**KARPAGAM ACADEMY OF HIGHER EDUCATION**
***(Deemed to be University Established Under Section 3 of UGC Act 1956)***
**Coimbatore – 641 021.**

**SYLLABUS**
**DEPARTMENT OF PHYSICS**

**STAFF NAME:** Mrs.A. SAHANA FATHIMA **CLASS:** II B.Sc., (PHYSICS)
**SUBJECT NAME:** DIGITAL SIGNAL PROCESSING **SUB.CODE:** 17PHU403
**SEMESTER:** IV

**Objective:** Digital signal processing has lot of applications in different fields of life. This objective of this paper is to give knowledge to students about the theory of signal processing and the different methods involved in it.

**UNIT- I**
**Discrete-Time Signals and Systems:** Classification of Signals, Transformations of the Independent Variable, Periodic and Aperiodic Signals, Energy and Power Signals, Even and Odd Signals, Discrete-Time Systems, System Properties. Impulse Response, Convolution Sum; Graphical Method; Analytical Method, Properties of Convolution; Commutative; Associative; Distributive; Shift; Sum Property System Response to Periodic Inputs, Relationship Between LTI System Properties and the Impulse Response; Causality; Stability; Invertibility, Unit Step Response.

**UNIT- II**
**Discrete-Time Fourier Transform:** Fourier Transform Representation of Aperiodic Discrete-Time Signals, Periodicity of DTFT, Properties; Linearity; Time Shifting; Frequency Shifting; Differencing in Time Domain; Differentiation in Frequency Domain; Convolution Property.

**UNIT-III**
**The *z*-Transform:** Bilateral (Two-Sided) *z*-Transform, Inverse *z*-Transform, Relationship Between *z*-Transform and Discrete-Time Fourier Transform, *z*-plane, Region-of-Convergence; Properties of ROC, Properties; Time Reversal; Differentiation in the *z*-Domain; Power Series Expansion Method (or Long Division Method); Analysis and Characterization of LTI Systems; Transfer Function and Difference-Equation System. Solving Difference Equations.

**UNIT-IV**
**Filter Concepts:** Phase Delay and Group delay, Zero-Phase Filter, Linear-Phase Filter, Simple FIR Digital Filters, Simple IIR Digital Filters, All pass Filters, Averaging Filters, Notch Filters.
**Discrete Fourier Transform:** Frequency Domain Sampling (Sampling of DTFT), The Discrete Fourier Transform (DFT) and its Inverse, DFT as a Linear transformation, Properties; Periodicity; Linearity; Circular Time Shifting; Circular Frequency Shifting.

**UNIT-V**

**Fast Fourier Transform:** Direct Computation of the DFT, Symmetry and Periodicity, Properties of the Twiddle factor (*WN*), Radix-2 FFT Algorithms; Decimation-In-Time (DIT) FFT Algorithm; Decimation-In-Frequency (DIF) FFT Algorithm, Inverse DFT Using FFT Algorithms. **Realization of Digital Filters:** Non Recursive and Recursive Structures, Canonic and Non Canonic Structures, Equivalent Structures (Transposed Structure), FIR Filter structures; Direct-Form; Cascade-Form; Basic structures for IIR systems; Direct-Form I.

**TEXT BOOKS**

1. Digital Signal Processing, Tarun Kumar Rawat, 2015, Oxford University Press, India, Digital Signal Processing, S. K. Mitra, McGraw Hill, India.
2. Lathi, B.P.Zhi Ding, Modern Digital and Analog Communication Systems, 2009, 3rd Edn. Oxford University Press.

**REFERENCE BOOKS:**

1. Fundamentals of Digital Signal processing using MATLAB, R.J. Schilling and S.L. Harris, 2005, Cengage Learning.
2. Fundamentals of signals and systems, P.D. Cha and J.I. Molinder, 2007, Cambridge University Press, Digital Signal Processing Principles Algorithm & Applications, J.G. Proakis and D.G. Manolakis, 2007, 4th Edn., Prentice Hall.
3. Fundamentals of Digital Signal processing using MATLAB, R.J. Schilling and S.L. Harris, 2011, Cengage Learning, Digital Signal Processing , J.G. Proakis and D.G. Manolakis, 2013., Prentice.

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**LECTURE PLAN**
**DEPARTMENT OF PHYSICS**

**STAFF NAME:** Mrs. A. SAHANA FATHIMA **SEMESTER:** IV
**SUBJECT NAME:** DIGITAL SIGNAL PROCESSING **SUB.CODE:** 17PHU403
**CLASS:** II B.Sc., (PHYSICS)

**UNIT I**

| | **Lecture Duration (hr.)** | **Topics to be covered** | **Support materials** |
|---|---|---|---|
| 1 | 1hr | Classification of Signals, Transformations of the Independent Variable, Periodic and Aperiodic Signals, Energy and Power Signals | T1(1.3-1.9), T1(1.36-1.41) |
| 2 | 1hr | Even and Odd Signals, Discrete-Time Systems, System Properties. Impulse Response, Convolution Sum | T1(1.36-1.48), T1(1.52-1.56) |
| 3 | 1hr | Graphical Method; Analytical Method, Properties of Convolution; Commutative | T1(1.58-1.61) |
| 4 | 1hr | Associative; Distributive; Shift | T1(1.62) |
| 5 | 1hr | Sum Property System Response to Periodic Inputs | T1(1.63) |
| 6 | 1hr | Relationship Between LTI System Properties and the Impulse Response | T1(1.71-1.75) |
| 7 | 1hr | Causality; Stability; Inevitability, Unit Step Response | T1(1.61-1.63), T1(1.80), T1(1.99) |
| 8 | 1hr | Revision | |
| Total no. of hours planned for unit –I | | 8 hr | |

**UNIT II**

| **S.No** | **Lecture Duration (hr.)** | **Topics to be covered** | **Support Materials** |
|---|---|---|---|
| 1 | 1hr | Fourier Transform Representation of Aperiodic Discrete-Time Signals | T1(1.107) |

| | | | |
|---|---|---|---|
| 2 | 1hr | Periodicity of DTFT, Properties; Linearity | T1(1.100-1.102) |
| 3 | 1hr | Time Shifting; Frequency Shifting | T1(1.111-1.112) |
| 4 | 1hr | Differencing in Time Domain; Differentiation in Frequency Domain | T1(1.112-1.113) |
| 5 | 1hr | Convolution Property | T1(1.114) |
| 6 | 1hr | Revision | |
| Total no. of hours planned for unit –II | | 6 hr | |

**UNIT III**

| S.No | Lecture Duration (hr.) | Topics to be covered | Support Materials |
|---|---|---|---|
| 1 | 1hr | Bilateral (Two-Sided) *z*-Transform, Inverse *z*-Transform | T1(2.1-2.3) |
| 2 | 1hr | Relationship Between *z*-Transform and Discrete-Time Fourier Transform, *z*-plane | T1(2.30-2.35) |
| 3 | 1hr | Region-of-Convergence; Properties of ROC, Properties; Time Reversal | T1(2.3-2.36) |
| 4 | 1hr | Differentiation in the *z*-Domain; Power Series Expansion Method | T1(2.17-2.32) |
| 5 | 1hr | Analysis and Characterization of LTI Systems | T1(2.3-2.45) |
| 6 | 1hr | Transfer Function and Difference-Equation System. Solving Difference Equations | T1(2.23-2.52), T1(2.58) |
| 7 | 1hr | Revision | |
| Total no. of hours planned for unit –III | | 7 hr | |

**UNIT IV**

| Si.No | Lecture Duration (hr.) | Topics to be covered | Support Materials |
|---|---|---|---|
| 1 | 1hr | Phase Delay and Group delay, Zero-Phase Filter | T1(1.139) |
| 2 | 1hr | Linear-Phase Filter, Simple FIR Digital Filters | T1(1.145-1.147) |
| 3 | 1hr | Simple IIR Digital Filters, All pass Filters | T1(1.153) |

| | | | |
|---|---|---|---|
| 4 | 1hr | Averaging Filters, Notch Filters | T1(1.156-1.181) |
| 5 | 1hr | Frequency Domain Sampling (Sampling of DTFT), The Discrete Fourier Transform (DFT) and its Inverse | T1(3.06), T1(3.1) |
| 6 | 1hr | DFT as a Linear transformation, Properties; Periodicity | T1(3.2) |
| 7 | 1hr | Linearity; Circular Time Shifting; Circular Frequency Shifting | T1(3.2-3.5) |
| 8 | 1hr | Revision | |
| Total no. of hours planned for unit –IV | | 8 hr | |

**UNIT V**

| Si.No | Lecture Duration (hr.) | Topics to be covered | Support Materials |
|---|---|---|---|
| 1 | 1hr | Direct Computation of the DFT, Symmetry and Periodicity | T1(4.1-4.3) |
| 2 | 1hr | Properties of the Twiddle factor (*WN*), Radix-2 FFT Algorithms; Decimation-In-Time (DIT) | T1(4.3-4.14) |
| 3 | 1hr | FFT Algorithm; Decimation-In-Frequency (DIF) FFT Algorithm | T1(4.19-4.30) |
| 4 | 1hr | Inverse DFT Using FFT Algorithms; Non Recursive and Recursive Structures | T1(5.54) |
| 5 | 1hr | Canonic and Non Canonic Structures, Equivalent Structures | T1(5.50-5.55) |
| 6 | 1hr | FIR Filter structures; Direct-Form; Cascade-Form | T1(5.55-5.58) |
| 7 | 1hr | Basic structures for IIR systems; Direct-Form I. | T1(5.58-5.65) |
| 8 | 1hr | Revision | |
| 9 | 1hr | Old question paper discussion | |
| 10 | 1hr | Old question Paper discussion | |
| 11 | 1hr | Old question Paper discussion | |
| Total no. of hours planned for unit –v | | 11 hr | |

**Suggested Reading Books:**

T1: Digital Signal Processing, 4th edition , Ramesh Babu, Sci.Tech

T2: Digital Signal Processing, Turun Kumar Rawat,2015, Oxford University Press, India.

**UNIT-I**

**SYLLABUS**

**Discrete-Time Signals and Systems:** Classification of Signals, Transformations of the Independent Variable, Periodic and Aperiodic Signals, Energy and Power Signals, Even and Odd Signals, Discrete-Time Systems, System Properties. Impulse Response, Convolution Sum; Graphical Method; Analytical Method, Properties of Convolution; Commutative; Associative; Distributive; Shift; Sum Property System Response to Periodic Inputs, Relationship Between LTI System Properties and the Impulse Response; Causality; Stability; Invertibility, Unit Step Response.

**Definition**

Anything that carries information can be called as signal. It can also be defined as a physical quantity that varies with time, temperature, pressure or with any independent variables such as speech signal or video signal.

The process of operation in which the characteristics of a signal (Amplitude, shape, phase, frequency, etc.) undergoes a change is known as signal processing.

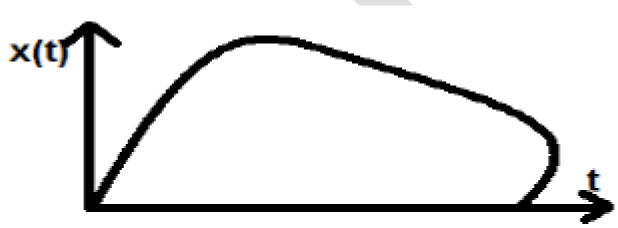
**Note** − Any unwanted signal interfering with the main signal is termed as noise. So, noise is also a signal but unwanted.

According to their representation and processing, signals can be classified into various categories details of which are discussed below.

CONTINUOUS TIME SIGNALS

Continuous-time signals are defined along a continuum of time and are thus, represented by a continuous independent variable. Continuous-time signals are often referred to as analog signals.

This type of signal shows continuity both in amplitude and time. These will have values at each instant of time. Sine and cosine functions are the best example of Continuous time signal.



Continuous Time signal

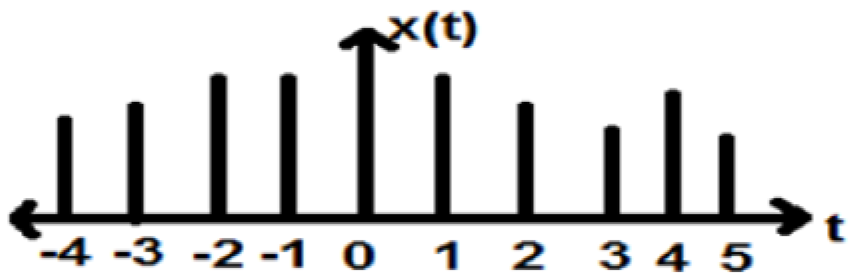
The signal shown above is an example of continuous time signal because we can get value of signal at each instant of time.

### DISCRETE TIME SIGNALS

The signals, which are defined at discrete times are known as discrete signals. Therefore, every independent variable has distinct value. Thus, they are represented as sequence of numbers.

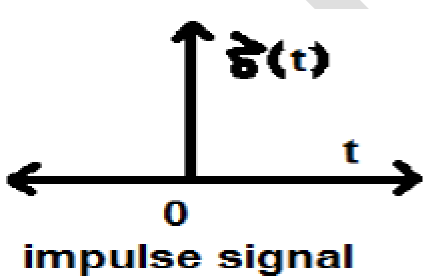
Although speech and video signals have the privilege to be represented in both continuous and discrete time format; under certain circumstances, they are identical. Amplitudes also show discrete characteristics. Perfect example of this is a digital signal; whose amplitude and time both are discrete.

The figure above depicts a discrete signal's discrete amplitude characteristic over a period of time. Mathematically, these types of signals can be formularized as;
$x=\{x[n]\},-\infty<n<\infty$
Where, *n* is an integer.
It is a sequence of numbers x, where $n^{th}$ number in the sequence is represented as x[n].




### UNIT IMPULSE OR DELTA FUNCTION

A signal, which satisfies the condition, δ(t)=limϵ→∞x(t)δ(t)=limϵ→∞x(t) is known as unit impulse signal. This signal tends to infinity when t = 0 and tends to zero when t ≠ 0 such that the area under its curve is always equals to one. The delta function has zero amplitude everywhere excunit_impulse.jpgept at t = 0.




### PROPERTIES OF UNIT IMPULSE SIGNAL

- δ(t) is an even signal.
- δ(t) is an example of neither energy nor power (NENP) signal.
- Area of unit impulse signal can be written as;

A=∫∞−∞δ(t)dt=∫∞−∞limϵ→0x(t)dt=limϵ→0∫∞−∞[x(t)dt]=1

- Weight or strength of the signal can be written as;

  y(t)=Aδ(t)

  Area of the weighted impulse signal can be written as

  y(t)=∫∞−∞y(t)dt=∫∞−∞Aδ(t)=A[∫∞−∞δ(t)dt]=A=1=Wigthedimpulse

UNIT STEP SIGNAL

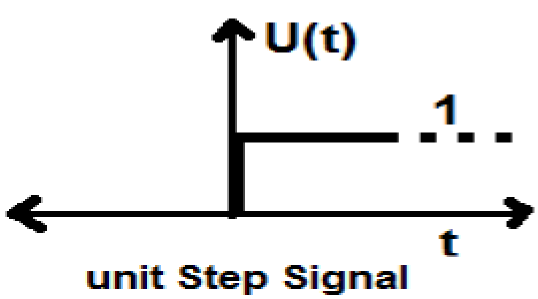
A signal, which satisfies the following two conditions

U(t)=1(whent≥0)and

U(t)=0(whent<0)

is known as a unit step signal.

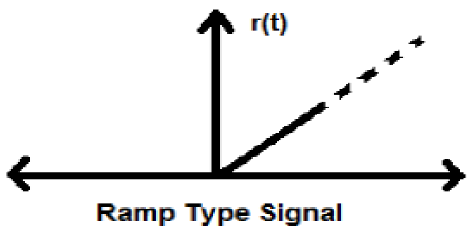
It has the property of showing discontinuity at t = 0. At the point of discontinuity, the signal value is given by the average of signal value. This signal has been taken just before and after the point of discontinuity (according to Gibb's Phenomena).




If we add a step signal to another step signal that is time scaled, then the result will be unity. It is a power type signal and the value of power is 0.5. The RMS (Root mean square) value is 0.707 and its average value is also 0.5

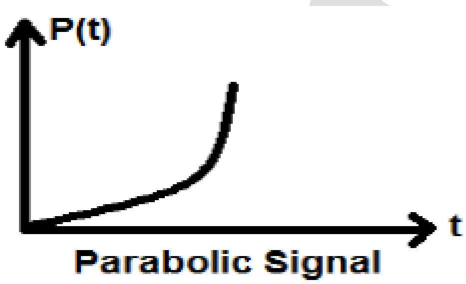
RAMP SIGNAL

Integration of step signal results in a Ramp signal. It is represented by r(t). Ramp signal also satisfies the condition r(t)=∫t−∞U(t)dt=tU(t)r(t)=∫−∞tU(t)dt=tU(t). It is neither energy nor power (NENP) type signal.




PARABOLIC SIGNAL

Integration of Ramp signal leads to parabolic signal. It is represented by p(t). Parabolic signal also satisfies he condition p(t)=∫t−∞r(t)dt=(t2/2)U(t)p(t)=∫−∞tr(t)dt=(t2/2)U(t) . It is neither energy nor Power (NENP) type signal.



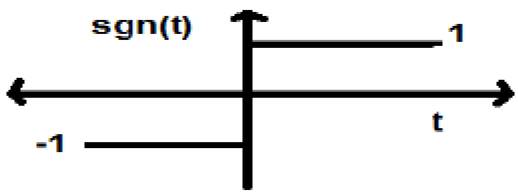

SIGNUM FUNCTION

This function is represented as

sgn(t)={1fort>0

−1 fort<0

It is a power type signal. Its power value and RMS (Root mean square) values, both are 1. Average value of signum function is zero.




signum function

### SINC FUNCTION

t is also a function of sine and is written as

SinC(t)=SinΠtΠT=Sa(Πt)

### PROPERTIES OF SINC FUNCTION

It is an energy type signal.
Sinc(0)=limt→0sinΠtΠt=1
Sinc(∞)=limt→∞sinΠ∞Π∞=0Sinc(∞)=limt→∞sinΠ∞Π∞=0(Range of sinπ∞ varies between -1 to +1 but anything divided by infinity is equal to zero)
If sinc(t)=0=>sinΠt=0
Πt=nΠ
t=n(n≠0)

### SINUSOIDAL SIGNAL

A signal, which is continuous in nature is known as continuous signal. General format of a sinusoidal signal is
x(t)=Asin(ωt+ϕ)
Here,

A = amplitude of the signal

ω = Angular frequency of the signal (Measured in radians)

φ = Phase angle of the signal (Measured in radians)

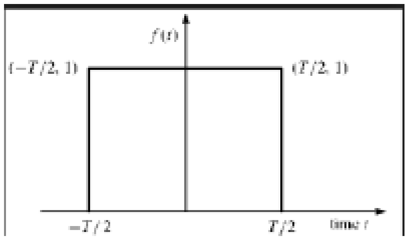
The tendency of this signal is to repeat itself after certain period of time, thus is called periodic signal. The time period of signal is given as;

T=2πωT=2πω

The diagrammatic view of sinusoidal signal is shown below.

### RECTANGULAR FUNCTION

A signal is said to be rectangular function type if it satisfies the following condition
π(t/τ)={1, fort≤τ/2
0, Otherwise




Being symmetrical about Y-axis, this signal is termed as even signal.

TRIANGULAR PULSE SIGNAL

Any signal, which satisfies the following condition, is known as triangular signal.

**Transformation of the Independent Variable**

**Signal Operation**

**Time Shifting**

Time shifting is, as the name suggests, the shifting of a signal in time. This is done by adding or subtracting the amount of the shift to the time variable in the function. Subtracting a fixed amount from the time variable will shift the signal to the right (delay) that amount, while adding to the time variable will shift the signal to the left (advance).

$$y(t) = x(t - t_0)$$

Here, the original signal x(t) is shifted by an amount $t_0$.

Rule: set $t - t_0=0$ and move the origin of x(t) to $t_0$.

Example 1-2-1: Given x(t) = u(t+2) - u(t-2), find $x(t-t_0)$ and $x(t+t_0)$.

**Time Scaling**

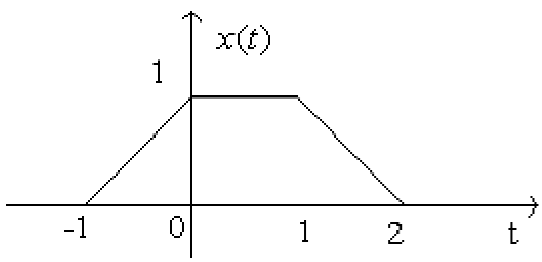
Time scaling compresses and dilates a signal by multiplying the time variable by some amount. If that amount is greater than one, the signal becomes narrower and the operation is called compression, while if the amount is less than one, the signal becomes wider and is called dilation. It often takes people quite a while to get comfortable with these operations, as people's intuition is often for the multiplication by an amount greater than one to dilate and less than one to compress.

The signal y(t) = x(at) is a time-scaled version of x(t).

If $|a| > 1$, we are SPEEDING UP x(t) by a factor of a.
If $|a| < 1$, we are SLOWING DOWN x(t) by a factor of a.

**Combinations of Scale and Shift**

Find x(2t+1) where x(t) is:




Method 1: Shift then scale: x(at+b)

(i) v(t)=x(t+b);

(ii) y(t) =v(at)= x(at+b).

v(t)=x(t+1)

y(t)=v(2t)

**Time Reversal**

A natural question to consider when learning about time scaling is: What happens when the time variable is multiplied by a negative number? The answer to this is time reversal. This operation is the reversal of the time axis, or flipping the signal over the y-axis.

We reverse a signal x(t) by flipping it over the vertical-axis to form a new signal y(t) = x(-t).

**Signal Characteristics**

**Periodic Functions**

How can we tell if a continuous- time signal x(t) is periodic? That is, given t and T, is there some period T >0 such that

$$x(t) = x(t + T).$$

If x(t) is periodic with period T, it is also periodic with period nT, that is:

$$x(t) = x(t + nT)$$

.

The minimum value of T that satisfies x(t) = x(t + T) is called the **fundamental period** of the signal and we denote it as $T_0$.

The fundamental frequency of the signal in hertz (cycles/second) is

and in radians/second, it is

If $x_1(t)$ is periodic with period $T_1$ and $x_2(t)$ is periodic with period $T_2$, then the sum of the two signals $x_1(t) + x_2(t)$ is periodic with period equal to the least common multiple($T_1$, $T_2$) if the ratio of the two periods is a rational number, i.e.:

Let $T' = k_1T_1 = k_2T_2$, and $z(t) = x_1(t) + x_2(t)$,

$$z(t + T') = x_1(t + k_1T_1) + x_2(t + k_2T_2) = x_1(t) + x_2(t) = z(t)$$

**Even and Odd Functions**

Any continuous time signal can be expressed as the sum of an even signal and an odd signal:

$$x(t) = x_e(t) + x_o(t)$$

Even: $x_e(t)= x_e(-t)$

Odd: $x_o(t)=- x_o(-t)$

An even signal is symmetric across the vertical axis.

An odd signal is anti-symmetric across the vertical axis.

$$x_e(t)=(x(t)+x(-t))/2$$

$$x_o(t)=(x(t)-x(-t))/2$$

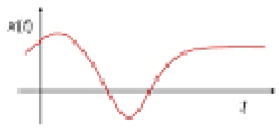
Example1-2-10: given the unit step function (a discontinuous continuous-time signal), find $u_e(t)$ and $u_o(t)$

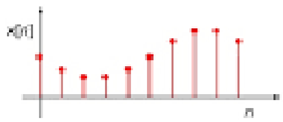
Signals are classified into the following categories:

- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

CONTINUOUS TIME AND DISCRETE TIME SIGNALS

A signal is said to be continuous when it is defined for all instants of time.





A signal is said to be discrete when it is defined at only discrete instants of time

DETERMINISTIC AND NON-DETERMINISTIC SIGNALS

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.

A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.

EVEN AND ODD SIGNALS

A signal is said to be even when it satisfies the condition x(t) = x(-t)

**Example 1:** t2, t4… cost etc.

Let x(t) = t2

x(-t) = (-t)2 = t2 = x(t)

∴,∴, t2 is even function

**Example:** t, t3 ... And sin t

Let x(t) = sin t

x(-t) = sin(-t) = -sin t = -x(t)

∴,∴, sin t is odd function.

Any function $f$(t) can be expressed as the sum of its even function $f_e$(t) and odd function $f_o$(t).

$f(t) = f_e(t) + f_0(t)$

where

$f_e(t) = ½[f(t) + f(-t)]$

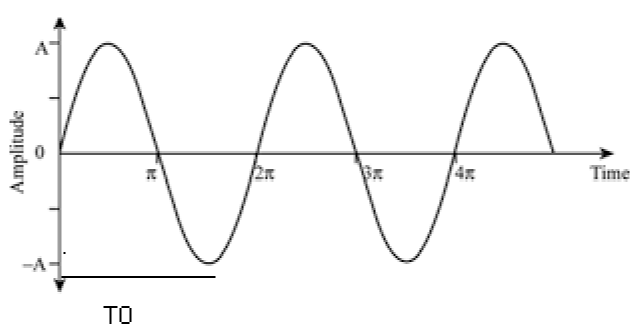
PERIODIC AND APERIODIC SIGNALS

A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N).

Where

T = fundamental time period,

1/T = f = fundamental frequency.




The above signal will repeat for every time interval $T_0$ hence it is periodic with period $T_0$.

ENERGY AND POWER SIGNALS

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2(t)dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0

Energy of power signal = ∞

REAL AND IMAGINARY SIGNALS

A signal is said to be real when it satisfies the condition x(t) = x*(t)

A signal is said to be odd when it satisfies the condition x(t) = -x*(t)

Example:

If x(t)= 3 then x*(t)=3*=3 here x(t) is a real signal.

If x(t)= 3j then x*(t)=3j* = -3j = -x(t) hence x(t) is a odd signal.

## Discrete-time systems

Discrete-time systems, "A set of connected parts or models which takes discrete-time signals as input, known as excitation, processes it under certain set of rules and algorithms to have a desired output of another discrete-time signal, known as response". In general, if a there is excitation x(n) and the response of the system is y(n), the we express the system as,

$$y(n) = T[x(n)]$$ or

$$x(n) \xrightarrow{T} y(n)$$

Where, ***T*** is the general rule or algorithm which is implemented on x(n) or the excitation to get the response y(n). For example, a few systems are represented as,

$$y(n) = -2x(n)$$

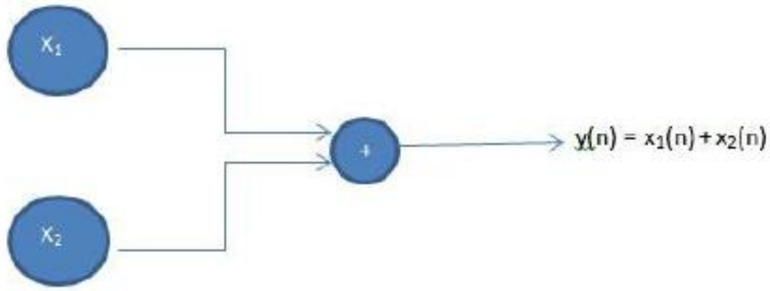
or, $$y(n) = x(n-1) + x(n) + x(n+1)$$

## Block Diagram representation of Discrete-time systems

Digital Systems are represented with blocks of different elements or entities connected with arrows which also fulfills the purpose of showing the direction of signal flow,


Some common elements of Discrete-time systems are:-

**Adder:** It performs the addition or summation of two signals or excitation to have a response. An adder is represented as,




**Constant Multiplier:** This entity multiplies the signal with a constant integer or fraction. And is represented as, in this example the signal x(n) is multiplied with a constant "a" to have the response of the system as y(n).

**Signal Multiplier:** This element multiplies two signals to obtain one.

**Unit-delay element:** This element delays the signal by one sample i.e. the response of the system is the excitation of previous sample. This can element is said to have a memory which stores the excitation at time n-1 and recalls this excitation at the time n form the memory. This element is represented as,


**Unit-advance element:** This element advances the signal by one sample i.e. the response of the current excitation is the excitation of future sample. Although, as we can see this element is not physically realizable unless the response and the excitation are already in stored or recorded form.

Discrete-time systems are classified on different principles to have a better idea about a particular system, their behavior and ultimately to study the response of the system.

**Relaxed system:** If $y(n_o-1)$ is the initial condition of a system with response y(n) and $y(n_o-1)=0$ , then the system is said to be initially relaxed i.e. if the system has no excitation prior to $n_o$ .

**Static and Dynamic systems:** A system is said to be a Static discrete-time system if the response of the system depends **at most** on the current or present excitation and not on the past or future excitation. If there is any other scenario then the system is said to be a Dynamic discrete-time system. The static systems are also said to be memory-less systems and on the other hand dynamic systems have either finite or infinite memory depending on the nature of the system. Examples below will clear any arising doubts regarding static and dynamic systems.

Static System

$$y(n) = 2x(n) + nx^2(n)$$

$$y(n) = ax(n)$$

Dynamic system with finite memory

$$y(n) = ax(n) + bx(n-1) + cx(n+1)$$

$$y(n) = \sum_{k=0}^{n} x(n-k)$$

Dynamic system with in -finite memory

$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

**Time-variant and Time-invariant system:** A discrete-time system is said to be time invariant if the input-output characteristics do not change with time, i.e. if the excitation is delayed by k units then the response of the system is also delayed by k units. Let there be a system,

$x(n) \xrightarrow{T} y(n) \quad \forall\, x(n)$

Then the relaxed system ***T*** is time-invariant if and only if,

$x(n-k) \xrightarrow{T} y(n-k) \quad \forall\, x(n)$ and k.

Otherwise, the system is said to be time-variant system if it does not follows the above specified set of rules. For example,

$y(n) = ax(n)$

time-invariant }

$y(n) = x(n) + x(n-3)$

time-invariant }

**Linear and non-Linear systems:** A system is said to be a linear system if it follows the superposition principle i.e. the sum of responses (output) of weighted individual excitations (input) is equal to the response of sum of the weighted excitations. Pay attention to the above specified rule, according to the rule the following condition must be fulfilled by the system in order to be classified as a Linear system,

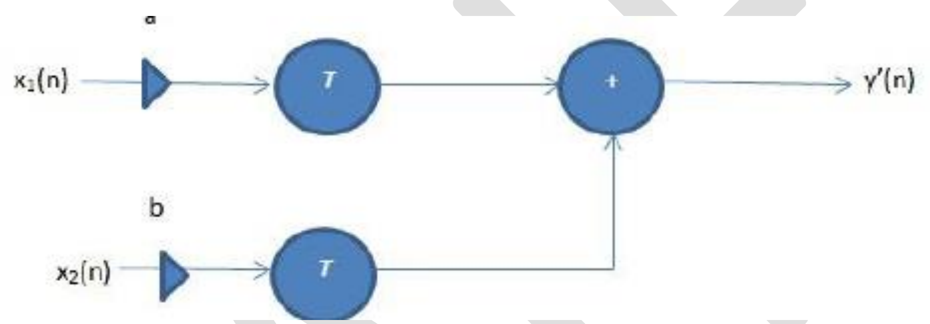
If, $y_1(n) = T[\ ax_1(n)\ ]$

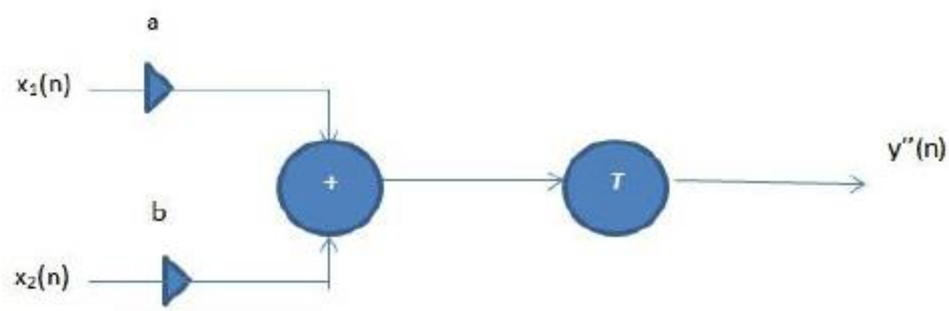
$y_2(n) = T[\ bx_2(n)\ ]$

and, $y(n) = T[ax_1(n) + bx_2(n)]$

Then, the system is said to be linear if ,

$$T[\ ax_1(n) + bx_2(n)] = T[\ ax_1(n)\ ] + T[\ bx_2(n)\ ]$$




So, iff y'(n) = y''(n) then the system is said to be linear. I the system does not fulfills this property then the system is a non-Linear system

**Causal and non-Causal systems:** A discrete-time system is said to be a causal system if the response or the output of the system at any time depends only on the present or past excitation or input and not on the future inputs. If the system ***T*** follows the following relation then the system is said to be causal otherwise it is a non-causal system.

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \qquad \{\ \text{Causal}\ \}$$

$y(n) = x(n) + x(n+1)$ {non-Causal }

**Stable and Unstable systems:** A system is said to be stable if the bounded input produces a bounded output i.e. the system is BIBO stable. If,

$x(n) = M \quad \forall \quad -\infty < M < \infty$

$y(n) = N \quad \forall \quad -\infty < N < \infty$

Then the system is said to be bounded system and if this is not the case then the system is unbounded or unstable.

# The Basics of the Convolution Sum

Consider a DT LTI system, L.

$$x(n) \longrightarrow \boxed{\text{L}} \longrightarrow y(n)$$

DT convolution is based on an earlier result where we showed that any signal $x(n)$ can be expressed as a sum of impulses.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

So let us consider $x(n)$ written in this form to be our input to the LTI system.

$$y(n) = \text{L}[x(n)] = \text{L}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

This looks like our general linear form with a scalar $x(k)$ and a signal in $n$, $\delta(n-k)$. Recall that for an LTI system:

- Linearity (L): $ax_1(n) + bx_2(n) \longrightarrow \boxed{\text{L}} \longrightarrow ay_1(n) + by_2(n)$
- Time Invariance (TI): $x(n-n_o) \longrightarrow \boxed{\text{L}} \longrightarrow y(n-n_o)$

We can use the property of linearity to distribute the system L over our input.

$$y(n) = \text{L}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)\text{L}[\delta(n-k)]$$

So now we wonder, what is $\text{L}[\delta(n-k)]$? Well, we can figure it out. Suppose we know how L acts on one impulse $\delta(n)$, and we call it

$$h(n) = \text{L}[\delta(n)]$$

then by time invariance we get our answer.

$$h(n-k) = \mathrm{L}\left[\delta(n-k)\right]$$

$$\delta(n-k) \longrightarrow \boxed{\mathrm{L}} \longrightarrow h(n-k)$$

This means that if we know *one* input-output pair for this system, namely

$$\delta(n) \longrightarrow \boxed{\mathrm{L}} \longrightarrow h(n)$$

then we can infer

$$x(n) \longrightarrow \boxed{\mathrm{L}} \longrightarrow y(n)$$

which gives us the following.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

This is the *convolution sum* for DT LTI systems.

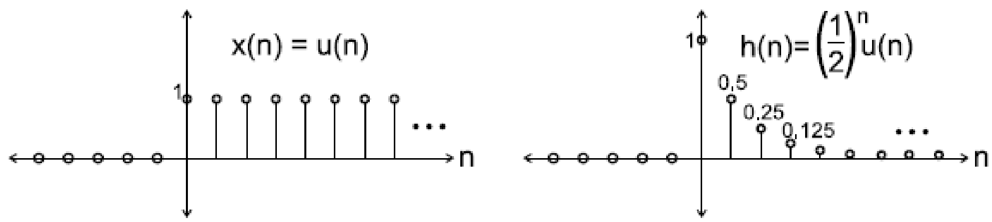
The convolution sum for $x(n)$ and $h(n)$ is usually written as shown here.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

### Example 2.1: DT Convolution: Step Response

Say we are given the following signal $x(n)$ and system impulse response $h(n)$.

$$x(n) = u(n) \quad \text{and} \quad h(n) = \left(\frac{1}{2}\right)^n u(n)$$




We wish to find the step response $s(n)$ of the system (i.e. the response of the system to the unit step input $x(n) = u(n)$. This is shown below.

$$s(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Thus the step response is as follows, found by substituting our actual signals into the general convolution sum.

$$s(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

Let's look at this step response in smaller ranges to see what happens.

- First, consider the case where $n < 0$.

$$
\begin{aligned}
s(n) &= \sum_{k=-\infty}^{\infty} u(k)\left(\frac{1}{2}\right)^{n-k} u(n-k) \\
&= \sum_{k=0}^{n} 1\cdot\left(\frac{1}{2}\right)^{n-k}\cdot 1 \\
&\quad \text{We can pull out any terms only in } n \\
&\quad \text{since that is not the summation variable.} \\
&= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n}\sum_{k=0}^{n} 2^{k}
\end{aligned}
$$

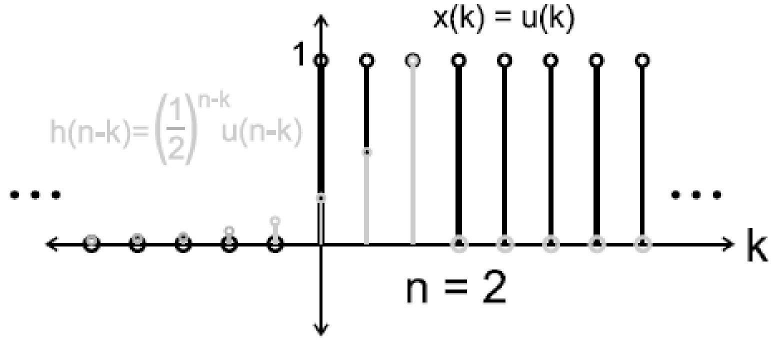
Now we have a form consistent with a geometric series. We can use that to solve.

$$
\text{Recall} \quad \sum_{k=0}^{n} 2^{k} = \frac{1-2^{n+1}}{1-2} = 2^{n+1}-1
$$

So we have $s(n)$ as follows.

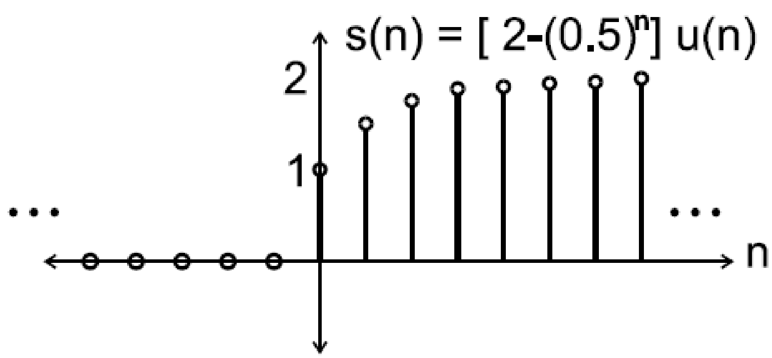
$$
\begin{aligned}
s(n) &= \left(\frac{1}{2}\right)^{n}\left(2^{n+1}-1\right) \\
&= \left(\frac{1}{2}\right)^{n}\left(2\cdot 2^{n}-1\right) \\
&= \left(\frac{1}{2}\right)^{n}\left(2\cdot\left(\frac{1}{2}\right)^{-n}-1\right) \\
&= 2\cdot\left(\frac{1}{2}\right)^{-n}\left(\frac{1}{2}\right)^{n} - 1\cdot\left(\frac{1}{2}\right)^{n} \\
s(n) &= 2-\left(\frac{1}{2}\right)^{n}
\end{aligned}
$$

We can visualize this, say for $n = 2$, as shown below. Note how the system output comes from the overlap of the input signal and the shifted and flipped impulse response.




So, overall, we have the following step response.

$$s(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n)$$




The $u(n)$ comes from our first case above since $s(n) = 0$ for $n < 0$, and obviously the other part comes from the expression found in the second case above.

# 3 Basic Properties of DT Convolution

Discrete-time convolution has several useful properties that allows us to solve systems more easily.

## 3.1 Commutativity

Convolution is a commutative operation, meaning signals can be convolved in any order.

$$x(n) * h(n) = h(n) * x(n)$$

This quite naturally is true of the convolution sums themselves, as well.

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

## 3.2 Associativity

Convolution is associative, meaning that convolution operations in series can be done in any order.

$$(x(n) * h(n)) * g(n) = x(n) * (h(n) * g(n))$$

This is significant because it means systems in series can be reordered.

Thus we have

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow \boxed{g(n)} \longrightarrow y(n)$$

is the same as

$$x(n) \longrightarrow \boxed{h(n) * g(n)} \longrightarrow y(n)$$

is the same as

$$x(n) \longrightarrow \boxed{g(n) * h(n)} \longrightarrow y(n)$$

is the same as

$$x(n) \longrightarrow \boxed{g(n)} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

and so the systems in series can be reordered.

### 3.3 Distributivity

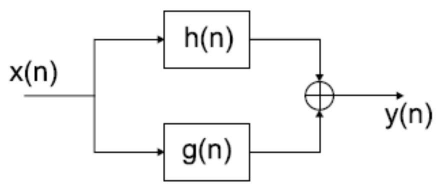
Convolution is distributive over addition.

$$x(n) * [h(n) + g(n)] = x(n) * h(n) + x(n) * g(n)$$

This is significant to all parallel connections because it means the following two arrangements are equivalent.

$$x(n) \longrightarrow \boxed{h(n) + g(n)} \longrightarrow y(n)$$

is the same as




### 3.4 Identity

We have previously established that $\delta(n)$ is the identity with respect to discrete-time convolution.

$$\text{Recall} \quad x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) = x(n) * \delta(n)$$

So $x(n) * \delta(n) = x(n)$.

This concept is quite easily extended, so $x(n) * \delta(n - n_o) = x(n - n_o)$ for $n_o \in \mathbb{Z}$ and $x(n - n_o) * \delta(n - n_1) = x(n - (n_o + n_1))$ for $n_o, n_1 \in \mathbb{Z}$.

## Impulse Response of Discrete Time System:

Discrete Time System is an algorithm, which operates on a discrete time signal called as input signal according to some well-defined rules/operation. Impulse Response of a system is the reaction to any discrete time system in response to some external changes. Impulse Response is generally denoted as **h(t)** or **h[n]**. The output y[n] of any discrete LTI system is depended on the input (i.e. x(n)) and system's response to unit impulse (i.e. h[n]).

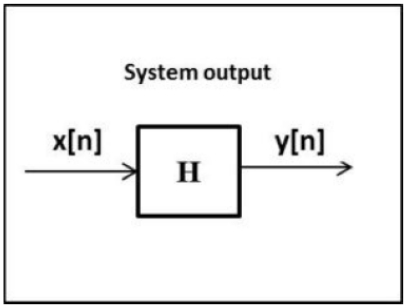
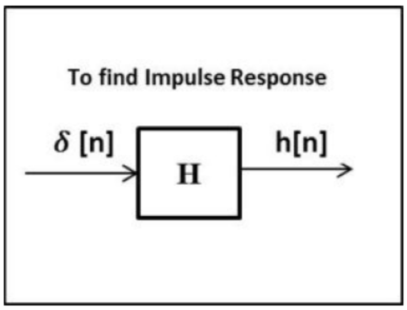



Figure 1




We can determine the systems output y[n], if we know system's impulse response, h[n], and the input, x[n]. To find the impulse response of the system we provide a **Unit impulse to the input** x[n].

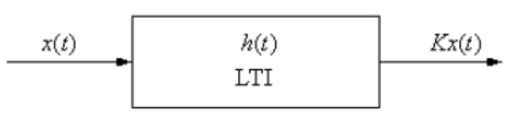
**Systems with memory**

In a memoryless system, the output $y(t)$ is a function of the input $x(t)$ at the time instant $t$ alone. It does not depend on either past or future inputs.

An LTI system that is memoryless can only have this form:

$$y(t) = x(t) * h(t) = Kx(t)$$

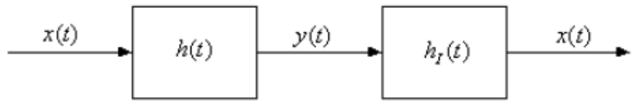
Here, $K$ is the system gain and it must be constant or else the system would vary with time.




For $y(t)= Kx(t)$, the impulse response $h(t)$ must be of the form of a unit impulse weighted by a constant $K$:

$$h(t) = K\delta(t)$$

## Invertible Systems




$$h(t) * h_I(t) = \delta(t)$$

A system is invertible if we can find $h_I(t)$ so that the original input $x(t)$ can be recovered from the output $y(t)$. For this to hold, the system must be *one-to-one*.

We will see how to do this when we study transforms.

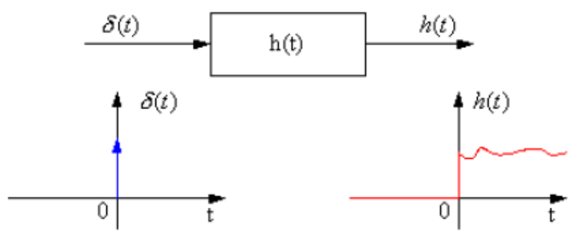
## Causality

We know that for a causal system, the output depends only on past or present inputs and not on future inputs.

Equivalently, a causal system does not respond to an input until it occurs (the output is not based on the future).

In other words, a response to an input at $t = t_0$, would occur only for $t \geq t_0$ and not before $t_0$.

We know that $h(t)$ is the system response to $\delta(t)$, and that $\delta(t)$ occurs at $t = 0$.




A system is causal, if $h(t) = 0, \quad t < 0$

<u>Another way to look at the causality condition:</u> Let's examine the convolution equation, flipping $h(t)$ instead of $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Causality: if $h(t)$ is causal then $h(t - \tau) = 0$, $t - \tau < 0$ or $t < \tau$.

So,

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau$$

which shows us that the output $y(t)$ depends only on values of the input $x(\tau)$ for $\tau \leq t$, i.e. it only depends on the past and present.

**Stability**

We can tell if an LTI system is BIBO stable from its impulse response.

$|x(t)| \leq B_1$, for all $t$, to determine if the system is BIBO stable, we need to determine if its output remains bounded for all time:

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right| \leq$$

$$\int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|d\tau \quad \text{Why ?}$$

$$= \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq \int_{-\infty}^{\infty} B_1|h(\tau)|d\tau = B_1\int_{-\infty}^{\infty}|h(\tau)|d\tau$$

$$\text{Therefore, } |y(t)| \leq B_1\int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty \text{ if } \int_{-\infty}^{\infty}|h(\tau)|d\tau < \infty$$

That is, the system is BIBO stable iff the impulse response $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau = G < \infty$$

In this case, the output will be bounded by a second constant: $|y(t)| \leq B_1G = B_2$ and thus, the system is BIBO stable.

**KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21**
**DEPARTMENT OF PHYSICS**
**II B.Sc PHYSICS (2016-2019)**
**DIGITAL SIGNAL PROCESSING (16PHU403)**

| QUESTIONS | CHOICE |
|---|---|
| **UNIT-I** | |
| An LTI system is said to be causal if and only if | Impulse r |
| The impulse response of a LTI system is h(n)={1,1,1}. What is the response of th | {1,3,6,3,1 |
| The system described by the equation y(n)=ay(n-1)+b x(n) is | a |
| $y(n)=-\sum_{k=1}^{N} a_k y(n-k)+\sum_{k=0}^{M} b_k x(n-k)$ Which of the following is a recursive form of a non-recursive system described | y(n)=y(n- |
| . If x(n) is a discrete-time signal, then the value of x(n) at non integer value of ‘ | 0 |
| The discrete time function defined as u(n)=n for n≥0;=0 for n<0 is an: | Unit samp |
| The phase function of a discrete time signal $x(n)=a^n$, where $a=r.e^{j\theta}$ is: | tan(nθ) |
| A real valued signal x(n) is called as anti-symmetric if: | x(n)=x(-n |
| The odd part of a signal x(t) is: | x(t)+x(-t) |
| Time scaling operation is also known as: | Down-sar |
| What is the condition for a signal $x(n)=Br^n$ where $r=e^{\alpha n}$ to be called as an decay | 0<r<∞ |
| The function given by the equation x(n)=1, for n=0;=0, for n≠0 is a: | Step funct |
| The output signal when a signal x(n)=(0,1,2,3) is processed through an ‘Identic | (3,2,1,0) |
| If a signal x(n) is passed through a system to get an output signal of y(n)=x(n+ | Delayed |
| What is the output y(n) when a signal x(n)=n*u(n)is passed through a accumu | $(n^2+n+1)$/ |
| The output signal when a signal x(n)=(0,1,2,3) is processed through an ‘Delay’ | (3,2,1,0) |
| The system described by the input-output equation $y(n)=nx(n)+bx^3(n)$ is a: | Static syst |
| Whether the system described by the input-output equations y(n)=x(n)-x(n-1 | time |
| The system described by the input-output equations $y(n)=x^2(n)$ is | Linear |
| If the output of the system of the system at any ‘n’ depends only the present or | Linear |
| The system described by the input-output equations y(n)=x(-n) | Linear |
| . If a system do not have a bounded output for bounded input, then the system | Causal |
| The impulse response of a LTI system is h(n)={1,1,1}. What is the response of t | {1,3,6,3,1 |
| Determine the output y(n) of a LTI system with impulse response $h(n)=a^n u(n)$ | $(1-a^{(n+1)})$/ |
| Determine the impulse response for the cascade of two LTI systems having im | $(1/2)^n$[2-( |
| An LTI system is said to be causal if and only if | Impulse r |
| $x(n)*\delta(n-n_0)$= | $x(n+n_0)$ |
| The discrete impulse function is defined by | δ(n) = 1, n |
| The computational procedure for Decimation in frequency algorithm takes | Log2 N sta |
| The anti causal sequences have ______ components in the left hand sequences. | Positive |
| The IIR filter designing involves | Designing |
| Which among the following represent/s the characteristic/s of an ideal filter? | Constant |
| FIR filters ________ | . are non- |
| In tapped delay line filter, the tapped line is also known as ________ | Pick-on n |
| How is the sensitivity of filter coefficient quantization for FIR filters? | Low |
| I I R digital filters are of the following nature | Recursive |
| In I I R digital filter the present output depends on | Present a |
| Which of the following is best suited for I I R filter when compared with the FII | Lower si |
| In the case of I I R filter which of the following is true if the phase distortion is | . More pa |
| A causal and stable I I R filter has | Linear p |
| Neither the Impulse response nor the phase response of the analog filter is Pre | The meth |
| Out of the given I I R filters the following filter is the efficient one | Circular |
| What is the disadvantage of impulse invariant method | Aliasing |
| . Which of the I I R Filter design method is antialiasing method? | a. The me |
| The nonlinear relation between the analog and digital frequencies is called | a. aliasing |
| The most common technique for the design of I I R Digital filter is | a. Direct |

| | |
|---|---|
| The I I R filter design method thatovercomes the limitation of applicability to c | a. Appro |
| The Fourier transform of a real valued time signal has | odd |
| A signal x(t) has a Fourier transform X(ω). If x(t) is a real and odd function of t | a real |
| The amplitude spectrum of a Gaussian pulse is | uniform |
| If a signal f(t) has energy E, the energy of the signal f(2t) is equal to | E |
| The trigonometric Fourier series of an even function does not have the | dc term |
| The Fourier series of an odd periodic function, contains only | odd |
| The trigonometric Fourier series of a periodic time function can have only | cosine |
| The trigonometric Fourier series of an even function of time does not have | cosine |
| A system with an input x(t) and output y(t) is described by the relation: y(t) = | linear |
| The input and output of a continuous time system are respectively denoted by | $y(t)=x(t$ |
| A discrete-time signal $x[n]=\sin(\pi 2\ n),n$ being an integer, is | Periodic |
| Convolution of $x(t+5)$ with impulse function $\delta(t-7)$is equal to | $x(t-12)$ |
| Two systems with impulse responses h1(t) and h2(t) are connected in | product |

## UNIT-II

| | |
|---|---|
| DTFT is the representation of | Periodic |
| The transforming relations performed by DTFT are | Linearity |
| The transforming relations performed by DTFT are | Modulati |
| The transforming relations performed by DTFT are | Shifting |
| The transforming relations performed by DTFT are | Convolut |
| DFT is preferred for | Removal |
| The DFT is preferred for | Its ability |
| As compared to the analog systems, the digital processing of signals allow | Programi |
| As compared to the analog systems, the digital processing of signals allow | Flexibility |
| As compared to the analog systems, the digital processing of signals allow | Cheaper |
| As compared to the analog systems, the digital processing of signals allow | More |
| The Nyquist theorem for sampling | Relates th |
| The Nyquist theorem for sampling | Gives |
| Roll-off factor is | The band |
| Frequency selectivity characteristics of DFT refers to | Ability to |
| Which term applies to the maintaining of a given signal level until the next | Holding |
| For a 4-bit DAC, the least significant bit (LSB) is | 6.25% |
| The DTFT transforms an infinite-length discrete signal in the time domain into | an finite- |
| As with continuous-time, convolution is represented by the symbol *, and can | y[n]=x[n |
| Let $f$ and $g$ be two functions with convolution $f*g$ .. Let $F$ be the Fourier | F(f*g)=F |
| Let $f$ and $g$ be two functions with convolution $f*g$ .. Let $F$ be the Fourier | $F(f\cdot g)=F$ |
| Inverse Fourier transform F−1, we can writ | f∗g=F−1( |
| The Fourier transform of a convolution is the pointwise product of | Fouriertr |
| convolution in one domain corresponds to point-wise ...... in the other | multiplic |
| Symmetry property deals with the effect on the frequency-domain representa | altered |
| a unit pulse with a very small duration, in time that becomes an infinite-length c | delta |
| Time shifting shows that a shift in time is equivalent to a | linear |
| . frequency content depends only on the shape of a signal, which is unchanged i | phasesp |
| convolution in time becomes……... in frequency | |
| convolution property is also another excellent example of …... between time and | symmetr |
| Convolution property is also another excellent example of symmetry between | time |
| | the |
| Continuous functions are sampled to form a | Fourier |
| 2D Fourier transform and its inverse are infinitely | aperiodi |
| Which property of delta function indicates the equality between the area under | Replicati |
| Which among the below specified conditions/cases of discrete time in terms of | a>1 |
| A system is said to be shift invariant only if______ | a shift in |
| Which condition determines the causality of the LTI system in terms of its impu | a. Only if t |
| An equalizer used to compensate the distortion in the communication system b | static |

| | |
|---|---|
| Which block of the discrete time systems requires memory in order to store the | adder |
| Which type/s of discrete-time system do/does not exhibit the necessity of any f | recursiv |
| Which type of system response to its input represents the zero value of its initia | Zero |
| Which among the following operations is/are not involved /associated with the | Folding O |
| A LTI system is said to be initially relaxed system only if ____ | zero inpu |
| What are the number of samples present in an impulse response called as? | string |
| Duality Theorem / Property of Fourier Transform states that ________ | a. Shape c |
| Which property of fourier transform gives rise to an additional phase shift of -2 | . Time Sca |
| What is/are the crucial purposes of using the Fourier Transform while analyzin | Transforr |
| What is the possible range of frequency spectrum for discrete time fourier serie | 0 to 2π |
| Which among the following assertions represents a necessary condition for the | Discrete ' |
| What is the nature of Fourier representation of a discrete & aperiodic signal? | Continuoı |
| Which property of periodic signal in DTFS gets completely clarified / identified | Conjugati |
| Which are the only waves that correspond/ support the measurement of phase | Sine wav |
| What does the signalling rate in the digital communication system imply ? | Number ( |
| As the signalling rate increases, _______ | Width of ( |
| Which phenomenon occurs due to an increase in the channel bandwidth during | Compress |
| What does the term y(-1) indicate especially in an equation that represents the | initial con |
| Damped sinusoids are | sinusoid |

| CHOICE | CHOICE3 | CHOICE4 | ANSWER |
|---|---|---|---|
| Impulse r | Impulse r | Impulse response i: | Impulse response is zero for negative values of n |
| {1,2,3,2,1 | {1,3,6,5,3 | {1,1,1,0,0} | {1,3,6,5,3} |
| causal | non-caus: | superpostion | a |
| ) y(n)=y(r | y(n)=y(n- | y(n)=y(n-1)+ 1/(M+ | y(n)=y(n-1)+ 1/(M+1)[x(n)-x(n-1-M)]. |
| positive | negative | not defined | not defined |
| Unit step | Unit ramp | None of the mentior | Unit ramp signal |
| nθ | tan $^{-1}$(nθ) | cosθ | tan $^{-1}$(nθ) |
| x(n)=-x(-r | x(n)=-x(n | x(n)=y(n) | x(n)=-x(-n) |
| x(t)-x(-t) | (1/2)*(x( | (1/2)*(x(t)-x(-t)) | (1/2)*(x(t)-x(-t)) |
| Up-sampl | Sampling | zer0 sampling | Down-sampling |
| 0<r<1 | r>1 | r<0 | 0<r<1 |
| Ramp fun | Triangula | Impulse function | Impulse function |
| (1,2,3,0) | (0,1,2,3) | (0,2) | (1,2,3,0) |
| Advanced | No operat | None of the mentior | None of the mentioned |
| (n(n+1))/ | (n+1) | (n+1)/2 | (n(n+1))/2 |
| (1,2,3,0) | (2,3,0) | (3,2,1,3) | (3,2,1,3) |
| Dynamic : | Identical : | ideal system | Static system |
| time in | delay | non-delay | time in |
| non | exponent | delay | non |
| Causal | Non-Line: | Non-causal | Causal |
| Causal | Non-Line: | Non-causal | Non-causal |
| Non-caus: | Stable | Non-stable | Non-stable |
| {1,2,3,2,1 | {1,3,6,5,3 | {1,1,1,0,0} | {1,3,6,5,3} |
| (1-a$^{(n+1)}$)/ | (1+a$^{(n+1)}$)/ | (1-a) | (1-a$^{(n+1)}$)/(1-a) |
| (1/2)$^{n}$[2-( | (1/2)$^{n}$[2+ | (1/2)$^{n}$[2+(1/2)$^{n}$], n: | (1/2)$^{n}$[2-(1/2)$^{n}$], n>0 |
| Impulse r | Impulse r | Impulse response is | Impulse response is zero for negative values of n |
| x(n-$n_0$) | x(-n-$n_0$) | x(-n+$n_0$) | x(n-$n_0$) |
| δ(n) = 1, | δ(n) = 1, r | δ(n) = 1, n ≤ 0,= 0, n | δ(n) = 1, |
| 2Log2 N s | Log2 N$^2$ s | Log2 N/2 stages | Log2 N stages |
| negative | not define | 0 | Positive |
| Designing | Designing | Designing of digital | Designing of digital filter in analog domain and tra |
| Zero gain | Linear Ph | All of the above | All of the above |
| . are recu | use feedb | linear | are non-recursive |
| Pick-off n | Pick-up n | Pick-down node | Pick-off node |
| Moderate | High | Unpredictable | Low |
| Non Rec | Reversiv | Non Reversive | Recursive |
| Present i | Present i | Present Input, Prev | Present Input, Previous input and output |
| Higher S | Lower si | No sidelobes in sto | Lower sidelobes in stopband |
| More me | Lower co | Higher computatio | Lower computational Complexity |
| No Linea | Linear ai | No Amplitude | No Linear phase |
| Impulse | Bilinear t | Matched Z - transfo | Bilinear transformation |
| Elliptical | Rectang | Chebyshev filter | Elliptical filter |
| one to on | anti alias d | warping | Aliasing |
| b. Impuls | c. Bilinea | d. Matched Z - trans | c. Bilinear transformation |
| b. warpir | c. prewa | d. antialiasing | b. warping |
| b. In dire | c. Recurs | d. non recursive me | b. In direct method |

| | | | |
|---|---|---|---|
| b. Impuls | c. Bilinea | d. Frequency sampl | b. Impulse Invariance |
| conjugat | even sym | no symmetry | conjugat |
| an | an imagin | a real and odd funct | an |
| Gaussian | a sine fun | An impulse functior | Gaussian |
| 2E | E/2 | 4E | E/2 |
| cosine | sine term | odd harmonic terms | sine terms |
| cosine | sine term | even harmonic term | sine terms |
| sine term | dc term | even harmonic term | cosine |
| sine term | dc term | even harmonic term | cosine |
| linear | non-linea | non-linear and time | linear |
| $y(t)=(t-$ | $y(t)=(t+4$ | $y(t)=(t+5)\ x(t+5)$ | $y(t)=(t+4)\ x(t-1)$ |
| Periodic | Periodic v | Not periodic | Not periodic |
| $x(t-2)$ | $x(t+12)$ | $x(t+2)$ | $x(t-2)$ |
| sum of | convoluti | subtraction of h2(t) | convolution of h1(t) and h2(t) |

| | | | |
|---|---|---|---|
| Aperiodi | Aperiodi | Periodic continuou | Aperiodic Discrete time signals |
| nonlinea | demodula | periodicity | Linearity |
| nonlinea | demodula | periodicity | Modulation |
| nonlinea | demodula | periodicity | Shifting |
| nonlinea | demodula | periodicity | Convolution |
| Filter de | demodula | periodicity | Filter design |
| Quantizat | demodula | periodicity | Quantization of signal |
| non | costly | not reliable | Programmable operations |
| non | costly | not reliable | Flexibility in the system design |
| non | costly | not reliable | Cheaper systems |
| non | costly | not reliable | More |
| Helps in | Gives | calculate bndwidth | Relates the conditions in time domain and frequen |
| Limits th | Helps in | calculate bndwidth | Limits the bandwidth requirement |
| The perf | Aliasing | sampling | The bandwidth occupied beyond the Nyquist Band |
| b. Ability | c. Ability | Ability to convert c | Ability to resolve different frequency components |
| Aliasing | Shannon | "Stair- | Holding |
| 0.625% | 12% of | 1.2% of | 6.25% |
| an finite- | an in | an | an finite- |
| y⌊n⌋=x⌊n | y⌊n⌋=x⌊n | y⌊n⌋=x⌊n | y⌊n⌋=x⌊n |
| F(f*∗g)= | F(fg)=F(f | F(f*∗g)= | F(f*g)=F |
| $F(f\cdot g)=F$ | $F(f\cdot * g)=$ | $F(f\cdot g)=F(f)/F(g)$ | $F(f\cdot g)=F(f)*F(g)$ |
| f∗g=F(F(f | fg=F−1(F | f∗g=F−1( | f∗g=F−1( |
| Fourier | infinite | FFT | Fouriertr |
| addition | subtracti | integrati | multiplic |
| constant | added | subtract | altered |
| impulse | ramp | step | delta |
| Non- | linear fre | linear frequency sh | linear |
| amplitud | time | frequenc | phasesp |
| addition | subtracti | integrati | |
| antisym | periodici | aperiodi | symmetr |
| time | phase | phase | time |
| the | the energ | the power of its Z t | the |
| fourierse | Ztransfo | digital | digital |
| periodic | linear | nonlinea | periodic |
| sampling | scaling | alising | sampling |
| a<1 | 0 | than a | than a |
| a shift in | c. a shifti | d. a shifting at input | a shift in the input signal also results in the corres |
| b. Only if | c. Only if t | d. Only if the value | a. Only if the value of an impulse response is zero f |
| dynamic | invertible | non- | invertible system |

| | | | |
|---|---|---|---|
| multplie | unit delay | unit | unit delay |
| nonrecur | linear | nonlinea | nonrecur |
| b. Zero in | c. Total re | d. Natural response | Zero |
| b. Shifting | c. Multipli | d. Integration Opera | d. Integration Operation |
| b. zero in | c. zero inp | d. none of the above | zero input produces zero output |
| b. array | c. length | d. element | c. length |
| b. Shape o | c. Shape o | d. Shape of signal in | a. Shape of signal in time domain & shape of spectr |
| Linearity | Time Shi | Duality | Time Shifting |
| Plotting o | Both a & | Transformation fro | Both a & b |
| -π to +π | Both a & b | 0 | Both a & b |
| Discrete ' | Discrete ' | Discrete Time Signa | Discrete Time Signal should be absolutely summa |
| Discrete a | Continuo | Discrete & periodic | Continuous & periodic |
| Time Shi | Frequenc | Time Reversal | Time Shifting |
| Cosine w | Triangula | Square wave | Cosine waves |
| Number o | Number o | Number of digital pu | Number of digital pulses transmitted per second |
| Width of | Width of | None of the above | Width of each pulse decreases |
| Expansion | Expansion | Compression in fre | Expansion in frequency domain |
| negative i | negative | response of the syst | initial condition of the system |
| sinusoid | sinusoid s | sinusoid signals div | sinusoid signals multiplied by decaying exponentia |

nsforming into digital domain

ıcy domain

lwidth of the filter
from input signal

ponding shift in the output
'or all negative values of time

rum can be interchangeable

ble

als

**UNIT-II**

**SYLLABUS**

**Discrete-Time Fourier Transform:** Fourier Transform Representation of Aperiodic Discrete-Time Signals, Periodicity of DTFT, Properties; Linearity; Time Shifting; Frequency Shifting; Differencing in Time Domain; Differentiation in Frequency Domain; Convolution Property

## DISCRETE TIME FOURIER TRANSFORM

A discrete-time signal can be considered as a continuous signal $x(t)$ sampled at a rate $F = 1/t_0$ or $\Omega = 2\pi/t_0$, where $t_0$ is the sampling period (time interval between two consecutive samples). The corresponding sampling function (comb function) is:

$$comb(t) = \sum_{m=-\infty}^{\infty} \delta(t - mt_0)$$

The sampling process can be represented by

$$x_s(t) = x(t)\ comb(t) = x(t) \sum_{m=-\infty}^{\infty} \delta(t - mt_0) = \sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)$$

where $x[m] = x(mt_0)$ is the value of $x(t)$ at $t = mt_0$. The Fourier transform of this discrete signal (treated as a special case of continuous signal) is:

$$X(j\omega) = \int_{\infty}^{\infty} x_s(t) e^{-j\omega t} dt = \int_{\infty}^{\infty} [\sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)]\ e^{-j\omega t} dt$$

$$= \sum_{m=-\infty}^{\infty} x[m] \int_{\infty}^{\infty} \delta(t - mt_0)\ e^{-j\omega t} dt = \sum_{m=-\infty}^{\infty} x[m]\ e^{-j\omega m t_0}$$

This is the forward Fourier transform (analysis) of a discrete signal $x_s(t)$. The spectrum $X(j\omega)$ is periodic with period $\Omega = 2\pi F = 2\pi/t_0$:

$$X(j(\omega+\Omega)) = \sum_{m=-\infty}^{\infty} x[m]e^{-j(\omega+\Omega))mt_0} = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega mt_0}e^{-j\Omega mt_0} = X(j\omega)$$

as

$$e^{-j\Omega mt_0} = e^{-j2m\pi} = 1$$

To get back the time signal $x[m]$ from its spectrum:

$$X(j\omega) = \sum_{m=-\infty}^{\infty} x[m]\ e^{-j\omega mt_0}$$

we multiply the equation by $e^{j\omega nt_0}/\Omega$ and integrate both sides with respect to $\omega$ over the period $\Omega = 2\pi F = 2\pi/t_0$ to obtain the inverse Fourier transform (synthesis):

$$\frac{1}{\Omega}\int_\Omega X(j\omega)e^{j\omega nt_0}d\omega = \frac{1}{\Omega}\int_\Omega [\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega mt_0}]e^{j\omega nt_0}d\omega$$

$$\sum_{m=-\infty}^{\infty} x[m]\frac{1}{\Omega}\int_\Omega e^{-j\omega(m-n)t_0}d\omega = \sum_{m=-\infty}^{\infty} x[m]\delta[m-n] = x[n]$$

Note that here we used

$$\frac{1}{\Omega}\int_\Omega e^{-j\omega(m-n)t_0}d\omega = \frac{1}{\Omega}\int_\Omega e^{-j(m-n)2\pi\omega/\Omega}d\omega = \delta[m-n] = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$$

which can be compared this with

$$\frac{1}{T}\int_T e^{j(m-n)\omega_0 t}dt = \frac{1}{T}\int_T e^{j(m-n)2\pi t/T}dt = \delta[m-n] = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$$

To summarize, the spectrum of a given discrete signal

$$x_s(t)=\sum_{m=-\infty}^{\infty} x[m]\delta(t-mt_0)$$

can be found by forward Fourier transform to be:

$$X_\Omega(j\omega)=\mathcal{F}[x[m]]=\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega mt_0}=\sum_{m=-\infty}^{\infty} x[m]e^{-j2\pi mt_0}$$

and the signal can be expressed by inverse Fourier transform:

$$x[m]=\mathcal{F}^{-1}[X_\Omega(j\omega)]=\frac{1}{\Omega}\int_\Omega X_\Omega(j\omega)e^{j\omega mt_0}d\omega=\int_F X_F(f)e^{j2\pi mt_0}df$$

It is interesting to compare this discrete time Fourier transform pair with the Fourier series expansion - the Fourier transform of a periodic signal:

$$x_T(t)=\mathcal{F}^{-1}[X[n]]=\sum_{n=-\infty}^{\infty} X[n]e^{jn\omega_0 t}=\sum_{n=-\infty}^{\infty} X[n]e^{j2\pi nf_0 t}$$

$$X[n]=\mathcal{F}[x_T(t)]=\frac{1}{T}\int_T x_T(t)e^{-jn\omega_0 t}dt=\frac{1}{T}\int_T x_T(t)e^{-j2\pi nf_0 t}dt$$

with discrete spectrum:

$$X(j\omega)=2\pi\sum_{n=-\infty}^{\infty} X[n]\delta(\omega-n\omega_0)\quad \text{or}\quad X(f)=\sum_{n=-\infty}^{\infty} X[n]\delta(f-nf_0)$$

We see symmetry between these two different forms of Fourier transform. If the signal $x(t)=x(t+T)$ is periodic, its spectrum $X(j\omega)$ is discrete, the coefficients of the

Fourier series with interval $\omega_0 = 2\pi/T$. On the other hand, if $x(t)$ is discrete with interval $t_0 = 2\pi/\Omega$, its spectrum $X(j\omega) = X(j\omega + \Omega)$ is periodic.

In particular, if the unit of time is so chosen that the sampling period is $t_0 = 1$, then $\Omega = 2\pi/t_0 = 2\pi$, and the forward Fourier transform of a discrete signal becomes:

$$X(j\omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-jm\omega} = \sum_{m=-\infty}^{\infty} x[m]e^{-jm2\pi f}$$

The inverse transform becomes:

$$x_s[m] = \frac{1}{2\pi}\int_0^{2\pi} X(j\omega)e^{jm\omega}d\omega = \int_0^1 X(j\omega)e^{jm2\pi f}df$$

The spectrum $X(j\omega) = X(j\omega + 2\pi)$ is periodic.

he spectrum of a time signal (continuous or discrete) can be denoted by $X(j\omega)$ or $X(f)$ to emphasize the fact that the spectrum represents how the energy contained in the signal is distributed as a function of frequency $\omega$ or $f$. Moreover, if $X(f)$ is used, the factor $1/2\pi$ in front of the inverse transform is dropped so that the transform pair takes a more symmetric form. On the other hand, as Fourier transform of discrete signal can be considered as a special case of Z transform when the real part of $s = \sigma + j\omega$ is zero, i.e., $z = e^s = e^{j\omega}$:

$$X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

it is also natural to denote the spectrum of $x[n]$ by $X(e^{j\omega})$

**DTFT Analysis of Discrete LTI Systems**

The input-output relationship of an LTI system is governed by a convolution process: y[n] = x[n] * h[n] where h[n] is the discrete time impulse response of the system

Then the frequency-response is simply the DTFT of h[n]:

**Properties of Discrete Fourier Transform**

$$x(n) \xleftrightarrow[N]{\text{DFT}} x(k)$$

As a special case of general Fourier transform, the discrete time transform shares all properties (and their proofs) of the Fourier transform discussed above, except now some of these properties may take different forms. In the following, we always assume $\mathcal{F}[x[m]] = X(e^{j\omega})$ and $\mathcal{F}[y[m]] = Y(e^{j\omega})$.

**Periodicity**

Let x(n) and x(k) be the DFT pair then if

x(n+N) = x(n) for all n then

X(k+N) = X(k) for all k

Thus periodic sequence xp(n) can be given as

$$xp(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

**Linearity**

$$\mathcal{F}[ax[m] + by[m]] = aX(e^{j\omega}) + bY(e^{j\omega})$$

The linearity property states that if

$$x1(n) \overset{DFT}{\longleftrightarrow} X1(k) \text{ And}$$

$$\overset{N}{} $$

$$x2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X2(k) \text{ Then}$$

Then

$$a1\,x1(n) + a2\,x2(n) \overset{DFT}{\longleftrightarrow} a1\,X1(k) + a2\,X2(k)\ N$$

DFT of linear combination of two or more signals is equal to the same linear combination of DFT of individual signals.

**Time Shifting**

$$\mathcal{F}[x[m - m_0]] = e^{-jm_0\omega}X(e^{j\omega})$$

**Proof:**

$$\mathcal{F}[x[m - m_0]] = \sum_{m=-\infty}^{\infty} x[m - m_0]e^{-j\omega m}$$

If we let $m' = m - m_0$, the above becomes

$$\mathcal{F}[x[m - m_0]] = \sum_{m=-\infty}^{\infty} x[m']e^{-j\omega(m'+m_0)} = e^{-j\omega m_0}X(e^{j\omega})$$

**Time Reversal**

$$\mathcal{F}[x[-m]] = X(e^{-j\omega})$$

**Frequency Shifting**

$$\mathcal{F}[x[m]e^{j\omega_0 m}] = X(e^{j(\omega-\omega_0)})$$

**Differencing**

Differencing is the discrete-time counterpart of differentiation.

$$\mathcal{F}[x[m] - x[m-1]] = (1 - e^{-j\omega})X(e^{j\omega})$$

**Proof:**

$$\mathcal{F}[x[m] - x[m-1]] = \mathcal{F}[x[m]] - \mathcal{F}[x[m-1]]$$

$$: \quad X(e^{j\omega}) - X(e^{j\omega})e^{-j\omega} = (1 - e^{-j\omega})X(e^{j\omega})$$

**Differentiation in frequency**

$$\mathcal{F}^{-1}[j\frac{d}{d\omega}X(e^{j\omega})] = m\ x[m]$$

**proof:** Differentiating the definition of discrete Fourier transform with respect to $\omega$, we get

$$\frac{d}{d\omega}X(e^{j\omega}) \ : \ \frac{d}{d\omega}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = \sum_{m=-\infty}^{\infty} x[m]\frac{d}{d\omega}e^{-j\omega m}$$

$$: \ \sum_{m=-\infty}^{\infty} -jmx[m]e^{-j\omega m}$$

**Convolution Theorems**

The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa:

$$\mathcal{F}[x[n] * y[n]] = X(e^{j\omega})\, Y(e^{j\omega}) \qquad (a)$$

$$\mathcal{F}[x[n]\ y[n]] = X(e^{j\omega}) * Y(e^{j\omega}) \qquad (b)$$

Recall that the convolution of periodic signals $x_T(t)$ and $y_T(t)$ is

$$x_T(t) * y_T(t) \stackrel{\triangle}{=} \frac{1}{T}\int_T x_T(\tau) y_T(t-\tau) d\tau$$

Here the convolution of periodic spectra $X(f)$ and $Y(f)$ is similarly defined as

$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{\Omega}\int_\Omega X(e^{j\omega'})Y(e^{j(\omega-\omega')})d\omega' = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega'})Y(e^{j(\omega-\omega')})d\omega'$$

**Proof of (a):**

$$\mathcal{F}[x[n] * y[n]] = \sum_{n=-\infty}^{\infty}[\sum_{m=-\infty}^{\infty} x[m]y[n-m]\,]e^{-jn\omega}$$

$$= \sum_{m=-\infty}^{\infty} x[m][\sum_{n=-\infty}^{\infty} y[n-m]e^{-j(n-m)\omega}]e^{-jm\omega}$$

$$= X(j\omega)\, Y(j\omega)$$

**Proof of (b):**

$$\mathcal{F}[x[n]y[n]] = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-jn\omega} = \sum_{n=-\infty}^{\infty}[\frac{1}{2\pi}\int_0^{2\pi} X(j\omega')e^{jn\omega'}d\omega']y[n]e^{-jn\omega}$$

$$: \frac{1}{2\pi}\int_0^{2\pi} X(j\omega')[\sum_{n=-\infty}^{\infty} e^{jn\omega'} y[n] e^{-jn\omega}] d\omega'$$

$$: \frac{1}{2\pi}\int_0^{2\pi} X(j\omega') \sum_{n=-\infty}^{\infty} y[n] e^{-jn(\omega-\omega')} d\omega'$$

$$: \frac{1}{2\pi}\int_0^{2\pi} X(j\omega') Y(j(\omega-\omega')) d\omega' = X(j\omega) * Y(j\omega)$$

**Parseval's Relation**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega$$

**Parseval'sTheorem**

The Parseval s theorem states

$$\sum_{n=0}^{N-1} X(n)\, y^*(n) = 1/N \sum_{n=0}^{N-1} x(k)\; y^*(k)$$

This equation give energy of finite duration sequence in terms of its frequency components.

**Example 1.** The spectrum of

$$x[n] = a^n u[n] \qquad (|a| < 1)$$

is

$$X(e^{j\omega}) = \mathcal{F}[x[n]] = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-jn\omega} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

If the signal is two-sided:

$$x[n] = a^{|n|} = a^n u[n] + a^{-n} u[-n] - \delta[n], \qquad (|a| < 1)$$

Due to the time reversal property, its spectrum is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{j\omega}} - 1 = \frac{1 - a^2}{1 - 2a\ cos\omega + a^2}$$

**Example 2.** Consider an LTI system with impulse response

$$h[n] = a^n u[n], \qquad (|a| < 1)$$

and input

$$x[n] = b^n u[n], \qquad (|b| < 1)$$

The output $y[n]$ can be found in either time domain by convolution or in frequency domain by multiplication. In time domain, we have

$$y[n] : h[n] * x[n] = \sum_{m=-\infty}^{\infty} a^{n-m} u[n-m] b^m u[m] = a^n \sum_{m=0}^{n} a^{-m} b^m$$

$$: a^n \frac{1 - (b/a)^{n+1}}{1 - (b/a)} u[n] = \frac{1}{a - b}(a^{n+1} - b^{n+1}) u[n]$$

When $a = b$, we have

$$y[n] = a^n \sum_{m=0}^{n} a^{-m} b^m = (n+1) a^n u[n]$$

In frequency domain, we first find the spectra of both $x[n]$ and $h[n]$ to be:

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}; \quad H(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

and the spectrum of the output is:

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}) = \frac{1}{(1 - be^{-j\omega})(1 - ae^{-j\omega})}$$

To find $y(n)$ in time domain by inverse transform of $Y(e^{j\omega})$, we use partial fraction expansion to rewrite the above as

$$Y(e^{j\omega}) = \frac{A}{1 - ae^{-j\omega}} + \frac{B}{1 - be^{-j\omega}} = \frac{A - Abe^{-j\omega} + B - aBe^{-j\omega}}{(1 - be^{-j\omega})(1 - ae^{-j\omega})}$$

By equating the coefficients of $e^{-j\omega}$ and the constants, we get

$$A + B = 1, \quad aB + bA = 0$$

which can be solved to get

$$A = \frac{a}{a - b}, \quad B = \frac{-b}{a - b}$$

In this form, $Y(j\omega)$ can be easily inverse transformed to yield

$$h[n] = [\frac{a}{a-b}a^n - \frac{b}{a-b}b^n]u[n] = \frac{1}{a-b}(a^{n+1} - b^{n+1})u[n]$$

same as the result from convolution. Again when $a = b$, we have

$$Y(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} = \frac{e^{j\omega}}{a} j \frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})$$

But since

$$\mathcal{F}^{-1}[\frac{1}{1-ae^{-j\omega}}] = a^n u[n]$$

by the frequency differentiation property, we have

$$\mathcal{F}^{-1}[j\frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})] = na^n u[n]$$

and the output in time domain is obtained as:

$$y[n] : \mathcal{F}^{-1}[Y(e^{j\omega})] = \mathcal{F}^{-1}[\frac{e^{j\omega}}{a} j \frac{d}{d\omega}(\frac{1}{1-ae^{-j\omega}})]$$

$$: \frac{1}{a}(n+1)a^{n+1}u[n+1] = (n+1)a^n u[n+1]$$

$$: (n+1)a^n u[n]$$

Note that the time-shifting property is used due to the factor $e^{j\omega}$. Also note that $u[n+1]$ (starting at $n=-1$) is replaced by $u[n]$ (starting at $n=0$) as $n+1=0$ when $n=-1$.

**Example 4.** The impulse response of a discrete LTI system is

$$h[m] = a^m u[m]$$

where $|a|<1$ so that the system is stable. The output $y[m]$ of the system with an input

$$x[m] = cos(\frac{2\pi m}{N}) = \frac{e^{j2\pi m/N} + e^{-j2\pi m/N}}{2}$$

can be found in three different ways.

- **Time domain convolution:** The output is the convolution of $x[m]$ and $h[m]$:

$$y[m] = h[m] * x[m] = \sum_{k=0}^{\infty} a^k \frac{e^{j2\pi(m-k)/N} + e^{-j2\pi(m-k)/N}}{2}$$

$$= \frac{1}{2} e^{j2\pi m/N} \sum_{k=0}^{\infty} a^k e^{-j2\pi k/N} + \frac{1}{2} e^{-j2\pi m/N} \sum_{k=0}^{\infty} a^k e^{j2\pi k/N}$$

$$= \frac{1}{2} e^{j2\pi m/N} \frac{1}{1-ae^{-j2\pi/N}} + \frac{1}{2} e^{-j2\pi m/N} \frac{1}{1-ae^{j2\pi/N}}$$

- 

- **The eigenequation method:** We first get the frequency response function from $h[m]$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = \sum_{k=0}^{\infty}(ae^{-j\omega})^k = \frac{1}{1-ae^{-j\omega}}$$

which is the eigenvalue of the system when the input is a complex exponential $e^{jn\omega}$. Now the system's response to

$$x[m] = cos(\frac{2\pi m}{N}) = \frac{e^{j2\pi m/N} + e^{-j2\pi m/N}}{2}$$

can be found to be

$$y[m] : \frac{1}{2}[H(e^{j2\pi/N})e^{j2\pi m/N} + H(e^{-j2\pi/N})e^{-j2\pi m/N}]$$

$$: \frac{1}{2}e^{j2\pi m/N}\frac{1}{1-ae^{-j2\pi/N}} + \frac{1}{2}e^{-j2\pi m/N}\frac{1}{1-ae^{j2\pi/N}}$$

- **Frequency domain multiplication:** If we find the spectra of both $h[m]$ and $x[m]$ in the frequency domain, the spectrum of $y[m]$ can be found by multiplication. We already know

$$H(e^{j\omega}) = \mathcal{F}[h[m]] = \frac{1}{1-ae^{-j\omega}}$$

We next find the spectrum of $x[m]$

$$X(e^{j\omega}) = \mathcal{F}[x[m]] = \sum_{m=-\infty}^{\infty} \frac{e^{j2\pi m/N} + e^{-j2\pi m/N}}{2} e^{-jm\omega}$$

$$= \pi \sum_{k=-\infty}^{\infty} [\delta(\omega - 2k\pi - 2\pi/N) + \delta(\omega - 2k\pi - 2\pi/N)]$$

Now the spectrum of the output $y[m]$ can be found

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{\pi}{1 - ae^{-j\omega}} \sum_{k=-\infty}^{\infty} [\delta(\omega - 2k\pi - 2\pi/N) + \delta(\omega - 2k\pi - 2\pi/N)]$$

and the output $y[m]$ is obtained by inverse Fourier transform:

$$y[m] = \frac{1}{2\pi} \int_0^{2\pi} [\frac{\pi}{1 - ae^{-j\omega}} \sum_{k=-\infty}^{\infty} [\delta(\omega - 2k\pi - \frac{2\pi}{N}) + \delta(\omega - 2k\pi - \frac{2\pi}{N})]] e^{jm\omega} d\omega$$

: $$\frac{1}{2}e^{j2\pi m/N}\frac{1}{1-ae^{-j2\pi/N}}+\frac{1}{2}e^{-j2\pi m/N}\frac{1}{1-ae^{j2\pi/N}}$$

The physical meaning of this result will be clear if we write $H(2\pi/N)$ in polar form:

$$H(e^{j2\pi/N})=\frac{1}{1-ae^{j2\pi/N}}=re^{j\theta}$$

and the output becomes

$$y[m]=r\ cos(\frac{2\pi}{N}m+\theta)$$

That is, the output of the system is also a sinusoidal signal of the same frequency as the input, but with different magnitude $r$ and a phase angle $\theta$. For example, if $N=4$, we have

$$H(e^{j\pi/2})=\frac{1}{1+ja}=\frac{1}{\sqrt{1+a^2}}e^{-j\ tan^{-1}(a)}$$

and the output is

$$y[m]=\frac{1}{\sqrt{1+a^2}}cos(\frac{\pi m}{2}-tan^{-1}(a))$$

**UNIT-II**

| | | | |
|---|---|---|---|
| DTFT is the representation of | Periodic | Aperiodi | Aperiodi |
| The transforming relations performed by DTFT are | Linearity | nonlinea | demodula |
| The transforming relations performed by DTFT are | Modulati | nonlinea | demodula |
| The transforming relations performed by DTFT are | Shifting | nonlinea | demodula |
| The transforming relations performed by DTFT are | Convolut | nonlinea | demodula |
| DFT is preferred for | Removal | Filter des | demodula |
| The DFT is preferred for | Its ability | Quantizat | demodula |
| As compared to the analog systems, the digital processing of signa | Programi | non | costly |
| As compared to the analog systems, the digital processing of signa | Flexibility | non | costly |
| As compared to the analog systems, the digital processing of signa | Cheaper : | non | costly |
| As compared to the analog systems, the digital processing of signa | More | non | costly |
| The Nyquist theorem for sampling | Relates th | Helps in ( | Gives |
| The Nyquist theorem for sampling | Gives | Limits th | Helps in ( |
| Roll-off factor is | The band | The perfo | Aliasing e |
| Frequency selectivity characteristics of DFT refers to | Ability to | b. Ability | c. Ability t |
| Which term applies to the maintaining of a given signal level | Holding | Aliasing | Shannon |
| For a 4-bit DAC, the least significant bit (LSB) is | 6.25% | 0.625% | 12% of |
| The DTFT transforms an infinite-length discrete signal in the time | an finite- | an finite- | an in |
| As with continuous-time, convolution is represented by the symb | y[n]=x[n | y[n]=x[n | y[n]=x[n |
| Let $f$ and $g$ be two functions with convolution $f*g$ .. Let $F$ be the | F(f*g)=F | F(f*∗g)= | F(fg)=F(f |
| Let $f$ and $g$ be two functions with convolution $f*g$ .. Let $F$ be the | $F(f\cdot g)$=F | $F(f\cdot g)$=F | $F(f\cdot *g)$= |
| Inverse Fourier transform F−1, we can writ | f∗g=F−1( | f∗g=F(F(f | fg=F−1(F |
| The Fourier transform of a convolution is the pointwise product c | Fouriertr | Fourier | infinite |
| convolution in one domain corresponds to point-wise ...... in | multiplic | addition | subtracti |
| Symmetry property deals with the effect on the frequency-domai | altered | constant | added |
| a unit pulse with a very small duration, in time that becomes an inf | delta | impulse | ramp |
| Time shifting shows that a shift in time is equivalent to a | linear | Non- | linear fre |
| . frequency content depends only on the shape of a signal, which is | phasesp | amplitud | time |
| convolution in time becomes…….. in frequency | | addition | subtracti |
| convolution property is also another excellent example of …... betw | symmetr | antisym | periodici |
| Convolution property is also another excellent example of symm | time | time | phase |
| | the | the | the energ |
| Continuous functions are sampled to form a | Fourier | fourierse | Ztransfo |
| 2D Fourier transform and its inverse are infinitely | aperiodi | periodic | linear |
| Which property of delta function indicates the equality between th | Replicati | sampling | scaling |
| Which among the below specified conditions/cases of discrete tim | a>1 | a<1 | 0 |
| A system is said to be shift invariant only if______ | a shift in | a shift in | c. a shiftir |
| Which condition determines the causality of the LTI system in terr | a. Only if t | b. Only if t | c. Only if t |
| An equalizer used to compensate the distortion in the communicat | static | dynamic | invertible |
| Which block of the discrete time systems requires memory in orde | adder | multplie | unit delay |
| Which type/s of discrete-time system do/does not exhibit the nece | recursiv | nonrecur | linear |
| Which type of system response to its input represents the zero valı | Zero | b. Zero in | c. Total re |
| Which among the following operations is/are not involved /associ | Folding O | b. Shifting | c. Multipli |
| A LTI system is said to be initially relaxed system only if ____ | zero inpu | b. zero inp | c. zero inp |
| What are the number of samples present in an impulse response c | string | b. array | c. length |
| Duality Theorem / Property of Fourier Transform states that ______ | a. Shape c | b. Shape c | c. Shape o |
| Which property of fourier transform gives rise to an additional pha | . Time Sca | Linearity | Time Shif |
| What is/are the crucial purposes of using the Fourier Transform w | Transforr | Plotting o | Both a & |
| What is the possible range of frequency spectrum for discrete time | 0 to 2π | -π to +π | Both a & b |
| Which among the following assertions represents a necessary con | Discrete ' | Discrete ' | Discrete ' |
| What is the nature of Fourier representation of a discrete & aperio | Continuoı | Discrete a | Continuo |
| Which property of periodic signal in DTFS gets completely clarifie | Conjugati | Time Shif | Frequenc |
| Which are the only waves that correspond/ support the measurem | Sine wav | Cosine w | Triangula |
| What does the signalling rate in the digital communication system | Number c | Number c | Number c |

| | | | |
|---|---|---|---|
| As the signalling rate increases, _______ | Width of ( | Width of | Width of |
| Which phenomenon occurs due to an increase in the channel bandv | Compress | Expansior | Expansior |
| What does the term y(-1) indicate especially in an equation that re | initial cor | negative i | negative |
| Damped sinusoids are | sinusoid | sinusoid | sinusoid s |

| | |
|---|---|
| Periodic continuou | Aperiodic Discrete time signals |
| periodicity | Linearity |
| periodicity | Modulation |
| periodicity | Shifting |
| periodicity | Convolution |
| periodicity | Filter design |
| periodicity | Quantization of signal |
| not reliable | Programmable operations |
| not reliable | Flexibility in the system design |
| not reliable | Cheaper systems |
| not reliable | More |
| calculate bndwidth | Relates the conditions in time domain and frequency domain |
| calculate bndwidth | Limits the bandwidth requirement |
| sampling | The bandwidth occupied beyond the Nyquist Bandwidth of the filter |
| Ability to convert cc | Ability to resolve different frequency components from input signal |
| "Stair- | Holding |
| 1.2% of | 6.25% |
| an | an finite- |
| y[n]=x[n | y[n]=x[n |
| F(f*∗g)= | F(f*g)=F |
| $F(f\cdot g)=F(f)/F(g)$ | $F(f\cdot g)=F(f)*F(g)$ |
| f∗g=F−1( | f∗g=F−1( |
| FFT | Fouriertr |
| integrati | multiplic |
| subtract | altered |
| step | delta |
| linear frequency sh | linear |
| frequenc | phasesp |
| integrati | |
| aperiodi | symmetr |
| phase | time |
| the power of its Z tı | the |
| digital | digital |
| nonlinea | periodic |
| alising | sampling |
| than a | than a |
| d. a shifting at input | a shift in the input signal also results in the corresponding shift in the |
| d. Only if the value c | a. Only if the value of an impulse response is zero for all negative value |
| non- | invertible system |
| unit | unit delay |
| nonlinea | nonrecur |
| d. Natural response | Zero |
| d. Integration Opera | d. Integration Operation |
| d. none of the above | zero input produces zero output |
| d. element | c. length |
| d. Shape of signal in | a. Shape of signal in time domain & shape of spectrum can be interchar |
| Duality | Time Shifting |
| Transformation fro | Both a & b |
| 0 | Both a & b |
| Discrete Time Signa | Discrete Time Signal should be absolutely summable |
| Discrete & periodic | Continuous & periodic |
| Time Reversal | Time Shifting |
| Square wave | Cosine waves |
| Number of digital pı | Number of digital pulses transmitted per second |

| | |
|---|---|
| None of the above | Width of each pulse decreases |
| Compression in fre | Expansion in frequency domain |
| response of the syst | initial condition of the system |
| sinusoid signals div | sinusoid signals multiplied by decaying exponentials |

output
s of time

ıgeable

# UNIT-III

## SYLLABUS

**The** *z***-Transform:** Bilateral (Two-Sided) *z*-Transform, Inverse *z*-Transform, Relationship Between *z*-Transform and Discrete-Time Fourier Transform, *z*-plane, Region-of-Convergence; Properties of ROC, Properties; Time Reversal; Differentiation in the *z*-Domain; Power Series Expansion Method (or Long Division Method); Analysis and Characterization of LTI Systems; Transfer Function and Difference-Equation System. Solving Difference Equations.

## Z-TRANSFORM

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

Concept of Z-Transform and Inverse Z-Transform

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots\ldots (1)$

If $Z = re^{j\omega}$ then equation 1 becomes

$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n)[re^{j\omega}]^{-n}$

$= \Sigma_{n=-\infty}^{\infty} x(n)[r^{-n}]e^{-j\omega n}$

$X(re^{j\omega}) = X(Z) = F.T[x(n)r^{-n}] \ldots\ldots (2)$

The above equation represents the relation between Fourier transform and Z-transform.

$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$

**INVERSE Z TRANSFORM**

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n \, F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \frac{1}{2\pi} \int X(re^j\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jz d\omega$$

$$d\omega = \frac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \frac{1}{2\pi} \int X(z) z^n \frac{1}{j} z^{-1} dz = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

Z-Transform has following properties:
Linearity Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

## Time Shifting Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \stackrel{\text{Z.T}}{\longleftrightarrow} z^{-m} X(Z)$$

## Multiplication by Exponential Sequence Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \, . \, x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z/a)$$

## Time Reversal Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

## Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

## Convolution Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

## Correlation Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$$

# Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

## Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

## Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z \to 1} [z - 1] X(z)$$

This is used to find the final value of the signal without taking inverse z-transform.

# Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

## Properties of ROC of Z-Transforms

ROC of z-transform is indicated with circle in z-plane.

ROC does not contain any poles.

If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.

If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.

If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.

If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.

If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

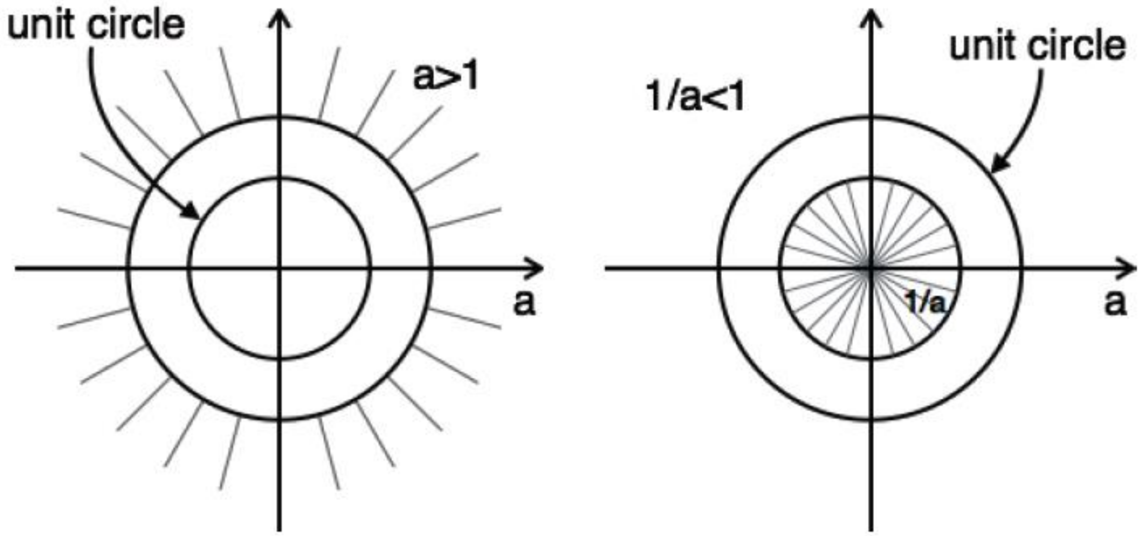
The concept of ROC can be explained by the following example:

**Example 1:** Find z-transform and ROC of $a^n u[n] + a^- nu[-n-1]$

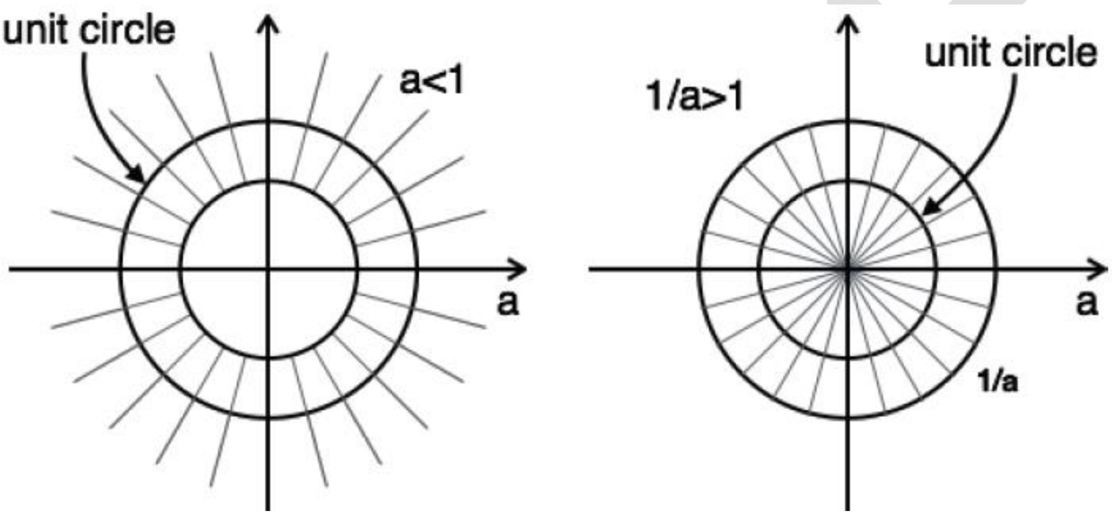
$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z\frac{-1}{a}}$$

$$ROC : |z| > a \qquad ROC : |z| < \frac{1}{a}$$

The plot of ROC has two conditions as a > 1 and a < 1,




In this case, there is no combination ROC.




Here, the combination of ROC is from $a < |z| < \frac{1}{a}$

Hence for this problem, z-transform is possible when a < 1.

# Causality and Stability

## Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

ROC is outside the outermost pole.

In The transfer function H[Z], the order of numerator cannot be grater than the order of denominator.

## Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

its system function H[Z] include unit circle |z|=1.

all poles of the transfer function lay inside the unit circle |z|=1.

**Power series expansion**

If the z-transform is given as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \ldots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots,$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of $z^{-1}$.

**Example: finite-length sequence**

The z-transform

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

By inspection, the corresponding sequence is therefore

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1].$$

**Example: power series expansion by long division**

Consider the transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore divide to get a power series in powers of $z^{-1}$:

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & \quad az^{-1} \\ & \quad az^{-1} - a^2z^{-2} \\ \hline & \qquad\quad a^2z^{-2} + \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots .$$

Therefore $x[n] = a^n u[n]$.

**Example: power series expansion for left-sided sequence**

Consider instead the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|.$$

Because of the ROC, the sequence is now a left-sided one. Thus we divide to obtain a series in powers of $z$:

$$\begin{array}{r l} & -a^{-1}z - a^{-2}z^2 - \cdots \\ \hline -a + z) & z \\ & z - a^{-1}z^2 \\ \hline & \quad az^{-1} \end{array}$$

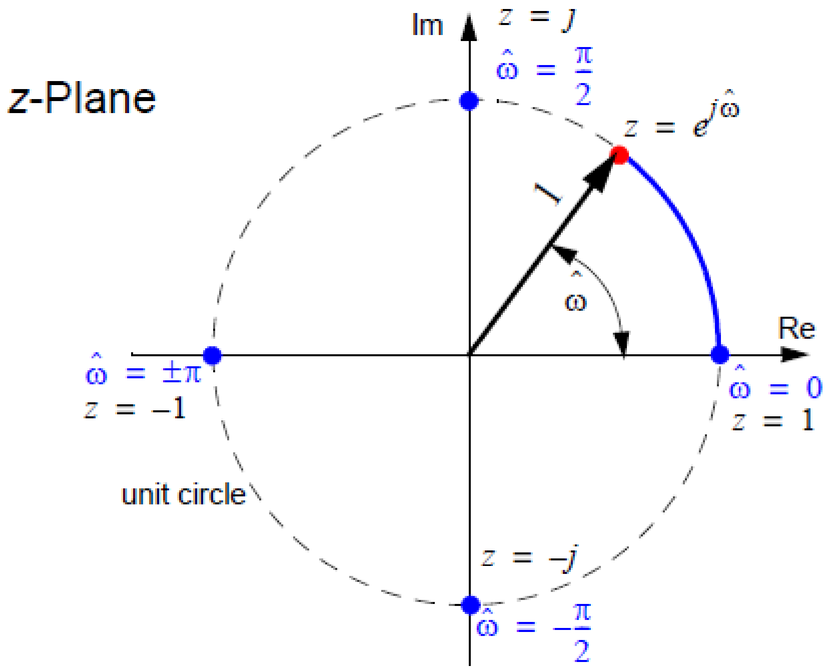
Thus $x[n] = -a^n u[-n-1]$.

# Z-Transform of Basic Signals

| x(t) | X[Z] |
|---|---|
| $\delta$ | 1 |
| $u(n)$ | $\frac{Z}{Z-1}$ |
| $u(-n-1)$ | $-\frac{Z}{Z-1}$ |
| $\delta(n-m)$ | $z^{-m}$ |
| $a^n u[n]$ | $\frac{Z}{Z-a}$ |
| $a^n u[-n-1]$ | $-\frac{Z}{Z-a}$ |
| $n\, a^n u[n]$ | $\frac{aZ}{\lvert Z-a\rvert^2}$ |
| $n\, a^n u[-n-1]$ | $-\frac{aZ}{\lvert Z-a\rvert^2}$ |
| $a^n \cos \omega n u[n]$ | $\frac{Z^2-aZ\cos\omega}{Z^2-2aZ\cos\omega+a^2}$ |
| $a^n \sin \omega n u[n]$ | $\frac{aZ\sin\omega}{Z^2-2aZ\cos\omega+a^2}$ |

**The *z*-Plane and the Unit Circle**

- If we consider the $z$-plane, we see that $H(e^{j\hat{\omega}})$ corresponds to evaluating $H(z)$ on the unit circle




- From this interpretation we also can see why $H(e^{j\hat{\omega}})$ is periodic with period $2\pi$
  - As $\hat{\omega}$ increases it continues to sweep around the unit circle over and over again

## The Zeros and Poles of $H(z)$

- Consider

$$H(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

where we have assumed that $b_0 = 1$

- Factoring $H(z)$ results in

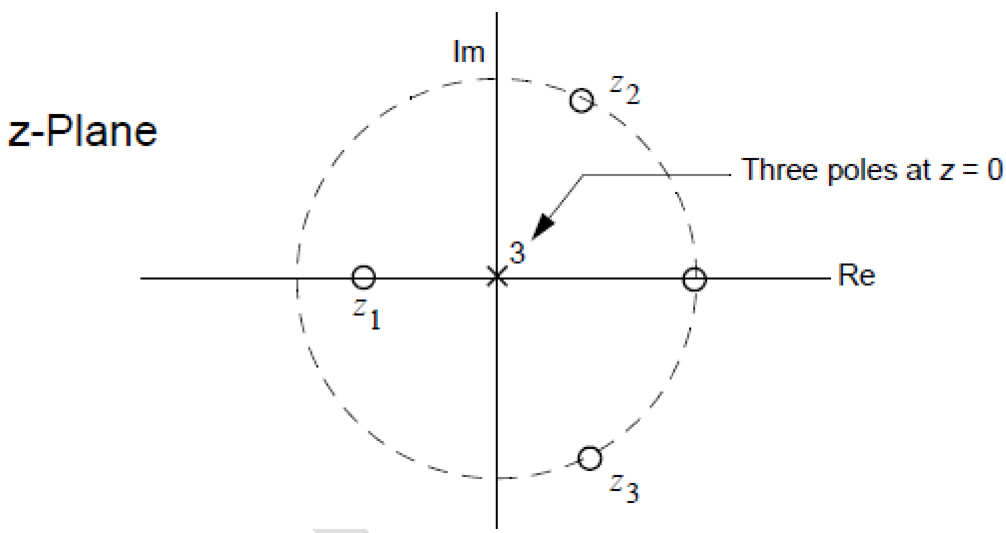
$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1})$$

- Multiplying by $z^3/z^3$ allows to write $H(z)$ in terms of positive powers of $z$

$$H(z) = \frac{z^3 + b_1 z^2 + b_2 z^1 + b_3 z^0}{z^3}$$

$$= \frac{(z - z_1)(z - z_2)(z - z_3)}{z^3}$$

- The *zeros* are the locations where $H(z) = 0$, i.e., $z_1, z_2, z_3$
- The *poles* are where $H(z) \to \infty$, i.e., $z \to 0$
- A *pole-zero plot* displays the pole and zero locations in the $z$-plane



## The Significance of the Zeros of $H(z)$

- The difference equation is the actual time domain means for calculating the filter output for a given filter input
- The difference equation coefficients are the polynomial coefficients in $H(z)$

For $x[n] = z_0^n$ we know that

$$y[n] = H(z_0)z_0^n,$$

so in particular if $z_0$ is one of the zeros of $H(z)$, $H(z_0) = 0$ and the output $y[n] = 0$

**Differentation in Z –Domain**

$$\mathcal{Z}[nx[n]] = -z\frac{d}{dz}X(z), \quad ROC = R_x$$

**Proof:**

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \frac{-1}{z}\sum_{n=-\infty}^{\infty} nx[n]z$$

i.e.,

$$\mathcal{Z}[nx[n]] = -z\frac{d}{dz}X(z)$$

**Example:** Taking derivative with respect to $z$ of the right side of

$$\mathcal{Z}[a^n u[n]] = \frac{1}{1-az^{-1}} \quad |z| > |a|$$

we get

$$\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{-az^{-2}}{(1-az^{-1})^2}$$

Due to the property of differentiation in z-domain, we have

$$\mathcal{Z}[na^n u[n]] = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > |a|$$

Note that for a different ROC $|z| < |a|$, we have

$$\mathcal{Z}[-na^n u[-n-1]] = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| < |a|$$

**Analysis and Characterization of LTI Systems Using z-Transform**

The z-transform plays a particularly important role in the analysis and representation of discrete-time LTI systems. Many properties of a system can be tied directly to characteristics of the poles, zeros, and region of convergence of the system function.

Due to its convolution property, the z-transform is a powerful tool to analyze LTI systems

$$y[n] = h[n] * x[n] \xrightarrow{\mathcal{Z}} Y(z) = H(z)X(z)$$

when the input is the eigenfunction of all LTI system, i.e., $x[n] = e^{sn} = z^n$, the

operation on this input by the system can be found by multiplying the system's eigenvalue $H(z)$ to the input:

$$y[n] = \mathcal{O}[z^n] = h[n] * z^n = H(z)z^n$$

**Causality**

A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, include infinity.

A discrete-time LTI system with rational system function H(z) is causal if and only if:
(a) the ROC is the exterior of a circle outside the outmost pole;
(b) with H(z) expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

**Stability**

An LTI system is stable if and only if the ROC of the system function H(z) includes the unit circle, |z|=1.

A causal LTI system with rational system function H(z) is stable if and only if all of the poles of H(z) lie inside the unit circle -i.e., they must all have magnitude smaller that 1.

The Transfer Function in the Z-domain

A LTI system is completely characterized by its impulse response $h[n]$ or equivalently the Z-transform of the impulse response $H(z)$ which is called the transfer function.

$$x[n] * h[n] \xrightarrow{\mathsf{Z}} X(z)H(z).$$

In case the impulse response is given to define the LTI system we can simply calculate the Z-transform to obtain :math:`H(z).

In case the system is defined with a difference equation we could first calculate the impulse response and then calculating the Z-transform. But it is far easier to calculate the Z-transform of both sides of the difference equation.

As an example consider the following difference equation:

$$y[n] = 1.5y[n-1] - 0.5y[n-2] + 0.5x[n].$$

Remember that `x[n-n_0]ztarrow z^{-n_0}X(z)$ and knowing that the Z-transform is a linear transform we can apply the Z-transform to both sides of the above equation and obtain:

$$Y(z) = 1.5z^{-1}Y(z) - 0.5z^{-2}Y(z) + 0.5X(z)$$

This can be rewritten as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.5}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{z^2}{2z^2 - 3z + 1}$$

**DIFFERENCE EQUATION**

A difference equation is an equation which expresses a relation between an independent variable and the successive values of the dependent variable or the successive differences of the dependent variable.

Difference equations arise in the situations in which the discrete values of the independent variable involve. Many practical phenomena are modelled with the help of difference equations.

Example

$$y_{x+3} + 2y_{x+2} - 3y_{x+1} + 5y_x = x^2$$

**Order of a Difference Equation :**

The difference between the largest and smallest arguments appearing in the difference equation is called its order.

**Solution of a Difference Equation :**

A solution of a difference equation is a relation between the independent variable and the dependent variable satisfying the equation.

*e.g.,* The relation $y(x) = ca^x$ is a solution of the difference equation $y(x + 1) – ay(x) = 0$, $a \neq 1$ where $c$ is an arbitrary constant.

The solution of a difference equation of order $n$ shall generally contain $n$ arbitrary constants.

A solution involving as many arbitrary constants as is the order of the equation, is called the **general solution.**

Any solution obtained from the general solution by assigning particular values to the arbitrary constants is called a **particular solution.**

In the above example, $y(x) = ca^x$ is the general solution and $y(x) = 3a^x$ is a particular solution.

A difference equation is formed by eliminating the arbitrary constants from a relation giving the order of the equation is equal to the number of arbitrary constants. The following examples illustrate the formation of difference equations :

**Example:** For the difference equation $y[n] - \frac{1}{2}y[n-1] = u[n]$ find $y[n]$ for $n \geq 0$.
Assume rest IC $y[-1] = 0$.
(Here $u[n]$ is the unit step function.)

**answer:** Rewrite the equation as $y[n] = u[n] + \frac{1}{2}y[n-1]$.

Make a table:

| $n$ | -1 | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|---|
| $u[n]$ | 0 | 1 | 1 | 1 | 1 | 1 | ... |
| $y[n]$ | 0 | 1 | 3/2 | 7/4 | 15/8 | 31/16 | ... |

We have already seen difference equations with Euler's formula. For example the IVP $y' = ky$; $y(0) = 1$ becomes the difference equation

$y_{n+1} = y_n + khy_n = (1 + kh)y_n \Leftrightarrow y_{n+1} - (1 + kh)y_n = 0.$

Here instead of $y[n]$ we wrote $y_n$

***Z*-transform** (analog of Laplace transform)

Let $x[n]$ be a sequence. Its $z$-transform is $X(z) = \sum_n x[n]z^{-n}$.

| UNIT-III | OPT 1 | OPT 2 | OPT 3 | OPT 4 |
|---|---|---|---|---|
| The z - transform F(z) of the function $f(nT) = a^{nT}$ is | $z/z{-}aT$ | $z/{-}a{-}T$ | $z/z{+}aT$ | Z/aT |
| The region of | \|z\|>1 | (Real part of | \|z\|<1 | (Real part of z) < 0 |
| Two discrete | $\delta[n-1] + \delta[n$ | $\delta[n-4]$ | $\delta[n-3]$ | $\delta[n-1]\ \delta[n-2]$ |
| For a system f | The zeros lie in | The zeros lie in | The poles lie in | The poles lie in right half of the |
| The s plane ar | $z = e^{sT}$ | $z = e^{2sT}$ | $z = 2e^{sT}$ | $z = e^{sT}/2$ |
| The similarity | Both convert fr | . Both convert d | . Both convert a | Both convert digital signal to a |
| The ROC of a s | range of z for w | range of freque | range of freque | range in which the signal is fre |
| For an expand | Inverse seque | Original seque | Negative valu | Positive values only |
| Which of the fo | x(n)+y(n) ↔X(z | x(n)+y(n) ↔X( | x(n)y(n) ↔X(z) | x(n)y(n) ↔X(z)Y(z) |
| What is the z-tr | 3/(1-2z-1)-4/(1 | 3/(1+2z-1)-4/( | 3/(1-2z)-4/(1- | None of the mentioned |
| According to Ti | zkX(z) | z-kX(z) | X(z-k) | X(z+k) |
| If X(z) is the z-t | X(az) | X(az-1) | X(a-1z) | None of the mentioned |
| If the ROC of X( | \|a\|r1<\|z\|<\|a\|r2 | \|a\|r1>\|z\|>\|a\|r2 | \|a\|r1<\|z\|>\|a\|r2 | \|a\|r1>\|z\|<\|a\|r2 |
| If X(z) is the z-t | X(-z) | X(z-1) | X-1(z) | X(Z) |
| X(z) is the z-tra | -z(dX(z))/dz | zdX(z)/dz | Z | d) z-1(dX(z))/dz |
| What is the set | Radius of conv | Radius of diver | Feasible solutio | None of the mentioned |
| What is the RO( | z=0 | z=∞ | Entire z-plane, | Entire z-plane, except at z=∞ |
| What is the | \|a\|<\|z\|<\|b\| | \|a\|>\|z\|>\|b\| | \|a\|>\|z\|<\|b\| | \|a\|<\|z\|>\|b\| |
| What is the RO | z=0 | z=∞ | Entire z-plane, | Entire z-plane, except at z=∞ |
| What is the | \|z\|>r1 | \|z\|<r1 | r2<\|z\|<r1 | z=1 |
| What is the RO( | Entire z-plane, | Entire z-plane, | Contain unit cir | contain ellipse |
| The ROC of z-tr | poles | zeros | ones | infinites |
| What is the | \|z\|<r1 | \|z\|>r1 | r2<\|z\|<r1 | Z=0 |
| If Z{x1(n)}=X1( | X1(z).X2(z) | X1(z)+X2(z) | X1(z)*X2(z) | X1(Z)-X2(Z) |
| What is the cor | {1,1,0,0,0,0,1,1 | {-1,-1,0,0,0,0,-1 | {-1,1,0,0,0,0,1,- | {1,-1,0,0,0,0,-1,1} |
| If Z{x1(n)}=X1( | X1(z).X2(z-1) | X1(z).X2(z-1) | X1(z).X2(z) | X1(z).X2(-z) |
| If x(n) is causal | x(-1) | x(1) | x(0) | Cannot be determined |
| What is the z-tr | zn0 | z-n0 | zn-n0 | zn+n0 |
| If X(z) is the z-t | X(z*) | X*(z) | X*(-z) | X*(z*) |
| If x(n) is an ima | 1/2[X(z)+X*(z* | 1/2[X(z)-X*(z* | 1/2[X(-z)-X*(z* | 1/2[X(-z)+X*(z*)]. |
| If x1(n)={1,2,3 | {1,2,3,1,1} | {1,2,3,4,5} | {1,3,5,6,2} | {1,2,6,5,3} |
| What are the va | Poles | Zeros | Solutions | None of the mentioned |
| What are the va | Poles | Zeros | Solutions | None of the mentioned |
| If X(z) has M fi | \|N-M\| poles at | \|N+M\| zeros at | \|N+M\| poles at | \|N-M\| zeros at origin(if N>M) |
| If X(z) has M fi | \|N-M\| poles at c | \|N+M\| zeros at | \|N+M\| poles at | \|N-M\| zeros at origin(if N < M) |
| The z-transform | One pole at z=0 | One pole at z=0 | One pole at z=a | One pole at z=a and one zero at |
| What are the va | Poles | Zeros | Solutions | None of the mentioned |
| If Y(z) is the z-t | (Y(z))/(X(z)) | (X(z))/(Y(z)) | Y(z).X(z) | None of the mentioned |
| What is the uni | 0.5(2)nu(n) | 2(0.5)nu(n) | 0.5(2)nu(-n) | 2(0.5)nu(-n) |
| Which of the fo | Counter integra | Expansion into | Partial fraction | All of the mentioned |
| For what kind c | All signals | Anti-causal sig | Causal signal | non-causal signal |
| What is the one | z2+2z+5+7z-1+ | 5+7z+z3 | z-2+2z-1+5+7z- | 5+7z-1+z-3 |
| What is the one | z-k | zk | 0 | 1 |
| What is the one | z-k | zk | 0 | 1 |
| The impulse re | 1/(1+a) | 1/(1-a) | a/(1+a) | a/(1-a) |
| If all the poles | Only causal | Only BIBO stab | BIBO stable an | neither BIBO stable and neither |
| If all the poles | Slow | Rapid | Constant | 0 |
| If one or more | Slow | Rapid | Constant | 0 |
| If the ROC of th | stable | Anti-causal sig | Causal signal | non-causal signal |
| A linear time in | Includes unit ci | Excludes unit c | Is an unit circle | circle |
| If all the poles | Only causal | Only BIBO stab | BIBO stable an | BIBO stable and non causal |
| If x(n) is a disc | Zero | Positive | Negative | Not defined |
| If the system is | Zero-state resp | Zero-input resp | Zero-condition | None of the mentioned |
| Zero-state resp | Zero-state resp | Forced respons | Natural respon | None of the mentioned |

| | | | | |
|---|---|---|---|---|
| The solution ol | General solutio | Particular solut | Homogenous sc | c) Complete solution |
| The total soluti | yp(n)-yh(n) | yp(n)+yh(n) | yh(n)-yp(n) | y[n]=x[n]h[n] |
| What is the par | 1/(1+a) u(n) | 1/(1+a) | 1/(1-a) u(n) | 1/(1-a) |
| The impulse res | {1,3,6,3,1} | {1,2,3,2,1} | {1,3,6,5,3} | {1,1,1,0,0} |

Answer
$z/z{-}aT$
$|z|>1$
$\delta[n-3]$
The poles lie in left half of the s plane
$z = e^{sT}$
Both convert discrete time domain to frequency spectrum domain
range of z for which the z transform converges
Inverse sequence values
x(n)+y(n) ↔X(z)+Y(z)
3/(1-2z-1)-4/(1-3z-1)
z-kX(z)
X(a-1z)
|a|r1<|z|<|a|r2
X(z-1)
-z(dX(z))/dz
Radius of convergence
Entire z-plane, except at z=0
|a|<|z|<|b|
Entire z-plane, except at z=∞
|z|>r1
Contain unit circle
poles
r2<|z|<r1
X1(z).X2(z)
{1,-1,0,0,0,0,-1,1}
X1(z).X2(z-1)
x(0)
z-n0
dX*(z*)
1/2[X(z)+X*(z*)].
{1,2,6,5,3}
Zeros
Poles
|N-M| zeros at origin(if N>M)
|N-M| poles at origin(if N < M)
One pole at z=a and one zero at z=0
Zeros
(Y(z))/(X(z))
2(0.5)nu(n)
All of the mentioned
Causal signal
5+7z-1+z-3
z-k
0
1/(1-a)
neither BIBO stable and neither causal
Rapid
Slow
Causal signal
Includes unit circle
BIBO stable and causal
Not defined
Zero-state response
Forced response

Homogenous solution
yp(n)+yh(n)
1/(1+a) u(n)
{1,3,6,5,3}

## UNIT-IV

## SYLLABUS

**Filter Concepts:** Phase Delay and Group delay, Zero-Phase Filter, Linear-Phase Filter, Simple FIR Digital Filters, Simple IIR Digital Filters, All pass Filters, Averaging Filters, Notch Filters.
**Discrete Fourier Transform:** Frequency Domain Sampling (Sampling of DTFT), The Discrete Fourier Transform (DFT) and its Inverse, DFT as a Linear transformation, Properties; Periodicity; Linearity; Circular Time Shifting; Circular Frequency Shifting.

## Unit IV

**Filter Concepts:** Phase Delay and Group delay, Zero-Phase Filter, Linear-Phase Filter, Simple FIR Digital Filters, Simple IIR Digital Filters, All pass Filters, Averaging Filters, Notch Filters.
**Discrete Fourier Transform:** Frequency Domain Sampling (Sampling of DTFT), The Discrete Fourier Transform (DFT) and its Inverse, DFT as a Linear transformation, Properties; Periodicity; Linearity; Circular Time Shifting; Circular Frequency Shifting.

### *DIGITAL* FILTER

A digital filter is just a filter that operates on digital signals, such as sound represented inside a computer. It is a computation which takes one sequence of numbers (the input signal) and produces a new sequence of numbers (the filtered output signal). The filters mentioned in the previous paragraph are not digital only because they operate on signals that are not digital. It is important to realize that a digital filter can do anything that a real-world filter can do. That is, all the filters alluded to above can be simulated to an arbitrary degree of precision digitally. Thus, a digital filter is only a formula for going from one digital signal to another. Digital filters are defined by their impulse response, h[n], or the filter output given a unit sample impulse input signal. A discrete-time unit impulse signal is defined by:

- Digital filters are often best described in terms of their *frequency response*. That is, how is a sinusoidal signal of a given frequency affected by the filter.
- The frequency response of a filter consists of its *magnitude* and *phase* responses. The magnitude response indicates the ratio of a filtered sine wave's output amplitude to its input amplitude. The phase reponse describes the phase ``offset'' or time delay experienced by a sine wave passing through a filter.

A *linear-phase filter* is typically used when a *causal* filter is needed to modify a signal'smagnitude-spectrum while preserving the signal's time-domain waveform as much as possible. Linear-phase filters have a *symmetric impulse response, e.g.*,

$$h(n) = h(N-1-n), \quad n = 0,1,2,\ldots,N-1.$$

The symmetric-impulse-response constraint means that *linear-phase filters must be FIR filters*, because a causal recursive filter cannot have a symmetric impulse response. Every real symmetric impulse response corresponds to a *real* frequency response times a *linear phase term* $e^{-j\alpha\omega T}$ where $\alpha = (N-1)/2$ is the *slope* of the linear phase. Linear phase is often ideal because a filter phase of the form $\Theta(\omega) = -\alpha\omega T$ corresponds to phase delay

$$P(\omega) \triangleq -\frac{\Theta(\omega)}{\omega} = -\frac{-\alpha\omega T}{\omega} = \alpha T = \frac{(N-1)T}{2}$$

and group delay

$$D(\omega) \triangleq -\frac{\partial}{\partial\omega}\Theta(\omega) = -\frac{\partial}{\partial\omega}(-\alpha\omega T) = \alpha T = \frac{(N-1)T}{2}.$$

That is, both the phase and group delay of a linear-phase filter are equal to $(N-1)/2$ samples of plain delay *at every frequency*.

ZERO-PHASE FILTERS

A *zero-phase filter* is a special case of a linear-phase filter in which the phase slope is $\alpha = 0$ .`The real impulse response $h(n)$ of a zero-phase filter is *even.* That is, it satisfies

$$h(n) = h(-n), \quad n \in \mathbf{Z}$$

Every even signal is symmetric, but not every symmetric signal is even. To be even, it must be symmetric about time 0. A *zero-phase filter cannot be causal.*

PHASE DELAY

The phase response $\Theta(\omega)$ of an LTI filter gives the radian phase shift added to the phase of each sinusoidal component of the input signal. It is often more intuitive to consider instead the *phase delay*, defined as

$$\boxed{P(\omega) \triangleq -\frac{\Theta(\omega)}{\omega}.} \qquad \text{(Phase Delay)}$$

The phase delay gives the *time delay* in seconds experienced by each sinusoidal component of the input signal.

For example the phase response was $\Theta(\omega) = -\omega T/2$ which corresponds to a phase delay $P(\omega) = T/2$ or one-half sample. Thus, we can say precisely that the filter $y(n) = x(n) + x(n-1)$ exhibits half a sample of time delay at every frequency.

From a sinewave-analysis point of view, if the input to a filter with frequency response is
$H(e^{j\omega T}) = G(\omega)e^{j\Theta(\omega)}$

Is $x(n) = \cos(\omega nT)$
then the output is

$$\begin{aligned} y(n) &= G(\omega)\cos[\omega nT + \Theta(\omega)] \\ &= G(\omega)\cos\{\omega[nT - P(\omega)]\} \end{aligned}$$

and it can be clearly seen in this form that the phase delay expresses the phase response as a time delay in seconds.
GROUP DELAY

A more commonly encountered representation of filter phase response is called the *group delay*, defined by

$$\boxed{D(\omega) \triangleq -\frac{d}{d\omega}\Theta(\omega).} \qquad \text{(Group Delay)}$$

For linear phase responses, *i.e.*, $\Theta(\omega) = -\alpha\omega$ for some constant $\alpha$ he group delay and thephase delay are identical, and each may be interpreted as time delay. If the phase response is nonlinear, then the relative phases of thesinusoidal signal components are generally altered by the filter. A nonlinear phase response normally causes a ``smearing" of attack transients such as in percussive sounds. Another term for this type of phase distortion is *phase dispersion*.
An example of a linear phase response is that of the simplest lowpass filter,
$\Theta(\omega) = -\omega T/2 \;\Rightarrow\; P(\omega) = D(\omega) = T/2$

Thus, both the phase delay and the group delay of the simplest lowpass filter are equal to half a sample at every frequency.

**LINEAR-PHASE FILTER**

**Linear phase** is a property of a filter, where the phase response of the filter is a linear function of frequency. The result is that all frequency components of the input signal are shifted in time (usually delayed) by the same constant amount (the slope of the linear function), which is referred to as the phase delay. And consequently, there is no phase distortion due to the time delay of frequencies relative to one another.

For discrete-time signals, perfect linear phase is easily achieved with a finite impulse response (FIR) filter. Approximations can be achieved with infinite impulse response (IIR) designs, which are more computationally efficient. Several techniques are**:**

- a Bessel transfer function which has a maximally flat group delay

- a maximally flat group delay approximation function
- a phase equalizer

If a discrete-time cosine signal

$$x_1(n) = \cos(\omega_1 n + \phi_1)$$

is processed through a discrete-time filter with frequency response

$$H^f(\omega) = A(\omega) \cdot e^{j\theta(\omega)}$$

then the output signal is given by

$$y_1(n) = A(\omega_1)\ \cos(\omega_1 n + \phi_1 + \theta(\omega_1))$$

or

$$y_1(n) = A(\omega_1)\ \cos\left(\omega_1 \left(n + \frac{\theta(\omega_1)}{\omega_1}\right) + \phi_1\right).$$

The LTI system has the effect of scaling the cosine signal and delaying it by $-\theta(\omega_1)/\omega_1$.

$$\Longrightarrow \qquad \frac{\theta(\omega)}{\omega} = \text{constant}$$

$$\Longrightarrow \qquad \theta(\omega) = K\,\omega$$

$$\Longrightarrow \qquad \text{The phase is linear}$$

The function $\theta(\omega)/\omega$ is called the *phase delay*. A linear phase filter therefore has constant phase delay.

Linear-phase FIR filter can be divided into four basic types.

| Type | impulse response | |
|---|---|---|
| I | symmetric | length is odd |
| II | symmetric | length is even |
| III | anti-symmetric | length is odd |
| IV | anti-symmetric | length is even |

DISCRETE FOURIER TRANSFORM-DFT
Like continuous time signal Fourier transform, discrete time Fourier Transform can be used to represent a discrete sequence into its equivalent frequency domain representation and LTI discrete time system and develop various computational algorithms.
X (jω) in continuous F.T, is a continuous function of x(n). However, DFT deals with representing x(n) with samples of its spectrum X(ω). Hence, this mathematical tool carries much importance computationally in convenient representation. Both, periodic and non-periodic sequences can be processed through this tool. The periodic sequences need to be sampled by extending the period to infinity.
Frequency Domain Sampling
From the introduction, it is clear that we need to know how to proceed through frequency domain sampling i.e. sampling X(ω). Hence, the relationship between sampled Fourier transform and DFT is established in the following manner. Similarly, periodic sequences can fit to this tool by extending the period N to infinity.
Let an Non periodic sequence be

$$X(n) = \lim_{N\to\infty} x_N(n)$$

Defining its Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn} X(K\delta\omega)$$

Here, X(ω) is sampled periodically, at every δω radian interval.

As X(ω) is periodic in 2π radians, we require samples only in fundamental range. The samples are taken after equidistant intervals in the frequency range 0≤ω≤2π. Spacing between equivalent intervals is $\delta\omega = \frac{2\pi}{N}k$

Now evaluating, $\omega = \frac{2\pi}{N}k$

$$X(\frac{2\pi}{N}k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N},$$

where k=0,1,......N-1

After subdividing the above, and interchanging the order of summation

$$X(\frac{2\pi}{N}k) = \sum_{n=0}^{N-1}[\sum_{l=-\infty}^{\infty} x(n-Nl)]e^{-j2\pi nk/N}$$

$\sum_{l=-\infty}^{\infty} x(n-Nl) = x_p(n) = a$ *periodic function of period N and its fourier series* $= \sum_{k=0}^{N-1} C_k e^{j2\pi nk/N}$

where, n = 0,1,.....,N-1; 'p'- stands for periodic entity or function

The Fourier coefficients are,

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi nk/N} \qquad k=0,1,...,N-1$$

Comparing equations 3 and 4, we get ;

$$NC_k = X(\frac{2\pi}{N}k) \qquad k=0,1,...,N-1$$

$$NC_k = X(\frac{2\pi}{N}k) = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x_p(n)e^{-j2\pi nk/N}$$

From Fourier series expansion,

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} NC_k e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi nk/N}$$

Where n=0,1,...,N-1

Here, we got the periodic signal from X(ω). $x(n)$ can be extracted from $x_p(n)$ only, if there is no aliasing in the time domain. $N \geq L$

N = period of $x_p(n)$ L= period of $x(n)$

$$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & Otherwise \end{cases}$$

The mapping is achieved in this manner.

The **inverse DFT** is given by:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi\frac{kn}{N}}.$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left\{\sum_{m=0}^{N-1} x(m)e^{-j2\pi\frac{km}{N}}\right\} e^{j2\pi\frac{kn}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) \underbrace{\left\{\frac{1}{N}\sum_{k=0}^{N-1} e^{-j2\pi\frac{k(m-n)}{N}}\right\}}_{\delta(m-n)} = x(n).$$

# Properties of DFT

## Linearity

It states that the DFT of a combination of signals is equal to the sum of DFT of individual signals. Let us take two signals $x_1(n)$ and $x_2(n)$, whose DFT s are $X_1(\omega)$ and $X_2(\omega)$ respectively. So, if

$$x_1(n) \rightarrow X_1(\omega) \quad \text{and} \quad x_2(n) \rightarrow X_2(\omega)$$

Then $$ax_1(n) + bx_2(n) \rightarrow aX_1(\omega) + bX_2(\omega)$$

where **a** and **b** are constants.

# Symmetry

The symmetry properties of DFT can be derived in a similar way as we derived DTFT symmetry properties. We know that DFT of sequence x(n) is denoted by X(K). Now, if x(n) and X(K) are complex valued sequence, then it can be represented as under

$$x(n) = x_R(n) + jx_1(n), 0 \leq n \leq N-1$$

And $$X(K) = X_R(K) + jX_1(K), 0 \leq K \leq N-1$$

## Duality Property

Let us consider a signal x(n), whose DFT is given as X(K). Let the finite duration sequence be X(N). Then according to duality theorem,

If, $$x(n) \longleftrightarrow X(K)$$

Then, $$X(N) \longleftrightarrow Nx[((-k))_N]$$

So, by using this theorem if we know DFT, we can easily find the finite duration sequence.

## Complex Conjugate Properties

Suppose, there is a signal x(n), whose DFT is also known to us as X(K). Now, if the complex conjugate of the signal is given as x*(n), then we can easily find the DFT without doing much calculation by using the theorem shown below.

If, $x(n) \longleftrightarrow X(K)$

Then, $x*(n) \longleftrightarrow X*((K))_N = X*(N-K)$

## Circular Frequency Shift

The multiplication of the sequence x(n) with the complex exponential sequence $e^{j2\Pi kn/N}$ is equivalent to the circular shift of the DFT by L units in frequency. This is the dual to the circular time shifting property.

If, $x(n) \longleftrightarrow X(K)$

Then, $x(n)e^{j2\Pi Kn/N} \longleftrightarrow X((K-L))_N$

## Multiplication of Two Sequence

If there are two signal $x_1(n)$ and $x_2(n)$ and their respective DFTs are $X_1(k)$ and $X_2(K)$, then multiplication of signals in time sequence corresponds to circular convolution of their DFTs.

If, $x_1(n) \longleftrightarrow X_1(K) \quad \& \quad x_2(n) \longleftrightarrow X_2(K)$

Then, $x_1(n) \times x_2(n) \longleftrightarrow X_1(K) © X_2(K)$

## Parseval's Theorem

For complex valued sequences x(n) and y(n), in general

If, $x(n) \longleftrightarrow X(K) \quad \& \quad y(n) \longleftrightarrow Y(K)$

Then, $\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(K)Y^*(K)$

DFT Circular Convolution

Let us take two finite duration sequences $x_1(n)$ and $x_2(n)$, having integer length as N. Their DFTs are $X_1(K)$ and $X_2(K)$ respectively, which is shown below −

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{j2\Pi kn}{N}} \quad k = 0, 1, 2 \ldots N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{\frac{j2\Pi kn}{N}} \quad k = 0, 1, 2 \ldots N-1$$

Now, we will try to find the DFT of another sequence $x_3(n)$, which is given as $X_3(K)$

$$X_3(K) = X_1(K) \times X_2(K)$$

By taking the IDFT of the above we get

$$x_3(n) = \frac{1}{N} \sum_{n=0}^{N-1} X_3(K) e^{\frac{j2\Pi kn}{N}}$$

After solving the above equation, finally, we get

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$$

$$m = 0, 1, 2 \ldots N-1$$

# Methods of Circular Convolution

Generally, there are two methods, which are adopted to perform circular convolution and they are –

Concentric circle method,

Matrix multiplication method.

## Concentric Circle Method

Let $x_1(n)$ and $x_2(n)$ be two given sequences. The steps followed for circular convolution of $x_1(n)$ and $x_2(n)$ are

Take two concentric circles. Plot N samples of $x_1(n)$ on the circumference of the outer circle (maintaining equal distance successive points) in anti-clockwise direction.

For plotting $x_2(n)$, plot N samples of $x_2(n)$ in clockwise direction on the inner circle, starting sample placed at the same point as $0^{th}$ sample of $x_1(n)$

Multiply corresponding samples on the two circles and add them to get output.

Rotate the inner circle anti-clockwise with one sample at a time.

## Matrix Multiplication Method

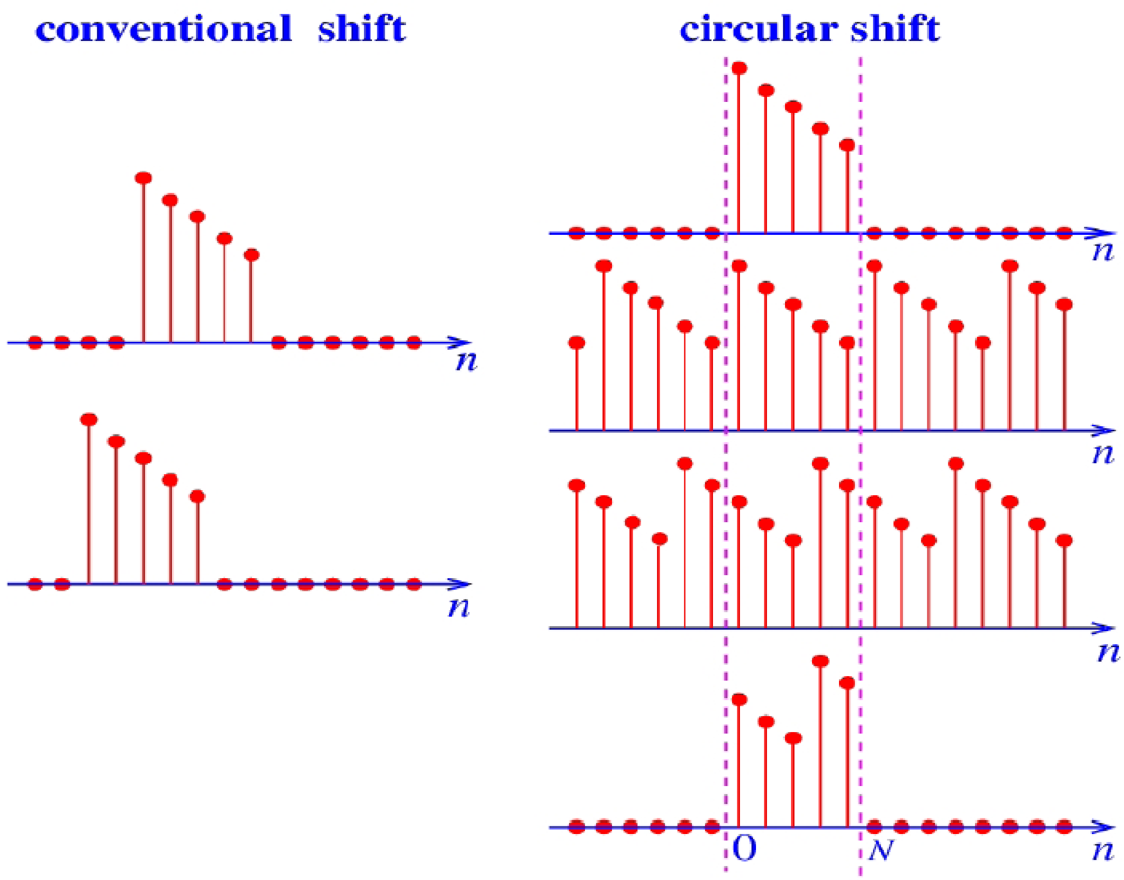
Matrix method represents the two given sequence $x_1(n)$ and $x_2(n)$ in matrix form.

One of the given sequences is repeated via circular shift of one sample at a time to form a N X N matrix.

The other sequence is represented as column matrix.

The multiplication of two matrices give the result of circular convolution.

# DFT: Circular Shift




$$\sum_{n=0}^{N-1} x((n-m)\text{mod}N)W^{kn}$$

$$= \quad W^{km}\sum_{n=0}^{N-1} x((n-m)\text{mod}N)W^{k(n-m)}$$

$$= W^{km} \sum_{n=0}^{N-1} x((n-m)\text{mod}N) W^{k(n-m)\text{mod}N}$$
$$= W^{km} X(k),$$

where we use the facts that $W^{k(l\text{mod}N)} = W^{kl}$ and that the order of summation in DFT does not change its result.

Similarly, if $X(k) = \mathcal{DFT}\{x(n)\}$, then

$$X((k-m)\text{mod}N) = \mathcal{DFT}\{x(n)e^{j2\pi\frac{mn}{N}}\}.$$

If

$$G[k] := W_N^{-mk} \cdot X[k]$$

then

$$g[n] = x[\langle n - m \rangle_N].$$

**Derivation:**

Begin with the Inverse DFT.

$$\begin{aligned} g[n] &= \frac{1}{N} \sum_{k=0}^{N-1} G[k] \, W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} \, X[k] \, W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{k(n-m)} \\ &= x[n-m] \\ &= x[\langle n - m \rangle_N]. \end{aligned}$$

Given an $N$-point signal $\{x[n],\ n \in \mathbb{Z}_N\}$, the signal

$$g[n] := x[\langle n-m \rangle_N]$$

represents a circular shift of $x[n]$ by $m$ samples to the right. For example, if

$$g[n] := x[\langle n-1 \rangle_N]$$

then

$$\begin{aligned} g[0] &= x[\langle -1 \rangle_N] = x[N-1] \\ g[1] &= x[\langle 0 \rangle_N] = x[0] \\ g[2] &= x[\langle 1 \rangle_N] = x[1] \\ &\vdots \\ g[N-1] &= x[\langle N-2 \rangle_N] = x[N-2] \end{aligned}$$

For example, if $x[n]$ is the 4-point signal

$$x[n] = (1, 3, 5, 2)$$

then

$$x[\langle n-1 \rangle_N] = (2, 1, 3, 5).$$

$x[\langle n-m \rangle_N]$ represents a *circular* shift by $m$ samples.

## circular shift in frequency

If

$$g[n] := W_N^{mn} \cdot x[n]$$

then

$$G[k] = X[\langle k - m \rangle_N].$$

**Derivation:**

Begin with the DFT.

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} g[n]\, W_N^{-nk} \\ &= \sum_{n=0}^{N-1} W_N^{mn}\, x[n]\, W_N^{-nk} \\ &= \sum_{n=0}^{N-1} x[n]\, W_N^{-n(k-m)} \\ &= X[k-m] \\ &= X[\langle k - m \rangle_N]. \end{aligned}$$

Verify Parseval's theorem of the sequence $x(n) = \frac{1^n}{4} u(n)$

**Solution –** $$\sum_{-\infty}^{\infty} |x_1(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X_1(e^{j\omega})|^2 d\omega$$

L.H.S $$\sum_{-\infty}^{\infty} |x_1(n)|^2$$

$$= \sum_{-\infty}^{\infty} x(n)x^*(n)$$

$$= \sum_{-\infty}^{\infty} (\frac{1}{4})^{2n} u(n) = \frac{1}{1-\frac{1}{16}} = \frac{16}{15}$$

R.H.S. $X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e-j\omega} = \frac{1}{1-0.25\cos\omega+j0.25\sin\omega}$

$\Longleftrightarrow X^*(e^{j\omega}) = \frac{1}{1-0.25\cos\omega-j0.25\sin\omega}$

Calculating, $X(e^{j\omega}).X^*(e^{j\omega})$

$$= \frac{1}{(1-0.25\cos\omega)^2+(0.25\sin\omega)^2} = \frac{1}{1.0625-0.5\cos\omega}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1.0625-0.5\cos\omega} d\omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1.0625-0.5\cos\omega} d\omega = 16/15$$

We can see that, LHS = RHS. (Hence Proved)

Compute the N-point DFT of $x(n) = 3\delta(n)$

**Solution** – We know that,

$$X(K) = \sum_{n=0}^{N-1} x(n)e^{\frac{j2\Pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} 3\delta(n)e^{\frac{j2\Pi kn}{N}}$$

$$= 3\delta(0) \times e^{0} = 1$$

So, $x(k) = 3, 0 \leq k \leq N-1$ ... Ans.

Compute the N-point DFT of $x(n) = 7(n - n_0)$

**Solution** – We know that,

$$X(K) = \sum_{n=0}^{N-1} x(n)e^{\frac{j2\Pi kn}{N}}$$

Substituting the value of x(n),

$$\sum_{n=0}^{N-1} 7\delta(n - n_0)e^{-\frac{j2\Pi kn}{N}}$$

$$= e^{-kj14\Pi kn_0/N}$$

CIRCULAR TIME SHIFTING

If

$$g[n] := x[\langle -n \rangle_N]$$

then

$$G[k] = X[\langle -k \rangle_N].$$

**Derivation:**

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} x[\langle -n \rangle_N] W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x[m] W_N^{-\langle -m \rangle_N k} \\ &= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\ &= X[-k] \\ &= X[\langle -k \rangle_N] \end{aligned}$$

where we used the change of variables $m = \langle -n \rangle_N$ (in which case $n = \langle -m \rangle_N$ for $0 \leq n \leq N-1$).

## UNIT-IV

| | OPT 1 | OPT 2 |
|---|---|---|
| In the Frequency Transformations of the analog domain the transformat | Low Pass | Lowpass t |
| In the Frequency Transformations of the analog domain the transformat | Low Pass | Lowpass t |
| The magnitude response of the following filter decreases monotonically | Butterwo | Chebyshe |
| The transition band is more in | Butterwo | Chebyshe |
| . The poles of Butterworth filter lies on | sphere | circle |
| I I R digital filters are of the following nature | Recursive | Non Recu |
| In I I R digital filter the present output depends on | Present a | Present ir |
| Which of the following is best suited for I I R filter when compared with t | Lower sid | Higher Sic |
| In the case of I I R filter which of the following is true if the phase distort | More para | More mer |
| A causal and stable I I R filter has | Linear ph | No Linear |
| Neither the Impulse response nor the phase response of the analog filte | The meth | Impulse ir |
| Out of the given I I R filters the following filter is the efficient one | Circular f | Elliptical f |
| What is the disadvantage of impulse invariant method | Aliasing | ne to one |
| Which of the I I R Filter design method is antialiasing method? | The meth | Impulse ir |
| The nonlinear relation between the analog and digital frequencies is call | aliasing | warping |
| The most common technique for the design of I I R Digital filter is | Direct M | In direct r |
| In the design a IIR Digital filter for the conversion of analog filter in to Di | The axis i | The Left H |
| The I I R filter design method thatovercomes the limitation of applicabili | Approxim | Impulse I |
| The direct form II for realisation involves | The realis | Realisatio |
| The direct form II for realisation involves | The realis | Realisatio |
| The direct form II for realisation involves | The realis | Realisatio |
| The direct form II for realisation involves | The realis | Realisatio |
| The cascade realisation of IIR systems involves | The trans | The trans |
| The cascade realisation of IIR systems involves | The trans | The trans |
| The advantage of using the cascade form of realisation is | It has sam | The num |
| The advantage of using the cascade form of realisation is | It hasdiffe | The num |
| Which among the following represent/s the characteristic/s of an ideal f | Constant | infinite ga |
| Which among the following represent/s the characteristic/s of an ideal f | zero gain | zero gain |
| Which among the following represent/s the characteristic/s of an ideal f | zero gain | constant |
| FIR filters ________ | are non-r | causal |
| FIR filters ________ | causal | do not ad |
| In tapped delay line filter, the tapped line is also known as | Pick-on n | Pick-off n |
| How is the sensitivity of filter coefficient quantization for FIR filters? | Low | Moderate |
| Decimation is a process in which the sampling rate is | enhanced | stable |
| Anti-imaging filter with cut-off frequency $\omega_c = \pi/ I$ is specifically used __ | Before | At the tin |
| The IIR filter designing involves | designing | Designing |
| IIR filter design by approximation of derivatives has the limitations | Used only | Used for l |
| IIR filter design by approximation of derivatives has the limitations | Used only | Used for |
| The filter that may not be realized by approximation of derivatives techn | Band pass | #NAME? |
| The filter that may not be realized by approximation of derivatives techn | Band pass | Band |
| In direct form for realisation of IIR filters, | Denomina | Multiplier |
| In direct form for realisation of IIR filters, | Denomina | Multiplier |
| Roll-off factor is | The band | The perfo |
| The DFT is preferred for | Its ability | Removal |
| The DFT is preferred for | Filter des | Removal o |
| Frequency selectivity characteristics of DFT refers to | Ability to | Ability to |
| DIT algorithm divides the sequence into | Positive a | Even and |
| The transformations are required for | Analysis | Quantizat |
| The transformations are required for | Easier op | Quantizat |
| The computational procedure for Decimation in frequency algorithm tak | Log2 N st | 2Log2 N s |
| Product of one even and one odd function is | even | odd |
| If ƒ(x,y) is imaginary, then its Fourier transform is | conjuga | |
| ƒ(0,0) is sometimes called | ac | dc |

| | | |
|---|---|---|
| Even functions are said to be | symme | antisy |
| Linear functions possesses property of | | homog |
| Continuous functions are sampled to form a | Fourier | Fourier |
| 2D Fourier transform and its inverse are infinitely | aperiod | periodi |
| Odd functions are said to be | symme | antisy |
| Gradient computation equation is | | \|Gx\|- |

| OPT 3 | OPT 4 | ANSWER |
|---|---|---|
| Lowpass t | Lowpass to Bandrej | Lowpass to Highpass |
| Lowpass t | Lowpass to Bandrej | Lowpass to Bandreject |
| Chebyshev type - 2 | | Butterworth Filter |
| Chebyshe | FIR Filter | Butterworth Filter |
| ellipse | parabola | circle |
| Reversive | Non Reversive | Recursive |
| Present in | Present Input, Previ | Present Input, Previous input and output |
| Lower sid | No sidelobes in stop | Lower sidelobes in stopband |
| Lower co | Higher computation | Lower computational Complexity |
| Linear an | No Amplitude | No Linear phase |
| Bilinear tr | Matched Z - transfo | Bilinear transformation |
| Rectangul | Chebyshev filter | Elliptical filter |
| anti aliasi | warping | Aliasing |
| Bilinear t | Matched Z - transfo | Bilinear transformation |
| prewarpi | antialiasing | warping |
| Recursive | non recursive meth | In direct method |
| The Left H | The Right Half Plane | The Left Half Plane(LHP) of the s - plane should map in to the |
| Bilinear T | Frequency sampling | Impulse Invariance |
| division o | subtraction of two t | The realisation of transfer function into two parts |
| division o | subtraction of two t | Realisation after fraction |
| Product o | subtraction of two t | Product of two transfer functions |
| division o | sum of two transfer | sum of two transfer functions |
| Factoring | integral of the trans | The transfer function broken into product of transfer functio |
| Factoring | integral of the trans | The transfer function divided into addition of transfer functi |
| The numb | Over all transfer fu | It has same number of poles and zeros as that of individual c |
| The num | Over all transfer fun | Over all transfer function may be determined |
| Non linea | finite band width | Constant gain in passband |
| Non linea | finite band width | zero gain in stop band |
| linear ph | finite band width | linear phase response |
| are recur | use feedback | are non-recursive |
| use feedb | are recursive | do not adopt any feedback |
| Pick-up n | Pick-down node | Pick-off node |
| High | Unpredictable | Low |
| reduced | unpredictable | reduced |
| After | All of the above | After |
| Designing | Designing of digital | Designing of digital filter in analog domain and transforming |
| band pass | filters having smalle | Used only for transforming analog high pass filters |
| Used only | Used for band pass | Used only for transforming analog low pass filters |
| Low pass | All pass filter | Band pass filters |
| Low pass | All pass filter | Band |
| Multiplier | all the above | Multipliers in the feedback paths are the negatives of the den |
| Numerato | all the above | Numerator coefficients are the multipliers in the feed forwar |
| Aliasing e | None of the above | The bandwidth occupied beyond the Nyquist Bandwidth of t |
| Quantizat | filter analysis | Its ability to determine the frequency component of the signa |
| Quantiza | sampling | Filter design |
| Ability to | None of the above | Ability to resolve different frequency components from inpu |
| Upper hig | Small and large san | Even and odd samples |
| Modulati | sampling | Analysis in time or frequency domain |
| Modulati | sampling | Easier operations |
| Log2 N^ s | Log2 N/2 stages | Log2 N stages |
| prime | aliasing | odd |
| antiher | symme | antiher |
| jaggy | | dc |

| | | |
|---|---|---|
| periodi | aperiod | symme |
| multipl | Both A | Both A |
| fast | digital | digital |
| linear | non | periodi |
| periodi | aperiod | antisy |
| \|Gx\|/\|G | \|Gx\|x\|G | \|Gx\|+\|G |

e unit circle in the Z -plane

ons
ions
omponents

g into digital domain

nominator coefficients
rd paths
he filter
al

ut signal

# UNIT-V

## SYLLABUS

**Fast Fourier Transform:** Direct Computation of the DFT, Symmetry and Periodicity, Properties of the Twiddle factor (*WN*), Radix-2 FFT Algorithms; Decimation-In-Time (DIT) FFT Algorithm; Decimation-In-Frequency (DIF) FFT Algorithm, Inverse DFT Using FFT Algorithms. **Realization of Digital Filters:** Non Recursive and Recursive Structures, Canonic and Non Canonic Structures, Equivalent Structures (Transposed Structure), FIR Filter structures; Direct-Form; Cascade-Form; Basic structures for IIR systems; Direct-Form I.

A fast Fourier transform (FFT) is any fast algorithm for computi
the DFT. The development of FFT algorithms had a tremendo
impact on computational aspects of signal processing and appli
science. The DFT of an $N$-point signal

$$\{x[n], 0 \le n \le N-1\}$$

is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{-kn}, \qquad 0 \le k \le N-1$$

where

$$W_N = e^{j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + j\,\sin\left(\frac{2\pi}{N}\right)$$

is the principal $N$-th root of unity.

**DIRECT DFT COMPUTATION**

\

Direct computation of $X[k]$ for $0 \leq k \leq N-1$ requires

$$(N-1)^2 \text{ complex multiplications}$$

$$N(N-1) \text{ complex additions}$$

# DFT as a Linear Transformation

- Matrix representation of DFT

Definition of DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0,1,\ldots,N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad n = 0,1,\ldots,N-1$$

where

Let

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix},$$

and

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Thus,

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \qquad N\text{ - point DFT}$$

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \qquad N\text{ - point IDFT}$$

$$= \frac{1}{N}\mathbf{W}_N^{*} \mathbf{X}_N$$

Because the matrix (transformation) $\mathbf{W}_N$ has a specific structure and because $W_N^k$ has particular values (for some $k$ and $n$), we can reduce the number of arithmetic operations for computing this transform.

*Example* $x[n] = [0\ 1\ 2\ 3]$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Only additions are needed to compute this specific transform.
(This is a well-known *radix-4 FFT*)

Thus, the DFT of $x[n]$ is $\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

# Fast Fourier Transform

-- Highly efficient algorithms for computing DFT

- General principle: *Divide-and-conquer*
- Specific properties of $W_N^k$
  - Complex conjugate symmetry: $W_N^{-kn} = (W_N^{kn})^*$
  - Symmetry: $W_N^{k+N/2} = -W_N^k$
  - Periodicity: $W_N^{k+N} = W_N^k$
  - Particular values of $k$ and $n$: e.g., radix-4 FFT (no multiplications)
- Direct computation of DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\cdot W_N^{kn}, \quad k = 0,1,\ldots,N-1$$
$$= \sum_{n=0}^{N-1} \left\{ \begin{array}{l} \left[\mathrm{Re}(x[n])\cdot \mathrm{Re}(W_N^{kn}) - \mathrm{Im}(x[n])\cdot \mathrm{Im}(W_N^{kn})\right] + \\ j\left[\mathrm{Re}(x[n])\cdot \mathrm{Im}(W_N^{kn}) + \mathrm{Im}(x[n])\mathrm{Re}(W_N^{kn})\right] \end{array} \right\}$$

For each $k$, we need $N$ complex multiplications and $N$-1 complex additions. → 4$N$ real multiplications and 4$N$-2 real additions.

We will show how to use the properties of $W_N^k$ to reduce computations.

- Radix-2 algorithms: Decimation-in-time; Decimation-in-frequency
- Composite $N$ algorithms: Cooley-Tukey; Prime factor
- Winograd algorithm
- Chirp transform algorithm

## RADIX-2 FFT

The radix-2 FFT algorithms are used for data vectors of lengths $N = 2^K$. They proceed by dividing the DFT into two DFTs of length $N/2$ each, and iterating. There are several types of radix-2 FFT algorithms, the most common being the *decimation-in-time* (DIT) and the *decimation-in-frequency* (DIF).

The development of the FFT will call on two properties of $W_N$.

The first property is:

$$\boxed{W_N^2 = W_{N/2}}$$

which is derived as

$$\begin{aligned} W_N^2 &= e^{-j\frac{2\pi}{N}\cdot 2} \\ &= e^{-j\frac{2\pi}{N/2}} \\ &= W_{N/2}. \end{aligned}$$

More generally, we have

$$W_N^{2nk} = W_{N/2}^{nk}.$$

The second property is:

$$\boxed{W_N^{k+\frac{N}{2}} = -W_N^k}$$

which is derived as

$$\begin{aligned} W_N^{k+\frac{N}{2}} &= e^{j\frac{2\pi}{N}(k+\frac{N}{2})} \\ &= e^{j\frac{2\pi}{N}k} \cdot e^{j\frac{2\pi}{N}(\frac{N}{2})} \\ &= e^{j\frac{2\pi}{N}k} \cdot e^{j\pi} \\ &= -e^{j\frac{2\pi}{N}k} \\ &= -W_N^k \end{aligned}$$

# Radix-2 Decimation-in-time Algorithms

-- Assume $N$-point DFT and $N = 2^{\nu}$

- Idea: $N$-point DFT → $N/2$-point DFT → $N/4$-point DFT

  $N/4$-point DFT

  $N/2$-point DFT → $N/4$-point DFT

  $N/4$-point DFT

- Sequence: $x[0]\ \ x[1]\ \ x[2]\ \ x[3]\ \ \cdots\ \ x[n/2]\ \ \cdots\ \ x[N-1]$

  Even index: $x[0]\ \ x[2]\ \ \cdots\ \ x[N-2]$

  Odd index: $x[1]\ \ x[3]\ \ \cdots\ \ x[N-1]$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0,1,\ldots,N-1$$

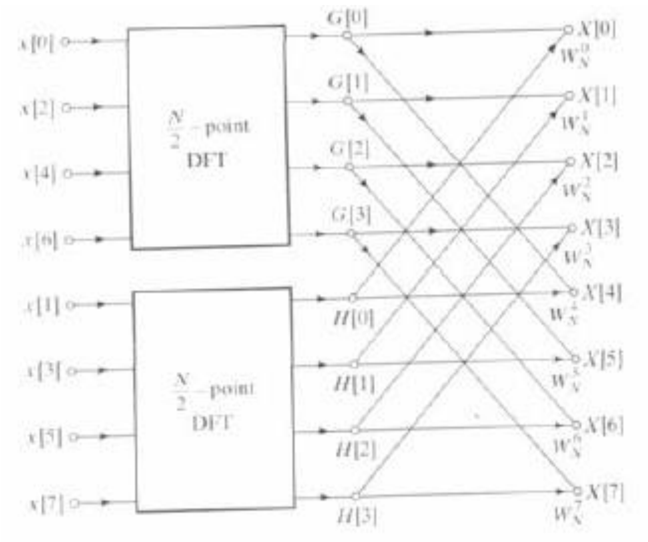
$$= \underbrace{\sum_{n\text{ even}} x[n] W_N^{kn}}_{n=2r} + \underbrace{\sum_{n\text{ odd}} x[n] W_N^{kn}}_{n=2r+1}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{(2r+1)k}$$

$$\because W_N^2 = e^{-2j\left(\frac{2\pi}{N}\right)} = e^{-2j\left(\frac{\pi}{N/2}\right)} = W_{N/2}$$

$$X[k] = \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}}$$

$$= G[k] + W_N^k H[k]$$




- Comparison:

(a) Direct computation of $N$-point DFT ($N$ frequency samples):

~ $N^2$ complex multiplications and $N^2$ complex adds

(b) Direct computation of $N/2$-point DFT:

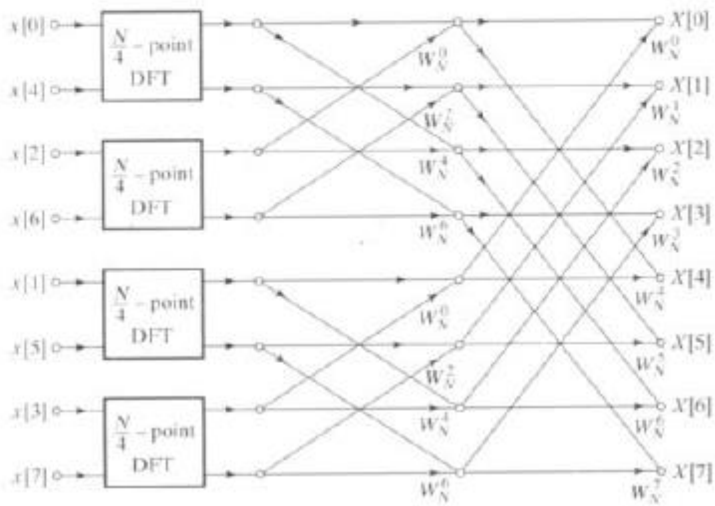
~ $\left(\frac{N}{2}\right)^2$ complex multiplications and $\left(\frac{N}{2}\right)^2$ complex adds

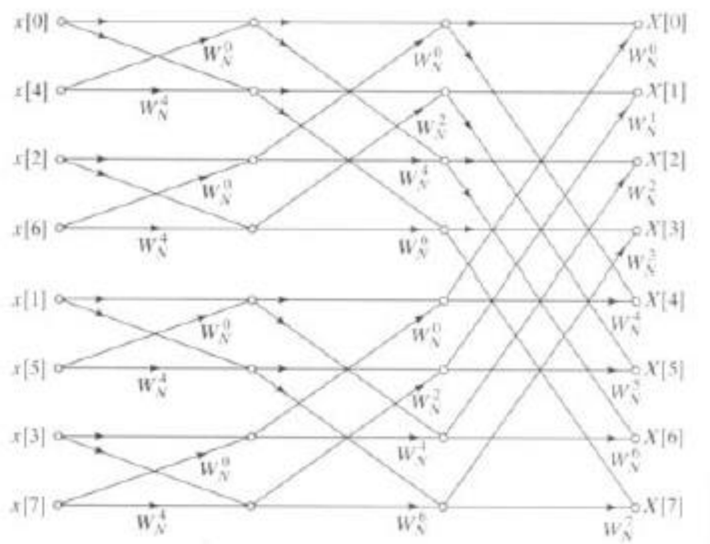
+ additional $N$ complex multis and $N$ complex adds

~ (Total:) $N+2\left(\frac{N}{2}\right)^2 = N+\frac{N^2}{2}$ complex multis and adds

(c) $\log_2 N$ -stage FFT

Since $N=2^{\nu}$, we can further break $N/2$-point DFT into two $N/4$-point DFT and so on.



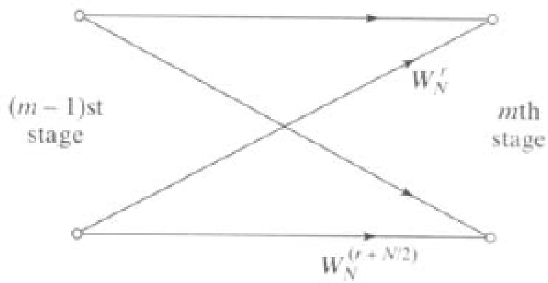




At each stage: ~ $N$ complex multis and adds

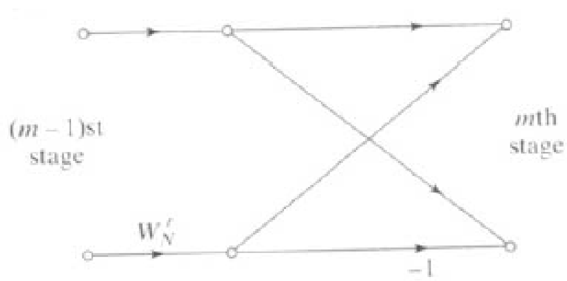
Total: ~ $N \log_2 N$ complex multis and adds (--> $\frac{N}{2}\log_2 N$)

**Butterfly**: Basic unit in FFT

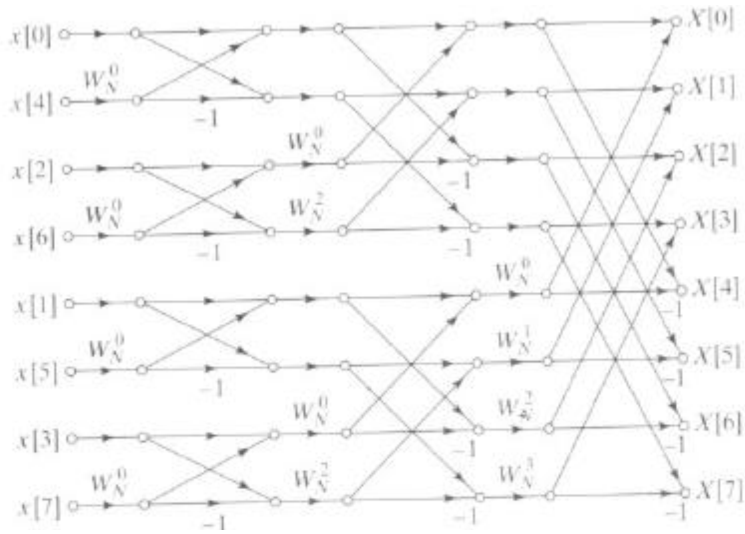
Two multiplications:




One multiplication:



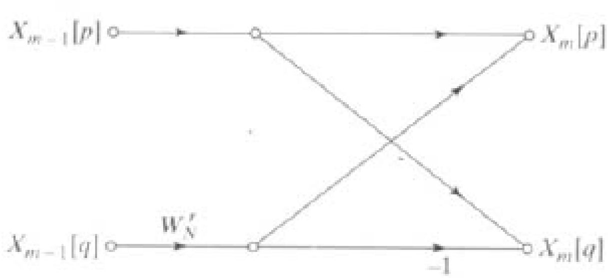

8-Point DFT




- **In-place** computations

  Only two registers are needed for computing a butterfly unit.

$$X_m[p] = X_{m-1}[p] + W_N^r X_{m-1}[q]$$
$$X_m[q] = X_{m-1}[p] - W_N^r X_{m-1}[q]$$



# Radix-2 Decimation-in-frequency Algorithms

- Dividing the output sequence $X[k]$ into smaller pieces.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0,1,\ldots,N-1$$

If $k$ is even, $k = 2r$.

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad r = 0,1,\cdots,\frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2nr} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2nr} \qquad n \leftarrow (n + N/2)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2nr} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] \cdot W_N^{2r\left(n+\frac{N}{2}\right)}$$

$$\because W_N^{2r[n+N/2]} = W_N^{2rn} W_N^{rN} = W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x\left[n + \frac{N}{2}\right] \right) \cdot W_N^{2nr}$$

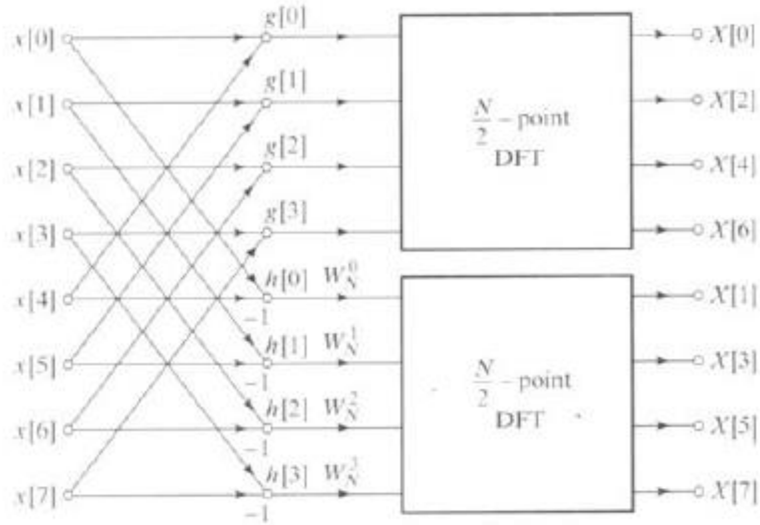
$$= \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x\left[n + \frac{N}{2}\right] \right) \cdot W_{N/2}^{nr}$$

Similarly, if $k$ is odd, $k = 2r + 1$.

$$X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] - x\left[n + \frac{N}{2}\right] \right) \cdot W_N^{n} \cdot W_{N/2}^{nr}$$

$$\begin{cases} X[2r] = \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x\left[n + \frac{N}{2}\right] \right) \cdot W_{N/2}^{nr} \\ X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] - x\left[n + \frac{N}{2}\right] \right) \cdot W_N^{n} \cdot W_{N/2}^{nr} \end{cases}$$

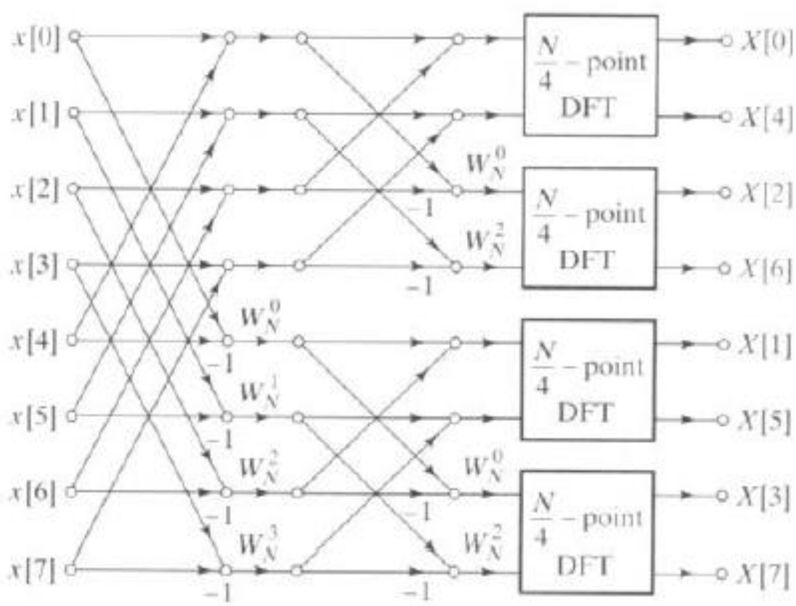
Let $\begin{cases} g[n] = x[n] + x\left[n + \frac{N}{2}\right] \\ h[n] = x[n] - x\left[n + \frac{N}{2}\right] \end{cases}$

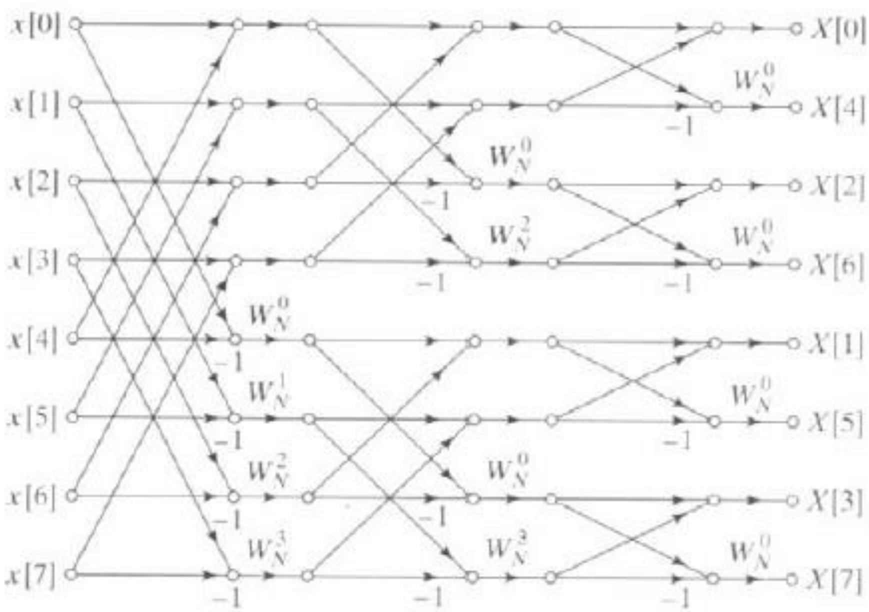



We can further break $X[2r]$ into even and odd groups …

Again, we can reduce the two-multiplication butterfly into one multiplication. Hence, the computational complexity is bout $\frac{N}{2}\log_2 N$. The in-place computation property holds if the outputs are in bit-reversed order (when inputs are in the normal order).

**Flow chart of decimation –in-frequency decomposition of an 8 -point DFT in to four 2-point DFT computations**






**Flow graph of complete decimation –in- frequency decomposition of an 8 point DFT computation**

# Inverse FFT

- IDFT: $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot W_N^{-kn} \qquad (*)$$

  DFT: $$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}$$

  Hence, take the conjugate of (*) :

$$\begin{aligned} x^*[n] &= \frac{1}{N}\left(\sum_{k=0}^{N-1} X[k] \cdot W_N^{-kn}\right)^* \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\left(X[k] \cdot W_N^{-kn}\right)^* \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\left(X^*[k] \cdot W_N^{kn}\right) \\ &= \frac{1}{N}\,\mathrm{DFT}\left[X^*(k)\right] \end{aligned}$$

  Take the conjugate of the above equation:

$$\begin{aligned} x[n] &= \frac{1}{N}\left(\mathrm{DFT}\left[X^*(k)\right]\right)^* \\ &= \frac{1}{N}\left(\mathrm{FFT}\left[X^*(k)\right]\right)^* \end{aligned}$$

Thus, we can use the FFT algorithm to compute the inverse DFT.

**Realization of Digital Filters:**

In signal processing, a **digital filter** is a system that performs mathematical operations on a sampled, discrete-time signal to reduce or enhance certain aspects of that signal. This is in contrast to the other major type of electronic filter, the analog filter, which is an electronic circuit operating on continuous-time analog signals.

Non Recursive and Recursive Structures, Canonic and Non Canonic Structures, Equivalent Structures (Transposed Structure), FIR Filter structures; Direct-Form; Cascade-Form; Basic structures for IIR systems; Direct-Form I.

## UNIT-V

| | OPT 1 | OPT 2 | OPT 3 | OPT 4 |
|---|---|---|---|---|
| Which of the following is true regarding the numb | N2 comp | N2 comp | N2 compl | N2 compl |
| Which of the following is true regarding the numb | 4N-2 real | 4N real n | 4N-2 real | 4N real m |
| WNk+N/2= | WNk | -WNk | WN-k | 0 |
| The computation of XR(k) for a complex valued x( | 2N2 evalu | 4N2 real r | 4N(N-1) : | All of the r |
| If the arrangement is of the form in which the first | n=l+mL | n=Ml+m | n=ML+l | n=0 |
| If N=LM, then what is the value of WNmqL? | WMmq | WLmq | WNmq | W |
| How many complex multiplications are performe | N(L+M+2 | N(L+M-2 | N(L+M-1 | N(L+M+1 |
| How many complex additions are performed in co | N(L+M+2 | N(L+M-2) | N(L+M-1 | N(L+M+1 |
| If we store the signal row wise and compute the L | WNlq | WNpq | WNlq | WNpm |
| If X(k) is the N/2 point DFT of the sequence x(n), | F1(k)+F2 | F1(k)- W | F1(k)+W | F1(k)/WN |
| How many complex multiplications are required | N(N+1) | N(N-1)/2 | N2/2 | N(N+1)/2 |
| The total number of complex multiplications requ | (N/2)log2 | Nlog2N | (N/2)logN | (N/2)lnN |
| The total number of complex additions required to | (N/2)log: | Nlog2N | (N/2)logN | (N/2)lnN |
| For a decimation-in-time FFT algorithm, which of | Both inpu | Both inpu | Input is sh | Input is in |
| . For a decimation-in-time FFT algorithm, which o | Both inpu | Both inpu | Input is s | Input is in |
| If x1(n) and x2(n) are two real valued sequences o | (x(n)-x*( | (x(n)+x*( | (x(n)+x*( | (x(n)-x*(n |
| . If X(k) is the DFT of x(n) which is defined as x(n)= | 1/2 [X*(k | ) 1/2 [X*( | 1/2j [X*(k | 1/2j [X*(k |
| If X(k) is the DFT of x(n) which is defined as x(n)= | (1/2) [X*( | (1/2) [X*( | (1/2j) [X | (1/2j) [X* |
| If g(n) is a real valued sequence of 2N points and | X1(k)-W2 | X1(k)+W | X1(k)+W | X1(k)-W2 |
| If g(n) is a real valued sequence of 2N points and | X1(k)-W2 | X1(k)+W | X1(k)+W2 | X1(k)-W2 |
| How many complex multiplications are need to be | (N/2)log: | Nlog2N | (N/2)log2 | (N/2)ln2N |
| How many complex additions are required to be p | (N/2)log: | 2Nlog2N | (N/2)log: | Nlog2N |
| How many complex multiplication are required p | [(N/2)log | [Nlog22N | [(N/2)log | [(N/2)log |
| Which of the following is used in the realization of | Delay ele | Multiplier | Adders | All of the |
| Computational complexity refers to the number o | Additions | Arithmet | Multiplica | division |
| Which of the following refers the number of memo | Computat | Finite wo | Memory r | bandwidt |
| Which of the following are called as finite word le | Paramete | Computa | Whether t | All of the |
| Which of the following is an method for implemen | Direct for | Cascade fo | Lattice str | All of the r |
| How many memory locations are used for storage | M+1 | M | M-1 | M/N |
| By combining two pairs of poles to form a fourth o | 25% | 30% | 40% | 50% |
| The desired frequency response is specified at a s | π/2M(k+ | π/M(k+α | 2π/M(k+ | 2π/M(k-α |
| The zeros of the system function of comb filter are | Inside un | On unit ci | Outside u | circle |
| If M and N are the orders of numerator and denom | M+N-1 | M+N | M+N+1 | M+N+2 |
| If M and N are the orders of numerator and denor | M+N-1 | M+N | M+N+1 | M+N+2 |
| If M and N are the orders of numerator and denor | M+N+1 | M+N | M+N-1 | M+N-2 |
| If M and N are the orders of numerator and denor | M+N+1 | M+N | Min [M,N | Max [M,N |
| What are the nodes that replace the adders in the | Source no | Sink node | Branch n | Summing |
| If we reverse the directions of all branch transmit | Direct for | Transpose | Direct for | sampling |
| In IIR Filter design by the Bilinear Transformation | Z-plane to | S-plane to | S-plane to | J-plane to |
| The state space or the internal description of the s | System va | Location | State var | variables |
| 3. Which of the following gives the complete defin | Amount o | Input sign | Input sign | Amount o |
| . If we interchange the rows and columns of the m | Identity s | Transpose | Diagonal | system |
| A closed form solution of the state space equation | Transpos | Symmetri | Identity | Diagonal |
| Which of the following is true regarding the | N˜ compl | | N˜ compl | |
| Which of the following is true regarding the numb | 4N-2 real | 4N real m | 4N-2 real | 4N real m |
| WNk+N/2= | WNk | -WNk | WN-k | W |
| The computation of XR(k) for a complex valued x( | 2N2 evalu | 4N2 real r | 4N(N-1) r | All of the r |
| If N=LM, then what is the value of WNmqL? | WMmq | WLmq | WNmq | W |
| What is the highest frequency that is contained in | 2Fs | Fs/2 | Fs | F |
| If {x(n)} is the signal to be analyzed, limiting the c | Kaiser wi | Hamming | Hanning v | Rectangul |
| Which of the following is the advantage of Hannin | More side | Less side | More wic | width of r |
| Which of the following is the disadvantage of Han | More side | Less side | More wic | width of m |
| If the input analog signal falls outside the range of | Granular | Overload | Particulat | Heavy no |

| | | | | |
|---|---|---|---|---|
| What is the abbreviation of SQNR? | Signal-to- | Signal-to- | Signal-to- | Signal-to- |
| What is the scale used for the measurement of SQI | DB | db | dB | All of the |
| In Overlap save method of long sequence filtering | L+M+1 | L+M | L+M-1 | L |
| Which of the following is true in case of Overlap a | M zeros a | M zeros a | M-1 zeros | M-1 zeros |
| What is the model that has been adopt for charact | Multiplic | Subtracti | Additive | noise moc |
| How many quantization errors are present in one | One | Two | Three | Four |
| What is the total number of quantization errors ir | 2N | 4N | 8N | 12N |
| How is the variance of the quantization error relat | Equal | Inversely | Square pr | Proportic |

| | ANSWER |
|---|---|
| lex additio | N2 complex multiplications and N(N-1) complex additions |
| ultiplicati | 4N real multiplications and 4N-2 real additions |
| | WNK |
| mentionec | All of the mentioned |
| | n=Ml+m |
| | WMmq |
| .) | N(L+M+1) |
| .) | N(L+M-2) |
| | WNpm |
| Nk F2(k) | F1(k)- WNk F2(k) |
| 2 | N(N+1)/2 |
| | (N/2)log2N |
| | Nlog2N |
| n order an | Input is shuffled and output is in order |
| order anc | Input is in order and output is shuffled |
| ))/2j | (x(n)-x*(n))/2j |
| )+X*(N-k] | 1/2 [X*(k)+X*(N-k)]. |
| (k)+X*(N- | (1/2j) [X*(k)-X*(N-k)]. |
| kX2(k) | X1(k)+W2kNX2(k) |
| 2kNX2(k) | X1(k)-W2kNX2(k) |
| N | (N/2)log2N |
| | 2Nlog2N |
| g2N]/L | [Nlog22N]/L |
| mentione | All of the mentioned |
| | Arithmetic operations |
| h requirer | Memory requirements |
| mentione | All of the mentioned |
| mentionec | All of the mentioned |
| | M-1 |
| | 50% |
| α) | 2π/M(k+α) |
| | On unit circle |
| | M+N+1 |
| | M+N |
| | M+N+1 |
| ]. | Max [M,N]. |
| node | Summing node |
| | Transposed form |
| Z-plane | S-plane to Z-plane |
| | State variables |
| of informa | Amount of information at n0+input signal x(n) for n≥n0 determines output signa |
| | Transposed system |
| | Diagonal |
| | |
| ultiplicati | 4N real multiplications and 4N-2 real additions |
| | -WNk |
| mentionec | All of the mentioned |
| | WMmq |
| | Fs/2 |
| ar windov | Rectangular window |
| main lobe | Less side lobes |
| nain lobe | More width of main lobe |
| ise | Overload noise |

| | |
|---|---|
| Quantizati | Signal-to-Quantization Noise Ratio |
| mentione | dB |
| | L+M-1 |
| are apper | M-1 zeros are appended at last of each data block |
| del | Additive white noise model |
| | Four |
| | 4N |
| onal | Proportional |

al for n≥n0