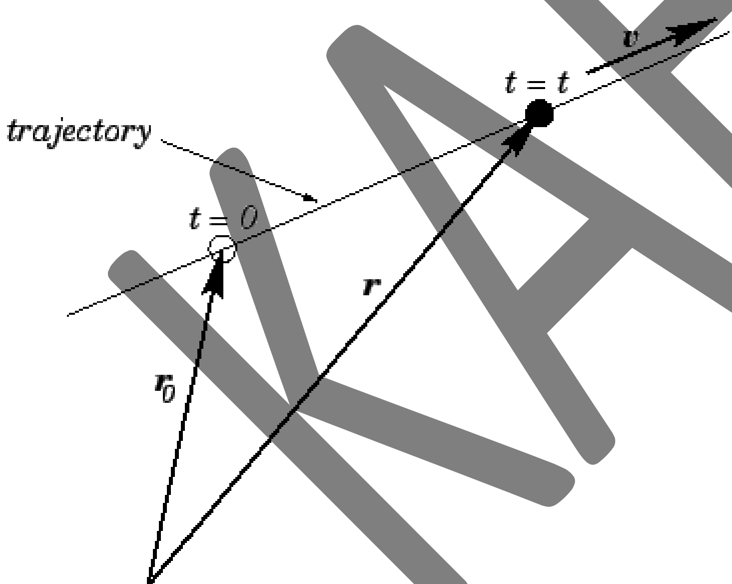


Figure. Body's trajectory

Nowadays, Newton's first law strikes us as almost a statement of the obvious. However, in Galileo's time this was far from being the case. From the time of the ancient Greeks, philosophers--observing that objects set into motion on the Earth's surface eventually come to rest--had concluded that the natural state of motion of objects was that they should remain at rest. Hence, they reasoned, any object which moves does so under the influence of an external influence, or *force*, exerted on it by some other object. It took the genius of Galileo to realize that an object set into motion on the Earth's surface eventually comes to rest under the influence of frictional forces, and that if these forces could somehow be abstracted from the motion then it would continue forever.

Newton's second law of motion

Newton used the word ``motion'' to mean what we nowadays call *momentum*. The momentum $\mathbf{p}$ of a body is simply defined as the product of its mass m and its velocity $\mathbf{v}$: *i.e.*,

$$\mathbf{p} = m\,\mathbf{v}.$$

Newton's second law of motion is summed up in the equation

$$\frac{d\mathbf{p}}{dt} = \mathbf{f},$$

where the vector $\mathbf{f}$ represents the net influence, or force, exerted on the object, whose motion is under investigation, by other objects. For the case of a object with *constant* mass, the above law reduces to its more conventional form

$$\mathbf{f} = m\,\mathbf{a}.$$

In other words, the net force exerted on a given object by other objects equals the product of that object's mass and its acceleration. Of course, this law is entirely devoid of content unless we have some independent means of quantifying the forces exerted between different objects.

Newton's third law of motion

Suppose, for the sake of argument, that there are only two bodies in the Universe. Let us label these bodies a and b. Suppose that body b exerts a force $\mathbf{f}_{ab}$ on body a. According to to Newton's third law of motion, body a must exert an *equal and opposite* force $\mathbf{f}_{ba} = -\mathbf{f}_{ab}$ on body b. See Fig. 22. Thus, if we label $\mathbf{f}_{ab}$ the ``action'' then, in Newton's language, $\mathbf{f}_{ba}$ is the equal and opposed ``reaction''.

Suppose, now, that there are many objects in the Universe (as is, indeed, the case). According to Newton's third law, if object j exerts a force $\mathbf{f}_{ij}$ on object i then object i must exert an equal and opposite force $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$ on object j. It follows that all of the forces acting in the Universe can ultimately be grouped into equal and opposite action-reaction pairs. Note, incidentally, that an action and its associated reaction always act on *different* bodies.

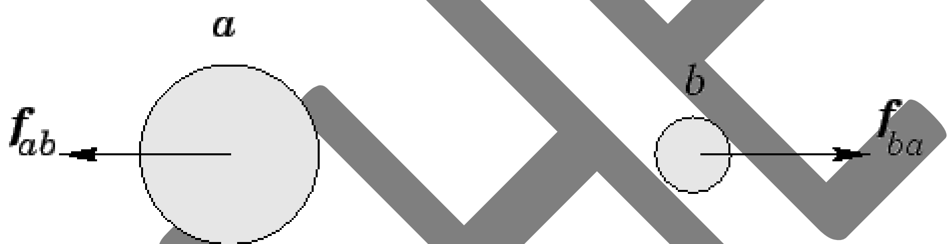


Figure 23: *Newton's third law*

Why do we need Newton's third law? Actually, it is almost a matter of common sense. Suppose that bodies a and b constitute an *isolated* system. If $\mathbf{f}_{ba} \neq -\mathbf{f}_{ab}$ then this system exerts a *non-zero net force* $\mathbf{f} = \mathbf{f}_{ab} + \mathbf{f}_{ba}$ on itself, without the aid of any external agency. It will, therefore, accelerate forever under its own steam. We know, from experience, that this sort of behaviour does not occur in real life. For instance, I cannot grab hold of my shoelaces and, thereby, pick myself up off the ground. In other words, I cannot self-generate a force which will spontaneously lift me into the air: I need to exert forces on other objects around me in order to achieve this. Thus, Newton's third law essentially acts as a guarantee against the absurdity of self-generated forces.

Application to the motion of a charge particle in external electric and magnetic fields

Consider a particle of mass m and electric charge q moving in the *uniform* electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. Suppose that the fields are ``crossed'' (*i.e.*, perpendicular to one another), so that $\mathbf{E}\cdot\mathbf{B}=0$.

The force acting on the particle is given by the familiar Lorentz law:

$$\mathbf{f}=q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}),$$

where $\mathbf{v}$ is the particle's instantaneous velocity. Hence, from Newton's second law, the particle's equation of motion can be written

$$m\,\frac{d\mathbf{v}}{dt}=q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

It turns out that we can eliminate the electric field from the above equation by transforming to a different inertial frame. Thus, writing

$$\mathbf{v}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}+\mathbf{v}',$$

Equation reduces to

$$m\,\frac{d\mathbf{v}'}{dt}=q\,\mathbf{v}'\times\mathbf{B},$$

where we have made use of a standard vector identity, as well as the fact that $\mathbf{E}\cdot\mathbf{B}=0$. Hence, we conclude that the addition of an electric field perpendicular to a given magnetic field simply causes the particle to drift perpendicular to both the electric and magnetic field with the fixed velocity

$$\mathbf{v}_{EB}=\frac{\mathbf{E}\times\mathbf{B}}{B^2},$$

irrespective of its charge or mass. It follows that the electric field has no effect on the particle's motion in a frame of reference which is co-moving with the so-called *E-cross-B velocity* given above.

Let us suppose that the magnetic field is directed along the z-axis. As we have just seen, in the $\mathbf{E}\times\mathbf{B}$ frame, the particle's equation of motion reduces to Equation, which can be written:

$$\frac{dv'_x}{dt} = \Omega\, v'_y,$$

$$\frac{dv'_y}{dt} = -\Omega\, v'_x,$$

$$\frac{dv'_z}{dt} = 0.$$

Here,

$$\Omega = \frac{q\,B}{m}$$

is the so-called *cyclotron frequency*. Equations can be integrated to give

$$v'_x = v_\perp\, \sin(\Omega\, t),$$

$$v'_y = v_\perp\, \cos(\Omega\, t)$$

$$v'_z = v_\parallel,$$

where we have judiciously chosen the origin of time so as to eliminate any phase offset in the arguments of the above trigonometrical functions. According to Equations, in the $\mathbf{E}\times\mathbf{B}$ frame, our charged particle gyrates at the cyclotron frequency in the plane perpendicular to the magnetic field with some fixed speed $v_\perp$, and drifts parallel to the magnetic field with some fixed speed $v_\parallel$. The fact that the cyclotron frequency is positive for positively charged particles, and negative for negatively charged particles, just means that oppositely charged particles gyrate in opposite directions in the plane perpendicular to the magnetic field.

Equations can be integrated to give

$$x' = -\rho \cos(\Omega t),$$

$$y' = \rho \sin(\Omega t)$$

$$z' = v_{\|} t,$$

where we have judiciously chosen the origin of our coordinate system so as to eliminate any constant offsets in the above equations. Here,

$$\rho = \frac{v_\perp}{\Omega}$$

is called the *Larmor radius*. Equations are the equations of a *spiral* of radius ρ, aligned along the direction of the magnetic field (*i.e.*, the z-direction).

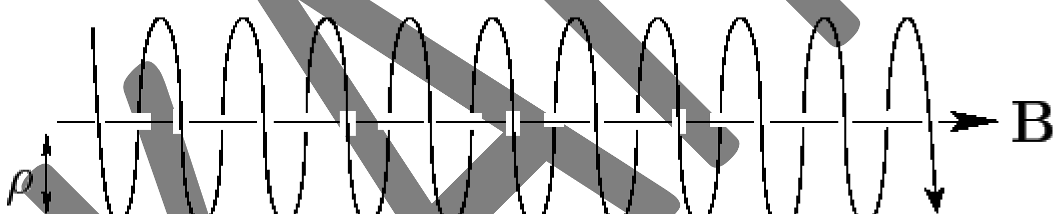
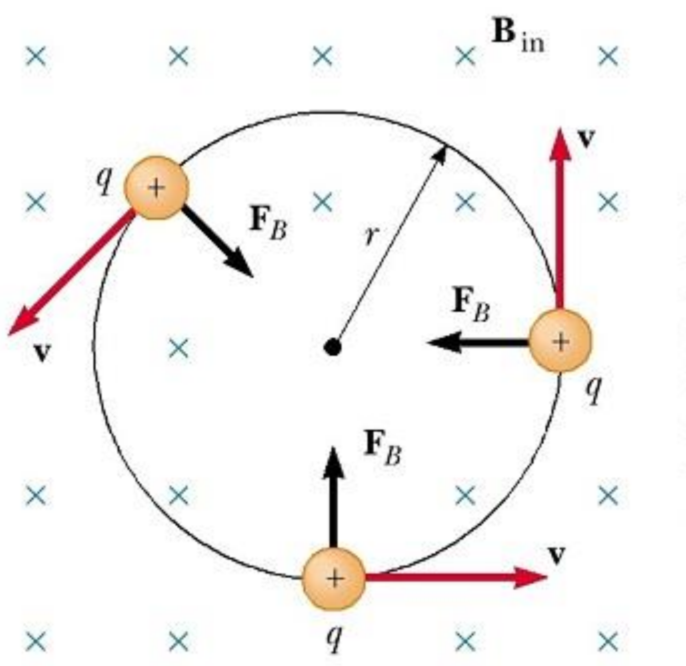


Figure 12: *The spiral trajectory of a negatively charged particle in a magnetic field.*

We conclude that the general motion of a charged particle in crossed electric and magnetic field is a combination of $\mathbf{E} \times \mathbf{B}$ drift and spiral motion aligned along the direction of the magnetic field

Motion of charged particle in uniform magnetic field

We know that the magnetic force acting on a charged particle moving in a magnetic field is perpendicular to the velocity of the particle and that consequently the work done on the particle by the magnetic force is zero. Let us now consider the special case of a positively charged particle moving in a uniform magnetic field with the initial velocity vector of the particle perpendicular to the field. Let us assume that the direction of the magnetic field is into the page. Figure shows that the particle moves in a circle in a plane perpendicular to the magnetic field.



The particle moves in this way because the magnetic force $\mathbf{F}_B$ is at right angles to **v** and **B** and has a constant magnitude qvB. As the force deflects the particle, the directions of **v** and $\mathbf{F}_B$ change continuously, as Figure 29.17 shows. Because $\mathbf{F}_B$ always points toward the center of the circle, **it changes only the direction of v and not its magnitude**. As Figure 29.17 illustrates, the rotation is counter-clockwise for a positive charge. If q were negative, the rotation would be clockwise. We can use Equation 6.1 to equate this magnetic force to the radial force required to keep the charge moving in a circle:

$$\sum F = ma_r$$

$$F_B = qvB = \frac{mv^2}{r}$$

$$r = \frac{mv}{qB}$$

That is, the radius of the path is proportional to the linear momentum mv of the particle and inversely proportional to the magnitude of the charge on the particle and to the magnitude of the magnetic field. The angular speed of the particle is

$$\omega = \frac{v}{r} = \frac{qB}{m}$$

The period of the motion (the time that the particle takes to complete one revolution) is equal to the circumference of the circle divided by the linear speed of the particle:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$$

These results show that the angular speed of the particle and the period of the circular motion do not depend on the linear speed of the particle or on the radius of the orbit. The angular speed ω is often referred to as the **cyclotron frequency** because charged particles circulate at this angular speed in the type of accelerator called a cyclotron.

Motion of Charged Particles in a Uniform Electric Field

When a particle of charge q and mass m is placed in an electric field **E**, the electric force exerted on the charge is q**E**. If this is the only force exerted on the particle, it must be the net force and so must cause the particle to accelerate. In this case, Newton's second law applied to the particle
gives

$$\mathbf{F}_e = q\mathbf{E} = m\mathbf{a}$$

The acceleration of the particle is

$$\mathbf{a} = \frac{q\mathbf{E}}{m}$$

therefore

If **E** is uniform (that is, constant in magnitude and direction), then the acceleration is constant. If the particle has a positive charge, then its acceleration is in the direction of the electric field. If the particle has a negative charge, then its acceleration is in the direction opposite the electric field.

Generalized Coordinates

Let the q_i, for $i = 1, \mathcal{F}$, be a set of coordinates which uniquely specifies the instantaneous configuration of some dynamical system. Here, it is assumed that each of the q_i can vary *independently*. The q_i might be Cartesian coordinates, or polar coordinates, or angles, or some mixture of all three types of coordinate, and are, therefore, termed *generalized coordinates*. A dynamical system whose instantaneous configuration is fully specified by $\mathcal{F}$ independent generalized coordinates is said to have $\mathcal{F}$ *degrees of freedom*. For instance, the instantaneous position of a particle moving freely in three dimensions is completely specified by its three Cartesian coordinates, x, y, and z. Moreover, these coordinates are clearly independent of one another. Hence, a dynamical system consisting of a single particle moving freely in three dimensions has three degrees of freedom. If there are two freely moving particles then the system has six degrees of freedom, and so on.

Suppose that we have a dynamical system consisting of N particles moving freely in three dimensions. This is an $\mathcal{F} = 3N$ degree of freedom system whose instantaneous configuration can be specified by $\mathcal{F}$ Cartesian coordinates. Let us denote these coordinates the x_j, for $j = 1, \mathcal{F}$. Thus, x_1, x_2, x_3 are the Cartesian coordinates of the first particle, x_4, x_5, x_6 the Cartesian coordinates of the second particle, *etc*. Suppose that the instantaneous configuration of the system can also be specified by $\mathcal{F}$ generalized coordinates, which we shall denote the q_i, for $i = 1, \mathcal{F}$. Thus, the q_i might be the spherical coordinates of the particles. In general, we expect the x_j to be functions of the q_i. In other words,

$$x_j = x_j(q_1, q_2, \cdots, q_{\mathcal{F}}, t)$$

for $j = 1, \mathcal{F}$. Here, for the sake of generality, we have included the possibility that the functional relationship between the x_j and the q_i might depend on the time, t, explicitly. This would be the case if the dynamical system were subject to time varying constraints. For instance, a system consisting of a particle constrained to move on a surface which is itself moving. Finally, by the chain rule, the variation of the x_j due to a variation of the q_i (at constant t) is given by

$$\delta x_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i}\,\delta q_i,$$

for $j = 1, \mathcal{F}$.

Euler- Lagrangian equation of motion

The Cartesian equations of motion of our system take the form

$$m_j\,\ddot{x}_j = f_j,$$

for $j = 1, \mathcal{F}$, where m_1, m_2, m_3 are each equal to the mass of the first particle, m_4, m_5, m_6 are each equal to the mass of the second particle, *etc.*Furthermore, the kinetic energy of the system can be written

$$K = \frac{1}{2} \sum_{j=1,\mathcal{F}} m_j \dot{x}_j^2.$$

Now, since $x_j = x_j(q_1, q_2, \cdots, q_{\mathcal{F}}, t)$, we can write

$$\dot{x}_j = \sum_{i=1,\mathcal{F}} \frac{\partial x_j}{\partial q_i} \dot{q}_i + \frac{\partial x_j}{\partial t},$$

for $j = 1, \mathcal{F}$. Hence, it follows that $\dot{x}_j = \dot{x}_j(\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, q_1, q_2, \cdots, q_{\mathcal{F}}, t)$.

According to the above equation,

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \frac{\partial x_j}{\partial q_i},$$

where we are treating the $\dot{q}_i$ and the q_i as *independent* variables.

Multiplying Equation by $\dot{x}_j$, and then differentiating with respect to time, we obtain

$$\frac{d}{dt}\left(\dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\dot{x}_j \frac{\partial x_j}{\partial q_i}\right) = \ddot{x}_j \frac{\partial x_j}{\partial q_i} + \dot{x}_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right).$$

Now,

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \sum_{k=1,\mathcal{F}} \frac{\partial^2 x_j}{\partial q_i \, \partial q_k} \dot{q}_k + \frac{\partial^2 x_j}{\partial q_i \, \partial t}.$$

Furthermore,

$$\frac{1}{2} \frac{\partial \dot{x}_j^2}{\partial \dot{q}_i} = \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i},$$

and

$$\frac{1}{2}\frac{\partial \dot{x}_j^2}{\partial q_i} = \dot{x}_j\,\frac{\partial \dot{x}_j}{\partial q_i} = \dot{x}_j\,\frac{\partial}{\partial q_i}\left(\sum_{k=1,\mathcal{F}}\frac{\partial x_j}{\partial q_k}\,\dot{q}_k + \frac{\partial x_j}{\partial t}\right)$$

$$= \dot{x}_j\left(\sum_{k=1,\mathcal{F}}\frac{\partial^2 x_j}{\partial q_i\,\partial q_k}\,\dot{q}_k + \frac{\partial^2 x_j}{\partial q_i\,\partial t}\right)$$

$$= \dot{x}_j\,\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right),$$

where use has been made of Equation . Thus, it follows from Equations that

$$\frac{d}{dt}\left(\frac{1}{2}\frac{\partial \dot{x}_j^2}{\partial \dot{q}_i}\right) = \ddot{x}_j\,\frac{\partial x_j}{\partial q_i} + \frac{1}{2}\frac{\partial \dot{x}_j^2}{\partial q_i}.$$

Let us take the above equation, multiply by m_j, and then sum over all j. We obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = \sum_{j=1,\mathcal{F}} f_j\,\frac{\partial x_j}{\partial q_i} + \frac{\partial K}{\partial q_i},$$

where use has been made of Equations. Thus, it follows from Equation that

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = Q_i + \frac{\partial K}{\partial q_i}.$$

Finally, making use of Equation, we get

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = -\frac{\partial U}{\partial q_i} + \frac{\partial K}{\partial q_i}.$$

It is helpful to introduce a function L, called the *Lagrangian*, which is defined as the difference between the kinetic and potential energies of the dynamical system under investigation:

$$L = K - U.$$

Since the potential energy U is clearly independent of the $\dot{q}_i$, it follows from Equation that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

for $i = 1, \mathcal{F}$. This equation is known as *Lagrange's equation*.

Hamilton's Principle

We can specify the instantaneous configuration of a conservative dynamical system with $\mathcal{F}$ degrees of freedom in terms of $\mathcal{F}$ independent generalized coordinates q_i, for $i = 1, \mathcal{F}$.

Let $K(q_1, q_2, \cdots, q_{\mathcal{F}},\ \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t)$ and $U(q_1, q_2, \cdots, q_{\mathcal{F}}, t)$ represent the kinetic and potential energies of the system, respectively, expressed in terms of these generalized coordinates. Here, $\dot{} \equiv d/dt$. The Lagrangian of the system is defined

$$L(q_1, q_2, \cdots, q_{\mathcal{F}}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t) = K - U.$$

Finally, the $\mathcal{F}$ Lagrangian equations of motion of the system take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$

for $i = 1, \mathcal{F}$.

Note that the above equations of motion have exactly the same mathematical form as the Euler-Lagrange equations. Indeed, it is clear, from Section that the $\mathcal{F}$ Lagrangian equations of motion can all be derived from a single equation: namely,

$$\delta \int_{t_1}^{t_2} L(q_1, q_2, \cdots, q_{\mathcal{F}}, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_{\mathcal{F}}, t)\, dt = 0.$$

In other words, the motion of the system in a given time interval is such as to maximize or minimize the time integral of the Lagrangian, which is known as the *action integral*.

Simple Harmonic Oscillator Equation

Suppose that a physical system possessing a single *degree of freedom*--that is, a system whose instantaneous state at time t is fully described by a single dependent variable, $s(t)$ --obeys the following time evolution equation [cf., Equation (2)],

$$\ddot{s} + \omega^2 s = 0,$$

where $\omega > 0$ is a constant. As we have seen, this differential equation is called the simple harmonic oscillator equation, and has the solution

$$s(t) = a \cos(\omega t - \phi),$$

where $a > 0$ and ϕ are constants. Moreover, this solution describes a type of oscillation characterized by a constant amplitude, a, and a constant angular frequency, ω. The phase angle, ϕ, determines the times at which the oscillation attains its maximum value. The frequency of the oscillation (in hertz) is $f = \omega/2\pi$, and the period is $T = 2\pi/\omega$. The frequency and period of the oscillation are both determined by the constant ω, which appears in the simple harmonic oscillator equation, whereas the amplitude, a, and phase angle, ϕ, are determined by the initial conditions. In fact, a and ϕ are the two arbitrary constants of integration of the second-order ordinary differential equation. Recall, from standard differential equation theory (Riley 1974), that the most general solution of an nth-order ordinary differential equation (i.e., an equation involving a single independent variable, and a single dependent variable, in which the highest derivative of the dependent with respect to the independent variable is n th-order, and the lowest zeroth-order) involves n arbitrary constants of integration. (Essentially, this is because we have to integrate the equation n times with respect to the independent variable to reduce it to zeroth-order, and so obtain the solution. Furthermore, each integration introduces an arbitrary constant. For example, the integral of $\dot{s} = a$, where a is a known constant, is $s = a\,t + b$, where b is an arbitrary constant.)

Multiplying Equation by $\dot{s}$, we obtain

$$\dot{s}\,\ddot{s} + \omega^2\,\dot{s}\,s = 0.$$

However, this can also be written

$$\frac{d}{dt}\left(\frac{1}{2}\,\dot{s}^2\right) + \frac{d}{dt}\left(\frac{1}{2}\,\omega^2\,s^2\right) = 0,$$

or

$$\frac{d\mathcal{E}}{dt} = 0,$$

where

$$\mathcal{E} = \frac{1}{2}\,\dot{s}^2 + \frac{1}{2}\,\omega^2\,s^2.$$

According to Equation, $\mathcal{E}$ is a conserved quantity. In other words, it does not vary with time. This quantity is generally proportional to the overall energy of the system. For instance, $\mathcal{E}$ would be the energy divided by the mass in the mass-spring system. The quantity $\mathcal{E}$ is either zero or positive, because neither of the terms on the right-hand side of Equation can be negative.

Let us search for an equilibrium state. Such a state is characterized by $s = \text{constant}$, so that $\dot{s} = \ddot{s} = 0$. It follows from Equation that $s = 0$, and from Equation that $\mathcal{E} = 0$. We conclude that the system can only remain permanently at rest when $\mathcal{E} = 0$. Conversely, the system can never permanently come to rest when $\mathcal{E} > 0$, and must, therefore, keep moving for ever. Because the equilibrium state is characterized by $s = 0$, we deduce that s represents a kind of ``displacement'' of the system from this state. It is also apparent, from Equation, that s attains it maximum value when $\dot{s} = 0$. In fact,

$$s_{\max} = \frac{\sqrt{2\,\mathcal{E}}}{\omega},$$

where $s_{\max} = a$ is the amplitude of the oscillation. Likewise, $\dot{s}$ attains its maximum value,

$$\dot{s}_{\max} = \sqrt{2\,\mathcal{E}},$$

when $s = 0$.

The simple harmonic oscillation specified by Equation can also be written in the form

$$s(t) = A\,\cos(\omega\,t) + B\,\sin(\omega\,t),$$

where $A = a\,\cos\phi$ and $B = a\,\sin\phi$. Here, we have employed the trigonometric identity $\cos(x-y) \equiv \cos x\,\cos y + \sin x\,\sin y$.Alternatively, Equation can be written

$$s(t) = a\,\sin(\omega\,t - \phi'),$$

where $\phi' = \phi - \pi/2$, and use has been made of the trigonometric identity $\cos\theta \equiv \sin(\theta + \pi/2)$. It follows that there are many different ways of representing a simple harmonic oscillation, but they all involve linear combinations of sine and cosine functions whose arguments take the form $\omega\,t + c$, where c is some constant. However, irrespective of its form, a general solution to the simple harmonic oscillator equation must always contain two arbitrary constants. For example, A and B in Equation, or a and ϕ' in Equation.

The simple harmonic oscillator equation, is a linear differential equation, which means that if $s(t)$ is a solution then so is $a\,s(t)$, where a is an arbitrary constant. This can be verified by multiplying the equation by a, and then making use of the fact that $a\,d^2s/dt^2 = d^2(a\,s)/dt^2$. Linear differential equations have the very important and useful property that their solutions are *superposable*. This means that if $s_1(t)$ is a solution to Equation , so that

$$\ddot{s}_1 = -\omega^2\,s_1,$$

and $s_2(t)$ is a different solution, so that

$$\ddot{s}_2 = -\omega^2\,s_2,$$

then $s_1(t) + s_2(t)$ is also a solution. This can be verified by adding the previous two equations, and making use of the fact that $d^2 s_1/dt^2 + d^2 s_2/dt^2 = d^2(s_1 + s_2)/dt^2$. Furthermore, it can be demonstrated that any linear combination of s_1 and s_2, such as $a\, s_1 + b\, s_2$, where a and b are constants, is also a solution. It is very helpful to know this fact. For instance, the special solution to the simple harmonic oscillator equation, with the simple initial conditions $s(0) = 1$ and $\dot{s}(0) = 0$ can be shown to be

$$s_1(t) = \cos(\omega\, t).$$

Likewise, the special solution with the simple initial conditions $s(0) = 0$ and $\dot{s}(0) = 1$ is

$$s_2(t) = \omega^{-1}\, \sin(\omega\, t).$$

Thus, because the solutions to the simple harmonic oscillator equation are superposable, the solution with the general initial conditions $s(0) = s_0$ and $\dot{s}(0) = \dot{s}_0$ becomes

$$s(t) = s_0\, s_1(t) + \dot{s}_0\, s_2(t),$$

or

$$s(t) = s_0\, \cos(\omega\, t) + \frac{\dot{s}_0}{\omega}\, \sin(\omega\, t).$$

KARPAGAM ACADEMY OF HIGHER EDUCATION					
DEPARTMENT OF PHYSICS					
III B.Sc., PHYSICS (2016-2019)					
CLASSICAL MECHANICS (16PHU504A)					
UNIT - I					
QUESTIONS	**Choice1**	**Choice2**	**Choice3**	**Choice4**	**ANSWER**
Canonical transformations are the transformations of	Phase space	Hillbert space	Minkowski space	Space phase	Phase space
The Hamilton's principle function is a generating function, which give rise to canonical transformation involving	both constant moments and co-ordinates	constant moments only	co-ordinates only	constant momenta and co-ordinates	both constant moments and co-ordinates
All function whose Poisson bracket with the Hamiltonian vanishes will be	constant of motion	constant of momentum	constant of co-ordinates	all the above	constant of motion
Let L and P represent the matrices of Lagrange and Poisson brackets respectively, then	LP = 1	LP = -1	LP = -1/2	LP = 1/2	LP = -1
The frequency of Harmonic oscillator is given by	$[1/2p(k/m)^{5/2}]$	$[1/2p(k/m)^{3/2}]$	$[1/2p(k/m)^{1/2}]$	[1/2p(k/m)]	$[1/2p(k/m)^{1/2}]$
The given transformation is not canonical when	[Q,P] = 1	[Q,P] = -1	[Q,P] = 1/2	[Q,P] = 0	[Q,P] = 0
The function p =1/Q and $q = PQ^2$ is	conjugate	canonical	identical	hyrebolic	canonical
In point transformation one set of co-ordinates q_j to a new set Q_j can be expressed as	$Q_j = Q_j (q_j, t)$	$Q_j = -Q_j (q_j, t)$	$Q_j = P_j (q_j, t)$	$Q_j = -P_j (q_j, t)$	$Q_j = Q_j (q_j, t)$
The problem consists on finding the path of a charged particle under the action if a central force is	Jacobi problem	cononical problem	Kepler problem	Poission problem	Kepler problem
Hamilton – Jacobi method is used to find the solution of problem in	Vibratory motion	periodic motion	circular mation	all the above	periodic motion
Hamilton equation of motion is	convergent	divergent	variant	invariant	invariant
Poisson and Lagrange brackets are _______ under Canonical Transformation	convergent	divergent	invariant	variant	invariant
Equation of motion in Poisson bracket from depends on	position	momentum	time	all the three	all the three
In Kepler problem, the path of the particle is	circular	parabolic	elliptical	zig-zag	elliptical

In Poisson bracket	[X,Y] = [Y,X]	[X,Y] = - [Y,X]	[X,Y] = 2[Y,X]	[X,Y] = - 2[Y,X]	[X,Y] = - [Y,X]
In Poisson bracket	[X,X] =0	[X,X] =1	[X,X] =2	[X,X] = -2	[X,X] =0
In Poisson bracket	[X,Y+Z] = [X,Y] - [X,Z]	[X,Y+Z] = [X,Y] * [X,Z]	[X,Y+Z] = [X,Y] + [X,Z]	[X,Y+Z] = [X,Y] / [X,Z]	[X,Y+Z] = [X,Y] + [X,Z]
In Poisson bracket	[X,YZ] = Y[X,Z] * [X,Y]Z	[X,YZ] = Y[X,Z] - [X,Y]Z	[X,YZ] = Y[X,Z] / [X,Y]Z	[X,YZ] = Y[X,Z] + [X,Y]Z	[X,YZ] = Y[X,Z] + [X,Y]Z
In Lagrange bracket	$[X,qj]_{Q,P} = -[qj,X]_{Q,P}$	$[X,qj]_{Q,P} = [qj,X]_{Q,P}$	$[X,qj]_{Q,P} = 2[qj,X]_{Q,P}$	$2[X,qj]_{Q,P} = -[qj,X]_{Q,P}$	$2[X,qj]_{Q,P} = -[qj,X]_{Q,P}$
In of Lagrange bracket	$[X,Y]_{Q,P} = -[X,Y]_{q,p}$	$[X,Y]_{Q,P} = [X,Y]_{q,p}$	$[X,Y]_{Q,P} = 2[X,Y]_{q,p}$	$[X,Y]_{Q,P} = -2[X,Y]_{q,p}$	$[X,Y]_{Q,P} = [X,Y]_{q,p}$
In of Lagrange bracket	$[X,X]_{q,p} = [X,X]_{Q,P} = 1$	$[X,X]_{q,p} = [X,X]_{Q,P} = 1$	$[X,X]_{q,p} = [X,X]_{Q,P} = 0$	$[X,X]_{q,p} = [X,X]_{Q,P} = 1/2$	$[X,X]_{q,p} = [X,X]_{Q,P} = 0$
Poisson bracket of two operator X and Y in quantum mechanics is given by	[X,Y] = - 2p/h[XY-YX]	[X,Y] = - 2p/h[XY+YX]	[X,Y] = - p/h[XY-YX]	[X,Y] = 2p/h[XY-YX]	[X,Y] = - 2p/h[XY-YX]
If the Lagrangian of the system does not contain a paricular co-ordinate q, then	cyclic co-ordinates	cylindrical co-ordinates	polar co-ordinates	spherical polar co-ordinates	cyclic co-ordinates
Lagrangian **L** =	T-V	T+V	$(T-V)^2$	$(T+V)^{1/2}$	T-V
Hamiltonian **H** =	T-V	T+V	$(T-V)^2$	$(T+V)^{1/2}$	T+V
Advantage of Action and Angle variable is that one can obtain the frequencies of	Vibratory motion	periodic motion	circular mation	all the above	periodic motion
For non-interacting particle in a quantum state the energy E is given by	p/2m	p^2/m	p/m	$p^2/2m$	$p^2/2m$
Co-ordinate transformation equations should not involve ___________ explicity.	position	momentum	time	force	time
Generating function have _____________ forms.	four	two	three	five	four
Hamilton's principal function is denoted by ______________.	H	K	P	S	S
Hamilton-Jacobi is a partial differential equation in ________ variables.	n	n+1	n-1	n+2	n+1
_____________ is a partial differential equation in (n+1) variables.	Hamilton-Jacobi equation	Lagrangian	Hamiltonian	Jacobian	Hamilton-Jacobi equation
Hamilton's characteristic function W is identified as _____________.	kinetic energy	potential energy	work	action A	action A

Hamilton's characteristic function is denoted by ______________.	S	K	W	H	W
The number of independent ways in which a mechanical system can move without violating any constraint which may be imposed is called the ______________.	action-angle variables	generalized variables	degrees of freedom	co-ordinates	degrees of freedom
The path adopted by the system during its motion can be represented by a space of ______________ dimensions.	3N	6N	9N	N	6N
Co-ordinate transformation equations should not involve ______________ explicitly.	time	position	momentum	velocity	time
Path in phase space almost refers to actual ____________path.	statistical	N	3N	dynamical	dynamical
The one way of obtaining the solution of mechanical problem is to transform _________ set of co-ordinates to __________ set of co-ordinates that are all cyclic.	old to new	new to old	new to new	old to old	old to new

UNIT II

Hamiltonain

Applications: Hamiltonian for a harmonic oscillator, solution of Hamilton's equation for Simple Harmonic Oscillations; particle in a central force field- conservation of angular momentum and energy.

Hamiltonian for a harmonic oscillator

A simple realization of the harmonic oscillator in classical mechanics is a particle which is acted upon by a restoring force proportional to its displacement from its equilibrium position. Considering motion in one dimension, this means

$$F=-kx(5.1)(5.1)F=-kx$$

Such a force might originate from a spring which obeys Hooke's law, as shown in Figure. According to Hooke's law, which applies to real springs for sufficiently small displacements, the restoring force is proportional to the displacement—either stretching or compression—from the equilibrium position.

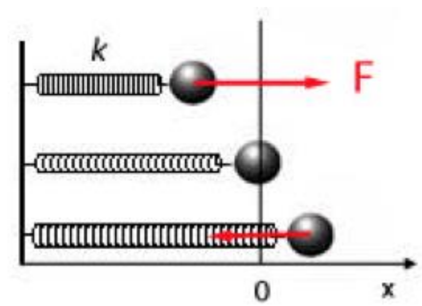


Figure: *Spring obeying Hooke's law.*

The *force constant* kk is a measure of the stiffness of the spring. The variable xx is chosen equal to zero at the equilibrium position, positive for stretching, negative for compression. The negative sign in Equation reflects the fact that FF is a *restoring* force, always in the opposite sense to the displacement xx.

Applying Newton's second law to the force from Equation, we find xx

$$F=-kx$$

where mm is the mass of the body attached to the spring, which is itself assumed massless. This leads to a differential equation of familiar form, although with different variables:

$$\ddot{x}(t)+\omega 2x(t)=0$$

with

$$\omega 2\equiv km$$

The dot notation (introduced by Newton himself) is used in place of primes when the independent variable is time. The general solution to Equation is

$$x(t)=A\sin\omega t+B\cos\omega t)$$

which represents periodic motion with a sinusoidal time dependence. This is known as *simple harmonic motion* and the corresponding system is known as a *harmonic oscillator*. The oscillation occurs with a constant angular frequency

$$\omega=km−−−\surd \text{radians per second}$$

This is called the *natural frequency* of the oscillator. The corresponding *circular (or angular) frequency* in Hertz (cycles per second) is

$$\nu=\omega 2\pi=12\pi km$$

The general relation between force and potential energy in a conservative system in one dimension is

$$F=-dVdx)$$

Thus the potential energy of a harmonic oscillator is given by

$$V(x)=12k\,x^2$$

which has the shape of a parabola, as drawn in Figure. A simple computation shows that the oscillator moves between positive and negative turning points $\pm xmax\pm xmax$ where the total energy EE equals the potential energy 12kx2max12kxmax2 while the kinetic energy is momentarily zero. In contrast, when the oscillator moves past $x=0$, the kinetic energy reaches its maximum value while the potential energy equals zero.

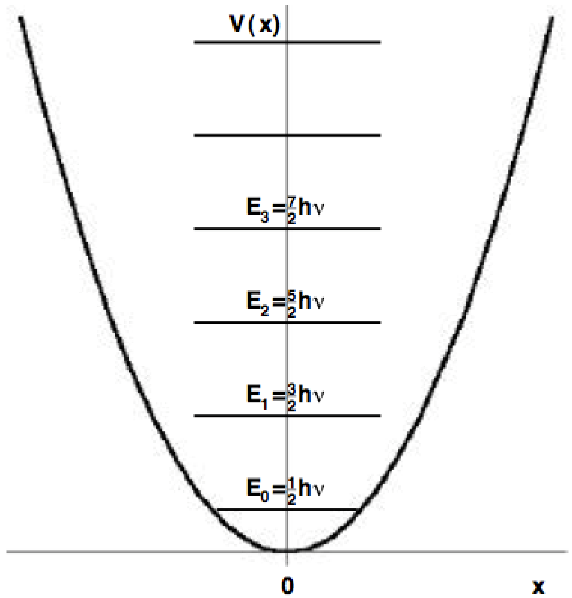


Figure. *Potential energy function and first few energy levels for harmonic oscillator.*

HAMILTONIAN FUNCTION:

Hamiltonian function, also called Hamiltonian, mathematical definition introduced in 1835 by Sir William Rowan Hamilton to express the rate of change in time of the condition of a dynamic physical system—one regarded as a set of moving particles. The Hamiltonian of a system specifies its total energy—*i.e.*, the sum of its kinetic energy (that of motion) and its potential energy (that of position)—in terms of the Lagrangian function derived in earlier studies of dynamics and of the position and momentum of each of the particles.

The Hamiltonian function originated as a generalized statement of the tendency of physical systems to undergo changes only by those processes that either minimize or maximize the abstract quantity called action. This principle is traceable to Euclid and the Aristotelian philosophers.

When, early in the 20th century, perplexing discoveries about atoms and subatomic particles forced physicists to search anew for the fundamental laws of nature, most of the old formulas became obsolete. The Hamiltonian function, although it had been derived from the obsolete formulas, nevertheless proved to be a more correct description of physical reality. With modifications, it survives to make the connection between energy and rates of change one of the centres of the new science.

HAMILTON'S VARIATIONAL PRINCIPLE:

Lagrange's equations have been shown to be the consequence of a variational principle, namely, the Hamilton's principle. Indeed the variational method has often proved to be the preferable method of deriving equations, for it is applicable to types of systems not usually comprised with in the scope of mechanics. It would be similarly advantageous if a variational principle could be found that leads directly to the Hamilton's equation of motion.

Hamilton's principle is stated as

$$\delta I = \delta \int_{t}^{t} L\, dt$$

Expressing L in terms of Hamiltonian by the expression by the expression

$$H = \sum_i p_i \dot{q}_i - L,$$

We find,

$$\delta I = \delta \int_{t_1}^{t_2} \left\{ \sum p_i \quad \underline{dq_i} \; H(q_i, p_i, t) \right\} dt$$

$$\delta \int_{t_1}^{t_2} \sum_i p_i \; dq_i \; - \; \delta \int_{t_1}^{t_2} H(q_i, p_i, t) dt = 0$$

The above equation is some times is referred as the modified Hamilton's principle. Although it will be used most frequently in connection with transformation theory ,the main interest is to show that the principle leads to the Hamilton's canonical equations of motions.

The modified Hamilton's principle is exactly of the form of the variational problems in a space of 2n dimensions as

$$\delta I = \delta \int_{t_1}^{t_2} f(q, \dot{q}, p, \dot{p}, t)\, dt = 0$$

For which the 2n Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} \qquad J=1,2,3\ldots.n$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} \qquad J=1,2,3\ldots.n$$

The integrand f as given as (2.29) contains $\dot{q}_j$ only through the $p_i\dot{q}_i$ term, q_j only in H. Hence equation (2.30) leads to

$$\dot{p}_j + \frac{\partial H}{\partial q_i} = 0$$

On the other hand there is no explicit dependence of the integrand in equation (2.30) on $\dot{p}_j$. The above equation therefore reduce simply to

$$\dot{q}_j - \frac{\partial H}{\partial p_i} = 0$$

The above two equations are exactly Hamilton's equations of motion .The Euler –Lagrange equations of the modified Hamilton's principle are thus the desired canonical equations of motion .From the above derivation of Hamilton's equations we can consider that Hamiltonian and Lagrangian formulation and therefore their respective variational principles, have the same physical content.

Hamilton's Equations:

The equations defined by

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{2}$$

where $\dot{p} \equiv dp/dt$ and $\dot{q} \equiv dq/dt$ is fluxion notation and H is the so-called Hamiltonian, are called Hamilton's equations. These equations frequently arise in problems of celestial mechanics.

The vector form of these equations is

$$\dot{q}_i = H_{p_i}(t, \mathbf{q}, \mathbf{p}) \tag{3}$$

$$\dot{p}_i = -H_{q_i}(t, \mathbf{q}, \mathbf{p}) \tag{4}$$

(Zwillinger 1997, p. 136; Iyanaga and Kawada 1980, p. 1005).

Another formulation related to Hamilton's equation is

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{5}$$

where L is the so-called Lagrangian.

HAMILTON'S CANONICAL EQUATIONS OF MOTION:

Theorem 6 : Define the Hamiltonian and hence derive the Hamilton's canonical equations of motion.

Proof : We know the Hamiltonian H is defined as

$$H = H\left(q_j, p_j, t\right) = \sum_j p_j \dot{q}_j - L. \qquad \ldots (1)$$

Consider $H = H\left(q_j, p_j, t\right).$ $\ldots (2)$

We find from equation (2) that

$$dH = \sum_j \frac{\partial H}{\partial p_j} dp_j + \sum_j \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial t} dt. \qquad \ldots (3)$$

Now consider $H = \sum_j p_j \dot{q}_j - L.$

Similarly we find

$$dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - dL,$$

$$\Rightarrow \quad dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt. \qquad \ldots (4)$$

We know the generalized momentum is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Hence equation (4) reduces to

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial t} dt \qquad \ldots (5)$$

Now comparing the coefficients of dp_j, dq_j and dt in equations (3) and (5) we get

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \qquad \ldots (6)$$

However, from Lagrange's equations of motion we have

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

Hence equations (6) reduce to

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad . \qquad \ldots (7)$$

These are the required Hamilton's canonical equations of motion. These are the set of 2n first order differential equations of motion and replace the n Lagrange's second order equations of motion.

PHYSICAL SIGNIFICANCE OF H:

1. For conservative scleronomic system the Hamiltonian H represents both a constant of motion and total energy.
2. For conservative rheonomic system the Hamiltonian H may represent a constant of motion but does not represent the total energy.

Proof : The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (1)$$

where L is the Lagrangian of the system and

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \ldots (2)$$

is the generalized momentum. This implies from Lagrange's equation of motion that

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad \ldots (3)$$

Differentiating equation (1) w. r. t. time t, we get

$$\frac{dH}{dt} = \sum_j \dot{p}_j \dot{q}_j + \sum_j p_j \ddot{q}_j - \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j - \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \frac{\partial L}{\partial t} \qquad \ldots (4)$$

On using equations (2) and (3) in equation (4) we readily obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \qquad \ldots (5)$$

Now if L does not contain time t explicitly, then from equation (5), we have

$$\frac{dH}{dt} = 0$$

This shows that H represents a constant of motion.

However, the condition L does not contain time t explicitly will be satisfied by neither the kinetic energy nor the potential energy involves time t explicitly.

Now there are two cases that the kinetic energy T does not involve time t explicitly.

1. **For the conservative and scleronomic system :**

In the case of conservative system when the constraints are scleronomic, the kinetic energy T is independent of time t and the potential energy V is only function of co-ordinates. Consequently, the Lagrangian L does not involve time t explicitly and hence from equation (5) the Hamiltonian H represents a constant of motion.

Further, for scleronomic system, we know the kinetic energy is a homogeneous quadratic function of generalized velocities.

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k . \qquad \ldots (6)$$

Hence by using Euler's theorem for the homogeneous quadratic function of generalized velocities we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T . \qquad \ldots (7)$$

For conservative system we have

$$p_j = \frac{\partial \mathrm{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} . \qquad \ldots (8)$$

Using (7) and (8) in the Hamiltonian H we get

$$H = 2T - (T - V),$$

$$H = T + V = E . \qquad \ldots (9)$$

where E is the total energy of the system. Equation (9) shows that for conservative scleronomic system the Hamiltonian H represents the total energy of the system.

2. For conservative and rheonomic system :

In the case of conservative rheonomic system, the transformation equations do involve time t explicitly, though some times the kinetic energy may not involve time t explicitly. Consequently, neither T nor V involves t, and hence L does not involve t. Hence in such cases the Hamiltonian may represent the constant of motion. However, in general if the system is conservative and rheonomic, the kinetic energy is a quadratic function of generalized velocities and is given by

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j a_j\dot{q}_j + a \qquad \ldots (10)$$

where

$$a_{jk} = \sum_i \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k},$$
$$a_j = \sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}, \qquad \ldots (11)$$
$$a = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2.$$

We see from equation (10) that each term is a homogeneous function of generalized velocities of degree two, one and zero respectively. On applying Euler's theorem for the homogeneous function to each term on the right hand side, we readily get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1. \qquad \ldots (12)$$

where

$$T_2 = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k,$$
$$T_1 = \sum_j a_j \dot{q}_j,$$
$$T_0 = a$$

are homogeneous function of generalized velocities of degree two, one and zero respectively. Substituting equation (12) in the Hamiltonian (1) we obtain

$$H = T_2 - T_0 + V$$

showing that the Hamiltonian H does not represent total energy. Thus for the conservative rheonomic systems H may represent the constant of motion but does not represent total energy.

APPLICATION OF HAMILTONIAN EQUATION OF MOTION TO

(i)SIMPLE PENDULUM:

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1-\cos\theta), \qquad \ldots (1)$$

where the generalized momentum is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{ml^2}. \qquad \ldots (2)$$

The Hamiltonian of the system is given by

$$H = p_\theta \dot{\theta} - L,$$
$$\Rightarrow \quad H = p_\theta \dot{\theta} - \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1-\cos\theta).$$

Eliminating $\dot{\theta}$ we obtain

$$H = \frac{p_\theta^2}{2ml^2} + mgl\left(1 - \cos\theta\right). \qquad \ldots (3)$$

Hamilton's canonical equations of motion are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

These equations give

$$\dot{\theta} = \frac{p_\theta}{ml^2}, \quad \dot{p}_\theta = -mgl\sin\theta. \qquad \ldots (4)$$

Now eliminating p_θ from these equations we get

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \qquad \ldots (5)$$

Now we claim that H represents the constant of motion.

Thus differentiating equation (3) with respect to t we get

$$\frac{dH}{dt} = \frac{p_\theta \dot{p}_\theta}{ml^2} + mgl\sin\theta\,\dot{\theta},$$

$$= ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta},$$

$$= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right),$$

$$\frac{dH}{dt} = 0.$$

This proves that H is a constant of motion. Now to see whether H represents total energy or not, we consider

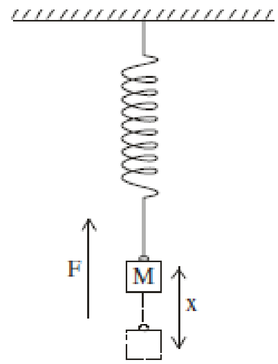
$$T + V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta).$$

Using equation (4) we eliminate $\dot{\theta}$ from the above equation, we obtain

$$T + V = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta). \qquad \ldots (6)$$

This is as same as the Hamiltonian H from equation (3). Thus Hamiltonian H represents the total energy of the pendulum.

(II)LINEAR HARMONIC OSCILLATOR:



Solution: The one dimensional harmonic oscillator consists of a mass attached to one end of a spring and other end of the spring is fixed. If the spring is pressed and released then on account of the elastic property of the spring, the spring exerts a force F on the body in the opposite direction. This is called restoring force. It is found that this force is proportional to the displacement of the body from its equilibrium position.

$$F \propto x$$
$$F = -kx$$

where k is the spring constant and negative sign indicates the force is opposite to the displacement. Hence the potential energy of the particle is given by

$$V = -\int F dx,$$
$$V = \int kx dx + c,$$
$$V = \frac{kx^2}{2} + c,$$

where c is the constant of integration. By choosing the horizontal plane passing through the position of equilibrium as the reference level, then V=0 at x=0. This gives c=0. Hence potential energy of the particle is

$$V = \frac{1}{2}kx^2. \qquad \ldots (1)$$

The kinetic energy of the one dimensional harmonic oscillator is

$$T = \frac{1}{2}m\dot{x}^2. \qquad \ldots (2)$$

Hence the Lagrangian of the system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \qquad \text{. . . (3)}$$

The Lagrange's equation motion gives

$$\ddot{x} + \omega^2 x = 0, \quad \omega^2 = \frac{k}{m}. \qquad \text{. . . (4)}$$

This is the equation of motion. ω is the frequency of oscillation.

The Hamiltonian H of the oscillator is defined as

$$H = \dot{x}p_x - L,$$

$$H = \dot{x}p_x - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2,$$

where

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \quad \dot{x} = \frac{p_x}{m}.$$

Substituting this in the above equation we get the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2. \qquad \text{. . . (5)}$$

Solving the Hamilton's canonical equations of motion we readily get the equation (4) as the equation of motion.

Energy Conservation

In the gravitational physics of orbits that we have been considering there are two important forms of energy that are being exchanged. GRAVITATIONAL POTENTIAL ENERGY and KINETIC ENERGY. The kinetic energy is the energy associated with a object's motion and is given by

$E_{kin}= M_b V^2/2$.

where M_b is the mass, say of a ball, and V is the magnitude of the velocity (the speed).

Now the gravitational potential energy is the energy that a body has which can subsequently be used to accelerate the body to a larger magnitude of velocity. For example, if I hold a ball at arms length at rest, and let the ball drop to the Earth, the ball will speed up before hitting the Earth. This potential energy, as I was holding the ball at rest, is given by

$E_{grav}=M_b g H$,

where H is the height of the ball above the Earth's surface, and g, the acceleration on the Earth is $g=(GM_e/R^2{}_e) = 9.8$ meters/s^2 (see the inset figure in the discussion of weight on our earlier packet of notes The Universal Law of Gravitation).

Now here's the deal: the gravitational potential energy of the ball at rest in my extended arm, is equal to the maximum kinetic energy that the ball can have just before it reaches the ground. As the ball falls, H decreases. Thus the gravitational energy decreases. Where does it go? Well, the speed of the ball increases. Thus the kinetic energy of the ball increases from the equation for kinetic energy above. Gravitational potential energy is being converted into kinetic energy. This is how energy is conserved.

It is also why you slow down and speed up as you travel up and down in a roller coaster.

Is it consistent with planets in elliptical orbits around the sun speeding up near the the perihelion and slowing down near the aphelion? and Kepler's second law?

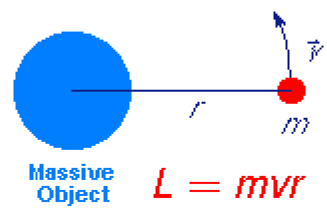
A bit more on the Ball

Back to the ball: note that when I drop the ball, it bounces back up it slows down as its gravitational potential energy is regained. Why does does the ball always return to a height slightly lower than that from which is was originally dropped? The reason is that there are other sources of energy loss: heat, compression, stresses on the ball itself which cannot be regained as gravitational energy. However, when all these energies are added up, their total is equal to the same as the initial gravitational potential energy.

Energy conservation is fundamental. Physics can describe to us only how energy in the Universe transforms from one form to another.

Angular Momentum Conservation

Objects executing motion around a point possess a quantity called ANGULAR MOMENTUM. This is an important physical quantity because all experimental evidence indicates that angular momentum is rigorously conserved in our Universe. It can be transferred, but it cannot be created or destroyed. For the simple case of a small mass executing uniform circular motion around a much larger mass (so that we can neglect the effect of the center of mass) the amount of angular momentum takes a simple form. As the adjacent figure illustrates the magnitude of the angular momentum in this case is



L = mvr

where L is the angular momentum, m is the mass of the small object, v is the magnitude of its velocity, and r is the separation between the small and large objects

KARPAGAM ACADEMY OF HIGHER EDUCATION					
DEPARTMENT OF PHYSICS					
III B.Sc., PHYSICS (2016-2019)					
CLASSICAL MECHANICS (16PHU504A)					
UNIT - II					
QUESTIONS	**Choice1**	**Choice2**	**Choice3**	**Choice4**	**ANSWER**
If the operators X, Y commute, then [X, Y] = _____________.	1	-1	0	-2	0
If [X, Y] = 0, then X and Y behave like __________ variables of classical mechanics.	statistical	dynamical	proportional	inversely proportional	dynamical
If Poisson bracket of two variables in classical mechanics is zero, then the operators which represent these variables in quantum theory should _____________.	vanish	be multiplied twice	proportional	commute	commute
The Lagrange's bracket is ____________ under canonical transformation.	invariant	variant	not applicable	exponentially variant	invariant
Lagrange's equation of motion are second order equations with __________ degrees of freedom.	n+1	n	2n+1	3n	2n+1
The greatest advantage of action and angle variable is that we can obtain the ______________ of periodic motion without finding a complete solution for the motion of the system.	displacement	frequencies	total time	accelerations	frequencies
The generalized co-ordinate conjugate to Jj are called ________________.	action variable	dynamic variable	statistical variable	angle variable	angle variable
Jj has the dimension of ____________.	angular momentum	angular velocity	linear momentum	linear velocity	angular momentum
If F does not involve time explicitly, then the Poisson bracket of F with H ______________.	is proportional with F	is proportional with K	Vanishes	exist	Vanishes
If the Poisson bracket of F with H vanishes then F will be a ________________.	positive value	constant of motion	negative value	same value	constant of motion
If Poisson bracket of momentum with H vanishes, then _________________ is conserved.	linear velocity	energy	angular momentum	linear momentum	linear momentum

If Poisson bracket of momentum with H vanishes, then the co-ordinate momenta is ______________.	cyclic	rotational	irrotational	spherical	cyclic
Lagrange's bracket does not obey the _____________law.	associative	kepler's	commutative	Hamilton's variational law	commutative
H = ______________.	T- V	T + V	T	V	T + V
L = __________.	T + V	T	V	T-V	T-V
In case of either of the set of conjugate variables with (q, p) or with (Q, P), the value of the Poisson bracket remains _____________.	same	proportional	inversely proportional	exponentially proportional	same
In new set of co-ordinates all Qj are _______________.	rotational	irrotational	cyclic	variable	cyclic
In new set of co-ordiantes all Pj are _________.	cyclic	constant	rotational	irrotational	constant
If H is conserved then the new Hamiltonian K is ___________.	same	variable	different	constant of motion	constant of motion
The matrix of Lagrange's bracket is the _____________ as the matrix of Poisson bracket with sign changed.	same	proportional	inversely proportional	exponentially proportional	same
61. An assembly of particles with _________inter-particle distance is called as rigid body	fixed	different	1 mm	2 mm	fixed
Degree of freedom to fix the configuration of a rigid body is	3	6	5	0	6
These are most useful set of generalised co-ordinates for a rigid body and are angles	Lagrangian angle	azimuthal angle	Euler's angle	Larmor's precession angle	Euler's angle
Angular momentum of a rigid body is	L = Iw/2	L = 2Iw	L = Iw2	L = Iw	L = Iw
A mathematical structure having nine components in 3-D is termed as tensor of rank	2	3	4	0	2
The rotation about space z-axis (angle f) is called _________	translation	precession	nutation	spin.	precession
Rotation about intermediate X1 axis (angle q) or line of nodes is called	translation	precession	nutation	spin.	nutation
The rotation about z' axis (angle Y) is called	translation	precession	nutation	spin.	spin.
The variation of angle q is referred as ________of the symmetry axis of the top and is	translation	precession	nutation	spin.	nutation

Precession can be	slow or fast	always slow	always fast	neither fast nor slow	always slow
______is ordinarily observed with a rapidly spinning top.	fast precession	slow precession	slow nutation	fast nutation	slow precession
In case of ______ top amplitude of nutation is small, nutation is sinusoidal,	slow	rotating	fast	both a & b	fast
The minimum spin angular velocity below which top cannot spin stably about vertical	wmin = (4mglI1/I32)	wmin = (4mglI1/I32)3/2	wmin = (4mglI1/I32)2	wmin = (4mglI1/I32)1/2	wmin = (4mglI1/I32)1/2
When wz < wmin then the top begins to	wobble	precesse	nutate	spin.	wobble
Angular velocity of a rigid body is given by	Vi = w2 x ri	Vi = (w x ri)1/2	Vi = w x ri	Vi = w3 x ri	Vi = w x ri
Angular momentum of a rigid body is L =	S m2(ri x Vi)	S m(ri x Vi)2	S m2(ri x Vi)2	S m(ri x Vi)	S m(ri x Vi)
The diagonal elements Ixx, Iyy, Izz of inertia _______I are moments of inertia	tensor	vector	scalar	donar	tensor
Tensor I is __________to principal axes	symmetric	antisymmetric	parallel	perpendicular	symmetric

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UNIT III

Small Amplitude Oscillations: Minima of potential energy and points of stable equilibrium, expansion of the potential energy around a minimum, small amplitude oscillations about the minimum, normal modes of oscillations example of N identical masses connected in a linear fashion to (N -1) - identical springs.

Minima of potential energy and points of stable equilibrium

Stable , Unstable and Neutral Equilibrium

Equilibrium can be further classified as stable, unstable and neutral equilibrium.

On being slightly disturbed from its equilibrium position, if a body

(i) tends to acquire the original configuration then the body is said to be in stable equilibrium.

(ii) tends to acquire a new position then the body is said to be in unstable equilibrium

(iii) remains at that position then the body is said to be in neutral equilibrium.

Stable, Unstable and neutral equilibrium in terms of potential energy

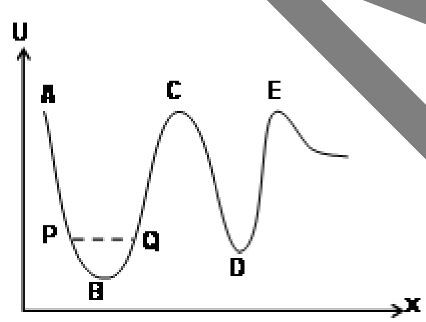
If potential energy of a body does not change with any change in its configuration then it is said to be in neutral equilibrium.

If potential energy of a body changes with change in its configuration then the body will have maximum potential energy at unstable equilibrium and minimum potential energy at stable equilibrium.

Stable Equilibrium And Oscillation :

Oscillation is intimately related with stable equilibrium.

To illustrate it, let us consider a typical curve between the position (x) of the particle and its potential energy (U) for a one dimensional particle motion in a conservative field.



Tangents drawn at B, C, D and E are parallel to the x-axis. This means, at these points, slope (dU/dx) is zero.

Recalling F = − (dU/dx) , we can further say that at B , C , D and E , force acting on the particle is zero i.e. these are equilibrium positions.

For portions BC and DE, an increase in the value of x corresponds to an increase in the value of U.

The slope of the curve at any point in this portion is positive and hence , force(F = − dU/dx) is negative.

It means, in BC and DE region, the force acting on the particle tends to pull it in a region of lower potential energy.

Similarly it can be shown that for the portions AB and CD (where slope is negative and hence force is positive) again the force pulls the particle in the region of lower potential energy.

Thus any slight displacement of the particle, either way from the position of minimum potential energy results into a force tending to bring the particle back to its original position.

This force is often referred to as restoring force and site of minimum potential energy, as recalled earlier, is the position of stable equilibrium.

Expansion of the potential energy around a minimum

If we make a Taylor expansion of the potential energy around the local minima (let's call it xminxmin) we obtain:

U(x)=U(xmin)+U′(xmin)(x−xmin)+12U″(xmin)(x−xmin)2+⋯

For small values of xxwe can use the first three terms of the Taylor expansion, and still get a pretty good approximation. Now, the term U′(xmin) is equal to zero, because the derivative of any function vanishes at minimum value, and let's assume U(xmin)=0, this doesn't alter our physical system: the shift of a potential energy doesn't alter the physics of the problem. We now obtain:

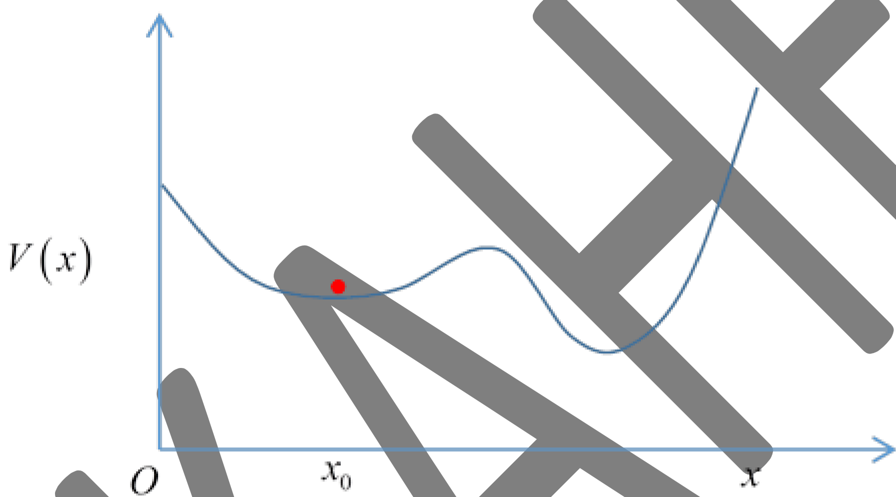
U(x)≈12U″(xmin)(x−xmin)2

Which looks (almost) exactly like the potential energy from Hooke's Law:

U(x)=12kx2

Small amplitude oscillations about the minimum

We begin with the one-dimensional case of a particle oscillating about a local minimum of the potential energy $V(x)$. We'll assume that near the minimum, call it x_0, the potential is well described by the leading second-order term, $V(x) = \frac{1}{2}V''(x_0)(x-x_0)^2$, so we're taking the zero of potential at x_0, assuming that the second derivative $V''(x_0) \neq 0$, and (for now) neglecting higher order terms.



To simplify the equations, we'll also move the x origin to x_0, so

$$m\ddot{x} = -V''(0)x = -kx,$$

replacing the second derivative with the standard "spring constant" expression.

This equation has solution

$$x = A\cos(\omega t + \delta), \text{ or } x = \mathrm{Re}\left(Be^{i\omega t}\right), \quad B = Ae^{i\delta}, \quad \omega = \sqrt{k/m}.$$

(This can, of course, also be derived from the Lagrangian, easily shown to be $L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2.$)

Normal modes of oscillations

The physical motion corresponding to the amplitudes eigenvector $(1,1)$ has two constants of integration (amplitude and phase), often written in terms of a single complex number, that is,

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}\operatorname{Re}Be^{i\omega_0 t}=\begin{pmatrix}A\cos(\omega_0 t+\delta)\\ A\cos(\omega_0 t+\delta)\end{pmatrix},\quad B=Ae^{i\delta}$$

with A,δ real.

Clearly, this is the mode in which the two pendulums are in sync, oscillating at their natural frequency, with the spring playing no role.

In physics, this mathematical eigenstate of the matrix is called a normal mode of oscillation. In a normal mode, all parts of the system oscillate at a single frequency, given by the eigenvalue.

The other normal mode,

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}\operatorname{Re}Be^{i\omega' t}=\begin{pmatrix}A\cos(\omega' t+\delta)\\ -A\cos(\omega' t+\delta)\end{pmatrix},\quad B=Ae^{i\delta}$$

where we have written $\omega'=\sqrt{\omega_0^2+2k}$. Here the system is oscillating with the single frequency ω', the pendulums are now exactly out of phase, so when they part the spring pulls them back to the center, thereby increasing the system oscillation frequency.

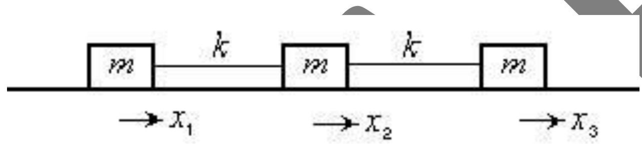
The matrix structure can be clarified by separating out the spring contribution:

$$\mathbf{M} = \begin{pmatrix} \omega_0^2 + k & -k \\ -k & \omega_0^2 + k \end{pmatrix} = \omega_0^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

All vectors are eigenvectors of the identity, of course, so the first matrix just contributes ω_0^2 to the eigenvalue. The second matrix is easily found to have eigenvalues are 0,2, and eigenstates $(1,1)$ and $(1,-1)$.

Example of N identical masses connected in a linear fashion to (N -1) - identical springs.

Consider the motion of three identical particles, each of mass *m*, connected as shown in the diagram by two springs, each of force constant *k*. We wish to find the normal modes of oscillation of the system, i.e. the frequencies with which all the masses oscillate.



Let x_1, x_2 and x_3 x_1, x_2 and x_3 be the displacements of the particles from their equilibrium positions. Note that the motion of the mass on the left compresses the spring attached to it by an amount x_1 x_1, whereas the motion of the mass in the middle extends it by an amount x_2 x_2. The net extension is therefore $x_2 - x_1$ $x_2 - x_1$. Similarly, the spring on the right is extended by an amount $x_3 - x_2$ $x_3 - x_2$. The equations of motion are therefore

$$m\ddot{x}_1 = k(x_2 - x_1) \tag{1.1}$$

$m\ddot{x}_1 = k(x_2 - x_1)$

$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) \tag{1.2}$$

$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2)$

$$m\ddot{x}_3 = -k(x_3 - x_2) \tag{1.3}$$

$m\ddot{x}_3 = -k(x_3 - x_2)$

These equations may be written in the form

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} k/m & -k/m & 0 \\ -k/m & 2k/m & -k/m \\ 0 & -k/m & k/m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} k/m & -k/m & 0 \\ -k/m & 2k/m & -k/m \\ 0 & -k/m & k/m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\,\omega^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad ; \quad \omega^2 = \frac{k}{m} \qquad (1.5)$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\,\omega^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad ; \quad \omega^2 = \frac{k}{m}$$

We see that the equations of motion (1.1) to (1.3) are coupled. This is indicated by the fact that the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad (1.6)$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

is not diagonal. Note, however, that **A** is Hermitian, and can therefore be diagonalized by a unitary, similarity transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{U}^{-1}\mathbf{A}\,\mathbf{U} \qquad (1.7)$$

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{U}^{-1}\mathbf{A}\,\mathbf{U}$$

where the diagonal elements of $\mathbf{A}'$ $\mathbf{A}'$ are the eigenvalues of **A**, and the orthonormalized eigenvectors of **A** are the columns of the unitary matrix **A**.

Eigenvalues of **A**

These are obtained from the relations

$$|\mathbf{A}-\lambda\mathbf{1}|=0$$ $|\mathbf{A}-\lambda\mathbf{1}|=0$ (2.1)

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$ $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$ (2.2)

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right]-(1-\lambda)=0$$ $(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right]-(1-\lambda)=0$ (2.3)

$$\lambda(1-\lambda)(\lambda-3)=0$$ $\lambda(1-\lambda)(\lambda-3)=0$ (2.4)

whence $$\lambda=0,\ 1,\ 3$$ $\lambda=0,\ 1,\ 3$ (2.5)

Normalized eigenfunctions of **A**

These are obtained from the eigenvalue equation

$$\mathbf{A}\psi=\lambda\psi$$ $\mathbf{A}\psi=\lambda\psi$ (2.6)

using the eigenvalues derived above. To simplify the notation, the eigenvalues will be denoted by λ_p λ_p and the corresponding eigenfunctions by $\psi^{(p)}$ $\psi^{(p)}$.

$\boxed{\lambda_1 = 0}$ $\boxed{\lambda_1 = 0}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \psi_3^{(1)} \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \psi_3^{(1)} \end{pmatrix} = 0$$

(2.7)

$$\psi_1^{(1)} - \psi_2^{(1)} = 0 \quad \Rightarrow \quad \psi_1^{(1)} = \psi_2^{(1)}$$

$\psi_1^{(1)} - \psi_2^{(1)} = 0 \quad \Rightarrow \quad \psi_1^{(1)} = \psi_2^{(1)}$ (2.8)

$$-\psi_1^{(1)} + 2\,\psi_2^{(1)} - \psi_3^{(1)} = 0$$

$-\psi_1^{(1)} + 2\,\psi_2^{(1)} - \psi_3^{(1)} = 0$ (2.9)

$$-\psi_2^{(1)} + \psi_3^{(1)} = 0 \quad \Rightarrow \quad \psi_2^{(1)} = \psi_3^{(1)}$$

$-\psi_2^{(1)} + \psi_3^{(1)} = 0 \quad \Rightarrow \quad \psi_2^{(1)} = \psi_3^{(1)}$ (2.10)

and the normalized eigenfunction is

$$\psi^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\psi^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (2.11)

$\boxed{\lambda_2 = 1}$ $\boxed{\lambda_2 = 1}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \psi_3^{(2)} \end{pmatrix} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \psi_3^{(2)} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \psi_3^{(2)} \end{pmatrix} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \psi_3^{(2)} \end{pmatrix}$$

(2.12)

$$\psi_1^{(2)} - \psi_2^{(2)} = \psi_1^{(2)} \quad \Rightarrow \quad \psi_2^{(2)} = 0$$

$\psi_1^{(2)} - \psi_2^{(2)} = \psi_1^{(2)} \quad \Rightarrow \quad \psi_2^{(2)} = 0$ (2.13)

$$-\psi_1^{(2)} + 2\,\psi_2^{(2)} - \psi_3^{(2)} = \psi_2^{(2)} \quad \Rightarrow \quad \psi_1^{(2)} = -\psi_3^{(2)}$$

2.14)

$-\psi_1^{(2)} + 2\,\psi_2^{(2)} - \psi_3^{(2)} = \psi_2^{(2)} \quad \Rightarrow \quad \psi_1^{(2)} = -\psi_3^{(2)}$

$$-\psi_2^{(2)}+\psi_3^{(2)}=\psi_3^{(2)}$$ $-\psi_2^{(2)}+\psi_3^{(2)}=\psi_3^{(2)}$ (2.15)

and a normalized eigenfunction, orthogonal to $\psi^{(1)}$ $\psi^{(1)}$ is

$$\psi^{(2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$ $\psi^{(2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$ (2.16)

$$\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}\psi_1^{(3)}\\\psi_2^{(3)}\\\psi_3^{(3)}\end{pmatrix}=3\begin{pmatrix}\psi_1^{(3)}\\\psi_2^{(3)}\\\psi_3^{(3)}\end{pmatrix}$$

$\boxed{\lambda_3=3}$ $\boxed{\lambda_3=3}$

$\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix}\begin{pmatrix}\psi_1^{(3)}\\\psi_2^{(3)}\\\psi_3^{(3)}\end{pmatrix}=3\begin{pmatrix}\psi_1^{(3)}\\\psi_2^{(3)}\\\psi_3^{(3)}\end{pmatrix}$ (2.17)

$$\psi_1^{(3)}-\psi_2^{(3)}=3\psi_1^{(3)} \quad\Rightarrow\quad \psi_2^{(3)}=-2\psi_1^{(3)}$$ (2.18)

$\psi_1^{(3)}-\psi_2^{(3)}=3\psi_1^{(3)} \quad\Rightarrow\quad \psi_2^{(3)}=-2\psi_1^{(3)}$

$$-\psi_1^{(3)}+2\,\psi_2^{(3)}-\psi_3^{(3)}=3\psi_2^{(3)} \quad\Rightarrow\quad \psi_1^{(3)}=-\left(\psi_2^{(3)}+\psi_3^{(3)}\right)$$ (2.19)

$-\psi_1^{(3)}+2\,\psi_2^{(3)}-\psi_3^{(3)}=3\psi_2^{(3)} \quad\Rightarrow\quad \psi_1^{(3)}=-\left(\psi_2^{(3)}+\psi_3^{(3)}\right)$

$$-\psi_2^{(3)}+\psi_3^{(3)}=3\psi_3^{(3)} \quad\Rightarrow\quad \psi_2^{(3)}=-2\psi_3^{(3)}$$ (2.20)

$-\psi_2^{(3)}+\psi_3^{(3)}=3\psi_3^{(3)} \quad\Rightarrow\quad \psi_2^{(3)}=-2\psi_3^{(3)}$

and the normalized eigenfunction, which is orthogonal to both $\psi^{(1)}$ **and** $\psi^{(2)}$ $\psi^{(1)}$ and $\psi^{(2)}$ is

$$\psi^{(3)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\psi^{(3)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$
(2.21)

The unitary matrix **U**, whose columns are the orthonormalized eigenfunctions of the matrix **A**, is therefore given by

$$\mathbf{U} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \;;\; \mathbf{U}^{\dagger} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix} \qquad (2.22)$$

$$\mathbf{U} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \;;\; \mathbf{U}^{\dagger} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix}$$

$$\mathbf{U}\mathbf{U}^{\dagger} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.23)$$

$$\mathbf{U}\mathbf{U}^{\dagger} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as we expect. It can also be shown that $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{1}$ $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{1}$.

Transformation to Normal Coordinates

Now, Equation (1.5) may be written in the form

$$\vec{G} = -\mathbf{A}\,\vec{F} \qquad (3.1)$$

$$\vec{G} = -\mathbf{A}\,\vec{F}$$

which gives the transformation of the vector

$$\vec{F} = \omega^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.2}$$

$$\vec{F} = \omega^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

into the vector

$$\vec{G} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} \tag{3.3}$$

$$\vec{G} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix}$$

in the unprimed coordinate system. Under a rotation of the coordinate system, the relation (1.31) becomes

$$\vec{G}' = -\mathbf{A}'\vec{F}' \tag{3.4}$$

$$\vec{G}' = -\mathbf{A}'\vec{F}'$$

(in the primed coordinate system), or

$$\vec{G}' = -\left(\mathbf{U}^{-1}\mathbf{A}\mathbf{U}\right)\vec{F}' = -\left(\mathbf{U}^{\dagger}\mathbf{A}\mathbf{U}\right)\vec{F}' \tag{3.5}$$

$$\vec{G}' = -\left(\mathbf{U}^{-1}\mathbf{A}\mathbf{U}\right)\vec{F}' = -\left(\mathbf{U}^{\dagger}\mathbf{A}\mathbf{U}\right)\vec{F}'$$

where we have used the fact that **U** is unitary. We thus obtain

$$\begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \\ \ddot{x}_3' \end{pmatrix} = -\frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \omega^2 \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \tag{3.6}$$

$$\begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \\ \ddot{x}_3' \end{pmatrix} = -\frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \omega^2 \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

and a little simplification shows that

$$\begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \\ \ddot{x}_3' \end{pmatrix} = -\omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \\ \ddot{x}_3' \end{pmatrix} = -\omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$ (3.7)

The matrix **A** has therefore been diagonalized , giving rise to the *uncoupled* equations

$$\ddot{x}_1' = 0$$ $\ddot{x}_1' = 0$ (3.8)

$$\ddot{x}_2' = -\omega^2 x_2'$$ $\ddot{x}_2' = -\omega^2 x_2'$ (3.9)

$$\ddot{x}_3' = -3\omega^2 x_3'$$ $\ddot{x}_3' = -3\omega^2 x_3'$ (3.10)

whose solutions are (for oscillatory behavior)

$$x_1' = 0$$ $x_1' = 0$ (3.11)

$$x_2' = C_2' \sin(\omega_2 t + \delta_2) \quad ; \quad \omega_2^2 = \omega^2 = k/m$$ (3.12)

$x_2' = C_2' \sin(\omega_2 t + \delta_2) \quad ; \quad \omega_2^2 = \omega^2 = k/m$

$$x_3' = C_3' \sin(\omega_3 t + \delta_3) \quad ; \quad \omega_3^2 = 3\omega^2 = 3k/m$$ (3.13)

$x_3' = C_3' \sin(\omega_3 t + \delta_3) \quad ; \quad \omega_3^2 = 3\omega^2 = 3k/m$

What we require, however, are the unprimed solutions x_1, x_2 and x_3 x_1, x_2 and x_3. Recall that

$$\vec{G} = -\mathbf{A}\,\vec{F} \quad \Rightarrow \quad \mathbf{U}^{-1}\vec{G} = -\left(\mathbf{U}^{-1}\mathbf{A}\,\mathbf{U}\right)\mathbf{U}^{-1}\,\vec{F} \qquad (3.14)$$

$$\vec{G} = -\mathbf{A}\,\vec{F} \quad \Rightarrow \quad \mathbf{U}^{-1}\vec{G} = -\left(\mathbf{U}^{-1}\mathbf{A}\,\mathbf{U}\right)\mathbf{U}^{-1}\,\vec{F}$$

Comparing Equations (3.5) and (3.14), we see that

$$\vec{F} \to \vec{F}' = \mathbf{U}^{-1}\vec{F} \quad \Rightarrow \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \mathbf{U}^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{U}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \qquad (3.15)$$

$$\vec{F} \to \vec{F}' = \mathbf{U}^{-1}\vec{F} \quad \Rightarrow \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \mathbf{U}^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{U}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \qquad (3.16)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

Using the solutions (3.11), (3.12) and (3.13), we get

$$x_1 = \frac{1}{\sqrt{2}}C_2' \sin\left(\omega_2 t + \delta_2\right) + \frac{1}{\sqrt{6}}C_3' \sin\left(\omega_3 t + \delta_3\right) \qquad (3.17)$$

$$x_1 = \frac{1}{\sqrt{2}}C_2' \sin\left(\omega_2 t + \delta_2\right) + \frac{1}{\sqrt{6}}C_3' \sin\left(\omega_3 t + \delta_3\right)$$

$$x_2 = -\frac{2}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right)$$ $$x_2 = -\frac{2}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right)$$ (3.18)

$$x_3 = -\frac{1}{\sqrt{2}} C_2' \sin\left(\omega_2 t + \delta_2\right) + \frac{1}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right)$$ (3.19)

$$x_3 = -\frac{1}{\sqrt{2}} C_2' \sin\left(\omega_2 t + \delta_2\right) + \frac{1}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right)$$

Now the normal modes refer to the common frequencies of vibration of all the masses. There are three possibilities:

$\boxed{C_2' = C_3' = 0}$ $\boxed{C_2' = C_3' = 0}$ This yields the result

$$x_1 = x_2 = x_3$$ $$x_1 = x_2 = x_3$$ (3.20)

which represents a pure translational motion of the particles (corresponding to an eigenfrequency $\omega = 0$ $\omega = 0$).

$\boxed{C_3' = 0\,;\, C_2' \neq 0}$ $\boxed{C_3' = 0\,;\, C_2' \neq 0}$ This gives

$$x_1 = -x_3 = \frac{1}{\sqrt{2}} C_2' \sin\left(\omega_2 t + \delta_2\right) \quad ; \quad x_2 = 0$$ (3.21)

$$x_1 = -x_3 = \frac{1}{\sqrt{2}} C_2' \sin\left(\omega_2 t + \delta_2\right) \quad ; \quad x_2 = 0$$

and the two outer masses oscillate 180° 180° out of phase, with the same amplitude, and with frequency given by $\omega_2 = \omega = \sqrt{k/m}$ $\omega_2 = \omega = \sqrt{k/m}$.

$\boxed{C_2' = 0\,;\, C_3' \neq 0}$ $\boxed{C_2' = 0\,;\, C_3' \neq 0}$ Here we get

$$x_1 = x_3 = \frac{1}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right) \quad ; \quad x_2 = -2x_3 \tag{3.22}$$

$$x_1 = x_3 = \frac{1}{\sqrt{6}} C_3' \sin\left(\omega_3 t + \delta_3\right) \quad ; \quad x_2 = -2x_3$$

The two outer masses oscillate in phase, with the same amplitude, but with frequency given by $\omega = \omega_3 = \sqrt{3k/m}$ $\omega = \omega_3 = \sqrt{3k/m}$. The mass in the middle oscillates 180° 180° out of phase with the other two, and with twice the amplitude.

It has been tacitly assumed in the foregoing analysis that the masses move on a frictionless horizontal surface. The total (horizontal) momentum of the system must therefore remain constant, as can easily be verified from Equations (3.21) and (3.22).

KARPAGAM ACADEMY OF HIGHER EDUCATION					
DEPARTMENT OF PHYSICS					
III B.Sc., PHYSICS (2016-2019)					
CLASSICAL MECHANICS (16PHU504A)					
UNIT - III					
QUESTIONS	**Choice1**	**Choice2**	**Choice3**	**Choice4**	**ANSWER**
Rotational kinetic energy of a rigid body is	½ w2 I2	w2 I	½ w2 I	2w2 I.	½ w2 I
In certain system of body axes with respect to which the off-diagonal elements	symmetric	antisymmetric	principal	perpendicular	principal
If wz = wz' > wmin, atop will spin with its axis vertical continuously , therefore it is	sleeping top	spinning top	rotating top	symmetric top	sleeping top
A rigid body with N particles have _____________degrees of freedom.	2N	3N	N	4N	3N
The configuration of a rigid body with respect to some cartesian co-ordinate system in space	momentum	inertia	orientation	angular momentum	orientation
The most useful set of generalised co-ordinates for a rigid body are ________ angles.	rotation	specified	auxillary	euler's	euler's
The transformation worked out through three _________ rotations performed only in a	successive	different	independent	dependent	successive
The distance between any two points of a rigid body is ___________	varied	fixed	proportional	exponentially proportional	fixed
A rigid body can possesses simultaneously the translational and _________ motion	arbitrary	circular	rotational	orbital	rotational
A mathematical structure having nine components in three dimensions is termed as a	tensor	matrix	covariant tensor	contravariant tensor	tensor
The products of inertia of all vanish when one of the axes of the body lies along the axis	rotation	vibration	motion	symmetry	symmetry
If the symmetry axis of the body is taken as axis of rotation and the origin of body axes lies	unsymmetry	rotational	symmetry	b and c	symmetry
The motion of a rigid body with one point fixed will take place under the action of torque N in	displacement	torque	time	rotational motion	torque

The assembly of particles with fixed inter-particle distance is called__________	fluid	vapor	colloidal	rigid body	rigid body
The orientation of the body by locating a cartesian set of co-ordinates fixed in the rigid	body set of axes	space set of axes	both a and b	rotational set of axes	body set of axes
The fixed point in the body which registers its translation and coincident with the center of	body set of axes	space or external set of axes	rotational set of axis	vibrational set of axes	space or external set of axes
The generation of body set of axes from the space set of axes through three successive	direction cosines	successive angles	rotational angles	Euler’s angles	Euler’s angles
The system of body axes in which off-diagonal elements disappear and the diagonal elements	principle axes	secondary axes	primary axes	catesian axes	secondary axes
The system of body axes in which off-diagonal elements disappear, and the diagonal elements	principle moment of inertia	secondary moment of inertia	moments of inertia	inertia	inertia
The secular equation of inertia tensor and its solution is called _____________	constant of motion	tensor of rank two	covariant tensor	eigen values	eigen values
A rigid body can possesses simulataneous the _________ and ____________ motion.	translation and rotational	linear and harmonic	periodic and non-harmonic	symmetrical around	translation and rotational
Rigid body possessing rotational and translational motion simulataneously will have	polar and cartesian	generalised and canonical	translation and rotational	both a and b	translation and rotational
If we consider three non-collinear points in a rigid body, then each particle will have	four	three	six	nine	three
Three non-collinear points in a rigid body will have the total of _________degrees of	six	three	nine	tweleve	nine
All the space set of axis if rotated wbout the space z-axis, then the yz plane takes _______	same	alternate	orthogonal	new	new
The inverse transformation matrix from body set of axes to space set of axes is given	AT	adj (A)	co-factor of A	determinant of A	AT
The position vector of any point p relative to the origin O of the body set of axes is	Different	constant	proportional	both a and c	constant

The configuration of a rigid body is completely specified by ________degrees of freedom.	two	three	six	nine	six
If a is the column matrix representing the co-ordinates having single frequency and aT is	0	I	a	1	1
If a is the column matrix representing the co-ordinates having single frequency and	0	I	1	a2	I
The generalised co-ordinate in which each one of them executing oscillations of one	normal co-ordinate	cartesain co-ordinate	polar co-ordinate	rectangular co-ordinate	normal co-ordinate
In parallel pendula the two pendula oscillates in ________	out or phase	phase	damped motion	undamped motion	phase
In parallel pendular, if the two pendula are independent i.e., there is no ________	unstretching	rarefying	transiting	stretching	stretching
In paralle pendula ___________ force due to spring will come into action.	impulsive	repulsive	restoring	attractive	restoring
If the system possesses two identical frequencies, then it is therefore said to be __________	degenerate	generate	distorted	in harmonic motion	degenerate
A continuous string has infinite number of normal modes and ________________	velocities	frequencies	vibrations	motion	frequencies
The use of nomal co-ordinate in the coupled system reduces it to one of a system of	dependent	single	independent	double	independent
A continuous string has a linear ____________	velocity	acceleration	displacement	mass density	mass density
If the system is in stable equilibrium, then the frequency wl2 should be a ______ quantity.	real	imaginary	complex	integer	real
If wl2 are real and positive, then all co-ordinate always remain _______ for any time.	infinite	same	different	finite	finite
If wl2 are not real and positive, then all the co-ordinate becomes ______ for any time.	infinite	finite	equal	exponential	infinite
The system is said to be unstable if the frequency wl2 are not _________	equal	finite	real	infinite	real

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UNIT IV

Special Theory of Relativity: Postulates of Special Theory of Relativity. Lorentz Transformations. Minkowski space. The invariant interval, light cone and world lines. Space-time diagrams. Time -dilation, length contraction and twin paradox. Four-vectors: space-like, time-like and light-like. Four-velocity and acceleration. Metric and alternating tensors. Four-momentum and energy-momentum relation. Doppler effect from a four-vector perspective. Concept of four-force. Conservation of four-momentum. Relativistic kinematics. Application to two-body decay of an unstable particle.

Special theory of relativity – Introduction

The Special Theory of Relativity was the result of developments in physics at the end of the nineteenth century and the beginning of the twentieth century. It changed our understanding of older physical theories such as Newtonian Physics and led to early Quantum Theory and General Relativity.

Special Relativity does not just apply to fast moving objects, it affects the everyday world directly through "relativistic" effects such as magnetism and the relativistic inertia that underlies kinetic energy and hence the whole of dynamics.

Special Relativity is now one of the foundation blocks of physics. It is in no sense a provisional theory and is largely compatible with quantum theory; it not only led to the idea of matter waves but is the origin of quantum 'spin' and underlies the existence of the antiparticles. Special Relativity is a theory of exceptional elegance, Einstein crafted the theory from simple postulates about the constancy of physical laws and of the speed of light and his work has been refined further so that the laws of physics themselves and even the constancy of the speed of light are now understood in terms of the most basic symmetries in space and time.

The Galilean Transformation Invariance of Newton's law of motion

Suppose there are two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure 1). In S, we have the co-ordinates $\{x, y, z, t\}$ and in S' we have the co-ordinates $\{x', y', z', t'\}$. S' is moving with respect to S with velocity v (as measured in S) in the x direction. The clocks in both systems were synchronised at time $t = 0$ and they run at the same rate.

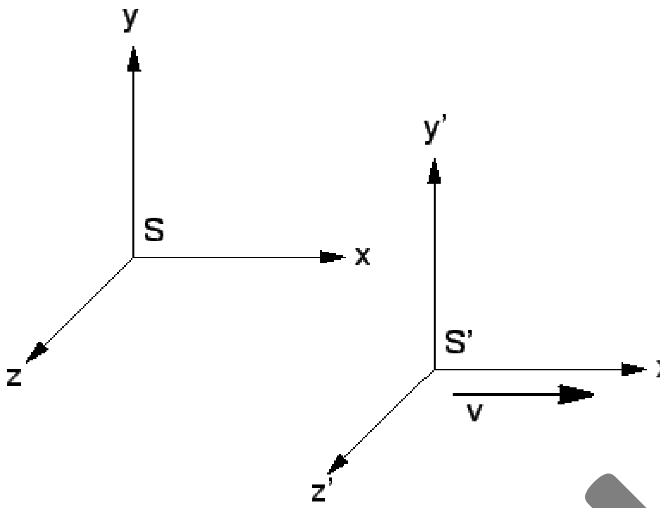


Figure 1: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

We have the intuitive relationships

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

This set of equations is known as the Galilean Transformation. They enable us to relate a measurement in one inertial reference frame to another. For example, suppose we measure the velocity of a vehicle moving in the in x-direction in system S, and we want to know what would be the velocity of the vehicle in S'.

$$v'_x = \frac{dx'}{dt'} = \frac{d(x-vt)}{dt} = v_x - v$$

The laws of physics to be the same in all inertial reference frames, as this is indeed our experience of nature. Physically, we should be able to perform the same experiments in different reference frames, and find always the same physical laws. Mathematically, these laws are expressed by equations. So, we should be able to ``transform'' our equations from one inertial reference frame to the other inertial reference frame, and always find the same answer.

Suppose we wanted to check that Newton's Second Law is the same in two different reference frames. We put one observer in the un-primed frame, and the other in the primed frame, moving with velocity v relative to the un-primed frame. Consider the vehicle of the previous case undergoing a constant acceleration in the x-direction,

$$
\begin{aligned}
f' = m'a' &= m'\frac{d^2x'}{dt'^2} \\
&= m'\frac{d}{dt'}\left(\frac{dx'}{dt'}\right) \\
&= m\frac{d}{dt}\left(\frac{d(x-vt)}{dt}\right) \\
&= m\frac{d(v_x - v)}{dt} \\
&= m\frac{dv_x}{dt} \\
&= ma = f
\end{aligned}
$$

Indeed, it does not matter which inertial frame we observe from, we recover the same Second Law of Motion each time. In the parlance of physics, we say the Second Law of Motion is invariant under the Galilean Transformation.

Non-variance of Maxwell's equation

Experiments on electric and magnetic fields, as well as induction of one type of field from changes in the other, lead to the collection of a set of equations, describing all these phenomena, known as Maxwell's Equations.

Maxwells Equations *in vacuo*

$$\begin{aligned} \nabla.\mathbf{B} &= 0, \\ \nabla.\mathbf{E} &= 0, \\ \nabla \times \mathbf{B} &= \epsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t}, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned}$$

Now, these equations are considered to be rock solid, arising from and verified by many experiments. Amazingly, they imply the existence of a previously not guessed at phenomenon. This is the electromagnetic wave. To see this in detail, take the time derivative of the second last equation and the curl of the last.

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}.$$

Now note that space and time derivatives commute

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \nabla \times \frac{\partial \mathbf{B}}{\partial t},$$

so

$$\nabla \times (\nabla \times \mathbf{E}) = -\epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Now, we use the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla\nabla.\mathbf{E} - \nabla^2\mathbf{E}.$$

The second term of the above equation drops out due to the vanishing of the divergence of the electric field (the second of Maxwell's Equations). So, we finally have the three dimensional wave equation

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

To see this is a wave equation, note the analogy in one dimension

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

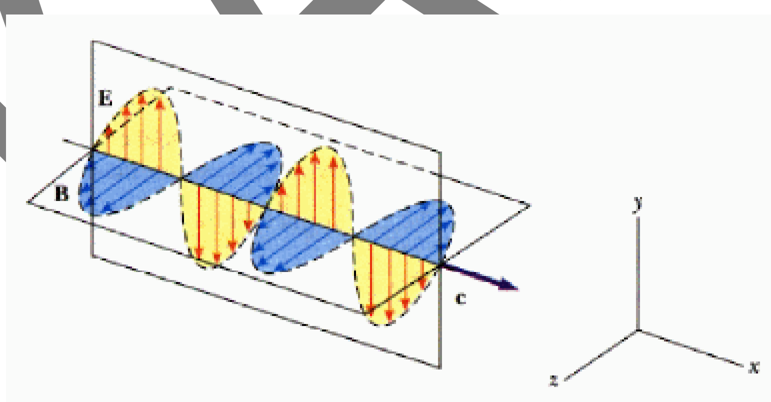
which is solved by the wave function

$$y(x,t) = \sin(x - ct),$$

which represents a wave traveling along the x axis with velocity c.

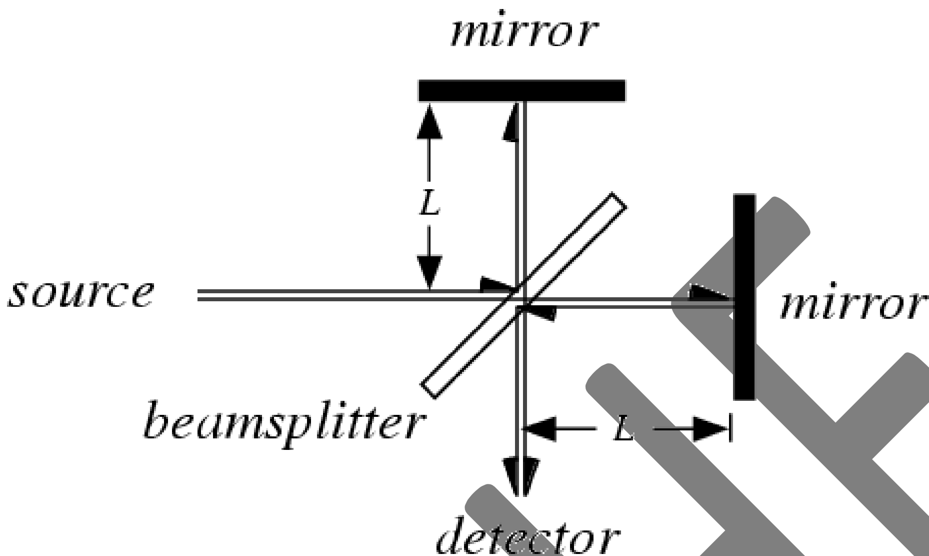
It is clear therefore that Maxwell's Equations are highly predictive.

1. A diversity is unified in a simplicity. The various phenomena of radiowaves, microwaves, infrared, visible and ultra-violet light, X-rays and gamma rays are all electromagnetic waves, differing only in their frequency.
2. They all travel at the same speed.
3. Even that speed is specified : $c = 1/\sqrt{\epsilon_0 \mu_0} = 2.997 \times 10^8$ m/s.
4. The speed appears independent of the source and the observer.



Michelson Morley experiment and explanation of the null result.

After the development of Maxwell's theory of electromagnetism, several experiments were performed to prove the existence of ether and its motion relative to the Earth. The most famous and successful was the one now known as the Michelson-Morley experiment, performed by Albert Michelson (1852-1931) and Edward Morley (1838-1923) in 1887.



Michelson and Morley built a Michelson interferometer, which essentially consists of a light source, a half-silvered glass plate, two mirrors, and a telescope. The mirrors are placed at right angles to each other and at equal distance from the glass plate, which is obliquely oriented at an angle of 45° relative to the two mirrors. In the original device, the mirrors were mounted on a rigid base that rotates freely on a basin filled with liquid mercury in order to reduce friction.

Prevailing theories held that ether formed an absolute reference frame with respect to which the rest of the universe was stationary. It would therefore follow that it should appear to be moving from the perspective of an observer on the sun-orbiting Earth. As a result, light would sometimes travel in the same direction of the ether, and others times in the opposite direction. Thus, the idea was to measure the speed of light in different directions in order to measure speed of the ether relative to Earth, thus establishing its existence.

Michelson and Morley were able to measure the speed of light by looking for interference fringes between the light which had passed through the two perpendicular arms of their apparatus. These would occur since the light would travel faster along an arm if oriented in the "same" direction as the ether was moving, and slower if oriented in the opposite direction. Since the two arms were perpendicular, the only way that light would travel at the same speed in both arms and therefore arrive simultaneous at the telescope would be if the instrument were motionless with respect to the ether. If not, the crests and troughs of the light waves in the two arms would arrive and interfere slightly out of synchronization, producing a diminution of intensity. (Of course, the same effect would be achieved if the arms of the interferometer were not of the same length, but these could be adjusted accurately by looking for the intensity peak as one arm was moved. Changing the orientation of the instrument should then show fringes.)

Although Michelson and Morley were expecting measuring different speeds of light in each direction, they found no discernible fringes indicating a different speed in any orientation or at any position of the Earth in its annual orbit around the Sun.

In 1895, Lorentz concluded that the "null" result obtained by Michelson and Morley was caused by a effect of contraction made by the ether on their apparatus and introduced the length contraction equation

$$L = L_0\sqrt{1-\frac{v^2}{c^2}},$$

where L is the contracted length, L_0 is the rest length, v is the velocity of the frame of reference, and c is the speed of light.

Concept of inertial frame of reference

A "frame of reference" is a standard relative to which motion and rest may be measured; any set of points or objects that are at rest relative to one another enables us, in principle, to describe the relative motions of bodies. A frame of reference is therefore a purely kinematical device, for the geometrical description of motion without regard to the masses or forces involved. A dynamical account of motion leads to the idea of an "inertial frame," or a reference frame relative to which motions have distinguished dynamical properties. For that reason an inertial frame has to be understood as a spatial reference frame together with some means of measuring time, so that uniform motions can be distinguished from accelerated motions.

The laws of Newtonian dynamics provide a simple definition: an inertial frame is a reference-frame with a time-scale, relative to which the motion of a body not subject to forces is always rectilinear and uniform, accelerations are always proportional to and in the direction of applied forces, and applied forces are always met with equal and opposite reactions. It follows that, in an inertial frame, the center of mass of a system of bodies is always at rest or in uniform motion. It also follows that any other frame of reference moving uniformly relative to an inertial frame is also an inertial frame. For example, in Newtonian celestial mechanics, taking the "fixed stars" as a frame of reference, we can determine an (approximately) inertial frame whose center is the center of mass of the solar system; relative to this frame, every acceleration of every planet can be accounted for (approximately) as a gravitational interaction with some other planet in accord with Newton's laws of motion.

Postulates of special theory of relativity

(i) Statement: "The laws of physics are the same in any inertial frame, regardless of position or velocity".

Physically, this means that there is no absolute spacetime, no absolute frame of reference with respect to which position and velocity are defined. Only relative positions and velocities between objects are meaningful.

(ii) Statement: "The speed of light *c* is a universal constant, the same in any inertial frame".

Simultaneity

Consider a rocket traveling at speed **v**, as shown in Fig. 4. There is an observer **O** at rest with respect to the rocket and an observer **O'** riding with the rocket. Two lightbulbs at the ends of the rocket were timed such that their flashes arrive at the observers at the same time. Light from the bulbs traveled towards the observers at the speed of light, **c**, in the reference frames of both observers. The figure shows how **O** and**O'** are lined up when the light arrives.

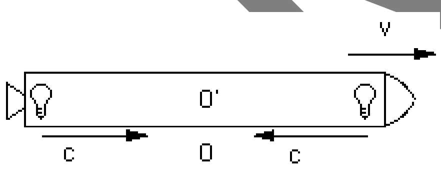


Fig. 4

For **O'** (on the rocket), the bulbs must have flashed simultaneously because **O'** is right in the middle. The bulbs are at rest in the frame of **O'**.

The other observer, **O**, draws a different conclusion. When the flashes were emitted, the rocket was not centered on **O**; it was to the left. The pulse from the bulb on the left must have been emitted first; it had farther to travel. Likewise, the pulse from the bulb on the right had a shorter distance to travel. Observer **O** concludes that the bulbs were not flashed simultaneously.

So, observer **O'** thinks the events (flashing of the bulbs) were simultaneous while observer **O** does not. Simultaneity is not independent of reference frame.

Length contraction

Moving rod contracts in length by factor of $\sqrt{1-\frac{v^2}{c^2}}$

i.e., $\begin{matrix}\text{length of a rod in}\\ \text{motion in a given}\\ \text{frame of reference}\end{matrix} = \begin{matrix}\text{length of the same rod}\\ \text{when at rest}\\ \text{in the given frame of}\\ \text{reference}\end{matrix} \times \sqrt{1-\frac{v^2}{c^2}}$

or $$l = l_0\sqrt{1-\frac{v^2}{c^2}} \quad ....(4)$$

Time dilation

Moving clock dilates in time interval measured by factor of

ie $\begin{matrix}\text{Time interval measured}\\ \text{by a clock in motion in}\\ \text{a given frame of reference}\end{matrix} = \begin{matrix}\text{Time interval measured}\\ \text{by the same clock when}\\ \text{at rest in the given frame}\\ \text{by reference.}\end{matrix} \times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

or $$\boxed{\tau = \frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}} \quad ...... (5)$$

Relativistic Law of Velocity Addition

If an object is in motion with velocity $\vec{u}'$ (u'_x, u'_y, u'_zcomponents) in frame S' and the velocity of the object measured in S is $\vec{u}$ (u_x, u_y, $u_z$components) then ,

$$u_x = \frac{u'_x + V}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(a)}$$

$$u_y = \frac{u'_y \sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(b)}$$

$$u_z = \frac{u'_z \sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(c)}$$

.............(6)

Relativistic Mass

The concept of 'Absolute Mass' of Newtonian Mechanics is no longer tenable in special Relativity; the requirement that Law of Conservation of momentum is a fundamental Law of nature imposes the relation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{........(7)}$$

then only consistency between the Lorentz-Transformations and Law of Conservation of momentum can be obtained. This expression given relativistic mass m in motion with Velocity V in a given frame of reference; in terms of the mass m0 called rest mass of the object when at rest in the given frame of reference.

The Experiment of Fizeau

In 1851, Fizeau carried out an experiment which tested for the aether convection coefficient. This was the first such test of Fresnel's formula, derived without experimental evidence, over twenty years earlier. Fresnel, in fact, had died more than twenty years before this experiment took place, a point of interest only because many texts derive Fresnel's formula based on the results of experiment, rather than the other way around. Experimental results, within the level of error available in the mid-1800's, are not sufficient to derive Fresnel's formula. These results can only confirm that, within error limits, the formula provides answers consistent with experiment. In fact, Fizeau's experimental results were so course that the only conclusion he could draw was that the displacement was less than should have been produced by the motion of the liquid if light were completely convected by the medium. From this, he assumed the validity of Fresnel's formula on the partial convection of the aether.

Fizeau's experiment involved passing light two ways through moving water (v ~ 7 m/s) and observing the interference pattern obtained, as illustrated in figure 1. The experiment was repeated by Michelson in 1886 with much more rigor, and quantitative results were obtained. Working backwards from the observed fringe shift, Michelson was able to calculate an apparent convection coefficient equivalent to Fresnel's formula. Varying the velocity and direction of the flow allowed for a variety of test points. By observing the change in interference pattern, the effective velocity of light through the moving medium, as measured in the lab frame, was calculated. Within experimental limits, the results obtained by measuring the fringe shift agreed with the results predicted by Fresnel's formula. However, Michelson neglected to take into account the Doppler effect of light from a stationary source interacting with moving water, and therefore concluded that the aether convection concept of Fresnel was essentially correct.

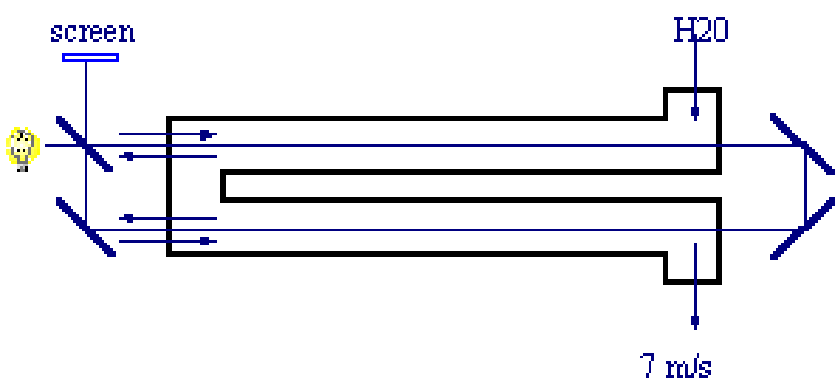


Figure 1. The experiment of Fizeau.

We now examine this experiment in a purely Galilean environment, taking into account the Doppler shift (change in wavelength) experienced by each beam of light. Michelson's paper gives an excellent analysis whereby the retarded velocity, b, seen in the water may be considered as due to the number of collisions with atoms, the "velocity of light through the atoms," and the width of the atoms. Since there will likely be objections to that analysis based on current understandings of the microscopic world, we present a more general approach. In what follows, the retarded velocity is again considered as due to the "collisions" (absorptions and re-emissions) of the photons in the medium, as it must be, but we do not require any assumptions as to "atom width," or "velocity through the atom."

For light traveling through a medium, the effective wavelength changes:

$$\lambda_1 = \frac{\lambda_0}{\eta} \quad (1)$$

The phase shift for light in such a medium is:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{\lambda_0} \quad (2)$$

The optical path length is defined from (2) as *lh*. The optical path difference between the medium and air is then:

$$l[\eta - 1] = l\eta[1 - \frac{1}{\eta}] \quad (3)$$

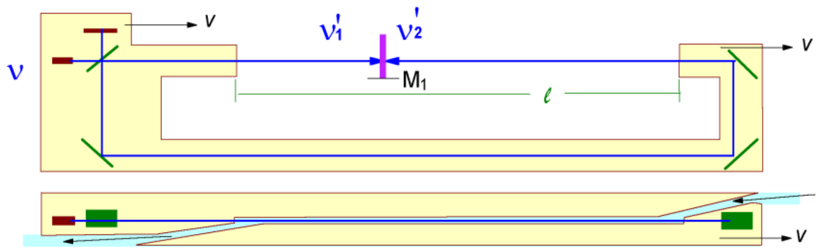
The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta}{\lambda_0}[\eta - 1] \quad (4)$$

In the Fizeau experiment we must consider Doppler effects. Since the water is moving with respect to the source, the two paths of light will experience Doppler shifts upon entering the water. Light moving in the opposite direction to the flow of water will be blue-shift (l_1). Light moving with the flow will be red shifted (l_2):

$$\lambda_1 = (1 - v/c)\lambda$$

$$\lambda_2 = (1 + v/c)\lambda \quad (8)$$



To see why the Doppler shift cannot be ignored in Fizeau's experiment, imagine the apparatus depicted in figure 2. All mirrors, the source and the observing screen are sealed in water filled containers. The water is not flowing, but is stationary in the containers. Alternatively, the containers could be made of solid glass, so long as the refractive index is different than air. The entire apparatus, with the exception of mirror (detector) M_1 moves through the lab frame at a velocity of *v*. Thus, air is moving through the gap, *l*, at a velocity of *v* in the equipment frame. To first order in *v/c*, the wavelengths of the light detected at M_1 is given by equation (8).

We now fill the apparatus containers with air and pass the entire apparatus through water. In the equipment frame, water is moving through the gap at a velocity *v*. The motion induced Doppler in the water, experienced by M_1, remains unchanged. If we, the observers, move along with the apparatus, this setup is indistinguishable from the actual Fizeau experiment. From our frame of reference, the equipment is at rest, water is moving through the gap at a velocity *v*, and the image on the screen reflects the fringe shift due to that motion. Thus we can replace the gap with a tube of flowing water, hold the rest of the apparatus stationary in the lab frame, and obtain a one-sided Fizeau experiment. Clearly, whatever analysis one uses to derive the formulas for the observed fringe shift, one must take into account the fact that the wavelength of the light in the moving medium is different from that of the source due to the motion induced Doppler effect of (8).

Substituting (8) into (2), we see that the phase shift including Doppler effects becomes:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{(1+v/c)\lambda_0} = \frac{l\eta c}{(c+v)\lambda_0} \quad (9)$$

The optical path length is defined from the above as:

$$\frac{l\eta c}{c+v} \quad (10)$$

The optical path difference between the medium and air is then:

$$\frac{lc}{c+v}[\eta - 1] = \frac{lc\eta}{c+v}[1 - \frac{1}{\eta}] \quad (11)$$

The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta c}{(c+v)\lambda_0}[1 - \frac{1}{\eta}] \quad (12)$$

For light traveling different paths and experiencing different Doppler effects, the total phase shift is given by:

$$\frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} = \frac{l_1\eta c}{(c+v_1)\lambda_0}[1 - \frac{1}{\eta}] - \frac{l_2\eta c}{(c+v_2)\lambda_0}[1 - \frac{1}{\eta}] \quad (13)$$

In the Fizeau experiment, l_1 and l_2 are given by (8). The path lengths l_1 and l_2 are respectively given below, where the factor of two is included because the light travels through two tubes of length *l*, and *b* is the velocity of light in the reference frame of the liquid.

$$bt_1 = 2l + vt_1, \quad or \quad t_1 = \frac{2l}{b-v}$$

$$l_1 = bt_1 = \frac{2lb}{b-v}, \qquad l_2 = \frac{2lb}{b+v} \quad (14)$$

Substituting these values into (13) for each path gives the following results:

$$\frac{\phi_1 - \phi}{2\pi} = \frac{\delta\phi_1}{2\pi} = \frac{2lb}{(b-v)} \cdot \frac{1}{\lambda_1}[\eta - 1] =$$

$$\frac{2lb}{(b-v)} \cdot \frac{c}{(c-v)\lambda}\eta[1 - \frac{1}{\eta}]$$

$$\frac{\phi_2 - \phi}{2\pi} = \frac{\delta\phi_2}{2\pi} = \frac{2lb}{(b+v)} \cdot \frac{1}{\lambda_2}[\eta - 1] =$$

$$\frac{2lb}{(b+v)} \cdot \frac{c}{(c+v)\lambda}\eta[1 - \frac{1}{\eta}] \qquad (15)$$

$$\delta N = \frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} \approx$$

$$\frac{2lbc\eta[1 - 1/\eta]}{[b-v][c-v]\lambda} - \frac{2lbc\eta[1 - 1/\eta]}{[b+v][c+v]\lambda} \approx$$

$$\frac{2l\eta[1 - 1/\eta]}{bc\lambda}[2vc + 2vb] \approx$$

$$\frac{4l\eta^2 v[1 - 1/\eta][1 + 1/\eta]}{\lambda c} \approx \frac{4l\eta^2 v}{\lambda c}[1 - \frac{1}{\eta^2}] \qquad (16)$$

Notice how these results were obtained without invoking "aether" drag, or relativistic velocity addition.

In the special relativistic analysis of this experiment, the velocity of light in the moving liquid as measured in the lab frame is no longer $b + v$, but is given by the relativistic velocity addition formula:

$$b' = \frac{b - v}{1 - \frac{vb}{c^2}} = \frac{b - v}{1 - \frac{v}{\eta c}} \qquad (17)$$

As a result, the path lengths derived in (14) become:

$$l_1 = \frac{2lb(1-\frac{v}{\eta c})}{b-v} \quad , \quad l_2 = \frac{2lb(1+\frac{v}{\eta c})}{b+v} \qquad (18)$$

The derivation of the total phase shift then becomes:

$$\frac{\phi_1 - \phi}{2\pi} = \frac{\delta\phi_1}{2\pi} = \frac{2lb(1-\frac{v}{\eta c})}{(b-v)} \cdot \frac{1}{\lambda_1}[\eta - 1]$$

$$= \frac{2lb(1-\frac{v}{\eta c})}{(b-v)} \cdot \frac{c}{(c-v)\lambda}\eta[1-\frac{1}{\eta}]$$

$$\frac{\phi_2 - \phi}{2\pi} = \frac{\delta\phi_2}{2\pi} = \frac{2lb(1+\frac{v}{\eta c})}{(b+v)} \cdot \frac{1}{\lambda_2}[\eta - 1]$$

$$= \frac{2lb(1+\frac{v}{\eta c})}{(b+v)} \cdot \frac{c}{(c+v)\lambda}\eta[1-\frac{1}{\eta}] \qquad (19)$$

$$\delta N = \frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} \approx$$

$$\frac{2l\eta[1-1/\eta]}{bc\lambda}[2vc + 2vb + 2\frac{vb}{\eta} + 2\frac{v^3}{\eta c}] \approx$$

$$\frac{4l\eta^2 v}{\lambda c}[1-1/\eta][1+1/\eta+1/\eta^2] \approx \frac{4l\eta^2 v}{\lambda c}[1-\frac{1}{\eta^3}] \qquad (20)$$

The two results, (16) and (20), differ in the exponent of the last h term. When Michelson and Morley performed the experiment, they obtained sixty one trials, using three different combinations of water velocity and tube length. The graph below shows the distribution of these results, normalized to a tube length of ten meters and a water velocity of one meter per second. The line marked RCM represents the value obtained from equation (16). The line marked SRT reflects the value obtained from (20). While there is a distribution of results, owing to experimental error, Michelson claimed an overall shift of 0.184 $\pm$ 0.02 fringe. This is completely consistent with (16), but eliminates the special relativistic result, with a value of 0.247, from consideration.

Summary

It is very difficult to find adequate tests between special relativity and other competing theories. Most theories overlap with SRT on a vast majority of the prediction made by each, yet are based on different underlying physical principles. Ultimately one must find a test that checks not only the results of the application of the mathematical theory, but also the underlying assumptions. The major conceptual difference between SRT and most competing theories is the idea of relative simultaneity—that distant events that are simultaneous for one observer will not be simultaneous for and observer in motion relative to the first. The relativistic velocity addition rule is a direct consequence of relativistic simultaneity, and the Fizeau experiment represents a direct test of the velocity addition formula. Regardless of what the correct theory is or may be, it is clear that SRT fails to give predictions consistent with results in this experiment—an experiment performed almost ten years before the development of SRT.

Four-vectors

Although the use of 4-vectors is not necessary for a full understanding of Special Relativity, they are a most powerful and useful tool for attacking many problems. A 4-vectors is just a 4-tuplet $A = (A_0, A_1, A_2, A_3)$ that transforms under a Lorentz Transformation in the same way as (cdt, dx, dy, dz) does. That is:

$A_0 = \gamma(A_0' + (v/c)A_1')$

$A_1 = \gamma(A_1' + (v/c)A_0')$

$A_2 = A_2'$

$A_3 = A_3'$

Lorentz transformations are very much like rotations in 4-dimensional spacetime. 4-vectors, then, generalize the concept of rotations in 3-space to rotations in 4-dimensions. Clearly, any constant multiple of (cdt, dx, dy, dz) is a 4-vector, but something like A = (cdt,mdx, dy, dz) (where m is just a constant) is not a 4-vector because the second component has to transform like mdxâÉá$A_1 = \gamma(A_1' + (v/c)A_0')$âÉá$\gamma((mdx') + vdt')$ from the definition of a 4-vector, but also like $mdx = m\gamma(dx' + (v/c)dt')$; these two expression are inconsistent. Thus we can transform a 4-vector either according to the 4- vector definition given above, or using what we know about how the dx_i transform to transform each A_i independently. There are only a few special vectors for which these two methods yield the same result. Several different 4-vectors are now discussed:

Velocity 4-vector

We can define a quantity $\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ which is called the proper time, and is invariant between frames. Dividing out original 4-vector ((cdt, dx, dx, dz)) by$d\tau$ gives:

$$V = \frac{1}{d\tau}(cdt, dx, dy, dz) = \gamma\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma c, \gamma \mathbf{v})$$

This arises because $\frac{dt}{d\tau} = \gamma$.

Energy-momentum 4-vector

If we multiply the velocity 4-vector by m we get:

$$P = mV = m(\gamma c, \gamma \mathbf{v}) = (\gamma \mathbf{mc}, \gamma \mathbf{mv}) = (\mathbf{E}/\mathbf{c}, \mathbf{p})$$

This is an extremely important 4-vector in Special Relativity.

Relation between momentum and kinetic energy

Sometimes it's desirable to express the kinetic energy of a particle in terms of the momentum. That's easy enough. Since $\mathbf{v} = \mathbf{p}/m$ and the kinetic energy $K = \frac{1}{2}mv^2$ so

$$K = \frac{1}{2}m(\frac{p}{m})^2 = \frac{p^2}{2m} \qquad (1.4)$$

Note that if a massive particle and a light particle have the same momentum, the light one will have a lot more kinetic energy. If a light particle and a heavy one have the same velocity, the heavy one has more kinetic energy.

KARPAGAM ACADEMY OF HIGHER EDUCATION					
DEPARTMENT OF PHYSICS					
III B.Sc., PHYSICS (2016-2019)					
CLASSICAL MECHANICS (16PHU504A)					
UNIT - IV					
QUESTIONS	**Choice1**	**Choice2**	**Choice3**	**Choice4**	**ANSWER**
When the forces acting on the particle vanishes, then the particle is said to be in	equilibrium	stable equlibrium	unstable	neutral equilibrium	equilibrium
Potential energy is minimum at stable equilibrium and ________at unstable	maximum	minimum	zero	infinity	maximum
In case of stable equilibrium the system undergoes bounded motion and in case of	same	unbounded	harmonic	distorted	unbounded
When a system at stable equilibrium is disturbed its potential energy increases and kinetic	increases	decreases	zero	constant	decreases
When a system at unstable equilibrium is disturbed its potential energy decreases and	increases	decreases	constant	neither increase nor decrease	increases
The example for stable equilibrium.	Bar pendulum at rest	compound pendulum at rest	simple pendulum at rest	pendulum in motion	Bar pendulum at rest
If a slight displacement of a system from its equilibrium results only in small	unstable	stable	neither stable nor unstable	neutral	stable
If a slight displacement of a system from its equilibrium results only in unbounded	unstable	stable	neither stable nor unstable	neutral	stable
The example for unstable equilibrium.	Rod standing on its one end	rod stretched on two ends	rod in motion	rod in simple harmonic motion	Rod standing on its one end
The two modes of motion involving a single frequency are called ________ modes	abnormal	normal	transverse	longitudinal	normal
The eigen frequency in case of oscillatory motion about the point of stable	imaginary	real	complex	whole number	real

The generalised co-ordinates each of them executing oscillations of one single	normal co-ordinates	genaral co-ordinates	spherical co-ordinates	polar co-ordinates	normal co-ordinates
Two pendula in parallel pendula oscillate in phase with frequency	w = (g/l)1/2	w = (g/l)1/3	w = (g/l)1/4	w = (g/l)	w = (g/l)
Two pendula in parallel pendula oscillate out of phase with frequency	w = (g/l +2k/m)	w = (g/l +2k/m)1/4	w = (g/l +2k/m)1/3	w = (g/l +2k/m)1/2	w = (g/l +2k/m)1/3
Triple pendulum is a	generate system	stable system	degenerate system	unstable system	unstable system
Triple pendulum is a degenerate system, since the two normal modes frequency	w1 = w2 = (g/l +2k/m)	w1 = w2 = (g/l +2k/m)1/3	w1 = w2 = (g/l +2k/m)1/4	w1 = w2 = (g/l +2k/m)1/2	w1 = w2 = (g/l +2k/m)1/3
Example for linear triatomic molecule is	HPO3	H2SO4	HNO3	CO2	HNO3
In case of linear triatomic molecule when w1 = 0 , the system undergoes	periodic motion	non-periodic motion	translatory motion	SHM motion	periodic motion
In case of linear triatomic molecule when w2 = (K/M)1/2 and	Oscillatory motion	translatory motion	periodic motion	SHM motion	Oscillatory motion
In case of linear triatomic molecule when ___________ the central atom does not	w = (K/M)1/2	w = (K/M)	w = (K/M)1/3	w = (K/M)1/4	w = (K/M)
In linear triatomic molecule when _____________ , the end atoms vibrate	w = { K/M(1+2 m/M)}	w = { K/M(1+2 m/M)}1/2	w = { K/M(1+2 m/M)}3	w = { K/M(1+2 m/M)}4	w = { K/M(1+2 m/M)}
The example for continuous system is	Continuous string	string stretched at one end	String stretched at two ends	String with load at one end	string stretched at one end
A continuous system has ________ number of normal modes of frequency.	Finite	infinite	Constant	Same	infinite
If the linear triatomic molecule is stretched symmetrically, the absorption band	Ultra-violet region	Infra-red region	Visible region	Microwave region	Visible region
A system of mutually interacting particles is called ____________	uncoupled system	Translatory system	Coupled system	harmonic system	uncoupled system
When the forces acting on a particle vanishes, the particle is said to be ________	Equilibrium	Stable equilibrium	unstable equilibrium	Neutral equilibrium	Stable equilibrium

The two modes of motion involving a single frequency are referred to as the _______	abnormal	normal	Damped	undamped	undamped
The system of two equal masses joined by identical springs to each other is called_______	Uncoupled	single coupled	Three-coupled	two-coupled	Uncoupled
A system of particles is said to be in stable equilibrium if all the particles _______	rest	periodic motion	damped motion	simple harmonic motion	damped motion
The system consists of two identical simple pendula, each of mass m, length l and coupled	series pendula	compound pendula	paralled pendula	complex pendula	complex pendula
All the other co-ordinates except one co-ordinate are zero for all times, then it corresponds	abnormal	standard	variable	normal	standard
If the motion for a given wl2 is completely oscillatory about the position of stable	imaginary	Real	complex	integer	imaginary
If the eigenfunctions is imaginary, then the motion is said to be ________ equilibrium	unstable	Stable	neutral	neither stable nor neutral	neither stable nor neutral
If the solution of equation of motion has one single frequency, then in such a case the	Cartesian	canonical	polar	normal	canonical
If the parallel pendula move in a vertical plane in equilibrium position, then the two	different	identical	relative to each other	Away from each other	relative to each other
In the two pendula it can vibrate as if they are independent i.e., there is no stretching or	rest	oscillate infinitely	action	neither action nor oscillate infinitely	neither action nor oscillate infinitely
In triple pendulum, if the system possesses two identical frequencies, then it is therefore	periodic	non-periodic	degenerate	harmonic	non-periodic
In linear triatomic molecule, the displacement of all the atoms are in the same direction and	unequal	equal	infinite	finite	unequal
The continuous string has infinite number of normal modes and _________________	vibrations	displacement	a & b together	frequencies	frequencies
A continuous string has a linear _________	momentum	volume density	mass density	specific density	volume density

The use of normal co-ordinates in the coupled system reduces it to one of a system of ____	dependent	harmonic	periodic	independent	periodic
The volume integral of the function of the Lagrangian functions within the braces	Hamiltonian	Lagrangian	linear	volume	Hamiltonian
Lagrangian density is a function of _______ and _____________ derivative of	space and time	angle and r	x and y co-ordiantes	y and z co-ordinates	x and y co-ordiantes
The system consists of two equal masses joined by identical springs to each other and to	damped	harmonic	periodic	undamped	harmonic
In case of two-coupled oscillators, the potential energy V of the system is the sum of	kinetic	potential	rest energy	a & b	a & b
The force tending to change any generalised co-ordinate depends on the _______ of	velocity	accelecration	displacement	momentum	displacement
If two pendula oscillate in phase, then the frequency of motion is	wl =Ög/l	wl =g/l	wl =1/2pÖg/l	wl =2pÖg/l	wl =Ög/l
In case of linear triatomic molecule there exists _______ bond between the central	Inelastic	covalent	Elastic	ionic	Elastic
The system consists of infinite chain of equal mass points spaced equally at a distance	Discontinus	continuous	harmonic	linear	continuous
The continuous system is a function of the continuous variables ____ and _____ to	w and t	x,y and z	r and w	x and t	x and t
In discrete system, the continuous variables changes only by_________	twice	thrice	unity	0	unity
The propagation velocity of the wave in continuous system is similar to that velocity	inelastic	elastic	damped	undamped	elastic
In linear triatomic molecule if the molecule is assymmetrically stretched, then __________	magnetic	quadrapole	oscillating dipole	both a & b	oscillating dipole
For small oscillation, the displacement of the particles are restricted to ___________	stable	periodic	non-periodic	small	small
The motion with imaginary frequency would give rise to an unbounded exponential rise	Uj	Vj	pj	qj	Uj

If the particle oscillates about the equilibrium point performing bound motion, then the	unstable	stable	neutral	neither neutral nor stable	stable
In the conservative force-field, generalised forces acting on each particle must __________	finite	infinite	vanish	a constant	vanish
The displacement of the generalised co-ordinates from their equilibrium value will be	Vj	wj	pj	Uj	Uj
If we transform set into another form of n equations, then it involves only a_________	Single	double	triple	more than three	Single

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UNIT V

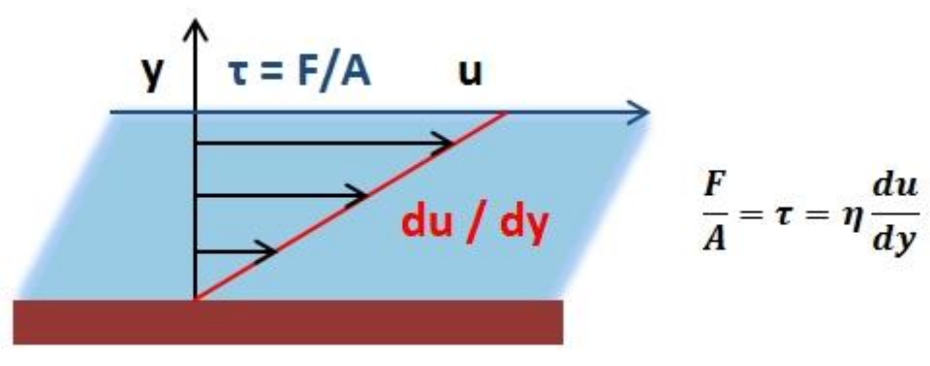
Fluid Dynamics: Density ρ and pressure P in a fluid, an element of fluid and its velocity, continuity equation and mass conservation, stream-lined motion, laminar flow, Poiseuille's equation for flow of a liquid through a pipe, Navier-Stokes equation, qualitative description of turbulence, Reynolds number.

Density ρ and pressure P in a fluid

In SI units, the unit of pressure is the Pascal (Pa), which is equal to a Newton / meter^2 (N/m^2). Other important units of pressure include the pound per square inch (psi) and the standard atmosphere (atm). The elementary mathematical expression for pressure is given by:

pressure=ForceArea=FApressure=ForceArea=FA

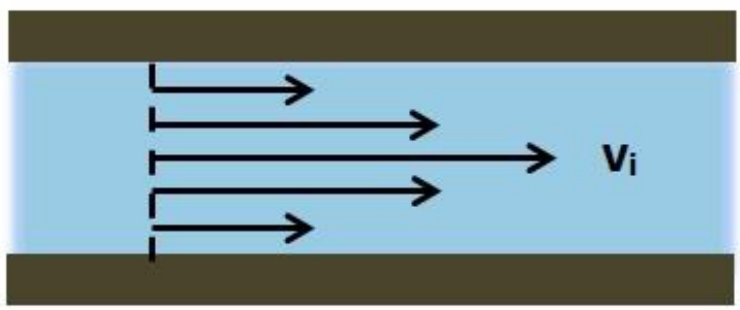
where p is pressure, F is the force acting perpendicular to the surface to which this force is applied, and A is the area of the surface. Any object that possesses weight, whether at rest or not, exerts a pressure upon the surface with which it is in contact. The magnitude of the pressure exerted by an object on a given surface is equal to its weight acting in the direction perpendicular to that surface, divided by the total surface area of contact between the object and the surface. shows the graphical representations and corresponding mathematical expressions for the case in which a force acts perpendicular to the surface of contact, as well as the case in which a force acts at angle θ relative to the surface.



Representation of Pressure: This image shows the graphical representations and corresponding mathematical expressions for the case in which a force acts perpendicular to the surface of contact, as well as the case in which a force acts at angle θ relative to the surface.

Pressure as a Function of Surface Area

Since pressure depends only on the force acting perpendicular to the surface upon which it is applied, only the force component perpendicular to the surface contributes to the pressure exerted by that force on that surface. Pressure can be increased by either increasing the force or by decreasing the area or can oppositely be decreased by either decreasing the force or increasing the area. illustrates this concept. A rectangular block weighing 1000 N is first placed horizontally. It has an area of contact (with the surface upon which it is resting) of 0.1 m^2, thus exerting a pressure of 1,000 Pa on that surface. That same block in a different configuration (also in Figure 2), in which the block is placed vertically, has an area of contact with the surface upon which it is resting of 0.01 m^2, thus exerting a pressure of 10,000 Pa—10 times larger than the first configuration due to a decrease in the surface area by a factor of 10.



Pressure as a Function of Surface Area: Pressure can be increased by either increasing the force or by decreasing the area or can oppositely be decreased by either decreasing the force or increasing the area.

A good illustration of this is the reason a sharp knife is far more effective for cutting than a blunt knife. The same force applied by a sharp knife with a smaller area of contact will exert a much greater pressure than a blunt knife having a considerably larger area of contact. Similarly, a person standing on one leg on a trampoline causes a greater displacement of the trampoline than that same person standing on the same trampoline using two legs—not because the individual exerts a larger force when standing on one leg, but because the area upon which this force is exerted is decreased, thus increasing the pressure on the trampoline. Alternatively, an object having a weight larger than another object of the same dimensionality and area of contact with a given surface will exert a greater pressure on that surface due to an increase in force. Finally, when considering a given force of constant magnitude acting on a constant area of a given surface, the pressure exerted by that force on that surface will be greater the larger the angle of that force as it acts upon the surface, reaching a maximum when that force acts perpendicular to the surface.

Liquids and Gases: Fluids

Just as a solid exerts a pressure on a surface upon which it is in contact, liquids and gases likewise exert pressures on surfaces and objects upon which they are in contact with. The pressure exerted by an ideal gas on a closed container in which it is confined is best analyzed on a molecular level. Gas molecules in a gas container move in a random manner throughout the volume of the container, exerting a force on the container walls upon collision. Taking the overall average force of all the collisions of the gas molecules confined within the container over a unit time allows for a proper measurement of the effective force of the gas molecules on the container walls. Given that the container acts as a confining surface for this net force, the gas molecules exert a pressure on the container. For such an ideal gas confined within a rigid container, the pressure exerted by the gas molecules can be calculated using the ideal gas law:

p=nRTVp=nRTV

where n is the number of gas molecules, R is the ideal gas constant (R = 8.314 J mol^{-1} K^{-1}), T is the temperature of the gas, and V is the volume of the container.

The pressure exerted by the gas can be increased by: increasing the number of collisions of gas molecules per unit time by increasing the number of gas molecules; increasing the kinetic energy of the gas by increasing the temperature; or decreasing the volume of the container. offers a representation of the ideal gas law, as well as the effect of varying the equation parameters on the gas pressure. Another common type of pressure is that exerted by a static liquid or hydrostatic pressure. Hydrostatic pressure is most easily addressed by treating the liquid as a continuous distribution of matter, and may be considered a measure of energy per unit volume or energy density.

<u>An element of fluid and its velocity</u>

Flow Rate

Volumetric flow rate is defined as

Q=v∗aQ,

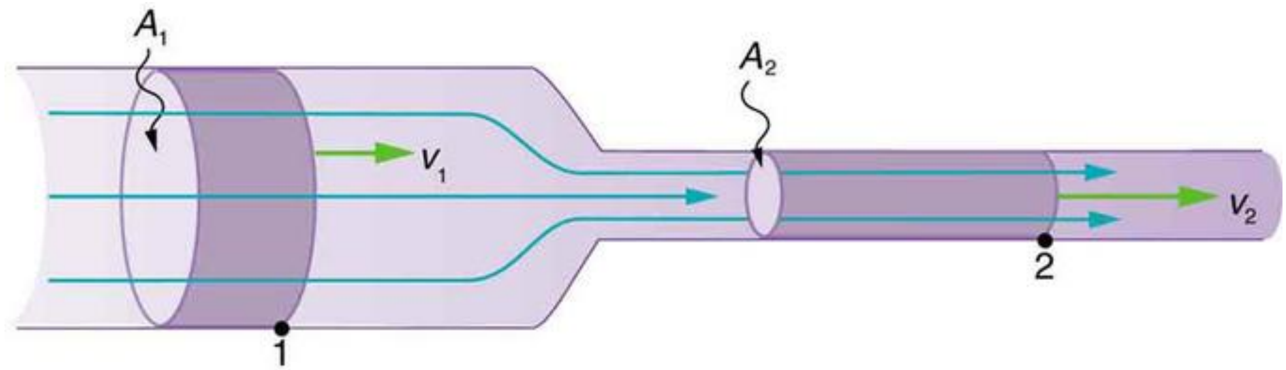
where Q is the flow rate, v is the velocity of the fluid, and a is the area of the cross section of the space the fluid is moving through. Volumetric flow rate can also be found with

Q=VtQ

where Q is the flow rate, V is the Volume of fluid, and t is elapsed time.

Continuity

The equation of continuity works under the assumption that the flow in will equal the flow out. This can be useful to solve for many properties of the fluid and its motion:



Flow in = Flow out: Using the known properties of a fluid in one condition, we can use the continuity equation to solve for the properties of the same fluid under other conditions.

Q1=Q2

This can be expressed in many ways, for example: A1∗v1=A2∗v2. The equation of continuity applies to any incompressible fluid. Since the fluid cannot be compressed, the amount of fluid which flows into a surface must equal the amount flowing out of the surface.

Continuity equation and mass conservation

When a fluid is in motion, it must move in such a way that mass is conserved. To see how mass conservation places restrictions on the velocity field, consider the steady flow of fluid through a duct (that is, the inlet and outlet flows do not vary with time). The inflow and outflow are one-dimensional, so that the velocity V and density \rho are constant over the area A (figure 14).

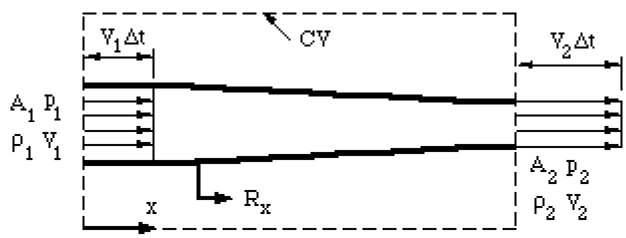


Figure 14. One-dimensional duct showing control volume.

Now we apply the principle of mass conservation. Since there is no flow through the side walls of the duct, what mass comes in over A_1 goes out of A_2, (the flow is steady so that there is no mass accumulation). Over a short time interval \Delta t,

$$\text{volume flow in over } A_1 = A_1 V_1 \Delta t$$
$$\text{volume flow out over } A_2 = A_2 V_2 \Delta t$$

Therefore

$$\text{mass in over } A = \rho A_1 V_1 \Delta t$$
$$\text{mass out over } A = \rho A_2 V_2 \Delta t$$

$$\text{So:} \quad \boxed{\rho A_1 V_1 = \rho A_2 V_2}$$

This is a statement of the principle of mass conservation for a steady, one-dimensional flow, with one inlet and one outlet. This equation is called the continuity equation for steady one-dimensional flow. For a steady flow through a control volume with many inlets and outlets, the net mass flow must be zero, where inflows are negative and outflows are positive.

Streamlines and Streamtubes

A streamline is a line that is tangential to the instantaneous velocity direction (velocity is a vector, and it has a magnitude and a direction). To visualize this in a flow, we could imagine the motion of a small marked element of fluid. For example, we could mark a drop of water with fluorescent dye and illuminate it using a laser so that it fluoresces. If we took a short exposure photograph as the drop moves according to the local velocity field (where the exposure needs to be short compared to the time it takes for the velocity to change appreciably), we would see a short streak, with a length V \Delta t, and with a direction tangential to the instantaneous velocity direction. If we mark many drops of water in this way, the streamlines in the flow will become visible. Since the velocity at any point in the flow has a single value (the flow cannot go in more than one direction at the same time), streamlines cannot cross. except at points where the velocity magnitude is zero, such as at a stagnation point.

There are other ways to make the flow visible. For example, we can trace out the path followed by our fluorescent drop using a long-exposure photograph. This line is called a pathline, and it is similar to what you see when you take a long-exposure photograph of car lights on a freeway at night. It is possible for pathlines to cross, as you can imagine from the freeway analogy: as a car changes lanes, the pathline traced out by its lights might cross another pathline traced out by an adjoining vehicle at a different time.

Another way to visualize flow patterns is by streaklines. A streakline is the line traced out by all the particles that passed through a particular point at some earlier time. For instance, if we issued fluorescent dye continuously from a fixed point, the dye makes up a streakline as it passes downstream. To continue the freeway analogy, it is the line made up of the lights on all the vehicles that passed through the same toll booth. If they all follow the same path (a steady flow), a single line results, but if they follow different paths (unsteady flow), it is possible for the line to cross over itself. In unsteady flow, streamlines, pathlines and streaklines are all different, but in steady flow, streamlines, pathlines and streaklines are identical.

Continuity Equation (Mass conservation):

Mass Conservation –

In the previous chapter we learned that Gauss' Theorem allows us to express surface area integrals in terms of the volume of the object of interest. In the notation of figure 1,

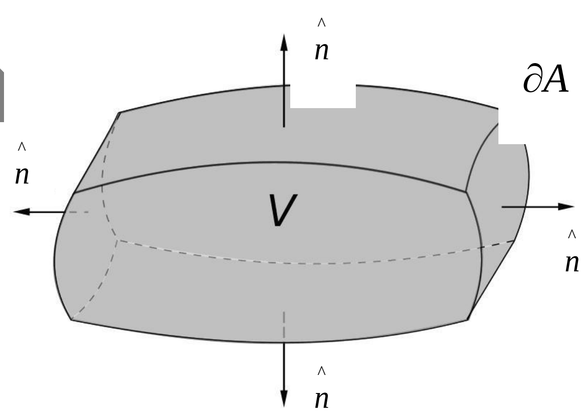


Figure 1. Diagram showing an arbitrary fixed volume V and associated surface area A with a unit vector $\hat{n}$ pointing outward normal to the surface. [1]

for a <u>fixed</u> volume element *V*, enclosed by a surface ∂A, Gauss' theorem is expressed as

$$\int_V \left(\nabla \cdot \vec{u}\right) dV = \oint_{\partial A} \vec{u} \cdot \hat{n}\, dA \qquad (1)$$

We will now see the usefulness of equation (1) by using it to derive conservation of mass.

For a fluid with density field, ρ, defined over a <u>fixed</u> volume, *V*, the change in mass, *M*, with respect to time is represented as $\frac{dM}{dt} = \frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$. Conservation of mass states that matter can neither be created or destroyed so

any increase or decrease in mass must be due to the *flux* of matter through the surface bounding the volume *V*. The density flux through the surface *A* bounding the volume V is defined as $\oint_{\partial A} \vec{u}\rho \cdot \hat{n}\, dA$. Now there is an ambiguity about whether the flux integral represents the outflow or inflow of mass through the surface. Utilizing the standard convention that the normal vector, $\hat{n}$, points outward on the closed surface ∂A as shown in figure 1, we can see that $-\oint_A \vec{u}\rho \cdot \hat{n}\, dA$ represents the outflow of mass through the surface. Thus conservation of mass shows us that

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t} dV = -\oint_{\partial A} \vec{u}\rho \cdot \hat{n}\, dA \qquad (3)$$

Using Gauss' theorem, we can write the final term in equation (3) as a volume integral as

$-\oint_A \vec{u}\rho \cdot \hat{n}\, dA = -\int_V \nabla \cdot \left(\vec{u}\rho\right) dV$ and equation (3) can then be written as:

$$\frac{dM}{dt} + \int_V \nabla \cdot \left(\vec{u}\rho\right) dV = \int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot \left(\vec{u}\rho\right) dV = \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\vec{u}\rho\right)\right) dV = 0$$

There are two possibilities for $\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\vec{u}\rho\right)\right) dV = 0$

I. The first is that there exists a unique boundary, shape or symmetry leading to the integral being zero. As an example of these unique symmetries, we notice that

$$\int_0^{2\pi} \sin(x)dx = 0$$

We know the above integral is zero because we are adding up an equal positive area to an equal negative area of the sine curve. This possibility that there is unique boundary leading to equation (5) being 0 is too restrictive to our analysis since we wish for our result to be true for any arbitrary volume or shape. This leads to our second possibility.

II. The second option is that the *integrand itself is equal to zero for the entire domain*.

This might seems like a trivial possibility but this leads to the exact result we are looking for:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{u}\right) = 0 \qquad (4)$$

Equation (4) is called the continuity equation and is the differential equation form of conservation of mass. Given the definition of the material derivative of the density field as $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho$, equation (4) can be expressed in the alternate form as

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u} = 0 \qquad (5)$$

Equation (5) shows that the fractional rate of change of the density (or volume) of a fluid element fluid is related to the divergence of the flow field.

Conditions under which incompressible flow is valid:

If variations in density are small compared to the background density the fluid is said to **be incompressible**. Equation (4) then takes the simple form:

$$\nabla \cdot \vec{u} = 0 \qquad (6)$$

The flow field is also said to be *solenoidal* under these circumstances. For most applications in the ocean and many in the atmosphere it is safe to assume that the fluid medium is approximately incompressible. To formally examine the necessary conditions in which it is safe to assume a medium is solenoidal; we need to perform dimensional analysis on equation (5). For a flow field scale, *U* which varies slightly over a length scale *L*, the requirement for incompressibility is

$$\left|\frac{1}{\rho}\frac{D\rho}{Dt}\right| << \left|\nabla\cdot\vec{u}\right| = \frac{U}{L}.$$

To gain further insight into these necessary conditions we must assume an equation of state. Now we are going to formally define the fluid as incompressible provided that the density does not depend on the pressure of the fluid medium. The details of the analysis will be seen next semester in SO414 but the result is still of interest here. What we find is that the medium can be considered incompressible provided that

$$\frac{U^2}{c^2} << 1 \qquad (7)$$

Where c is the sound speed of the fluid medium and U is an approximation of the fluid flow. We can see from equation (7) that most examples that we consider in both the ocean *and* atmosphere allows us to use the incompressibility requirement.

POISEUILLES EQUATION FOR FLOW OF LIQUID THROUGH A PIPE

The Poiseuille's law states that the flow of liquid depends on following factors like the pressure gradient (**ΔP**), the length of the narrow tube (L) of radius (r) and the viscosity of the fluid **(η)** alongwith relationship among them.

The entire relation or the Poiseuille's Law formula is given by,

Q = ΔPπr4 / 8ηl

Wherein,

The Pressure Gradient (ΔP) : Shows the difference in the pressure between the two ends of the tube, determined by the fact any fluid will always flow from high pressure (p1) to low pressure region(p2) and the flow rate is determined by the pressure gradient (P1 – P2)

Radius of tube: The liquid flow varies directly with the radius to the power 4.

Viscosity (η): The flow of the fluid varies inversely with the viscosity of the fluid and as the viscosity of the fluid increases, the flow decreases vice versa.

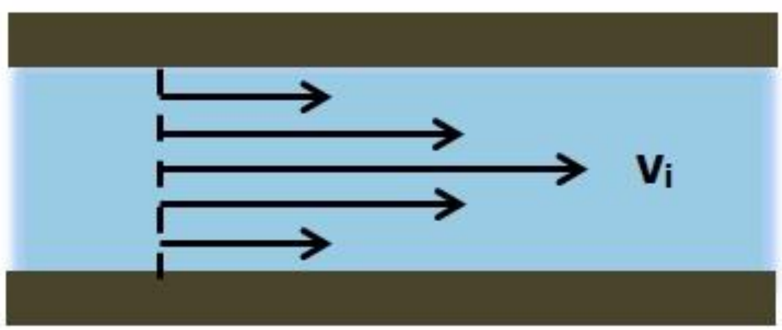
Length of the Tube (L): The liquid flow is inversely proportional to the length of the tube, therefore longer the tube, greater is the resistance to the flow.

Resistance(R): The resistance is described by 8Ln / πr4 and therefore the Poiseuille's law becomes

Q= (ΔP) R

Laminar Flow

Laminar flow consists of a regular-flow pattern with constant-flow velocity throughout the fluid volume and is much easier to analyze than turbulent flow.



Relative Magnitudes of Velocity Vectors: Laminar fluid flow in a circular pipe at the same direction.

Laminar flow is often encountered in common hydraulic systems, such as where fluid flow is through an enclosed, rigid pipe; the fluid is incompressible, has constant viscosity, and the Reynolds number is below this lower critical threshold value. It is characterized by the flow of a fluid in parallel layers, in which there is no disruption or interaction between the different layers, and in which each layer flows at a different velocity along the same direction. The variation in velocity between adjacent parallel layers is due to the viscosity of the fluid and resulting shear forces.

This figure (see) gives a representation of the relative magnitudes of the velocity vectors of each of these layers for laminar fluid flow through a circular pipe, in a direction parallel to the pipe axis.

$$\Delta p = \frac{8\eta Q \Delta x}{\pi r^4}$$

Poiseuille's Equation: Can be used to determine the pressure drop of a constant viscosity fluid exhibiting laminar flow through a rigid pipe.

Considering laminar flow of a constant density, incompressible fluid such as for a Newtonian fluid traveling in a pipe, with a Reynolds number below the upper limit level for fully laminar flow, the pressure difference between two points along the pipe can be found from the volumetric flow rate, or vice versa. For such a system with a pipe radius of r, fluid viscosity η, distance between the two points along the pipe $\Delta x = x_2 - x_1$, and the volumetric flow rate Q, of the fluid, the pressure difference between the two points along the pipe Δp is given by Poiseuille's equation (see).

This equation is valid for laminar flow of incompressible fluids only, and may be used to determine a number of properties in the hydraulic system, if the others are known or can be measured. In practice, Poiseuille's equation holds for most systems involving laminar flow of a fluid, except at regions where features disrupting laminar flow, such as at the ends of a pipe, are present.

Poiseuille's equation as given in this example is analogous to Ohm 's equation for determining the resistance in an electronic circuit and is of great practical use in hydraulic-circuit analysis.

$$R_e = \frac{V}{I} \quad \rightarrow \quad \begin{cases} R_h \rightarrow R_e \\ \Delta p \rightarrow V \\ I \rightarrow Q \end{cases} \quad \rightarrow \quad \Delta p = \frac{8\eta Q \Delta x}{\pi r^4} \quad \rightarrow \quad R_h = \frac{\Delta p}{Q} = \frac{8\eta \Delta x}{\pi r^4}$$

Poiseuille's Equation: Analogous to Ohm's Law Analogy

Turbulent Flow:

Turbulent flow is a type of fluid (gas or liquid) flow in which the fluid undergoes irregular fluctuations, or mixing, in contrast to laminar flow, in which the fluid moves in smooth paths or layers. In turbulent flow the speed of the fluid at a point is continuously undergoing changes in both magnitude and direction. The flow of wind and rivers is generally turbulent in this sense, even if the currents are gentle. The air or water swirls and eddies while its overall bulk moves along a specific direction.

Most kinds of fluid flow are turbulent, except for laminar flow at the leading edge of solids moving relative to fluids or extremely close to solid surfaces, such as the inside wall of a pipe, or in cases of fluids of high viscosity (relatively great sluggishness) flowing slowly through small channels. Common examples of turbulent flow are blood flow in arteries, oil transport in pipelines, lava flow, atmosphere and ocean currents, the flow through pumps and turbines, and the flow in boat wakes and around aircraft-wing tips.

Navier-Stokes equation

In **fluid dynamics**, the **Navier-Stokes equations** are equations, that describe the **three-dimensional motion** of **viscous fluid** substances. These equations are named after Claude-Louis Navier (1785-1836) and George Gabriel Stokes (1819-1903). In situations in which there are no strong temperature gradients in the fluid, these equations provide a **very good approximation of reality**.

The **Navier-Stokes equations consists** of a time-dependent continuity equation forconservation of mass, three time-dependent conservation of **momentum equations** and a time-dependent conservation of energy equation. There are four independent variables in the problem, the x, y, and z spatial coordinates of some domain, and the time t.

Navier-Stokes Equations

Continuity Equation

$$\nabla \cdot \vec{V} = 0$$

Momentum Equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Total derivative

=

$$\rho[\frac{\partial V}{\partial t} + (V.\nabla)V]$$

Change of velocity with time

Convective term

Pressure gradient

Fluid flows in the direction of largest change in pressure.

Body force term

External forces, that act on the fluid (gravitational force or electromegnetic).

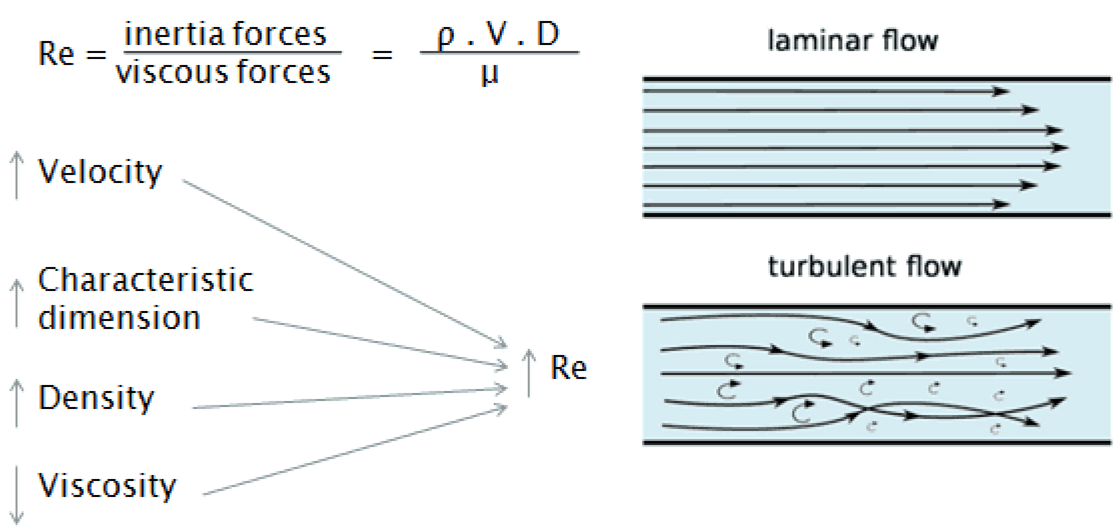
Diffusion term

For a Newtonian fluid, viscosity operates as a diffusion of momentum.

As can be seen, the **Navier-Stokes equations** are second-order nonlinear partial differential equations, their solutions have been found to a variety of interesting viscous flow problems. They may be used to model the weather, ocean currents, air flow around an airfoil and water flow in a pipe or in a reactor. The **Navier–Stokes equations** in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of nuclear reactors and many other things.

Reynolds number

The Reynolds number is the ratio of **inertial forces**to **viscous forces** and is a convenient parameter for predicting if a flow condition will be **laminar or turbulent**. It can be interpreted that when the **viscous forces** are dominant (slow flow, low Re) they are sufficient enough to keep all the fluid particles in line, then the flow is laminar. Even very low Re indicates viscous creeping motion, where inertia effects are negligible. When the **inertial forces dominate** over the viscous forces (when the fluid is flowing faster and Re is larger) then the flow is turbulent.



It is a dimensionless number comprised of the physical characteristics of the flow. An increasing Reynolds number indicates an increasing turbulence of flow.

It is defined as:

$$Re_D = \frac{\rho V D}{\mu} = \frac{V D}{\nu}$$

where:

V is the flow velocity,

D is a **characteristic linear dimension**, (travelled length of the fluid; hydraulic diameter etc.)

ρ fluid density (kg/m^3),

μ dynamic viscosity (Pa.s),

ν kinematic viscosity (m^2/s); $\nu = \mu / \rho$.

KARPAGAM ACADEMY OF HIGHER EDUCATION					
DEPARTMENT OF PHYSICS					
III B.Sc., PHYSICS (2016-2019)					
CLASSICAL MECHANICS (16PHU504A)					
UNIT - IV					
QUESTIONS	**Choice1**	**Choice2**	**Choice3**	**Choice4**	**ANSWER**
The mass of 70 kg man moving in car at 66kmh is	70 kg	100 kg	infinite	zero	70 kg
Special theory of relativity treats problems involving	inertial frame of reference	non-inertial frame of reference	non-accelerated frame of reference	accelerated frame of reference	inertial frame of reference
According to special theory of relativity which one is not an absolute quantity	time	mass	height	both a and b	both a and b
Length contraction happens only	perpendicular to direction of motion	along the direction of motion	parallel to direction of motion	both a and b	along the direction of motion
Conversion of solar energy into carbohydrates and starch by leaf of a plant is an example for	energy into mass	mass in to energy	momentum into velocity	velocity into momentum	energy into mass
A reference frame attached to the earth:	is an inertial frame by definition	is an inertial frame because Newton’s laws are applicable in the frame	Cannot be an inertial frame because the earth is revolving round the sun	Cannot be an inertial frame because the earth is rotating about its own axis.	is an inertial frame by definition

Michelson and Morley experiment showed that	Newtonian mechanics is correct for all low and high velocities	There is an absolute ether frame	There is no absolute ether frame, but all frames are relative	Velocity of light is relative in all cases.	There is no absolute ether frame, but all frames are relative
Two photons approach each other, their relative velocity will be	c/2	Zero (	c/8	c	c
An inertial frame is	Accelerated	decelarated	Moving with uniform velocity or at rest.	May be accelerated, decelerated or moving with constant velocity	Moving with uniform velocity or at rest.
Michelson-Morley experiment proved that	speed of light is relative	there is no preferred frame like ether	earth is an inertial frame.	earth is a non-inertial frame	there is no preferred frame like ether
All the inertial frames are equivalent" this statement is called the principle of -----------	relative motion	equivalence	inertia	Correspondence.	relative motion
Special theory of relativity deals with the events in the frames of reference which move with constant-----------------	speed	velocity	acceleration	momentum.	velocity
Michelson-Morley experiment to detect the presence of either is based on the phenomenon of:	interference	diffraction	polarization	dispersion	interference

According to relativity, the length of a rod in motion:	is same as its rest length	is more than its rest length	is less than its rest length	may be more or less than or equal to rest length depending on the speed of rod.	is less than its rest length
If v = c, the length of a rod in motion is:	zero	equal to proper length	less than proper length	more than proper length.	zero
According to special theory of relativity:	speed of light is relative	speed of light is same in all inertial frames	time is relative	mass is relative	speed of light is same in all inertial frames
James travels at high speed from the Earth to the star Alpha Centauri, four light years away. In James's frame	the trip takes more time than it does in the Earth's frame.	James travels to Alpha Centauri over a length that is shorter than four light years.	clocks on Earth and on Alpha Centauri are synchronized.	Alpha Centauri travels to James over a length that is shorter than four light years.	Alpha Centauri travels to James over a length that is shorter than four light years.
Relativity mechanics is applicable for a particle which is moving with a velocityà	Greater than that of light	Less than that of light	Comparable to that of light	equal to velocity of light	Comparable to that of light

The relativistic measurement depends uponà	The state of motion of the observer as well as upon the quality that is being measured.	The state of motion of the observer only	The quantity that is being measured	absolute motion	The quantity that is being measured
A frame which is moving with zero acceleration is called	Non-inertial frame	Inertial frame	rest frame	decelerated frame	Inertial frame
When we specific the place of occurrence of a phenomenon as well as the time of occurrence it is considered as	a point	an event	an incident	an accident	an event
Newton’s law’s remain unchanged or invariant	Under Galilean transformation	under lorentz transformation	cartesean transformation	new transformation	Under Galilean transformation
The laws of mechanics in all initial frame of reference are	same	different	none	variable	same
The acceleration of a particle under Galilean transformation is	invariant	non-variant	none	variable	invariant
Michelson-Morley experiment proves	The existence of ether medium	The non-existence of ether medium (i.e. absolute rest frame)	None	Ether pervades	The non-existence of ether medium (i.e. absolute rest frame)
Michelson-Morley experiment proves that	The speed of light in free space in invariant	The speed of light is changing	None	variable light velocity	The speed of light in free space in invariant
The special theory of relativity was proposed by	Einstein	newton	eigen	galileo	Einstein
The mass energy relation was proposed by	Newton	Einstein	Kepler	Michelson	Einstein

The Lorentz transformation will converted to Galilean transformation when the relative velocity v between two inertial frames will satisfy the condition	v>>c	v=c	v<<c	v=0	v<<c
the length of an object is maximum in a reference frame in which it is	at rest	in motion	neither rest nor in motion	varying speed	at rest
the length of a rod in uniform motion relative to an observer	appears to be shortened when it at rest w.r.t. to the observer	appears to be lengthened when it is at rest w.r.t. to the observer	equal to aboslute length	invariant length	appears to be shortened when it at rest w.r.t. to the observer
The time interval between two event in a reference in a reference frame which is in motion is	Maximum	minimum	zero	varying speed	Maximum
A moving clock	Runs slower than a stationary identical clock	Runs than a stationary identical clock	neither slow nor fast	very fast	Runs slower than a stationary identical clock
If the velocity of a moving particle is comparable to velocity of light then the mass of the moving object is	Greater than when it is rest	Smaller than when it is at rest	Equal	very smaller	Greater than when it is rest
Einistein's mass energy equation E=mc2 implies that	Energy disappears to reappear as mass	Mass disappears to reappear as energy	All the above statements are correct except d	nothing can be done	All the above statements are correct except d
How fast a particle must travel so that its mass becomes twice its rest mass?	0.5 c	2 c	0.866 c	0.9c	0.866 c
Relative velocity of two particles moving with velocity of light of light in opposite direction is	0	2c	c	3c	c
For a photon particle which is moving with a velocity of light, the rest mass is	0	1	2	3	0

The fictitious force, which acts on particle in motion relative to a rotating frame of reference is called	Coriolis force	Newtonian force	Pseudo force	centripetal force	Coriolis force
If the particle is at rest relative to the rotating frame of reference the coriolis force is	0	1	10	2	0
When the particle is at a non-rotating of reference the Coriolis force	1	0	2	3	0
The Coriolis acceleration on a freely falling body under the action of gravitational force is	Directed towards the east	Directed towards the west	directed towards north	directed towards south	Directed towards the east

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