KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF PHYSICS

STAFF NAME : **Dr. S. KARUPPUSAMY** SUBJECT NAME: **MATHEMATICAL PHYSICS-II**
SUB.CODE: **18PHU203** SEMESTER: **II**
CLASS: **I B.Sc., (PHYSICS)**

Sl.No.	Lecture Duration Period	Topics to be covered	Support Material/Page Nos.
UNIT I			
1.	1 hr	Fourier series, Periodic functions	T1(527-528)
2.	1 hr	Orthogonality of sine and cosine functions, Dirichlet Conditions	T1(527-530)
3.	1 hr	Expansion of periodic functions in a series of sine and cosine functions and determination of Fourier coefficients	T1(527-529)
4.	1 hr	Complex representation of Fourier series	T1(540)
5.	1 hr	Expansion of functions with arbitrary period.	T1(541)
6.	1 hr	Expansion of non-periodic functions over an interval	T1(544)
7.	1 hr	Even and odd functions and their Fourier expansions ,Application	T1(528)
8.	1 hr	Application	T1(528)
9.	1 hr	Revision	
Total Number of Hours Planned For Unit I = 9hr			

UNIT II			
10.	1 hr	Bisection method	T2(69-72)
11.	1 hr	Method of successive approximations	T2(75-80)
12.	1 hr	RegulaFalsi method	T2(81-88)
13.	1 hr	Newton-Raphson method	T2(88-98)
14.	1 hr	Horner's method	T2(98-101)
15.	1 hr	Euler's method	T2(369-370)
16.	1 hr	Modified Euler's method	T2(371-375)
17.	1 hr	RungeKutta method (II & IV)	T2(379-395)
18.	1 hr	Revision	
Total Number of Hours Planned For Unit II = 9hr			
UNIT III			
19.	1 hr	Gauss elimination method	T2(112-114)
20.	1 hr	Gauss-Jordan method	T2(114-120)
21.	1 hr	Gauss-Seidel method	T2(147-158)
22.	1 hr	Computation of inverse of a matrix using Gauss elimination method	T2(122-126)
23.	1 hr	Method of triangularisation	T2(126-132)
24.	1 hr	Trapezoidal rule	T2(300-306)
25.	1 hr	Simpson's 1/3 rule	T2(303-305)
26.	1 hr	Simpson's 3/8 rule	T2(305-307)
27.	1 hr	Revision	
Total Number of Hours Planned For Unit III = 9hr			
UNIT IV			

28.	1 hr	Arithmetic mean, Median	T1(766-768)
29.	1 hr	Quartiles, Deciles , Percentiles, Mode	T1(768-769)
30.	1 hr	Empirical relation between mean	T1(773-774)
31.	1 hr	Empirical relation between median and mode	T1(774-775)
32.	1 hr	Geometric mean, harmonic mean	T1(767)
33.	1 hr	Relation between arithmetic mean, geometric mean and harmonic mean	T1(769-770)
34.	1 hr	Range, Range meanor average deviation	T1(770-771)
35.	1 hr	Standard deviation	T1(771-772)
36.	1 hr	Variance and mean square deviation	T1(776)
37.	1 hr	Revision	
Total Number of Hours Planned For Unit IV = 10hr			
UNIT V			
38.	1 hr	Solutions to partial differential equations, using separation of variables	T1(566-567)
39.	1 hr	Laplace's Equation in problems of rectangular	T1(573)
40.	1 hr	Cylindrical symmetry	T1(576)
41.	1 hr	Spherical symmetry	T1(574)
42.	1 hr	Wave equation	T1(575)
43.	1 hr	Solution for vibrational modes of a stretched string	T1(624-627)
44.	1 hr	Rectangular membranes	T1(602)
45.	1 hr	Circular membranes	T1(608-617)
46.	1 hr	Diffusion Equation	T1(582)
47.	1 hr	Revision	

48.	1 hr	Old Question Paper Revision	
49.	1 hr	Old Question Paper Revision	
50.	1 hr	Old Question Paper Revision	
Total Number of Hours Planned For Unit V = 13hr			

Suggested Reading Books

T1 : Mathematical Physics by Sathya prakash, S.Chand & company, New Delhi.

T2 : Numerical Methods by Dr.P.Kandasamy, Dr.K.Thilagavathy, Dr.K.Gunavathi, S.Chand & company, New Delhi.

SYLLABUS

Fourier Series: Periodic functions. Orthogonality of sine and cosine functions, Dirichlet Conditions (Statement only). Expansion of periodic functions in a series of sine and cosine functions and determination of Fourier coefficients. Complex representation of Fourier series. Expansion of functions with arbitrary period. Expansion of non-periodic functions over an interval. Even and odd functions and their Fourier expansions. Application.

Fourier series

Periodic functions

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Examples of successive approximations to common functions using Fourier series are illustrated above.

In particular, since the superposition principle holds for solutions of a linear homogeneous ordinary differential equation, if such an equation can be solved in the case of a single sinusoid, the solution for an arbitrary function is immediately available by expressing the original function as a Fourier series and then plugging in the solution for each sinusoidal component. In some special cases where the Fourier series can be summed in closed form, this technique can even yield analytic solutions.

Any set of functions that form a complete orthogonal system have a corresponding generalized Fourier series analogous to the Fourier series. For example, using orthogonality of the roots of a Bessel function of the first kind gives a so-called Fourier-Bessel series.

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(m x) \sin(n x)\, dx = \pi\delta_{mn} \tag{1}$$

$$\int_{-\pi}^{\pi} \cos(m x) \cos(n x)\, dx = \pi\delta_{mn} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(m x) \cos(n x)\, dx = 0 \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(m x)\, dx = 0 \tag{4}$$

$$\int_{-\pi}^{\pi} \cos(m x)\, dx = 0 \tag{5}$$

for $m, n \neq 0$, where δ_{mn} is the Kronecker delta.

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = \cos x$ and $f_2(x) = \sin x$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x), \tag{6}$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \tag{7}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n x)\, dx \tag{8}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n x)\, dx \tag{9}$$

and $n = 1$, 2, 3, ... Note that the coefficient of the constant term a_0 has been written in a special form compared to the general form for a generalized Fourier series in order to preserve symmetry with the definitions of a_n and b_n.

Orthogonal Series Expansion

Let $\{\varphi_n(x)\}$ be an infinite orthonormal set of functions on interval [a,b]. and f(x) be a function defined on [a,b]. Then f(x) can be written as $f(x)=c_o\varphi_o(x)+c_1\varphi_2(x)+c_2\varphi_2(x)+ - - - - +c_n\varphi_n(x)+ - - - -$ (6.22)

where $$c_n = \frac{\int_a^b f(x)\varphi_n(x)dx}{\int_a^b |\varphi_n(x)|^2\, dx} = 1 \qquad (6.23)$$

n=0, 1,2,3 …

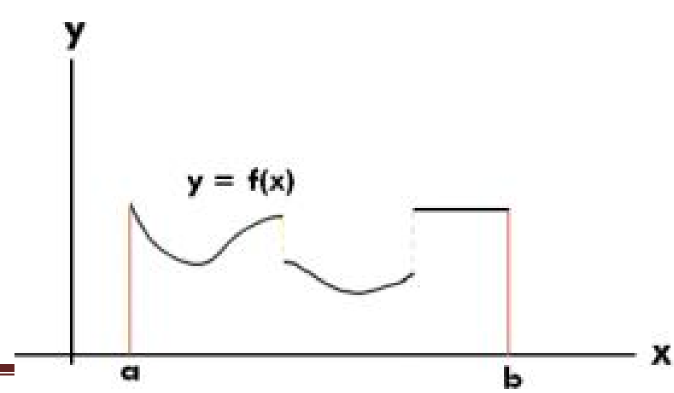
The series on the right hand side of (6.22) is called orthogonal expansion of f(x) defined on [a,b] in terms of the orthonormal set of functions $\{\varphi_n(x)\}$ defined on [a,b]. c_n's given by (6.23) are called coefficients of orthogonal expansion of f. If orthonormal set of Example 6.6 is considered we get cosine Fourier expansion of f(x), that is, (6.22) will be cosine Fourier series and (6.23) will give cosine Fourier coefficients. One can consider expansion of a function in terms of Bessel's orthonormal set of functions and Legendre's orthonormal set of functions.

Dirichlet conditions

A piecewise regular function that

1. Has a finite number of finite discontinuities and

2. Has a finite number of extrema can be expanded in a Fourier series which converges to the function at continuous points and the mean of the positive and negative limits at points of discontinuity.



Def. Sectionally continuous (or piecewise continuous) function. A function $f(x)$ is said to be **sectionally continuous** (or **piecewise continuous**) on an interval a $\leq$ x $\leq$ b if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits. See Figure The requirement that a function be sectionally continuous on some interval [a, b] is equivalent to the requirement that it meet the **Dirichlet conditions** on the interval.

Fourier series. Let $f(x)$ be a sectionally continuous function defined on an interval c < x < c + 2L. It can then be represented by the **Fourier series**

1) $$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + a_3 \cos\frac{3\pi x}{L} + \dots + b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + b_3 \sin\frac{3\pi x}{L} + \dots$$

Where

2) $$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \dots$$
$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \dots$$

At a point of discontinuity $f(x)$ is given a value equal to its mean value at the discontinuity i.e. if x = a is a point of discontinuity, $f(x)$ is given the value

$$f(x) = \frac{\lim_{x\to a_-} f(x) + \lim_{x\to a_+} f(x)}{2}$$

Complex form of Fourier series

We show how a Fourier series can be expressed more concisely if we introduce the complex number i where $i^2 = -1$. By utilising the Euler relation:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

We can replace the trigonometric functions by complex exponential functions. By also combining the Fourier coefficients a_n and b_n into a complex coefficient c_n through

$C_n = (a_n - ib_n)$

–

We find that, for a given periodic signal, both sets of constants can be found in one operation. We also obtain Parseval's theorem which has important applications in electrical engineering. The complex formulation of a Fourier series is an important precursor of the Fourier transforms which attempts to Fourier analyse non-periodic functions.

So far we have discussed the trigonometric form of a Fourier series i.e. we have represented functions of period T in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right\}$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

We obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals.

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega_0 t \; dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin n\omega_0 t \; dt \qquad n = 1, 2, \ldots$$

An alternative, more concise form, of a Fourier series is available using complex quantities. This form is quite widely used by engineers, for example in Circuit Theory and Control Theory, and leads naturally into the Fourier Transform which is the subject of

Fourier series in the interval (0, T)

We assume that the function f(x) is piecewise continuous on the interval [0,T]. Using the substitution x= Lyπ (−π≤x≤π), we can transform it into the function

$$F(y)=f(Ly/\pi)$$

which is defined and integrable on [−π,π]. Fourier series expansion of this function F(y) can be written as

$$F(y)=f(Ly/\pi)=a0/2+\sum (a_n \cos ny+b_n \sin ny).$$

Even functions

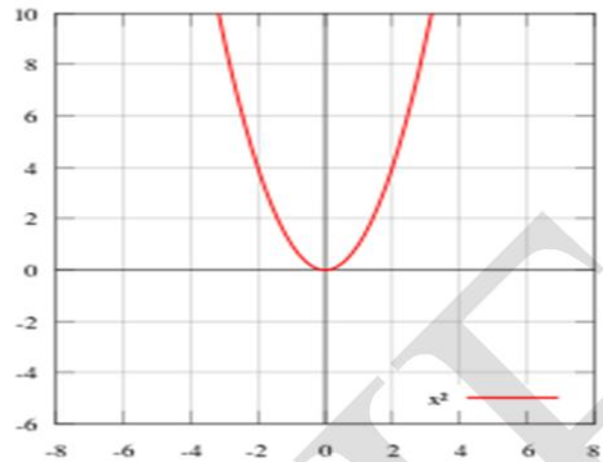
$f(x) = x^2$ is an example of an even function.

Let $f(x)$ be a real-valued function of a real variable. Then f is **even** if the following equation holds for all x and $-x$ in the domain of f:[1]

or

Geometrically speaking, the graph face of an even function is symmetric with respect to the *y*-axis, meaning that its graph remains unchanged after reflection about the *y*-axis.

Examples of even functions are $|x|$, x^2, x^4, cos(x), cosh(x), or any linear combination of these.

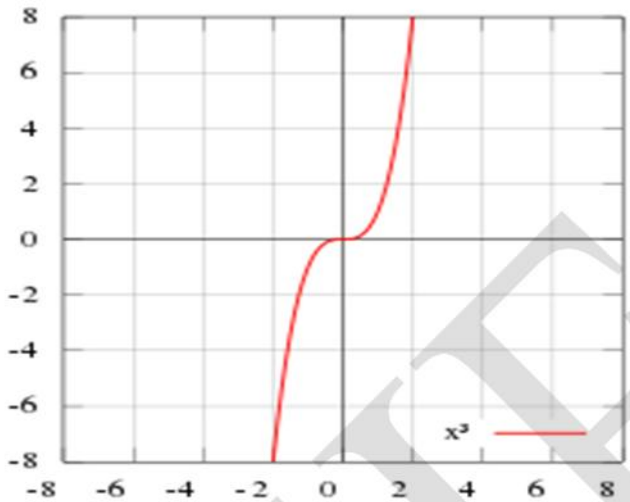


Odd functions

$f(x) = x^3$ is an example of an odd function.

Again, let *f*(*x*) be a real-valued function of a real variable. Then *f* is **odd** if the following equation holds for all *x* and -*x* in the domain of *f*. or Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

Examples of odd functions are *x*, x^3, sin(*x*), sinh(*x*), erf(*x*), or any linear combination of these.



Orthogonality of sines and cosines.

In this section we shall show that certain sequences of sine and cosine functions are orthogonal on certain intervals. The resulting expansions

$$f = \sum_{j=1}^{\infty} c_j¤_j$$

using these sines and cosines become the Fourier series expansions of the function *f*. First, we just consider the functions $_n(x) = \cos nx$. These are orthogonal on the interval $0 < x < ®$. The resulting expansion (1) is called the Fourier cosine series expansion of *f* and will be considered in more detail in section 1.5.

Proposition 1. The functions $_0(x) = 1$, $_1(x) = \cos x$, $_2(x) = \cos 2x$, $_3(x) = \cos 3x$, …, $_n(x) = \cos nx$, … are orthogonal on the interval $0 < x < ®$. Furthermore $|\ _0|^2 = .®$ and $|\ _n|^2 = \frac{®}{2}$ for $n = 1, 2, \ldots$.

Proof. Using the first identity in (8) of section 1.3 one has ($_n(x)$, $_m(x)) = \int_0^{®} \cos(nx)\cos(mx)\,dx$ $= \int_0^{®} [\frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x]\,dx$ = **Error!** = 0 so the $_n$ are orthogonal. The fact that | $_0$ $|^2$ = ® is an easy verification. | $_n$ $|^2 = \int_0^{®} \cos^2(nx)\,dx = \int_0^{®} \frac{1}{2}[1 + \cos 2nx]\,dx = \frac{1}{2}[x + \frac{1}{2n}\sin 2nx]\Big|_0^{®} = \frac{®}{2}$.

//

Next, we just consider the functions $\cdot_n(x) = \sin nx$. These are also orthogonal on the interval 0 < x < ®. The resulting expansion (1) is called the Fourier sine series expansion of f and will be considered.

Proposition 2. The functions $\cdot_1(x) = \sin x$, $\cdot_2(x) = \sin 2x$, $\cdot_3(x) = \sin 3x$, …, $\cdot_n(x) = \sin nx$, … are orthogonal on the interval 0 < x < ®. Furthermore, | $\cdot_n$ $|^2 = \frac{®}{2}$ for n = 1, 2, … .

Proof. Using the second identity in (8) of section 1.3 one has ($\cdot_n(x)$, $\cdot_m(x)) = \int_0^{®} \sin(nx)\sin(mx)\,dx$ $= \int_0^{®} [\frac{1}{2}\cos(n+m)x - \frac{1}{2}\cos(n-m)x]\,dx = [\frac{1}{2(n+m)}\sin(n+m)x - \frac{1}{2(n-m)}\sin(n-m)x]\Big|_0^{®}$ = 0 so the $_n$ are orthogonal. | $\cdot_n$ $|^2 = \int_0^{®} \sin^2(nx)\,dx = \int_0^{®} \frac{1}{2}[1 - \cos 2nx]\,dx = \frac{1}{2}[x - \frac{1}{2n}\sin 2nx]\Big|_0^{®} = \frac{®}{2}$. //

Finally, we consider the functions $_n(x) = \cos nx$ and $\cdot_n(x) = \sin nx$. These are orthogonal on the interval - ® < x < ®.

Uses of Fourier series

Fourier series and frequencies

The basic idea of Fourier series is that we try to express the given function as a combination of oscillations, starting with one whose frequency is given by the given function (either its periodicity or the length of the bounded interval on which it is given) and then taking multiples of this frequency, that is, using fractional periods. When we look at coefficients of the resulting "infinite linear combination", we can expect that if some of them are markedly larger then the rest, then this frequency plays an important role in the phenomenon described by the given function. This detection of hidden periodicity can be very useful in analysis, since not every periodicity can be readily seen by looking at a function. In particular, this is true if there are several periods that interact.

Imagine that a function *f* describes temperatures at time *t* over many many years. There is one period that should be easily visible, namely seasonal changes with period one year. We also expect another period going over this basic yearly period, namely 1-day period of cold nights and warm days. Now the interesting question is whether there are also other periods. This is very useful to know, since such knowledge would tell us something important about what is happening with weather and climate. Frequency analysis offers a useful tool for such an investigation, looking over long data sequences it may point out cold ages and other long term changes in climate.

There are areas where decomposition into waves comes naturally, for instance storage of sound. When we are given a sound sample, Fourier transform allows us to decompose it into basic waves and store it in this way. Apart from data compression we also get further memory savings by simply ignoring coefficients that correspond to frequencies that a typical human ear does not hear. This is the basis of the mp3 format (it uses transform that is something like a fourth generation offspring of cosine Fourier series).

Fourier decomposition can be also generalized to more dimensions and then it can be quite powerful in storing visual information - it is for instance the heart of the system used by F.B.I. to store their fingerprint database. Since this decomposition is so useful, one important aspect is the speed at which we can find the coefficients. This inspired further development and today we do not usually use the standard Fourier series but its more powerful offspring, for instance something called Fast Fourier Transform (FFT). Here also hardware helps, there are devices (integrators) that have this wired in, roughly speaking one feeds it a function and the device spits out a Fourier coefficient.

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KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE-21
DEPARTMENT OF PHYSICS
CLASS : I B.SC PHYSICS
BATCH: 2018-2021
PART A : MULTIPLE CHOICE QUESTIONS (ONLINE EXAMINATIONS)
SUBJECT : MATHEMATICAL PHYSICS - II
SUBJECT CODE : 18PHU203
UNIT I

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
Which of the following is an even function?	x^3	cos x	sin x	tan X	cos x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	b)f(x) = - f(x)	f(-x) = - f(x)	None	f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	b)f(x) = - f(x)	f(-x) = - f(x)	None	f(-x) = f(x)
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient a_n	sine	coefficient a_0	cosine	sine
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0	coefficient a_n	cosine
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace	None	Dirichlet
In dirichlet condition, the function f(x) has only a finite number of finite dis continuities and no ------- discontinuities	semi finite	continuous	infinite	finite	infinite
If f(x) is even, then it's Fourier co- efficient --------is zero.	a_0	a_n	b_n	none	bn
If the periodic function f(x) is odd, then it's Fourier co- efficient -------is zero.	a_0	a_n	b_n	none	an
The period of cos nx where n is the positive integer is	π/n	π/2n	2π	nπ	2π
The Fourier co efficient a_0 in f(x) = x for 0< x£ π is	π	π/2	2π	0	π/2
If the function f (x) = -π in the interval -π x< 0, the coefficient a_0 is	$\pi^2/3$	$2\pi^2/3$	2π/3	(- π/2)	(- π/2)
If the function f(x) = x sin x, the Fourier coefficient	bn = 0	$a_0 = 1$	$a_0 = \pi^2/3$	$a_0 = -1$	bn = 0
For a function f(x) = x^3, the Fourier coefficient	$b_n = 0$	$a_n = 0$	$a_0 = 0$	$a_n = b_n = 0$	an = 0
The function x sin x be a ------- function.	even	odd	continuous	None	even
The function x cos x be a ------- function.	even	odd	continuous	None	odd
Which of the following is an odd function?	sin x	cos x	x^2	$\sin^2 x$	sin x
Which of the following is an even function	x^3	cos x	sin x	$\sin^2 x$	cos x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	3	f(-x) = - f(x)
The function f(x) is said to be an EVEN function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	1	f(-x) = f(x)
If a periodic function f(x) is odd, Fourier expansion contains no ------ terms	cosine	sine	coefficient a_0	coefficient a_n	sine
If a periodic function f(x) is even, Fourier expansion contains no ------ terms	cosine	sine	coefficient a_0	coefficient a_n	cosine
In Fourier series, the function f(x) has only a finite number of maxima and minima	Dirichlet	Kuhn Tucker	Laplace	None	Dirichlet
In dirichlet condition, the function f(x) has no --------discontinuities	semi finite	continuous	infinite	finite	infinite
If f(x) is even, then it's Fourier co- efficient --------is zero.	a_0	a_n	b_n	none	bn
If f(x) is odd, then it's Fourier co- efficient --------is zero.	a_0	a_n	b_n	none	an
The period of cos nx where n is the positive integer is	π/n	π/2n	2π	nπ	2π
The Fourier co efficient a_0 in f(x) = x for 0< x£ π is	π	π/2	2π	2	π/2
If the function f (x) = -π in the interval -π x< 0, the coefficient a_0 is	$\pi^2/3$	$2\pi^2/3$	2π/3	(-π/2)	(-π/2)
If the function f(x) = x sin x, the Fourier coefficient	bn = 0	$a_0 = 1$	$a_0 = \pi^2/3$	$a_0 = -1$	bn = 0
For a function f(x) = x^3, the Fourier coefficient	$b_n = 0$	$a_n = 0$	$a_0 = 0$	None	bn =0
Which kind of frequency spectrum/spectra is/are obtained from the line spectrum of a continuous signal on the basis of Polar Fourier Series Method	Continuous in nature	Discrete in nature	Sampled in nature	All of the above	Discrete in nature

Which type/s of Fourier Series allow/s to represent the negative frequencies by plotting the double-sided spectrum for the analysis of periodic signals ?	Trigonometric Fourier Series	Polar Fourier Series	Exponential Fourier Series	All of the above	Exponential Fourier Series
Duality Theorem / Property of Fourier Transform states that _________	Shape of signal in time domain & shape of spectrum can be interchangeable	Shape of signal in frequency domain & shape of spectrum can be interchangeable	Shape of signal in time domain & shape of spectrum can never be	Shape of signal in time domain & shape of spectrum can never be	Shape of signal in time domain & shape of spectrum can be interchangeable
Which property of fourier transform gives rise to an additional phase shift o -2π ft_d for the generated time delay in the communication system without affecting an amplitude spectrum ?	Time Scaling	Linearity	Time Shifting	Duality	Time Shifting
The exponential form of a complex number is	$z = re^{iq}$	$z = e^{iq}$	z = cos q / r	z = r / cos q	$z = re^{iq}$
Which is the analytic function of complex variable z = x + iy	|Z|	Re Z	Z^{-1}	Log Z	Z^{-1}
Which is the analytic function of complex variable Z=x +iy	| Z |	Sin Z	Log z	Re Z	Sin Z
Which is the analytic function of complex variable z=X+iY	|Z|	e^{sinz}	log Z	Re Z	e^{sinz}
Which is not the analytic function of complex variable z=X+iY	z^{-1}	Z	e^{sinz}	Sin Z	Z
Which is not the analytic function of complex variable z=X+iY	Z^{-1}	$e^{Sin Z}$	Re Z	SinZ	Re Z
Which is not the analytic function of complex variable z=X+iY	Z^{-1}	log Z	$e^{Sin Z}$	SinZ	log Z
Which of the following functions has the period 2p?	cos nx	sin nx	tan nx	tan x	sinnx
If f(x) = -x for -p< x£ 0then its Fourier coefficient a_0 is	$p^2/2$	p/4	p/3	p/2	p/2
The function f(x) is said to be an EVEN function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)	1	f(-x) = f(x)
If f(x) is even, then it's Fourier co- efficient --------is zero.	a_0	a_n	b_n	none	bn
The Fourier co efficient a_0 in f(x) = x for 0< x£ π is	π	π/2	2π	0	π/2
The period of cos nx where n is the positive integer is	π/n	π/2n	2π	nπ	2π
In dirichlet condition, the function f(x) has no --------discontinuities	semi finite	continuous	infinite	finite	infinite
Which is the analytic function of complex variable z=X+iY	|Z|	e^{sinz}	log Z	Re Z	e^{sinz}
Which of the following is an odd function?	sin x	cos x	x^2	sin^2x	sin x

UNIT-II

SYLLABUS

Bisection method - method of successive approximations - RegulaFalsi method - Newton-Raphson method - Horner's method - Euler's method - modified Euler's method - RungeKutta method (II & IV).

BISECTION METHOD:

Let us suppose we have an equation of the form f(x) = 0 in which solution lies between in the range (a,b).

Also f(x) is continuous and it can be algebraic or transcendental. If f(a) and f(b) are opposite signs, then there existatleast one real root between a and b.Let f(a) be positive and f(b) negative. Which implies atleast one root exits between a and b. We assume that root to be x_o= (a+b)/2. Check the sign of f(x_o). If f(x_0)is negative , the root lies between a and x_o. If f(x_0)is positive , the root lies between x_o and b. Subsequently any one of this case occur.

X_1= X_0+a /2 (or) x_0+b/2

When f(x_1) is negative , the root lies between xo and x1 and let the root be x_2=(x_0 +x_1) / 2 .

Again f(x_2) negative then the root lies between x_o and x_2, let x_3 = (x_0+x_2)/2 and so on.

Repeat the process x_o, x_1,x_2, …. Whose limit of convergence is the exact root.

Steps:

1. Find a and b in which f(a) and f(b) are opposite signs for the given equation using trial and error method.
2. Assume initial root as x_o= (a+b)/2.
3. If f(x_0)is negative , the root lies between a and x_o and take the root as x_1 = (x_o+a)/2.
4. If f(x_o) is positive , then the root lies between x_o and b and take the root as x_1= (x_o +b)/ 2.
5. If f(x_1) is negative , the root lies between x_o and $x_1$and let the root be x_2= (x_0 +x_1) / 2 .
6. If f(x_2) is negative , the root lies between x_o and $x_1$and let the root be x_3= (x_0 +x_2) / 2 .
7. Repeat the process until any two consecutive values are equal and hence the root.

Advantages of bisection method:

- The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
- As iterations are conducted, the interval gets halved. So one can guarantee the error in the solution of the equation.

Drawbacks of bisection method:

- The convergence of the bisection method is slow as it is simply based on halving the interval.

- If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

Example 1:

Find the positive root of $x^3 - x = 1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 - x - 1$

$f(0) = 0^3 - 0 - 1 = -1 = -ve$

$f(1) = 1^3 - 1 - 1 = -1 = -ve$

$f(2) = 2^3 - 2 - 1 = 5 = +ve$

So root lies between 1 and 2 , we can take (1+2) /2 as initial root and proceed.

i.e., f(1.5) = 0.8750 = +ve

and f(1) = -1 = -ve

So root lies between 1 and 1.5 ,

Let xo = (1+1.5) /2 as initial root and proceed.

f(1.25) = - 0.2969

So root lies between x1 between 1.25 and 1.5

Now x_1 = (1.25 + 1.5) /2 = 1.3750

f(1.375) = 0.2246 = +ve

So root lies between x_2 between 1.25 and 1.375

Now x_2 =(1.25 + 1.375)/2 = 1.3125

f(1.3125) = -0.051514 = -ve

Therefore, root lies between 1.375and 1.3125

Now x_3 = (1.375 + 1.3125) / 2 = 1.3438

f(1.3438) = 0.082832 = +ve

So root lies between 1.3125 and 1.3438

Now x_4 = (1.3125 + 1.3438) / 2 = 1.3282

f(1.3282) = 0.014898 = +ve

So root lies between 1.3125 and 1.3282

Now x_5 = (1.3125 + 1.3282) / 2 = 1.3204

f(1.3204) = - 0.018340 = -ve

So root lies between 1.3204 and 1.3282

Now x_6 = (1.3204 + 1.3282) / 2 = 1.3243

f(1.3243) = -ve

So root lies between 1.3243 and 1.3282

Now x_7 = (1.3243 + 1.3282) / 2 = 1.3263

f(1.3263) = +ve

So root lies between 1.3243 and 1.3263

Now x_8 = (1.3243 + 1.3263) / 2 = 1.3253

f(1.3253) = +ve

So root lies between 1.3243 and 1.3253
Now x_9 = (1.3243 + 1.3253) / 2 = 1.3248
f(1.3248) = +ve
So root lies between 1.3243 and 1.3248
Now x_{10} = (1.3243 + 1.3248) / 2 = 1.3246
f(1.3246) = -ve
So root lies between 1.3248 and 1.3246
Now x_{11} = (1.3248 + 1.3246) / 2 = 1.3247
f(1.3247) = -ve
So root lies between 1.3247 and 1.3248
Now x_{12} = (1.3247 + 1.3247) / 2 = 1.32475
Therefore , the approximate root is 1.32475

Example 2:
Find the positive root of x – cos x = 0 by bisection method.
Solution :
Let f(x) = x – cos x
f(0) = 0 -cos (0) = 0 -1 = -1 = -ve
f(0.5) = 0.5 –cos (0.5) = -0.37758 = -ve
f(1) = 1 –cos (1) = 0.42970 = +ve
So root lies between 0.5 and 1
Let xo = (0.5 +1) /2 as initial root and proceed.
f(0.75) = 0.75 – cos (0.75) = 0.018311 = +ve
So root lies between 0.5 and 0.75
x_1= (0.5 +0.75) /2 =0.625
f(0.625) = 0.625 – cos (0.625) = - 0.18596
So root lies between 0.625 and 0.750
x_2= (0.625 +0.750) /2 = 0.6875
f(0.6875) = - 0.085335
So root lies between 0.6875 and 0.750
x3= (0.6875 +0.750) /2 = 0.71875
f(0.71875) = 0.71875-cos(0.71875) = - 0.033879
So root lies between 0.71875 and 0.750
x4= (0.71875 +0.750) /2 = 0.73438
f(0.73438) = -0.0078664 = -ve
So root lies between 0.73438 and 0.750
x5= 0.742190
f(0.742190) = 0.0051999 = + ve
x6= (0.73438 +0.742190) /2 = 0.73829
f(0.73829) = -0.0013305
So root lies between 0.73829 and 0.74219
x7= (0.73829+0.74219) = 0.7402
f(0.7402) = 0.7402-cos(0.7402) = 0.0018663

So root lies between 0.73829 and 0.7402
x8= 0.73925
f(0.73925) = 0.00027593
x9= 0.7388
The root is 0.7388.

Example 3:
You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.
The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993\times10^{-4} = 0$$

Use the bisection method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of each iteration.
Solution:
From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$, where
$R =$ radius of the ball,
that is
$0 \le x \le 2R$
$0 \le x \le 2(0.055)$
$0 \le x \le 0.11$

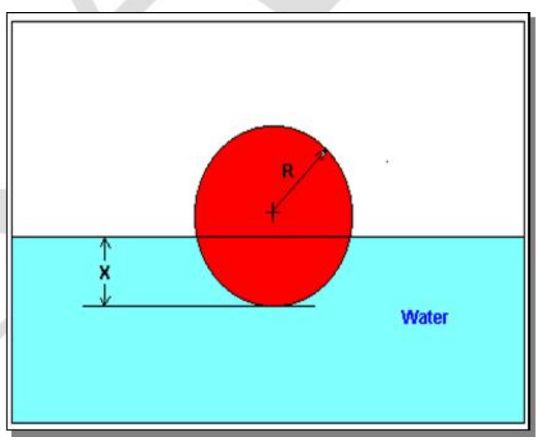


Figure : Floating ball problem

Lets us assume
$x_\ell = 0, x_u = 0.11$

Check if the function changes sign between x_ℓ and x_u.

$f(x_\ell) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$

$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$

Hence

$$f(x_\ell)f(x_u) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

So there is at least one root between x_ℓ and x_u, that is between 0 and 0.11.

Iteration 1

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0+0.11}{2}$$
$$= 0.055$$

$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$

$f(x_\ell)f(x_m) = f(0)f(0.055) = (3.993\times10^{-4})(6.655\times10^{-4}) > 0$

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limit of the new bracket is

$x_\ell = 0.055,\ x_u = 0.11$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0.055+0.11}{2}$$
$$= 0.0825$$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times10^{-4} = -1.622\times10^{-4}$$
$$f(x_\ell)f(x_m) = f(0.055)f(0.0825) = (6.655\times10^{-5})\times(-1.622\times10^{-4}) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825. So the lower and upper limit of the new bracket is

$x_\ell = 0.055,\ x_u = 0.0825$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$

$$= \left| \frac{0.0825 - 0.055}{0.0825} \right| \times 100$$
$$= 33.33\%$$

None of the significant digits are at least correct in the estimated root of $x_m = 0.0825$ because the absolute relative approximate error is greater than 5%.

Iteration 3

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0.055 + 0.0825}{2}$$
$$= 0.06875$$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$
$$f(x_\ell) f(x_m) = f(0.055) f(0.06875) = (6.655 \times 10^{5}) \times (-5.563 \times 10^{-5}) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.06875. So the lower and upper limit of the new bracket is

$x_\ell = 0.055,\ x_u = 0.06875$

The absolute relative approximate error $|\in_a|$ at the ends of Iteration 3 is

$$|\in_a| = \left| \frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}} \right| \times 100$$
$$= \left| \frac{0.06875 - 0.0825}{0.06875} \right| \times 100$$
$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

Seven more iterations were conducted and these iterations are shown in Table 1.

Table 1 Root of $f(x) = 0$ as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert\in_a\rvert$ %	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

At the end of 10th iteration,

$|\in_a| = 0.1721\%$

Hence the number of significant digits at least correct is given by the largest value of m for which

$|\in_a| \le 0.5\times10^{2-m}$

$0.1721 \le 0.5\times10^{2-m}$

$0.3442 \le 10^{2-m}$

$\log(0.3442) \le 2-m$

$m \le 2-\log(0.3442) = 2.463$

So

$m = 2$

The number of significant digits at least correct in the estimated root of 0.06241 at the end of the 10^{th} iteration is 2.

REGULAFALSI METHOD OR METHOD OF FALSE POSITION:

Consider the equation *f(x) = 0* and *f(a)* and *f(b) are* of opposite signs. Also let a< b. The graph *y = f(x) will* Meet the x-axis at some point between A(a, f(a)) and B (b,f(b)). The equation of the chord joining the two points A(a, f(a)) and

B (b,f(b)) is

$$\frac{y-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

The x- Coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the of f(x) = 0. Taking y = 0 in the chord equation, we get

$$\frac{-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

$x[f(a) - f(b)] - a f(a) + a f(b) = -a f(a) + b f(b)$

$x[f(a) - f(b)] = b f(a) - a f(b)$

This x_1 gives an approximate value of the root f(x) = 0. ($a < x_1 < b$)

Now $f(x_1)$ and f(a)are of opposite signs or $f(x_1)$ and f(b) are opposite signs.

If $f(x_1)$, f(a) <0 . then x2 lies between x_1 and a.

Therefore $x_2 = a f(x_1) - x_1 f(b) / f(x_1) - f(a)$

This process of calculation of ($x_3, x_4, x_5, \ldots$) is continued till any two successive values are equal and subsequently we get the solution of the given equation.

Steps:

1. Find *a* and *b* in which *f(a)* and *f(b)* are opposite signs for the given equation using trial and error method.
2. Therefore root lies between *a* and *b* if *f(a)* is very close to zero select and compute x_1 by using the following formula:

$$x_1 = a\,f(b) - b\,f(a) / f(b) - f(a)$$

3. If $f(x_1), f(a) < 0$. then root lies between x_1 and *a*. Compute x_2 by using the following formula:

$$x_2 = a\,f(x_1) - x_1\,f(b) / f(x_1) - f(a)$$

4. Calculate the values of ($x_3, x_4, x_5, \ldots$) by using the above formula until any two successive values are equal and subsequently we get the solution of the given equation.

Example:

Solve for a positive root of $x^3-4x+1=0$ by and RegulaFalsi method

Solution :

Let $f(x) = x^3-4x+1 = 0$

$f(0) = 0^3-4(0)+1 = 1 = +ve$

$f(1) = 1^3-4(1)+1 = -2 = -ve$

So a root lies between 0 and 1

We shall find the root that lies between 0 and 1.

Here a=0, b=1

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{(0 \times f(1) - 1 \times f(0))}{(f(1) - f(0))}$$

$$= \frac{-1}{(-2 - 1)}$$

$$= 0.333333$$

$f(x_1) = f(1/3) = (1/27)-(4/3) +1 = -0.2963$

Now f(0) and f(1/3) are opposite in sign.

Hence the root lies between 0 and 1/3.

$$x_2 = \frac{(0 \times f(1/3) - 1/3 \times f(0))}{(f(1/3) - f(0))}$$

$x_2 = (-1/3)/(-1.2963) = 0.25714$

Now $f(x_2) = f(0.25714) = -0.011558 = -ve$

So the root lies between 0 and 0.25714

$x_3 = (0 \times f(0.25714) - 0.25714 \times f(0))/ (f(0.25714) - f(0))$

= -0.25714/-1.011558 = 0.25420

$f(x_3)$ = f(0.25420) = -0.0003742

So the root lies between 0 and 0.25420

x_4= (0xf(0.25420)- 0.25420 xf(0))/ (f(0.25420) – f(0))

= -0.25420 / -1.0003742 = 0.25410

$f(x_4)$ = f(0.25410) = - 0.000012936

The root lies between 0 and 0.25410

x_5= (0xf(0.25410) - 0.25410 xf(0))/ (f(0.25410) – f(0))

= -0.25410 / -1.000012936 = 0.25410

Hence the root is 0.25410.

NEWTON-RAPSON METHOD:

Let us suppose we have an equation of the form f(x) = 0 in which solution is lies between in the range (a,b). Also f(x) is continuous and it can be algebraic or transcendental. If f(a) and f(b) are opposite signs, then there exist
atleast one real root between a and b.

Let f(a) be positive and f(b) negative. Which implies at least one root exits between a and b. We assume that root to be either a or b, in which the value of f(a) or f(b) is very close to zero. That number is assumed to be initial root. Then we iterate the process by using the following formula until the value is converges.

$$X_{n+1} = Xn- \frac{f(X_n)}{f'(Xn)}$$

Steps:

1. Find a and b in which f(a) and f(b) are opposite signs for the given equation using trial and error method.
2. Assume initial root as Xo= ai.e., if f(a)is very close to zero or Xo = b if f(a)is very close to zero
3. Find X1 by using the formula

$$X1 = Xo - \frac{f(Xo)}{f'(Xo)}$$

4. Find X2 by using the following formula

$$X2 = X1 - \frac{f(X1)}{f'(X1)}$$

5. Find X3,X4, …Xn until any two successive values are equal.

Example 1 :

Find the positive root of f(x) = 2x3- 3x-6 =0 by Newton – Raphson method correct to five decimal places.

Solution:

Let $f(x) = 2x^3-3x – 6$; $f'(x) = 6x^2 – 3$

f(1) = 2-3-6 = -7 = -ve
f(2) = 16 – 6-6 = 4 = +ve

So, a root between 1 and 2 . In which 4 is closer to 0 Hence we assume initial root as 2.
Consider $x_0 = 2$

So $X_1 = X_0 - f(X_0)/f'(X_0)$

$= X_0 - ((2X_0 3 - 3X_0 - 6) / 6\alpha_0 - 3) = (4X_0 3 + 6)/(6X_0 2 - 3)$

$X_{i+1} = (4X_i 3 + 6)/(6X_i 2 - 3)$

$X_1 = (4(2)^2+6)/(6(2)^2 - 3) = 38/21 = 1.809524$

$X_2 = (4(1.809524)^3+6)/(6(1.809524)^2 - 3) = 29.700256/16.646263 = 1.784200$

$X_3 = (4(1.784200)^3+6)/(6(1.784200)^2 - 3) = 28.719072/16.100218 = 1.783769$

$X_4 = (4(1.783769)^3+6)/(6(1.783769)^2 - 3) = 28.702612/16.090991 = 1.783769$

Example 2:

Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$ correct to 5 decimal places.

Solution :

Let $f(x) = x^3 - 6x + 4$; $f(0) = 4 = +ve$; $f(1) = -1 = -ve$

So a root lies between 0 and 1

f(1) is nearer to 0. Therefore we take initial root as $X_0 = 1$

$$f'(x) = 3x^2 - 6$$

$$= x - \frac{f(x)}{f'(x)}$$

$$= x - (3x^3 - 6x + 4)/(3x^2 - 6)$$

$$= (2x^3 - 4)/(3x^2 - 6)$$

$$X_1 = (2X_0 3 - 4)/(3X_0 2 - 6) = (2-4)/(3-6) = 2/3 = 0.$$

$X_2 = (2(2/3)^3 - 4)/(3(2/3)^2 - 6) = 0.73016$

$X_3 = (2(0.73015873)^3 - 4)/(3(0.73015873)^2 - 6)$
$= (3.22145837/ 4.40060469)$
$= 0.73205$

$X_4 = (2(0.73204903)^3 - 4)/(3(0.73204903)^2 - 6)$
$= (3.21539602/ 4.439231265)$
$= 0.73205$

The root is 0.73205 correct to 5 decimal places.

HORNER'S METHOD:

This numerical methods is employed to determine both the commensurable and the incommensurable real roots of a numerical polynomial equation. Firstly, we find the integral part of the root and then by every iteration. We find each decimal place value in succession.
Suppose a positive root of f(x) = 0 lies between a and a+1.

Let that root be a,a1a2a3….

First diminish the root of f(x)-0 by the integral part a and let ϕ1(x) = 0 possess the root 0.a1a2a3…
Secondly , multiply the roots of ϕ1(x) = 0 by 10 and let ϕ2(x) = 0 possess the root a1.a2a3…as a root.

Thirdly, find the value od a1 and then diminish the roots by a1 and let ϕ3(x) = 0 possess a root 0.a2a3…
Now repeating the process we find a2,a3,a4…. each time.

Example:
Find the positive root of $x^3 + 3x - 1 = 0$, correct to two decimal places by Horner's method.
Solution:
Let $f(x) = x^3 + 3x - 1 = 0$
f(0) = -ve f(1) = * ve
The positive root lies between 0 and 1.
Let it be 0.a1a2a3….

Since the integral part is zero, diminishing the root by the integral part is not necessary.
Therefore multiply the roots by 10.
Therefore $\phi1(x) = x^3 + 300x - 1000 = 0$ has root a1.a2a3…
ϕ1(3) = -ve, ϕ1(4) = +ve
Therefore a1=3

Now, the root is 3.a2a3…
Therefore, diminish root of ϕ1(x) = 0 by 3
By synthetic division method, we get
$\phi2(x) = x^3 + 9x^2 + 327x - 73 = 0$ has root 0.a2a3…
Multiply the roots of ϕ2(x) = 0 by 10.
$\phi3(x) = x^3 + 90x^2 + 32700x - 73000 = 0$ has root a2.a3a4…
Now, ϕ3(2) = -ve, ϕ3(3) = +ve

Therefore a2=2
Now diminish the roots of ϕ3(x) by 2.
By synthetic division method, we get
$\phi4(x) = x^3 + 96x^2 + 33072x - 7232 = 0$ has root 0.a3a4…
Multiply the roots of ϕ4(x) = 0 by 10.
$\phi5(x) = x^3 + 960x^2 + 3307200x - 7232000 = 0$ has root a3.a4…
Now, ϕ5(2) = -ve, ϕ5(3) = +ve

Therefore a3=2

Hence the root is 0.322.

EULER'S METHOD:

Take the Taylor series to 1st order, and let the interval $h = x_1 - x_0$, then

$$y_1 = y(x_0) + f(x_0, y_0)h + \frac{y''(\P)}{2}h^2 .$$

The error for a time step (the local error) is $O(h^2)$. The global error, after many steps, is $O(h)$. Then

$$y_1 = y_0 + f(x_0, y_0)h \text{ where } y_0 = y(x_0),$$
$$y_2 = y_1 + f(x_1, y_1)h \text{ where } x_1 = x_0 + h,$$
$$\ldots$$
$$y_{N+1} = y_N + f(x_N, y_N)h \text{ where } x_{N+1} = x_N + h.$$

Example:

$$\frac{dy}{dx} = x + y, \; y(0) = 1$$

The exact solution can be found from

$$\frac{dy}{dx} - y = x .$$

Let $y = y_c + y_p$ where $\frac{dy_c}{dx} - y_c = 0$, or $y_c = Ce^{rx}$. Then $rCe^{rx} - Ce^{rx} = 0$ for all x,

or $r = 1$, and $y_c = Ce^x$. Since the right hand side is linear in x try $y_p = Ax + B$. Then

$\frac{dy_p}{dx} = A$ and $\frac{dy_p}{dx} - y_p = x$ becomes $A - Ax - B = x$ which must hold for all x. Hence

$A = -1$, and B=-1, making $y_p = -(x+1)$, and since $y = y_c + y_p$ then

$$y = Ce^x - (x+1).$$

But @ $x = 0$, $y = 1$ or $C - 1 = 1$, and $C = 2$. Making the complete solution

$$y = 2e^{x} - (x+1).$$

Using Euler's method and taking $h = 0.02$

$y_0 = 1, x_0 = 0 \Rightarrow x_1 = 0.02, y_1 = 1 + 1 \cdot 0.02 = 1.02$, since $y_0' = 1$. In general,

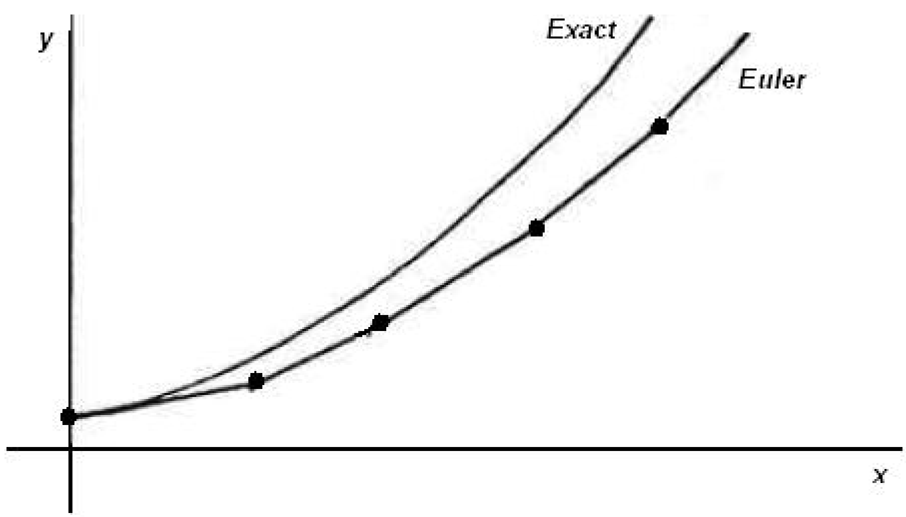
$$y_{n+1} = y_n + y_n' h; x_{n+1} = x_n + h$$

n	x_n	y_n	y_{exact}
0	0.00	1.0000	1.0000
1	0.02	1.0200	1.0204
2	0.04	1.0408	1.0416
3	0.06	1.0624	1.0637
4	0.08	1.0848	1.0866
5	0.10	1.1081	1.1103

For the error, $\Delta y_{Euler} = y_5 - y_0 = 0.1081$, $\Delta y_{Exact} = y_5 - y_0 = 0.1103$, can be defined as

$$Relative\ Error = \frac{\Delta y_{Euler} - \Delta y_{Exact}}{\frac{1}{2}(\Delta y_{Euler} + \Delta y_{Exact})} = \frac{0.0022}{0.1092} \approx 2\%$$

The results plot as



It would be better to use the slope at the beginning and end of the increment (e.g., the average at each end), and although we don't know the slope at the end we can approximate it.

MODIFIED EULER'S METHOD:

Let $y_n' = f(x_n, y_n)$. Then an approximation for y at the end of the increment is

$$\tilde{y}_{n+1} = y_n + y_n' h$$

and an estimate for the slope at the end of the increment is $\tilde{y}'_{n+1} = f(x_{n+1}, \tilde{y}_{n+1})$.

We can now set

$$y_{n+1} = y_n + \frac{1}{2}\left(y_n' + \tilde{y}'_{n+1}\right)h .$$

The error can be found from

$$y_{n+1} = y_n + y_n' h + \frac{1}{2} y_n'' h^2 + O\left(h^3\right)$$

and since

$$y_{n+1} = y_n + y_n' h + \frac{1}{2}\left[\frac{y'_{n+1} - y'_n}{h} + O(h)\right]h^2 + O\left(h^3\right)$$

or

$$y_{n+1} = y_n + \left(\frac{y'_{n+1} + y'_n}{2}\right)h + O\left(h^3\right).$$

Hence the local error is $O\left(h^3\right)$ and the global error is $O\left(h^2\right)$. Another way to write our results is

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h, y_n + k_1)$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

The previous example now can include modified Euler

n	x_n	y_{euler}	$y_{modified}$	y_{exact}
0	0.00	1.0000	1.0000	1.0000
1	0.02	1.0200	1.0204	1.0204
2	0.04	1.0408	1.0416	1.0416
3	0.06	1.0624	1.0637	1.0637
4	0.08	1.0848	1.0866	1.0866
5	0.10	1.1081	1.1104	1.1103

which is much better.

RUNGE KUTTA II ORDER:

Working Rule :

To solve $dy/dx = f(x,y)$, $y(x_0)=y_0$

Calculate $k_1=hf(x_0,y_0)$

$K_2=hf(x_0+1/2h, y_0+1/2k_1)$

$K_3= hf(x_0+1/2h, y_0+1/2k_2)$

$K_4=hf(x_0+h, y_0+k_3)$

and $\Delta y= 1/6 (k_1+2k_2+2k_3+k_4)$

where $\Delta x=h$

Now $y_1=y_0+ \Delta y$

Now starting from (x_1,y_1) and repeating the process, we get (x_2,y_2) etc.,

Example

Obtain the values of y at x=0.1, 0.2 using R.K method of second order for the differential equation y'=-y, given y(0)=1.

Solution :Here $f(x,y)=-y, x_0=0, y_0=1, x_1=0.1, x_2=0.2$

$k_1=hf(x_0,y_0)=(0.1)(-y_0)= - 0.1$

$k_2=hf(x_0+ ½ h, y_0+ ½ k_1) = (0.1) f(0.05,0.95)$

$= -0.1(x0.95)= - 0.095= \Delta y$

$y_1=y_0+\Delta y=1-0.095=0.905$

$y_1=y(0.1)=0.905$

Again starting from (0.1, 0.905) replacing (x_0,y_0) by (x_1,y_1)we get

$k_1=(0.1) f(x_1,y_1)=(0.1) (-0.905)= - 0.0905$

$k_2=hf(x_1+ ½ h, y_1+ ½ k_1)$

$=(0.1)[f(0.15,0.85975)]=(0.1)(-0.85975)=-0.085975$

$\Delta y=k_2$ $y_2=y(0.2)=y_1+\Delta y=0.819025$

RUNGE KUTTA IV ORDER:

What is the Runge-Kutta 4th order method?

Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1:
Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2:
Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y), \ y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y}, \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

The Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h \quad (1)$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y = y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

Equation (1) is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i, y_i}(x_{i+1} - x_i)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i, y_i}(x_{i+1} - x_i)^4 \quad (2)$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 + \frac{1}{3!} f''(x_i, y_i)h^3 + \frac{1}{4!} f'''(x_i, y_i)h^4 \quad (3)$$

Based on equating Equation (2) and Equation (3), one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \quad (4)$$

$$k_1 = f(x_i, y_i) \quad (5a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \quad (5b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \quad (5c)$$

$$k_4 = f(x_i + h, y_i + k_3 h) \quad (5d)$$

Example 3:

A ball at 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right), \theta(0) = 1200\,\mathrm{K}$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method. Assume a step size of $h = 240$ seconds.

Solution:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f(t, \theta) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200\mathrm{K}$

$$k_1 = f(t_0, \theta_0)$$
$$= f(0, 1200)$$
$$= -2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right)$$
$$= -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579) \times 240\right)$$

$$= f(120, 653.05)$$
$$= -2.2067 \times 10^{-12}(653.05^4 - 81 \times 10^8)$$
$$= -0.38347$$
$$k_3 = f\left(t_0 + \frac{1}{2}h, {}^{-}{}_0 + \frac{1}{2}k_2 h\right)$$
$$= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347) \times 240\right)$$
$$= f(120, 1154.0)$$
$$= -2.2067 \times 10^{-12}(1154.0^4 - 81 \times 10^8)$$
$$= -3.8954$$
$$k_4 = f(t_0 + h, {}^{-}{}_0 + k_3 h)$$
$$= f(0 + 240, 1200 + (-3.894) \times 240)$$
$$= f(240, 265.10)$$
$$= -2.2067 \times 10^{-12}(265.10^4 - 81 \times 10^8)$$
$$= 0.0069750$$
$${}^{-}{}_1 = {}^{-}{}_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= 1200 + \frac{1}{6}(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750))240$$
$$= 1200 + (-2.1848) \times 240$$
$$= 675.65\,\text{K}$$

${}^{-}{}_1$ is the approximate temperature at

$$t = t_1$$
$$= t_0 + h$$
$$= 0 + 240$$
$$= 240$$
$${}^{-}{}_1 = {}^{-}(240)$$
$$\approx 675.65\,\text{K}$$

For $i = 1, t_1 = 240, {}^{-}{}_1 = 675.65\,\text{K}$

$$k_1 = f(t_1, {}^{-}{}_1)$$
$$= f(240, 675.65)$$
$$= -2.2067 \times 10^{-12}(675.65^4 - 81 \times 10^8)$$
$$= -0.44199$$
$$k_2 = f\left(t_1 + \frac{1}{2}h, {}^{-}{}_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.44199)240\right)$$
$$= f(360,622.61)$$
$$=-2.2067\times10^{-12}\left(622.61^{4}-81\times10^{8}\right)$$
$$=-0.31372$$

$$k_3 = f\left(t_1+\frac{1}{2}h,{}^{-}{}_{1}+\frac{1}{2}k_2h\right)$$
$$= f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.31372)\times240\right)$$
$$= f(360,638.00)$$
$$=-2.2067\times10^{-12}\left(638.00^{4}-81\times10^{8}\right)$$
$$=-0.34775$$

$$k_4 = f\left(t_1+h,{}^{-}{}_{1}+k_3h\right)$$
$$= f\left(240+240,675.65+(-0.34775)\times240\right)$$
$$= f(480,592.19)$$
$$= 2.2067\times10^{-12}\left(592.19^{4}-81\times10^{8}\right)$$
$$=-0.25351$$

$${}^{-}{}_{2} = {}^{-}{}_{1}+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$$
$$=675.65+\frac{1}{6}\left(-0.44199+2(-0.31372)+2(-0.34775)+(-0.25351)\right)\times240$$
$$=675.65+\frac{1}{6}(-2.0184)\times240$$
$$=594.91\text{K}$$

${}^{-}{}_{2}$ is the approximate temperature at

$$t=t_2$$
$$=t_1+h$$
$$=240+240$$
$$=480$$
$${}^{-}{}_{2}={}^{-}(480)$$
$$\approx 594.91\text{K}$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method with different step sizes.

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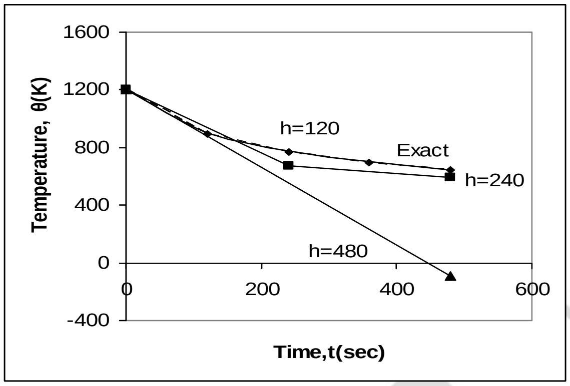

Figure 1: Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.
Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 480$ seconds.

TABLE 1 Value of temperature at time, $t = 480$ s for different step sizes

Step size, h	¯(480)	E_t	$\|£_t\|\%$
480	-90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

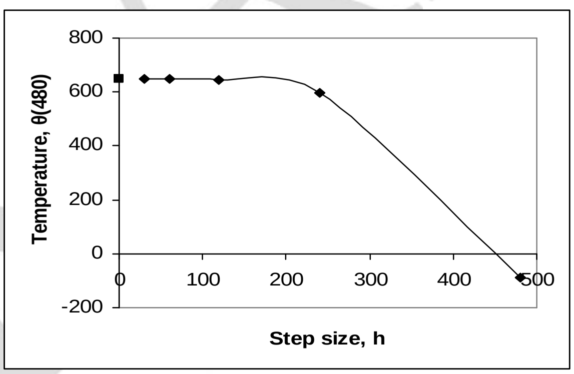

Figure 2: Effect of step size in Runge-Kutta 4th order method.
In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method), and Runge-Kutta 4th order method. The formula described in this chapter was developed by Runge. This formula is same as Simpson's 1/3 rule, if $f(x, y)$ were only a function of x. There are other versions of the 4th order method just like there are several versions of the second order methods. The formula developed by Kutta is

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h \qquad (6)$$

where

$$k_1 = f(x_i, y_i) \qquad (7a)$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right) \qquad (7b)$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right) \qquad (7c)$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3) \qquad (7d)$$

This formula is the same as the Simpson's 3/8 rule, if $f(x, y)$ is only a function of x.

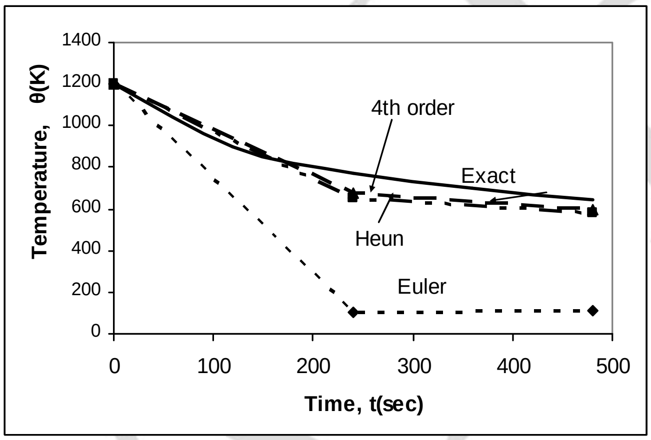

Figure 3: Comparison of Runge-Kutta methods of 1st (Euler), 2nd, and 4th order.

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KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE-21
DEPARTMENT OF PHYSICS
CLASS : I B.SC PHYSICS
BATCH: 2018-2021
PART A : MULTIPLE CHOICE QUESTIONS (ONLINE EXAMINATIONS)
SUBJECT : MATHEMATICAL PHYSICS - II
SUBJECT CODE : 18PHU203
UNIT II

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
The ------------ Method is based on the repeated application of the intermediate value theorem.	Gauss Seidal	Secant	Bisection	Chebyshev	Bisection
The formula for Newton Raphson method is ------------------.	x_{n+1} = f (x_n)/f'(xn)	xn+1 = xn + f (xn)/f '(xn)	$x_{n+1} = x_n$ – f (x_n)/ f '(xn)	$x_{n+1} = x_n$ – f '(x_n)/f (xn)	$x_{n+1} = x_n$ – f (x_n)/ f '(xn)
The order of convergence of Newton Raphson method is ----------.	4	2	1	A	2
Graffees root squaring method is useful to find -----------------.	Complex roots	single root	unequal roots	polynomial roots	polynomial roots
The approximate value of the root of f(x) given by the bisection method is ---.	x_0 = a + b	x0 = f(a) + f(b)	x0 = (a + b)/ 2	x0 = (f(a) + f(b))/2	x0 = (a + b)/ 2
In Newton Raphson method, the error at any stage is proportional to the ----- of the error in the previous stage.	Cubic	square	square root	same as that	square
In case of bisection method, the convergence is -------------.	Linear	quadratic	very rapid	h2	very rapid
The order of convergence of Regula falsi method may be assumed to ---------.	1	1.618	0	0.5	1.618
The formula for Regula falsi method is ----------------------.	$x_n + x_{n+1}$ = 1	x = af(b) – bf(a) /f(b) – f(a)	x = af(a) – bf(b) / f(a) – f(b)	$x_n - x_{n+1}$ = 1	x = af(b) – bf(a) /f(b) – f(a)
The ------------- Method is also called as Method of tangents.	Gauss Seidal	Secant	Bisection	Newton Rapson	Newton Rapson
If f (x) contains some functions like exponential, trigonometric, logarithmic etc., then f (x) is called -------------- equation.	Algebraic	transcendental	numerical	polynomial	transcendental
A polynomial in x of degree n is called an algebraic equation of degree n if ---	f (x) = 0	f (x) = 1	f (x) < 1	f (x) > 1	f (x) = 0
The method of false position is also known as ------------- method.	Gauss Seidal	Secant	Bisection	Regula falsi	Regula falsi
The Newton Rapson method fails if ---------------.	f '(x) = 0	f (x) = 0	f (x) = 1	f '(x) = 1	f '(x) = 0
The bisection method is simple but ------------------.	Slowly convergent	fast convergent	slowly divergent	fastly divergent	Slowly convergent
________ method is also called as Bolzano's method	Bisection	False position	Newton Rapson	Euler	Bisection
If the initial approximation to the root is not given, choose two values of x say 'a' and 'b', such that f(a) and f(b) are of opposite signs. If $\lvert f(a)\rvert < \lvert f(b)\rvert$ then take ---- as the initial approximation.	'a'	'b'	0	1	'a'
Graeffe's root squaring method has a great advantage over other methods in that it does not require prior information about the ---------------------.	Initial value	approximate values	final value	mid value	Initial value
If we choose the initial approximation x0 ------------ to the root then we get the root of the equation very quickly.	Close	far	average	very far	Close
In Newton Rapson method when f '(x) is very large and the interval h will b--- then the root can be calculated in even less time.	Small	large	average of the roots	negative	Small
The order of convergence in --------------- method is two.	Bisection	Regula falsi	False position	Newton Rapson	Newton Rapson
The approximate value x0 = (a + b)/ 2 of the root of f (x) is given by the --------- method.	Bisection	Regula falsi	Newton Rapson	Graffes root squaring	Bisection
If f (x1) and f (a) are of opposite signs, then the actual roots of the equation f(x)=0 in False position method lie between ------------------.	'a' and 'b'	'b' and 'x1'	'a' and 'x1'	'x1' and 'x2'	'a' and 'x1'
The iterative procedure is repeated till the --------- is found to the desired degree of accuracy.	Initial value	approximate value	root	0	root
The ----------- Method is the method to find the root of algebraic or transcendental equation.	Graffe's method	Regula falsi	Root squaring	Bisection	Regula falsi
If we equate a function f(x) to zero, then f(x) = 0 will represent an ----- equation	polynomial	transcedental	algebraic	cubic	algebraic
The equation 3x – cosx – 1 = 0 is known as ------------- equation.	polynomial	transcedental	algebraic	cubic	transcedental
If f(a) and f(b) have opposite signs then the root of f(x) = 0 lies between -----.	0 and a	a and b	b and 0	1 and –1	a and b
The error at any stage is proportional to the square of the ------------------.	error in the previous stage	error in the next stage	error in the last stage	error in the first stage	error in the previous stage
The convergence of iteration method is ----------------.	zero	polynomial	quadratic	linear	linear
The method of successive Approximation is also called as -----------.	Bisection method	Iteration method	Regula falsi method	Root squaring	Iteration method
The sufficient condition for convergence of iterations is ----------------.	$\lvert\Phi'(x)\rvert = 1$	$\lvert\Phi'(x)\rvert > 1$	$\lvert\Phi'(x)\rvert < 1$	$\lvert\Phi'(x)\rvert < 0$	$\lvert\Phi'(x)\rvert < 1$
Solution of an equation f(x) = 0 means we have to find its -----------.	roots or zeros	initial values	final values	approximate values	roots or zeros
Assuming that a root of $x^3 - 9x + 1 = 0$ lies between 2 and 4. Find the initial approximation root value of bisection method.	2	3	4	3.5	3
In Newton Rapson method if -----------, then 'a' is taken as the initial approximation to the root.	$\lvert f(a)\rvert \neq \lvert f(b)\rvert$	$\lvert f(a)\rvert = \lvert f(b)\rvert$	$\lvert f(a)\rvert > \lvert f(b)\rvert$	$\lvert f(a)\rvert < \lvert f(b)\rvert$	$\lvert f(a)\rvert < \lvert f(b)\rvert$
In iteration method the given equation is taken in the form of -------------.	y = Φ(x)	x = Φ(x)	x = Φ(y)	Φ	x = Φ(x)
The convergence of the sequence is not guaranteed always unless the choice of ---------------- is properly chosen.	x0	y0	x2	y2	x0
The sequence will converge rapidly in Iteration method, if $\lvert\Phi'(x)\rvert$ is -----------.	zero	very large	very small	one	very small
In Iteration method if the convergence is linear then the convergence is of order ------------------.	four	three	two	one	one
By Regula Falsi method, the positive root of first approximation of x3– 4x +1=0 lies between ---------------.	0 and 1	1 and 2	–1 and 2	0 and –1	1 and 2

The values of x which makes f(x) as ----------- are known as roots or zeros of the function f(x).	Zero	one	f '(x)	f ''(x)	Zero
In Iteration method if the convergence is ---------- then the convergence is of order one.	Cubic	quadratic	linear	zero	linear
The order of convergence of --------------- method may be assumed to 1.618.	Bisection	Regula falsi	Newton Raphson	Graeffe	Regula falsi
In Newton Raphson method the choice of --------------- is very impartant for the convergence.	initial value	final value	intermediate value	approximate value	initial value
If f(a) and f(b) are of opposite signs, a root of f(x) = 0 lies between -----------.	'0' and 'b'	'a' and '0'	'a' and 'b'	'0' and '1'	'a' and 'b'
If f(a) and f(b) are of opposite signs, a root of f(x) = 0 lies between 'a' and 'b'. This idea can be used to fix an ---------------------.	Approximate root	actual root	intermediate root	none	none
If f (–1) and f (–2) are of opposite signs, then the negative roots of the equation f(x)=0 in False position method lie between ------------------.	–1 and –2	–1 and 1	1 and –2	1 and 2	–1 and –2
The -------------------- method fails if f '(x) = 0.	Bisection	False Position	Newton Raphson	Gauss seidal method	Newton Raphson
In which of the following method, we approximate the curve of solution by the tangent in each interval.	Picard's method	Euler's method	Newton's method	Runge Kutta method	Euler's method
The convergence of which of the following method is sensitive to starting value?	False position	Gauss seidal method	Newton-Raphson method	regula falsi	Newton-Raphson method

UNIT-III

SYLLABUS

Gauss elimination method - Gauss-Jordan method - Gauss-Seidel method - computation of inverse of a matrix using Gauss elimination method - method of triangularisation. Trapezoidal rule - Simpson's 1/3 rule and 3/8 rule

GAUSS ELIMINATION:

Gaussian elimination is one popular method of solving linear equations. We illustrate this technique by means of an example.

How is a set of equations solved numerically?

One of the most popular techniques for solving simultaneous linear equations is the Gaussian elimination method. The approach is designed to solve a general set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nn}x_n = b_n$$

Gaussian elimination consists of two steps

Forward Elimination of Unknowns: In this step, the unknown is eliminated in each equation starting with the first equation. This way, the equations are reduced to one equation and one unknown in each equation.

Back Substitution: In this step, starting from the last equation, each of the unknowns is found.

Forward Elimination of Unknowns:

In the first step of forward elimination, the first unknown, x_1 is eliminated from all rows below the first row. The first equation is selected as the pivot equation to eliminate x_1. So, to eliminate x_1 in the second equation, one divides the first equation by a_{11} (hence called the pivot element) and then multiplies it by a_{21}. This is the same as multiplying the first equation by a_{21}/a_{11} to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + ... + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Now, this equation can be subtracted from the second equation to give

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + ... + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$a'_{22}x_2 + ... + a'_{2n}x_n = b'_2$

where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$
$$\vdots$$
$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

This procedure of eliminating x_1, is now repeated for the third equation to the n^{th} equation to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$
$$a'_{32}x_2 + a'_{33}x_3 + ... + a'_{3n}x_n = b'_3$$
$$\vdots$$
$$a'_{n2}x_2 + a'_{n3}x_3 + ... + a'_{nn}x_n = b'_n$$

This is the end of the first step of forward elimination. Now for the second step of forward elimination, we start with the second equation as the pivot equation and a'_{22} as the pivot element. So, to eliminate x_2 in the third equation, one divides the second equation by a'_{22} (the pivot element) and then multiply it by a'_{32}. This is the same as multiplying the second equation by a'_{32}/a'_{22} and subtracting it from the third equation. This makes the coefficient of x_2 zero in the third equation. The same procedure is now repeated for the fourth equation till the n^{th} equation to give

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$
$$a''_{33}x_3 + ... + a''_{3n}x_n = b''_3$$
$$\vdots$$
$$a''_{n3}x_3 + ... + a''_{nn}x_n = b''_n$$

The next steps of forward elimination are conducted by using the third equation as a pivot equation and so on. That is, there will be a total of $n-1$ steps of forward elimination. At the end of $n-1$ steps of forward elimination, we get a set of equations that look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \dots + a''_{3n}x_n = b''_3$$
$$\vdots \qquad \vdots$$
$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution:
Now the equations are solved starting from the last equation as it has only one unknown.

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then the second last equation, that is the $(n-1)^{\text{th}}$ equation, has two unknowns: x_n and x_{n-1}, but x_n is already known. This reduces the $(n-1)^{\text{th}}$ equation also to one unknown. Back substitution hence can be represented for all equations by the formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, n-2, \dots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Example 1:
Solve the system

$$x_1 + x_2 + 3x_3 = 5$$
$$2x_2 - x_3 = 1$$
$$-3x_1 + 2x_2 + 2x_3 = 1$$

Solution: Now applying the operation $R_3 = r_3 + 3r_1$ we have the following

$$x_1 + x_2 + 3x_3 = 5$$
$$2x_2 - x_3 = 1$$
$$5x_2 + 11x_3 = 16$$

Applying $R_2 = 1/2\, r_2$ we have

$$x_1 + x_2 + 3x_3 = 5$$
$$x_2 - .5\,x_3 = .5$$
$$5x_2 + 11x_3 = 16$$

And by $R_3 = r_3 - 5r_2$

$$\begin{aligned} x_1 + \; x_2 + \; 3x_3 &= 5 \\ x_2 - .5x_3 &= .5 \\ 13.5x_3 &= 13.5 \end{aligned}$$

Finally we the following by applying $R_3 = r_2 / 13.5$

$$\begin{aligned} x_1 + \; x_2 + \; 3x_3 &= 5 \\ x_2 - .5x_3 &= .5 \\ x_3 &= 1 \end{aligned}$$

We now have that $x_3 = 1$, and other unknowns can easily be found by backward substitution into second and first equations. We have the solution $(x_1, x_2, x_3) = (1, 1, 1)$. This method is called the Gaussian Elimination method.

Example 2. Find *x*, *y* and *z* that satisfy the following three equations at the same time.

(1)
$$\begin{aligned} x - y + 3z &= 4 \\ 2x - y + 2z &= 6 \\ 3x + y - 2z &= 9 \end{aligned}$$

Before discussing the details of Gaussian elimination, let's look at two ways to reformulate a system of linear equations. Both ways begin by putting the equations in vector form. For the equations above this is the following.

(2)
$$\begin{pmatrix} x - y + 3z \\ 2x - y + 2z \\ 3x + y - 2z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

The left side we can write as the matrix of coefficients times the vector of unknowns.

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

or

(3) $Au = b$

where

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & -2 \end{pmatrix} \qquad u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

So the original equations (1) are equivalent to (3). In general the problem of solving a system of linear equations is equivalent to solving $Au = b$ where A is the matrix of coefficients, b is the vector of numbers on the right side and u is the vector of unknowns.
The second reformulation of the equations starts with (2) and writes the vector on the left as the sum of three vectors where each term contains the terms with one of the variables. We get

$$\begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix} + \begin{pmatrix} -y \\ -y \\ y \end{pmatrix} + \begin{pmatrix} 3z \\ 2z \\ -2z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

Now we factor the variables out of each of the vectors on the left to get

$$x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix}$$

or

$$xv_1 + yv_2 + zv_3 = b$$

where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

So the original equations (1) are equivalent to writing b as a linear combination of v_1, v_2 and v_3. In general the problem of solving a system of linear equations is equivalent to writing b as a linear combination of the vectors that are the coefficients of each of the variables.

Now let's look at solving linear equations using Gaussian elimination. We shall look at two methods to keep track of our calculations. One is with the equations themselves. The other is by means of another matrix which is just the coefficient matrix A and right hand side b of the equation combined. It is called the *augmented matrix*. For the equations in Example 1 it is.

$$M = \left(\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 2 & -1 & 2 & 6 \\ 3 & 1 & -2 & 9 \end{array}\right)$$

Note that we draw a line separating the last column which contains b from the rest which contains A. To start out we have the original equations and the corresponding M.

Equations Augmented matrix

Error! $M =$ **Error!**

The idea behind Gaussian elimination is to add or subtract multiples of the first equation from the other two in order to eliminate x from the second and third equations. In this case we can subtract two times the first equation from the second and three times the first equation from the third. In terms of the augmented matrix we subtract two times the first row from the second and three times the first row from the third. This gives us the following.

Equations Augmented matrix

(2) **Error!** M_1 = **Error!**

Note that the new set of equations have the same solutions as the original equatons. It is clear that any solution to the equations is a solution to the new set because we obtained the equations in the new set by adding multiples of the original equations. However, the original equations can be obtained from the new set by adding two time the first equation to the second and three times the first equation to the third. Therefore any solution to the new equations is also a solution to the original equations.

There is another way of looking at the process of going from the original augmented matrix to the new augmented matrix that will be useful as we go along. One has

$$M_1 = E_1 M$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

The reason this is true is because when we multiply M on the right by E_1 the rows of the product E_1M are linear combinations of the rows of M using the entries of the corresponding row of E_1 as the multipliers. So, in particular, the second row of E_1M is -2 times the first row of M plus the second row of M which is how the second row of M_1 is formed. By a similar argument one can see that

$$M = F_1 M_1$$

where

$$F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

and

$$I = F_1 E_1 \qquad I = E_1 F_1$$

A pair of matrices A and B satisfying $AB = I$ and $BA = I$ are said to be inverse to each other and we write $B = A^{-1}$ and $A = B^{-1}$. So $F_1 = (E_1)^{-1}$.

Note that in the new set of equations (2) the second and third equations only involve *y* and *z*. So we concentrate on them. Now we eliminate *z* from the third equation by adding or subtracting a multiple of the second equation. In this case we can subtract 4 times the second equation from the third. In terms of the augmented matrix we subtract 4 times the second row from the third. We get

Equations Augmented matrix

(3) **Error!** M_2 = **Error!**

Again this set of equations has the same solution as the original set. Also, note that

$$M_2 = E_2M_1 = E_2E_1M$$

where

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

In (3) the third equation only involves *z*. All we have to do is divide this equation by 5 to get *z*. In terms of the augmented matrix we divide the third row by 5. This gives

Equations Augmented matrix

(4) **Error!** M_3 = **Error!**

Also note that

$$M_3 = E_3M_2 = E_3E_2E_1M$$

where

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

At this point we could substitute $z = 1$ in the second equation and solve for *y*. However, an equivalent thing to do is add 4 times the third equation to the second to eliminate *z*. At the same time we can subtract 3 times the third equation from the first to eliminate *z* from it also. In terms of the augmented matrix we are subtracting 3 times the third row from the first and adding 4 times the third row to the second. We get

Equations Augmented matrix

(5) **Error!** M_4 = **Error!**

Also

$$M_4 = E_4 M_3 = E_4 E_3 E_2 E_1 M$$

where

$$E_4 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

The last step is to add equation 2 to equation 1 to eliminate y from equation 1. In terms of the augmented matrix we add row 2 to row 1. This gives

Equations Augmented matrix

(6) **Error!** M_5 = **Error!**

Also

(7) $M_5 = E_5 M_4 = E_5 E_4 E_3 E_2 E_1 M$

where

$$E_5 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When we reach the point (6) we have the solution. In terms of the augmented matrix the solution is the last column. The part of the augmented matrix to the left of the vertical line is the identity matrix. If we were to ignore the last column of the augmented matrix, then the relation (7) says

(8) $I = E_5 E_4 E_3 E_2 E_1 A$

It turns out that $A^{-1} = (E_5 E_4 E_3 E_2 E_1)^{-1}$. We shall show that in the next chapter.

There is one other operation on the equations that we sometimes need to use or want to use. That is interchanging two equations. This corresponds to interchanging two rows of the augmented matrix. For example, suppose the original equations were

Equations Augmented matrix

$$\begin{array}{rcrcrcr} & & y & - & 4z & = & -2 \\ x & - & y & + & 3z & = & 4 \\ 3x & + & y & - & 2z & = & 9 \end{array} \qquad M = \begin{pmatrix} 0 & 1 & -4 & | & -2 \\ 1 & -1 & 3 & | & 4 \\ 3 & 1 & -2 & | & 9 \end{pmatrix}$$

Remark. These equations are equivalent to $\begin{pmatrix} 0 & 1 & -4 \\ 1 & -1 & 3 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 9 \end{pmatrix}$ or $Au = b$ where $A = \begin{pmatrix} 0 & 1 & -4 \\ 1 & -1 & 3 \\ 3 & 1 & -2 \end{pmatrix}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 9 \end{pmatrix}$. They are also equivalent to $x\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + z\begin{pmatrix} -4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 9 \end{pmatrix}$ or $xv_1 + yv_2 + zv_3 = b$ where $v_1 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -4 \\ 3 \\ -2 \end{pmatrix}$. In other words we are trying to write b as a superposition of v_1, v_2 and v_3.

The first step would be to interchange the first equation with either the second or the third. If we interchange the first and second equations we get

Equations Augmented matrix

(9)
$$\begin{array}{rcrcrcr} x & - & y & + & 3z & = & 4 \\ & & y & - & 4z & = & -2 \\ 3x & + & y & - & 2z & = & 9 \end{array} \qquad M_1 = \begin{pmatrix} 1 & -1 & 3 & | & 4 \\ 0 & 1 & -4 & | & -2 \\ 3 & 1 & -2 & | & 9 \end{pmatrix}$$

Note that

$$M_1 = E_1 M$$

where

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rest of the solution is similar to Example 1. Subtract 3 times the first equation from the third giving

Equations Augmented matrix

Error! M_2 = **Error!**

One has $M_2 = E_2M_1 = E_2E_1M$ where $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$. Subtract 4 times the second equation from the third giving

Equations Augmented matrix

Error! M_3 = **Error!**

One has $M_3 = E_3M_2 = E_3E_2E_1M$ where $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. Divide equation 3 by 5 giving

Equations Augmented matrix

Error! M_4 = **Error!**

One has $M_4 = E_4M_3 = E_4E_3E_2E_1M$ where $E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$. Add 4 times the third equation to the second and subtract 3 times the third equation from the first. We get

Equations Augmented matrix

Error! M_5 = **Error!**

Also $M_5 = E_5M_4 = E_5E_4E_3E_2E_1M$ where $E_5 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. Finally, add equation 2 to equation 1 giving

Equations Augmented matrix

Error! M_6 = **Error!**

Also $M_6 = E_6M_5 = E_6E_5E_4E_3E_2E_1M$ where $E_6 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. One has $I = E_5E_4E_3E_2E_1A$ so $A^{-1} = (E_5E_4E_3E_2E_1)^{-1}$.

To summarize, to solve a set of *n* equations and *n* unknowns

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ &\cdots \\ &\cdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= b_n \end{aligned} \qquad (10)$$

We form the augmented matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & | & b_n \end{pmatrix}$$

Using the following operations

(11) Add or subtract multiples of one row to another
(12) Multiply or divide a row by a non-zero constant
(13) Interchange two rows

we transform *M* to the form

$$M' = \begin{pmatrix} 1 & 0 & \cdots & 0 & | & c_1 \\ 0 & 1 & \cdots & 0 & | & c_2 \\ \cdots \\ 0 & 0 & \cdots & 1 & | & c_m \end{pmatrix} \qquad (14)$$

i.e. we have the identity matrix to the left of the vertical line. The solution is the last column, i.e.

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ &\cdots \\ x_n &= c_n \end{aligned} \qquad (15)$$

The row operations (11), (12) and (13) are called *elementary row operations*. If it is possible to transform *M* to the form (14) by elementary row operations then the system of equations (10) has one and only one solution which is (15). This is equivalent to being able to transform *A* to the identity *I* by the elementary row operations. If it is not possible to transform *M* to the form (14) by elementary row operations, then either there is no solution, or if there is a solution then there is more than one.

GAUSS-JORDAN METHOD:
Step 1: Form the augmented matrix corresponding to the system of linear equations.
Step 2: Transform the augmented matrix to the matrix in reduced row echelon form via elementary row operations.
Step 3: Solve the linear system corresponding to the matrix in reduced row echelon form. The solution(s) are also for the system of linear equations in step 1.

Example 1:
Solve for the following linear system:

$$x_1 + x_2 + 2x_3 - 5x_4 = 3$$
$$2x_1 + 5x_2 - x_3 - 9x_4 = -3$$
$$2x_1 + x_2 - x_3 + 3x_4 = -11$$
$$x_1 - 3x_2 + 2x_3 + 7x_4 = -5$$

Solution:
The Gauss-Jordan reduction is as follows:
Step 1:
The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{bmatrix}.$$

Step 2:
After elementary row operations, the matrix in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 3:
The linear system corresponding to the matrix in reduced row echelon form is

$$\begin{aligned} x_1 \qquad\qquad + 2x_4 &= -5 \\ x_2 \qquad - 3x_4 &= 2 \\ x_3 - 2x_4 &= 3 \end{aligned}$$

The solutions are

$$x_1 = -5 - 2t,\ x_2 = 2 + 3t,\ x_3 = 3 + 2t,\ x_4 = t,\ t \in R$$

$$\Leftrightarrow \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5-2t \\ 2+3t \\ 3+2t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \\ 1 \end{bmatrix} t$$

Number of solutions of a system of linear equations:

- For any system of linear equations, precisely one of the following is true.
- The system has exactly one solution.
- The system has an infinite number of solutions.
- The system has no solution.

Note: the linear system with at least one solution is called consistent and the linear system with no solution is called inconsistent.

Example 2:
Exactly one solution:
Solve for the following system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 \qquad - x_3 &= 3 \end{aligned}$$

Solution:
The Gauss-Jordan reduction is as follows:
Step 1:
The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{bmatrix}.$$

Step 2:
The matrix in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 3:
The solution is

$$x_1 = 2,\ x_2 = -1,\ x_3 = 3$$

Example 3:
Infinite number of solutions:
Solve for the following system:

$$2x_1 + 4x_2 - 2x_3 = 0$$
$$3x_1 + 5x_2 = 1$$

Solution:
The Gauss-Jordan reduction is as follows:
Step 1:
The augmented matrix is

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}$$

Step 2:
The matrix in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Step 3:
The linear system corresponding to the matrix in reduced row echelon form is

$$x_1 + 5x_3 = 2$$
$$x_2 - 3x_3 = -1$$

The solutions are

$$x_1 = 2 - 5t,\ x_2 = -1 + 3t,\ x_3 = t,\ t \in R$$

$$\Leftrightarrow \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-5t \\ -1+3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} t$$

Example 4:
No solution:
Solve for the following system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 4x_4 &= 5 \\ x_1 + 3x_2 + 5x_3 + 7x_4 &= 1 \\ x_1 \qquad - x_3 - 2x_4 &= -6 \end{aligned}$$

Solution:
The Gauss-Jordan reduction is as follows:
Step 1:
The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{bmatrix}$$

Step 2:
The matrix in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3:
The linear system corresponding to the matrix in reduced row echelon form is

$$\begin{aligned} x_1 \qquad - x_3 - 2x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \\ 0 &= 1 \end{aligned}$$

Since $0 \neq 1,$ there is no solution.

GAUSS-SEIDAL METHOD:

Why do we need another method to solve a set of simultaneous linear equations?

In certain cases, such as when a system of equations is large, iterative methods of solving equations are more advantageous. Elimination methods, such as Gaussian elimination, are prone to large round-off errors for a large set of equations. Iterative methods, such as the Gauss-Seidel method, give the user control of the round-off error. Also, if the physics of the problem are well known, initial guesses needed in iterative methods can be made more judiciously leading to faster convergence.

What is the algorithm for the Gauss-Seidel method? Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$
$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

.
.
.

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j \neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j \neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i. At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$\left|\in_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

where x_i^{new} is the recently obtained value of x_i, and x_i^{old} is the previous value of x_i.
When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example 1:

The upward velocity of a rocket is given at three different times in the following table

TABLE 1 Velocity vs. time data.

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = a_1 t^2 + a_2 t + a_3, \qquad 5 \le t \le 12$$

Find the values of a_1, a_2, and a_3 using the Gauss-Seidel method. Assume an initial guess of the solution as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and conduct two iterations.

Solution:

The polynomial is going through three data points $(t_1, v_1), (t_2, v_2)$, and (t_3, v_3) where from the above table

$$t_1 = 5, \quad v_1 = 106.8$$
$$t_2 = 8, \quad v_2 = 177.2$$
$$t_3 = 12, \quad v_3 = 279.2$$

Requiring that $v(t) = a_1 t^2 + a_2 t + a_3$ passes through the three data points gives

$$v(t_1) = v_1 = a_1 t_1^2 + a_2 t_1 + a_3$$
$$v(t_2) = v_2 = a_1 t_2^2 + a_2 t_2 + a_3$$
$$v(t_3) = v_3 = a_1 t_3^2 + a_2 t_3 + a_3$$

Substituting the data $(t_1, v_1), (t_2, v_2)$, and (t_3, v_3) gives

$$a_1(5^2) + a_2(5) + a_3 = 106.8$$
$$a_1(8^2) + a_2(8) + a_3 = 177.2$$
$$a_1(12^2) + a_2(12) + a_3 = 279.2$$

Or

$$25a_1 + 5a_2 + a_3 = 106.8$$
$$64a_1 + 8a_2 + a_3 = 177.2$$
$$144a_1 + 12a_2 + a_3 = 279.2$$

The coefficients a_1, a_2, and a_3 for the above expression are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Rewriting the equations gives

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Iteration #1

Given the initial guess of the solution vector as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

we get

$$a_1 = \frac{106.8 - 5(2) - (5)}{25}$$
$$= 3.6720$$
$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8}$$
$$= -7.8150$$
$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1}$$
$$= -155.36$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{3.6720 - 1}{3.6720}\right| \times 100$$
$$= 72.76\%$$
$$|\in_a|_2 = \left|\frac{-7.8510 - 2}{-7.8510}\right| \times 100$$
$$= 125.47\%$$
$$|\in_a|_3 = \left|\frac{-155.36 - 5}{-155.36}\right| \times 100$$
$$= 103.22\%$$

At the end of the first iteration, the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

and the maximum absolute relative approximate error is 125.47%.

ITERATION #2

The estimate of the solution vector at the end of Iteration #1 is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

Now we get

$$a_1 = \frac{106.8 - 5(-7.8510) - (-155.36)}{25}$$
$$= 12.056$$
$$a_2 = \frac{177.2 - 64(12.056) - (-155.36)}{8}$$
$$= -54.882$$
$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1}$$
$$= -798.34$$

The absolute relative approximate error for each x_i then is

$$|\in_a|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| \times 100$$
$$= 69.543\%$$
$$|\in_a|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| \times 100$$
$$= 85.695\%$$
$$|\in_a|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| \times 100$$
$$= 80.540\%$$

At the end of the second iteration the estimate of the solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

and the maximum absolute relative approximate error is 85.695%.
Conducting more iterations gives the following values for the solution vector and the corresponding absolute relative approximate errors.

Iteration	a_1	$\|\in_a\|_1$ %	a_2	$\|\in_a\|_2$ %	a_3	$\|\in_a\|_3$ %
1	3.6720	72.767	–7.8510	125.47	–155.36	103.22
2	12.056	69.543	–54.882	85.695	–798.34	80.540
3	47.182	74.447	–255.51	78.521	–3448.9	76.852
4	193.33	75.595	–1093.4	76.632	–14440	76.116
5	800.53	75.850	–4577.2	76.112	–60072	75.963
6	3322.6	75.906	–19049	75.972	–249580	75.931

As seen in the above table, the solution estimates are not converging to the true solution of

$$a_1 = 0.29048$$
$$a_2 = 19.690$$
$$a_3 = 1.0857$$

Example 2:
Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess and conduct two iterations.

Solution:
The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$
$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$
$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.
Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$
$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Iteration #1

$$x_1 = \frac{1 - 3(0) + 5(1)}{12}$$
$$= 0.50000$$

$$x_2 = \frac{28-(0.50000)-3(1)}{5}$$
$$= 4.9000$$
$$x_3 = \frac{76-3(0.50000)-7(4.9000)}{13}$$
$$= 3.0923$$

The absolute relative approximate error at the end of the first iteration is

$$|\in_a|_1 = \left|\frac{0.50000-1}{0.50000}\right| \times 100$$
$$= 100.00\%$$
$$|\in_a|_2 = \left|\frac{4.9000-0}{4.9000}\right| \times 100$$
$$= 100.00\%$$
$$|\in_a|_3 = \left|\frac{3.0923-1}{3.0923}\right| \times 100$$
$$= 67.662\%$$

The maximum absolute relative approximate error is 100.00%

ITERATION #2

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12}$$
$$= 0.14679$$
$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5}$$
$$= 3.7153$$
$$x_3 = \frac{76-3(0.14679)-7(3.7153)}{13}$$
$$= 3.8118$$

At the end of second iteration, the absolute relative approximate error is

$$|\in_a|_1 = \left|\frac{0.14679-0.50000}{0.14679}\right| \times 100$$
$$= 240.61\%$$
$$|\in_a|_2 = \left|\frac{3.7153-4.9000}{3.7153}\right| \times 100$$
$$= 31.889\%$$
$$|\in_a|_3 = \left|\frac{3.8118-3.0923}{3.8118}\right| \times 100$$
$$= 18.874\%$$

The maximum absolute relative approximate error is 240.61%. This is greater than the value of 100.00% we obtained in the first iteration. Is the solution diverging? No, as you conduct more iterations, the solution converges as follows.

Iteration	x_1	$\|\in_a\|_1 \%$	x_2	$\|\in_a\|_2 \%$	x_3	$\|\in_a\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.874
3	0.74275	80.236	3.1644	17.408	3.9708	4.0064
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

This is close to the exact solution vector of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 3:
Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$12x_1 + 3x_2 - 5x_3 = 1$$

find the solution using the Gauss-Seidel method. Use

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

as the initial guess.

Solution
Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

the next six iterative values are given in the table below.

Iteration	x_1	$\left\|\in_a\right\|_1 \%$	x_2	$\left\|\in_a\right\|_2 \%$	x_3	$\left\|\in_a\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	–196.15	110.71	14.421	94.453	–462.30	110.96
3	1995.0	109.83	–116.02	112.43	4718.1	109.80
4	–20149	109.90	1204.6	109.63	–47636	109.90
5	2.0364×10^5	109.89	–12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^6	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$[A]=\begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \le |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the Gauss-Seidel method may or may not converge.
However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.
Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases. For example, the following set of equations

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + 4x_3 = 9$$
$$x_1 + 7x_2 + x_3 = 9$$

cannot be rewritten to make the coefficient matrix diagonally dominant.
In this method, we can write the iterative scheme of the system of equations
A**x** = **b** as follows:

$$a_{11}x_1^{(k+1)} = -a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_n x_n^{(k)} + b_1$$

$$a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} = -a_{23}x_3^{(k)} - \ldots - a_{2n}x_n^{(k)} + b_2$$

.

.

$$a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} \ldots + a_{nn}x_n^{(k+1)} = + b_n$$

In matrix form, this system can be written as (D + L) $\mathbf{x}^{(k+1)} = -\mathrm{U}\,\mathbf{x}^{(k)} + \mathbf{b}$ with the same notation as adopted in Jacobi method.

From the above, we get

$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b$$

$$= T\boldsymbol{x}^{(k)} + \boldsymbol{c}_n$$

i.e. T = $-$ (D+L) $^{-1n}$ U and **c** = (D + L)$^{-1}$**b**

This iteration method is also known as the method of successive displacement.

For computation point of view, we rewrite (A x) as

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{x} a_{ij}x_j^{(k)} - b_i\right]$$

i = 1, 2, …, n

Also in this case, if A is diagonally dominant, then iteration method always converges. In general Gauss-Seidel method will converge if the Jacobi method converges and will converge at a faster rate. You can observe this in the following example. We have not considered the problem: How many iterations are needed to have a reasonably good approximation to **x**? This needs the concept of matrix norm.

Example 6: Solve the linear system A**x** = **b** given in Example 4 by Gauss-Seidel method rounded to four decimal places. The equations can be written as follows:

$$x_1^{(k+1)} = \frac{1}{10}x_2^{(k)} - \frac{1}{3}x_3^{(k)} + \frac{3}{5}$$

$$x_2^{(k+1)} = \frac{1}{11}x_1^{(k+1)} + \frac{1}{11}x_3^{k} - \frac{3}{11}x_4^{(k)} + \frac{25}{11}$$

$$x_3^{(k+1)} = -\frac{1}{3}x_1^{(k+1)} + \frac{1}{10}x_2^{(k+1)} + \frac{1}{10}x_4^{(k)} - \frac{11}{10}$$

$$x_4^{(k+1)} = -\frac{3}{8}x_2^{(k+1)} + \frac{1}{8}x_3^{(k+1)} + \frac{15}{8}.$$

Letting $\mathbf{x}^{(0)}$ = (0, 0, 0, 0)^T we have from first equation

$$x_1^{(1)} = 0.6000$$

$$x_2^{(1)} = \frac{0.6000}{3} + \frac{25}{11} = 2.3273$$

$$x_3^{(1)} = -\frac{0.6000}{3} + \frac{1}{10}(2.3273) - \frac{11}{10} = -0.1200 + 0.2327 - 1.1000 = -0.9873$$

$$x_4^{(1)} = -\frac{3}{8}(2.3273) + \frac{1}{8}(-0.9873) + \frac{15}{8}$$

= – 0.8727 – 0.1234 + 1.8750

= 0.8789

Using $\mathbf{x}^{(1)}$ we get

$$\mathbf{x}^{(2)} = (1.0300, 2.037, -1.014, 0.9844)^T$$

and we can check that

$$\mathbf{x}^{(5)} = (1.0001, 2.0000, -1.0000, 1.0000)^T$$

Note that $\mathbf{x}^{(5)}$ is a good approximation to the exact solution. Here are a few exercises for you to solve.

COMPUTATION OF INVERSE OF A MATRIX USING GAUSS ELIMINATION METHOD:

For a given non-singular matrix *A*, the inverse matrix $B = A^{-1}$ exists such that $AB = BA = I$, where *I* is an identity matrix of order same as *A* or *B*.
A matrix ***A*** is non-singular iff $\det(A) = |A| \neq 0$

To find inverse of a nonsingular matrix using calculator:
Step 1. Input the matrix say *A*
Step 2. Call matrix *A* and hit x^{-1} in your calculator then hit MATH and select 1 : ▷Frac to get the matrix along with determinant value.

Example 1. Find the inverse of a two by two matrix by hand:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Now verify that $AA^{-1} = A^{-1}A = I$

Example 2. Show that $(AB)^{-1} = B^{-1}A^{-1}$

Solution: We can consider that

$$AB(AB)^{-1} = I$$
$$A^{-1}AB(AB)^{-1} = A^{-1}I = A^{-1}, \text{ multiplying by } A^{-1}$$
$$B^{-1}IB(AB)^{-1} = B^{-1}A^{-1}, \text{ using } A^{-1}A = I \text{ and multiplying by } B^{-1}$$
$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$
$$(AB)^{-1} = B^{-1}A^{-1}$$

Example 3. Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ using $(AB)^{-1} = B^{-1}A^{-1}$

Example 4. Solve the following system of equations by matrix inverse:

$$\begin{cases} x + y - z = 12 \\ 2x - y + 2z = -3 \\ x + 2y - z = 6 \end{cases}$$

Solution: We have the following matrix system

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix}, \text{ check for det} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} = 4$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ -4 & 0 & 4 \\ -5 & 1 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{27}{4} \\ -6 \\ -\frac{45}{4} \end{pmatrix}$$

METHOD OF TRIANGULARIZATION (OR METHOD OF FACTORIZATION) (DIRECT METHOD) :

This method is also called as decomposition method. In this method, the coefficient matrix A of the system AX = B, decomposed or factorized into theproduct of a lower triangular matrix L and an upper triangular matrixU. we will explain this method in the case of three equations in three unknowns.

Consider the system of equations

a11x1+ a12x2+ a13x3 = b1
a21x1+ a22x2+ a23x3 = b2
a31x1+ a32x2+ a33x3 = b3

This system is equivalent to AX = B

Where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

And an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$ so that

LUX = B Let UX = Y And hence LY = B

That is, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

∴ $y_1 = b$, $l_{21}y_1+y_2 = b_2$, $l_{31}y_1+l_{32}y_2+y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4), $$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$, $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known. Now L and U can be found from LU = A

i.e., $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3 l's and 6 u's.

That is, L and U re known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first rows of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$ $l_{21}u_{12}+u_{22} = a_{22}$ $l_{21}u_{13}+u_{23} = a_{23}$

$$l_{21} = \frac{a_{21}}{a_{11}},\quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.a_{13}$$

again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12}+l_{32}u_{22} = a_{32}$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = a_{33}$

solving, $l31 = \frac{a_{31}}{a_{11}}$, $l32 = \frac{a_{32} - \frac{a_{21}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}}$

$$u33 = \left[a_{33} - \frac{a_{31}}{a_{11}}\right] a13 - \left[\frac{a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}}\right] a_{33} - \frac{a_{31}}{a_{11}} . a13$$

Therefore L and U are known.

Example 2

By the method of triangularization, solve the following system.

5x – 2y + z = 4, 7x + y – 5z = 8, 3x + 7y + 4z = 10.

Solution.The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

A X = B

Now, let LU = A

That is, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$$

Multiplying and equating coefficients,

u11 = 5, u12 = −2, u13 = 1

l21u11 = 7 l21u12+u22 = 1 l21u13+u23 = −5

$l21 = \frac{7}{5}$, $u22 = 1 - \frac{7}{5}.(-2) = \frac{19}{5}$ and

$u23 = -5 - \frac{7}{5}.(1) = -\frac{32}{5}$

Again equating elements in the third row,

l31u11 = 3, l31u12+l32u22 = 7 and l31u13+l32u23+u33 = 4

$$\therefore \quad l31 = \frac{3}{5}, \quad l32 = \frac{7 - \frac{3}{5}.(-2)}{\frac{19}{5}} = \frac{41}{19}$$

$$u33 = 4 - \frac{3}{5}.(1) - \frac{41}{19}\left(-\frac{32}{5}\right) = 4 - \frac{3}{5} + \frac{1312}{95}$$

$$= \frac{1635}{95} = \frac{327}{19}$$

Now L and U are known.Since LUX = B, LY = B where UX =Y.

From LY = B,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

y1 = 4, $\frac{7}{5}$ y1+y2 = 8 , $\frac{3}{5}$y1+ $\frac{41}{19}$ y2+y3 = 10

$$y_2 = 8 - \frac{28}{5} = \frac{12}{5}$$

$$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \times \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$$

UX =Y gives

$$\begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$$

$5x - 2y + z = 4$

$$\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$$

$$\frac{327}{19}z = \frac{46}{19}$$

$$z = \frac{46}{327}$$

$$\frac{19}{5}y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$$

$$y = \frac{284}{327}$$

$$5x = 4 + 2y - z = 4 + 2\left(\frac{568}{327}\right) - \frac{46}{327}$$

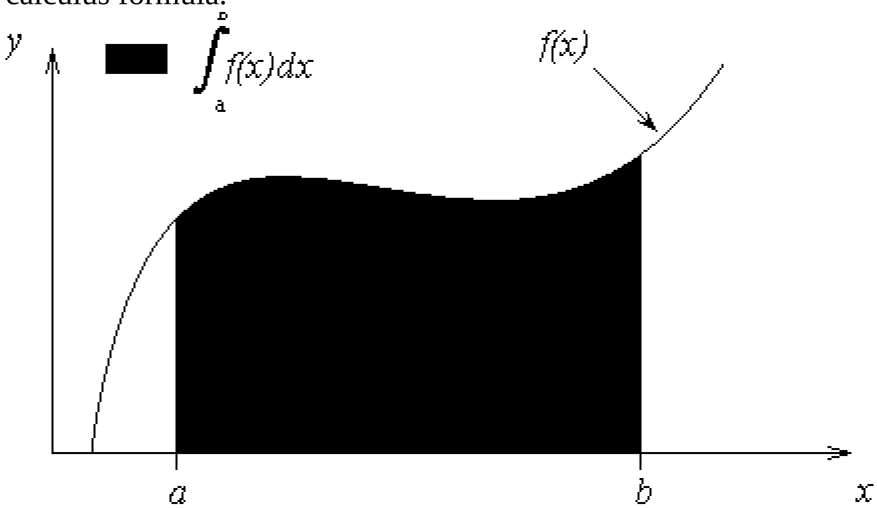
$$\therefore \quad x = \frac{366}{327}$$

$$\therefore \quad x = \frac{366}{327}, \quad y = \frac{284}{327}, \quad z = \frac{46}{327}$$

TRAPEZOIDAL RULE:

What is the trapezoidal rule?

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial. Integrating polynomials is simple and is based on the calculus formula.



$$\int_a^b x^n dx = \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right),\ n \neq -1 \qquad (1)$$

So if we want to approximate the integral

$$I = \int_a^b f(x)dx \qquad (2)$$

to find the value of the above integral, one assumes

$$f(x) \approx f_n(x) \qquad (3)$$

where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n. \qquad (4)$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

Derivation of the Trapezoidal Rule

Method 1: Derived from Calculus

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

$$= \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) \quad (5)$$

But what is a_0 and a_1? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b,

$$f(a) = f_1(a) = a_0 + a_1 a \quad (6)$$
$$f(b) = f_1(b) = a_0 + a_1 b \quad (7)$$

Solving the above two equations for a_1 and a_0,

$$a_1 = \frac{f(b) - f(a)}{b-a}$$
$$a_0 = \frac{f(a)b - f(b)a}{b-a} \quad (8a)$$

Hence from Equation (5),

$$\int_a^b f(x)dx \approx \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b)-f(a)}{b-a}\frac{b^2-a^2}{2} \quad (8b)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \quad (9)$$

Method 2: Also Derived from Calculus

$f_1(x)$ can also be approximated by using Newton's divided difference polynomial as

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \quad (10)$$

Hence

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

$$= \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]dx$$

$$= \left[f(a)x + \frac{f(b)-f(a)}{b-a}\left(\frac{x^2}{2} - ax\right)\right]_a^b$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab + \frac{a^2}{2}\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{1}{2}(b-a)^2$$

$$= f(a)b - f(a)a + \frac{1}{2}\left(f(b)-f(a)\right)(b-a)$$

$$= f(a)b - f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a - \frac{1}{2}f(a)b + \frac{1}{2}f(a)a$$

$$= \frac{1}{2}f(a)b - \frac{1}{2}f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad (11)$$

This gives the same result as Equation (10) because they are just different forms of writing the same polynomial.

Method 3: Derived from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid. The integral

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \frac{1}{2}(\text{Sum of length of parallel sides})(\text{Perpendicular distance between parallel sides})$$

$$= \frac{1}{2}\left(f(b)+f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad (12)$$

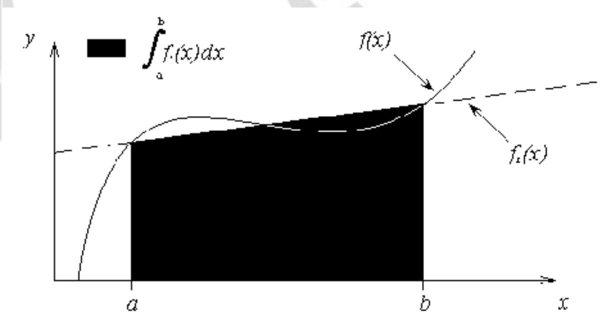


FIGURE 2 Geometric representation of trapezoidal rule.

Method 4: Derived from Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients. The formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \quad (13)$$

$$= \sum_{i=1}^{2} c_i f(x_i)$$

where

$c_1 = \frac{b-a}{2}$

$c_2 = \frac{b-a}{2}$

$x_1 = a$

$x_2 = b$

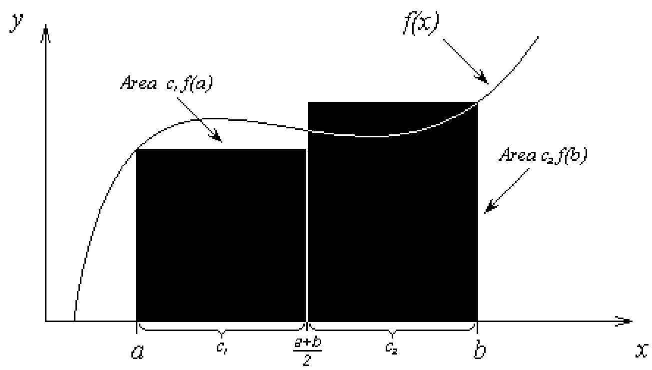


FIGURE 3 Area by method of coefficients.

The interpretation is that $f(x)$ is evaluated at points a and b, and each function evaluation is given a weight of $\frac{b-a}{2}$. Geometrically, Equation (12) is looked at as the area of a trapezoid, while Equation (13) is viewed as the sum of the area of two rectangles, as shown in Figure 3. How can one derive the trapezoidal rule by the method of coefficients?

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \quad (14)$$

Let the right hand side be an exact expression for integrals of $\int_a^b 1dx$ and $\int_a^b xdx$, that is, the formula will then also be exact for linear combinations of $f(x) = 1$ and $f(x) = x$, that is, for $f(x) = a_0(1) + a_1(x)$.

$$\int_a^b 1dx = b - a = c_1 + c_2 \qquad (15)$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b \qquad (16)$$

Solving the above two equations gives

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2} \qquad (17)$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \qquad (18)$$

Method 5: Another approach on the Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients by another approach

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \qquad (19)$$

Let the right hand side be exact for integrals of the form

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left(a_0 x + a_1 \frac{x^2}{2} \right)_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \qquad (20)$$

But we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \qquad (21)$$

to give the same result as Equation (20) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \qquad (22)$$

Hence from Equations (20) and (22),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary for a general straight line

$$c_1 + c_2 = b - a$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \qquad (23)$$

Again, solving the above two equations (23) gives

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2} \qquad (24)$$

Therefore

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \qquad (25)$$

Example 1

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$$

Use the single segment trapezoidal rule to find the distance covered for $t = 8$ *to* $t = 30$ *seconds.*

Find the true error, E_t *for part (a).*

Find the absolute relative true error for part (a).

Solution

a) $I \approx (b-a)\left[\frac{f(a) + f(b)}{2}\right]$, where

$a = 8$

$b = 30$

$$f(t) = 2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000 - 2100(8)}\right] - 9.8(8)$$

$$= 177.27 \text{ m/s}$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right]-9.8(30)$$
$$= 901.67 \text{ m/s}$$
$$I \approx (30-8)\left[\frac{177.27+901.67}{2}\right]$$
$$= 11868 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$
$$= 11061 \text{ m}$$

so the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$
$$= 11061 - 11868$$
$$= -807 \text{ m}$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{11061-11868}{11061}\right| \times 100$$
$$= 7.2958\%$$

Multiple-Segment Trapezoidal Rule

In Example 1, the true error using a single segment trapezoidal rule was large. We can divide the interval $[8,30]$ into $[8,19]$ and $[19,30]$ intervals and apply the trapezoidal rule over each segment.

$$f(t) = 2000\ln\left(\frac{140000}{140000-2100t}\right)-9.8t$$
$$\int_{8}^{30} f(t)dt = \int_{8}^{19} f(t)dt + \int_{19}^{30} f(t)dt$$
$$\approx (19-8)\left[\frac{f(8)+f(19)}{2}\right]+(30-19)\left[\frac{f(19)+f(30)}{2}\right]$$
$$f(8) = 177.27 \text{ m/s}$$
$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right)-9.8(19) = 484.75 \text{ m/s}$$
$$f(30) = 901.67 \text{ m/s}$$

Hence

$$\int_8^{30} f(t)dt \approx (19-8)\left[\frac{177.27+484.75}{2}\right]+(30-19)\left[\frac{484.75+901.67}{2}\right]$$

$$=11266 \text{ m}$$

The true error, E_t is

$$E_t = 11061-11266$$

$$= -205\,\text{m}$$

The true error now is reduced from 807 m to 205 m. Extending this procedure to dividing $[a,b]$ into n equal segments and applying the trapezoidal rule over each segment, the sum of the results obtained for each segment is the approximate value of the integral.

Divide $(b-a)$ into n equal segments as shown in Figure 4. Then the width of each segment is

$$h = \frac{b-a}{n} \tag{26}$$

The integral I can be broken into h integrals as

$$I = \int_a^b f(x)dx$$

$$= \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx \tag{27}$$

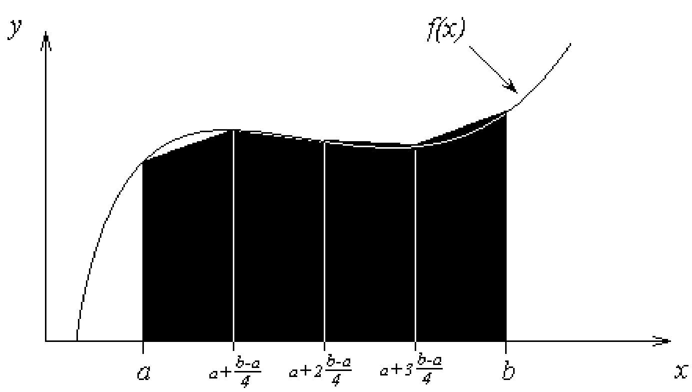


FIGURE 4 Multiple ($n=4$) segment trapezoidal rule

Applying trapezoidal rule Equation (27) on each segment gives

$$\int_a^b f(x)dx = [(a+h)-a]\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+\left[(a+2h)-(a+h)\right]\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+\ldots\ldots\ldots\ldots\ldots\; +\left[\left(a+(n-1)h\right)-\left(a+(n-2)h\right)\right]\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+\left[b-\left(a+(n-1)h\right)\right]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+f(a+h)}{2}\right]+h\left[\frac{f(a+h)+f(a+2h)}{2}\right]+\ldots\ldots\ldots\ldots$$

$$+h\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]+h\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+2f(a+h)+2f(a+2h)+\ldots+2f(a+(n-1)h)+f(b)}{2}\right]$$

$$=\frac{h}{2}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

$$=\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right] \qquad (28)$$

Example 2

The vertical distance covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$x=\int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Use the two-segment trapezoidal rule to find the distance covered from $t=8$ to $t=30$ seconds.
Find the true error, E_t for part (a).

Find the absolute relative true error for part (a).

Solution

a) The solution using 2-segment Trapezoidal rule is

$$I\approx\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

$n=2$

$a=8$

$b=30$

$$h=\frac{b-a}{n}$$

$$=\frac{30-8}{2}$$

$$=11$$

$$I \approx \frac{30-8}{2(2)}\left[f(8)+2\left\{\sum_{i=1}^{2-1} f(8+11i)\right\}+f(30)\right]$$

$$=\frac{22}{4}\left[f(8)+2f(19)+f(30)\right]$$

$$=\frac{22}{4}\left[177.27+2(484.75)+901.67\right]$$

$$=11266 \text{ m}$$

b) The exact value of the above integral is

$$x=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

$$=11061 \text{ m}$$

so the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$

$$=11061-11266$$

$$=-205 \text{ m}$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$=\left|\frac{11061-11266}{11061}\right| \times 100$$

$$=1.8537\%$$

TABLE 1 Values obtained using multiple-segment trapezoidal rule for

$$x=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

n	Approximate Value	E_t	$\|\in_t\|\%$	$\|\in_a\|\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482

8	11074	-12.9	0.1165	0.03560

Example 3

Use the multiple-segment trapezoidal rule to find the area under the curve

$$f(x)=\frac{300x}{1+e^{x}}$$

from $x=0$ to $x=10$.

Solution

Using two segments, we get

$$h=\frac{10-0}{2}=5$$

$$f(0)=\frac{300(0)}{1+e^{0}}=0$$

$$f(5)=\frac{300(5)}{1+e^{5}}=10.039$$

$$f(10)=\frac{300(10)}{1+e^{10}}=0.136$$

$$I\approx\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

$$=\frac{10-0}{2(2)}\left[f(0)+2\left\{\sum_{i=1}^{2-1}f(0+5)\right\}+f(10)\right]$$

$$=\frac{10}{4}\left[f(0)+2f(5)+f(10)\right]$$

$$=\frac{10}{4}\left[0+2(10.039)+0.136\right]\quad=50.537$$

So what is the true value of this integral?

$$\int_0^{10}\frac{300x}{1+e^{x}}dx=246.59$$

Making the absolute relative true error

$$\left|\in_t\right|=\left|\frac{246.59-50.535}{246.59}\right|\times100$$

$$=79.506\%$$

Why is the true value so far away from the approximate values? Just take a look at Figure 5. As you can see, the area under the “trapezoids” (yeah, they really look like triangles now) covers a small portion of the area under the curve. As we add more segments, the approximated value quickly approaches the true value.

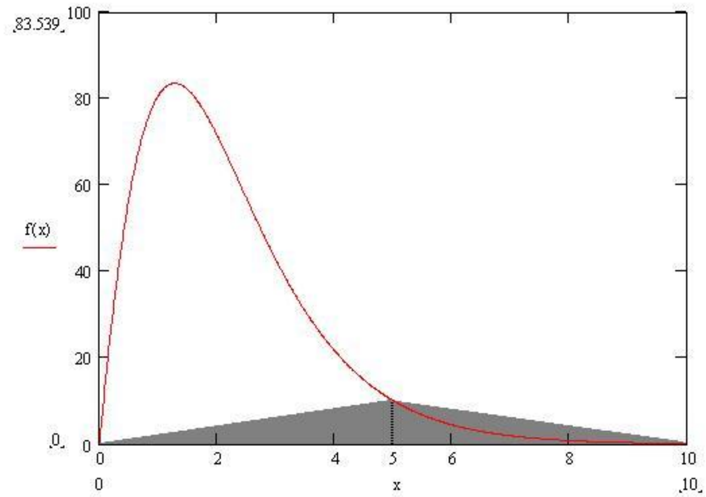


FIGURE 5 2-segment trapezoidal rule approximation.

TABLE 2 Values obtained using multiple-segment trapezoidal rule for $\int_0^{10} \frac{300x}{1+e^x} dx$.

n	Approximate Value	E_t	$\lvert\in_t\rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Example 4:
Use multiple-segment trapezoidal rule to find

$$I = \int_0^2 \frac{1}{\sqrt{x}} dx$$

Solution
We cannot use the trapezoidal rule for this integral, as the value of the integrand at $x = 0$ is infinite. However, it is known that a discontinuity in a curve will not change the area under it.

We can assume any value for the function at $x = 0$. The algorithm to define the function so that we can use the multiple-segment trapezoidal rule is given below.

Function $f(x)$
If $x = 0$ Then $f = 0$
If $x \neq 0$ Then $f = x\wedge(-0.5)$
End Function

Basically, we are just assigning the function a value of zero at $x = 0$. Everywhere else, the function is continuous. This means the true value of our integral will be just that—true. Let's see what happens using the multiple-segment trapezoidal rule.
Using two segments, we get

$$h = \frac{2-0}{2} = 1$$

$$f(0) = 0$$

$$f(1) = \frac{1}{\sqrt{1}} = 1$$

$$f(2) = \frac{1}{\sqrt{2}} = 0.70711$$

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$= \frac{2-0}{2(2)}\left[f(0) + 2\left\{\sum_{i=1}^{2-1} f(0+1)\right\} + f(2)\right]$$

$$= \frac{2}{4}\left[f(0) + 2f(1) + f(2)\right]$$

$$= \frac{2}{4}\left[0 + 2(1) + 0.70711\right]$$

$$= 1.3536$$

So what is the true value of this integral?

$$\int_0^2 \frac{1}{\sqrt{x}}dx = 2.8284$$

Thus making the absolute relative true error

$$|\in_t| = \left|\frac{2.8284 - 1.3536}{2.8284}\right| \times 100$$

$$= 52.145\%$$

TABLE 3 Values obtained using multiple-segment trapezoidal rule for $\int_0^2 \frac{1}{\sqrt{x}}dx$.

n	Approximate Value	E_t	$\lvert\in_t\rvert$
2	1.354	1.474	52.14%
4	1.792	1.036	36.64%
8	2.097	0.731	25.85%
16	2.312	0.516	18.26%
32	2.463	0.365	12.91%
64	2.570	0.258	9.128%
128	2.646	0.182	6.454%
256	2.699	0.129	4.564%
512	2.737	0.091	3.227%
1024	2.764	0.064	2.282%
2048	2.783	0.045	1.613%
4096	2.796	0.032	1.141%

Error in Multiple-segment Trapezoidal Rule
The true error for a single segment Trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12} f''(\xi), \quad a < \xi < b$$

Where ξ is some point in $[a,b]$.
What is the error then in the multiple-segment trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment trapezoidal rule. The error in each segment is

$$E_1 = -\frac{[(a+h)-a]^3}{12} f''(\xi_1), \quad a < \xi_1 < a+h$$

$$= -\frac{h^3}{12} f''(\xi_1)$$

$$E_2 = -\frac{[(a+2h)-(a+h)]^3}{12} f''(\xi_2), \quad a+h < \xi_2 < a+2h$$

$$= -\frac{h^3}{12} f''(\xi_2)$$

.
.
.

$$E_i = -\frac{[(a+ih)-(a+(i-1)h)]^3}{12} f''(\xi_i), \quad a+(i-1)h < \xi_i < a+ih$$

$$= -\frac{h^3}{12} f''(\xi_i)$$

.

.
.

$$E_{n-1} = -\frac{\left[\{a+(n-1)h\}-\{a+(n-2)h\}\right]^3}{12} f''(\xi_{n-1}), \quad a+(n-2)h < \xi_{n-1} < a+(n-1)h$$

$$= -\frac{h^3}{12} f''(\xi_{n-1})$$

$$E_n = -\frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\xi_n), \quad a+(n-1)h < \xi_n < b$$

$$= -\frac{h^3}{12} f''(\xi_n)$$

Hence the total error in the multiple-segment trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i$$

$$= -\frac{h^3}{12}\sum_{i=1}^{n} f''(\xi_i)$$

$$= -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

$$= -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

The term $\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$ is an approximate average value of the second derivative $f''(x),\ a < x < b$.

Hence

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

In Table 4, the approximate value of the integral

$$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

is given as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

TABLE 4 Values obtained using multiple-segment trapezoidal rule for

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt.$$

n	Approximate Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.853	5.343
4	11113	-52	0.4701	0.3594
8	11074	-13	0.1175	0.03560
16	11065	-4	0.03616	0.00401

For example, for the 2-segment trapezoidal rule, the true error is -205, and a quarter of that error is -51.25. That is close to the true error of -48 for the 4-segment trapezoidal rule.

SIMPSON'S 1/3 RULE:

Simpson's 1/3rd Rule/Formula

In this case the integral is evaluated over two intervals at a time, say $[x_0, x_1]$ and $[x_1, x_2]$. The function f(x) is approximated by a quadratic passing through the points (x_0, y_0) and (x_1, y_1) and

(x_2, y_2). From Lagrange's formula we may write the quadratic as,

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Integrating term by term we get,

$$\int_{x_0}^{x_2} \frac{(x - x_1)(x - x_2)}{(-h)(-2h)} dx = \frac{1}{2h^2}\left[(x - x_1)\frac{(x - x_2)^2}{2} - \frac{(x - x_2)^3}{6}\right]_{x_0}^{x_2} = \frac{h}{3}$$

$$\int_{x_0}^{x_2} \frac{(x - x_0)(x - x_2)}{h(-h)} dx = \frac{1}{h^2}\left[(x - x_0)\frac{(x - x_2)^2}{2} - \frac{(x - x_2)^3}{6}\right]_{x_0}^{x_2} = \frac{4}{3} h$$

$$\int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} dx = \frac{1}{2h^2}\left[(x - x_0)\frac{(x - x_1)^2}{2} - \frac{(x - x_1)^3}{6}\right]_{x_0}^{x_2} = \frac{h}{3}$$

Hence we get,

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} y(x)\,dx = \frac{h}{3} y_0 + \frac{4h}{3} y_1 + \frac{h}{3} y_2$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Applying this formula over next two intervals and then next two and so on for $\frac{n}{2}$ times and adding we get

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} y(x)\,dx = \int_{x_0}^{x_2} y(x)\,dx = \int_{x_2}^{x_4} y(x)\,dx + \ldots + \int_{x_{n-2}}^{x_n} y(x)\,dx$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

Obviously n should be chosen as a multiple of 2 i.e. an even number for applying this formula.

Example 1:

Evaluate the integral $I = \int_0^1 \frac{dx}{\sqrt{1+x^2}}$ by trapezoidal rule dividing the interval [0, 1] into five equal parts. Compute upto five decimals.

Solution

$n = 5;\ h = \frac{1-0}{5} = 0.2$

i	0	1	2	3	4	5
x	0	0.2	0.4	0.6	0.8	1.0
$y = \frac{1}{\sqrt{1+x^2}}$	1.0	0.98058	0.92848	0.85749	0.78087	0.70711

From Trapezoidal Rule;

$$I = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}[1.0 + 0.70711 + 2(0.98058 + 0.92848 + 0.85749 + 0.78087)]$$

$$= 0.1[1.70711 + 2 \times 3.54742]$$

$$= 0.88016$$

Example 2:

Evaluate the integral $I = \int_0^{0.8} \frac{dx}{\sqrt{1+x}}$ by Simpson's $1/3^{rd}$ rule dividing the interval [0, 0.8] to 4 equal sub-intervals. Compute up to five places of decimal only.

Solution

$n = 4;\ h = \frac{0.8-0}{4} = 0.2$

i	0	1	2	3	4
x	0	0.2	0.4	0.6	0.8
$y = \frac{1}{\sqrt{1+x^2}}$	1.0	0.91287	0.84515	0.79057	0.74536

From Simpson's $1/3^{rd}$ Rule

$$I = \int_0^{0.8} y\,dx = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4)]$$

$$= \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2 \times y_2)]$$
$$= \frac{0.2}{3}[1.0 + 0.74536 + 4(0.91287 + 0.79051) + 2 \times 0.84515)]$$
$$= \frac{0.2}{3}[1.74536 + 4 \times 1.70344 + 1.69030]$$
$$= 0.68329$$

Note : The maximum error in various integration formulas in the evaluation of the integral $I = \int_a^b f(x)\,dx$ is

(i) Rectangular Rule : $\frac{(b-a)}{2} f'(\xi)$

(ii) Trapezoidal Rule : $-\frac{(b-a)\,h^2}{12} f''(\xi)$

(iii) Simpson's 1/3rd Rule : $-\frac{(b-a)\,h^4}{180} f^{iv}(\xi)$

where $x = \xi$ is some point in $[a, b]$ for which $f'(x)$ or $f''(x)$ or $f^{(iv)}(x)$ has maximum numerical value.

Example 3:
All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1-\sim) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2®}} e^{-\frac{x^2}{2}} dx$$

a) Use Simpson's 1/3 Rule to find the frequency.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$, for part (a).

SOLUTION

a) $$(1-\sim) \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$a = -2.15$
$b = 2.9$

$$\frac{a+b}{2} = 0.37500$$

$$f(x) = \frac{1}{\sqrt{2®}} e^{-\frac{x^2}{2}}$$

$$f(-2.15) = \frac{1}{\sqrt{2®}} e^{-\frac{(-2.15)^2}{2}}$$

$$= 0.039550$$

$$f(2.9) = \frac{1}{\sqrt{2®}} e^{-\frac{(2.9)^2}{2}}$$

$$= 0.0059525$$

$$f(0.375) = \frac{1}{\sqrt{2®}} e^{-\frac{(0.375)^2}{2}}$$

$$= 0.37186$$

$$(1-\sim) \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\approx \left(\frac{(2.9-(-2.15)}{6}\right)\left[f(-2.15) + 4f(0.37500) + f(2.9)\right]$$

$$\approx \left(\frac{5.05}{6}\right)\left[0.039550 + 4(0.37186) + 0.0059525\right]$$

$$\approx 1.2902$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value.

$$(1-\sim) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2®}} e^{-\frac{x^2}{2}} dx$$

$$= 0.98236$$

So the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$= 0.98236 - 1.2902$$

$$= -0.30785$$

Absolute Relative true error,

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$

$$= \left|\frac{-0.30785}{0.98236}\right| \times 100\,\%$$

$$= 31.338\,\%$$

Example 4:
All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1-\sim)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2®}}e^{-\frac{x^2}{2}}dx$$

a) Use four segment Simpson's 1/3 Rule to find the frequency.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

SOLUTION
a) Using n segment Simpson's 1/3 Rule,

$$(1-\sim)\approx\frac{b-a}{3n}\left[f(x_0)+4\sum_{\substack{i=1\\ i=odd}}^{n-1}f(x_i)+2\sum_{\substack{i=2\\ i=even}}^{n-2}f(x_i)+f(x_n)\right]$$

$$n=4$$
$$a=-2.15$$
$$b=2.9$$
$$h=\frac{b-a}{n}$$
$$=\frac{2.9-(-2.15)}{4}$$
$$=1.2625$$
$$f(x)=\frac{1}{\sqrt{2®}}e^{-\frac{x^2}{2}}$$

So

$$f(x_0)=f(-2.15)$$
$$f(-2.15)=\frac{1}{\sqrt{2®}}e^{-\frac{(-2.15)^2}{2}}$$
$$=0.03955$$

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$$f(x_1) = f(-2.15 + 1.265)$$
$$= f(-0.8875)$$
$$= \frac{1}{\sqrt{2®}} e^{-\frac{(-0.8875)^2}{2}}$$
$$= 0.26907$$

$$f(x_2) = f(-0.8875 + 1.2625)$$
$$= f(0.375)$$
$$= \frac{1}{\sqrt{2®}} e^{-\frac{(0.375)^2}{2}}$$
$$= 0.37186$$

$$f(x_3) = f(0.375 + 1.2625)$$
$$= f(1.6375)$$
$$= \frac{1}{\sqrt{2®}} e^{-\frac{(1.6375)^2}{2}}$$
$$= 0.10439$$

$$f(x_4) = f(x_n)$$
$$= f(2.9)$$
$$= \frac{1}{\sqrt{2®}} e^{-\frac{(2.9)^2}{2}}$$
$$= 0.0059525$$

$$(1-\sim) \approx \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\approx \frac{2.9-(-2.15)}{3(4)}\left[f(-2.15) + 4\sum_{\substack{i=1\\ i=odd}}^{3} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{2} f(x_i) + f(2.9) \right]$$

$$\approx \frac{5.05}{12}\left[f(-2.15) + 4f(x_1) + 4f(x_3) + 2f(x_2) + f(2.9) \right]$$

$$\approx \frac{5.05}{12}\left[f(-2.15) + 4f(-0.8875) + 4f(1.6375) + 2f(0.375) + f(2.9) \right]$$

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$$\approx \frac{5.05}{12}\left[0.03955 + 4(0.26907) + 4(0.10439) + 2((0.37186)) + 0.0059525\right]$$
$$\approx 0.96079$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value.

$$(1-\sim) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2®}} e^{-\frac{x^2}{2}} dx$$
$$= 0.98236$$

So the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 0.98236 - 0.96079$$
$$= 0.021568$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{0.021568}{0.98236}\right| \times 100\,\%$$
$$= 2.1955\,\%$$

TABLE 1 Values of Simpson's 1/3 Rule for Example 2 with multiple segments.

n	Approximate Value	E_t	$\lvert\in_t\rvert\,\%$
2	1.2902	0.30785	31.338
4	0.96079	0.021568	2.1955
6	0.98168	0.00068166	0.069391
8	0.98212	0.00023561	0.023984
10	0.98226	0.000092440	0.0094101

SIMPSON'S 3/8 RULE:

Putting n = 3 in Newton – cotes formula

$$= \frac{3h}{8}(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\ldots.+y_{n-1})+2(y_3+ y_6+y_9+\ldots.+y_n) \quad \ldots..(2)$$

Equation (2) is called *Simpson's three – eighths rule* which is applicable only when n is a multiple of 3.Truncation error in simpson's rule is of the order h

Example:

Evaluate $\int_{-3}^{3} x^4 dx$ by using (1) trapezoidal rule (2)simpson's rule. Verify your results by actual integration.

Solution.

Here $y(x) = x^4$. Interval length$(b - a) = 6$. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

We form below the table

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

(i) By trapezoidal rule:

$\int_{-3}^{3} y\,dx = \frac{h}{2}$ [(sum of the first and the last ordinates) +

2(sum of the remaining ordinates)]

$= \frac{1}{2}$ [(81+81)+2(16+1+0+1+16)]

=115

(ii) By simpson's one - third rule (since number of ordinates is odd)**:**

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 2(1+1) + 4(16+0+16)]

= 98.

(iii) Since $n = 6$, (multiple of three), we can also use **simpson's three - eighths rule.** By this rule,

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 3(16+1+1+16) + 2(0)]

= 99

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KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE-21

DEPARTMENT OF PHYSICS

CLASS : I B.SC PHYSICS

BATCH: 2018-2021

PART A : MULTIPLE CHOICE QUESTIONS (ONLINE EXAMINATIONS)

SUBJECT : MATHEMATICAL PHYSICS - II

SUBJECT CODE : 18PHU203

UNIT III

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
By putting n = 3 in Newton cote's formula we get ----------- rule.	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Simpson's ⅜ rule
By putting n = 2 in Newton cote's formula we get ----------- rule.	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Simpson's 1/3 rule
I = h/2 [y0 + 2(y1 + y2 + y3 +……….+ yn – 1) + yn] is known as ------------- rule.	Simpson's 1/3	Simpson's ⅜	Trapezoidal	Romberg	Trapezoidal
By putting n = 1 in Newton cote's formula we get ----------- rule.	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Trapezoidal rule
I = (3h / 8) { $(y_0 + y_n) + 3 (y_1 + y_2 + y_4 + y_5 + \ldots..)+2(y_3 + y_6 + y_9 + \ldots.)$ } is known as ------------------.	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Simpson's ⅜ rule
I = (h / 3) { (y0 + yn) + 2 (y2+ y4 + y6 + y8 + …..)+ 4(y1 + y3 + y5 + …..) } is known as ------------------.	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Simpson's 1/3 rule
If the given integral is approximated by the sum of 'n' trapezoids, then the rule is called as ----------------.	Newton's method	Trapezoidal rule	Simpson's rule	weddles	Trapezoidal rule
Simpson's rule is exact for a ----------------- even though it was derived for a	Cubic	less than cubic	linear	quadratic	linear
Simpson's ⅜ rule can be applied only if the number of sub interval is in ------ .	Equal	even	multiple of three	unequal.	multiple of three
Simpson's one-third rule on numerical integration is called a --------- formula.	Closed	open	semi closed	semi opened	Closed
While evaluating the definite integral by Trapezoidal rule, the accuracy can be increased by taking	large number of sub-intervals	even number of sub intervals	h=4	has a multiple of 3	large number of sub-intervals
In application of Simpson's 1/3 rule the interval for closer approximation should be	even	small	odd	even and small	even
Numerical integration when applied to a function of a single variable, it is	maxima	minima	quadrature	quadrant	quadrature
Two point Gaussian Quadrature formula is exact for polynomial uo to degree	3	5	2	4	3
Three point Gaussian quadrature formula is exact for polynomial up to degree	1	4	3	5	5
The two-segment trapezoidal rule of integration is exact for integrating at most _____ order polynomials	first	second	third	fourth	first
The highest order of polynomial integrand for which simpson's 1/3 rule of integration is exact is ___	first	second	third	fourth	third
While applying Simpsons 3/8 rule the number of subintervals should be	odd	8	even	multiple of 3	multiple of 3
Trapezoidal and simpson's rules can be used to evaluate	double integrals	differentiation	multiple integrals	divided difference	multiple integrals
The value of integral ex is evaluated from 0 to 0.4 by the following formula. Which method will give the least error?	Trapezoidal rule with h=0.2	Trapezoidal rule with h=0.1	Simpson's 1/3 rule with h=0.1	Simpson's 1/3 rule with h=0.2	Simpson's 1/3 rule with h=0.1
The results obtained by using Simpsons rule will be greater than those obtained by using the trapezoidal rule	in all the cases	provided the intervals are small	provided the boundary is concave towards the base lin	provided the boundary is convex towards the base line	provided the boundary is convex towards the base line
If the determinant of coefficients is not very small, Gaussian elimination	gives no solution	gives incorrect solution	gives solution	sometime give solution	gives solution
Using the trapezoidal rule, what is the area under the curve y =√x from x = 1 to x = 3, using 4 subintervals	2.61	2.793	2.797	2.8	2.793
If four equal subdivisions of [– 2, 6] are used, what is the trapezoidal approximation of ∫ln(x2+1)dx ?	4ln5 + 2 ln17	2ln5 + 2 ln 17 + 2 ln 37	3 ln 5 + 2 ln 17 + ln 37	6 ln 5 + 3 ln 17 + 2 ln 37	3 ln 5 + 2 ln 17 + ln 37
Using the trapezoidal rule, what is the area under the curve y= 2x- x2 from x = 1 to x = 2, using 4 subintervals?	0.53125	0.65625	0.66667	0.67187	0.65625
Using the trapezoidal rule, what is the area under the curve y = sinx from x = 0 to x = π , using 4 subintervals?	1.896	1.948	2	2.052	1.896
Using the trapezoidal rule, what is the area under the curve y=x2+ x from x = 0 to x = 3, using 6 subintervals?	10.625	13.4375	13.5	13.625	13.625
What is an approximation for the area under the curve y =3/1+x2 on the interval [0, 3] using the trapezoidal rule with 5 subintervals?	2.932	3.742	3.747	3.75	3.742
What is an approximation for the area under the curve y =1/x on the interval [2, 5] using the trapezoidal rule with 9 subintervals?	0.868	0.915	0.916	0.918	0.918
The formula using (2n) coefficients polynomial of degree (2n-1) is called as	Gauss-Legendre quadrature formula	trapexoidal formul	weddle's rule	taylor's rule	Gauss-Legendre quadrature formula
In simpson's one third rule y(x) is a polynomial of degree	1	2	3	4	2
In simpson's one third rule the number of ordinates must be	even	odd	even or odd	0	odd
In simpson's three-eigth rule y(x) is a polynomial of degree	1	2	3	4	3
In Weddle's rule y(x) is a polynomial of degree	2	4	6	8	6
In Weddle's rule the number of ordinates must be	2	4	6	7	7
While applying Weddle's rule the number of intervals should be	even	multiple of 6	multiple of three	odd	multiple of 6
If there are only 7 ordinatesin weddle's rule the coefficients are	1,5,1,6,1,5,1	1,6,1,5,1,5,1	5,1,6,1,5,1	1,5,1,6,1	1,5,1,6,1,5,1
The coefficients of first group in weddle's rule is	1,5,1,6,1,5	5,1,6,1,5	1,5,1,5	1,1,6,1,5	1,5,1,6,1,5
In weddle's rule the coefficients may be remembered in groups of	2	3	6	9	6
h2 is the order of error in _____	Simpson's 1/3 rule	Simpson's ⅜ rule	Trapezoidal rule	weddles	Trapezoidal rule
The order of error in simpsons 1/3 rule is	h	h2	h3	h4	h4
The accuracy of the result can be increased by ___	repetition	number of intervals	step-by step	0	number of intervals
The accuracy of the result can be increased by ___	decreasing the value of h	increasing the value of h	repetition	step-by step	decreasing the value of h

Though y2 has suffix evven, it is _________ ordinate	first	second	third	fourth	third
Which rule is applicable only when n is a multiple of 3?	Simpson’s 1/3 rule	Simpson’s ⅜ rule	Trapezoidal rule	weddles	Simpson’s ⅜ rule
By putting n = 6 in Newton cote’s formula we get ----------- rule.	Simpson’s 1/3 rule	Simpson’s ⅜ rule	Trapezoidal rule	weddles	weddles
The approximate value of ∫0^1=dx/1+x^2	$\log_e 2$	π	e	$\log_{10} 2$	π
Interpolating polynomial is also called as _______	collocation polynomial	smoothing function	extrpolation	interpolating formula	collocation polynomial
In Newton’s forward interpolation formula, the first _________ terms will give the linear interpolation	2	3	4	5	2
In Newton’s forward interpolation formula, the ________terms will give the parabolic interpolation	2	3	4	5	3
The elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal is called ------------.	Gauss elimination	Gauss Jordan	Gauss Jacobi	Gauss Seidal	Gauss Jordan
In Gauss Jordan method, we get the solution ---------------------.	without using back substitution method	by using back substitution method	by using forward substitution method	without using forward substitution method	without using back substitution method
If the coefficient matrix is diagonally dominant, then ---------- method converges quickly.	Gauss elimination	Gauss Jordan	direct	Gauss Seidal	Gauss Seidal
Which is the condition to apply Jocobi’s method to solve a system of equations.	First row is dominant	First column is dominant	Diagonally dominant	upper triangular matrix	Diagonally dominant
Iterative method is a ------------ method.	Direct	indirect	step by step	difficult	indirect

UNIT-IV

SYLLABUS

Arithmetic mean - Median - Quartiles - Deciles - Percentiles - Mode - Empirical relation between mean, median and mode - Geometric mean, harmonic mean - Relation between arithmetic mean, geometric mean and harmonic mean - Range - Range meanor average deviation - Standard deviation - Variance and mean square deviation.

ARITHMETIC MEAN:

The arithmetic mean of a set of values is the quantity commonly called "the" mean or the average. Given a set of samples $\{x_i\}$, the arithmetic mean is

$$\bar{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i. \quad (1)$$

It can be computed in the Wolfram Language using Mean[list].

The arithmetic mean is the special case M_1 of the power mean and is one of the Pythagorean means.

When viewed as an estimator for the mean of the underlying distribution (known as the population mean), the arithmetic mean of a sample is called the sample mean.

For a continuous distribution function, the arithmetic mean of the population, denoted μ, $\bar{x}$, $\langle x \rangle$, or $A(x)$ and called the population mean of the distribution, is given by

$$\mu \equiv \int_{-\infty}^{\infty} P(x) f(x)\, dx, \quad (2)$$

where $\langle x \rangle$ is the expectation value. Similarly, for a discrete distribution,

$$\mu \equiv \sum_{n=1}^{N} P(x_n) f(x_n). \quad (3)$$

The arithmetic mean satisfies

$$\langle f(x) + g(x) \rangle = \langle f(x) \rangle + \langle g(x) \rangle \quad (4)$$

$$\langle c\, f(x) \rangle = c \langle f(x) \rangle, \quad (5)$$

and

$$\langle f(x)\, g(y) \rangle = \langle f(x) \rangle \langle g(y) \rangle \quad (6)$$

if x and y are independent statistics. The "sample mean," which is the mean estimated from a statistical sample, is an unbiased estimator for the population mean.

Hoehn and Niven (1985) show that

$$A(a_1 + c, a_2 + c, \ldots, a_n + c) = c + A(a_1, a_2, \ldots, a_n) \quad (7)$$

for any constant c. For positive arguments, the arithmetic mean satisfies

$$A \geq G \geq H, \quad (8)$$

where G is the geometric mean and H is the harmonic mean (Hardy et al. 1952, Mitrinović 1970, Beckenbach and Bellman 1983, Bullen et al. 1988, Mitrinović et al. 1993, Alzer 1996). This can be shown as follows. For $a, b > 0$,

$$\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right)^2 \geq 0 \qquad (9)$$

$$\frac{1}{a} - \frac{2}{\sqrt{a\,b}} + \frac{1}{b} \geq 0 \qquad (10)$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{a\,b}} \qquad (11)$$

$$\sqrt{a\,b} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} \qquad (12)$$

$$G \geq H, \qquad (13)$$

with equality iff $b = a$. To show the second part of the inequality,

$$\left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{a\,b} + b \geq 0 \qquad (14)$$

$$\frac{a+b}{2} \geq \sqrt{a\,b} \qquad (15)$$

$$A \geq G, \qquad (16)$$

with equality iff $a = b$. Combining (◇) and (◇) then gives (◇).

Given n independent random normally distributed variates X_i, each with population mean $\mu_i = \mu$ and variance $\sigma_i^2 = \sigma^2$,

$$\bar{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i \qquad (17)$$

$$\langle \bar{x} \rangle = \frac{1}{N}\left\langle \sum_{i=1}^{N} x_i \right\rangle \qquad (18)$$

$$= \frac{1}{N}\sum_{i=1}^{N} \langle x_i \rangle \qquad (19)$$

$$= \frac{1}{N}\sum_{i=1}^{N} \mu \qquad (20)$$

$$= \frac{1}{N}(N\,\mu) \qquad (21)$$

$$= \mu, \qquad (22)$$

so the sample mean is an unbiased estimator of the population mean. However, the distribution of $\bar{x}$ depends on the sample size. For large samples, $\bar{x}$ is approximately normal. For small samples, Student's t-distribution should be used.

The variance of the sample mean is independent of the distribution, and is given by

$$\text{var}(\bar{x}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{N} x_i\right) \qquad (23)$$

$$= \frac{1}{N^2} \operatorname{var}\left(\sum_{i=1}^{N} x_i\right) \quad (24)$$

$$= \frac{1}{N^2} \sum_{i=1}^{n} \operatorname{var}(x_i) \quad (25)$$

$$= \left(\frac{1}{N^2}\right) \sum_{i=1}^{N} \sigma^2 \quad (26)$$

$$= \frac{\sigma^2}{N}. \quad (27)$$

For small samples, the sample mean is a more efficient estimator of the population mean than the statistical median, and approximately $\pi/2$ less (Kenney and Keeping 1962, p. 211). Here, an estimator of a parameter of a probability distribution is said to be more efficient than another one if it has a smaller variance. In this case, the variance of the sample mean is generally less than the variance of the sample median. The relative efficiency of two estimators is the ratio of this variance.
A general expression that often holds approximately is

$$\text{mean} - \text{mode} \approx 3\,(\text{mean} - \text{median}) \quad (28)$$

MEAN, MEDIAN, MODE AND RANGE:
Mean, median, and mode are three kinds of "averages". There are many "averages" in statistics, but these are, I think, the three most common, and are certainly the three you are most likely to encounter in your pre-statistics courses, if the topic comes up at all.
The "mean" is the "average" you're used to, where you add up all the numbers and then divide by the number of numbers. The "median" is the "middle" value in the list of numbers. To find the median, your numbers have to be listed in numerical order from smallest to largest, so you may have to rewrite your list before you can find the median. The "mode" is the value that occurs most often. If no number in the list is repeated, then there is no mode for the list.
The "range" of a list a numbers is just the difference between the largest and smallest values.

Example:
Find the mean, median, mode, and range for the following list of values:
13, 18, 13, 14, 13, 16, 14, 21, 13
The mean is the usual average, so I'll add and then divide:
(13 + 18 + 13 + 14 + 13 + 16 + 14 + 21 + 13) ÷ 9 = 15
Note that the mean, in this case, isn't a value from the original list. This is a common result. You should not assume that your mean will be one of your original numbers.
The median is the middle value, so first I'll have to rewrite the list in numerical order:
13, 13, 13, 13, 14, 14, 16, 18, 21
There are nine numbers in the list, so the middle one will be the (9 + 1) ÷ 2 = 10 ÷ 2 = 5th number:
13, 13, 13, 13, 14, 14, 16, 18, 21
So the median is 14.

The mode is the number that is repeated more often than any other, so 13 is the mode.
The largest value in the list is 21, and the smallest is 13, so the range is 21 – 13 = 8.
mean: 15
median: 14
mode: 13
range: 8

Note: The formula for the place to find the median is "([the number of data points] + 1) ÷ 2", but you don't have to use this formula. You can just count in from both ends of the list until you meet in the middle, if you prefer, especially if your list is short. Either way will work.

Example:
Find the mean, median, mode, and range for the following list of values:
1, 2, 4, 7
The mean is the usual average:
(1 + 2 + 4 + 7) ÷ 4 = 14 ÷ 4 = 3.5
The median is the middle number. In this example, the numbers are already listed in numerical order, so I don't have to rewrite the list. But there is no "middle" number, because there are an even number of numbers. Because of this, the median of the list will be the mean (that is, the usual average) of the middle two values within the list. The middle two numbers are 2 and 4, so:
(2 + 4) ÷ 2 = 6 ÷ 2 = 3
So the median of this list is 3, a value that isn't in the list at all.
The mode is the number that is repeated most often, but all the numbers in this list appear only once, so there is no mode.
The largest value in the list is 7, the smallest is 1, and their difference is 6, so the range is 6.
mean: 3.5
median: 3
mode: none
range: 6
The values in the list above were all whole numbers, but the mean of the list was a decimal value. Getting a decimal value for the mean (or for the median, if you have an even number of data points) is perfectly okay; don't round your answers to try to match the format of the other numbers.

Example:
Find the mean, median, mode, and range for the following list of values:
8, 9, 10, 10, 10, 11, 11, 11, 12, 13
The mean is the usual average, so I'll add up and then divide:
(8 + 9 + 10 + 10 + 10 + 11 + 11 + 11 + 12 + 13) ÷ 10 = 105 ÷ 10 = 10.5
The median is the middle value. In a list of ten values, that will be the (10 + 1) ÷ 2 = 5.5-th value; the formula is reminding me, with that "point-five", that I'll need to average the fifth and sixth numbers to find the median. The fifth and sixth numbers are the last 10 and the first 11, so:
(10 + 11) ÷ 2 = 21 ÷ 2 = 10.5

The mode is the number repeated most often. This list has two values that are repeated three times; namely, 10 and 11, each repeated three times.
The largest value is 13 and the smallest is 8, so the range is 13 – 8 = 5.
mean: 10.5
median: 10.5
modes: 10 and 11
range: 5
As you can see, it is possible for two of the averages (the mean and the median, in this case) to have the same value. But this is not usual, and you should not expect it.

Example:
A student has gotten the following grades on his tests: 87, 95, 76, and 88. He wants an 85 or better overall. What is the minimum grade he must get on the last test in order to achieve that average?
The minimum grade is what I need to find. To find the average of all his grades (the known ones, plus the unknown one), I have to add up all the grades, and then divide by the number of grades. Since I don't have a score for the last test yet, I'll use a variable to stand for this unknown value: "x". Then computation to find the desired average is:
$(87 + 95 + 76 + 88 + x) \div 5 = 85$
Multiplying through by 5 and simplifying, I get:
$87 + 95 + 76 + 88 + x = 425$
$346 + x = 425$
$x = 79$
He needs to get at least a 79 on the last test.

QUARTILES:
Quartiles are values that divide a sample of data into four equal parts. With them you can quickly evaluate a data set's spread and central tendency, which are important first steps in understanding your data.

Quartile	Description
1st quartile (Q1)	25% of the data are less than or equal to this value.
2nd quartile (Q2)	The median. 50% of the data are less than or equal to this value.
3rd quartile (Q3)	75% of the data are less than or equal to this value.
Interquartile range	The distance between the 1st and 3rd quartiles (Q3-Q1); thus, it spans the middle 50% of the data.

For example, for the following data: 7, 9, 16, 36, 39, 45, 45, 46, 48, 51

Q1 = 14.25
Q2 (median) = 42
Q3 = 46.50
Interquartile range = 14.25 to 46.50, or 32.25
Note:
Quartiles are calculated values, not observations in the data. It is often necessary to interpolate between two observations to calculate a quartile accurately.
Because they are not affected by extreme observations, the median and interquartile range are a better measure of central tendency and spread for highly skewed data than are the mean and standard deviation.

DECILES:
Deciles are the partition values which divide the set of observations into ten equal parts. There are nine deciles: D1,D2,D3,…,D9D1,D2,D3,…,D9. The first decile is D1D1, which is a point which has 10% of the observations below it.

D1=Value of (n+110)thitem
D2=Value of 2(n+110)thitem
D3=Value of 3(n+110)thitem
⋮
D9=Value of 9(n+110)thitem
Quartile for a Frequency Distribution (Discrete Data)
D1=Value of (n+110)thitem(n=∑f)
D2=Value of 2(n+110)thitem
D3=Value of 3(n+110)thitem
⋮
D9=Value of 9(n+110)thitem
Quartile for Grouped Frequency Distribution
D1 = l+hf(n10−c) (n=∑f)
D2=l+hf(2n10−c)
D3=l+hf(3n10−c)
⋮
D9=l+hf(9n10−c)D1 = l+hf(n10−c) (n=∑f)D2=l+hf(2n10−c)D3=l+hf(3n10−c)⋮D9=l+hf(9n10−c)

PERCENTILES:
Percentiles are the points which divide the set of observations into one hundred equal parts. These points are denoted by P1,P2,P3,…,P99P1,P2,P3,…,P99, and are called the first, second, third... ninety ninth percentile. The percentiles are calculated for a very large number of observations like workers in factories and the populations in provinces or countries. Percentiles are usually calculated for grouped data. The first percentile denoted by P1P1 is calculated as P1=Value of (n100)thitemP1=Value of (n100)thitem. We find the group in which the (n100)th(n100)th item lies and then P1P1 is interpolated from the formula.

P1 = l+hf(n100−c) (n=∑f)
P2=l+hf(2n100−c)
P3=l+hf(3n100−c)
⋮
P99=l+hf(99n100−c)

GEOMETRIC MEAN:

The geometric mean of a sequence $\{a_i\}_{i=1}^{n}$ is defined by

$$G(a_1, \ldots, a_n) \equiv \left(\prod_{i=1}^{n} a_i\right)^{1/n}. \quad (1)$$

Thus,

$$G(a_1, a_2) = \sqrt{a_1 a_2} \quad (2)$$

$$G(a_1, a_2, a_3) = (a_1 a_2 a_3)^{1/3}, \quad (3)$$

and so on.

The geometric mean of a list of numbers may be computed using GeometricMean[list] in the Wolfram Languagepackage DescriptiveStatistics` .

For $n = 2$, the geometric mean is related to the arithmetic mean A and harmonic mean H by

$$G = \sqrt{A H} \quad (4)$$

(Havil 2003, p. 120).

The geometric mean is the special case M_0 of the power mean and is one of the Pythagorean means.

Hoehn and Niven (1985) show that

$$G(a_1 + c, a_2 + c, \ldots, a_n + c) > c + G(a_1, a_2, \ldots, a_n) \quad (5)$$

for any positive constant c.

HARMONIC MEAN:

The harmonic mean $H(x_1, \ldots, x_n)$ of n numbers x_i (where $i = 1, \ldots, n$) is the number H defined by

$$\frac{1}{H} \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}. \quad (1)$$

The harmonic mean of a list of numbers may be computed in the Wolfram Language using HarmonicMean[list].

The special cases of $n = 2$ and $n = 3$ are therefore given by

$$H(x_1, x_2) = \frac{2 x_1 x_2}{x_1 + x_2} \quad (2)$$

$$H(x_1, x_2, x_3) = \frac{3 x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3}, \quad (3)$$

and so on.

The harmonic means of the integers from 1 to n for $n = 1$, 2, ... are 1, 4/3, 18/11, 48/25, 300/137, 120/49, 980/363, ... (OEIS A102928 and A001008).

For $n = 2$, the harmonic mean is related to the arithmetic mean A and geometric mean G by

$$H = \frac{G^2}{A} \tag{4}$$

The harmonic mean is the special case M_{-1} of the power mean and is one of the Pythagorean means. In older literature, it is sometimes called the subcontrary mean.
The volume-to-surface area ratio for a cylindrical container with height h and radius r and the mean curvature of a general surface are related to the harmonic mean.
Hoehn and Niven (1985) show that

$$H(a_1 + c, a_2 + c, \ldots, a_n + c) > c + H(a_1, a_2, \ldots, a_n) \tag{5}$$

for any positive constant c .

RELATION BETWEEN ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN:

For two numbers x and y, let x, a, y be a sequence of three numbers. If x, a, y is an arithmetic progression then 'a' is called arithmetic mean. If x, a, y is a geometric progression then 'a' is called geometric mean. If x, a, y form a harmonic progression then 'a' is called harmonic mean.

Let AM = arithmetic mean, GM = geometric mean, and HM = harmonic mean. The relationship between the three is given by the formula

AM×HM=GM2AM×HM=GM2

Below is the derivation of this relationship.
Derivation of AM × HM = GM2
Arithmetic mean:
x,AM,yx,AM,y → arithmetic progression

Taking the common difference of arithmetic progression,
AM−x=y−AMAM−x=y−AM
x+y=2AMx+y=2AM → Equation (1)

Geometric Progression
x,GM,yx,GM,y → geometric progression

The common ratio of this geometric progression is
GMx=yGMGMx=yGM
xy=GM2xy=GM2 → Equation (2)
Harmonic Progression
x,HM,yx,HM,y → harmonic progression
1x,1HM,1y1x,1HM,1y → the reciprocal of each term will form an arithmetic progression

The common difference is
1HM−1x=1y−1HM1HM−1x=1y−1HM

2HM=1y+1x2HM=1y+1x
2HM=x+yxy2HM=x+yxy → Equation (3)

Substitute x + y = 2AM from Equation (1) and xy = GM2 from Equation (2) to Equation (3)
2HM=2AMGM22HM=2AMGM2
GM2=AM×HMGM2=AM×HM

Range mean or average deviation , Standard deviation , Variance and mean square deviation:
Mean is a measure of central tendency. It measures what the majority of the data are doing toward the middle of a set. The mean is often referred to as the average of a data set. As an example, an algebra class has 10 students. Their grades on the last test were 85, 90, 87, 93, 100, 53, 78, 85, 99 and 82. What is the average grade for the students? To find mean, simply add all the numbers in a data set and divide by the number of items in the set:
85 + 90 + 87 + 93 + 100 + 53 + 78 + 85 + 99 + 82 = 852 852 / 10 = 85.2
The average, or mean, test grade in the class is 85.2.
Mode Occurs Most
Mode is another measure of central tendency. The mode is just the number that occurs most frequently. It's easy to remember because mode and most sound alike. Using the algebra class example, what grade occurred most frequently among the students? To answer, put the values in order:
53, 78, 82, 85, 85, 87, 90, 93, 99, 100
The only grade that occurred more than once is 85. Since 85 occurred most, the mode is 85.
Median Is the Middle, Range Is the Spread
Median is another measure of central tendency. The median is simply the middlenumber of a set. Put the numbers in order and look for one in the middle. If there is no middle number, add the two in the center and divide by 2. In the algebra class example, what is the median grade? To answer, put the values in order:
53, 78, 82, 85, 85, 87, 90, 93, 99, 100
Since there are an even number of test grades, there is no middle number. The two test grades in the middle are 85 and 87. Add them and divide by 2:
85 + 87 = 172 172 / 2 = 86
The median, or middle grade, is 86.
Range is a quick calculation. Range is simply the largest value minus the smallest. It shows you how spread out the numbers are. For these grades, subtract 53 from 100 to get the range of 47.

STANDARD DEVIATION:
Standard deviation is the square root of the variance, so you must find the variance first. Variance is the average of the squared difference of each number from the mean. That may sound confusing, but it's pretty simple to do. Take each number in the set and subtract if from the mean. Then square it. Add those values together, and divide by the number of items in your set. Working with the algebra class grades again, subtract each one from the mean:
85.2 - 53 = 32.2 85.2 - 78 = 7.2 85.2 - 82 = 3.2 85.2 - 85 = 0.2 85.2 - 85 = 0.2 85.2 - 87 = -1.8 85.2 - 90 = -4.8 85.2 - 93 = -7.8 85.2 - 99 = -13.8 85.2 - 100 = 14.8

Square each of those values, then add them together:
1,036.84 + 51.84 + 10.24 + 0.04 + 0.04 + 3.24 + 23.04 + 60.84 + 190.44 + 219.04 = 1,595.6
Finally, divide that sum by the number of items in the set, in this case 10:
1,595.6 / 10 = 159.56
The variance for this data set is 159.56.
Standard Deviation Measures Spread
Standard deviation is the measure of how spread out the numbers are from the center of a data set. A small standard deviation means a lot of the numbers are grouped around the middle of the set. A large standard deviation means that the number are spread out with some very high and low numbers. With the algebra grades, use this equation:
square root (159.56) = 12.63
The standard deviation for this data set is 12.63.

KAHE

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE-21
DEPARTMENT OF PHYSICS
CLASS : I B.SC PHYSICS
BATCH: 2018-2021
PART A : MULTIPLE CHOICE QUESTIONS (ONLINE EXAMINATIONS)
SUBJECT : MATHEMATICAL PHYSICS - II
SUBJECT CODE : 18PHU203
UNIT IV

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
Arithmetic mean between a and 10 is 30, value of ‘a’ should be	60	34	56	50	50
hree arithmetic means between 11 and 19 are	12, 14, 15	13, 15, 17	13, 14, 15	14, 15, 17	13, 15, 17
Arithmetic mean between √5 and 3√5 is	2√5	4√5	√5	5	2√5
rithmetic mean between −4 and 12 is	5	4	-4	3	-4
Arithmetic mean between x -1 and x + 7 is	X+4	X-3	X+3	X-4	X+3
he mean of eight numbers is 25. If five is subtracted from each number, what will be the new mean?	30	29	67	20	20
The mean of 14 numbers is 6. If 3 is added to every number, what will be the new mean?	8	9	7	6	9
The heights of five runners are 160 cm, 137 cm, 149 cm, 153 cm and 161 cm respectively. Find the mean height per runner.	152	153	156	123	152
Find the mean of the first five prime numbers.	8.9	6.5	5.6	3.7	5.6
Find the arithmetic mean of the first 7 natural numbers.	5	6	7	4	4
Find the mean of the first six multiples of 4.	12	13	14	15	14
If the mean of 9, 8, 10, x, 12 is 15, find the value of x.	23	36	45	63	36
he mean of 40 numbers was found to be 38. Later on, it was detected that a number 56 was misread as 36. Find the correct mean of given numbers.	45	36	38.5	43	38.5
Median of 7, 6, 4, 8, 2, 5, 11 is	6	12	11	4	6
Number which occurs most frequently in a set of numbers is	mean	median	mode	none of the above	mode
Mode of 12, 17, 16, 14, 13, 16, 11, 14 is	13	11	14	14 and 16	14 and 16
If mean of 6 numbers is 41 then sum of these numbers is	250	246	134	456	246
If mean of 6 numbers is 17 then sum of numbers is	102	103	150	120	102
Difference of mode and mean is equal to	3(mean-median)	2(mean-median)	3(mean-mode)	2(mode mean)	3(mean-median)
If mean is 11 and median is 13 then value of mode is	15	13	11	17	17
Distribution in which values of median, mean and mode are not equal is considered as	experimental distribution	asymmetrical distribution	symmetrical distribution	exploratory distribution	asymmetrical distribution
If value of three measures of central tendencies median, mean and mode then distribution is considered as	negatively skewed modal	triangular model	unimodel	bimodel	unimodel
If value of mode is 14 and value of arithmetic mean is 5 then value of median is	12	18	8	14	8
The mean of a distribution is 14 and the standard deviation is 5. What is the value of the coecient of variation?	60.40%	48.30%	35.70%	27.80%	35.70%
Most frequent observation in a data set is called	mode	median	mode	range	mode
Summary statistics which measure middle or center of data are called	logarithms	measures of central tendency	measures of dispersion	proportions	measures of central tendency
Sum of deviations of values from their mean is always	0	1	2	3	0
Average of all observations in a set of data is known as	mean	mode	range	median	mean
Median in set 6, 4, 2, 3, 4, 5, 5, 4 would be	3	4	5	6	4
Find the median of the set of numbers: 1,2,3,4,5,6,7,8,9 and 10.	55	10	1	5.5	5.5
Find the median of the set of numbers: 21, 3, 7, 17, 19, 31, 46, 20 and 43.	19	20	3	167	20
Find the median of the set of numbers: 100, 200, 450, 29, 1029, 300 and 2001.	300	29	7	4080	300
The following represents age distribution of students in an elementary class. Find the mode of the values: 7, 9, 10, 13, 11, 7, 9, 19, 12, 11, 9, 7, 9, 10, 11.	7	9	10	11	9
Find the mode from these test results: 90, 80, 77, 86, 90, 91, 77, 66, 69, 65, 43, 65, 75, 43, 90.	43	77	65	90	90
Find the mode from these test results: 17, 19, 18, 17, 18, 19, 11, 17, 16, 19, 15, 15, 15, 17, 13, 11.	15	11	17	19	17
Find the mean of these set of numbers: 100, 1050, 320, 600 and 150.	333	444	440	320	444
The following numbers represent the ages of people on a bus: 3, 6, 27, 13, 6, 8, 12, 20, 5, 10. Calculate their mean of their ages.	11	6	9	110	11
These numbers are taken from the number of people that attended a particular church every Friday for 7 weeks: 62, 18, 39, 13, 16, 37, 25. Find the mean.	25	210	62	30	30
Median, mode, deciles and percentiles are all considered as measures of	mathematical averages	population averages	sample averages	averages of position	averages of position
Quartiles, median, percentiles and deciles are measures of central tendency classified as	paired average	deviation averages	positioned averages	central averages	positioned averages
According to percentiles, median to be measured must lie in	80^{th}	40^{th}	50^{th}	100^{th}	50^{th}
Percentile and moment system are two groups of	skewness measures	central tendencies measures	quartile measures	percentile measures	central tendencies measures
Harmonic mean, arithmetic mean and geometric mean are all considered as	mathematical averages	population averages	sample averages	averages of position	mathematical averages
If arithmetic mean is 25 and harmonic mean is 15 then geometric mean is	17.36	16.36	15.36	19.36	19.36
Manner in which geometric mean, harmonic mean and arithmetic mean are related is as	A.M>G.M>H.M	A.M>G.M<H.M	A.M<G.M<H.M	A.M<G.M>H.M	A.M>G.M>H.M
For individual observations, reciprocal of arithmetic mean is called	geometric mean	harmonic mean	deviation square mean	paired mean	harmonic mean
If arithmetic mean is 20 and harmonic mean is 30 then geometric mean is	14.94	24.94	34.94	44.94	24.94
Value of Σfd is 250, A= 25, number of observations are 12 and width of class interval is 6 then arithmetic mean is	25	250	150	275	150
In measures of skewness, absolute skewness is equal to	mean+mode	mean-mode	mean+median	mean-median	mean-mode
In a negative skewed distribution, order of mean, median and mode is as	mean<median>mode	mean>median>mode	mean<median<mode	mean>median<mode	mean<median<mode

SYLLABUS

Partial Differential Equations: Solutions to partial differential equations, using separation of variables: Laplace's Equation in problems of rectangular, cylindrical and spherical symmetry. Wave equation and its solution for vibrational modes of a stretched string, rectangular and circular membranes. Diffusion Equation.

An Introduction

A partial differential equation (PDE) is an equation involving an unknown function u of two or more variables and some or all of its partial derivatives. The partial differential equation is usually a mathematical representation of problems arising in nature, around us. The process of understanding physical systems can be divided in to three stages:

(i) Modelling the problem or deriving the mathematical equation (in our case it would be formulating PDE). The derivation process is usually a result of conservation laws or balancing forces.

(ii) Solving the equation (PDE). What do we mean by a solution of the PDE?

(iii) Studying properties of the solution. Usually, we do not end up with a definite formula for the solution. Thus, how much information about the solution can one extract without any knowledge of the formula?

Definitions

Recall that the ordinary differential equations (ODE) dealt with functions of one variable, $u : \Omega \subset R \to R$. The subset Ω could have the interval form (a, b). The derivative of u at $x \in \Omega$ is defined as

$$u'(x) := \lim_{h\to 0} \frac{u(x+h) - u(x)}{h},$$

provided the limit exists. The derivative gives the slope of the tangent line at x ∈ Ω. How to generalise this notion of derivative to a function u : Ω ⊂ R n → R These concepts are introduced in a course on multi-variable calculus. However, we shall jump directly to concepts necessary for us to begin this course.

Let Ω be an open subset of R n and let u : Ω → R be a given function. We denote the directional derivative of u at x ∈ Ω, along a vector ξ ∈ R n , as

$$\frac{\partial u}{\partial \xi}(x) = \lim_{h\to 0} \frac{u(x+h\xi) - u(x)}{h},$$

provided the limit exists. The directional derivative of u at x ∈ Ω, along the standard basis vectors ei = (0, 0, . . . , 1, 0, . . . , 0) is called the i-th partial derivative of u at x and is given as

$$u_{x_i} = \frac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \frac{u(x+he_i) - u(x)}{h}.$$

The order of the PDE is the order of the highest (partial) differential coefficient in the equation.

As with ordinary differential equations (ODEs) it is important to be able to distinguish between linear and nonlinear equations.

A linear equation is one in which the equation and any boundary or initial conditions do not include any product of the dependent variables or their derivatives; an equation that is not linear is a nonlinear equation.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{first order linear PDE (simplest wave equation)},$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Phi(x, y), \quad \text{second order linear PDE (Poisson)}.$$

A nonlinear equation is semilinear if the coefficients of the highest derivative are functions of the independent variables only.

$$(x+3)\frac{\partial u}{\partial x} + xy^2\frac{\partial u}{\partial y} = u^3,$$

$$x\frac{\partial^2 u}{\partial x^2} + (xy + y^2)\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = u^4.$$

A nonlinear PDE of order m is quasilinear if it is linear in the derivatives of order m with coefficients depending only on x, y, . . . and derivatives of order $< m$.

$$\left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y}\,\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial y^2} = 0.$$

Principle of superposition:

A linear equation has the useful property that if u1 and u2 both satisfy the equation then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in R$. This is often used in constructing solutions to linear equations (for example, so as to satisfy boundary or initial conditions; c.f. Fourier series methods). This is not true for nonlinear equations, which helps to make this sort of equations more interesting, but much more difficult to deal with.

Wave Equations

Waves on a string, sound waves, waves on stretch membranes, electromagnetic waves, etc.,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

or more generally

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

where c is a constant (wave speed).

PARTIAL DIFFERENTIAL EQUATIONS:

Method of Separation of Variables for Solving partial Differential Equations

Method of separation of variables is a powerful method for solving partial differential equations of the type

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (1)$$

under certain situations.

The basic idea of this method is to transform a partial differential equation into as many differential equations as the number of independent variables in the partial differential equation by representing the solution as a product of functions of each independent variable. After these ordinary differential equations are solved, the method reduces to solving eigenvalue problems and constructing the general solution as an eigenfunction expansion, where the coefficients are evaluated by using the boundary and initial conditions.

Let u (x,y) = X(x) Y(y)

(2) be a solution of (1) then (1) may be written in the form

$$\frac{1}{X(x)} f(D_x)\ X = \frac{1}{Y(y)}\ g\ (D_y) Y \qquad (3)$$

where $f(D_x)$, $g(D_y)$ are quadratic functions of $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$ respectively. In this situation we say that (1) is separable in the variables x,y. The derivation of a solution of the equation is straight forward. For the left hand side of (3) is a function of x alone, and right-hand is a function of y-alone, and the two can be equal only if each is equal to a constant, say λ. The problem of finding solutions of the form (2) of (1) therefore reduces to solving the pair of second order linear ordinary differential equations

$$f(D)\,X = \lambda\, X(x), \quad g(D)\,Y = \lambda\, Y(y) \qquad (4)$$

Application to Heat Equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = X(x)\,T(t)$

be a solution of the heat equation. Then the last equation can be written as

$$\frac{1}{T}\frac{dT}{dt} = \frac{k}{X}\frac{d^2X}{dx^2} \qquad (5)$$

$$\left[\text{Since } u = X(x)T(t), \text{ we have } \frac{\partial u}{\partial x} = X'(x)\,T(t), \frac{\partial^2 u}{\partial x} = X''(x)\,T(t) = \left(\frac{d^2X}{dx^2}\right) T(t)\right.$$

and $\frac{\partial u}{\partial t} = X(x)\,T'(t) = X(x)\,\frac{dT}{dt}$

Putting these values in the heat equation we get equation 5]. The pair of ordinary differential equations corresponding to (4) is

$$\frac{d^2X}{dx^2} = \lambda X, \quad \frac{dT}{dt} = k\lambda T$$

or
$$\frac{d^2X}{dx^2} - \lambda X = 0 \text{ and } \frac{dT}{dt} - k\lambda T = 0 \qquad (6)$$

Let $\lambda = -n^2$ then by the method discussed in 2.1 we find that $T(t) = Ke^{-kn^2 t}$ is a general solution of the second equation of (6), where K is a constant of integration which can be determined by given initial and boundary conditions. The general solution of the first equation of (6) is given in Section 6.7.

Application to Wave Equation

$$\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{dx^2}$$

Let u(x,t) = X(x) T(t), then

$$\frac{\partial u}{\partial t} = X(x)\ T'(t)\ ,\ \frac{\partial^2 u}{\partial t^2} = X(x)\ T''(t)$$

$$\frac{\partial u}{\partial x} = X'(x)\ T(t)\ ,\ \frac{\partial^2 u}{\partial x^2} = X''(x)\ T(t)$$

Putting these values in the equation we get

$$X(x)\ T''(t) = c^2 X''(x)\ T(t)$$

or $$\frac{1}{c^2}\frac{T''(t)}{T(t)} = -\lambda,\ \frac{X''(x)}{X(x)} = -\lambda$$

or $$T''(t) + c^2 \lambda T = 0,\ X''(x) + \lambda X = 0$$

Application to Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let u(x,y) = X(x) Y(y) be a solution of the equation. Then

$$\frac{\partial u}{\partial x} = X'(x) Y(y)\ ,\ \frac{\partial^2 u}{\partial x^2} = X''(x)\ Y''(y)$$

$$\frac{\partial u}{\partial y} = X(x) Y'(y)\ ,\ \frac{\partial^2 u}{\partial y^2} = X(x)\ Y''(y)$$

Putting these values in the equation we get

$$X''(x)\ Y(y) + X(x)\ Y''(y) = 0$$

or $$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = -n^2$$

or $X''(x) + n^2X=0,\ Y''(y) - n^2Y+0$

Solutions of Partial Differential Equations with Boundary Conditions

In this section we present solutions of the wave, heat and Laplace equations with boundary and initial conditions. We briefly discuss how a physical situation can be written in the form of the wave equation.

The Wave Equation with Initial and Boundary Conditions

Modeling of a Physical Situation

Vibrations in a membrane or drumhead, or oscillations induced in a guitar or violin string, are governed by a partial differential equation called the wave equation. We will derive this equation in a simple setting.

Consider an elastic string stretched between two pegs, as on a guitar. We want to describe the motion of the string if it is given a small displacement and released to vibrate in a plane.

Place the string along the x axis from 0 to l and assume that it vibrates in the x, y plane. We want a function u(x,t) such that at any time t>0, the graph of the function u=u(x,t) of x is the shape of the string at that time. Thus, u (x,t) allows us to take a snapshot of the string at any time, showing it as a curve in the plane. For this reason u(x,t) is called the **position function** for the string. Figure 12.1 shows a typical configuration.

To begin with a simple case, neglect damping forces such as air resistance and the weight of the string and assume that the tension **T**(x,t) in the string always acts tangentially to the string and that individual particles of the string move only vertically. Also assume that the mass ρ per unit length is constant.

Now consider a typical segment of string between x and $x+\Delta x$ and apply Newton's second law of motion to write

Net force on this segment due to the tension

= acceleration of the center of mass of the segment times its mass.

This is a vector equation. For Δx small, the vertical component of this equation (Figure 12.2) gives us approximately.

$$T(x+\Delta x,t)\sin(\theta+\Delta\theta)-T(x,t)\sin(\theta) = \rho\Delta x \frac{\partial^2 u}{\partial t^2}(\bar{x},t),$$

where $\bar{x}$ is the center of mass of the segment and $T(x,t) = \|\mathbf{T}(x,t)\|$=magnitude of **T**.

String Profile at time =t.

Then

$$\frac{T(x+\Delta x,t)\sin(\theta+\Delta\theta)-T(x,t)\ \sin(\theta)}{\Delta x} = \rho\frac{\partial^2 u}{\partial t^2}(\bar{x},t).$$

Now v(x,t)= T(x,t) sin (θ) is the vertical component of the tension, so the last equation becomes

$$\frac{v(x+\Delta x,t)-v(x,t)}{\Delta x} = \rho\frac{\partial^2 u}{\partial t^2}(\bar{x}\,t).$$

In the limit as $\Delta x \to 0$, we also have $\bar{x} \to x$ and the last equation becomes

$$\frac{\partial v}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}. \qquad (7)$$

The horizontal component of the tension is h(x,t) =T(x,t) cos(θ), so

$$v(x,t)=h(x,t)\tan(\theta)=h(x,t)\frac{\partial u}{\partial x}$$

Substitute this into equation (7) to get

$$\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2}(x,t). \qquad (8)$$

To compute the left side of this equation, use the fact that the horizontal component of the tension of the segment is zero, so

$$h(x+\Delta x,t)-h(x,t)=0.$$

Thus h is independent of x and equation 8 can be written

$$h\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

Letting $c^2 = h/\rho$, this equation is often written

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

This is the one-dimensional (1-space dimension) wave equation.

In order to model the string's motion, we need more than just the wave equation. We must also incorporate information about constraints on the ends of the string and about the initial velocity and position of the string, which will obviously influence the motion.

If the ends of the string are fixed, then

$u(0,t)=u(l,t)=0 \quad \text{for } t \geq 0.$

These are the **boundary conditions**.

The **initial conditions** specify the initial (at time zero) position

$u(x,0)=f(x) \quad \text{for } 0 \leq x \leq l$

and the initial velocity

$\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 < x < l,$

in which f and g are given functions satisfying certain compatibility conditions. For example, if the string is fixed at its ends, then the initial position function must reflect this by satisfying

$f(0)=f(l)=0.$

If the initial velocity is zero (the string is released from rest), then $g(x)=0$.

The wave equation, together with the boundary and initial conditions, constitute a boundary value problem for the position function $u(x,t)$ of the string. These provide enough information to uniquely determine the solution $u(x,t)$.

If there is an external force of magnitude F units of force per unit length acting on the string in the vertical direction, then this derivation can be modified to obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho} F.$$

Again, the boundary value problem consists of this wave equation and the boundary and initial conditions.

In 2-space dimensions the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad (9)$$

This equation governs vertical displacements u(x,y,t) of a membrane covering a specified region of the plane (for example, vibrations of a drum surface).

Again, boundary and initial conditions must be given to determine a unique solution. Typically, the frame is fixed on a boundary (the rim of the drum surface), so we would have no displacement of points on the boundary:

u(x,y,t)= 0 for (x,y) on the boundary of the region and t>0.

Further, the initial displacement and initial velocity must be given. These initial conditions have the form

$$u(x,y,0) = f(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

with f and g given.

Sometimes polar coordinates formulation is more convenient. We present below this form. Let

x=r cos(θ), y=r sin(θ).

Then

$r=\sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$.

Let

u(x,y)= u (r cos(θ), r sin (θ)) = v (r, θ).

Compute

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial r}-\frac{y}{x^2+y^2}\frac{\partial v}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial v}{\partial r}-\frac{y}{r^2}\frac{\partial v}{\partial \theta}$$

Then

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial v}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right)-\frac{\partial v}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right)+\frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial r}\right)-\frac{y}{r^2}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \theta}\right)$$

$$= \frac{y^2}{r^3}\frac{\partial v}{\partial r} + \frac{2xy}{r^4}\frac{\partial v}{\partial \theta} + \frac{x^2 \partial^2 v}{r^2 \partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{y^2}{r^4}\frac{\partial^2 v}{\partial \theta^2}.$$

By a similar calculation, we get

$$\frac{\partial u}{\partial y} = \frac{y}{r}\frac{\partial v}{\partial r} + \frac{x}{r^2}\frac{\partial v}{\partial \theta}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3}\frac{\partial v}{\partial r} - \frac{2xy}{r^4}\frac{\partial v}{\partial \theta} + \frac{y^2 \partial^2 v}{r^2 \partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 v}{\partial r \partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 v}{\partial \theta^2}.$$

Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}.$$

Therefore, in polar coordinates, the two-dimensional wave equation (9) is

$$\frac{\partial^2 v}{\partial t^2} = c^2 \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \right). \qquad (10)$$

in which v(r,θ,t) is the vertical displacement of the membrane from the x, y plane at point (r, θ) and time t.

LAPLACE EQUATION IN PROBLEMS OF RECTANGULAR, CYLINDRICAL AND SPHERICAL SYMMETRY:

Laplace's Equation--Spherical Coordinates

In spherical coordinates, the scale factors are $h_r = 1$, $h_\theta = r \sin\phi$, $h_\phi = r$, and the separation functions are $f_1(r) = r^2$, $f_2(\theta) = 1$, $f_3(\phi) = \sin\phi$, giving a Stäckel determinant of $S = 1$.

The Laplacian is

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\right). \qquad (1)$$

To solve Laplace's equation in spherical coordinates, attempt separation of variables by writing

$$F(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi). \qquad (2)$$

Then the Helmholtz differential equation becomes

$$\frac{d^2 R}{d r^2}\Phi\Theta + \frac{2}{r}\frac{d R}{d r}\Phi\Theta + \frac{1}{r^2 \sin^2\phi}\frac{d^2\Theta}{d\theta^2}\Phi R + \frac{\cos\phi}{r^2\sin\phi}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{d^2\Phi}{d\phi^2}\Theta R = 0. \quad (3)$$

Now divide by $R\,\Theta\Phi$,

$$\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi\Theta\frac{d^2 R}{d r^2} + \frac{2}{r}\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi\Theta\frac{d R}{d r} + \frac{1}{r^2\sin^2\phi}\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi R\frac{d^2\Theta}{d\theta^2} +$$
$$\frac{\cos\phi}{r^2\sin\phi}\frac{r^2\sin^2\phi}{\Phi\Theta R}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{r^2\sin^2\phi}{\Phi R\Theta}\frac{d^2\Phi}{d\phi^2}\Theta R = 0 \quad (4)$$

$$\left(\frac{r^2\sin^2\phi}{R}\frac{d^2 R}{d r^2} + \frac{2 r\sin^2\phi}{R}\frac{d R}{d r}\right) + \left(\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}\right)$$
$$+\left(\frac{\cos\phi\sin\phi}{\Phi}\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2}\right) = 0. \quad (5)$$

The solution to the second part of (5) must be sinusoidal, so the differential equation is

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -m^2, \quad (6)$$

which has solutions which may be defined either as a complex function with , ..., ∞

$$\Theta(\theta) = A_m e^{i m\theta}, \quad (7)$$

or as a sum of real sine and cosine functions with $m = -\infty$, ..., ∞

(8)

Plugging (6) back into (7),

$$\frac{r^2}{R}\frac{d^2 R}{d r^2} + \frac{2 r}{R}\frac{d R}{d r} + \frac{1}{\sin^2\phi}\left(\frac{\cos\phi\sin\phi}{\Phi}\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2} - m^2\right) = 0. \quad (9)$$

The radial part must be equal to a constant

$$\frac{r^2}{R}\frac{d^2 R}{d r^2} + \frac{2 r}{R}\frac{d R}{d r} = l(l+1) \quad (10)$$

$$r^2\frac{d^2 R}{d r^2} + 2 r\frac{d R}{d r} = l(l+1) R. \quad (11)$$

But this is the Euler differential equation, so we try a series solution of the form

$$R = \sum_{n=0}^{\infty} a_n r^{n+c}. \quad (12)$$

Then

$$r^2 \sum_{n=0}^{\infty} (n+c)(n+c-1)\, a_n\, r^{n+c-2} + 2r \sum_{n=0}^{\infty} (n+c)\, a_n\, r^{n+c-1} - l(l+1) \sum_{n=0}^{\infty} a_n\, r^{n+c} = 0 \quad (13)$$

$$\sum_{n=0}^{\infty} (n+c)(n+c-1)\, a_n\, r^{n+c} + 2 \sum_{n=0}^{\infty} (n+c)\, a_n\, r^{n+c} - l(l+1) \sum_{n=0}^{\infty} a_n\, r^{n+c} = 0 \quad (14)$$

$$\sum_{n=0}^{\infty} [(n+c)(n+c+1) - l(l+1)]\, a_n\, r^{n+c} = 0. \quad (15)$$

This must hold true for all powers of r. For the r^c term (with $n=0$),

$$c(c+1) = l(l+1), \quad (16)$$

which is true only if $c = l, -l-1$ and all other terms vanish. So $a_n = 0$ for $n \neq l$, $-l-1$. Therefore, the solution of the R component is given by

$$R_l(r) = A_l\, r^l + B_l\, r^{-l-1}. \quad (17)$$

Plugging (17) back into (◇),

$$l(l+1) - \frac{m^2}{\sin^2\phi} + \frac{\cos\phi}{\sin\phi}\frac{1}{\Phi}\frac{d\Phi}{d\phi} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \quad (18)$$

$$\Phi'' + \frac{\cos\phi}{\sin\phi}\Phi' + \left[l(l+1) - \frac{m^2}{\sin^2\phi}\right]\Phi = 0, \quad (19)$$

which is the associated Legendre differential equation for $x = \cos\phi$ and , ..., l. The general complex solution is therefore

where

are the (complex) spherical harmonics. The general real solution is

Some of the normalization constants of can be absorbed by and C_m, so this equation may appear in the form

$$\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(A_l\,r^l + B_l\,r^{-l-1}\right)P_l^m(\cos\phi)\,[S_l^m \sin(m\,\theta) + C_l^m \cos(m\,\theta)]$$
$$\equiv \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(A_l\,r^l + B_l\,r^{-l-1}\right)\times\left[S_l^m\,Y_l^{m(o)}(\theta,\phi) + C_l^m\,Y_l^{m(e)}(\theta,\phi)\right], \qquad (23)$$

where

$$Y_l^{m(o)}(\theta,\phi) \equiv P_l^m(\cos\phi)\sin(m\,\theta) \qquad (24)$$

$$Y_l^{m(e)}(\theta,\phi) \equiv P_l^m(\cos\phi)\cos(m\,\theta) \qquad (25)$$

are the even and odd (real) spherical harmonics. If azimuthal symmetry is present, then $\Theta(\theta)$is constant and the solution of the Φcomponent is a Legendre polynomial $P_l(\cos\phi)$. The general solution is then

$$F(r,\phi) = \sum_{l=0}^{\infty}\left(A_l\,r^l + B_l\,r^{-l-1}\right)P_l(\cos\phi). \qquad (26)$$

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DEPARTMENT OF PHYSICS
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SUBJECT : MATHEMATICAL PHYSICS - II
SUBJECT CODE : 18PHU203
UNIT V

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWER
The Laplace transform of f(t) is denoted by	L { F(s) }	L { f (t) }	L { F(t)	L { f(s) }	L { f (t) }
.L (e^{-at}) = ---	a)1/s+a	1/s-a	1/s * a	1/s	1/s+a
.L (cos h at) = ----	a/s^2 - a^2	s/s^2 - a^2	a/s^2 + a^2	s/s^2 +a^2	s/s^2 - a^2
L (sinh at) = ----	a/s^2 - a^2	s/s^2 - a^2	a/s^2 + a^2	s/s^2 +a^2	a/s^2 - a^2
.L (cosat) = ----	s/s^2 -a^2	a/s^2 +a^2	a/s^2 - a^2	s/s^2 +a^2	s/s^2 +a^2
.L (sinat) = ----	s/s^2 -a^2	a/s^2 +a^2	a/s^2 - a^2	s/s^2 +a^2	a/s^2 +a^2
L (t^n) = ----	é(n+1)/s^{n+1}	é(n-1)/s^{n+1}	é(n+1)/s^{n-1}	none	é(n+1)/sn+1
L(1) = ----	1	s	1/s	0	1/s
L(t) = -----	1/s	1/s^2	t	1/t^2	1/s2
L(t^2) = -----	2/s^3	1/t^2	2/t^3	1/s^2	2/s3
é1/2 = ---	ÖP/2	ÖP/4	ÖP	ÖP/8	ÖP
L (e^{at}) = ----	1/s+a	1/s-a	1/s*a	None	1/s-a
L (t sinat) = -----	2as/ (s^2-a^2)	2as/ ($s^2$$a^2$)	2as/ (s^2+a^2)	None	2as/ (s2+a2)
L (tcos at) = -----	s^2-a^2/(s^2+a^2)^2	s^2 +a^2/(s^2+a^2)^2	s^2-a^2/(s^2+a^2)^2	None	s2-a2/(s2+a2)2
If L^{-1}{1/(s+a)^2 }= ------	t e^{at}	t e^{-at}	$e^{a t}$	None	t e -at
L^{-1} (1 /(s^2 + 4)) is equal to	e^{-4t}	cos2t/2	sin2t/2	e^{4t}	sin2t/2
L^{-1} (1/s) = --------	1	0	t	none.	1
L^{-1} [1/(s+a)] = ----------	$e^{s t}$	$e^{a t}$	$e^{-s t}$	$e^{- a t}$	e s t
The function x sin x be a ------- function.	even	odd	continuous	None	0
The function x cos x be a ------- function.	even	odd	continuous	None	x
The exponential form of a complex number is	z = re^{iq}	z = e^{iq}	continuous	z = r / cos q	z = reiq
L(1) = ----	1	s	1/s	0	1/s
L(t) = -----	1/s	1/s^2	t	1/t^2	1/s2
L(t^2) = -----	2/s^3	1/t^2	2/t^3	1/s^2	2/s3
é1/2 = ---	ÖP/2	ÖP/4	ÖP	ÖP/8	ÖP
L (e^{at}) = ----	1/s+a	1/s-a	1/s*a	None	1/s-a
L (t sinat) = -----	2as/ (s^2-a^2)	2as/ ($s^2$$a^2$)	2as/ (s^2+a^2)	None	2as/ (s2+a2)
L (tcos at) = -----	s^2-a^2/(s^2+a^2)^2	s^2 +a^2/(s^2+a^2)^2	s^2-a^2/(s^2+a^2)^2	None	s2-a2/(s2+a2)2
If L^{-1}{1/(s+a)^2 }= ------	t e^{at}	t e^{-at}	$e^{a t}$	None	t e -at
L^{-1} (1 /(s^2 + 4)) is equal to	e^{-4t}	cos2t/2	sin2t/2	e^{4t}	sin2t/2
L^{-1} (1/s) = --------	1	0	t	none.	1
L^{-1} [1/(s+a)] = ----------	$e^{s t}$	$e^{a t}$	$e^{-s t}$	$e^{- a t}$	e s t
L (cosat) = ----	s/s^2 -a^2	a/s^2 +a^2	a/s^2 - a^2	s/s^2 +a^2	s/s2 +a2
L (sinat) = ----	s/s^2 +a^2	a/s^2 +a^2	a/s^2 - a^2	s/s^2 -a^2	a/s2 +a2
L (t^n) = ----	é(n+1)/s^{n+1}	é(n-1)/s^{n+1}	é(n+1)/s^{n-1}	None	é(n+1)/sn+1
é(n+1) = ----	(n-1)!	n!	(n+1)!	None	(n+1)!
L(1) = ----	1	s	1/s	0	1/s
The sum of n^{th} roots of unity are ------	0	1	2	3	0
Singular points are of __________ types	1	2	3	4	2
(sin ∏ /3+ i cos ∏ /3)^3 is equal to	-1	1	–i	i	i