KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be Univerty)
(Established Under Section 3 of UGC Act 1956)
(For the candidates admitted from 2017 onwards)
Coimbatore - 641021.

DEPARTMENT OF PHYSICS

SEMESTER : I
SUBJECT CODE: 17PHP103 **CLASS : I M.Sc. PHYSICS**
SUBJECT : CLASSICAL MECHANICS AND RELATIVITY

Scope: Study of Classical Mechanics gives an idea about how classical physics deal with matter and energy. Even though classical physics cannot explain many observed phenomena in the case of microparticles and relativistic velocities, it is still valid in the case of macro objects at non-relativistic velocities.

Objective: The objective of this course is to give an insight into the classical methods of physics.

UNIT - I

Conservation laws: Mechanics of a system of particles – Conservation laws: linear momentum, angular momentum, energy – Constraints, Degrees of freedom – Generalised co-ordinates – Generalized notations – Brachistocrone problems – Atwood's machine.

Hamilton's variational principle – Lagrange's equation of motion from Hamilton's principle, D'Alembert's principle – Applications of Lagrange's equation of motion – particle moving under a central force – particle moving on the surface of earth– Superiority of Langrange's approach over Newtonian's approach.

UNIT – II

Phase space: Hamiltonian – Hamilton's canonical equations of motion – Physical significance of H – Advantage of Hamiltonian approach – Hamilton's canonical equation of motion in different coordinate systems – Hamilton-Jacobi method – Kepler's problem solution by Hamilton-Jacobi method – Action and angle variables – Solution of Harmonic oscillator by action angle variable method – canonical or contact transformation – Condition for a transformation to be canonical.

UNIT – III

General features of central force motion : General features of orbits – Centre of mass and laboratory coordinates – Virial theorem – Stable and unstable equilibrium – Properties of T, V and ω for small oscillations .

Generalized coordinates for rigid body motion : Euler's angles – Angular velocity, momentum of rigid body – moment and products of inertia – Principal axis transformation – rotational kinetic energy of a rigid body – Moment of inertia of a rigid body – motion of a symmetric top under action of gravity.

UNIT - IV

Special Theory of Relativity: Introduction – Galilean transformation and invariance of Newton's laws of motion – Non variance of Maxwell's equations – Michelson Morley experiment and explanation of the null result.

Concept of inertial frame – Postulates of special theory – simultaneity – Lorentz transformation along one of the axes – length contraction – time dilatation and velocity addition theorem – Fizeau's experiment – Four vectors – Relativistic dynamics – Variation of mass with velocity – Energy momentum relationship.

UNIT - V

General theory of Relativity: Introduction – Limitation of special theory of relativity and need for a relativity theory in non-inertial frames of reference Limitation of special theory of relativity and need for a relativity theory in non-inertial frames of reference. Concept of gravitational and inertial mass and the basic postulate of GTR, gravitation & acceleration and their relation to non-inertial frames of reference – principle of equivalence of principle of general co-variance – Minkowski space and Lorentz transformation.

TEXT BOOKS:

1. Goldstein.H.A. 2000, Classical Mechanics, 2nd Edition, Wesley Publishing Company,

London.

2. Gupta. S. L., V.Kumar and H.V.Sharma, 2008, Classical Mechanics, 19th Edition, Pragati Prakashan, Meerut.
3. Banerji Sriranjan and Asit Banerjee, 2nd Edition 2013, The Special Theory of Relativity, Printice-Hall of India, New Delhi
4. Aruldhas G.,1st edition, 2008, Classical Mechanics, Printice Hall of India, New Delhi

REFERENCES:

1. Sardesai D.L., 1st edition, 2004, A Primer of Special Relativity, New Age International Publishers, New Delhi
2. Hartle B. James, 1st edition ,2009, Gravity, An Introduction to Einstein's General Relativity, Dorling Kindersley (India) Pvt. Ltd., Delhi

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S.No	Lecture Duration (Hr)	Topics to be covered	Support material /Page no.
		UNIT - I	
1	1	Mechanics of a system of particles, Conservation laws: linear momentum	T2(5-7)
2	1	Angular momentum, energy , Constraints, Degrees of freedom	T2(8-17)
3	1	Generalised co-ordinates – Generalized notations	T2(17-26)
4	1	Brachistocrone problems – Atwood's machine	T2(34-36) T2(66-67)
5	1	Hamilton's variational principle . Lagrange's equation of motion from Hamilton's principle	T2(37-39) T2(39-41)
6	1	D'Alembert's principle Application of Lagrange equation of motion	T2(41-42) T2(60-62)
7	1	particle moving under a central force, particle moving on the surface of earth	T2(62), T(60-61)
8	1	Superiority of Langrange's approach over Newtonian's approach	T2(83-84)
9	1	Revision	
		Total no.of Hours planned for unit –I	9

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S.No	Lecture Duration (Hr)	Topics to be covered	Support material/Page no.
		UNIT - II	
1	1	Hamiltonian – Hamilton's canonical equations of motion	T2(113-114)
2	1	Physical significance of H Advantage of Hamiltonian approach	T2(114-116)
3	1	Hamilton's Canonical equation of motion in different systems	T2(117-120)
4	1	Hamilton Jacobi method	T2(153)
5	1	Kepler's problem solution by Hamilton Jacobi method	T2(163-165)
6	1	Action and angle variables Solution of Harmonic oscillator by action angle variable method	T2(171-173) T2(173-174)
7	1	Canonical or contact transformation Condition for a transformation to be canonical	T2(137-139) T2-147
8	1	Revision	
Total no.of Hours planned for unit –II			8

S.No	Lecture Duration (Hr)	Topics to be covered	Support material Page no.
		UNIT - III	
1	1	General features of orbits Centre of mass and laboratory coordinates	T1(665-672) T1(673-676)
2	1	Virial theorem	T1(680-683)
3	1	Stable and unstable equilibrium	T1(689-697)
4	1	Properties of T, V and ω for small oscillations	T1(689-697) T1 (701-702)
5	1	Euler's angles Angular velocity, momentum of rigid body	T1(709) T1(712-714)
6	1	moment and products of inertia	T2(304)
7	1	Principal axis transformation	T2(309)
8	1	rotational kinetic energy of a rigid body, Moment of inertia of a rigid body	T2(310)
9	1	motion of a symmetric top under action of gravity	T2(312)
10	1	Revision	
Total no.of Hours planned for unit –III			10

S.No	Lecture Duration (Hr)	Topics to be covered	Support material /Page no.
		UNIT - IV	
1	1	Introduction – Galilean transformation Invariance of Newton's laws of motion	T2(337-338) T2(420-421)
2	1	Non variance of Maxwell's equations Michelson Morley experiment and explanation of the null result	T2-(422-424) T2(435-437)
3	1	Concept of inertial frame	T2(440-442)
4	1	Postulates of special theory simultaneity	T2(451-452)
5	1	Lorentz transformation along one of the axes	T2(453-454)
6	1	length contraction – time dilatation and velocity addition theorem	T2(455-457)
7	1	Fizeau's experiment – Four vectors –	T2(457-458)
8	1	Relativistic dynamics -Variation of mass with velocity	T2(520-525)
9	1	Energy momentum relationship.	T2(526-530)
10	1	Revision	
Total no.of Hours planned for unit –IV			10

S.No	Lecture Duration (Hr)	Topics to be covered	Support material Page no.
		UNIT - V	
1	1	Introduction – Limitation of special theory of relativity	R1(95-96)
2	1	Need for a relativity theory in non-inertial frames of reference	R1(97-99)
3	1	Concept of gravitational and inertial mass	R1 (99-102)
4	1	The basic postulate of GTR	R1(103-104)
5	1	The basic postulate of gravitation	R1(105-106)
6	1	The basic postulate of acceleration	R1(107-108)
7	1	Relation to non-inertial frames of reference	R1(109-110)
8	1	principle of equivalence	R1(111-114)
1	1	principle of general co-variance Minkowski space and Lorentz transformation	R1(114-116) R1(117-120)
10	1	Revision	
11	1	Five years question discussion	
Total no.of Hours planned for unit –V			11

TEXT BOOKS:

1. Goldstein.H.A. 2000, Classical Mechanics, 2nd Edition, Wesley Publishing Company, London.
2. Gupta. S. L., V.Kumar and H.V.Sharma, 2008, Classical Mechanics, 19th Edition, Pragati Prakashan, Meerut.
3. Banerji Sriranjan and Asit Banerjee, 2nd Edition 2013, The Special Theory of Relativity, Printice-Hall of India, New Delhi
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UNIT-I

SYLLABUS

Conservation laws: Mechanics of a system of particles – Conservation laws: linear momentum, angular momentum, energy – Constraints, Degrees of freedom – Generalised co-ordinates – Generalized notations – Brachistocrone problems – Atwood's machine.Hamilton's variational principle – Lagrange's equation of motion from Hamilton's principle, D'Alembert's principle – Applications of Lagrange's equation of motion – particle moving under a central force – particle moving on the surface of earth– Superiority of Langrange's approach over Newtonian's approach.

Mechanics of a system of particles

Mechanics is the study of the motion of physical bodies .The possible and actual motions of physical objects, whether large or small, fall under the domain of mechanics. In the present century the term "Classical mechanics" has come in to wide to denote this branch of physics in the contradiction to the newer theories especially quantum mechanics. "*Classical mechanics has been customarily used to denote that part of the mechanics which deals with the description and explanation of the motion of the objects, neither too big so there exists a close agreement between theory and experiment nor too small interacting objects, more precisely like the systems on molecular or subatomic scale.*" We shall follow this usage, interpreting theories the name to include the type of mechanics. Classical mechanics may be classified in to three subsections (i) Kinematics (ii) Dynamics (iii) Statics.

In this unit we deals with the structure and law of mechanics with the applications, starting from basic fundamental concepts .Having established the essential pre-requisites, the Lagrangian formulation known for its mathematical elegance.

CONSTRAINTS

Constraints are the geometrical or kinematical restrictions on the motion of the particle or system of the particles. Systems with such constraints of motion are called as

Constrained systems and their motion is known as **constrained or restricted motion**. Some examples of restricted motions are-

- The motion of the rigid body is restricted to the condition that the distance between any two particles remains unchanged.
- The motion of the gas molecules with in the container is restricted by the walls of the vessels.
- A particle placed on the surface of a solid sphere is restricted so that it can only move either on the surface or outside the surface.

Classification of Constraints

The constraints can be classified in to the following categories:

(i) Holonomic and non-holomonic constraints (ii) Scleronomic and rhenomic constraints

Holonomic constraints:-Constraints are said to be holomonic if the conditions of all the constraints can be expressed as equations connecting the coordinates of the particles and possible time in the form

$f\,(r_1,r_2,r_3……..,r_n,t) =0$ (1.1)

Where $\vec{r}_1, \vec{r}_2, \vec{r}_3……..\vec{r}_n$ represent the position vectors of the particles of a system and *t* the time. In Cartesian coordinates equation (1.1) can be written as,

$f\,(x_1, y_1, z_1; x_2, y_2, z_2,……… x_n, y_n, z_n,t) =0$ (1.2)

Examples of holonomic constraints:-

1. The constraints involved in the motion of rigid bodies. In rigid bodies, the distance between any two particles is always constant and the condition of constraints are expressed as-
   $|\vec{r}_i - \vec{r}_j|^2 - C_{ij}^2 = 0$ (1.3)
2. Constraints involved in the motion of the point mass of a simple pendulum.
3. The constraints involved when a particle is restricted to move along any curve (circle or ellipse) or in a given surface.

Non-holonomic constraints: - If the conditions of the constraints can not be expressed as equations connecting the coordinates of particles as in case of holomonic, they are called as non-holomonic constraints. The conditions of these constraints are expressed in the form of inequalities. The motion of the particle placed on the surface of sphere under theaction of the gravitational force is bound by non-holonomic constraints, for it can be expressed as an inequality, $r^2 - a^2 \geq 0$.

Examples of non-holonomic constraints

1. Constraints involved in the motion of a particle placed on the surface of a solid sphere
2. An object rolling on the rough surface without slipping.
3. Constraints involved in the motion of gas molecules in a container.

(ii) Scleronomic and Rhenomic Constraints: - The constraints which are independent of time are called Scleronomic constraints and the constraints which contain time explicitly, called rhenomic constraints

Examples: - A bead sliding on a rigid curved wire fixed in space is obviously subjected to Scleronomic constraints and if the wire is moving is prescribed fashion the constraints become Rhenomic.

GENERALISED COORDINATES

Generalised co-ordinates:- These are the coordinates which are used to eliminate the dependent coordinates and can be expressed in another way by the introduction of (3N-p) independent coordinates of variables called the Generalised coordinates, where N represent the number of particles of a system and p represent the holonomic constraints. Thus any '*q*' quantities which completely define the configuration of the system having '*f*' degree of freedom are called Generalised co-ordinates of the system and are denoted by $q_1, q_2, q_3, \ldots\ldots q_f$, or just q_i (i=1,2,3,4…*f*)

Principles for the choosing a suitable set of Generalised co-ordinates - For this three principles are used –

1. They should specify the configuration of the system.
2. They may be varied arbitrarily and independently of each other, with out violating the constraints on the system.
3. There is no uniqueness in the choice of the generalised coordinates

It may be noted that generalised co-ordinates need not to have the dimensions of length or angles. Generalised co-ordinates need not to be Cartesian co-ordinates of the particles and the condition of the problem may render some other choice of co-ordinates which may be more convenient.

Generalised Notations

(i) **Generalised Displacement –** A small displacement of an N particle system is defined by changes $\delta \vec{r}_i$ in position co-ordinates $\vec{r}_i$ (i =1,2,3….,N) with time 't' held fixed. An arbitrary virtual displacement $\delta \vec{r}_i$, remembering that $\vec{r}_i$'s are function of generalised co-ordinates i.e. $\vec{r}_i = \vec{r}_i (q_1, q_2, \ldots.. q_{3N}, t)$, can be written by using Euler's theorem as,

$$\delta r_i = \sum_{j=1}^{3N} \frac{\partial r_i}{\partial q_j} \delta q_i \tag{1.5}$$

δq_j is called the generalised displacement or virtual displacement. If q_j is an angle co-ordinate, δq_j is an angular displacement.

(ii) **Generalised velocity –** The time derivative of the generalised q_k ,is called generalised velocity associated with particular co-ordinates q_k for an unconstrained system,

$$\vec{r}_i = \vec{r}_i (q_1, q_2, \ldots.. q_{3N}, t),$$

Then,

$$\dot{\vec{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_i + \frac{\partial \vec{r}_i}{\partial t} \tag{1.6}$$

If N-particle system contains k-constraints, the number of generalised co-ordinates are 3N-k=*f* and,

$$\vec{r}_i = \sum_{j=1}^{f} \frac{\partial \vec{r}_i}{\partial q_j} q_i + \frac{\partial \vec{r}_i}{\partial t} \tag{1.7}$$

(iii) **Generalised Acceleration-** components of generalised acceleration are obtained by differentiating equation (1.6) or (1.7) w.r.t. time and finally we obtain the expression

$$\ddot{\vec{r}}_i = \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \ddot{q}_i + \sum_{j=1}^{3N}\sum_{k=1}^{3N} \frac{\partial^2 r_i}{\partial q_j\, \partial q_k} \dot{q}_j \dot{q}_k + 2\sum_{j=1}^{3N} \frac{\partial^2 r_i}{\partial q_j\, \partial t} \dot{q}_j + \frac{\partial^2 r_i}{\partial^2 t} \tag{1.8}$$

From the above equation it is clear that the cartesian components are not linear functions of components of generalised acceleration $\ddot{q}_j$ alone, but depend quadratically and linearly on generalised velocity component $\dot{q}_j$ as well.

(iv) **Generalised Force –** Let us consider the amount of work done δW by the force $\sum_i \vec{F}_i$ during an arbitrary small displacement $\sum_i \delta \vec{r}_i$ of the system

$$\sum_{i}^{N} \quad \sum_{i=1}^{N} \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \quad \sum_{i=1}^{N} \left(\sum_{j=1}^{3N} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j} \right)$$

$$\delta W = \vec{F}_i . \vec{\delta r}_i = \vec{F}_i . \qquad \delta q_j = \qquad F_i . \qquad \delta q_j$$

$$= \sum_{i=1}^{3N} Q_i . \delta q_i \tag{1.9}$$

Where,

$$Q_j = \left(\sum_{j=1}^{N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j} \right) \tag{1.10}$$

Here we note that Q_j depends on the force acting on the particles and on the co-ordinate q_j and possibly on time t. Therefore, *Q_j is called the generalised force.*

Advantages of Generalised co-ordinates

The main advantage in the formulating laws of mechanics in terms of generalised co-ordinates and the associated mechanical quantities is that the equation of motion looks simpler and can be solved independently of each other since generalised co-ordinates are all independent and constraints have no effect on them. The equations of motion are then called Lagrange's equation of motion.

D'ALEMBERT'S PRINCIPLE

This method is based on the principle of virtual work. The system is subjected to an infinitesimal displacement consistent with the forces and constraints imposed on the system at a given time t. This change in the configuration of the system is not associated with a change in time i.e., there is no actual displacement during which forces and constraints may change and hence the displacement is termed virtual displacement.

From the principle of virtual work

$$\sum_{i}^{N} \vec{F}_i^{\,a} . \vec{\delta r}_i = 0$$

(1.11)

Here $\vec{F}_i^a$ represent the applied force and $\delta\vec{r}_i$ denote the virtual displacement.

To interpret the equilibrium of the systems, D'Alembert adopted an idea of reverse force. He conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\vec{F}_i$ plus reversed effective force $\dot{\vec{p}}_i$. Thus

$$\vec{F}_i + (-\dot{\vec{p}}_i) = 0 \tag{1.12}$$

or, $$\vec{F}_i - \dot{\vec{p}}_i = 0$$

Thus the principle of virtual work takes the form,

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0$$

Again writing $\vec{F}_i = \vec{F}_i^a + \vec{f}_i$

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i).\delta\vec{r}_i + \vec{f}_i.\delta\vec{r}_i = 0$$

Dealing with the systems for which the virtual work of the forces of constraints is zero, we write

$$\sum_i (\vec{F}^a_i - \dot{\vec{p}}_i)\delta\vec{r}_i = 0$$

Since force of constraints are no more in picture, it is better to drop the superscript 'a'. Thus

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \tag{1.13}$$

The equation (1.13) is called D'Alembert principle. To satisfy the above equation, we can not equalate the coefficient of $\delta\vec{r}_i$ to zero since $\delta\vec{r}_i$ are not independent of each other and

hence it is necessary to transform $\delta\vec{r}_i$ into generalised co-ordinates , δq_j which are independent of each other .The coefficient of δq_j will then equated to zero.

DERIVATION OF LAGRANGE'S EQUATION

The Lagrange's equations can be obtained from Hamilton's variational principle, velocity dependent potentials and also by Rayleigh's dissipation function. In the present article we shall discuss the derivation of Lagrange's equations from velocity dependent potential and by Rayleigh's dissipation function.

Lagrange's Equations from velocity dependent potential

The co-ordinate transformation equations are

$\vec{r}_i = \vec{r}_i\,(q_1,q_2\ldots\ldots,q_{n,t})$

So that,

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{d\vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \ldots\ldots\ldots + \frac{d\vec{r}_i}{\partial t}\frac{dt}{dt}$$

So that

$$\vec{v}_i = \sum_j \frac{\partial\vec{r}_i}{\partial qj}\dot{q}_j + \frac{\partial\vec{r}_i}{\partial t} \qquad (1.14)$$

Further infinitesimal displacement $\delta\vec{r}_i$ can be connected with δq_i

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial qj}\delta q_j + \frac{\partial\vec{r}_i}{\partial t}\delta t$$

But the last term is zero since in virtual displacement only co-ordinate displacement is considered and not that of time. Therefore,

$$\delta\vec{r}_i = \sum_j \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j$$

Now we write equation (1.13) as,

$$\sum_i(\vec{F}_i - \dot{\vec{p}}_i)\sum_j \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j = 0,$$

$$\sum_{i,j} \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j \qquad (1.15)$$

We define $\sum \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j = Q_i$ as the component of generalised force. So the above equation becomes

$$\sum_j Q_j\,\delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j = 0 \qquad (1.16)$$

Lagrangian Mechanics

The evaluation of second term in equation (1.16) gives the expansion as

$$\sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial\vec{r}_i}{\partial qj}\,\delta q_j = \sum_j \left(\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_i}\left(\sum_i (½)\, m_i v_i^2\right)\right\} - \left\{\frac{\partial}{\partial q_i}\left(\Sigma(½)\, m_i v_i^2\right)\right\}\right)\delta q_j \qquad (1.17)$$

With this substitution equation (1.16) becomes

$$\sum_j Q_j\,\delta q_j - \sum_j \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\delta q_j = 0$$

Where $\Sigma(1/2)\, m_i v_i^2 = T$, is written since it represents the total kinetic energy of the system, further the above equation may be

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

Since the constraints are holonomic, q_j are independent of each other and hence to satisfy above equation the coefficient of each δq_j should necessary vanish, i.e.

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right) = Q_j \tag{1.18}$$

As j ranges 1 to n, there will be 'n' such second order equations.

If potential are velocity dependent, called generalised potentials, then through the system is not conservative, yet the above form Lagrange's equations can be obtained provided Q_j, the components of the generalised force, are obtained from a function $U(q_j,\dot{q}_j)$ such that

$$Q_j = -\frac{\partial U}{\partial qj} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}j}\right) \tag{1.19}$$

Hence the from equation (1.18) and equation (1.19) ,we have

$$\frac{d}{dt}\left[\frac{\partial\,(T-U)}{\partial \dot{q}_j}\right] - \frac{\partial\,(T-U)}{\partial q_j} = 0$$

If we take L = T-U, the Lagrangian function, where U is generalised potential, then above equation becomes

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0 \tag{1.20}$$

Which are the Lagrangian equations for holonomic constraints systems.

Lagrange's equations from Rayleigh's dissipation function

It can be shown that if a system involves frictional forces or dissipative forces, then in suitable circumstance, such a system can also be described in terms of extended Lagrangian formulation. Frictional forces are found to be proportional to the velocity of the particle so that in cartesian co-ordinates components are,

$$F_j^d = - k_i\dot{x}_j \, , \tag{1.21}$$

Where k_j are constants. Such frictional forces are defined in terms of a new quantity called Rayleigh dissipation function given as,

$$\Im = (1/2)\Sigma k_i \dot{x}^2_j$$

Which yields

$$F_j^d = - \frac{\partial \Im}{\partial \dot{x}_i} \tag{1.22}$$

Writing equation (1.18) in cartesian co-ordinates, assuming that this still holds for such a system,

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right) = Q_j$$

Where L contains the potential of conservative forces as described earlier; Q_j represents the forces which do not arise from a potential, i.e.

$$Q_j^d = F_j^d = - \frac{\partial \Im}{\partial \dot{x}_i} \tag{1.23}$$

Thus equation (1.18) can be written as,

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} \right) + \frac{\partial \Im}{\partial \dot{x}_i} = 0$$

The above equation may be expressed as in terms of generalised co-ordinates q_j

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right) + \frac{\partial \Im}{\partial \dot{q}_i} = 0 \qquad (1.24)$$

Thus for such a system, to obtain equations of motion, two scalar L and $\Im$ are to be specified.

VARIATIONAL PRINCIPLE

This principle state that the integral $\int_{t_1}^{t_2} (T-V)\,dt$ shall have a stationary value or extremum value, where T, kinetic energy of the mechanical system, is a function of co-ordinates and their derivatives and V is the potential energy of the mechanical system, is a function of co-ordinate only. Such a system for which V is purely a function of co-ordinates is called conservative system.

Statement: The variational principle for the conservative system is stated as follows

"The motion of the system from time t_1 to time t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} (T-V)\,dt = \int_{t_1}^{t_2} L\,dt,$$

is extemum for the path of motion" .Here L=T-V is the Lagrangian function .

EULER –LAGRANGE EQUATION

The integral I, representing a path between the two points 1 and 2 will be written as

$$I = \int_{t_1}^{t_2} f\,[y_1(x)\ y_2(x),\dots\dots\ \dots\dots\dot{y}_1(x)\dot{y}_2(x)\dots\dots\dots,x]dx$$

(1.25)

Now to account for all possible curves between the two points1,2,we assign different values of a parameter α to these curves, so that y_j will also be a function of α, i.e. curves being represented by y_j (x, α).The family of the curves may be represented as

$$y_1(x,\alpha) = y_1(x,0) + \alpha\eta_1(x)$$
$$y_2(x,\alpha) = y_2(x,0) + \alpha\eta_2(x)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...$$

Where η_1 and η_2 etc. are completely arbitrary functions of x,which vanishes at end points and the curves $y_1(x,0)$, $y_2(x,0)$ etc. for $\alpha=0$ are paths for which the integral I is extemum

The integral I will be the function of α and hence its variation can be represented as

$$\frac{\partial\, I(\alpha)}{\partial\,(\alpha)}d\alpha = \int_{t_1}^{t_2}\sum_j\left(\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial\,(\alpha)}d\alpha + \frac{\partial f}{\partial \dot{y}_j}\frac{\partial \dot{y}_j}{\partial\,(\alpha)}d\alpha\right)dx$$

Integrating by parts the second term of the integrand we get,

$$\frac{\partial\, I(\alpha)}{\partial\,(\alpha)}d\alpha = \int_{t_1}^{t_2}\sum_j\left(\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial\,(\alpha)}d\alpha\right)dx + \sum_j\frac{\partial f}{\partial \dot{y}_j}\frac{\partial \dot{y}_j}{\partial\,(\alpha)}d\alpha\,\Bigg|_1^2 - \int_{t_1}^{t_2}\sum_j\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right)\frac{\partial y_j}{\partial \alpha}d\alpha\, dx$$

(1.26)

Lagrangian and Hamiltonian Mechanics

Since at end points, which are held fixed, all paths meet,so $\frac{\partial y_j}{\partial \alpha}\Big|_1^2 = 0$. Therefore equation (1.26) becomes

$$\frac{\partial\, I(\alpha)}{\partial\,(\alpha)}d\alpha = \int_{t_1}^{t_2}\sum_j\left(\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial\,(\alpha)}d\alpha\right)dx - \int_{t_1}^{t_2}\sum_j\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right)\frac{\partial y_j}{\partial \alpha}d\alpha\, dx$$

$$= \int_{t_1}^{t_2}\sum_j\left(\frac{\partial f}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right)\right)\frac{\partial y_j}{\partial \alpha}d\alpha dx$$

$$\frac{\partial f}{\partial y_j}$$

Let us put

$$\frac{\partial I}{\partial \alpha} d\alpha = \delta I \quad \& \quad \frac{\partial y_j}{\partial \alpha} d\alpha = \delta y_j$$

So that

$$\delta I = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial f}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right) \right) \delta y_j \, dx$$

For the integral to be extremum

$$\delta I = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial f}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right) \right) \delta y_j \, dx = 0$$

Since δy_j are independent of each other, coefficient of δy_j should separately vanish if above equation is to be satisfied. Thus.

$$\left(\frac{\partial f}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_j}\right) \right) = 0, \; j=1,2,3,\ldots n \qquad (1.27)$$

The set of differential equations represented by equation (1.27)are known as Euler-Lagrange differential equations. Thus solutions of Euler-Lagrange equation represent those curves for which the integral assumes an extremum value $I = \int_1^2 f(y_j, y_j, x)dx$

1.8 DERIVATION OF LAGRANGE'S EQUATION FROM HAMILTON'S PRINCIPLE

According to Hamiltonian's variational principle, motion of a conservative system from time t_1 to time t_2 is such that the variation of the line integral

$$I = \int_{t_1}^{t_2} L\,[q_j(t),\ \dot{q}_j(t),\ t]dt \quad , \text{ is zero}$$

i.e.

$$\delta I = \delta\int_{t_1}^{t_2} L\,[q_j(t),\ \dot{q}_j(t),\ t]dt = 0 \tag{1.28}$$

Now we shall show that Lagrange's equations of motion follow directly from Hamilton's principle. If we account for all possible paths of motion of the system in configuration space and label each with a value of a parameter α,then since paths are being represented by $q_j(t,\alpha)$,I also becomes a function of α so that we can writ,

$$I(\alpha) = \int_{t_1}^{t_2} L\,[q_j(t,\alpha),\ \dot{q}_j(t,\alpha),\ t]dt \tag{1.29}$$

So that, $$\frac{\partial I(\alpha)}{\partial(\alpha)} = \int_{t_1}^{t_2}\sum_j\left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial\alpha} + \frac{\partial L}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial\alpha} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial\alpha}\right)dt$$

Since in δ variation, there is no time variation along any path and also at end points and hence $(\partial I/\partial\alpha)$ is zero along all paths. Therefore, on multiplying by $d\alpha$, above equation is

$$\frac{\partial I(\alpha)}{\partial(\alpha)}d\alpha = \int_{t_1}^{t_2}\sum_j\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial\alpha}d\alpha\, dt + \int_{t_1}^{t_2}\sum_j\frac{\partial L}{\partial\dot{q}_j}\frac{\partial\dot{q}_j}{\partial\alpha}d\alpha\, dt \tag{1.30}$$

Integrating second term of L.H.S. by parts

$$= \int_{t_1}^{t_2}\sum_j\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial\alpha}d\alpha\, dt + \sum_j\frac{\partial L}{\partial\dot{q}_j}\frac{\partial q_j}{\partial\alpha}d\alpha\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_j\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_j}\right)\frac{\partial q_j}{\partial\alpha}d\alpha\, dt$$

The middle term is zero since δ variation involves fixed end points.

$$\frac{\partial I(\alpha)}{\partial(\alpha)}d\alpha = \int_{t_1}^{t_2}\sum_j\left(\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial(\alpha)}d\alpha\right)dt - \int_{t_1}^{t_2}\sum_j\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_j}\right)\frac{\partial q_j}{\partial t}d\alpha\, dt$$

So,

$$= \int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right) \frac{\partial q_j}{\partial t} \, dt \qquad (1.31)$$

Since q_j are independent of each other, the variations δq_j will be independent. Hence $\partial I(\alpha)=0$ if and only if the coefficients of δq_j separately vanish, i.e.

$$\left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right) = 0 \qquad (1.32)$$

Which are Lagrange equations of motions for a conservative system. It is obvious that these equations follow directly from Hamilton's principle.

Application Of Lagrange's Equation Of Motion:

Simple Pendulum:

Consider a simple pendulum of mass m which is deflected by an angle θθ from its mean position. Let l be the length of the pendulum and x be its linear displacement fro equilibrium position.

From fig we have,

$$X=l\theta$$

$$X^{\cdot}=l\theta^{\cdot}$$

The kinetic energy of the system is,

$$T=12mx^{\cdot 2}$$

$$=12ml^2\theta^2$$

The pendulum gains height AC at extreme position so that its potential energy is,

$$V=mgAC$$

$$=mg(OA-OC)$$

$$=mg(l-l\cos\theta)$$

$$V=mgl(1-\cos\theta)$$

The Lagrangian of the pendulum is,

$$L=T-V=1/2ml^2\dot{\theta}^2-mgl(1-\cos\theta)$$

The equation of motion is given by,

$$d/dt(\delta L/\delta\dot{\theta})-\delta L/\delta\theta=0$$

Here, $\delta L/\delta\dot{\theta}=ml^2\dot{\theta}$ and $\delta L/\delta\theta=-mgl\sin\theta$

So, equation of motion becomes,

$$ddt(ml2\dot{\theta})+mgl\sin\theta=0$$

$$ml2\ddot{\theta}+mgl\sin\theta=0$$

$$l\ddot{\theta}+g\sin g\theta=0$$

$$\ddot{\theta}+gl\sin g\theta=0$$

For small angle θ , $\sin\theta=\theta$

$$\ddot{\theta}+\omega 2\theta=0$$

where, $\omega^2=gl\omega^2=g/l$

and $T=2\pi/\omega=2\pi\sqrt{1/g}$,which is the equation of motion of simple pendulum.

Compound Pendulum:

Compound pendulum is a rigid object capable of oscillating in a vertical plane about horizontal axis.

Consider a compound pendulum of mass m oscillating in xy plane. In the figure the point 'o' is the point of suspension through which the horizontal axis passes and C is the center of mass.

Now the kinetic energy of system is

$$T=1/2I\omega^2$$
$$=1/2I\dot{\theta}^2\cdots(1)$$

Where $\dot{\theta}$ is the generalized co-ordinate for the system.

and potential energy $(v)=-mgl\cos\theta\cdots(2)$

So Lagrangian of system is

$$L=T-V$$
$$=1/2I\dot{\theta}^2+mgl\cos\theta$$

We have, lagrangian equation of motion is

$$d/dt(\delta L/\delta \dot{q}j)-\delta L/\delta qj=0$$

In this case, $d/dt(\delta L/\delta\dot{\theta})-\delta L/\theta\theta=0$

so,

$$\delta L/\delta\theta=-mgl\sin\theta$$

and

$$d/dt(\delta L/\theta\dot{\theta})=I\ddot{\theta}$$

Now the Lagrangian equation of motion reduces to

$I\ddot{\theta}+mgl\sin\theta=0 I\ddot{\theta}+mgl\sin\theta=0$

$$I\ddot{\theta}+mgl\theta=0 \quad [\because \text{For small}\,\theta]]$$

$$\ddot{\theta}+mgl\theta I=0\cdots(3)$$

IN equation (3) mgl/I refers to ω^2

$$\omega^2=mgl/I$$

$$T=2\pi\sqrt{I/mgl}----\sqrt{}\cdots(4)$$

Equation (4) gives the time period of compound pendulum.

POSSIBLE QUESTIONS:

PART B: (6 MARK)

- What are Constraints? Explain its various types of Constraints.
- Explain about the Degrees of freedom.
- What are the Generalized co-ordinates? Derive the various notation for the momentum, force ,potential.
- Explain the concept of D'Alembert's principle.
- Derive the lagrangian differential equation from D'Alembert's principle
- Describe the application of Lagrangian equation of motion to linear harmonic oscillator.
- Simple pendulum.
- Compound pendulum.

S.No	QUESTIONS	Option A	Option B	Option C	Option D	ANSWER
1	Total energy of body is sum of	kinetic energies	potential energies.	forces.	both a and b.	both a and b.
2	Energy can neither be created nor be destroyed, but it can be changed from one form to another. This law is known as	kinetic energy.	potential energies.	conservation of energy	conservation principle.	conservation of energy
3	An artificial Satellite revolves round the Earth in circular orbit, which quantity remains constant?	Angular Momentum	Linear Momentum	Angular Displacement	None of these	Angular Momentum
4	A man presses more weight on earth at :	Sitting position	Standing Position	Lying Position	None of these	Standing Position
5	The rotational effect of a force on a body about an axis of rotation is described in terms of the	Centre of gravity	Centripetal force	Centrifugal force	Moment of force	Moment of force

6	If no external force acts on a system of bodies, the total linear momentum of the system of bodies remains constant. Which law states that ?	Newton's first law	Newton's Second Law	Newton's Third Law	Principle of conservation of linear momentum	Principle of conservation of linear momentum
7	Which law is also called the law of inertia ?	Newton's first law	Newton's Second Law	Newton's Third Law	All of these	Newton's first law
8	Energy possessed by a body in motion is called	kinetic energy.	potential energies.	conservation of energy	conservation principle.	kinetic energy.
9	Lagrangian L =	T-V	T+V	(T-V)2	(T+V)1/2	T-V
10	The path adopted by the system during its motion can be represented by a space of ________________ dimensions.	3N	6N	9N	N	6N
11	Co-ordinate transformation equations should	time	position	momentum	velocity	time

	not involve ______________ explicitly.					
12	The frequency of Harmonic oscillator is given by	[1/2p(k/m)5/2]	[1/2p(k/m)3/2]	[1/2p(k/m)1/2]	[1/2p(k/m)]	[1/2p(k/m)1/2]
13	If the total energy of the particle is conserved then,	T+V =constant	b. T-V=0	c. T-V =constant	None of these	T+V =constant
14	Constraint relations do not depend on time is	scleronomic	b. rheonomic	c. unilateral	None of these	scleronomic
15	Constraint relations depend on time is	scleronomic	b. rheonomic	c. unilateral	None of these	rheonomic
16	Constraint relations can be made independent of velocities	scleronomic	b. rheonomic	c. unilateral	.d holonomic	unilateral
17	The Branchistochrone problem is to find	shape of a curve	blength of a curve c.	elasticity of a curve electrons	None of these	shape of a curve

18	"If no external torque is applied on a body, then total angular momentum remains constant" stated law is called	A. law of conservation of angular velocity.	A. law of conservation of angular acceleration.	A. law of conservation of angular momentum.	A. law of conservation of angular speed.	A. law of conservation of angular momentum.
19	Which one of the following choices is an example of a non-conservative force?	elastic spring force	kinetic frictional force	torque	gravitational force	kinetic frictional force
20	Which one of the following choices is an example of a conservative force?	elastic spring force	kinetic frictional force	torque	gravitational force	elastic spring force
21	A man of mass 50 kg jumps to a height of 1 m. His potential energy at the highest point is (g = 10 m/s2)	50J	500J	12J	30J	500J
22	The type of energy possessed by a simple pendulum, when it is at the mean	KE	PE	KE+PE	KE-PE	KE

	position is					
23	If air resistance is negligible, the sum total of potential and kinetic energies of a freely falling body	increases	increases	becomes zero	remains the same	remains the same
24	Name the physical quantity which is equal to the product of force and velocity.	WORK	ENERGY	POWER	ACCELERATION	POWER
25	The P.E. of a body at a certain height is 200 J. The kinetic energy possessed by it when it just touches the surface of the earth is	>PE	<PE	P.E	Not Known	>PE
26	The point, through which the whole weight of the body acts, irrespective of its position, is known as	centre of mass	centre of percussion	moment of inertia	centre of gravity	centre of gravity

27	According to the law of moments, if a number of coplaner forces acting on a particle are in equilibrium, then	the algebraic sum of their moments about any point in their plane is zero	the algebraic sum of their moments about any point is equal to the moment of their resultant force about the same point.	their lines of action are at equal distances	their algebraic sum is zero	the algebraic sum of their moments about any point in their plane is zero
28	The motion of a particle round a fixed axis is	translatory as well as rotatry	translatory	rotary	circular	circular
29	The principle of transmissibility of forces states that, when a force acts upon a body, its effect is	different at different points on its line of action	maximum, if it acts at the centre of gravity of the body	minimum, if it acts at the centre of gravity of the body	same at every point on its line of action	minimum, if it acts at the centre of gravity of the body
30	The centre of gravity of a semi-circle lies at a distance of ___________ from its base measured along the vertical radius.	3r/4π	4r/ 3π	3r/ 8	8r/3	4r/ 3π

31	Concurrent forces are those forces whose lines of action	meet on the same plane	lie on the same line	meet at one point	none of these	meet at one point
32	The velocity ratio in case of an inclined plane inclined at angle θ to the horizontal and weight being pulled up the inclined plane by vertical effort is	cos θ	sin θ	tan θ	cot θ	sin θ
33	One complete round trip of a vibrating body about it's mean position is	frequency	time period	amplitude	vibration	vibration
34	Potential energy of mass attached to spring at mean position is	maximum	moderate	zero	minimum	zero
35	Velocity of bob in SHM becomes zero at	mean position	in air	extreme position	middle of mean and extreme position	extreme position

36	If potential energies and kinetic energies are equal then displacement of an object in SHM is	4	0	3	1	0
37	Kinetic energy of mass attached to spring at extreme position is	maximum	moderate	zero	minimum	zero
38	Potential energy of mass attached to spring at extreme position is	maximum	moderate	zero	minimum	maximum
39	Hamiltonian H =	T-V	T+V	(T-V)2	(T+V)1/2	T+V
40	Advantage of Action and Angle variable is that one can obtain the frequencies of	Vibratory motion	periodic motion	circular motion	all the above	periodic motion
41	For non-interacting particle in a quantum state the energy E is given by	p/2m	p2/m	p/m	p2/2m	p2/2m

42	Co-ordinate transformation equations should not involve ___________ explicity.	position	momentum	time	force	time
43	Generating function have _____________ forms.	four	two	three	five	four
44	Hamilton's principal function is denoted by ______________.	H	K	P	S	S
45	Hamilton's characteristic function W is identified as _____________.	kinetic energy	potential energy	work	action A	action A
46	Hamilton's characteristic function is denoted by _______________.	S	K	W	H	W
47	The number of independent ways in which a mechanical system	action-angle variables	generalized variables	degrees of freedom	co-ordinates	degrees of freedom

	can move without violating any constraint which may be imposed is called the _______________.					

UNIT-II

SYLLABUS

Phase space: Hamiltonian – Hamilton's canonical equations of motion – Physical significance of H – Advantage of Hamiltonian approach – Hamilton's canonical equation of motion in different coordinate systems – Hamilton-Jacobi method – Kepler's problem solution by Hamilton-Jacobi method – Action and angle variables – Solution of Harmonic oscillator by action angle variable method – canonical or contact transformation – Condition for a transformation to be canonical

PHASE SPACE:

The origin of the term phase space is somewhat murky. For the purpose of this explanation let's just say that in 1872 the term was used in the context of classical and statistical mechanics. It refers to to the positions and momenta as the Bewegungsphase in German - phase motion. It is often erroneously cited that the term was first used by Liouville in 1838.

In classical mechanics, the phase space is the space of all possible states of a system; the state of a mechanical system is defined by the constituent positions p and momenta q. p and q together determine the future behavior of that system. In other words if you know p and q at time t you will be able to calculate the p and q at time t+1 using the theorems of classical mechanics - Hamilton's equations.

To describe the motion of a single particle you will need 6 variables, 3 positions and 3 momenta. You can imagine a 6 dimensional space; three positions and three momenta. Each point in this 6 dimensional space is a possible description of the particles' possible states, of course constraint by the laws of classical mechanics.

If you have N particles to describe the system, you have a 6N-dimensional phase space.

Let's make a simple example. The Pendulum. The Pendulum consists of a single particle mass that swings in a plane. The pendulum is thus fully described by one position and one momentum. Its momentum is zero at the top and maximum at bottom. The position perhaps is denoted by angle and varies between plus/minus a. If you draw states p and a in a Cartesian plane coordinate system you will get an ellipsoid (or if chose adequate coordinates a circle) that fully describes all possible states of the pendulum.

In quantum mechanics the term phase re-appeared: it refers to the complex phase of the complex numbers that wave functions take values in.

In quantum mechanics, the coordinates *p* and *q* of phase space normally become operators in a Hilbert space.

A quantum mechanical state does not necessarily have a well-defined position or a well-defined momentum (and never can have both according to Heisenberg's uncertainty principle). The notion of phase space and of a Hamiltonian H, can be viewed as a crucial link between what otherwise looks like two very different theories. A state is now not a point in phase space, but is instead a complex valued wave function. The Hamiltonian H becomes an operator and describes the observable quantity.

HAMILTONIAN FUNCTION:

Hamiltonian function, also called Hamiltonian, mathematical definition introduced in 1835 by Sir William Rowan Hamilton to express the rate of change in time of the condition of a dynamic physical system—one regarded as a set of moving particles. The Hamiltonian of a system specifies its total energy—*i.e.*, the sum of its kinetic energy (that of motion) and its potential energy (that of position)—in terms of the Lagrangian function derived in earlier studies of dynamics and of the position and momentum of each of the particles.

The Hamiltonian function originated as a generalized statement of the tendency of physical systems to undergo changes only by those processes that either minimize or maximize the abstract quantity called action. This principle is traceable to Euclid and the Aristotelian philosophers.

When, early in the 20th century, perplexing discoveries about atoms and subatomic particles forced physicists to search anew for the fundamental laws of nature, most of the old formulas became obsolete. The Hamiltonian function, although it had been derived from the obsolete formulas, nevertheless proved to be a more correct description of physical reality. With modifications, it survives to make the connection between energy and rates of change one of the centres of the new science.

HAMILTON'S VARIATIONAL PRINCIPLE:

Lagrange's equations have been shown to be the consequence of a variational principle, namely, the Hamilton's principle. Indeed the variational method has often proved to be the preferable method of deriving equations, for it is applicable to types of systems not usually comprised with in the scope of mechanics. It would be similarly advantageous if a variational principle could be found that leads directly to the Hamilton's equation of motion.

Hamilton's principle is stated as

$$\delta I = \delta \int_{t}^{t} L\, dt$$

Expressing L in terms of Hamiltonian by the expression by the expression

$$H = \sum_i p_i \dot{q}_i - L,$$

We find,

$$\delta I = \delta \int_{t_1}^{t_2} \left(\sum p_i \quad \frac{dq_i}{} \; H(q_i, p_i, t) \right) dt$$

$$\delta \int_{t_1}^{t_2} \sum_i p_i \; dq_i \; - \; \delta \int_{t_1}^{t_2} H(q_i, p_i, t) dt = 0$$

The above equation is some times is referred as the modified Hamilton's principle. Although it will be used most frequently in connection with transformation theory ,the main interest is to show that the principle leads to the Hamilton's canonical equations of motions.

The modified Hamilton's principle is exactly of the form of the variational problems in a space of 2n dimensions as

$$\delta I = \delta \int_{t_1}^{t_2} f(q, \dot{q}, p, \dot{p}, t)\, dt = 0$$

For which the 2n Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} \qquad J=1,2,3\ldots.n$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} \qquad J=1,2,3\ldots.n$$

The integrand f as given as (2.29) contains $\dot{q}_j$ only through the $p_i\dot{q}_i$ term, q_j only in H. Hence equation (2.30) leads to

$$\dot{p}_j + \frac{\partial H}{\partial q_i} = 0$$

On the other hand there is no explicit dependence of the integrand in equation (2.30) on $\dot{p}_j$. The above equation therefore reduce simply to

$$\dot{q}_j - \frac{\partial H}{\partial p_i} = 0$$

The above two equations are exactly Hamilton's equations of motion .The Euler –Lagrange equations of the modified Hamilton's principle are thus the desired canonical equations of motion .From the above derivation of Hamilton’s equations we can consider that Hamiltonian and Lagrangian formulation and therefore their respective variational principles, have the same physical content.

Hamilton's Equations:

The equations defined by

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{2}$$

where $\dot{p} \equiv dp/dt$ and $\dot{q} \equiv dq/dt$ is fluxion notation and H is the so-called Hamiltonian, are called Hamilton's equations. These equations frequently arise in problems of celestial mechanics.

The vector form of these equations is

$$\dot{q}_i = H_{p_i}(t, \mathbf{q}, \mathbf{p}) \tag{3}$$

$$\dot{p}_i = -H_{q_i}(t, \mathbf{q}, \mathbf{p}) \tag{4}$$

(Zwillinger 1997, p. 136; Iyanaga and Kawada 1980, p. 1005).

Another formulation related to Hamilton's equation is

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{5}$$

where L is the so-called Lagrangian.

HAMILTON'S CANONICAL EQUATIONS OF MOTION:

Theorem 6 : Define the Hamiltonian and hence derive the Hamilton's canonical equations of motion.

Proof : We know the Hamiltonian H is defined as

$$H = H\left(q_j, p_j, t\right) = \sum_j p_j \dot{q}_j - L. \qquad \ldots(1)$$

Consider $H = H\left(q_j, p_j, t\right)$. $\ldots(2)$

We find from equation (2) that

$$dH = \sum_j \frac{\partial H}{\partial p_j} dp_j + \sum_j \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial t} dt. \qquad \ldots(3)$$

Now consider $H = \sum_j p_j \dot{q}_j - L$.

Similarly we find

$$dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - dL,$$

$$\Rightarrow \quad dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt. \qquad \ldots(4)$$

We know the generalized momentum is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Hence equation (4) reduces to

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial t} dt \qquad \ldots (5)$$

Now comparing the coefficients of dp_j, dq_j and dt in equations (3) and (5) we get

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \qquad \ldots (6)$$

However, from Lagrange's equations of motion we have

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

Hence equations (6) reduce to

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad . \qquad \ldots (7)$$

These are the required Hamilton's canonical equations of motion. These are the set of 2n first order differential equations of motion and replace the n Lagrange's second order equations of motion.

PHYSICAL SIGNIFICANCE OF H:

1. For conservative scleronomic system the Hamiltonian H represents both a constant of motion and total energy.
2. For conservative rheonomic system the Hamiltonian H may represent a constant of motion but does not represent the total energy.

Proof : The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (1)$$

where L is the Lagrangian of the system and

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \ldots (2)$$

is the generalized momentum. This implies from Lagrange's equation of motion that

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad \ldots (3)$$

Differentiating equation (1) w. r. t. time t, we get

$$\frac{dH}{dt} = \sum_j \dot{p}_j \dot{q}_j + \sum_j p_j \ddot{q}_j - \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j - \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \frac{\partial L}{\partial t} \qquad \ldots (4)$$

On using equations (2) and (3) in equation (4) we readily obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \qquad \ldots (5)$$

Now if L does not contain time t explicitly, then from equation (5), we have

$$\frac{dH}{dt} = 0$$

This shows that H represents a constant of motion.

However, the condition L does not contain time t explicitly will be satisfied by neither the kinetic energy nor the potential energy involves time t explicitly.

Now there are two cases that the kinetic energy T does not involve time t explicitly.

1. For the conservative and scleronomic system :

In the case of conservative system when the constraints are scleronomic, the kinetic energy T is independent of time t and the potential energy V is only function of co-ordinates. Consequently, the Lagrangian L does not involve time t explicitly and hence from equation (5) the Hamiltonian H represents a constant of motion.

Further, for scleronomic system, we know the kinetic energy is a homogeneous quadratic function of generalized velocities.

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k \, . \qquad \ldots (6)$$

Hence by using Euler's theorem for the homogeneous quadratic function of generalized velocities we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \, . \qquad \ldots (7)$$

For conservative system we have

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \, . \qquad \ldots (8)$$

Using (7) and (8) in the Hamiltonian H we get

$$H = 2T - (T - V) \, ,$$

$$H = T + V = E \, . \qquad \ldots (9)$$

where E is the total energy of the system. Equation (9) shows that for conservative scleronomic system the Hamiltonian H represents the total energy of the system.

2. For conservative and rheonomic system :

In the case of conservative rheonomic system, the transformation equations do involve time t explicitly, though some times the kinetic energy may not involve time t explicitly. Consequently, neither T nor V involves t, and hence L does not involve t. Hence in such cases the Hamiltonian may represent the constant of motion. However, in general if the system is conservative and rheonomic, the kinetic energy is a quadratic function of generalized velocities and is given by

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j a_j\dot{q}_j + a \qquad \ldots (10)$$

where

$$a_{jk} = \sum_i \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}, \qquad a_j = \sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}, \qquad \ldots (11)$$

$$a = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2.$$

We see from equation (10) that each term is a homogeneous function of generalized velocities of degree two, one and zero respectively. On applying Euler's theorem for the homogeneous function to each term on the right hand side, we readily get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1. \qquad \ldots (12)$$

where

$$T_2 = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k,$$
$$T_1 = \sum_j a_j\dot{q}_j,$$
$$T_0 = a$$

are homogeneous function of generalized velocities of degree two, one and zero respectively. Substituting equation (12) in the Hamiltonian (1) we obtain

$$H = T_2 - T_0 + V$$

showing that the Hamiltonian H does not represent total energy. Thus for the conservative rheonomic systems H may represent the constant of motion but does not represent total energy.

APPLICATION OF HAMILTONIAN EQUATION OF MOTION TO

(i)SIMPLE PENDULUM:

$$L=\frac{1}{2}ml^2\dot{\theta}^2-mgl\left(1-\cos\theta\right), \qquad \ldots(1)$$

where the generalized momentum is given by

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta}\Rightarrow \quad \dot{\theta}=\frac{p_\theta}{ml^2}. \qquad \ldots(2)$$

The Hamiltonian of the system is given by

$$H = p_\theta\dot{\theta}-L,$$

$$\Rightarrow \quad H = p_\theta\dot{\theta}-\frac{1}{2}ml^2\dot{\theta}^2+mgl\left(1-\cos\theta\right).$$

Eliminating $\dot{\theta}$ we obtain

$$H=\frac{p_\theta^2}{2ml^2}+mgl\left(1-\cos\theta\right). \qquad \ldots(3)$$

Hamilton's canonical equations of motion are

$$\dot{q}_j=\frac{\partial H}{\partial p_j}, \quad \dot{p}_j=-\frac{\partial H}{\partial q_j}.$$

These equations give

$$\dot{\theta}=\frac{p_\theta}{ml^2}, \quad \dot{p}_\theta=-mgl\sin\theta. \qquad \ldots(4)$$

Now eliminating p_θ from these equations we get

$$\ddot{\theta}+\frac{g}{l}\sin\theta=0. \qquad \ldots(5)$$

Now we claim that H represents the constant of motion.

Thus differentiating equation (3) with respect to t we get

$$\frac{dH}{dt} = \frac{p_\theta \dot{p}_\theta}{ml^2} + mgl\sin\theta\,\dot{\theta},$$

$$= ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta},$$

$$= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right),$$

$$\frac{dH}{dt} = 0.$$

This proves that H is a constant of motion. Now to see whether H represents total energy or not, we consider

$$T + V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta).$$

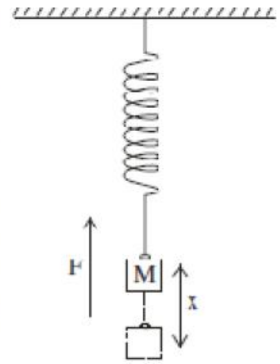
Using equation (4) we eliminate $\dot{\theta}$ from the above equation, we obtain

$$T + V = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos\theta). \qquad \ldots(6)$$

This is as same as the Hamiltonian H from equation (3). Thus Hamiltonian H represents the total energy of the pendulum.

(II)LINEAR HARMONIC OSCILLATOR:

Solution: The one dimensional harmonic oscillator consists of a mass attached to one end of a spring and other end of the spring is fixed. If the spring is pressed and released then on account of the elastic property of the spring, the spring exerts a force F on the body in the opposite direction. This is called restoring force. It is found that this force is proportional to the displacement of the body from its equilibrium position.



$$F \propto x$$
$$F = -kx$$

where k is the spring constant and negative sign indicates the force is opposite to the displacement. Hence the potential energy of the particle is given by

$$V = -\int F dx,$$
$$V = \int kx dx + c,$$
$$V = \frac{kx^2}{2} + c,$$

where c is the constant of integration. By choosing the horizontal plane passing through the position of equilibrium as the reference level, then V=0 at x=0. This gives c=0. Hence potential energy of the particle is

$$V = \frac{1}{2}kx^2. \qquad \ldots (1)$$

The kinetic energy of the one dimensional harmonic oscillator is

$$T = \frac{1}{2}m\dot{x}^2. \qquad \ldots (2)$$

Hence the Lagrangian of the system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \qquad \ldots (3)$$

The Lagrange's equation motion gives

$$\ddot{x} + \omega^2 x = 0, \quad \omega^2 = \frac{k}{m}. \qquad \ldots (4)$$

This is the equation of motion. ω is the frequency of oscillation.

The Hamiltonian H of the oscillator is defined as

$$H = \dot{x}p_x - L,$$
$$H = \dot{x}p_x - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2,$$

where

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \quad \dot{x} = \frac{p_x}{m}.$$

Substituting this in the above equation we get the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2. \qquad \ldots (5)$$

Solving the Hamilton's canonical equations of motion we readily get the equation (4) as the equation of motion.

POSSIBLE QUESTIONS:

PART B: (6 MARK)

- Define Phase Space. Explain?
- Derive an expression for Hamilton's variational principle.
- Describe the Hamilton's canonical equations of motion.
- What are the physical significance of H.
- Give any two application of Hamiltonian equation of motion.
- Discuss about the simple pendulum.
- Linear harmonic oscillator.

S.No	QUESTIONS	Option A	Option B	Option C	Option D	ANSWER
1	Canonical transformations are the transformations of	Phase space	Hillbert space	Minkowski space	Space phase	Phase space
2	The Hamilton's principle function is a generating function, which give rise to canonical transformation involving	both constant moments and co-ordinates	constant moments only	co-ordinates only	constant momenta and co-ordinates	both constant moments and co-ordinates
3	All function whose Poisson bracket with the Hamiltonian vanishes will be	constant of motion	constant of momentum	constant of co-ordinates	all the above	constant of motion
4	Let L and P represent the matrices of Lagrange and Poisson brackets respectively, then	LP = 1	LP = -1	LP = -1/2	LP = 1/2	LP = -1
6	The given transformation is not canonical when	[Q,P] = 1	[Q,P] = -1	[Q,P] = 1/2	[Q,P] = 0	[Q,P] = 0
7	The function p =1/Q and q = PQ2 is	conjugate	canonical	identical	hyrebolic	canonical
8	In point transformation one set of co-ordinates qj to a new set	Qj = Qj (qj, t)	Qj = -Qj (qj, t)	Qj = Pj (qj, t)	Qj = -Pj (qj, t)	Qj = Qj (qj, t)

	Qj can be expressed as					
9	The problem consists on finding the path of a charged particle under the action if a central force is	Jacobi problem	cononical problem	Kepler problem	Poission problem	Kepler problem
10	Hamilton – Jacobi method is used to find the solution of problem in	Vibratory motion	periodic motion	circular mation	all the above	periodic motion
11	Hamilton equation of motion is	convergent	divergent	variant	invariant	invariant
12	Poisson and Lagrange brackets are _______ under Canonical Transformation	convergent	divergent	invariant	variant	invariant
13	Equation of motion in Poisson bracket from depends on	position	momentum	time	all the three	all the three
15	In Kepler problem, the path of the particle is	circular	parabolic	elliptical	zig-zag	elliptical
16	In Poisson bracket	[X,Y] = [Y,X]	[X,Y] = - [Y,X]	[X,Y] = 2[Y,X]	[X,Y] = -2[Y,X]	[X,Y] = - [Y,X]
17	In Poisson bracket	[X,X] =0	[X,X] =1	[X,X] =2	[X,X] = -2	[X,X] =0
18	In Poisson bracket	[X,Y+Z] = [X,Y] - [X,Z]	[X,Y+Z] = [X,Y] * [X,Z]	[X,Y+Z] = [X,Y] + [X,Z]	[X,Y+Z] = [X,Y] / [X,Z]	[X,Y+Z] = [X,Y] + [X,Z]

19	In Poisson bracket	[X,YZ] = Y[X,Z] * [X,Y]Z	[X,YZ] = Y[X,Z] - [X,Y]Z	[X,YZ] = Y[X,Z] / [X,Y]Z	[X,YZ] = Y[X,Z] + [X,Y]Z	[X,YZ] = Y[X,Z] + [X,Y]Z
20	In Lagrange bracket	[X,qj]Q,P = - [qj,X]Q,P	[X,qj]Q,P = [qj,X]Q,P	[X,qj]Q,P = 2 [qj,X]Q,P	2[X,qj]Q,P = - [qj,X]Q,P	2[X,qj]Q,P = - [qj,X]Q,P
21	In of Lagrange bracket	[X,Y]Q,P = - [X,Y]q,p	[X,Y]Q,P = [X,Y]q,p	[X,Y]Q,P = 2[X,Y]q,p	[X,Y]Q,P = - 2[X,Y]q,p	[X,Y]Q,P = [X,Y]q,p
22	In of Lagrange bracket	[X,X]q,p = [X,X]Q,P = 1	[X,X]q,p = [X,X]Q,P = -1	[X,X]q,p = [X,X]Q,P = 0	[X,X]q,p = [X,X]Q,P = 1/2	[X,X]q,p = [X,X]Q,P = 0
23	Poisson bracket of two operator X and Y in quantum mechanics is given by	[X,Y] = - 2p/h[XY-YX]	[X,Y] = - 2p/h[XY+YX]	[X,Y] = - p/h[XY-YX]	[X,Y] = 2p/h[XY-YX]	[X,Y] = - 2p/h[XY-YX]
24	If the Lagrangian of the system does not contain a paricular co-ordinate q, then	cyclic co-ordinates	cylindrical co-ordinates	polar co-ordinates	spherical polar co-ordinates	cyclic co-ordinates

25	Hamilton-Jacobi is a partial differential equation in ________ variables.	n	n+1	n-1	n+2	n+1
26	_____________ is a partial differential equation in (n+1) variables.	Hamilton-Jacobi equation	Lagrangian	Hamiltonian	Jacobian	Hamilton-Jacobi equation
27	Hamilton's characteristic function W is identified as ______________.	kinetic energy	potential energy	work	action A	action A
28	Hamilton's characteristic function is denoted by ________________.	S	K	W	H	W
29	The number of independent ways in which a mechanical system can move without violating any constraint which may be imposed is called the _______________.	action-angle variables	generalized variables	degrees of freedom	co-ordinates	degrees of freedom
32	Path in phase space almost refers to actual _____________path.	statistical	N	3N	dynamical	dynamical
33	The one way of obtaining the solution of mechanical problem is to transform __________ set of co-ordinates to ___________ set of co-ordinates that are all	old to new	new to old	new to new	old to old	old to new

	cyclic.					
34	If the operators X, Y commute, then [X, Y] = _____________.	1	-1	0	-2	0
35	If [X, Y] = 0, then X and Y behave like __________ variables of classical mechanics.	statistical	dynamical	proportional	inversely proportional	dynamical
36	If Poisson bracket of two variables in classical mechanics is zero, then the operators which represent these variables in quantum theory should _____________.	vanish	be multiplied twice	proportional	commute	commute
37	The Lagrange's bracket is ____________ under canonical transformation.	invariant	variant	not applicable	exponentially variant	invariant
38	Lagrange's equation of motion are second order equations with __________ degrees of freedom.	n+1	n	2n+1	3n	2n+1

39	The greatest advantage of action and angle variable is that we can obtain the _______________ of periodic motion without finding a complete solution for the motion of the system.	displacement	frequencies	total time	accelerations	frequencies
40	The generalized co-ordinate conjugate to Jj are called ________________.	action variable	dynamic variable	statistical variable	angle variable	angle variable
41	Jj has the dimension of _____________.	angular momentum	angular velocity	linear momentum	linear velocity	angular momentum
42	If F does not involve time explicitly, then the Poisson bracket of F with H _______________.	is proportional with F	is proportional with K	Vanishes	exist	Vanishes
43	If the Poisson bracket of F with H vanishes then F will be a ________________.	positive value	constant of motion	negative value	same value	constant of motion
44	If Poisson bracket of momentum with H vanishes, then _________________ is conserved.	linear velocity	energy	angular momentum	linear momentum	linear momentum

45	If Poisson bracket of momentum with H vanishes, then the co-ordinate momenta is ______________.	cyclic	rotational	irrotational	spherical	cyclic
46	Lagrange's bracket does not obey the _____________law.	associative	kepler's	commutative	Hamilton's variational law	commutative
47	H = _____________.	T- V	T + V	T	V	T + V
48	L = __________.	T + V	T	V	T-V	T-V
49	In case of either of the set of conjugate variables with (q, p) or with (Q, P), the value of the Poisson bracket remains _____________.	same	proportional	inversely proportional	exponentially proportional	same
50	In new set of co-ordinates all Qj are ________________.	rotational	irrotational	cyclic	variable	cyclic
51	In new set of co-ordiantes all Pj are _________.	cyclic	constant	rotational	irrotational	constant
52	If H is conserved then the new Hamiltonian K is ___________.	same	variable	different	constant of motion	constant of motion
53	An assembly of particles with _________inter-particle distance is called as rigid body	fixed	different	1 mm	2 mm	fixed

54	Degree of freedom to fix the configuration of a rigid body is	3	6	5	0	6
55	These are most useful set of generalised co-ordinates for a rigid body and are angles	Lagrangian angle	azimuthal angle	Euler's angle	Larmor's precession angle	Euler's angle
56	Angular momentum of a rigid body is	L = Iw/2	L = 2Iw	L = Iw2	L = Iw	L = Iw
57	A mathematical structure having nine components in 3-D is termed as tensor of rank	2	3	4	0	2
58	The rotation about space z-axis (angle f) is called ________	translation	precession	nutation	spin.	precession
59	Rotation about intermediate X1 axis (angle q) or line of nodes is called	translation	precession	nutation	spin.	nutation
60	The rotation about z' axis (angle Y) is called	translation	precession	nutation	spin.	spin.
61	The variation of angle q is referred as ________of the symmetry axis of the top and is	translation	precession	nutation	spin.	nutation
62	Precession can be	slow or fast	always slow	always fast	neither fast nor slow	always slow

63	______is ordinarily observed with a rapidly spinning top.	fast precession	slow precession	slow nutation	fast nutation	slow precession
64	In case of ______ top amplitude of nutation is small, nutation is sinusoidal,	slow	rotating	fast	both a & b	fast
65	The minimum spin angular velocity below which top cannot spin stably about vertical	wmin = (4mglI1/I32)	wmin = (4mglI1/I32)3/2	wmin = (4mglI1/I32)2	wmin = (4mglI1/I32)1/2	wmin = (4mglI1/I32)1/2
66	When wz < wmin then the top begins to	wobble	precesse	nutate	spin.	wobble
67	Angular velocity of a rigid body is given by	Vi = w2 x ri	Vi = (w x ri)1/2	Vi = w x ri	Vi = w3 x ri	Vi = w x ri
68	Angular momentum of a rigid body is L =	S m2(ri x Vi)	S m(ri x Vi)2	S m2(ri x Vi)2	S m(ri x Vi)	S m(ri x Vi)
69	The diagonal elements Ixx, Iyy, Izz of inertia _______I are moments of inertia	tensor	vector	scalar	donar	tensor
70	Tensor I is __________to principal axes	symmetric	antisymmetric	parallel	perpendicular	symmetric

UNIT-III

SYLLABUS

General features of central force motion : General features of orbits – Centre of mass and laboratory coordinates – Virial theorem – Stable and unstable equilibrium – Properties of T, V and ω for small oscillations .
Generalized coordinates for rigid body motion : Euler's angles – Angular velocity, momentum of rigid body – moment and products of inertia – Principal axis transformation – rotational kinetic energy of a rigid body – Moment of inertia of a rigid body – motion of a symmetric top under action of gravity.

General features of central force motion

In classical mechanics, a **central force** is a force whose magnitude only depends on the distance r of the object from the origin and is directed along the line joining them: [1]

$$\vec{F} = \mathbf{F}(\mathbf{r}) = F(||\mathbf{r}||)\hat{\mathbf{r}}$$

where $\vec{F}$ is the force, **F** is a vector valued force function, F is a scalar valued force function, **r** is the position vector, ||**r**|| is its length, and $\hat{\mathbf{r}}$ = **r**/||**r**|| is the corresponding unit vector.

Equivalently, a force field is central if and only if it is spherically symmetric.

A central force is a conservative field, that is, it can always be expressed as the negative gradient of a potential:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}), \text{ where } V(\mathbf{r}) = \int_{|\mathbf{r}|}^{+\infty} F(r)\,\mathrm{d}r$$

(the upper bound of integration is arbitrary, as the potential is defined up to an additive constant).

In a conservative field, the total mechanical energy (kinetic and potential) is conserved:

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) = \text{constant}$$

(where ṙ denotes the derivative of **r** with respect to time, that is the velocity), and in a central force field, so is the angular momentum:

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = \text{constant}$$

because the torque exerted by the force is zero. As a consequence, the body moves on the plane perpendicular to the angular momentum vector and containing the origin, and obeysKepler's second law. (If the angular momentum is zero, the body moves along the line joining it with the origin.)

As a consequence of being conservative, a central force field is irrotational, that is, its curl is zero, except at the origin:

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}.$$

General features of orbit

The essential elements of the object are described by a set, and the symmetries of the object are described by the symmetry group of this set, which consists of bijectivetransformations of the set. In this case, the group is also called a **permutation group** (especially if the set is finite or not a vector space) or **transformation group** (especially if the set is a vector space and the group acts like linear transformations of the set).

A group action is an extension to the definition of a symmetry group in which every element of the group "acts" like a bijective transformation (or "symmetry") of some set, without being identified with that transformation. This allows for a more comprehensive description of the symmetries of an object, such as a polyhedron, by allowing the same group to act on several different sets of features, such as the set of vertices, the set of edges and the set of faces of the polyhedron.

If G is a group and X is a set then a group action may be defined as a group homomorphism from G to the symmetric group of X. The action assigns a permutation of X to each element of the group in such a way that the permutation of X assigned to:

- The identity element of G is the identity transformation of X;
- A product gh of two elements of G is the composite of the permutations assigned to g and h.

Since each element of G is represented as a permutation, a group action is also known as a **permutation representation**.

The abstraction provided by group actions is a powerful one, because it allows geometrical ideas to be applied to more abstract objects. Many objects in mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves. Despite this generality, the theory of group actions contains wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several fields.

Laboratory Frame and the Center-of-Mass Frame

When the potential is central, the problem can be reduced to the one we have just studied; this can be achieved through the separation of the motion of the center of mass.

Let us assume that we have two particles with masses m_1 and m_2, at coordinates $\vec{r}_1$ and $\vec{r}_2$, interacting through a central potential. The equations for the motion can be written as

$$m_1\ddot{\vec{r}}_1 = -\vec{\nabla}_1 V(|\vec{r}_1 - \vec{r}_2|),$$

$$m_2\ddot{\vec{r}}_2 = -\vec{\nabla}_2 V(|\vec{r}_1 - \vec{r}_2|),$$

where $\vec{\nabla}$ is the gradient operator, which has the following form in spherical coordinates

$$\vec{\nabla}_i = \hat{r}_i \frac{\partial}{\partial r_i} + \frac{\hat{\theta}_i}{r_i}\frac{\partial}{\partial \theta_i} + \frac{\hat{\phi}_i}{r_i \sin\theta_i}\frac{\partial}{\partial \phi_i} \qquad i = 1, 2.$$

Since the potential energy depends only on the relative separation of the two particles, let us define the variables:

$$\vec{r} = \vec{r}_1 - \vec{r}_2,$$

$$\vec{R}_{\rm CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2},$$

where $\vec{r}$ denotes the coordinate of m_1 relative to m_2, and $\vec{R}_{\rm CM}$ defines the coordinate of the center-of-mass of the system (see Fig. 1.5). From Eqs. (1.42) and (1.44) we can easily obtain the following:

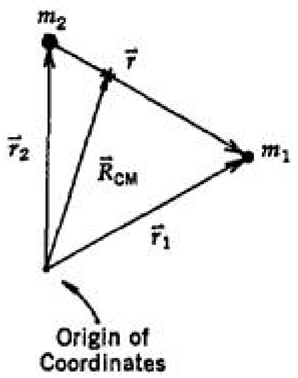
$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} \equiv \mu\ddot{\vec{r}} = -\vec{\nabla}V(|\vec{r}|) = -\frac{\partial V(|\vec{r}|)}{\partial r}\hat{r},$$

$$(m_1 + m_2)\ddot{\vec{R}}_{\rm CM} = M\ddot{\vec{R}}_{\rm CM} = 0, \quad \text{or} \quad \dot{\vec{R}}_{\rm CM} = \text{constant} \times \hat{R},$$

where we have used the fact that $V(\vec{r}) = V(r)$ depends only on the radial coordinate r, and not on the angular variables associated with $\vec{r}$, and where we have defined

$$M = m_1 + m_2 = \text{ total mass of the system},$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{ “reduced” mass of the system.}$$



$\vec{r}_1 = r_1\,\hat{r}_1$
$\vec{r}_2 = r_2\,\hat{r}_2$
$\vec{r} = r\,\hat{r}$
$\vec{R}_{CM} = R_{CM}\,\hat{R}_{CM}$

Virial theorem

A theorem in classical mechanics which relates the kinetic energy of a system to the virial of Clausius, as defined below. The theorem can be generalized to quantum mechanics and has widespread application. It connects the average kinetic and potential energies for systems in which the potential is a power of the radius. Since the theorem involves integral quantities such as the total kinetic energy, rather than the kinetic energies of the individual particles that may be involved, it gives valuable information on the behavior of complex systems. For example, in statistical mechanics the virial theorem is intimately connected to the equipartition theorem; in astrophysics it may be used to connect the internal temperature, mass, and radius of a star and to discuss stellar stability.

The virial theorem makes possible a very easy derivation of the counterintuitive result that as a star radiates energy and contracts it heats up rather than cooling down. The virialtheorem states that the time-averaged value of the kinetic energy in a confined system (that is, a system in which the velocities and position vectors of all the particles remain finite) is equal to the virial of Clausius. The virial of Clausius is defined to equal −½ times the time-averaged value of a sum over all the particles in the system. The term in this sum associated with a particular particle is the dot product of the particle's position vector and the force acting on the particle. Alternatively, this term is the product of the distance, r, of the particle from the origin of coordinates and the radial component of the force acting on the particle.

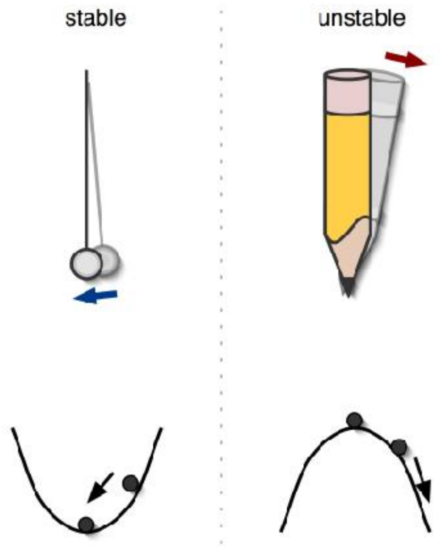
In the common case that the forces are derivable from a power-law potential, V, proportional to r^k, where k is a constant, the virial is just −k/2 times the potential energy. Thus, in this case the virial theorem simply states that the kinetic energy is k/2 times the potential energy. For a system connected by Hooke's-law springs, k = 2, and the average kinetic and potential energies are equal. For k = 1, that is, for gravitational or Coulomb forces, the potential energy is minus twice the kinetic energy.

Stable and unstable equilibrium

Equilibrium is a state of a system in which the variables which describe the system are not changing (note that a system can be in a dynamic equilibrium where things might be moving or changing, but some variable(s) which describe the system as a whole is(are) constant). One example you are all familiar with is a mechanical system in equilibium where positions of objects are not changing (ie. no net forces acting).

In a **Stable equilibrium** if a small perturbation away from equilibrium is applied, the system will return itself to the equilibrium state. A good example of this is a pendulum hanging straight down. If you nudge the pendulum slightly, it will experience a force back towards the equilibrium position. It may oscillate around the equilibrium position for a bit, but it will return to its equilibrium position.

In an **Unstable equilibrium** if a small perturbation away from equilibrium is applied, the system will move farther away from its equilibrium state. A good example of this is a pencil balanced on it's end. If you nudge the pencil slightly, it will experience a force moving it away from equilibrium. It will simply fall to lying flat on a surface.



Properties of T, V and ω for small oscillations

Consider a small mass on the free end of a spring. If we displace the mass slightly away from equilibrium, the elastic force will accelerate it back toward its equilibrium position. When it reaches equilibrium, however, it has a nonzero momentum and overshoots that position. The elastic force now accelerates the mass in the opposite direction, back toward the equilibrium position. This periodic motion is called oscillation.

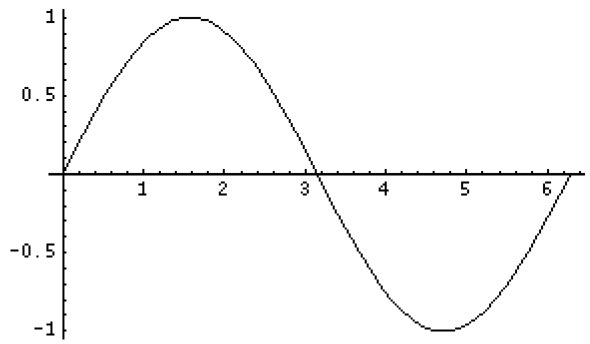
If we combine Hooke's Law with Newton's Law, we find that

$m a = - k x,$

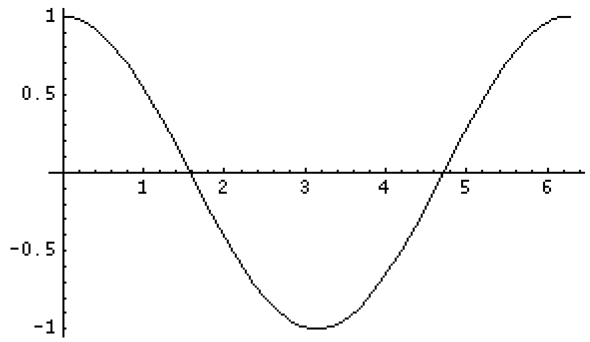
or

$a = (-k / m) x.$

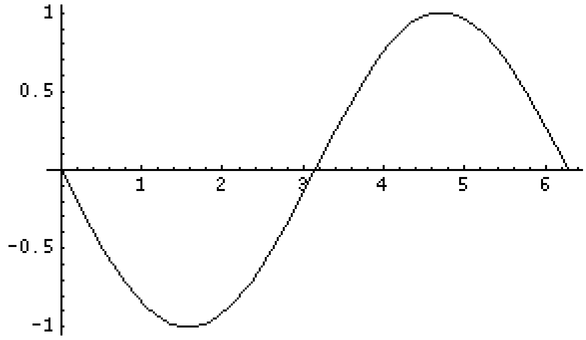
In words, this means that the rate of change of the rate of change of position is proportional to the position. Graphically, we can try to picture the position as an oscillatory function of time: perhaps a sine function:



At each point on that graph, the slope gives us the velocity of the mass at that time. The velocity graph must also be an oscillatory function of time:



and the slope at any point on this graph gives us the acceleration of the mass. Clearly, the graph of acceleration versus time must also be oscillatory, and to satisfy our equation, every point on it must be proportional to the value of the position at that time, but reflected about the x axis because of the minus sign:



Both the sine and cosine functions have this property: the slope of the slope of the function at any point is proportional to the negative of the function. To be specific, our simple harmonic oscillator could be described by either

x(t) = sin (ω t)

or

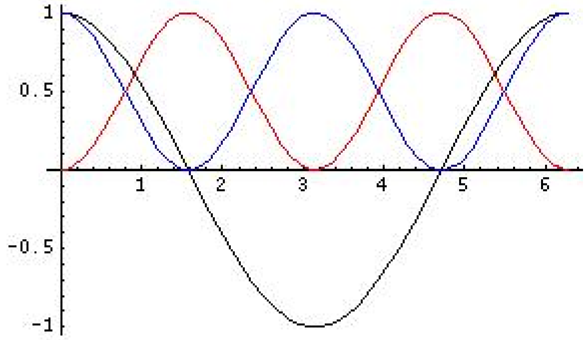
x(t) = cos (ω t),

where

ω = (k / m)1/2.

In this case, we choose the cosine function, because at time t = 0 the mass was displaced a small distance from the origin; since sin (t) is zero at time zero, only the cosine can describe these oscillations.

When we plot the position in black, the square of the velocity, which is proportional to the mass' kinetic energy, in red, and the square of the position, which is proportional to its potential energy, in blue:



The kinetic energy is always a maximum at the equilibrium position, where the potential energy is zero, and the potential energy is always a maximum at the extremes of the oscillation, where the velocity (and kinetic energy) is zero. Conservation of energy then tells us that the total energy of the oscillator is just the potential energy at the maximum displacement from equilibrium. This displacement is called the amplitude A, and the total energy is

k A2 / 2.

In fact, the position and velocity of this oscillator are

A cos (ω t) and

-A ω sin (ω t),

so that the sum of the kinetic and potential energies is

m (-A ω sin (ω t))2 / 2 + k (A cos (ω t))2 / 2,

= m ω2 A2 sin (ω t)2 / 2 + k A2 cos (ω t)2 / 2

= k A2 (sin (ω t)2 + cos (ω t)2) / 2

= k A2 / 2

at every point along its trajectory.

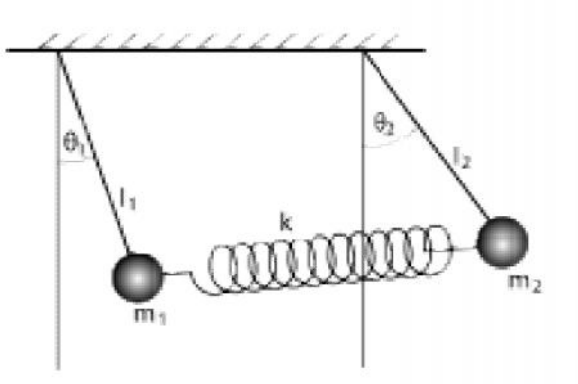
The arguments of trigonometric functions must always be unitless. The variable ω (which in rotational motion was used to denote the angular velocity) is called the angular frequency and has units of 1 / s, so that the argument of the cosine function is indeed unitless. Dividing ω by 2π we find the frequency ν (the Greek letter nu) which is the number of oscillations or cycles per second from the maximum amplitude through zero to the minimum amplitude and back to the maximum again (each of the graphs above was one cycle). The inverse of the frequency is the period T, which is the time in seconds for one oscillation (and is therefore always positive).

In general, the argument of an oscillatory function is called the phase. The phase can also be a function of x: k x. This k is called the wave number and has units of 1 / m. The wavelength λ (the Greek letter lambda) is 2π / k and is analogous to the period. Like the period, the wavelength is always positive.

The phase can also include a term which is a unitless number (denoted by the Greek letter delta), so in its most general form the phase is written

k x +- ω t +- δ.δ is called the phase angle, and effectively allows us to specify the relative starting point of the oscillator at time zero. By experimenting with various values of δ (ie., 0, π/2, π, 3π/2, 2π), we see that we can produce oscillations which have any given initial value (between -A and A)attimezero.

Parallel pendula



This shows a parallel pendula of lengths ℓ_1, ℓ_2 and masses m_1, m_2 are not equal and/or the equilibrium length of the spring is not equal to the horizontal distance between the pendulum supports. So let's make the simplifying assumptions that $\ell_1 = \ell_2 = \ell$, $m_1 = m_2 = m$, and the relaxed spring length $X_0 = d$ the distance between the supports. Then the small oscillations Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\ell^2\left(\dot\theta_1^2 + \dot\theta_2^2\right) - \frac{1}{2}mg\ell\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{2}k\ell^2\left(\theta_1 - \theta_2\right)^2$$

and the force and mass matrices are

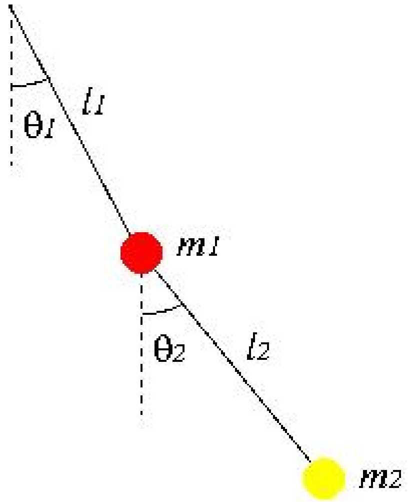
$$\mathbf{K} = mg\ell\begin{pmatrix} 1+\epsilon & -\epsilon \\ -\epsilon & 1+\epsilon \end{pmatrix}, \qquad \mathbf{M} = m\ell^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \epsilon = \frac{k\ell}{mg}$$

This system has two normal modes with frequencies

$$\omega_1^2 = \frac{g}{\ell}, \qquad \omega_2^2 = \frac{g}{\ell}(1 + 2\epsilon)$$

Double pendulum

Let us consider a double pendulum as shown in a below figure.



Although the double pendulum is often introduced in many textbooks of the classical mechanics, its dynamics are seldom analyzed in them. Actually, it is known that it easily exhibits chaotic behaviors. In the double pendulum, the effect of the friction around the axis of rotation is not considered. Therefore, the energy of the system is conserved, and such a system is called a Hamiltonian system or a conservative system.

The energy E of the system is a sum of the kinetic energy K and the potential energy U written as

$$\begin{aligned} K &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2), \\ U &= (m_1+m_2)gl_1(1-\cos\theta_1) - m_2 g l_2(1-\cos\theta_2), \\ E &= K+U. \end{aligned}$$

Using the Lagrange differential equation, a set of differential equations which governs the dynamics of the double pendulum is obtained, and it is written as

$$\begin{aligned} \ddot{\theta}_1 + Ml\ddot{\theta}_2 \cos\Delta\theta + Ml\dot{\theta}_2^2 \sin\Delta\theta + \omega^2 \sin\theta_1 &= 0, \\ \ddot{\theta}_1 \cos\Delta\theta + l\ddot{\theta}_2 - \dot{\theta}_1^2 \sin\Delta\theta + \omega^2 \sin\theta_2 &= 0, \end{aligned}$$

$$\Delta\theta \equiv \theta_1 - \theta_2$$

$$M \equiv m_2/(m_1 + m_2)$$

$$l \equiv l_2/l_1$$

$$\omega^2 \equiv g/l_1$$

From the above equations, the second derivatives of angles are obtained as follows.

$$\begin{aligned} \ddot{\theta}_1 &= \frac{\omega^2 l(-\sin\theta_1 + M\cos\Delta\theta \sin\theta_2) - Ml(\dot{\theta}_1^2 \cos\Delta\theta + l\dot{\theta}_2^2)\sin\Delta\theta}{l - Ml\cos^2\Delta\theta}, \\ \ddot{\theta}_2 &= \frac{\omega^2 \cos\Delta\theta \sin\theta_1 - \omega^2 \sin\theta_2 + (\dot{\theta}_1^2 + Ml\dot{\theta}_2^2 \cos\Delta\theta)\sin\Delta\theta}{l - Ml\cos^2\Delta\theta}. \end{aligned}$$

Regarding the above differential equations as a differential equation $\dot{x} = f(x)$ for a vector $x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$, behaviors of a double pendulum can be analyzed.

Because the double pendulum is a Hamiltonian system (a conservative system) where the energy of the system is conserved, one must use numerical integration methods which conserve the energy.

Here we used the fourth order implicit Gaussian method written as

$$\begin{aligned} k_1 &= dt\, f(x_0 + a_{11}k_1 + a_{12}k_2), \\ k_2 &= dt\, f(x_0 + a_{21}k_1 + a_{22}k_2), \\ x_1 &= x_0 + (k_1 + k_2)/2, \\ a_{11} &= 1/4, \\ a_{12} &= 1/4 - \sqrt{3}/6, \\ a_{21} &= 1/4 + \sqrt{3}/6, \\ a_{22} &= 1/4. \end{aligned}$$

Triatomic Molecule

Consider the simple model of a linear triatomic molecule (*e.g.*, carbon dioxide) illustrated in Figure. The molecule consists of a central atom of mass M flanked by two identical atoms of mass m. The atomic bonds are represented as springs of spring constant k. The linear displacements of the flanking atoms are q_1 and q_2, whilst that of the central atom is q_3. Let us investigate the linear modes of oscillation our model molecule.

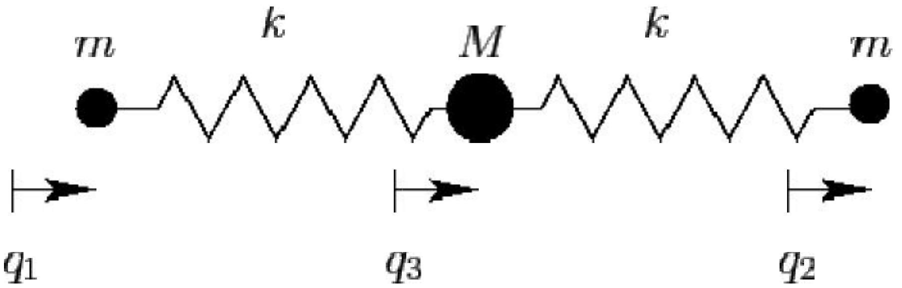


Figure 38: *A model triatomic molecule.*

The kinetic energy of the molecule is written

$$K = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{M}{2}\dot{q}_3^2,$$

whereas the potential energy takes the form

$$U = \frac{k}{2}(q_3 - q_1)^2 + \frac{k}{2}(q_2 - q_3)^2.$$

Clearly, we have a three degree of freedom dynamical system. However, we can reduce this to a two degree of freedom system by only considering *oscillatory* modes of motion, and, hence, neglecting*translational* modes. We can achieve this by demanding that the center of mass of the system remains stationary. In other words, we require that

$$m\,(q_1 + q_2) + M\,q_3 = 0.$$

This constraint can be rearranged to give

$$q_3 = -\frac{m}{M}\,(q_1 + q_2).$$

Eliminating q_3 from Equations, we obtain

$$K = \frac{m}{2}\left[(1+\alpha)\,\dot{q}_1^{\,2} + 2\,\alpha\,\dot{q}_1\,\dot{q}_2 + (1+\alpha)\,\dot{q}_2^{\,2}\right],$$

and

$$U = \frac{k}{2}\left[(1+2\,\alpha+2\,\alpha^2)\,q_1^{\,2} + 4\,\alpha\,(1+\alpha)\,q_1\,q_2 + (1+2\,\alpha+2\,\alpha^2)\,q_2^{\,2}\right],$$)

respectively, where $\alpha = m/M$.

A comparison of the above expressions with the standard forms and yields the following expressions for the mass matrix, $\mathbf{M}$, and the force matrix, $\mathbf{G}$:

$$\mathbf{M} \;=\; m\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & 1+\alpha \end{pmatrix},$$

$$\mathbf{G} \;=\; -k\begin{pmatrix} 1+2\,\alpha+2\,\alpha^2 & 2\,\alpha\,(1+\alpha) \\ 2\,\alpha\,(1+\alpha) & 1+2\,\alpha+2\,\alpha^2 \end{pmatrix}.$$

Now, the equation of motion of the system takes the form

$$(\mathbf{G} - \lambda\,\mathbf{M})\,\mathbf{x} = \mathbf{0},$$

where $\mathbf{x}$ is the column vector of the q_1 and q_2 values. The solubility condition for the above equation is

$$|\mathbf{G} - \lambda \mathbf{M}| = 0,$$

which yields the following quadratic equation for the eigenvalue λ:

$$(1 + 2\alpha)\left[m^2 \lambda^2 + 2\, m\, k\, (1 + \alpha)\, \lambda + k^2 (1 + 2\alpha)\right] = 0.$$

The two roots of the above equation are

$$\lambda_1 = -\frac{k}{m},$$

$$\lambda_2 = -\frac{k(1 + 2\alpha)}{m}.$$

The fact that the roots are negative implies that both normal modes are indeed *oscillatory* in nature. The characteristic oscillation frequencies are

$$\omega_1 = \sqrt{-\lambda_1} = \sqrt{\frac{k}{m}},$$

$$\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{k\,(1 + 2\,\alpha)}{m}}.$$

Equation can now be solved, subject to the normalization condition to give the two eigenvectors:

$$\mathbf{x}_1 = (2m)^{-1/2}(1, -1),$$

$$\mathbf{x}_2 = (2m)^{-1/2}(1 + 2\,\alpha)^{-1/2}(1, \quad 1).$$

Thus, we conclude from Equations that our model molecule possesses two normal modes of oscillation.

The first mode oscillates at the frequency ω_1, and is an *anti-symmetric* mode in which $q_1 = -q_2$ and $q_3 = 0$. In other words, in this mode of oscillation, the two end atoms move in opposite directions whilst the central atom remains stationary. The second mode oscillates at the frequency ω_2, and is a mixed symmetry mode in which $q_1 = q_2$ but $q_3 = -2\,\alpha\, q_1$. In other words, in this mode of oscillation, the two end atoms move in the same direction whilst the central atom moves in the opposite direction.

Finally, it is easily demonstrated that the normal coordinates of the system are

$$\eta_1 = \sqrt{\frac{m}{2}}\,(q_1 - q_2),$$

$$\eta_2 = \sqrt{\frac{m\,(1+2\,\alpha)}{k}}\,(q_1 + q_2).$$

When expressed in terms of these coordinates, K and U reduce to

$$K = \frac{1}{2}\,(\dot{\eta}_1^2 + \dot{\eta}_2^2),$$

$$U = \frac{1}{2}\,(\omega_1^2\,\eta_1^2 + \omega_2^2\,\eta_2^2),$$

respectively.

Rigid Body

A macroscopic object can often be approximated by a "particle", which has a mass and position in space. A particle has one physical parameter, its mass, and three translational degress of freedom because it can move in 3-dimensional space.

The equations of motion of a particle can be generalized to a system of N particles. Such a system is defined by N mass parameters, and has $3N$ translational degrees of freedom. Its configuration at any time can be represented by N points in 3-dimensional space, or by a single point in $3N$ dimensional configuration space.

When the size and shape of a macroscopic object matters, it can often be approximated by a "rigid body". A rigid body is a system of particles in which every pair of particles has fixed relative displacement. This is an approximation because the smallest parts of objects are atoms which do not have definite positions according to quantum theory. It is also an approximation because a change in position of one particle cannot affect the position of another particle instantaneously according to the theory of relativity.

Suppose that the rigid body is made of N particles or "atoms", how many degrees of freedom does it have, and how many physical parameters are needed to describe it? Its location and orientation are completely fixed by specifying the positions in space of any three non-collinear particles. A rigid triatomic molecule, which can translate and rotate but not vibrate, has 6 degrees of freedom, 3 translational and 3 rotational. Therefore a rigid body also has 6 degrees of freedom. The configuration space of a rigid body is the product space of a 3-dimensional Euclidean space of translational motion with a 3-dimensional closed ball of radius π with antipodal points identified. The rotations of a rigid body belong to the rotation group SO(3), which is an extremely important concept in physics.

The number of physical parameters required to describe a rigid body approximated by N particles is N masses plus $3N-6$ parameters to specify the fixed relative locations of all the particles.

Generalized coordinates

Eulerian Angles and Euler's Equations

The description of a rigid body is simplest in the body-fixed reference frame which uses the principal axes coordinate system. The moment of inertia tensor is diagonal and constant. The equations of motion are easily expressed in terms of the angular velocity components $\omega_1, \omega_2, \omega_3$ along the principal axes directions.

Rigid bodies are usually observed from a space-fixed inertial reference frame. The moment of inertia tensor is not diagonal in general, and its components change with time. We would like to write the equations of motion in terms of vector components in the inertial reference frame.

Euler introduced a very convenient notation for relating quantities in the two frames in terms of Euler angles.

To focus on rotational motion, suppose that the origin of coordinates in the inertial frame is chosen to coincide with the origin in the body-fixed frame at a particular instant of time t, and that the inertial frame is moving with the same instantaneous velocity as the rigid body at this time t. Of course this will change with time if the body is accelerating, but we just want to obtain the form of the equations in the fixed frame at this instant: by Galilean invariance, this form will hold in all inertial frames.

Figure shows a standard definition of the Euler angles ϕ, θ, ψ. The intersection of the inertial and body-fixed x - y planes is called the line of nodes. The coordinate systems are both right-handed, θ is a polar angle in the range $[0, \pi]$, and ϕ, ψ are azimuthal angles in the range $[0, 2\pi]$.

The figure also shows the instantaneous angular velocity $\boldsymbol{\omega}$ of the rigid body about the origin. As the body rotates, the Euler angles will change with rates $\dot{\phi}, \dot{\theta}, \dot{\psi}$ about the space-fixed z axis, the line of nodes, and the body-fixed z' axis, respectively:

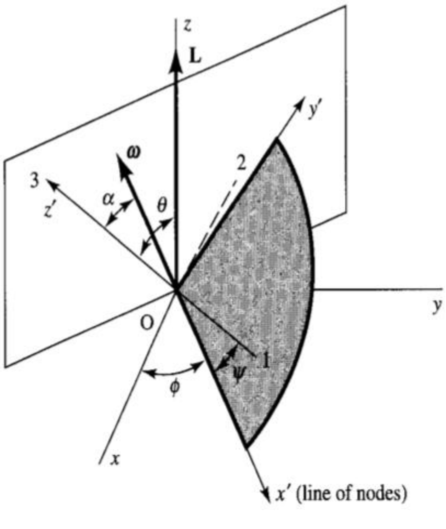
$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{1}} + \omega_2 \hat{\mathbf{2}} + \omega_3 \hat{\mathbf{3}} = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{n}} + \dot{\psi}\hat{\mathbf{3}}$$

where $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ are principal axes unit vectors, and $\hat{\mathbf{n}}$ is the unit vector along the line of nodes.

$$\omega_1 = \dot{\phi}\, \hat{\mathbf{1}}\cdot\hat{\mathbf{z}} + \dot{\theta}\, \hat{\mathbf{1}}\cdot\hat{\mathbf{n}} + \dot{\psi}\, \hat{\mathbf{1}}\cdot\hat{\mathbf{3}} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi}\, \hat{\mathbf{2}}\cdot\hat{\mathbf{z}} + \dot{\theta}\, \hat{\mathbf{2}}\cdot\hat{\mathbf{n}} + \dot{\psi}\, \hat{\mathbf{2}}\cdot\hat{\mathbf{3}} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_3 = \dot{\phi}\, \hat{\mathbf{3}}\cdot\hat{\mathbf{z}} + \dot{\theta}\, \hat{\mathbf{3}}\cdot\hat{\mathbf{n}} + \dot{\psi}\, \hat{\mathbf{3}}\cdot\hat{\mathbf{3}} = \dot{\phi} \cos\theta + \dot{\psi}$$



The dot products above are most easily evaluated by noting that the z axis direction has polar angle θ and azimuthal angle $90° - \psi$ with respect to the principal axes

$$\hat{\mathbf{z}} = \cos(90° - \psi) \sin\theta\, \hat{\mathbf{1}} + \sin(90° - \psi) \sin\theta\, \hat{\mathbf{2}} + \cos\theta\, \hat{\mathbf{3}}$$

and that

$$\hat{\mathbf{n}} = \cos\psi\,\hat{\mathbf{1}} - \sin\psi\,\hat{\mathbf{2}}$$

Moment and products of Inertia

The symmetric rank-2 tensor

$$\mathbf{I} = \sum_i m_i \left(r_i'^2 \mathbf{1} - \mathbf{r}_i' \tilde{\mathbf{r}}_i' \right) = \sum_i m_i \begin{pmatrix} y_i'^2 + z_i'^2 & -x_i' y_i' & -x_i' z_i' \\ -y_i' x_i' & x_i'^2 + z_i'^2 & -y_i' z_i' \\ -z_i' x_i' & -z_i' y_i' & x_i'^2 + y_i'^2 \end{pmatrix}$$

where $\mathbf{1}$ is the unit 3×3 matrix, represents the moment of inertia tensor of the rigid body relative to the body-fixed coordinate system. The kinetic energy of the rigid body, which is a scalar, is compactly represented in tensor notation:

$$T = \frac{1}{2} M \tilde{\mathbf{v}}_{\rm cm} \mathbf{v}_{\rm cm} + \frac{1}{2} \tilde{\boldsymbol{\omega}} \mathbf{I} \boldsymbol{\omega}$$

An important theorem of linear algebra states that a real symmetric matrix can be diagonalized by an orthogonal transformation:

$$\mathbf{I} = \mathcal{O} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \mathcal{O}^{-1}$$

where the orthogonal matrix $\mathcal{O}$ transforms from the body-fixed coordinate system to a "principal axes" coordinate system. The constants I_1, I_2, I_3 are called the "principal moments of inertia" of the rigid body.

The moment of inertia tensor is defined relative to a point in space. A very simple and useful formula relates the moment of inertia tensor $\mathbf{I}$ about the origin of coordinates defined above to the moment of inertia tensor $\mathbf{I}^{\rm cm}$ defined relative to the center of mass of the rigid body.

$$\mathbf{I} = \mathbf{I}^{\rm cm} + M \left({r'_{\rm cm}}^2 \mathbf{1} - \mathbf{r}'_{\rm cm} \tilde{\mathbf{r}}'_{\rm cm} \right)$$

where

$$\mathbf{r}'_{\rm cm} = \frac{\sum_i m_i \mathbf{r}'_i}{M}$$

is the position of the center of mass relative to the body-fixed coordinate system.

To prove this result write

$$\mathbf{r}'_i = (\mathbf{r}'_i - \mathbf{r}'_{\rm cm}) + \mathbf{r}'_{\rm cm} = \bar{\mathbf{r}}'_i + \mathbf{r}'_{\rm cm}$$

where $\bar{\mathbf{r}}'_i$ is the position of m_i relative to the center of mass. Then

$$\sum_i m_i r_i'^2 = \sum_i m_i \bar{r}_i'^2 + 2\mathbf{r}'_{\rm cm}\cdot \sum_i m_i \bar{\mathbf{r}}'_i + r'_{\rm cm}{}^2 \sum_i m_i$$
$$= \sum_i m_i \bar{r}_i'^2 + M r'_{\rm cm}{}^2$$

because

$$\sum_i m_i \bar{\mathbf{r}}'_i = \sum_i m_i (\mathbf{r}'_i - \mathbf{r}'_{\rm cm}) = M \frac{\sum_i m_i \mathbf{r}'_i}{M} - \mathbf{r}'_{\rm cm} \sum_i m_i = 0$$

and

$$\sum_i m_i \mathbf{r}'_i \tilde{\mathbf{r}}'_i = \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\bar{\mathbf{r}}}'_i + \mathbf{r}'_{\rm cm} \sum_i m_i \bar{\bar{\mathbf{r}}}'_i + \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\mathbf{r}}'_{\rm cm} + \mathbf{r}'_{\rm cm} \tilde{\mathbf{r}}'_{\rm cm} \sum_i m_i$$
$$= \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\bar{\mathbf{r}}}'_i + M \mathbf{r}'_{\rm cm} \tilde{\mathbf{r}}'_{\rm cm}$$

Rotational Kinetic Energy of the rigid body

The equations of motion can be derived from the Lagrangian of the system $L = T - V$. The kinetic energy is given by

$$T = \frac{1}{2} \sum_i m_i \mathbf{v}_i^2 = \frac{1}{2} \sum_i m_i \left(\mathbf{V}_0 + \boldsymbol{\omega} \times \mathbf{r}'_i\right)^2$$
$$= \frac{1}{2} \sum_i m_i \mathbf{V}_0^2 + \mathbf{V}_0 \cdot \boldsymbol{\omega} \times \sum_i m_i \mathbf{r}'_i + \frac{1}{2} \sum_i m_i \left(\boldsymbol{\omega} \times \mathbf{r}'_i\right)^2$$

The middle term is zero if we choose the body-fixed origin at the center of mass of the rigid body

$$\mathbf{R}_0 = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \mathbf{r}_{\rm cm} \ , \qquad \sum_i m_i \mathbf{r}'_i = 0$$

The third term can simplified using

$$\left(\boldsymbol{\omega} \times \mathbf{r}'_i\right)^2 = \omega^2 r_i'^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}'_i\right)^2$$

to obtain

$$T = \frac{1}{2} M \mathbf{v}_{\rm cm}^2 + \frac{1}{2} \sum_i m_i \left[\omega^2 r_i'^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}'{}_i\right)^2\right]$$

Angular Momentum of a rigid body

The angular momentum of the system of particles comprising the rigid body about the origin of the inertial space-fixed coordinate system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i m_i \left(\mathbf{R}_0 + \mathbf{r}'_i\right) \times \left(\mathbf{V}_0 + \boldsymbol{\omega} \times \mathbf{r}'_i\right)$$

$$= \sum_i m_i \mathbf{R}_0 \times \mathbf{V}_0 + \mathbf{R}_0 \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}'_i\right) + \sum_i m_i \mathbf{r}'_i \times \mathbf{V}_0$$

$$+ \sum_i m_i \mathbf{r}'_i \times \left(\boldsymbol{\omega} \times \mathbf{r}'_i\right)$$

$$= M\mathbf{r}_{\text{cm}} \times \mathbf{v}_{\text{cm}} + \sum_i m_i \left[r_i'^2 \boldsymbol{\omega} - \mathbf{r}'_i \left(\mathbf{r}'_i \cdot \boldsymbol{\omega}\right)\right]$$

$$= \mathbf{L}_{\text{cm}} + \mathbf{I}\boldsymbol{\omega}$$

where the location of the body-fixed origin at the center of mass, and the vector triple product identity

$$\sum_i m_i \mathbf{r}'_i = 0 \ , \qquad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$$

have been used. The angular momentum of the rigid body is the sum of an "orbital" angular momentum of a equivalent particle of mass M , and an internal "spin" angular momentum about its center of mass

$$\mathbf{L}_{\text{spin}} = \mathbf{I}\boldsymbol{\omega} = \begin{pmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{x'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{pmatrix} \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}$$

Using Lagrange's equations of motion we see that orbital and spin angular momentum of a rigid body are separately conserved in the absence of external forces:

$$\frac{d}{dt}\mathbf{L}_{\text{cm}} = M(\dot{\mathbf{r}}_{\text{cm}} \times \mathbf{v}_{\text{cm}} + \mathbf{r}_{\text{cm}} \times \dot{\mathbf{v}}_{\text{cm}}) = 0 \ , \qquad \frac{d}{dt}\mathbf{L}_{\text{spin}} = \mathbf{I}\dot{\boldsymbol{\omega}} = 0$$

Moment of inertia of rigid body

Consider a rigid body rotating with angular velocity ω around a certain axis. The body consists of N point masses m_i whose distances to the axis of rotation are denoted r_i. Each point mass will have the speed $v_i = \omega r_i$, so that the total kinetic energy T of the body can be calculated as

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\omega^2 \left(\sum_{i=1}^{N} m_i r_i^2\right).$$

In this expression the quantity in parentheses is called the **moment of inertia** of the body (with respect to the specified axis of rotation). It is a purely geometric characteristic of the object, as it depends only on its shape and the position of the rotation axis. The moment of inertia is usually denoted with the capital letter *I*:

$$I = \sum_{i=1}^{N} m_i r_i^2 \ .$$

It is worth emphasizing that r_i here is the distance from a point to the *axis of rotation*, not to the origin. As such, the moment of inertia will be different when considering rotations about different axes.

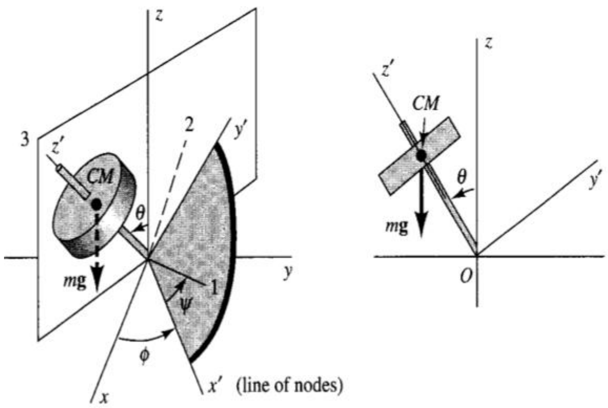
Similarly, the **moment of inertia** of a continuous solid body rotating about a known axis can be calculated by replacing the summation with the integral:

$$I = \int_V \rho(\mathbf{r})\, d(\mathbf{r})^2 \mathrm{d}V(\mathbf{r}),$$

where **r** is the radius vector of a point within the body, $\rho(\mathbf{r})$ is the mass density at point **r**, and $d(\mathbf{r})$ is the distance from point **r** to the axis of rotation. The integration is evaluated over the volume *V* of the body.

Motion of Symmetric Top under action of gravity

Consider a symmetric top spinning about a tip of its symmetric axis as shown in Figure



Note that its center of mass is a distance ℓ from the tip. The moments of inertia about the tip are

$$I_1 = I_2 = I^{\mathrm{cm}} + m\ell^2 \ , \quad I_3 = I_3^{\mathrm{cm}} = I_s$$

The rotational kinetic energy of a rigid body with axis of symmetry $I_1 = I_2 = I, I_3 = I_s$ in terms of Euler angles is

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} I(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_s \omega_3^2 \\ &= \frac{1}{2} I \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_s \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 \end{aligned}$$

The gravitational potential energy relative to the level of the tip is

$$V = mg\ell \sin\theta$$

and the Lagrangian function is

$$\mathcal{L} = T - V = \frac{1}{2} I \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_s \left(\dot{\phi}\cos\theta + \dot{\psi} \right)^2 - mg\ell\cos\theta$$

Note that the Lagrange function does not depend on ϕ and θ The Lagrange equations of motion for ϕ and ψ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt} L_z = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = \frac{d}{dt} L_3 = \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

show that the angular momentum components along the vertical and symmetric directions are conserved

$$L_z = (I\sin^2\theta + I_s\cos^2\theta)\dot{\phi} + I_s\dot{\psi}\cos\theta = \text{constant}$$

$$L_3 = I_s(\dot{\phi}\cos\theta + \dot{\psi}) = \text{constant}$$

These equations can be solved for

$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{I\sin^2\theta}$$

$$\dot{\psi} = \frac{L_3}{I_s} - \frac{L_z - L_3\cos\theta}{I\sin^2\theta}\cos\theta$$

The equation of motion for θ is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$I\ddot{\theta} = I\dot{\phi}^2\sin\theta\cos\theta - I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}\sin\theta + mg\ell\sin\theta$$

General features of central force motion

In classical mechanics, a **central force** is a force whose magnitude only depends on the distance r of the object from the origin and is directed along the line joining them: [1]

$$\vec{F} = \mathbf{F}(\mathbf{r}) = F(||\mathbf{r}||)\hat{\mathbf{r}}$$

where $\vec{F}$ is the force, **F** is a vector valued force function, F is a scalar valued force function, **r** is the position vector, ||**r**|| is its length, and $\hat{\mathbf{r}}$ = **r**/||**r**|| is the corresponding unit vector.

Equivalently, a force field is central if and only if it is spherically symmetric.

A central force is a conservative field, that is, it can always be expressed as the negative gradient of a potential:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}), \text{ where } V(\mathbf{r}) = \int_{|\mathbf{r}|}^{+\infty} F(r)\,\mathrm{d}r$$

(the upper bound of integration is arbitrary, as the potential is defined up to an additive constant).

In a conservative field, the total mechanical energy (kinetic and potential) is conserved:

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) = \text{constant}$$

(where $\dot{r}$ denotes the derivative of **r** with respect to time, that is the velocity), and in a central force field, so is the angular momentum:

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = \text{constant}$$

because the torque exerted by the force is zero. As a consequence, the body moves on the plane perpendicular to the angular momentum vector and containing the origin, and obeysKepler's second law. (If the angular momentum is zero, the body moves along the line joining it with the origin.)

As a consequence of being conservative, a central force field is irrotational, that is, its curl is zero, except at the origin:

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}.$$

General features of orbit

The essential elements of the object are described by a set, and the symmetries of the object are described by the symmetry group of this set, which consists of bijectivetransformations of the set. In this case, the group is also called a **permutation group** (especially if the set is finite or not a vector space) or**transformation group** (especially if the set is a vector space and the group acts like linear transformations of the set).

A group action is an extension to the definition of a symmetry group in which every element of the group "acts" like a bijective transformation (or "symmetry") of some set, without being identified with that transformation. This allows for a more comprehensive description of the symmetries of an object, such as a polyhedron, by allowing the same group to act on several different sets of features, such as the set ofvertices, the set of edges and the set of faces of the polyhedron.

If G is a group and X is a set then a group action may be defined as a group homomorphism from G to the symmetric group of X. The action assigns a permutation of X to each element of the group in such a way that the permutation of X assigned to:

- The identity element of G is the identity transformation of X;
- A product gh of two elements of G is the composite of the permutations assigned to g and h.

Since each element of G is represented as a permutation, a group action is also known as a **permutation representation**.

The abstraction provided by group actions is a powerful one, because it allows geometrical ideas to be applied to more abstract objects. Many objects in mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves. Despite this generality, the theory of group actions contains wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several fields.

Laboratory Frame and the Center-of-Mass Frame

When the potential is central, the problem can be reduced to the one we have just studied; this can be achieved through the separation of the motion of the center of mass.

Let us assume that we have two particles with masses m_1 and m_2, at coordinates $\vec{r}_1$ and $\vec{r}_2$, interacting through a central potential. The equations for the motion can be written as

$$m_1\ddot{\vec{r}}_1 = -\vec{\nabla}_1 V(|\vec{r}_1 - \vec{r}_2|),$$

$$m_2\ddot{\vec{r}}_2 = -\vec{\nabla}_2 V(|\vec{r}_1 - \vec{r}_2|),$$

where $\vec{\nabla}$ is the gradient operator, which has the following form in spherical coordinates

$$\vec{\nabla}_i = \hat{r}_i \frac{\partial}{\partial r_i} + \frac{\hat{\theta}_i}{r_i}\frac{\partial}{\partial \theta_i} + \frac{\hat{\phi}_i}{r_i \sin\theta_i}\frac{\partial}{\partial \phi_i} \qquad i = 1, 2.$$

Since the potential energy depends only on the relative separation of the two particles, let us define the variables:

$$\vec{r} = \vec{r}_1 - \vec{r}_2,$$

$$\vec{R}_{\mathrm{CM}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2},$$

where $\vec{r}$ denotes the coordinate of m_1 relative to m_2, and $\vec{R}_{CM}$ defines the coordinate of the center-of-mass of the system (see Fig. 1.5). From Eqs. (1.42) and (1.44) we can easily obtain the following:

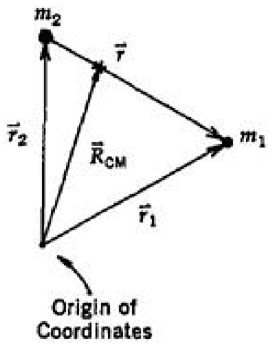
$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} \equiv \mu\ddot{\vec{r}} = -\vec{\nabla}V(|\vec{r}|) = -\frac{\partial V(|\vec{r}|)}{\partial r}\hat{r},$$

$$(m_1 + m_2)\ddot{\vec{R}}_{CM} = M\ddot{\vec{R}}_{CM} = 0, \quad \text{or} \quad \dot{\vec{R}}_{CM} = \text{constant} \times \hat{R},$$

where we have used the fact that $V(\vec{r}) = V(r)$ depends only on the radial coordinate r, and not on the angular variables associated with $\vec{r}$, and where we have defined

$$M = m_1 + m_2 = \text{total mass of the system},$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{“reduced” mass of the system.}$$



$\vec{r}_1 = r_1\,\hat{r}_1$
$\vec{r}_2 = r_2\,\hat{r}_2$
$\vec{r} = r\,\hat{r}$
$\vec{R}_{CM} = R_{CM}\,\hat{R}_{CM}$

Virial theorem

A theorem in classical mechanics which relates the kinetic energy of a system to the virial of Clausius, as defined below. The theorem can be generalized to quantum mechanics and has widespread application. It connects the average kinetic and potential energies for systems in which the potential is a power of the radius. Since the theorem involves integral quantities such as the total kinetic energy, rather than the kinetic energies of the individual particles that may be involved, it gives valuable information on the behavior of complex systems. For example, in statistical mechanics the virial theorem is intimately connected to the equipartition theorem; in astrophysics it may be used to connect the internal temperature, mass, and radius of a star and to discuss stellar stability.

The virial theorem makes possible a very easy derivation of the counterintuitive result that as a star radiates energy and contracts it heats up rather than cooling down. The virialtheorem states that the time-averaged value of the kinetic energy in a confined system (that is, a system in which the velocities and position vectors of all the particles remain finite) is equal to the virial of Clausius. The virial of Clausius is defined to equal −½ times the time-averaged value of a sum over all the particles in the system. The term in this sum associated with a particular particle is the dot product of the particle's position vector and the force acting on the particle. Alternatively, this term is the product of the distance, r, of the particle from the origin of coordinates and the radial component of the force acting on the particle.

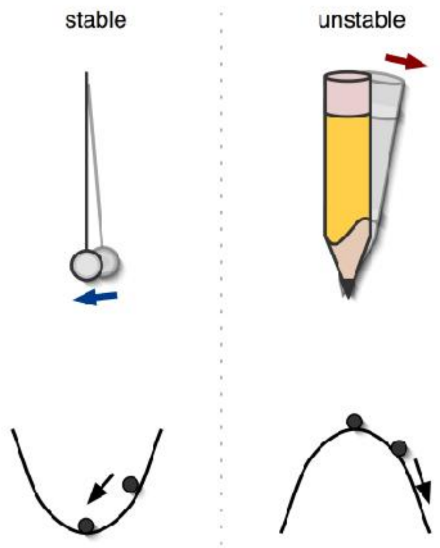
In the common case that the forces are derivable from a power-law potential, V, proportional to r^k, where k is a constant, the virial is just −k/2 times the potential energy. Thus, in this case the virial theorem simply states that the kinetic energy is k/2 times the potential energy. For a system connected by Hooke's-law springs, k = 2, and the average kinetic and potential energies are equal. For k = 1, that is, for gravitational or Coulomb forces, the potential energy is minus twice the kinetic energy.

Stable and unstable equilibrium

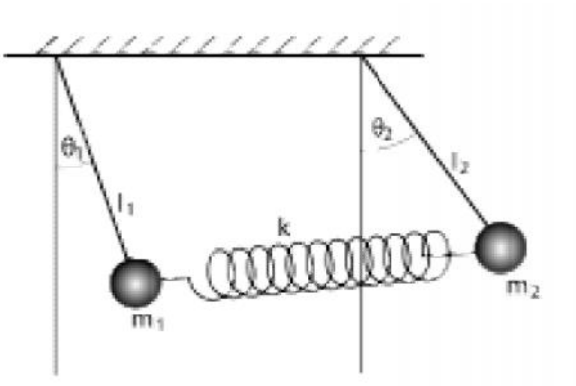
Equilibrium is a state of a system in which the variables which describe the system are not changing (note that a system can be in a dynamic equilibrium where things might be moving or changing, but some variable(s) which describe the system as a whole is(are) constant). One example you are all familiar with is a mechanical system in equilibium where positions of objects are not changing (ie. no net forces acting).

In a **Stable equilibrium** if a small perturbation away from equilibrium is applied, the system will return itself to the equilibrium state. A good example of this is a pendulum hanging straight down. If you nudge the pendulum slightly, it will experience a force back towards the equilibrium position. It may oscillate around the equilibrium position for a bit, but it will return to its equilibrium position.

In an **Unstable equilibrium** if a small perturbation away from equilibrium is applied, the system will move farther away from its equilibrium state. A good example of this is a pencil balanced on it's end. If you nudge the pencil slightly, it will experience a force moving it away from equilibrium. It will simply fall to lying flat on a surface.



Parallel pendula



This shows a parallel pendula of lengths ℓ_1, ℓ_2 and masses m_1, m_2 are not equal and/or the equilibrium length of the spring is not equal to the horizontal distance between the pendulum supports. So let's make the simplifying assumptions that $\ell_1 = \ell_2 = \ell$, $m_1 = m_2 = m$, and the relaxed spring length $X_0 = d$ the distance between the supports. Then the small oscillations Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\ell^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}mg\ell\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{2}k\ell^2\left(\theta_1 - \theta_2\right)^2$$

and the force and mass matrices are

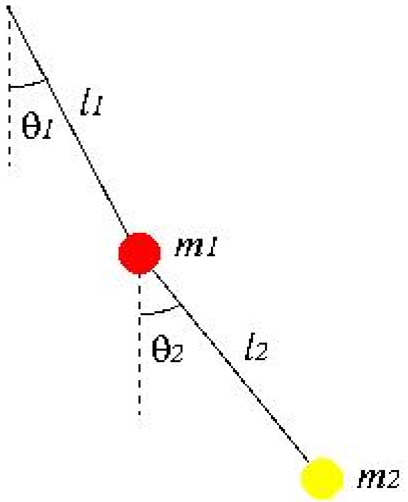
$$\mathbf{K} = mg\ell\begin{pmatrix} 1+\epsilon & -\epsilon \\ -\epsilon & 1+\epsilon \end{pmatrix}, \qquad \mathbf{M} = m\ell^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \epsilon = \frac{k\ell}{mg}$$

This system has two normal modes with frequencies

$$\omega_1^2 = \frac{g}{\ell}, \qquad \omega_2^2 = \frac{g}{\ell}(1+2\epsilon)$$

Double pendulum

Let us consider a double pendulum as shown in a below figure.



Although the double pendulum is often introduced in many textbooks of the classical mechanics, its dynamics are seldom analyzed in them. Actually, it is known that it easily exhibits chaotic behaviors. In the double pendulum, the effect of the friction around the axis of rotation is not considered. Therefore, the energy of the system is conserved, and such a system is called a Hamiltonian system or a conservative system.

The energy E of the system is a sum of the kinetic energy K and the potential energy U written as

$$\begin{aligned} K &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2), \\ U &= (m_1+m_2)gl_1(1-\cos\theta_1) - m_2 g l_2(1-\cos\theta_2), \\ E &= K+U. \end{aligned}$$

Using the Lagrange differential equation, a set of differential equations which governs the dynamics of the double pendulum is obtained, and it is written as

$$\ddot{\theta}_1 + Ml\ddot{\theta}_2 \cos\Delta\theta + Ml\dot{\theta}_2^2 \sin\Delta\theta + \omega^2 \sin\theta_1 = 0,$$

$$\ddot{\theta}_1 \cos\Delta\theta + l\ddot{\theta}_2 - \dot{\theta}_1^2 \sin\Delta\theta + \omega^2 \sin\theta_2 = 0,$$

$$\Delta\theta \equiv \theta_1 - \theta_2$$

$$M \equiv m_2/(m_1 + m_2)$$

$$l \equiv l_2/l_1$$

$$\omega^2 \equiv g/l_1$$

From the above equations, the second derivatives of angles are obtained as follows.

$$\ddot{\theta}_1 = \frac{\omega^2 l(-\sin\theta_1 + M\cos\Delta\theta\sin\theta_2) - Ml(\dot{\theta}_1^2\cos\Delta\theta + l\dot{\theta}_2^2)\sin\Delta\theta}{l - Ml\cos^2\Delta\theta},$$

$$\ddot{\theta}_2 = \frac{\omega^2\cos\Delta\theta\sin\theta_1 - \omega^2\sin\theta_2 + (\dot{\theta}_1^2 + Ml\dot{\theta}_2^2\cos\Delta\theta)\sin\Delta\theta}{l - Ml\cos^2\Delta\theta}.$$

Regarding the above differential equations as a differential equation $\dot{x} = f(x)$ for a vector $x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$, behaviors of a double pendulum can be analyzed.

Because the double pendulum is a Hamiltonian system (a conservative system) where the energy of the system is conserved, one must use numerical integration methods which conserve the energy.

Here we used the fourth order implicit Gaussian method written as

$$\begin{aligned}
k_1 &= dt f(x_0 + a_{11}k_1 + a_{12}k_2),\\
k_2 &= dt f(x_0 + a_{21}k_1 + a_{22}k_2),\\
x_1 &= x_0 + (k_1 + k_2)/2,\\
a_{11} &= 1/4,\\
a_{12} &= 1/4 - \sqrt{3}/6,\\
a_{21} &= 1/4 + \sqrt{3}/6,\\
a_{22} &= 1/4.
\end{aligned}$$

Triatomic Molecule

Consider the simple model of a linear triatomic molecule (*e.g.*, carbon dioxide) illustrated in Figure. The molecule consists of a central atom of mass M flanked by two identical atoms of mass m. The atomic bonds are represented as springs of spring constant k. The linear displacements of the flanking atoms are q_1 and q_2, whilst that of the central atom is q_3. Let us investigate the linear modes of oscillation our model molecule.

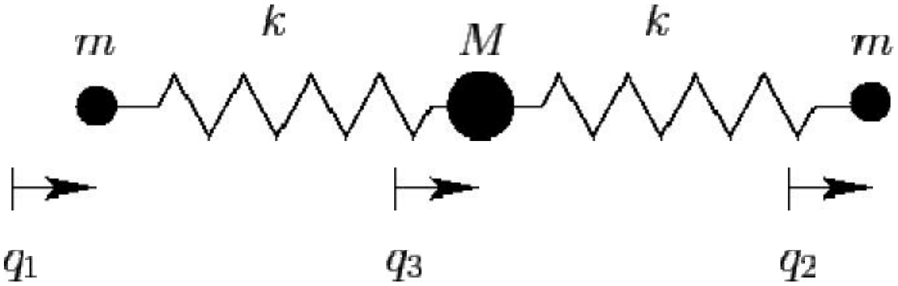


Figure 38: *A model triatomic molecule.*

The kinetic energy of the molecule is written

$$K = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{M}{2}\dot{q}_3^2,$$

whereas the potential energy takes the form

$$U = \frac{k}{2}(q_3 - q_1)^2 + \frac{k}{2}(q_2 - q_3)^2.$$

Clearly, we have a three degree of freedom dynamical system. However, we can reduce this to a two degree of freedom system by only considering *oscillatory* modes of motion, and, hence, neglecting *translational* modes. We can achieve this by demanding that the center of mass of the system remains stationary. In other words, we require that

$$m\,(q_1 + q_2) + M\,q_3 = 0.$$

This constraint can be rearranged to give

$$q_3 = -\frac{m}{M}\,(q_1 + q_2).$$

Eliminating q_3 from Equations, we obtain

$$K = \frac{m}{2}\left[(1+\alpha)\,\dot{q}_1^{\,2} + 2\,\alpha\,\dot{q}_1\,\dot{q}_2 + (1+\alpha)\,\dot{q}_2^{\,2}\right],$$

and

$$U = \frac{k}{2}\left[(1+2\,\alpha+2\,\alpha^2)\,q_1^{\,2} + 4\,\alpha\,(1+\alpha)\,q_1\,q_2 + (1+2\,\alpha+2\,\alpha^2)\,q_2^{\,2}\right], \qquad)$$

respectively, where $\alpha = m/M$.

A comparison of the above expressions with the standard forms and yields the following expressions for the mass matrix, $\mathbf{M}$, and the force matrix, $\mathbf{G}$:

$$\mathbf{M} = m\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & 1+\alpha \end{pmatrix},$$

$$\mathbf{G} = -k\begin{pmatrix} 1+2\,\alpha+2\,\alpha^2 & 2\,\alpha\,(1+\alpha) \\ 2\alpha\,(1+\alpha) & 1+2\,\alpha+2\,\alpha^2 \end{pmatrix}.$$

Now, the equation of motion of the system takes the form

$$(\mathbf{G} - \lambda\,\mathbf{M})\,\mathbf{x} = \mathbf{0},$$

where $\mathbf{x}$ is the column vector of the q_1 and q_2 values. The solubility condition for the above equation is

$$|\mathbf{G} - \lambda \mathbf{M}| = 0,$$

which yields the following quadratic equation for the eigenvalue λ:

$$(1+2\alpha)\left[m^2\lambda^2 + 2\,m\,k\,(1+\alpha)\,\lambda + k^2\,(1+2\alpha)\right] = 0.$$

The two roots of the above equation are

$$\lambda_1 = -\frac{k}{m},$$

$$\lambda_2 = -\frac{k\,(1+2\alpha)}{m}.$$

The fact that the roots are negative implies that both normal modes are indeed *oscillatory* in nature. The characteristic oscillation frequencies are

$$\omega_1 = \sqrt{-\lambda_1} = \sqrt{\frac{k}{m}},$$

$$\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{k\,(1+2\,\alpha)}{m}}.$$

Equation can now be solved, subject to the normalization condition to give the two eigenvectors:

$$\mathbf{x}_1 = (2\,m)^{-1/2}\,(1,-1),$$

$$\mathbf{x}_2 = (2\,m)^{-1/2}\,(1+2\,\alpha)^{-1/2}\,(1,\quad 1).$$

Thus, we conclude from Equations that our model molecule possesses two normal modes of oscillation.

The first mode oscillates at the frequency ω_1, and is an *anti-symmetric* mode in which $q_1 = -q_2$ and $q_3 = 0$. In other words, in this mode of oscillation, the two end atoms move in opposite directions whilst the central atom remains stationary. The second mode oscillates at the frequency ω_2, and is a mixed symmetry mode in which $q_1 = q_2$ but $q_3 = -2\,\alpha\,q_1$. In other words, in this mode of oscillation, the two end atoms move in the same direction whilst the central atom moves in the opposite direction.

Finally, it is easily demonstrated that the normal coordinates of the system are

$$\eta_1 = \sqrt{\frac{m}{2}}\,(q_1 - q_2),$$

$$\eta_2 = \sqrt{\frac{m\,(1+2\,\alpha)}{k}}\,(q_1 + q_2).$$

When expressed in terms of these coordinates, K and U reduce to

$$K = \frac{1}{2}\,(\dot{\eta}_1^2 + \dot{\eta}_2^2),$$

$$U = \frac{1}{2}\,(\omega_1^2\,\eta_1^2 + \omega_2^2\,\eta_2^2),$$

respectively.

Rigid Body

A macroscopic object can often be approximated by a "particle", which has a mass and position in space. A particle has one physical parameter, its mass, and three translational degress of freedom because it can move in 3-dimensional space.

The equations of motion of a particle can be generalized to a system of N particles. Such a system is defined by N mass parameters, and has $3N$ translational degrees of freedom. Its configuration at any time can be represented by N points in 3-dimensional space, or by a single point in $3N$ dimensional configuration space.

When the size and shape of a macroscopic object matters, it can often be approximated by a "rigid body". A rigid body is a system of particles in which every pair of particles has fixed relative displacement. This is an approximation because the smallest parts of objects are atoms which do not have definite positions according to quantum theory. It is also an approximation because a change in position of one particle cannot affect the position of another particle instantaneously according to the theory of relativity.

Suppose that the rigid body is made of N particles or "atoms", how many degrees of freedom does it have, and how many physical parameters are needed to describe it? Its location and orientation are completely fixed by specifying the positions in space of any three non-collinear particles. A rigid triatomic molecule, which can translate and rotate but not vibrate, has 6 degrees of freedom, 3 translational and 3 rotational. Therefore a rigid body also has 6 degrees of freedom. The configuration space of a rigid body is the product space of a 3-dimensional Euclidean space of translational motion with a 3-dimensional closed ball of radius π with antipodal points identified. The rotations of a rigid body belong to the rotation group SO(3), which is an extremely important concept in physics.

The number of physical parameters required to describe a rigid body approximated by N particles is N masses plus $3N-6$ parameters to specify the fixed relative locations of all the particles.

Generalized coordinates

Eulerian Angles and Euler's Equations

The description of a rigid body is simplest in the body-fixed reference frame which uses the principal axes coordinate system. The moment of inertia tensor is diagonal and constant. The equations of motion are easily expressed in terms of the angular velocity components $\omega_1, \omega_2, \omega_3$ along the principal axes directions.

Rigid bodies are usually observed from a space-fixed inertial reference frame. The moment of inertia tensor is not diagonal in general, and its components change with time. We would like to write the equations of motion in terms of vector components in the inertial reference frame.

Euler introduced a very convenient notation for relating quantities in the two frames in terms of Euler angles.

To focus on rotational motion, suppose that the origin of coordinates in the inertial frame is chosen to coincide with the origin in the body-fixed frame at a particular instant of time t, and that the inertial frame is moving with the same instantaneous velocity as the rigid body at this time t. Of course this will change with time if the body is accelerating, but we just want to obtain the form of the equations in the fixed frame at this instant: by Galilean invariance, this form will hold in all inertial frames.

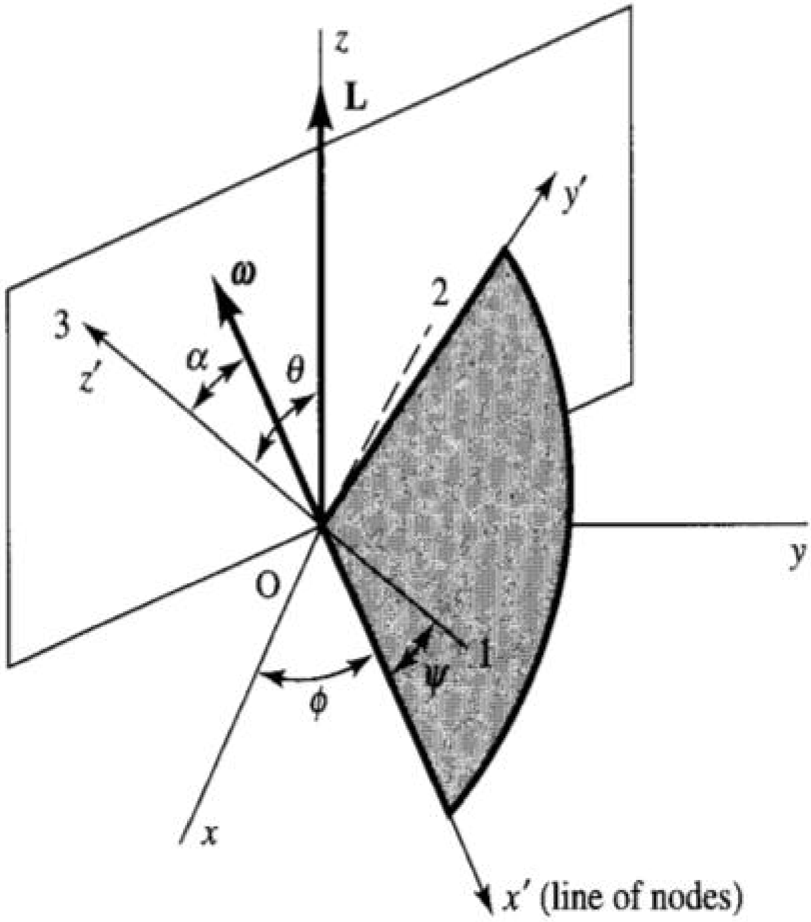


Figure shows a standard definition of the Euler angles ϕ, θ, ψ. The intersection of the inertial and body-fixed x - y planes is called the line of nodes. The coordinate systems are both right-handed, θ is a polar angle in the range $[0, \pi]$, and ϕ, ψ are azimuthal angles in the range $[0, 2\pi]$.

The figure also shows the instantaneous angular velocity $\boldsymbol{\omega}$ of the rigid body about the origin. As the body rotates, the Euler angles will change with rates $\dot{\phi}, \dot{\theta}, \dot{\psi}$ about the space-fixed z axis, the line of nodes, and the body-fixed z' axis, respectively:

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{1}} + \omega_2 \hat{\mathbf{2}} + \omega_3 \hat{\mathbf{3}} = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{n}} + \dot{\psi}\hat{\mathbf{3}}$$

where $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ are principal axes unit vectors, and $\hat{\mathbf{n}}$ is the unit vector along the line of nodes.

$$\omega_1 = \dot\phi\, \hat{\mathbf{1}}\cdot\hat{\mathbf{z}} + \dot\theta\, \hat{\mathbf{1}}\cdot\hat{\mathbf{n}} + \dot\psi\, \hat{\mathbf{1}}\cdot\hat{\mathbf{3}} = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_2 = \dot\phi\, \hat{\mathbf{2}}\cdot\hat{\mathbf{z}} + \dot\theta\, \hat{\mathbf{2}}\cdot\hat{\mathbf{n}} + \dot\psi\, \hat{\mathbf{2}}\cdot\hat{\mathbf{3}} = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_3 = \dot\phi\, \hat{\mathbf{3}}\cdot\hat{\mathbf{z}} + \dot\theta\, \hat{\mathbf{3}}\cdot\hat{\mathbf{n}} + \dot\psi\, \hat{\mathbf{3}}\cdot\hat{\mathbf{3}} = \dot\phi \cos\theta + \dot\psi$$

The dot products above are most easily evaluated by noting that the z axis direction has polar angle θ and azimuthal angle $90^\circ - \psi$ with respect to the principal axes

$$\hat{\mathbf{z}} = \cos(90^\circ - \psi) \sin\theta\, \hat{\mathbf{1}} + \sin(90^\circ - \psi) \sin\theta\, \hat{\mathbf{2}} + \cos\theta\, \hat{\mathbf{3}}$$

and that

$$\hat{\mathbf{n}} = \cos\psi\, \hat{\mathbf{1}} - \sin\psi\, \hat{\mathbf{2}}$$

Moment and products of Inertia

The symmetric rank-2 tensor

$$\mathbf{I} = \sum_i m_i \left(r_i'^2 \mathbf{1} - \mathbf{r}_i' \tilde{\mathbf{r}}_i' \right) = \sum_i m_i \begin{pmatrix} y_i'^2 + z_i'^2 & -x_i' y_i' & -x_i' z_i' \\ -y_i' x_i' & x_i'^2 + z_i'^2 & -y_i' z_i' \\ -z_i' x_i' & -z_i' y_i' & x_i'^2 + y_i'^2 \end{pmatrix}$$

where $\mathbf{1}$ is the unit 3×3 matrix, represents the moment of inertia tensor of the rigid body relative to the body-fixed coordinate system. The kinetic energy of the rigid body, which is a scalar, is compactly represented in tensor notation:

$$T = \frac{1}{2} M \tilde{\mathbf{v}}_{\mathrm{cm}} \mathbf{v}_{\mathrm{cm}} + \frac{1}{2} \tilde{\boldsymbol{\omega}} \mathbf{I} \boldsymbol{\omega}$$

An important theorem of linear algebra states that a real symmetric matrix can be diagonalized by an orthogonal transformation:

$$\mathbf{I} = \mathcal{O} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \mathcal{O}^{-1}$$

where the orthogonal matrix $\mathcal{O}$ transforms from the body-fixed coordinate system to a "principal axes" coordinate system. The constants I_1, I_2, I_3 are called the "principal moments of inertia" of the rigid body.

The moment of inertia tensor is defined relative to a point in space. A very simple and useful formula relates the moment of inertia tensor $\mathbf{I}$ about the origin of coordinates defined above to the moment of inertia tensor $\mathbf{I}^{\mathrm{cm}}$ defined relative to the center of mass of the rigid body.

$$\mathbf{I} = \mathbf{I}^{\mathrm{cm}} + M\left({r'_{\mathrm{cm}}}^2\mathbf{1} - \mathbf{r}'_{\mathrm{cm}}\tilde{\mathbf{r}}'_{\mathrm{cm}}\right)$$

where

$$\mathbf{r}'_{\mathrm{cm}} = \frac{\sum_i m_i \mathbf{r}'_i}{M}$$

is the position of the center of mass relative to the body-fixed coordinate system.

To prove this result write

$$\mathbf{r}'_i = (\mathbf{r}'_i - \mathbf{r}'_{\mathrm{cm}}) + \mathbf{r}'_{\mathrm{cm}} = \bar{\mathbf{r}}'_i + \mathbf{r}'_{\mathrm{cm}}$$

where $\bar{\mathbf{r}}'_i$ is the position of m_i relative to the center of mass. Then

$$\sum_i m_i {r'_i}^2 = \sum_i m_i {\bar{r}'_i}{}^2 + 2\mathbf{r}'_{\mathrm{cm}}\cdot \sum_i m_i \bar{\mathbf{r}}'_i + {r'_{\mathrm{cm}}}^2 \sum_i m_i$$
$$= \sum_i m_i {\bar{r}'_i}{}^2 + M{r'_{\mathrm{cm}}}^2$$

because

$$\sum_i m_i \bar{\mathbf{r}}'_i = \sum_i m_i(\mathbf{r}'_i - \mathbf{r}'_{\mathrm{cm}}) = M\frac{\sum_i m_i \mathbf{r}'_i}{M} - \mathbf{r}'_{\mathrm{cm}}\sum_i m_i = 0$$

and

$$\sum_i m_i \mathbf{r}'_i \tilde{\mathbf{r}}'_i = \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\bar{\mathbf{r}}}'_i + \mathbf{r}'_{\mathrm{cm}}\sum_i m_i \bar{\bar{\mathbf{r}}}'_i + \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\mathbf{r}}'_{\mathrm{cm}} + \mathbf{r}'_{\mathrm{cm}}\tilde{\mathbf{r}}'_{\mathrm{cm}}\sum_i m_i$$
$$= \sum_i m_i \bar{\mathbf{r}}'_i \tilde{\bar{\mathbf{r}}}'_i + M\mathbf{r}'_{\mathrm{cm}}\tilde{\mathbf{r}}'_{\mathrm{cm}}$$

Rotational Kinetic Energy of the rigid body

The equations of motion can be derived from the Lagrangian of the system $L = T - V$. The kinetic energy is given by

$$T = \frac{1}{2}\sum_i m_i \mathbf{v}_i^2 = \frac{1}{2}\sum_i m_i\left(\mathbf{V}_0 + \boldsymbol{\omega}\times\mathbf{r}'_i\right)^2$$
$$= \frac{1}{2}\sum_i m_i \mathbf{V}_0^2 + \mathbf{V}_0\cdot\boldsymbol{\omega}\times\sum_i m_i\mathbf{r}'_i + \frac{1}{2}\sum_i m_i\left(\boldsymbol{\omega}\times\mathbf{r}'_i\right)^2$$

The middle term is zero if we choose the body-fixed origin at the center of mass of the rigid body

$$\mathbf{R}_0 = \frac{\sum_i m_i\mathbf{r}_i}{\sum_i m_i} = \mathbf{r}_{\mathrm{cm}}\ , \qquad \sum_i m_i \mathbf{r}'_i = 0$$

The third term can simplified using

$$(\boldsymbol{\omega} \times \mathbf{r}_i')^2 = \omega^2 r_i'^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_i')^2$$

to obtain

$$T = \frac{1}{2} M \mathbf{v}_{\mathrm{cm}}^2 + \frac{1}{2} \sum_i m_i \left[\omega^2 r_i'^2 - (\boldsymbol{\omega} \cdot \mathbf{r}'_i)^2 \right]$$

Angular Momentum of a rigid body

The angular momentum of the system of particles comprising the rigid body about the origin of the inertial space-fixed coordinate system is

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i m_i (\mathbf{R}_0 + \mathbf{r}_i') \times (\mathbf{V}_0 + \boldsymbol{\omega} \times \mathbf{r}_i') \\
&= \sum_i m_i \mathbf{R}_0 \times \mathbf{V}_0 + \mathbf{R}_0 \times \left(\boldsymbol{\omega} \times \sum_i m_i \mathbf{r}_i' \right) + \sum_i m_i \mathbf{r}_i' \times \mathbf{V}_0 \\
&\quad + \sum_i m_i \mathbf{r}_i' \times (\boldsymbol{\omega} \times \mathbf{r}_i') \\
&= M \mathbf{r}_{\mathrm{cm}} \times \mathbf{v}_{\mathrm{cm}} + \sum_i m_i \left[r_i'^2 \boldsymbol{\omega} - \mathbf{r}_i' (\mathbf{r}_i' \cdot \boldsymbol{\omega}) \right] \\
&= \mathbf{L}_{\mathrm{cm}} + \mathbf{I} \boldsymbol{\omega}
\end{aligned}$$

where the location of the body-fixed origin at the center of mass, and the vector triple product identity

$$\sum_i m_i \mathbf{r}_i' = 0 , \qquad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B})$$

have been used. The angular momentum of the rigid body is the sum of an "orbital" angular momentum of a equivalent particle of mass M , and an internal "spin" angular momentum about its center of mass

$$\mathbf{L}_{\mathrm{spin}} = \mathbf{I} \boldsymbol{\omega} = \begin{pmatrix} I_{x'x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'y'} & I_{x'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'z'} \end{pmatrix} \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}$$

Using Lagrange's equations of motion we see that orbital and spin angular momentum of a rigid body are separately conserved in the absence of external forces:

$$\frac{d}{dt} \mathbf{L}_{\mathrm{cm}} = M (\dot{\mathbf{r}}_{\mathrm{cm}} \times \mathbf{v}_{\mathrm{cm}} + \mathbf{r}_{\mathrm{cm}} \times \dot{\mathbf{v}}_{\mathrm{cm}}) = 0 , \qquad \frac{d}{dt} \mathbf{L}_{\mathrm{spin}} = \mathbf{I} \dot{\boldsymbol{\omega}} = 0$$

Moment of inertia of rigid body

Consider a rigid body rotating with angular velocity ω around a certain axis. The body consists of N point masses m_i whose distances to the axis of rotation are denoted r_i. Each point mass will have the speed $v_i = \omega r_i$, so that the total kinetic energy T of the body can be calculated as

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \omega^2 \left(\sum_{i=1}^{N} m_i r_i^2 \right).$$

In this expression the quantity in parentheses is called the **moment of inertia** of the body (with respect to the specified axis of rotation). It is a purely geometric characteristic of the object, as it depends only on its shape and the position of the rotation axis. The moment of inertia is usually denoted with the capital letter I:

$$I = \sum_{i=1}^{N} m_i r_i^2 \ .$$

It is worth emphasizing that r_i here is the distance from a point to the *axis of rotation*, not to the origin. As such, the moment of inertia will be different when considering rotations about different axes.

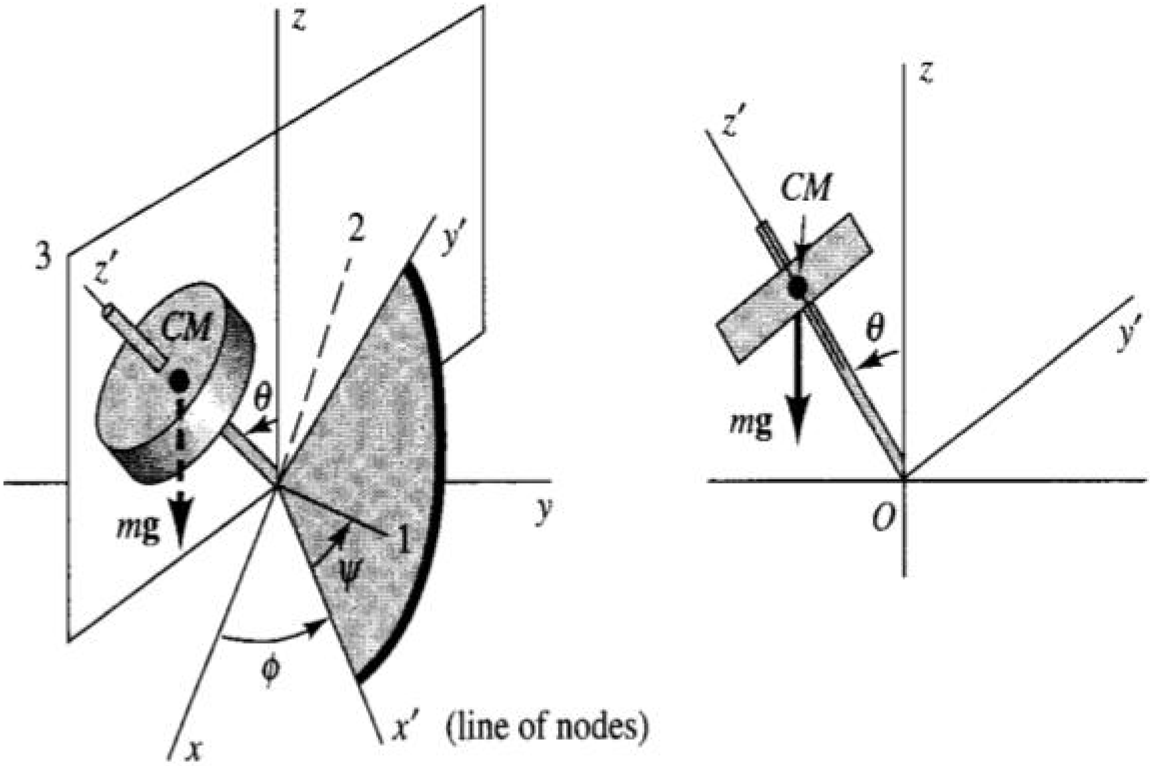
Similarly, the **moment of inertia** of a continuous solid body rotating about a known axis can be calculated by replacing the summation with the integral:

$$I = \int_V \rho(\mathbf{r})\, d(\mathbf{r})^2 \mathrm{d}V(\mathbf{r}),$$

where $\mathbf{r}$ is the radius vector of a point within the body, $\rho(\mathbf{r})$ is the mass density at point $\mathbf{r}$, and $d(\mathbf{r})$ is the distance from point $\mathbf{r}$ to the axis of rotation. The integration is evaluated over the volume V of the body.

Motion of Symmetric Top under action of gravity

Consider a symmetric top spinning about a tip of its symmetric axis as shown in Figure



Note that its center of mass is a distance ℓ from the tip. The moments of inertia about the tip are

$$I_1 = I_2 = I^{\rm cm} + m\ell^2 \ , \quad I_3 = I_3^{\rm cm} = I_s$$

The rotational kinetic energy of a rigid body with axis of symmetry $I_1 = I_2 = I, I_3 = I_s$ in terms of Euler angles is

$$\begin{aligned} T_{\rm rot} &= \frac{1}{2}I(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_s\omega_3^2 \\ &= \frac{1}{2}I\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}I_s\left(\dot\phi\cos\theta + \dot\psi\right)^2 \end{aligned}$$

The gravitational potential energy relative to the level of the tip is

$$V = mg\ell\sin\theta$$

and the Lagrangian function is

$$\mathcal{L} = T - V = \frac{1}{2}I\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}I_s\left(\dot\phi\cos\theta + \dot\psi\right)^2 - mg\ell\cos\theta$$

Note that the Lagrange function does not depend on ϕ and θ The Lagrange equations of motion for ϕ and ψ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{d}{dt}L_z = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = \frac{d}{dt}L_3 = \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

show that the angular momentum components along the vertical and symmetric directions are conserved

$$L_z = (I\sin^2\theta + I_s\cos^2\theta)\dot{\phi} + I_s\dot{\psi}\cos\theta = \text{constant}$$

$$L_3 = I_s(\dot{\phi}\cos\theta + \dot{\psi}) = \text{constant}$$

These equations can be solved for

$$\dot{\phi} = \frac{L_z - L_3\cos\theta}{I\sin^2\theta}$$

$$\dot{\psi} = \frac{L_3}{I_s} - \frac{L_z - L_3\cos\theta}{I\sin^2\theta}\cos\theta$$

The equation of motion for θ is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$I\ddot{\theta} = I\dot{\phi}^2\sin\theta\cos\theta - I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}\sin\theta + mg\ell\sin\theta$$

S.NO	Question	Option A	Option B	Option C	Option D	Answer
1	Rotational kinetic energy of a rigid body is	½ w2 I2	w2 I	½ w2 I	2w2 I.	½ w2 I
2	In certain system of body axes with respect to which the off-diagonal elements	symmetric	antisymmetric	principal	perpendicular	principal
3	If wz = wz' > wmin, atop will spin with its axis vertical continuously , therefore it is	sleeping top	spinning top	rotating top	symmetric top	sleeping top
4	A rigid body with N particles have ______________degrees of freedom.	2N	3N	N	4N	3N
5	The configuration of a rigid body with respect to some cartesian co-ordinate system in space	momentum	inertia	orientation	angular momentum	orientation

6	The most useful set of generalised co-ordinates for a rigid body are ________ angles.	rotation	specified	auxillary	euler's	euler's
7	The transformation worked out through three _________ rotations performed only in a	successive	different	independent	dependent	successive
8	The distance between any two points of a rigid body is ____________	varied	fixed	proportional	exponentially proportional	fixed
9	A rigid body can possesses simultaneously the translational and _________ motion	arbitrary	circular	rotational	orbital	rotational
10	A mathematical structure having nine components in three dimensions is termed as a	tensor	matrix	covariant tensor	contra variant tensor	tensor
11	The products of inertia of all vanish when one of the axes of the body lies along the axis	rotation	vibration	motion	symmetry	symmetry

12	If the symmetry axis of the body is taken as axis of rotation and the origin of body axes lies	unsymmetry	rotational	symmetry	b and c	symmetry
13	The motion of a rigid body with one point fixed will take place under the action of torque N in	displacement	torque	time	rotational motion	torque
14	The assembly of particles with fixed inter-particle distance is called_________	fluid	vapor	colloidal	rigid body	rigid body
15	The orientation of the body by locating a cartesian set of co-ordinates fixed in the rigid	body set of axes	space set of axes	both a and b	rotational set of axes	body set of axes
16	The fixed point in the body which registers its translation and coincident with the center of	body set of axes	space or external set of axes	rotational set of axis	vibrational set of axes	space or external set of axes
17	The generation of body set of axes from the space set of axes through three successive	direction cosines	successive angles	rotational angles	Euler's angles	Euler's angles

18	The system of body axes in which off-diagonal elements disappear and the diagonal elements	principle axes	secondary axes	primary axes	catesian axes	secondary axes
19	The system of body axes in which off-diagonal elements disappear, and the diagonal elements	principle moment of inertia	secondary moment of inertia	moments of inertia	inertia	inertia
20	The secular equation of inertia tensor and its solution is called _____________	constant of motion	tensor of rank two	covariant tensor	eigen values	eigen values
21	A rigid body can possesses simulataneous the ________ and ____________ motion.	translation and rotational	linear and harmonic	periodic and non-harmonic	symmetrical around	translation and rotational
22	Rigid body possessing rotational and translational motion simulataneously will have	polar and cartesian	generalised and canonical	translation and rotational	both a and b	translation and rotational
23	If we consider three non-collinear points in a rigid body, then each particle will have	four	three	six	nine	three

24	Three non-collinear points in a rigid body will have the total of ________degrees of	six	three	nine	twelve	nine
25	All the space set of axis if rotated wbout the space z-axis, then the yz plane takes ______	same	alternate	orthogonal	new	new
26	The inverse transformation matrix from body set of axes to space set of axes is given	AT	adj (A)	co-factor of A	determinant of A	AT
27	The position vector of any point p relative to the origin O of the body set of axes is	Different	constant	proportional	both a and c	constant
28	The configuration of a rigid body is completely specified by ________degrees of freedom.	two	three	six	nine	six
29	If a is the column matrix representing the co-ordinates having single frequency and aT is	0	l	a	1	1

30	If a is the column matrix representing the co-ordinates having single frequency and	0	l	1	a2	l
31	The generalised co-ordinate in which each one of them executing oscillations of one	normal co-ordinate	cartesain co-ordinate	polar co-ordinate	rectangular co-ordinate	normal co-ordinate
32	In parallel pendula the two pendula oscillates in ________	out or phase	phase	damped motion	undamped motion	phase
33	In parallel pendular, if the two pendula are independent i.e., there is no ________	unstretching	rarefying	transiting	stretching	stretching
34	In paralle pendula ___________ force due to spring will come into action.	impulsive	repulsive	restoring	attractive	restoring
35	If the system possesses two identical frequencies, then it is therefore said to be __________	degenerate	generate	distorted	in harmonic motion	degenerate
36	A continuous string has infinite number of normal modes and	velocities	frequencies	vibrations	motion	frequencies

	________________					
37	The use of nomal co-ordinate in the coupled system reduces it to one of a system of	dependent	single	independent	double	independent
38	A continuous string has a linear ____________	velocity	acceleration	displacement	mass density	mass density
39	If the system is in stable equilibrium, then the frequency wl2 should be a ______ quantity.	real	imaginary	complex	integer	real
40	If wl2 are real and positive, then all co-ordinate always remain _______ for any time.	infinite	same	different	finite	finite
41	If wl2 are not real and positive, then all the co-ordinate becomes ______ for any time.	infinite	finite	equal	exponential	infinite
42	The system is said to be unstable if the frequency wl2 are not _________	equal	finite	real	infinite	real

UNIT-IV

SYLLABUS

Special Theory of Relativity: Introduction – Galilean transformation and invariance of Newton's laws of motion – Non variance of Maxwell's equations – Michelson Morley experiment and explanation of the null result.
Concept of inertial frame – Postulates of special theory – simultaneity – Lorentz transformation along one of the axes – length contraction – time dilatation and velocity addition theorem – Fizeau's experiment – Four vectors – Relativistic dynamics – Variation of mass with velocity – Energy momentum relationship.

Special theory of relativity – Introduction

The Special Theory of Relativity was the result of developments in physics at the end of the nineteenth century and the beginning of the twentieth century. It changed our understanding of older physical theories such as Newtonian Physics and led to early Quantum Theory and General Relativity.

Special Relativity does not just apply to fast moving objects, it affects the everyday world directly through "relativistic" effects such as magnetism and the relativistic inertia that underlies kinetic energy and hence the whole of dynamics.

Special Relativity is now one of the foundation blocks of physics. It is in no sense a provisional theory and is largely compatible with quantum theory; it not only led to the idea of matter waves but is the origin of quantum 'spin' and underlies the existence of the antiparticles. Special Relativity is a theory of exceptional elegance, Einstein crafted the theory from simple postulates about the constancy of physical laws and of the speed of light and his work has been refined further so that the laws of physics themselves and even the constancy of the speed of light are now understood in terms of the most basic symmetries in space and time.

The Galilean Transformation Invariance of Newton's law of motion

Suppose there are two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure 1). In S, we have the co-ordinates $\{x, y, z, t\}$ and in S' we have the co-ordinates $\{x', y', z', t'\}$. S' is moving with respect to S with velocity v (as measured in S) in the x direction. The clocks in both systems were synchronised at time $t = 0$ and they run at the same rate.

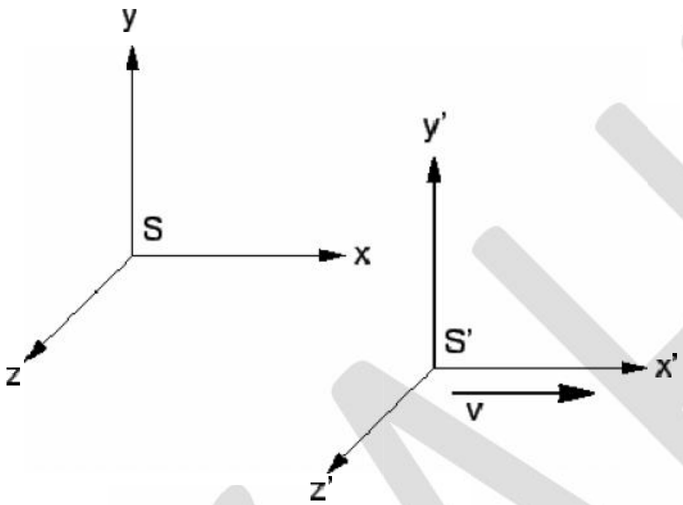


Figure 1: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

We have the intuitive relationships

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

This set of equations is known as the Galilean Transformation. They enable us to relate a measurement in one inertial reference frame to another. For example, suppose we measure the velocity of a vehicle moving in the in x-direction in system S, and we want to know what would be the velocity of the vehicle in S'.

$$v'_x = \frac{dx'}{dt'} = \frac{d(x - vt)}{dt} = v_x - v$$

The laws of physics to be the same in all inertial reference frames, as this is indeed our experience of nature. Physically, we should be able to perform the same experiments in different reference frames, and find always the same physical laws. Mathematically, these laws are expressed by equations. So, we should be able to ``transform'' our equations from one inertial reference frame to the other inertial reference frame, and always find the same answer.

Suppose we wanted to check that Newton's Second Law is the same in two different reference frames. We put one observer in the un-primed frame, and the other in the primed frame, moving with velocity v relative to the un-primed frame. Consider the vehicle of the previous case undergoing a constant acceleration in the x-direction,

$$\begin{aligned} f' = m'a' &= m'\frac{d^2x'}{dt'^2} \\ &= m'\frac{d}{dt'}\left(\frac{dx'}{dt'}\right) \\ &= m\frac{d}{dt}\left(\frac{d(x-vt)}{dt}\right) \\ &= m\frac{d(v_x - v)}{dt} \\ &= m\frac{dv_x}{dt} \\ &= ma = f \end{aligned}$$

Indeed, it does not matter which inertial frame we observe from, we recover the same Second Law of Motion each time. In the parlance of physics, we say the Second Law of Motion is invariant under the Galilean Transformation.

Non-variance of Maxwell's equation

Experiments on electric and magnetic fields, as well as induction of one type of field from changes in the other, lead to the collection of a set of equations, describing all these phenomena, known as Maxwell's Equations.

Maxwells Equations *in vacuo*

$$\begin{aligned} \nabla.\mathbf{B} &= 0, \\ \nabla.\mathbf{E} &= 0, \\ \nabla\times\mathbf{B} &= \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}. \end{aligned}$$

Now, these equations are considered to be rock solid, arising from and verified by many experiments. Amazingly, they imply the existence of a previously not guessed at phenomenon. This is the electromagnetic wave. To see this in detail, take the time derivative of the second last equation and the curl of the last.

$$\frac{\partial}{\partial t}\nabla\times\mathbf{B} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2},$$

$$\nabla\times(\nabla\times\mathbf{E}) = -\nabla\times\frac{\partial\mathbf{B}}{\partial t}.$$

Now note that space and time derivatives commute

$$\frac{\partial}{\partial t}\nabla\times\mathbf{B} = \nabla\times\frac{\partial\mathbf{B}}{\partial t},$$

so

$$\nabla\times(\nabla\times\mathbf{E}) = -\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Now, we use the identity

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\nabla.\mathbf{E} - \nabla^2\mathbf{E}.$$

The second term of the above equation drops out due to the vanishing of the divergence of the electric field (the second of Maxwell's Equations). So, we finally have the three dimensional wave equation

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

To see this is a wave equation, note the analogy in one dimension

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

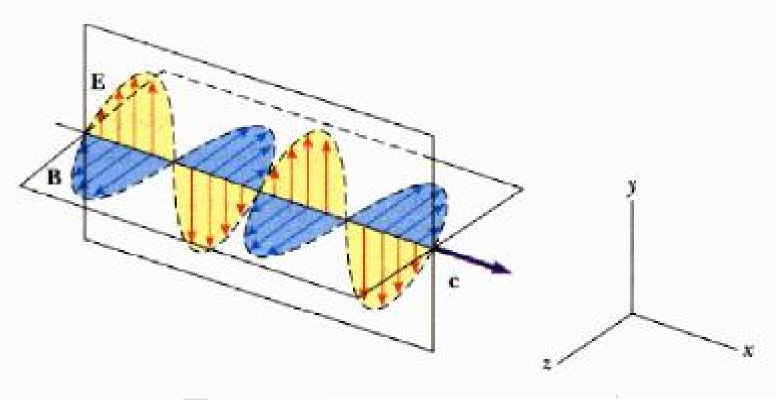
which is solved by the wave function

$$y(x,t) = \sin(x - ct),$$

which represents a wave traveling along the x axis with velocity c.

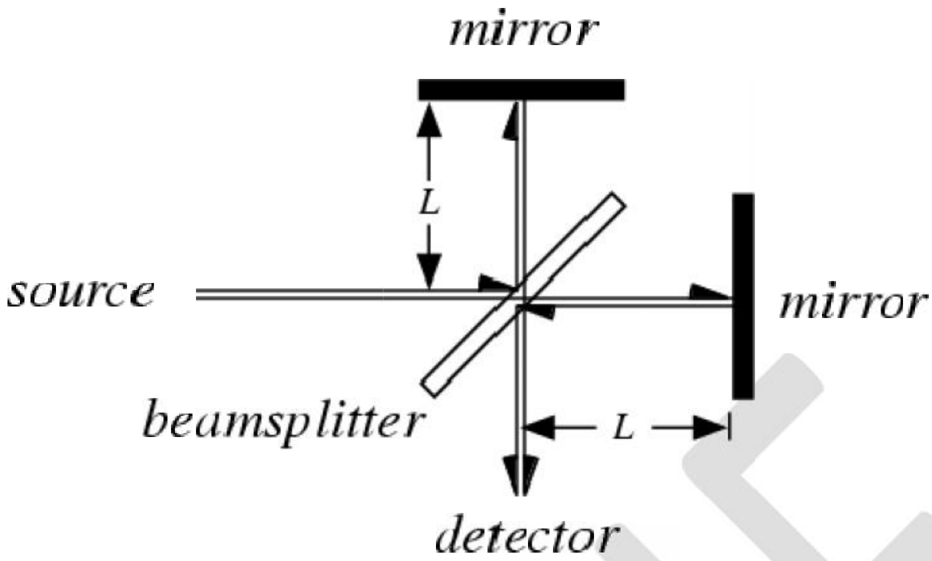
It is clear therefore that Maxwell's Equations are highly predictive.

1. A diversity is unified in a simplicity. The various phenomena of radiowaves, microwaves, infrared, visible and ultra-violet light, X-rays and gamma rays are all electromagnetic waves, differing only in their frequency.
2. They all travel at the same speed.
3. Even that speed is specified : $c = 1/\sqrt{\epsilon_0\mu_0} = 2.997 \times 10^8$ m/s.
4. The speed appears independent of the source and the observer.



Michelson Morley experiment and explanation of the null result.

After the development of Maxwell's theory of electromagnetism, several experiments were performed to prove the existence of ether and its motion relative to the Earth. The most famous and successful was the one now known as the Michelson-Morley experiment, performed by Albert Michelson (1852-1931) and Edward Morley (1838-1923) in 1887.



Michelson and Morley built a Michelson interferometer, which essentially consists of a light source, a half-silvered glass plate, two mirrors, and a telescope. The mirrors are placed at right angles to each other and at equal distance from the glass plate, which is obliquely oriented at an angle of 45° relative to the two mirrors. In the original device, the mirrors were mounted on a rigid base that rotates freely on a basin filled with liquid mercury in order to reduce friction.

Prevailing theories held that ether formed an absolute reference frame with respect to which the rest of the universe was stationary. It would therefore follow that it should appear to be moving from the perspective of an observer on the sun-orbiting Earth. As a result, light would sometimes travel in the same direction of the ether, and others times in the opposite direction. Thus, the idea was to measure the speed of light in different directions in order to measure speed of the ether relative to Earth, thus establishing its existence.

Michelson and Morley were able to measure the speed of light by looking for interference fringes between the light which had passed through the two perpendicular arms of their apparatus. These would occur since the light would travel faster along an arm if oriented in the "same" direction as the ether was moving, and slower if oriented in the opposite direction. Since the two arms were perpendicular, the only way that light would travel at the same speed in both arms and therefore arrive simultaneous at the telescope would be if the instrument were motionless with respect to the ether. If not, the crests and troughs of the light waves in the two arms would arrive and interfere slightly out of synchronization, producing a diminution of intensity. (Of course, the same effect would be achieved if the arms of the interferometer were not of the same length, but these could be adjusted accurately by looking for the intensity peak as one arm was moved. Changing the orientation of the instrument should then show fringes.)

Although Michelson and Morley were expecting measuring different speeds of light in each direction, they found no discernible fringes indicating a different speed in any orientation or at any position of the Earth in its annual orbit around the Sun.

In 1895, Lorentz concluded that the "null" result obtained by Michelson and Morley was caused by a effect of contraction made by the ether on their apparatus and introduced the length contraction equation

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}},$$

where L is the contracted length, L_0 is the rest length, v is the velocity of the frame of reference, and c is the speed of light.

Concept of inertial frame of reference

A “frame of reference” is a standard relative to which motion and rest may be measured; any set of points or objects that are at rest relative to one another enables us, in principle, to describe the relative motions of bodies. A frame of reference is therefore a purely kinematical device, for the geometrical description of motion without regard to the masses or forces involved. A dynamical account of motion leads to the idea of an “inertial frame,” or a reference frame relative to which motions have distinguished dynamical properties. For that reason an inertial frame has to be understood as a spatial reference frame together with some means of measuring time, so that uniform motions can be distinguished from accelerated motions.

The laws of Newtonian dynamics provide a simple definition: an inertial frame is a reference-frame with a time-scale, relative to which the motion of a body not subject to forces is always rectilinear and uniform, accelerations are always proportional to and in the direction of applied forces, and applied forces are always met with equal and opposite reactions. It follows that, in an inertial frame, the center of mass of a system of bodies is always at rest or in uniform motion. It also follows that any other frame of reference moving uniformly relative to an inertial frame is also an inertial frame. For example, in Newtonian celestial mechanics, taking the “fixed stars” as a frame of reference, we can determine an (approximately) inertial frame whose center is the center of mass of the solar system; relative to this frame, every acceleration of every planet can be accounted for (approximately) as a gravitational interaction with some other planet in accord with Newton's laws of motion.

Postulates of special theory of relativity

(i) Statement: "The laws of physics are the same in any inertial frame, regardless of position or velocity".

Physically, this means that there is no absolute spacetime, no absolute frame of reference with respect to which position and velocity are defined. Only relative positions and velocities between objects are meaningful.

(ii) Statement: "The speed of light *c* is a universal constant, the same in any inertial frame".

Simultaneity

Consider a rocket traveling at speed **v**, as shown in Fig. 4. There is an observer **O** at rest with respect to the rocket and an observer **O'** riding with the rocket. Two lightbulbs at the ends of the rocket were timed such that their flashes arrive at the observers at the same time. Light from the bulbs traveled towards the observers at the speed of light, **c**, in the reference frames of both observers. The figure shows how **O** and**O'** are lined up when the light arrives.

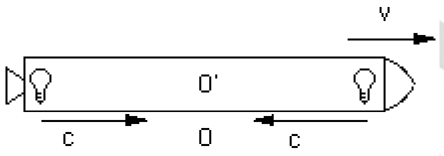


Fig. 4

For **O'** (on the rocket), the bulbs must have flashed simultaneously because **O'** is right in the middle. The bulbs are at rest in the frame of **O'**.

The other observer, **O**, draws a different conclusion. When the flashes were emitted, the rocket was not centered on **O**; it was to the left. The pulse from the bulb on the left must have been emitted first; it had farther to travel. Likewise, the pulse from the bulb on the right had a shorter distance to travel. Observer **O** concludes that the bulbs were not flashed simultaneously.

So, observer **O'** thinks the events (flashing of the bulbs) were simultaneous while observer **O** does not. Simultaneity is not independent of reference frame.

Length contraction

Moving rod contracts in length by factor of $\sqrt{1-\frac{v^2}{c^2}}$

i.e., length of a rod in motion in a given frame of reference = length of the same rod when at rest in the given frame of reference $\times \sqrt{1-\frac{v^2}{c^2}}$

$$\text{or} \quad l = l_0\sqrt{1-\frac{v^2}{c^2}} \quad \text{.......(4)}$$

Time dilation

Moving clock dilates in time interval measured by factor of

ie Time interval measured by a clock in motion in a given frame of reference = Time interval measured by the same clock when at rest in the given frame by reference. $\times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\text{or} \quad \boxed{\tau = \frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{...... (5)}$$

Relativistic Law of Velocity Addition

If an object is in motion with velocity $\vec{u'}$ (u'_x, u'_y, u'_zcomponents) in frame S' and the velocity of the object measured in S is $\vec{u}$ (u_x, u_y, $u_z$components) then ,

$$\left.\begin{aligned} u_x &= \frac{u'_x + V}{1 + \frac{u'_x V}{C^2}} && \text{.......(a)} \\ u_y &= \frac{u'_y\sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} && \text{.......(b)} \\ u_z &= \frac{u'_z\sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} && \text{.......(c)} \end{aligned}\right] \text{.............(6)}$$

Relativistic Mass

The concept of 'Absolute Mass' of Newtonian Mechanics is no longer tenable in special Relativity; the requirement that Law of Conservation of momentum is a fundamental Law of nature imposes the relation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{.......(7)}$$

then only consistency between the Lorentz-Transformations and Law of Conservation of momentum can be obtained. This expression given relativistic mass m in motion with Velocity V in a given frame of reference; in terms of the mass m0 called rest mass of the object when at rest in the given frame of reference.

The Experiment of Fizeau

In 1851, Fizeau carried out an experiment which tested for the aether convection coefficient. This was the first such test of Fresnel's formula, derived without experimental evidence, over twenty years earlier. Fresnel, in fact, had died more than twenty years before this experiment took place, a point of interest only because many texts derive Fresnel's formula based on the results of experiment, rather than the other way around. Experimental results, within the level of error available in the mid-1800's, are not sufficient to derive Fresnel's formula. These results can only confirm that, within error limits, the formula provides answers consistent with experiment. In fact, Fizeau's experimental results were so course that the only conclusion he could draw was that the displacement was less than should have been produced by the motion of the liquid if light were completely convected by the medium. From this, he assumed the validity of Fresnel's formula on the partial convection of the aether.

Fizeau's experiment involved passing light two ways through moving water (v ~ 7 m/s) and observing the interference pattern obtained, as illustrated in figure 1. The experiment was repeated by Michelson in 1886 with much more rigor, and quantitative results were obtained. Working backwards from the observed fringe shift, Michelson was able to calculate an apparent convection coefficient equivalent to Fresnel's formula. Varying the velocity and direction of the flow allowed for a variety of test points. By observing the change in interference pattern, the effective velocity of light through the moving medium, as measured in the lab frame, was calculated. Within experimental limits, the results obtained by measuring the fringe shift agreed with the results predicted by Fresnel’s formula. However, Michelson neglected to take into account the Doppler effect of light from a stationary source interacting with moving water, and therefore concluded that the aether convection concept of Fresnel was essentially correct.

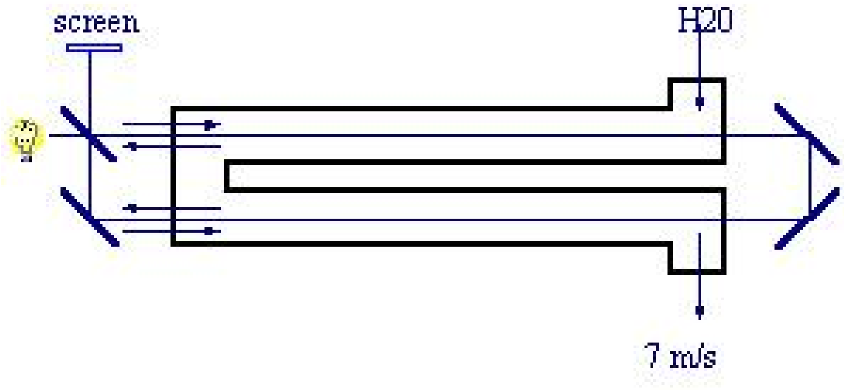


Figure 1. The experiment of Fizeau.

We now examine this experiment in a purely Galilean environment, taking into account the Doppler shift (change in wavelength) experienced by each beam of light. Michelson’s paper gives an excellent analysis whereby the retarded velocity, *b*, seen in the water may be considered as due to the number of collisions with atoms, the "velocity of light through the atoms," and the width of the atoms. Since there will likely be objections to that analysis based on current understandings of the microscopic world, we present a more general approach. In what follows, the retarded velocity is again considered as due to the "collisions" (absorptions and re-emissions) of the photons in the medium, as it must be, but we do not require any assumptions as to "atom width," or "velocity through the atom."

For light traveling through a medium, the effective wavelength changes:

$$\lambda_1 = \frac{\lambda_0}{\eta} \quad (1)$$

The phase shift for light in such a medium is:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{\lambda_0} \quad (2)$$

The optical path length is defined from (2) as *lh*. The optical path difference between the medium and air is then:

$$l[\eta - 1] = l\eta[1 - \frac{1}{\eta}] \quad (3)$$

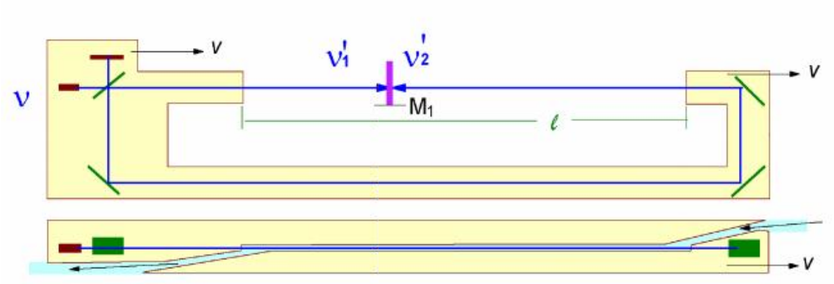
The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta}{\lambda_0}[\eta - 1] \quad (4)$$

In the Fizeau experiment we must consider Doppler effects. Since the water is moving with respect to the source, the two paths of light will experience Doppler shifts upon entering the water. Light moving in the opposite direction to the flow of water will be blue-shift (l_1). Light moving with the flow will be red shifted (l_2):

$$\lambda_1 = (1 - v/c)\lambda$$

$$\lambda_2 = (1 + v/c)\lambda \quad (8)$$



To see why the Doppler shift cannot be ignored in Fizeau's experiment, imagine the apparatus depicted in figure 2. All mirrors, the source and the observing screen are sealed in water filled containers. The water is not flowing, but is stationary in the containers. Alternatively, the containers could be made of solid glass, so long as the refractive index is different than air. The entire apparatus, with the exception of mirror (detector) M_1 moves through the lab frame at a velocity of *v*. Thus, air is moving through the gap, *l*, at a velocity of *v* in the equipment frame. To first order in *v/c*, the wavelengths of the light detected at M_1 is given by equation (8).

We now fill the apparatus containers with air and pass the entire apparatus through water. In the equipment frame, water is moving through the gap at a velocity *v*. The motion induced Doppler in the water, experienced by M_1, remains unchanged. If we, the observers, move along with the apparatus, this setup is indistinguishable from the actual Fizeau experiment. From our frame of reference, the equipment is at rest, water is moving through the gap at a velocity *v*, and the image on the screen reflects the fringe shift due to that motion. Thus we can replace the gap with a tube of flowing water, hold the rest of the apparatus stationary in the lab frame, and obtain a one-sided Fizeau experiment. Clearly, whatever analysis one uses to derive the formulas for the observed fringe shift, one must take into account the fact that the wavelength of the light in the moving medium is different from that of the source due to the motion induced Doppler effect of (8).

Substituting (8) into (2), we see that the phase shift including Doppler effects becomes:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{(1 + v/c)\lambda_0} = \frac{l\eta c}{(c+v)\lambda_0} \quad (9)$$

The optical path length is defined from the above as:

$$\frac{l\eta c}{c+v} \quad (10)$$

The optical path difference between the medium and air is then:

$$\frac{lc}{c+v}[\eta - 1] = \frac{lc\eta}{c+v}[1 - \frac{1}{\eta}] \quad (11)$$

The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta c}{(c+v)\lambda_0}[1 - \frac{1}{\eta}] \quad (12)$$

For light traveling different paths and experiencing different Doppler effects, the total phase shift is given by:

$$\frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} = \frac{l_1\eta c}{(c+v_1)\lambda_0}[1 - \frac{1}{\eta}] - \frac{l_2\eta c}{(c+v_2)\lambda_0}[1 - \frac{1}{\eta}] \quad (13)$$

In the Fizeau experiment, l_1 and l_2 are given by (8). The path lengths l_1 and l_2 are respectively given below, where the factor of two is included because the light travels through two tubes of length *l*, and *b* is the velocity of light in the reference frame of the liquid.

$$bt_1 = 2l + vt_1, \quad or \quad t_1 = \frac{2l}{b-v}$$

$$l_1 = bt_1 = \frac{2lb}{b-v}, \qquad l_2 = \frac{2lb}{b+v} \quad (14)$$

Substituting these values into (13) for each path gives the following results:

$$\frac{\phi_1 - \phi}{2\pi} = \frac{\delta\phi_1}{2\pi} = \frac{2lb}{(b-v)} \cdot \frac{1}{\lambda_1}[\eta - 1] =$$

$$\frac{2lb}{(b-v)} \cdot \frac{c}{(c-v)\lambda}\eta[1 - \frac{1}{\eta}]$$

$$\frac{\phi_2 - \phi}{2\pi} = \frac{\delta\phi_2}{2\pi} = \frac{2lb}{(b+v)} \cdot \frac{1}{\lambda_2}[\eta - 1] =$$

$$\frac{2lb}{(b+v)} \cdot \frac{c}{(c+v)\lambda}\eta[1 - \frac{1}{\eta}] \qquad (15)$$

$$\delta N = \frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} \approx$$

$$\frac{2lbc\eta[1 - 1/\eta]}{[b-v][c-v]\lambda} - \frac{2lbc\eta[1 - 1/\eta]}{[b+v][c+v]\lambda} \approx$$

$$\frac{2l\eta[1 - 1/\eta]}{bc\lambda}[2vc + 2vb] \approx$$

$$\frac{4l\eta^2 v[1 - 1/\eta][1 + 1/\eta]}{\lambda c} \approx \frac{4l\eta^2 v}{\lambda c}[1 - \frac{1}{\eta^2}] \qquad (16)$$

Notice how these results were obtained without invoking "aether" drag, or relativistic velocity addition.

In the special relativistic analysis of this experiment, the velocity of light in the moving liquid as measured in the lab frame is no longer $b + v$, but is given by the relativistic velocity addition formula:

$$b' = \frac{b - v}{1 - \frac{vb}{c^2}} = \frac{b - v}{1 - \frac{v}{\eta c}} \qquad (17)$$

As a result, the path lengths derived in (14) become:

$$l_1 = \frac{2lb(1 - \frac{v}{\eta c})}{b - v} \quad , \quad l_2 = \frac{2lb(1 + \frac{v}{\eta c})}{b + v} \qquad (18)$$

The derivation of the total phase shift then becomes:

$$\frac{\phi_1-\phi}{2\pi}=\frac{\delta\phi_1}{2\pi}=\frac{2lb(1-\frac{v}{\eta c})}{(b-v)}\cdot\frac{1}{\lambda_1}[\eta-1]$$

$$=\frac{2lb(1-\frac{v}{\eta c})}{(b-v)}\cdot\frac{c}{(c-v)\lambda}\eta[1-\frac{1}{\eta}]$$

$$\frac{\phi_2-\phi}{2\pi}=\frac{\delta\phi_2}{2\pi}=\frac{2lb(1+\frac{v}{\eta c})}{(b+v)}\cdot\frac{1}{\lambda_2}[\eta-1]$$

$$=\frac{2lb(1+\frac{v}{\eta c})}{(b+v)}\cdot\frac{c}{(c+v)\lambda}\eta[1-\frac{1}{\eta}] \quad (19)$$

$$\delta N=\frac{\delta\phi_1}{2\pi}-\frac{\delta\phi_2}{2\pi}\approx$$

$$\frac{2l\eta[1-1/\eta]}{bc\lambda}[2vc+2vb+2\frac{vb}{\eta}+2\frac{v^3}{\eta c}]\approx$$

$$\frac{4l\eta^2v}{\lambda c}[1-1/\eta][1+1/\eta+1/\eta^2]\approx\frac{4l\eta^2v}{\lambda c}[1-\frac{1}{\eta^3}] \quad (20)$$

The two results, (16) and (20), differ in the exponent of the last h term. When Michelson and Morley performed the experiment, they obtained sixty one trials, using three different combinations of water velocity and tube length. The graph below shows the distribution of these results, normalized to a tube length of ten meters and a water velocity of one meter per second. The line marked RCM represents the value obtained from equation (16). The line marked SRT reflects the value obtained from (20). While there is a distribution of results, owing to experimental error, Michelson claimed an overall shift of 0.184 $\pm$ 0.02 fringe. This is completely consistent with (16), but eliminates the special relativistic result, with a value of 0.247, from consideration.

Summary

It is very difficult to find adequate tests between special relativity and other competing theories. Most theories overlap with SRT on a vast majority of the prediction made by each, yet are based on different underlying physical principles. Ultimately one must find a test that checks not only the results of the application of the mathematical theory, but also the underlying assumptions. The major conceptual difference between SRT and most competing theories is the idea of relative simultaneity—that distant events that are simultaneous for one observer will not be simultaneous for and observer in motion relative to the first. The relativistic velocity addition rule is a direct consequence of relativistic simultaneity, and the Fizeau experiment represents a direct test of the velocity addition formula. Regardless of what the correct theory is or may be, it is clear that SRT fails to give predictions consistent with results in this experiment—an experiment performed almost ten years before the development of SRT.

Four-vectors

Although the use of 4-vectors is not necessary for a full understanding of Special Relativity, they are a most powerful and useful tool for attacking many problems. A 4-vectors is just a 4-tuplet A = (A_0, A_1, A_2, A_3) that transforms under a Lorentz Transformation in the same way as (cdt, dx, dy, dz) does. That is:

$A_0 = \gamma(A_0' + (v/c)A_1')$

$A_1 = \gamma(A_1' + (v/c)A_0')$

$A_2 = A_2'$

$A_3 = A_3'$

Lorentz transformations are very much like rotations in 4-dimensional spacetime. 4-vectors, then, generalize the concept of rotations in 3-space to rotations in 4-dimensions. Clearly, any constant multiple of (cdt, dx, dy, dz) is a 4-vector, but something like A = (cdt,mdx, dy, dz) (where m is just a constant) is not a 4-vector because the second component has to transform like mdxâÉá$A_1 = \gamma(A_1' + (v/c)A_0')$âÉáγ((mdx') +vdt') from the definition of a 4-vector, but also like mdx = mγ(dx' + (v/c)dt') ; these two expression are inconsistent. Thus we can transform a 4-vector either according to the 4- vector definition given above, or using what we know about how the dx_i transform to transform each A_i independently. There are only a few special vectors for which these two methods yield the same result. Several different 4-vectors are now discussed:

Velocity 4-vector

We can define a quantity τ = $\sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ which is called the proper time, and is invariant between frames. Dividing out original 4-vector ((cdt, dx, dx, dz)) bydτ gives:

V = $\frac{1}{d\tau}$ (cdt, dx, dy, dz) = γ $\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$ = (γc, γ $\mathbf{v})$

This arises because $\frac{dt}{d\tau}$ = γ .

Energy-momentum 4-vector

If we multiply the velocity 4-vector by m we get:

P = mV = m(γc, γ $\mathbf{v}) = (\gamma \mathbf{mc}, \gamma \mathbf{mv}) = (\mathbf{E}/\mathbf{c}, \mathbf{p})$

This is an extremely important 4-vector in Special Relativity.

Relation between momentum and kinetic energy

Sometimes it's desirable to express the kinetic energy of a particle in terms of the momentum. That's easy enough. Since $\mathbf{v} = \mathbf{p}/m$ and the kinetic energy $K = \frac{1}{2}mv^2$ so

$$K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} \qquad (1.4)$$

Note that if a massive particle and a light particle have the same momentum, the light one will have a lot more kinetic energy. If a light particle and a heavy one have the same velocity, the heavy one has more kinetic energy.

UNIT-IV

SYLLABUS

Special Theory of Relativity: Introduction – Galilean transformation and invariance of Newton's laws of motion – Non variance of Maxwell's equations – Michelson Morley experiment and explanation of the null result.
Concept of inertial frame – Postulates of special theory – simultaneity – Lorentz transformation along one of the axes – length contraction – time dilatation and velocity addition theorem – Fizeau's experiment – Four vectors – Relativistic dynamics – Variation of mass with velocity – Energy momentum relationship.

Special theory of relativity – Introduction

The Special Theory of Relativity was the result of developments in physics at the end of the nineteenth century and the beginning of the twentieth century. It changed our understanding of older physical theories such as Newtonian Physics and led to early Quantum Theory and General Relativity.

Special Relativity does not just apply to fast moving objects, it affects the everyday world directly through "relativistic" effects such as magnetism and the relativistic inertia that underlies kinetic energy and hence the whole of dynamics.

Special Relativity is now one of the foundation blocks of physics. It is in no sense a provisional theory and is largely compatible with quantum theory; it not only led to the idea of matter waves but is the origin of quantum 'spin' and underlies the existence of the antiparticles. Special Relativity is a theory of exceptional elegance, Einstein crafted the theory from simple postulates about the constancy of physical laws and of the speed of light and his work has been refined further so that the laws of physics themselves and even the constancy of the speed of light are now understood in terms of the most basic symmetries in space and time.

The Galilean Transformation Invariance of Newton's law of motion

Suppose there are two reference frames (systems) designated by S and S' such that the co-ordinate axes are parallel (as in figure 1). In S, we have the co-ordinates $\{x, y, z, t\}$ and in S' we have the co-ordinates $\{x', y', z', t'\}$. S' is moving with respect to S with velocity v (as measured in S) in the x direction. The clocks in both systems were synchronised at time $t = 0$ and they run at the same rate.

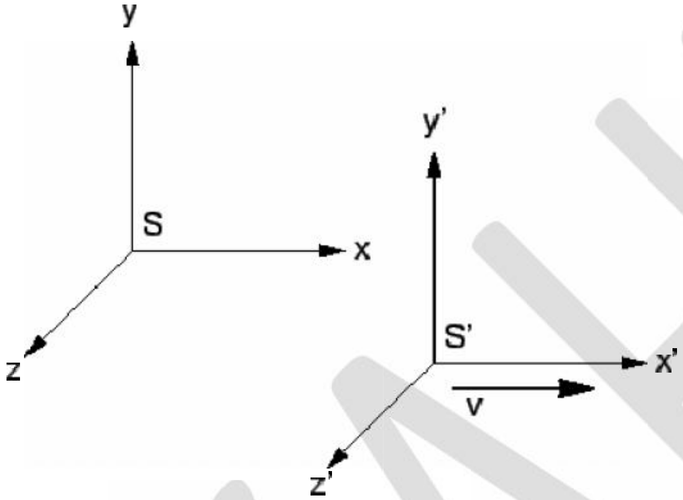


Figure 1: Reference frame S' moves with velocity v (in the x direction) relative to reference frame S.

We have the intuitive relationships

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

This set of equations is known as the Galilean Transformation. They enable us to relate a measurement in one inertial reference frame to another. For example, suppose we measure the velocity of a vehicle moving in the in x-direction in system S, and we want to know what would be the velocity of the vehicle in S'.

$$v'_x = \frac{dx'}{dt'} = \frac{d(x - vt)}{dt} = v_x - v$$

The laws of physics to be the same in all inertial reference frames, as this is indeed our experience of nature. Physically, we should be able to perform the same experiments in different reference frames, and find always the same physical laws. Mathematically, these laws are expressed by equations. So, we should be able to ``transform" our equations from one inertial reference frame to the other inertial reference frame, and always find the same answer.

Suppose we wanted to check that Newton's Second Law is the same in two different reference frames. We put one observer in the un-primed frame, and the other in the primed frame, moving with velocity v relative to the un-primed frame. Consider the vehicle of the previous case undergoing a constant acceleration in the x-direction,

$$\begin{aligned} f' = m'a' &= m'\frac{d^2x'}{dt'^2} \\ &= m'\frac{d}{dt'}\left(\frac{dx'}{dt'}\right) \\ &= m\frac{d}{dt}\left(\frac{d(x-vt)}{dt}\right) \\ &= m\frac{d(v_x - v)}{dt} \\ &= m\frac{dv_x}{dt} \\ &= ma = f \end{aligned}$$

Indeed, it does not matter which inertial frame we observe from, we recover the same Second Law of Motion each time. In the parlance of physics, we say the Second Law of Motion is invariant under the Galilean Transformation.

Non-variance of Maxwell's equation

Experiments on electric and magnetic fields, as well as induction of one type of field from changes in the other, lead to the collection of a set of equations, describing all these phenomena, known as Maxwell's Equations.

Maxwells Equations *in vacuo*

$$\begin{aligned} \nabla.\mathbf{B} &= 0, \\ \nabla.\mathbf{E} &= 0, \\ \nabla\times\mathbf{B} &= \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}. \end{aligned}$$

Now, these equations are considered to be rock solid, arising from and verified by many experiments. Amazingly, they imply the existence of a previously not guessed at phenomenon. This is the electromagnetic wave. To see this in detail, take the time derivative of the second last equation and the curl of the last.

$$\frac{\partial}{\partial t}\nabla\times\mathbf{B} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2},$$

$$\nabla\times(\nabla\times\mathbf{E}) = -\nabla\times\frac{\partial\mathbf{B}}{\partial t}.$$

Now note that space and time derivatives commute

$$\frac{\partial}{\partial t}\nabla\times\mathbf{B} = \nabla\times\frac{\partial\mathbf{B}}{\partial t},$$

so

$$\nabla\times(\nabla\times\mathbf{E}) = -\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Now, we use the identity

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\nabla.\mathbf{E} - \nabla^2\mathbf{E}.$$

The second term of the above equation drops out due to the vanishing of the divergence of the electric field (the second of Maxwell's Equations). So, we finally have the three dimensional wave equation

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

To see this is a wave equation, note the analogy in one dimension

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

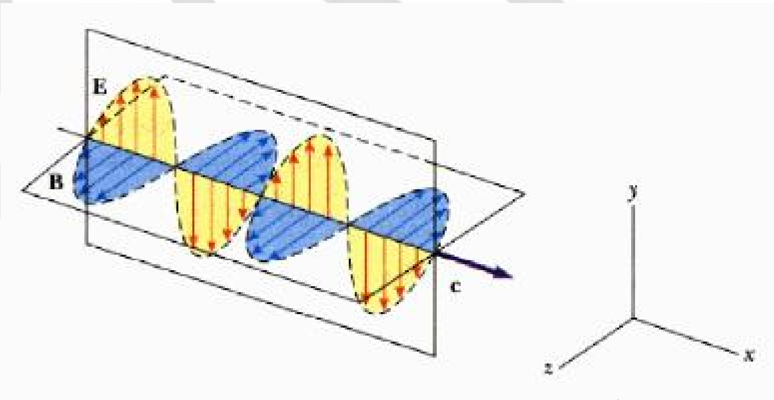
which is solved by the wave function

$$y(x,t) = \sin(x - ct),$$

which represents a wave traveling along the x axis with velocity c.

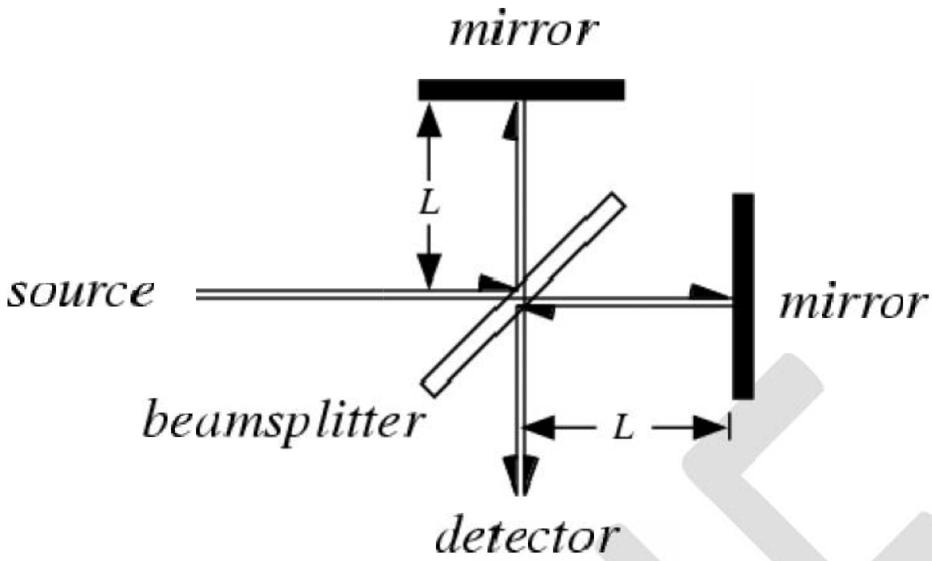
It is clear therefore that Maxwell's Equations are highly predictive.

1. A diversity is unified in a simplicity. The various phenomena of radiowaves, microwaves, infrared, visible and ultra-violet light, X-rays and gamma rays are all electromagnetic waves, differing only in their frequency.
2. They all travel at the same speed.
3. Even that speed is specified : $c = 1/\sqrt{\epsilon_0 \mu_0} = 2.997 \times 10^8$ m/s.
4. The speed appears independent of the source and the observer.



Michelson Morley experiment and explanation of the null result.

After the development of Maxwell's theory of electromagnetism, several experiments were performed to prove the existence of ether and its motion relative to the Earth. The most famous and successful was the one now known as the Michelson-Morley experiment, performed by Albert Michelson (1852-1931) and Edward Morley (1838-1923) in 1887.



Michelson and Morley built a Michelson interferometer, which essentially consists of a light source, a half-silvered glass plate, two mirrors, and a telescope. The mirrors are placed at right angles to each other and at equal distance from the glass plate, which is obliquely oriented at an angle of 45° relative to the two mirrors. In the original device, the mirrors were mounted on a rigid base that rotates freely on a basin filled with liquid mercury in order to reduce friction.

Prevailing theories held that ether formed an absolute reference frame with respect to which the rest of the universe was stationary. It would therefore follow that it should appear to be moving from the perspective of an observer on the sun-orbiting Earth. As a result, light would sometimes travel in the same direction of the ether, and others times in the opposite direction. Thus, the idea was to measure the speed of light in different directions in order to measure speed of the ether relative to Earth, thus establishing its existence.

Michelson and Morley were able to measure the speed of light by looking for interference fringes between the light which had passed through the two perpendicular arms of their apparatus. These would occur since the light would travel faster along an arm if oriented in the "same" direction as the ether was moving, and slower if oriented in the opposite direction. Since the two arms were perpendicular, the only way that light would travel at the same speed in both arms and therefore arrive simultaneous at the telescope would be if the instrument were motionless with respect to the ether. If not, the crests and troughs of the light waves in the two arms would arrive and interfere slightly out of synchronization, producing a diminution of intensity. (Of course, the same effect would be achieved if the arms of the interferometer were not of the same length, but these could be adjusted accurately by looking for the intensity peak as one arm was moved. Changing the orientation of the instrument should then show fringes.)

Although Michelson and Morley were expecting measuring different speeds of light in each direction, they found no discernible fringes indicating a different speed in any orientation or at any position of the Earth in its annual orbit around the Sun.

In 1895, Lorentz concluded that the "null" result obtained by Michelson and Morley was caused by a effect of contraction made by the ether on their apparatus and introduced the length contraction equation

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}},$$

where *L* is the contracted length, L_0 is the rest length, *v* is the velocity of the frame of reference, and *c* is the speed of light.

Concept of inertial frame of reference

A "frame of reference" is a standard relative to which motion and rest may be measured; any set of points or objects that are at rest relative to one another enables us, in principle, to describe the relative motions of bodies. A frame of reference is therefore a purely kinematical device, for the geometrical description of motion without regard to the masses or forces involved. A dynamical account of motion leads to the idea of an "inertial frame," or a reference frame relative to which motions have distinguished dynamical properties. For that reason an inertial frame has to be understood as a spatial reference frame together with some means of measuring time, so that uniform motions can be distinguished from accelerated motions.

The laws of Newtonian dynamics provide a simple definition: an inertial frame is a reference-frame with a time-scale, relative to which the motion of a body not subject to forces is always rectilinear and uniform, accelerations are always proportional to and in the direction of applied forces, and applied forces are always met with equal and opposite reactions. It follows that, in an inertial frame, the center of mass of a system of bodies is always at rest or in uniform motion. It also follows that any other frame of reference moving uniformly relative to an inertial frame is also an inertial frame. For example, in Newtonian celestial mechanics, taking the "fixed stars" as a frame of reference, we can determine an (approximately) inertial frame whose center is the center of mass of the solar system; relative to this frame, every acceleration of every planet can be accounted for (approximately) as a gravitational interaction with some other planet in accord with Newton's laws of motion.

Postulates of special theory of relativity

(i) Statement: "The laws of physics are the same in any inertial frame, regardless of position or velocity".

Physically, this means that there is no absolute spacetime, no absolute frame of reference with respect to which position and velocity are defined. Only relative positions and velocities between objects are meaningful.

(ii) Statement: "The speed of light *c* is a universal constant, the same in any inertial frame".

Simultaneity

Consider a rocket traveling at speed **v**, as shown in Fig. 4. There is an observer **O** at rest with respect to the rocket and an observer **O'** riding with the rocket. Two lightbulbs at the ends of the rocket were timed such that their flashes arrive at the observers at the same time. Light from the bulbs traveled towards the observers at the speed of light, **c**, in the reference frames of both observers. The figure shows how **O** and**O'** are lined up when the light arrives.

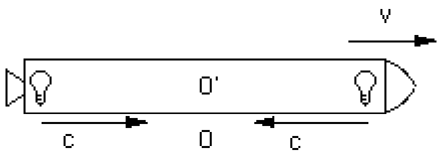


Fig. 4

For **O'** (on the rocket), the bulbs must have flashed simultaneously because **O'** is right in the middle. The bulbs are at rest in the frame of **O'**.

The other observer, **O**, draws a different conclusion. When the flashes were emitted, the rocket was not centered on **O**; it was to the left. The pulse from the bulb on the left must have been emitted first; it had farther to travel. Likewise, the pulse from the bulb on the right had a shorter distance to travel. Observer **O** concludes that the bulbs were not flashed simultaneously.

So, observer **O'** thinks the events (flashing of the bulbs) were simultaneous while observer **O** does not. Simultaneity is not independent of reference frame.

Length contraction

Moving rod contracts in length by factor of $\sqrt{1-\frac{v^2}{c^2}}$

i.e., length of a rod in motion in a given frame of reference = length of the same rod when at rest in the given frame of reference $\times \sqrt{1-\frac{v^2}{c^2}}$

or $$l = l_0\sqrt{1-\frac{v^2}{c^2}} \quad \text{.......(4)}$$

Time dilation

Moving clock dilates in time interval measured by factor of

ie Time interval measured by a clock in motion in a given frame of reference = Time interval measured by the same clock when at rest in the given frame by reference. $\times \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

or $$\boxed{\tau = \frac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \text{...... (5)}$$

Relativistic Law of Velocity Addition

If an object is in motion with velocity $\vec{u}'$ (u'_x, u'_y, u'_zcomponents) in frame S' and the velocity of the object measured in S is $\vec{u}$ (u_x, u_y, $u_z$components) then ,

$$\left.\begin{aligned} u_x &= \frac{u'_x + V}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(a)} \\ u_y &= \frac{u'_y\sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(b)} \\ u_z &= \frac{u'_z\sqrt{1 - V^2/C^2}}{1 + \frac{u'_x V}{C^2}} \quad \text{.......(c)} \end{aligned}\right] \quad \text{..............(6)}$$

Relativistic Mass

The concept of 'Absolute Mass' of Newtonian Mechanics is no longer tenable in special Relativity; the requirement that Law of Conservation of momentum is a fundamental Law of nature imposes the relation

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \ldots\ldots(7)$$

then only consistency between the Lorentz-Transformations and Law of Conservation of momentum can be obtained. This expression given relativistic mass m in motion with Velocity V in a given frame of reference; in terms of the mass m0 called rest mass of the object when at rest in the given frame of reference.

The Experiment of Fizeau

In 1851, Fizeau carried out an experiment which tested for the aether convection coefficient. This was the first such test of Fresnel's formula, derived without experimental evidence, over twenty years earlier. Fresnel, in fact, had died more than twenty years before this experiment took place, a point of interest only because many texts derive Fresnel's formula based on the results of experiment, rather than the other way around. Experimental results, within the level of error available in the mid-1800's, are not sufficient to derive Fresnel's formula. These results can only confirm that, within error limits, the formula provides answers consistent with experiment. In fact, Fizeau's experimental results were so course that the only conclusion he could draw was that the displacement was less than should have been produced by the motion of the liquid if light were completely convected by the medium. From this, he assumed the validity of Fresnel's formula on the partial convection of the aether.

Fizeau's experiment involved passing light two ways through moving water (v ~ 7 m/s) and observing the interference pattern obtained, as illustrated in figure 1. The experiment was repeated by Michelson in 1886 with much more rigor, and quantitative results were obtained. Working backwards from the observed fringe shift, Michelson was able to calculate an apparent convection coefficient equivalent to Fresnel's formula. Varying the velocity and direction of the flow allowed for a variety of test points. By observing the change in interference pattern, the effective velocity of light through the moving medium, as measured in the lab frame, was calculated. Within experimental limits, the results obtained by measuring the fringe shift agreed with the results predicted by Fresnel's formula. However, Michelson neglected to take into account the Doppler effect of light from a stationary source interacting with moving water, and therefore concluded that the aether convection concept of Fresnel was essentially correct.

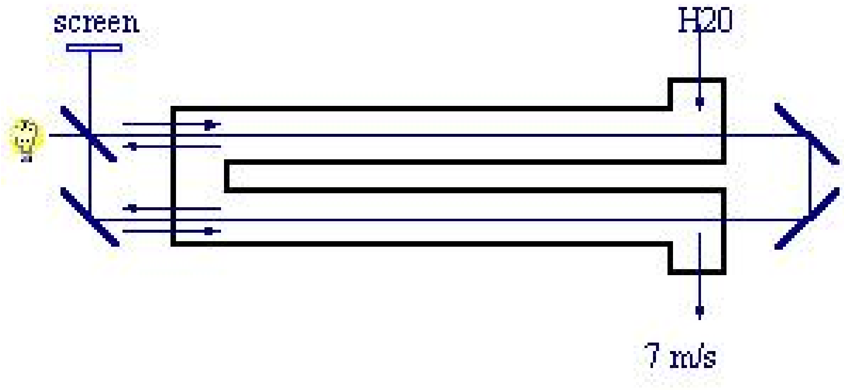


Figure 1. The experiment of Fizeau.

We now examine this experiment in a purely Galilean environment, taking into account the Doppler shift (change in wavelength) experienced by each beam of light. Michelson's paper gives an excellent analysis whereby the retarded velocity, *b*, seen in the water may be considered as due to the number of collisions with atoms, the "velocity of light through the atoms," and the width of the atoms. Since there will likely be objections to that analysis based on current understandings of the microscopic world, we present a more general approach. In what follows, the retarded velocity is again considered as due to the "collisions" (absorptions and re-emissions) of the photons in the medium, as it must be, but we do not require any assumptions as to "atom width," or "velocity through the atom."

For light traveling through a medium, the effective wavelength changes:

$$\lambda_1 = \frac{\lambda_0}{\eta} \quad (1)$$

The phase shift for light in such a medium is:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{\lambda_0} \quad (2)$$

The optical path length is defined from (2) as *lh*. The optical path difference between the medium and air is then:

$$l[\eta - 1] = l\eta[1 - \frac{1}{\eta}] \quad (3)$$

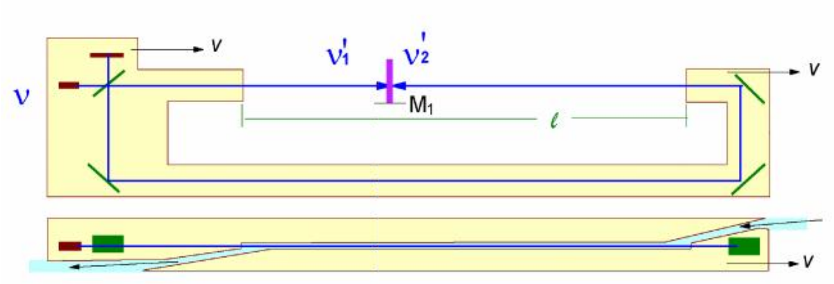
The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta}{\lambda_0}[\eta - 1] \quad (4)$$

In the Fizeau experiment we must consider Doppler effects. Since the water is moving with respect to the source, the two paths of light will experience Doppler shifts upon entering the water. Light moving in the opposite direction to the flow of water will be blue-shift (l_1). Light moving with the flow will be red shifted (l_2):

$$\lambda_1 = (1 - v/c)\lambda$$

$$\lambda_2 = (1 + v/c)\lambda \quad (8)$$



To see why the Doppler shift cannot be ignored in Fizeau's experiment, imagine the apparatus depicted in figure 2. All mirrors, the source and the observing screen are sealed in water filled containers. The water is not flowing, but is stationary in the containers. Alternatively, the containers could be made of solid glass, so long as the refractive index is different than air. The entire apparatus, with the exception of mirror (detector) M_1 moves through the lab frame at a velocity of *v*. Thus, air is moving through the gap, *l*, at a velocity of *v* in the equipment frame. To first order in *v/c*, the wavelengths of the light detected at M_1 is given by equation (8).

We now fill the apparatus containers with air and pass the entire apparatus through water. In the equipment frame, water is moving through the gap at a velocity *v*. The motion induced Doppler in the water, experienced by M_1, remains unchanged. If we, the observers, move along with the apparatus, this setup is indistinguishable from the actual Fizeau experiment. From our frame of reference, the equipment is at rest, water is moving through the gap at a velocity *v*, and the image on the screen reflects the fringe shift due to that motion. Thus we can replace the gap with a tube of flowing water, hold the rest of the apparatus stationary in the lab frame, and obtain a one-sided Fizeau experiment. Clearly, whatever analysis one uses to derive the formulas for the observed fringe shift, one must take into account the fact that the wavelength of the light in the moving medium is different from that of the source due to the motion induced Doppler effect of (8).

Substituting (8) into (2), we see that the phase shift including Doppler effects becomes:

$$\frac{\delta\phi}{2\pi} = \frac{l}{\lambda_1} = \frac{l\eta}{(1+v/c)\lambda_0} = \frac{l\eta c}{(c+v)\lambda_0} \quad (9)$$

The optical path length is defined from the above as:

$$\frac{l\eta c}{c+v} \quad (10)$$

The optical path difference between the medium and air is then:

$$\frac{lc}{c+v}[\eta - 1] = \frac{lc\eta}{c+v}[1-\frac{1}{\eta}] \quad (11)$$

The phase difference compared with the same path in air is:

$$\frac{\delta\phi}{2\pi} = \frac{l\eta c}{(c+v)\lambda_0}[1-\frac{1}{\eta}] \quad (12)$$

For light traveling different paths and experiencing different Doppler effects, the total phase shift is given by:

$$\frac{\delta\phi_1}{2\pi} - \frac{\delta\phi_2}{2\pi} = \frac{l_1\eta c}{(c+v_1)\lambda_0}[1-\frac{1}{\eta}] - \frac{l_2\eta c}{(c+v_2)\lambda_0}[1-\frac{1}{\eta}] \quad (13)$$

In the Fizeau experiment, l_1 and l_2 are given by (8). The path lengths l_1 and l_2 are respectively given below, where the factor of two is included because the light travels through two tubes of length *l*, and *b* is the velocity of light in the reference frame of the liquid.

$$bt_1 = 2l + vt_1, \quad or \quad t_1 = \frac{2l}{b-v}$$

$$l_1 = bt_1 = \frac{2lb}{b-v}, \qquad l_2 = \frac{2lb}{b+v} \quad (14)$$

Substituting these values into (13) for each path gives the following results:

$$\frac{\phi_1-\phi}{2\pi}=\frac{\delta\phi_1}{2\pi}=\frac{2lb}{(b-v)}\cdot\frac{1}{\lambda_1}[\eta-1]=$$

$$\frac{2lb}{(b-v)}\cdot\frac{c}{(c-v)\lambda}\eta[1-\frac{1}{\eta}]$$

$$\frac{\phi_2-\phi}{2\pi}=\frac{\delta\phi_2}{2\pi}=\frac{2lb}{(b+v)}\cdot\frac{1}{\lambda_2}[\eta-1]=$$

$$\frac{2lb}{(b+v)}\cdot\frac{c}{(c+v)\lambda}\eta[1-\frac{1}{\eta}] \qquad (15)$$

$$\delta N=\frac{\delta\phi_1}{2\pi}-\frac{\delta\phi_2}{2\pi}\approx$$

$$\frac{2lbc\eta[1-1/\eta]}{[b-v][c-v]\lambda}-\frac{2lbc\eta[1-1/\eta]}{[b+v][c+v]\lambda}\approx$$

$$\frac{2l\eta[1-1/\eta]}{bc\lambda}[2vc+2vb]\approx$$

$$\frac{4l\eta^2v[1-1/\eta][1+1/\eta]}{\lambda c}\approx\frac{4l\eta^2v}{\lambda c}[1-\frac{1}{\eta^2}] \qquad (16)$$

Notice how these results were obtained without invoking "aether" drag, or relativistic velocity addition.

In the special relativistic analysis of this experiment, the velocity of light in the moving liquid as measured in the lab frame is no longer $b + v$, but is given by the relativistic velocity addition formula:

$$b'=\frac{b-v}{1-\frac{vb}{c^2}}=\frac{b-v}{1-\frac{v}{\eta c}} \qquad (17)$$

As a result, the path lengths derived in (14) become:

$$l_1=\frac{2lb(1-\frac{v}{\eta c})}{b-v}\quad,\quad l_2=\frac{2lb(1+\frac{v}{\eta c})}{b+v} \qquad (18)$$

The derivation of the total phase shift then becomes:

$$\frac{\phi_1-\phi}{2\pi}=\frac{\delta\phi_1}{2\pi}=\frac{2lb(1-\frac{v}{\eta c})}{(b-v)}\cdot\frac{1}{\lambda_1}[\eta-1]$$

$$=\frac{2lb(1-\frac{v}{\eta c})}{(b-v)}\cdot\frac{c}{(c-v)\lambda}\eta[1-\frac{1}{\eta}]$$

$$\frac{\phi_2-\phi}{2\pi}=\frac{\delta\phi_2}{2\pi}=\frac{2lb(1+\frac{v}{\eta c})}{(b+v)}\cdot\frac{1}{\lambda_2}[\eta-1]$$

$$=\frac{2lb(1+\frac{v}{\eta c})}{(b+v)}\cdot\frac{c}{(c+v)\lambda}\eta[1-\frac{1}{\eta}] \quad (19)$$

$$\delta N=\frac{\delta\phi_1}{2\pi}-\frac{\delta\phi_2}{2\pi}\approx$$

$$\frac{2l\eta[1-1/\eta]}{bc\lambda}[2vc+2vb+2\frac{vb}{\eta}+2\frac{v^3}{\eta c}]\approx$$

$$\frac{4l\eta^2 v}{\lambda c}[1-1/\eta][1+1/\eta+1/\eta^2]\approx\frac{4l\eta^2 v}{\lambda c}[1-\frac{1}{\eta^3}] \quad (20)$$

The two results, (16) and (20), differ in the exponent of the last h term. When Michelson and Morley performed the experiment, they obtained sixty one trials, using three different combinations of water velocity and tube length. The graph below shows the distribution of these results, normalized to a tube length of ten meters and a water velocity of one meter per second. The line marked RCM represents the value obtained from equation (16). The line marked SRT reflects the value obtained from (20). While there is a distribution of results, owing to experimental error, Michelson claimed an overall shift of 0.184 $\pm$ 0.02 fringe. This is completely consistent with (16), but eliminates the special relativistic result, with a value of 0.247, from consideration.

Summary

It is very difficult to find adequate tests between special relativity and other competing theories. Most theories overlap with SRT on a vast majority of the prediction made by each, yet are based on different underlying physical principles. Ultimately one must find a test that checks not only the results of the application of the mathematical theory, but also the underlying assumptions. The major conceptual difference between SRT and most competing theories is the idea of relative simultaneity—that distant events that are simultaneous for one observer will not be simultaneous for and observer in motion relative to the first. The relativistic velocity addition rule is a direct consequence of relativistic simultaneity, and the Fizeau experiment represents a direct test of the velocity addition formula. Regardless of what the correct theory is or may be, it is clear that SRT fails to give predictions consistent with results in this experiment—an experiment performed almost ten years before the development of SRT.

Four-vectors

Although the use of 4-vectors is not necessary for a full understanding of Special Relativity, they are a most powerful and useful tool for attacking many problems. A 4-vectors is just a 4-tuplet A = (A_0, A_1, A_2, A_3) that transforms under a Lorentz Transformation in the same way as (cdt, dx, dy, dz) does. That is:

$A_0 = \gamma(A_0' + (v/c)A_1')$

$A_1 = \gamma(A_1' + (v/c)A_0')$

$A_2 = A_2'$

$A_3 = A_3'$

Lorentz transformations are very much like rotations in 4-dimensional spacetime. 4-vectors, then, generalize the concept of rotations in 3-space to rotations in 4-dimensions. Clearly, any constant multiple of (cdt, dx, dy, dz) is a 4-vector, but something like A = (cdt,mdx, dy, dz) (where m is just a constant) is not a 4-vector because the second component has to transform like mdxâÉá$A_1 = \gamma(A_1' + (v/c)A_0')$âÉáγ((mdx') +vdt') from the definition of a 4-vector, but also like mdx = mγ(dx' + (v/c)dt') ; these two expression are inconsistent. Thus we can transform a 4-vector either according to the 4- vector definition given above, or using what we know about how the dx_i transform to transform each A_i independently. There are only a few special vectors for which these two methods yield the same result. Several different 4-vectors are now discussed:

Velocity 4-vector

We can define a quantity τ = $\sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ which is called the proper time, and is invariant between frames. Dividing out original 4-vector ((cdt, dx, dx, dz)) bydτ gives:

$V = \frac{1}{d\tau}(cdt, dx, dy, dz) = \gamma\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma c, \gamma \mathbf{v})$

This arises because $\frac{dt}{d\tau} = \gamma$.

Energy-momentum 4-vector

If we multiply the velocity 4-vector by m we get:

$P = mV = m(\gamma c, \gamma \mathbf{v}) = (\gamma \mathbf{mc}, \gamma \mathbf{mv}) = (\mathbf{E}/\mathbf{c}, \mathbf{p})$

This is an extremely important 4-vector in Special Relativity.

Relation between momentum and kinetic energy

Sometimes it's desirable to express the kinetic energy of a particle in terms of the momentum. That's easy enough. Since $\mathbf{v} = \mathbf{p}/m$ and the kinetic energy $K = \frac{1}{2}mv^2$ so

$$K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} \qquad (1.4)$$

Note that if a massive particle and a light particle have the same momentum, the light one will have a lot more kinetic energy. If a light particle and a heavy one have the same velocity, the heavy one has more kinetic energy.

UNIT V

SYLLABUS

General theory of Relativity: Introduction – Limitation of special theory of relativity and need for a relativity theory in non-inertial frames of reference. Concept of gravitational and inertial mass and the basic postulate of GTR, gravitation & acceleration and their relation to non-inertial frames of reference – principle of equivalence of principle of general co-variance – Minkowski space and Lorentz transformation

General relativity - Introduction

Prior to the 20^{th} century all physics theories assumed space and time to be absolutes. Together they formed a background within which matter moved. The role of a physical theory was to describe how different kinds of matter would interact with each other and, by doing so, predict their motions. With the development of special and later general relativity theory in the early 20^{th} century, the role of space and time in our theories of physics changed dramatically. Instead of being a passive background, space and time came to be viewed as dynamic actors in physics, capable of being changed by the matter within them and in turn changing the way that matter behaves.

In GR, spacetime becomes *curved* in response to the effects of matter. I will discuss below what it means for spacetime to be curved, but just to give a flavor of this idea I can note here that in a curved spacetime the laws of Euclidean geometry no longer hold: the angles of a triangle do not in general add up to 180°, the ratio of the circumference of a circle to its diameter is in general not p, and so on. This curvature in turn affects the behavior of matter. In Newtonian physics a particle with nothing pushing or pulling it (no forces acting on it) will move in a straight line. In a curved spacetime what used to be straight lines are now twisted and bent, and particles with no forces acting on them are seen to move along curved paths.

Limitations of special theory of relativity

γ can be expanded into a Taylor series or binomial series for $\frac{v^2}{c^2} < 1$, obtaining:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{(2k-1)}{2k} \frac{v^2}{c^2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \dots$$

and consequently

$$E - mc^2 = \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \frac{5}{16}\frac{mv^6}{c^4} + \dots;$$

$$\mathbf{p} = m\mathbf{v} + \frac{1}{2}\frac{mv^2\mathbf{v}}{c^2} + \frac{3}{8}\frac{mv^4\mathbf{v}}{c^4} + \frac{5}{16}\frac{mv^6\mathbf{v}}{c^6} + \cdots .$$

For velocities much smaller than that of light, one can neglect the terms with c^2 and higher in the denominator. These formulas then reduce to the standard definitions of Newtonian kinetic energy and momentum. This is as it should be, for special relativity must agree with Newtonian mechanics at low velocities.

Inertial and Gravitational Mass

Mass, from the traditional physics viewpoint, arises from two sources, its inertia and the gravitational attraction of other masses. This has led in physics to a distinction between inertial mass and gravitational mass -- a distinction which can be easily demonstrated in a simple Experiment. One can be thought of as resistive force to change in motion (speed and/or direction), while the other stems from an attractive force between masses.

According to Newton's second law of motion (F = ma), the mass of a body can be determined by measuring the acceleration produced in it by a constant force. (i.e) m = F/a. Intertial mass of a body is a measure of the ability of a body to oppose the production of acceleration in it by an external force.

Inertial mass

According to Newton's second law of motion (F = ma), the mass of a body can be determined by measuring the acceleration produced in it by a constant force. (i.e) m = F/a. Intertial mass of a body is a measure of the ability of a body to oppose the production of acceleration in it by an external force.

If a constant force acts on two masses m_A and m_B and produces accelerations a_A and a_B respectively, then, $F= m_A a_A = m_B a_B$

$m_A / m_B = a_A / a_B$

The ratio of two masses is independent of the constant force. If the same force is applied on two different bodies, the inertial mass of the body is more in which the acceleration produced is less.

If one of the two masses is a standard kilogram, the unknown mass can be determined by comparing their accelerations.

Gravitational mass

According to Newton's law of gravitation, the gravitational force on a body is proportional to its mass. We can measure the mass of a body by measuring the gravitational force exerted on it by a massive body like Earth. Gravitational mass is the mass of a body which determines the magnitude of gravitational pull between the body and the Earth. This is determined with the help of a beam balance.

If F_A and F_B are the gravitational forces of attraction on the two bodies of masses m_A and m_B due to the Earth, then

$F_A = Gm_AM /R^2$

$F_B = Gm_BM /R^2$

where M is mass of the Earth, R is the radius of the Earth and G is the gravitational constant.

$m_A/ m_B = F_A /F_B$

If one of the two masses is a standard kilogram, the unknown mass can be determined by comparing the gravitational forces.

Gravitation seems a simple concept, wherein two objects with mass are attracted to each other, dependent upon the inverse square of the distance between them, their respective masses, and a Gravitational Constant. These masses are attracted to each other is never really addressed, while the means by which a force connects them -- what physics thinks of as “action-at-a-distance” -- has been under debate since Galileo. Or longer. Thus gravity, while experientially easy to deal with -- i.e. no support, one fall down! -- the physics itself is still in flux. (To add insult to injury, there is evidence from such a diverse field as Hyperdimensional Physics to suggest that the Gravitational Constant... is not a constant, and has changed notably over the eons. Even some 0.06% in the last twenty years or so! It may be just a matter of time before “what goes up... stays”, i.e. Levitation and/or the worst fears of the Anti-Gravity Defamation League come true.)

Inertia’s case, on the other hand, is even more difficult. Galileo’s attempt was to define inertia as a property of matter that kept an object in uniform motion, unless acted upon by a force external to the object. Sir Isaac Newton formalized this in his Principia, and in his first and second laws. His first law is actually a special case of the second, the latter which states that the acceleration (a) -- change in velocity (speed and direction) is proportional to the force (F) applied on the object, and that the constant of proportionality is the mass (m). I.e. F = ma. Inertial mass can thus be viewed as the resistance of an object to being accelerated by an external force. When there is no force, or when the force ceases, the acceleration is zero, and the object moves in uniform motion (maintaining the same speed and same direction). Massive objects are therefore assumed to resist acceleration because such resistance is an innate property of matter.

Postulates of a general theory

The general theory is based on a seemingly common observation about gravity and accelerations. The two postulates of Einstein's general theory of relativity are:

- All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.
- In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. This is the principle of equivalence, which forms the basis of the general theory of relativity.

Mass have seemingly different properties: a gravitational attraction and an inertial property that resists acceleration. To designate these two attributes, we use the subscripts g and i and write:

Gravitational property $F_g = m_g g$

Inertial property $\sum F = m_i a$.

The second postulate implies that gravitational mass and inertial mass are completely equivalent, not just proportional.

Gravitation and acceleration

It is easy to verify that, if air resistence is negligable, all objects accelerate towards the earth at the same rate. This mystery, first verified experimentally by Galileo, is at least partially explained by Newton's law of gravity. The ``reason'' is that the gravitational force on an object is proportional to its inertial mass. According to Newton's second law, in order to calculate the acceleration of an object caused by gravity, we must take the gravitational force on that object and divide by the inertial mass. Thus, the inertial mass of the object cancels out of the resulting expression for the acceleration. In fact the acceleration of any object at the Earth's surface is determined by the distance of the object form the center of the Earth (R_E), Newton's constant (G) and the mass of the Earth:

$$g = \frac{GM_E}{R_E^2}$$

If you put the value of Newton's constant, the radius of the Earth (6×10^6 meters) and the mass of the Earth (6×10^{24}kg) into the above expression you will get approximately 9.8m/s^2, which is the rate at which all objects accelerate downwards at the surface of the Earth.

Although the magnitude of the acceleration due to gravity, g, is the same everywhere on the Earth's surface, its direction changes depending on where you are. It is a vector that always points towards the center of the Earth, so, for example, it is in the opposite direction at the North Pole than at the South Pole. This effect is not very relevant to us because the Earth is so big. If we move from one end of the room to another, or even one end of the city to another, we are only moving across a very small fraction of the total circumference of the Earth, so the direction of the gravitational acceleration changes very little. Our notion of ``down'' only changes significantly when we travel very large distances. However, as we will see later, if you happen to be near a very massive, but small object, such as a black hole, the fact that gravitational acceleration changes direction depending on your location becomes very significant indeed: it gives rise to so-called tidal gravitational forces that can tear a spaceship apart in microseconds.

Lorentz transformation

In flat space in three dimensions, the distance between two points is given by ds where

ds^2 = dx^2 + dy^2 + dx^2

This expression tells us how we measure distances and is called the "metric". If we can write the metric in the form above, we say that space if Euclidean (or "flat"). Let's say you want to change to another coordinate system (X,Y,Z) where you can make the same combination of dX and dY and dZ and come up with the same distance (this will be useful since distances are invariant in Newtonian physics)

Then ds^2 = dX^2 + dY^2 + dZ^2

If you calculate the transformations allowed from {x,y,z} to {X,Y,Z} you find that they're the orthogonal transformations, which just describe rotations in R^3.

Now suppose you're not interested in keeping how you measure distances constant, but you want the speed of light, c, constant. Then it's not difficult to show that you want a system of coordinates in which

ds^2 = -c^t dt^2 + dx^2 + dy^2 + dz^2 =0

and

ds^2 = -c^t dt^2 + dx^2 + dy^2 + dz^2 is a constant between two events.

This describes a space (space-time really) which is obviously different from Euclidean space, and we call the space-time Minkowski space-time. So Minkowski space-time is a space-time where we can set up coordinates (t,x,y,z) so that light travels along lines where

-c^2 dt^2 + dx^2 + dy^2 + dz^2 = 0.

Now you can ask yourself, if I have a set of coordinates {t,x,y,z}, what transformations am I allowed from {t,x,y,z} to {T,X,Y,Z} that keep the form of the Minkowski metric. Those transformations are the Lorentz transformations. They generalise the rotations in R^3.

So Lorentz transformations describe how we can change coordinates systems between two inertial frames

Minkowski Diagrams

A Minkowski diagram or spacetime diagram is a convenient way of graphically representing the lorentz transformations between frames as a transformation of coordinates. They are especially useful for gaining a qualitative understanding of relativistic problems. We make a spacetime diagram by representing frame F as the coordinate axes x (horizontal) and ct(vertical). We are ignoring the y and z directions, since they are uninteresting. The plot of an object's x - position versus time on the Minkowski diagram is called its worldline. Notice that light, traveling one unit of ct for every unit of xwill follow the line x = ct , inclined at a 45 ^{o} angle.

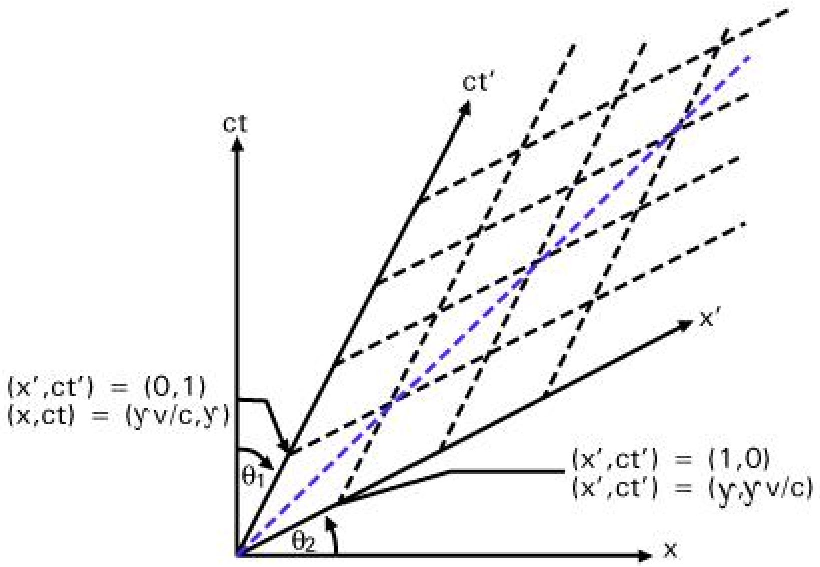


Figure %: Minkowski or spacetime diagram.

What do the axes of F' , moving with velocity v along the x -axis of F look like? Take the point (x', ct') = (0, 1) . From the lorentz transformations we can find that this point transforms to (x, ct) = (γv/c, γ) . As shown in the angle between the ct' and ct axes is given by: tanθ $_1$ = x/ct = v/c . Actually, the ct' axis is just the worldline of the origin of F' . The point (x, ct) = (γv/c, γ) is a distance $\sqrt{\gamma^2 + gamma^2 v^2/c^2}$ = γ $\sqrt{1 + v^2/c^2}$ from the origin, so the ratio of units on the ct' axis to those on the ct axis is just this value, namely:

$$\frac{ct'}{ct} = \sqrt{\frac{1+v^2/c^2}{1-v^2/c^2}}$$

This approaches infinity as $v \rightarrow c$ and is one if $v = 0$. Similar analysis shows that the x' axis is an equal angle from the x -axis and that the ratio of units $\frac{x'}{x}$ is also equal (see). Thus, the faster F' relative to F , the more its coordinates are squished towards the x = ct line.

The advantage of a Minkowski diagram is that the same worldline applies to both sets of coordinate axes (that is, to x and ct , as well as to x' and ct'). The Lorentz transformation is made by changing the coordinate system underneath the worldline rather than the worldline itself. In many situations this allows us to visualize the perspectives of the different observers more easily. If we had a very detailed and accurate Minkowski diagram we could use it to read off the values for Δx , Δct , $\Delta x'$, and $\Delta ct'$. To find the spacetime coordinates of an event in F , one can read the value off the x and ct axes; to find the coordinates in a moving frame the x' and ct' axes corresponding to the appropriate velocity can be constructed (using the angle formulas explained above), and the value read off using the units derived for x' and ct' , above.

S.No	Questions	Option A	Option B	Option C	Option D	Answer
1	The mass of 70 kg man moving in car at 66kmh is	70 kg	100 kg	infinite	zero	70 kg
2	Special theory of relativity treats problems involving	inertial frame of reference	non-inertial frame of reference	non-accelerated frame of reference	accelerated frame of reference	inertial frame of reference
3	According to special theory of relativity which one is not an absolute quantity	time	mass	height	both a and b	both a and b
4	Conversion of solar energy into carbohydrates and starch by leaf of a plant is an example for	energy into mass	mass in to energy	momentum into velocity	velocity into momentum	energy into mass

5	A reference frame attached to the earth:	is an inertial frame by definition	is an inertial frame because Newton's laws are applicable in the frame	Cannot be an inertial frame because the earth is revolving round the sun	Cannot be an inertial frame because the earth is rotating about its own axis.	is an inertial frame by definition
6	Two photons approach each other, their relative velocity will be	c/2	Zero	c/8	c	c
7	An inertial frame is	Accelerated	decelarated	Moving with uniform velocity or at rest.	May be accelerated, decelerated or moving with constant velocity	Moving with uniform velocity or at rest.
8	All the inertial frames are equivalent" this statement is called the principle of -----------	relative motion	equivalence	inertia	Correspondence.	relative motion
9	According to relativity, the length of a rod in motion:	is same as its rest length	is more than its rest length	is less than its rest length	may be more or less than or equal to rest	is less than its rest length

					length depending on the speed of rod.	
10	If v = c, the length of a rod in motion is:	zero	equal to proper length	less than proper length	more than proper length.	zero
11	According to special theory of relativity:	speed of light is relative	speed of light is same in all inertial frames	time is relative	mass is relative	speed of light is same in all inertial frames
12	James travels at high speed from the Earth to the star Alpha Centauri, four light years away. In James's frame	the trip takes more time than it does in the Earth's frame.	James travels to Alpha Centauri over a length that is shorter than four light years.	clocks on Earth and on Alpha Centauri are synchronized.	Alpha Centauri travels to James over a length that is shorter than four light years.	Alpha Centauri travels to James over a length that is shorter than four light

						years.
13	Relativity mechanics is applicable for a particle which is moving with a velocityà	Greater than that of light	Less than that of light	Comparable to that of light	equal to velocity of light	Comparable to that of light
14	The relativistic measurement depends uponà	The state of motion of the observer as well as upon the quality that is being measured.	The state of motion of the observer only	The quantity that is being measured	absolute motion	The quantity that is being measured
15	A frame which is moving with zero acceleration is called	Non-inertial frame	Inertial frame	rest frame	decelerated frame	Inertial frame
16	When we specific the place of occurrence of a phenomenon as well as the time of occurrence it is considered as	a point	an event	an incident	an accident	an event

17	Newton's law's remain unchanged or invariant	Under Galilean transformation	under lorentz transformation	cartesean transformation	new transformation	Under Galilean transformation
18	The laws of mechanics in all initial frame of reference are	same	different	none	variable	same
19	The acceleration of a particle under Galilean transformation is	invariant	non-variant	none	variable	invariant
20	The mass energy relation was proposed by	Newton	Einstein	Kepler	Michelson	Einstein
21	The Lorentz transformation will converted to Galilean transformation when the relative velocity v between two inertial frames will satisfy the condition	v>>c	v=c	v<<c	v=0	v<<c

22	the length of an object is maximum in a reference frame in which it is	at rest	in motion	neither rest nor in motion	varying speed	at rest
23	the length of a rod in uniform motion relative to an observer	appears to be shortened when it at rest w.r.t. to the observer	appears to be lengthened when it is at rest w.r.t. to the observer	equal to aboslute length	invariant length	appears to be shortened when it at rest w.r.t. to the observer
24	The time interval between two event in a reference in a reference frame which is in motion is	Maximum	minimum	zero	varying speed	Maximum

25	A moving clock	Runs slower than a stationary identical clock	Runs than a stationary identical clock	neither slow nor fast	very fast	Runs slower than a stationary identical clock
26	If the velocity of a moving particle is comparable to velocity of light then the mass of the moving object is	Greater than when it is rest	Smaller than when it is at rest	Equal	very smaller	Greater than when it is rest
27	Einistein's mass energy equation E=mc2 implies that	Energy disappears to reappear as mass	Mass disappears to reappear as energy	All the above statements are correct except d	nothing can be done	All the above statements are correct except d

28	How fast a particle must travel so that its mass becomes twice its rest mass?	0.5 c	2 c	0.866 c	0.9c	0.866 c
29	Relative velocity of two particles moving with velocity of light of light in opposite direction is	0	2c	c	3c	c
30	For a photon particle which is moving with a velocity of light, the rest mass is	0	1	2	3	0
31	The fictitious force, which acts on particle in motion relative to a rotating frame of reference is called	Coriolis force	Newtonian force	Pseudo force	centripetal force	Coriolis force
32	If the particle is at rest relative to the rotating frame of reference the coriolis force is	0	1	10	2	0

33	When the particle is at a non-rotating of reference the Coriolis force	1	0	2	3	0
34	The Coriolis acceleration on a freely falling body under the action of gravitational force is	Directed towards the east	Directed towards the west	directed towards north	directed towards south	Directed towards the east
35	According to theory of relative mass of an object is	depends on particles.	speed of light.	volume of object.	area of object.	speed of light.
36	Radiation with energy that is easily detected as quanta	1 eV.	1 keV.	1 MeV.	. 10-10 eV.	1 MeV.
37	lf the kinetic energy of a body becomes four times its initial value, the new momentum will be	Three times the initial value	Four times the initial value	Two times the initial value	unchanged	Two times the initial value

38	Lorentz transformation equations hold for	Non-relativistic velocities only	Relativistic velocities only	. All velocities: relativistic & non-relativistic	Photons only	All velocities: relativistic & non-relativistic
39	lf the kinetic energy of a body becomes four times its initial value, the new momentum will be	Three times the initial value	Four times the initial value	. Two times the initial value	unchanged	Two times the initial value
40	If the radius of the earth were to shrink, its mass remaining the same, the value of acceleration due to gravity at the pole and at the equator will	Increase and decrease respectively	Decrease and increase respectively	Increase at both places	Decrease at both places	Increase at both places

41	What do we mean by the straightest possible path between two points on Earth's surface?	a path that actually is a perfectly straight line	a path that follows a circle of longitude	a path that follows a circle of latitude	the shortest path between the two points	the shortest path between the two points
42	Which of the following statements is not a prediction of the general theory of relativity?	Time runs slightly slower on the surface of the Sun than on the surface of Earth.	The Universe has no boundaries and no center.	The curvature of spacetime can distort the appearance of distant objects.	Different observers can disagree about the fundamental structure of spacetime.	Different observers can disagree about the fundamental structure of spacetime.

43	What does the equivalence principle say?	Gravity is the same thing as curvature of spacetime.	The effects of gravity are exactly equivalent to the effects of acceleration.	All observers must always measure the same (equivalent) weights for moving objects.	The effects of relativity are exactly equivalent to those predicted by Newton's laws of motion.	The effects of gravity are exactly equivalent to the effects of acceleration.

44	Each of the following is a prediction of the theory of relativity. Which one is crucial to understanding how the Sun provides light and heat to Earth?	If you observe someone moving by you, you'll see their time running slowly.	Gravity is curvature of spacetime.	E = mc2	Time runs slower on the surface of the Sun than on Earth.	E = mc2

45	According to general relativity, how is time affected by gravity?	Time is not affected by gravity.	Time is stopped by any gravitational field.	Time runs slower in stronger gravitational fields.	Time is stopped by any gravitational field.	Time runs slower in stronger gravitational fields.
46	According to general relativity, a black hole is	an object that cannot be seen.	a hole in the observable universe	a place where there is no gravity.	a place where light travels faster than the normal speed of light	a hole in the observable universe

47	According to general relativity, why does Earth orbit the Sun?	Earth is following the straightest path possible, but spacetime is curved in such a way that this path goes around the Sun	Earth is following the straightest path possible, but spacetime is curved in such a way that this path goes around the Sun	The mysterious force that we call gravity holds Earth in orbit.	The mysterious force that we call centripetalforce holds Earth in orbit.	Earth is following the straightest path possible, but spacetime is curved in such a way that this path goes around the Sun
48	If you draw a spacetime diagram, the worldline of an object that is accelerating	vertical.	curved.	horizontal.	slanted.	curved.

	away from you is					
49	If you draw a spacetime diagram, the worldline of an object that is traveling by you at constant speed is	vertical.	curved.	horizontal.	slanted.	slanted.
50	If you draw a spacetime diagram, the worldline of an object that is stationary in your reference frame is	vertical.	curved.	horizontal.	slanted.	vertical.
51	What do we mean by dimension in the context of relativity?	the size of an object	the number of independent directions in which movement is possible	the letter used to represent length mathematically	the height of an object	the number of independent directions in which movement is possible

52	Suppose you claim that you are feeling the effects of a gravitational field. How can you explain the fact that Al is weightless?	She is weightless because she is moving at constant velocity.	She is weightless because she is in a free-float frame.	She is weightless because she is in free-fall.	If you are in a gravitational field, then she cannot be weightless	She is weightless because she is in free-fall.
53	Einstein's Theory of General Relativity states that	gravity and acceleration are equivalent.	the speed of light is constant.	physics for accelerated and nonaccelerated frames are not the same.	physics for nonmoving and moving frames are not the same.	gravity and acceleration are equivalent.

54	Einstein said that gravity exists because	massive objects warp space.	massive objects attract one another.	light moves randomly throughout the universe	of the existence of black holes.	massive objects warp space
55	According to Einstein, what is considered the fourth dimension?	horizontal dimension	curled dimension	me dimension	space dimension	me dimension
56	Einstein's famous equation E = mc2 states that	mass is always greater than energy.	energy and mass are equivalent.	energy and the speed of light are equivalent.	mass and the speed of light are equivalent.	energy and mass are equivalent.
57	A person is riding a moped that is traveling at 20.0 m/s. What is the	20.0 m/s	3.00 × 108 m/s	24.0 m/s	3.00 × 108 m/s + 20.0 m/s	24.0 m/s

	speed of a ball if the moped rider throws a ball forward at 4.00 m/s while riding the moped?					
58	A beam of light travels at 3.00 × 108 m/s. If a moped moving at 20.0 m/s turns on its headlight, how fast does the light travel?	20.0 m/s	3.00 × 108 m/s	3.00 × 108 m/s + 20.0 m/s	3.00 × 108 m/s – 20.0 m/s	3.00 × 108 m/s
59	Einstein's Second Postulate of Special Relativity states that the speed of light	is constant regardless of the speed of the observer or the light source.	can increase if the speed of the light source increases.	can decrease if the speed of the observer decreases.	randomly changes depending upon its original light source.	is constant regardless of the speed of the observer or the light source.
60	A particular task requires 3.46 J of energy. Using E = mc2, how much mass is	3.11 × 1017 kg	3.84 × 10–17 kg	3.46 × 10–8 kg	1.15 × 10–8 kg	3.84 × 10⁻17 kg

	needed to accomplish this task?					
61	Mass of 700 N man moving in car at 66 kmh^{-1} is	70 kg.	100Kg	0	10Kg	70 kg.
62	Special theory of relativity treats problems involving	inertial frame of reference.	non-inertial frame of reference.	non-accelerated frame of reference.	accelerated frame of reference.	inertial frame of reference.