UNIT-I

Curvature in Cartesian coordinates-centre and radius of curvature in Cartesian and polar forms-Total differentiation

CURVATURE

CURVATURE OF THE CURVE:

A curve has a definite at every point on it. At any particular point, the direction of the curve is the same as that of the tangent to the curve at the point. The direction usually changes from point to point and the tangent line rotates has the point moves along the curve.

Let S denote the length of arc AP measured from some fixed point A on the curve and ψ the angle which the tangent makes with x-axis.

As P moves along the curve S and ψ very and the rate at which ψ increases relative to S.

That is $\frac{d\psi}{dS}$ is called the curvature of the curve at the point P

Curvature of the curve at the point P

Curvature is the rate of change of the direction of the tangent at P

We say that the curvature is the rate at which the curve curves and its sign indicates the direction in which the tangent is turning has S increases.

Curvature of a circle:

Let ABC be any given circle of radius r.

Draw AQ the tangent at A

Let O be the center of the circle join OA.

Select any point P on the circle.

Draw PM the tangent at P cutting AQ at an angle ψ.

Measure the length of the arc of the circle from A. So that the arc AP is S.

Therefore, $< AOP = \psi$

$S = r\psi$

Differentiating with respect to ψ

$$\frac{dS}{d\psi} = r$$

$$\frac{d\psi}{dS} = \frac{1}{r}$$

Ie curvature of the circle is the reciprocal of its radius.

CARTESIAN FORMULA FOR THE RADIUS OF CURVATURE:

Let $y = f(x)$ be any curve,

Let P be any point on the curve.

Let ψ be any angle.

Which the tangent makes with x-axis

We know that $\frac{dy}{dx} = \tan\psi$

Differentiating with respect to x $\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx}$

$$\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx}.\frac{dS}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dS}.\sec\psi$$

$$\frac{d^2y}{dx^2} = \sec^3\psi \frac{d\psi}{dS}$$

$$\frac{dS}{d\psi} = \frac{\sec^3\psi}{d^2y/dx^2}$$

$$\frac{dS}{d\psi} = \frac{(\sec^2\psi)^{3/2}}{d^2y/dx^2}$$

$$\frac{dS}{d\psi} = \frac{(1+\tan^2\psi)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1+\left(dy/dx\right)^2\right)^{3/2}}{d^2y/dx^2}$$

Problems:

1. What is the radius of the curve $x^4 + y^4 = 2$ at the point (1,1)

Solution:

Given $x^4 + y^4 = 2$ at the point (1,1)

$x^4 + y^4 = 2$ With respect to x, we get

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$4y^3 \frac{dy}{dx} = -4x^3$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3}$$

$$\frac{dy}{dx} = \frac{-x^3}{y^3}$$

Differentiating again with respect to x

$$\frac{d^2y}{dx^2} = \frac{y^3(-3x^2) - (-x^3)3y^2 \frac{dy}{dx}}{(y^3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{3\left[y^3x^2 + x^3y^2\left(\frac{-x^3}{y^3}\right)\right]}{y^6}$$

$$\frac{d^2y}{dx^2} = \frac{3\left[-y^3x^2 + x^3\left(\frac{-x^3}{y}\right)\right]}{y^6}$$

$$\frac{d^2y}{dx^2} = \frac{3\left[-x^2y^3 - \left(\frac{x^6}{y}\right)\right]}{y^6}$$

At the point (1,1) $\frac{dy}{dx} = \frac{-1}{1}$

$$\frac{dy}{dx} = -1$$

$$\left(\frac{d^2y}{dx^2}\right)_2 = \frac{3\left[-1-\left(\frac{1}{1}\right)\right]}{1}$$
$$= 3(-2)$$
$$= -6$$
$$\left(\frac{d^2y}{dx^2}\right)_2 = -6$$

The radius of curve

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left(1+(-1)^2\right)^{3/2}}{-6}$$

$$\rho = \frac{(2)^{3/2}}{-6}$$

$$\rho = \frac{2\sqrt{2}}{-6}$$

$$\rho = -\frac{\sqrt{2}}{3}$$

2. What is the radius of the curve $y^2 = x^3 + 8$ at the point (-2,0)

Solution:

Given $y^2 = x^3 + 8$ at the point (-2,0)

$y^2 = x^3 + 8$ With respect to x, we get

$$2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

Differentiating again with respect to x

$$\frac{d^2y}{dx^2} = \frac{2y(6x) - (3x)^2 2\frac{dy}{dx}}{(2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy + 6x^2\left(\frac{3x^2}{2y}\right)}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy + \left(\frac{18x^4}{2y}\right)}{4y^2}$$

At the point (-2,0) $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 0$$

$$\left(\frac{d^2y}{dx^2}\right)_2 = \frac{12(-2)(0) - \frac{18(-2)^4}{2(0)}}{0}$$

$= 0$

$$\left(\frac{d^2y}{dx^2}\right)_2 = 0$$

The radius of curve

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{(1+0)^{3/2}}{0}$$

$$\rho = 0$$

3. Prove that the radius of curvature at any point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

Solution:

We know that $x^{2/3} + y^{2/3} = a^{2/3}$ ……….(1)

Differentiating (1) with respect to x,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\left(\frac{dy}{dx}\right)_{(a\cos^3\theta, a\sin^3\theta)} = -\frac{(a\sin^3\theta)^{1/3}}{(a\cos^3\theta)^{1/3}}$$

$$\left(\frac{dy}{dx}\right)_{(a\cos^3\theta, a\sin^3\theta)} = -\frac{a\sin\theta}{a\cos\theta}$$

$$\left(\frac{dy}{dx}\right)_{(a\cos^3\theta, a\sin^3\theta)} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta)$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(\tan\theta)\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta}{3a\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^4\theta}{3a\sin\theta}$$

Radius of curvature

$$\rho = \frac{\left(1+\left(dy/dx\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{(1+\tan^2\theta)^{3/2}}{\sec^4\theta/3a\sin\theta}$$

$$\rho = \frac{\left(\sec^2\theta\right)^{3/2}}{\sec^4\theta / 3a\sin\theta}$$

$$\rho = 3a\sin\theta\cos\theta$$

The co-ordinates of the centre of curvature:

Let $y = f(x)$ be a curve. Let $p(x, y)$ be any point on the curve.

Let the center of curvature of the curve $y = f(x)$ corresponding to the point $p(x, y)$ be x and y.

$$X = x - \frac{y_1(1+y_1^{\,2})}{y_2}$$

$$Y = y + \frac{(1+y_1^{\,2})}{y_2}$$

Where $y_1 = \frac{dy}{dx}; y_2 = \frac{d^2y}{dx^2}$

The locus of center of curvatures for a curve is called the evolutes of the curve.

1. Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$, $X = 2a + 3at^2, Y = -2at^3$ Reduce the equation of the evolutes.

Solution:

We know that,

$$X = at^2, Y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\frac{dy}{dx}=2a\times\frac{1}{2at}$$

$$\frac{dy}{dx}=\frac{1}{t}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{t^2}\times\frac{1}{2at}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2at^3}$$

The radius of curvature

$$\rho=\frac{\left(1+\left({}^{dy}\!/\!{}_{dx}\right)^2\right)^{3/2}}{{}^{d^2y}\!/\!{}_{dx^2}}$$

$$\rho=\frac{\left(1+\left({}^{1}\!/\!{}_{t}\right)^2\right)^{3/2}}{-{}^{1}\!/\!{}_{2at^3}}$$

$$\rho=\frac{\left(1+\left({}^{1}\!/\!{}_{t^2}\right)\right)^{3/2}}{-{}^{1}\!/\!{}_{2at^3}}$$

$$\rho=\left(\frac{t^2+1}{t^2}\right)^{3/2}\left(-2at^3\right)$$

$$\rho = \left(t^2+1\right)^{3/2}\left(-2at\right)$$

$$\rho = -2at\left(t^2+1\right)^{3/2}$$

$$X = x - \frac{y_1(1+y_1^{\,2})}{y_2}$$

$$X = at^2 - \frac{1/t\,(1+1/t^2)}{-1/2at^3}$$

$$X = at^2 - 1/t \times (1+1/t^2) \times -1/2at^3$$

$$X = at^2 + 2a(t^2+1)$$

$$X = at^2 + 2at^2 + 2a$$

$$X = 3at^2 + 2a \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$Y = y + \frac{(1+y_1^{\,2})}{y_2}$$

$$Y = 2at - \frac{(1+1/t^2)}{-1/2at^3}$$

$$Y = 2at - 2at^3 - 2at$$

$$Y = -2at^3 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Eliminating t from (1)&(2)

$$(1) \Rightarrow X = 3at^2 + 2a$$

$$3at^2 = X - 2a$$

$$t^2 = \frac{X-2a}{3a}$$

$$t = \left(\frac{X-2a}{3a}\right)^{1/2}$$

Sub the value of t in (2)

$$Y = -2a\left(\frac{X-2a}{3a}\right)^{3/2}$$

Squaring on both sides

$$Y^2 = 4a^2\left(\frac{X-2a}{3a}\right)^3$$

$$Y^2 = 4a^2\frac{(X-2a)^3}{27a^3}$$

$$Y^2 = 4\frac{(X-2a)^3}{27a}$$

$$27aY^2 = 4(X-2a)^3$$

The locus of (X,Y) is $27aY^2 = 4(X-2a)^3$ is called semi cubical parabola.

2. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution:

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{b\cos\theta}{d - a\sin\theta}$$

$$y_1 = \frac{-b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-b}{a}\cot\theta\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{-b}{a}\cot\theta\right)\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a}\frac{d}{d\theta}(\cot\theta)\frac{1}{-a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2}(\cos ec^2\theta)\cos ec\theta$$

$$y_2 = \frac{-b}{a^2}\cos ec^3\theta$$

$$X = x - \frac{y_1(1 + {y_1}^2)}{y_2}$$

$$X = a\cos\theta - \frac{\frac{-b}{a}\cot\theta\left(1 + \left(\frac{-b}{a}\cot\theta\right)^2\right)}{\frac{-b}{a^2}\cos ec^3\theta}$$

$$X = a\cos\theta + \frac{\frac{-b}{a}\cot\theta\left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right)}{\frac{b}{a^2}\cos ec^3\theta}$$

$$X = a\cos\theta - \frac{b}{a}\cot\theta\left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right)\frac{a^2}{b}\frac{1}{\cos ec^3\theta}$$

$$X = a\cos\theta - \frac{b}{ab}\cot\theta\left(a^2 + b^2\cot^2\theta\right)\sin^3\theta$$

$$X = a\cos\theta - \frac{1}{a}\frac{\cos\theta}{\sin\theta}\left(a^2 + b^2\cot^2\theta\right)\sin^3\theta$$

$$X = a\cos\theta - \frac{1}{a}\cos\theta\left(a^2 + b^2\cot^2\theta\right)\sin^2\theta$$

$$X = \frac{a^2\cos\theta - \cos\theta\left(a^2 + b^2\cot^2\theta\right)\sin^2\theta}{a}$$

$$X = \frac{a^2\cos\theta - \cos\theta\sin^2\theta\left(a^2 + b^2\cot^2\theta\right)}{a}$$

$$X = \frac{a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cos\theta\sin^2\theta\frac{\cos^2\theta}{\sin^2\theta}}{a}$$

$$X = \frac{a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cos^3\theta}{a}$$

$$X = \frac{a^2\cos\theta(1 - \sin^2\theta) - b^2\cos^3\theta}{a}$$

$$X = \frac{a^2\cos\theta(\cos^2\theta) - b^2\cos^3\theta}{a}$$

$$X = \frac{a^2\cos^3\theta - b^2\cos^3\theta}{a}$$

$$Y = y + \frac{(1+{y_1}^2)}{y_2} \quad X = \frac{(a^2 - b^2)\cos^3\theta}{a} \text{.............(1)}$$

$$Y = b\sin\theta + \frac{\left(1 + \left(\frac{-b}{a}\cot\theta\right)^2\right)}{\frac{-b}{a^2}\cos ec^3\theta}$$

$$Y = b\sin\theta - \frac{\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{\frac{b}{a^2}\cos ec^3\theta}$$

$$Y = b\sin\theta - \left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right) \times \frac{a^2}{b}\sin^3\theta$$

$$Y = b\sin\theta - \left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right) \times \frac{a^2}{b}\sin^3\theta$$

$$Y = b\sin\theta - \left(a^2 + b^2\cot^2\theta\right) \times \frac{1}{b}\sin^3\theta$$

$$Y = b\sin\theta - \left(a^2 + b^2\frac{\cos^2\theta}{\sin^2\theta}\right) \times \frac{1}{b}\sin^3\theta$$

$$Y = b\sin\theta - \frac{1}{b}\left(a^2\sin^3\theta + b^2\frac{\cos^2\theta\sin^3\theta}{\sin^2\theta}\right)$$

$$Y = \frac{b^2\sin\theta - b^2\sin\theta\cos^2\theta - a^2\sin^3\theta}{b}$$

$$Y = \frac{b^2\sin\theta(1-\cos^2\theta) - a^2\sin^3\theta}{b}$$

$$Y = \frac{b^2 \sin\theta \sin^2\theta - a^2 \sin^3\theta}{b}$$

$$Y = \frac{b^2 \sin^3\theta - a^2 \sin^3\theta}{b}$$

$$Y = \frac{(b^2 - a^2)\sin^3\theta}{b} ....(2)$$

$$(1) \Rightarrow X = \frac{(a^2 - b^2)\cos^3\theta}{a}$$

$$\cos^3\theta = \frac{Xa}{(a^2 - b^2)}$$

$$\cos\theta = \left(\frac{Xa}{(a^2 - b^2)}\right)^{1/3}$$

$$(2) \Rightarrow \frac{(b^2 - a^2)\sin^3\theta}{b} = Y$$

$$\sin^3\theta = \frac{Yb}{(b^2 - a^2)}$$

$$\sin\theta = \left(\frac{Yb}{(b^2 - a^2)}\right)^{1/3}$$

$$\cos^2\theta + \sin^2\theta = \left(\frac{Xb}{(a^2 - b^2)}\right)^{2/3} + \left(\frac{Yb}{(b^2 - a^2)}\right)^{2/3}$$

$$1 = \left(\frac{Xb}{(a^2 - b^2)}\right)^{2/3} + \left(\frac{Yb}{(b^2 - a^2)}\right)^{2/3}$$

The locus of (X,Y)is

$$\left(\frac{Xb}{(a^2-b^2)}\right)^{2/3}+\left(\frac{Yb}{(b^2-a^2)}\right)^{2/3}=1$$

$$\frac{(aX)^{2/3}+(bX)^{2/3}}{(a^2-b^2)^{2/3}}=1$$

$$(aX)^{2/3}+(bX)^{2/3}=(a^2-b^2)^{2/3}$$

This curve is a four cusped hypocycloid.

INVOLUTES:

If the evolute if self the regarded as the original curve, the curve of which it is the evaluate is called an involuate.

De can see that in a curve there can be one evolute but an infinite number of involutes.

RADIUS OF CURVATURE:

The radius of curvature for polar co-ordinates

The radius of curvature when the curvature is given in polar co-ordinates

$$\rho=\frac{\left(r^2+\left(dr/d\theta\right)^2\right)^{3/2}}{r^2+2\left(dr/d\theta\right)^2-r\left(d^2r/d\theta^2\right)}$$

PROBLEMS:

1. Find the radius of curvature of the cardiod $r = a(1-\cos\theta)$

Solution:

$$r = a(1-\cos\theta)$$

$$\frac{dr}{d\theta} = a(-(-\sin\theta))$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{\left(r^2 + \left(dr/d\theta\right)^2\right)^{3/2}}{r^2 + 2\left(dr/d\theta\right)^2 - r\left(d^2r/d\theta^2\right)}$$

Now, $\left(r^2 + \left(dr/d\theta\right)^2\right)^{3/2}$

$$= \left((a(1-\cos\theta))^2 + a^2\sin^2\theta\right)^{3/2}$$

$$= \left((a^2(1-2\cos\theta+\cos^2\theta) + a^2\sin^2\theta\right)^{3/2}$$

$$= \left(a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta\right)^{3/2}$$

$$= \left(a^2 - 2a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)\right)^{3/2}$$

$$= \left(a^2 - 2a^2\cos\theta + a^2\right)^{3/2}$$

$$= \left(2a^2 - 2a^2\cos\theta\right)^{3/2}$$

$$= \left(2a^2(1-\cos\theta)\right)^{3/2}$$

$$= (2a.a(1-\cos\theta))^{3/2}$$

$$= (2a.r)^{3/2}$$

$$= 2\sqrt{2}(ar)^{3/2}$$

Also, $r^2 + 2\left(dr/d\theta\right)^2 - r\left(d^2r/d\theta^2\right)$

$$= a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - (a(1-\cos\theta)a\cos\theta)$$

$$= a^2(1-2\cos\theta+\cos^2\theta) + 2a^2\sin^2\theta - (a(1-\cos\theta)a\cos\theta)$$

$$= a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2 - 3a^2\cos\theta + 2a^2\cos^2\theta + 2a^2\sin^2\theta$$

$$= a^2 - 3a^2\cos\theta + 2a^2(\cos^2\theta + \sin^2\theta)$$

$$= a^2 - 3a^2\cos\theta + 2a^2$$

$$= 3a^2 - 3a^2\cos\theta$$

$$= 3a^2(1-\cos\theta)$$

$$= 3aa(1-\cos\theta)$$

$$= 3ar$$

$$\rho = \frac{2\sqrt{2}(ar)^{3/2}}{3ar}$$

$$\rho = \frac{2\sqrt{2}(ar)^{3/2-1}}{3}$$

$$\rho = \frac{2\sqrt{2}(a)^{3/2-1} r^{3/2-1}}{3}$$

$$\rho = \frac{2\sqrt{2ar}}{3}$$

2. Find the radius of curvature of the cardiod $r = a(1+\cos\theta)$

Solution:

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$\rho = \frac{\left(r^2 + \left(dr/d\theta\right)^2\right)^{3/2}}{r^2 + 2\left(dr/d\theta\right)^2 - r\left(d^2r/d\theta^2\right)}$$

Now, $\left(r^2 + \left(dr/d\theta\right)^2\right)^{3/2}$

$$= \left((a(1+\cos\theta))^2 + a^2\sin^2\theta\right)^{3/2}$$

$$= \left((a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta\right)^{3/2}$$

$$= \left(a^2 + 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta\right)^{3/2}$$

$$= \left(a^2 + 2a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)\right)^{3/2}$$

$$= \left(a^2 + 2a^2\cos\theta + a^2\right)^{3/2}$$

$$= \left(2a^2 + 2a^2\cos\theta\right)^{3/2}$$

$$= \left(2a^2(1+\cos\theta)\right)^{3/2}$$

$$= \left(2a.a(1+\cos\theta)\right)^{3/2}$$

$$= \left(2a.r\right)^{3/2}$$

$$= 2\sqrt{2}\left(ar\right)^{3/2}$$

Also, $r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)$

$$= a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - (a(1-\cos\theta)a\cos\theta)$$

$$= a^2(1+2\cos\theta+\cos^2\theta) + 2a^2\sin^2\theta - (a(1-\cos\theta)(-a\cos\theta))$$

$$= a^2 + 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta + a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2 + 3a^2\cos\theta + 2a^2\cos^2\theta + 2a^2\sin^2\theta$$

$$= a^2 + 3a^2\cos\theta + 2a^2(\cos^2\theta + \sin^2\theta)$$

$$= a^2 + 3a^2\cos\theta + 2a^2$$

$$= 3a^2 + 3a^2\cos\theta$$

$$= 3a^2(1+\cos\theta)$$

$$= 3aa(1+\cos\theta)$$

$$= 3ar$$

$$\rho = \frac{2\sqrt{2}(ar)^{3/2}}{3ar}$$

$$\rho = \frac{2\sqrt{2}(ar)^{3/2-1}}{3}$$

$$\rho = \frac{2\sqrt{2}(a)^{3/2-1} r^{3/2-1}}{3}$$

$$\rho = \frac{2\sqrt{2ar}}{3}$$

$$\rho^2 = \left(\frac{2\sqrt{2ar}}{3}\right)^2$$

$$\rho^2 = \frac{4(2ar)}{9}$$

$$\rho^2 = \frac{8ar}{9}$$

$$\frac{\rho^2}{r} = \frac{8}{9}a$$

Hence $\frac{\rho^2}{r}$ is a constant.

P-R EQUATION OF CURVE

OR

PEDEL EQUATION

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Or

$$\frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2$$

PROBLEMS:

1. Prove that pedel equation of the cardiod $r = a(1-\cos\theta)$ is $p^2 = \frac{r^3}{2a}$

Solution:

Given that $r = a(1-\cos\theta)$ …………………(1)

Diff w.r.t θ

$$\frac{dr}{d\theta} = a\sin\theta$$

The p-r equation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(a\sin\theta)^2$$

$$\frac{1}{p^2} = \frac{r^2 + a^2 \sin^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 (1-\cos\theta)^2 + a^2 \sin^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 (1-2\cos\theta+\cos^2\theta) + a^2 \sin^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2 \sin^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 - 2a^2\cos\theta + a^2(\cos^2\theta + \sin^2 \theta)}{r^4}$$

$$\frac{1}{p^2} = \frac{a^2 - 2a^2\cos\theta + a^2}{r^4}$$

$$\frac{1}{p^2} = \frac{2a^2(1-\cos\theta)}{r^4}$$

$$\frac{1}{p^2} = \frac{2ar}{r^4}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$p^2 = \frac{r^3}{2a}$$

2. From the polar equation of a parabola show that $p^2 = ar$

Solution:

The given equation of a parabola is given by $\frac{2a}{r} = 1 - \cos\theta$............(1)

$$2a\left(\frac{-1}{r^2}\right)\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r^2}\frac{dr}{d\theta} = \frac{-\sin\theta}{2a}$$

p-r equation is

$$\frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{-\sin\theta}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{(1-\cos\theta)^2}{r^2} + \left(\frac{-\sin\theta}{2a}\right)^2$$

$$\frac{1}{p^2} = \frac{(1-\cos\theta)^2}{4a^2} + \frac{\sin^2\theta}{4a^2}$$

$$\frac{1}{p^2} = \frac{(1-\cos\theta)^2 + \sin^2\theta}{4a^2}$$

$$\frac{1}{p^2} = \frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{4a^2}$$

$$\frac{1}{p^2} = \frac{1-2\cos\theta+1}{4a^2}$$

$$\frac{1}{p^2} = \frac{2-2\cos\theta}{4a^2}$$

$$\frac{1}{p^2} = \frac{2(1-\cos\theta)}{4a^2}$$

$$\frac{1}{p^2} = \frac{2\left(2a/r\right)}{4a^2}$$

$$\frac{1}{p^2} = \frac{\left(4a/r\right)}{4a^2}$$

$$\frac{1}{p^2} = \frac{1}{ar}$$

$$p^2 = ar$$

Hence the proof

TOTAL DIFFERENTIAL CO-EFFICIENT:

1. Find $\frac{du}{dt}$ when $u = x^2 + y^2 + z^2, x = e^t, y = e^t \sin t, z = e^t \cos t$

Solution:

Given,

$$u = x^2 + y^2 + z^2$$

Differentiating,

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 2z$$

Also, $x = e^t$

$y = e^t \sin t$

$z = e^t \cos t$

Differentiating

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = e^t \cos t + e^t \sin t$$

$$\frac{dy}{dt} = e^t (\cos t + \sin t)$$

$$\frac{dz}{dt} = -e^t \sin t + e^t \cos t$$

$$\frac{dy}{dt} = e^t (\cos t - \sin t)$$

$$\frac{du}{dt} = \frac{du}{dx}.\frac{dx}{dt} + \frac{du}{dy}.\frac{dy}{dt} + \frac{du}{dz}.\frac{dz}{dt}$$

$$\frac{du}{dt} = 2xe^t + 2ye^t (\cos t + \sin t) + 2ze^t (\cos t - \sin t)$$

$$\frac{du}{dt} = 2e^t (x + y(\cos t + \sin t) + z(\cos t - \sin t))$$

$$\frac{du}{dt} = 2e^t (e^t + e^t \sin t(\cos t + \sin t) + e^t \cos t(\cos t - \sin t))$$

$$\frac{du}{dt} = 2e^t (e^t + e^t \sin t \cos t + e^t \sin^2 t + e^t \cos^2 t - e^t \cos t \sin t)$$

$$\frac{du}{dt} = 2e^t(e^t + e^t \sin^2 t + e^t \cos^2 t)$$

$$\frac{du}{dt} = 2e^t(e^t + e^t(\sin^2 t + \cos^2 t))$$

$$\frac{du}{dt} = 2e^t(e^t + e^t(1))$$

$$\frac{du}{dt} = 2e^t(e^t + e^t)$$

$$\frac{du}{dt} = 2e^t(2e^t)$$

$$\frac{du}{dt} = 4e^{2t}$$

Possible Questions

2 Mark questions

1. Define the Curvature.
2. What is the radius of curvature of the curve $x^2 + y^2 = 25$ at (3,4).
3. Find the radius of curvature of the curve $x^4+y^4=2$ at the point (1,1).
4. Verify Euler's theorem for the function $f = x^3 - 2x^2y + 3xy^2 + y^3$.
5. Find the radius of curvature of the parabola $y^2 = 4ax$ at any point.
6. Find the (p-r) equation of the curve $r^2 = a^2 \sin 2\theta$ and hence find the radius of curvature.
7. Find the radius of curvature of the curve $y = alog\sec\left(\frac{x}{a}\right)$.
8. Find the radius of curvature of the curve $y = c\cosh\left(\frac{x}{c}\right)$ at (0, c).
9. Write down the formula for radius of curvature in Cartesian form.

6 Mark questions

1. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at$\left(\frac{a}{4}, \frac{a}{4}\right)$.
2. Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1\text{-}\cos\theta)$ is $4a\cos\frac{\theta}{2}$.
3. Show that the radius of curvature at the point θ on the curve
   $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.
4. Prove that the radius of curvature at a point $(acos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.
5. Find the centre of curvature of the curve $xy = c^2$ at (c,c)
6. Find the radius of curvature at$(a, 0)$ on the curve $xy^2 = a^3 - x^3$.
7. Find the equation of the evolute of the parabola $y^2 = 4ax$.
8. Find ρ for the curve $y = 4sinx - \sin 2x$ at the point $x = \frac{\pi}{2}$.
9. Find the radius of the curvature of the curve given by
   $x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0$.
10. Find the radius of curvature at the point (x,y) for the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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Subject: Mathematics-I **Subject Code: 18PHU304**

Class : II - B.Sc. Physics **Semester : III**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The curvature of a straight line is --------------	0	1	pi	infinity	0
What is the value of slope of tangent to the curve y=f(x)-----------	tanφ	ϕ	sinφ	cosφ	tanφ
The parametric equation of circle is-----------	x=rcosθ,y=sinθ	x=cosθ,y=sinθ	x=rcosθ,y=rsinθ	x=cosθ,y=rsinθ	x=rcosθ,y=rsinθ
The curvature of a circle of radius r at any point is---------	r	r^2	1/r^2	1/r	1/r
Let f(x,y) be the implicit form of the given curve then dy/dx is-------------	f_x/f_y	-f_x/f_y	0	f_y/f_x	-f_x/f_y
What is the radius of curvature of the curve x^2+y^2=49 at (0,0)-------------	49	7	0	1	7
The radius of curvature value is ρ then the curvature value is----------	ρ	ρ^2	1/ρ	0	1/ρ
What is the value of dy/dx to the function x^3+y^3=3axy----------	(x^2-ax)/(y^2-ax)	(x^2-ay)/(y^2-ax)	- (x^2-ay)/(y^2-ax)	-(x^2-ax)/(y^2-ax)	-(x^2-ax)/(y^2-ax)
What is the value of e^x+e^(-x) -------------	coshx	sinhx	2coshx	2sinhx	coshx
of the following functions could be f(x)------------	ax+b	sinx	cosx	ax^2+b	ax+b

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The radius of curvature at any point of the curve r=e^θ is ---------------	2r	√(2+r)	2√2-r	r√(2)	r√(2)
What is the value of cosnπ ---------------	0	1	-1	(-1)^n	(-1)^n
The reciprocal of the curvature of a curve at any point is called -------------	circle	radius of curvature	circle of curvature	centre of curvature	radius of curvature
The centre of curvature of the curve y=x^2 at the origin is--------------------	(0,0)	(0,1/2)	(1/2,1)	(0,-1/2)	(0,1/2)
Let u=x^y then ∂u/∂y is ----------	x^y	e^(ylogx) logx	e^ylogx	logx	e^(ylogx) logx
If f(x,y)=x^3+xyz+z, Find fx at(1,1,1) is--------	0	1	-1	3	3
The radius of curvature of the curve x^2+y^2=16 at (1,1) is ---------------	16	1	2	4	4
The curvature value of the circle in all points are ----------	0	equal	notequal	infinity	equal
The value of slope of tangent to the curve y=f(x)-----------	dy/dx	x	dx/dy	dx	dy/dx
The parametric equation of probola y^2=4ax is -----------	x=at,y=a	x=at,y=at^3	x=at^2,y=at	x=at,y=at^2	x=at,y=at^2
What is the name of the curves x=a(θ+sinθ),y=a(1-cosθ) is-----------	cardioid	astroid	cycloid	ellipse	cycloid
What is the name of the curves r=a(1-cosθ) is-----------	cardioid	astroid	cycloid	ellipse	cardioid
If the equation x=rcosθ,y=rsinθ is called--------------	cartesian form	polar form	perametric form	implicit form	perametric form
If the equation x^3+3xy^2+5x^2+7y^2-6x=0 is called---------------	quatertic form	polar form	perametric form	implicit form	implicit form
If the equation form only x and y varibles is called---------------	cartesian form	polar form	perametric form	sperical polar	cartesian form
If the equation form only r and θ varibles is called---------------	cartesian form	polar form	perametric form	sperical polar	polar form
What is the value of sinnπ where n belongs to the integers then---------------	0	1	-1	infinity	0

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
What is the name of the curves r=a(1+cosθ) is-----------	cardioid	astroid	cycloid	ellipse	cardioid
The curvature value is 1/ρ then the curvature value is----------	ρ	ρ^2	1/ρ	0	ρ
In the polar form, which is analogous to the origin---------	r	θ	pole	argument	
The radius of curvature of the curve in polar form is----------	(r^2+r1^2)^(3/2) /(r^2+2r1^2-rr2)	(r1^2+r1^2)^(3/2) /(r^2+2r1^2-rr2)	(r2^2+r2^2)^(3/2) /(r^2+2r1^2-rr2)	(r^2-r1^2)^(3/2) /(r^2+2r1^2-rr2)	(r^2+r1^2)^(3/2) /(r^2+2r1^2-rr2)
The center of curvature is ……. for all the points in curve.	equal	origin	not equal	(1,1)	not equal
What is the value of e^x-e^(-x) -------------	coshx	sinhx	2coshx	2sinhx	2sinhx
The radius of curvature of the curve in cartesian form is -------------	((1+(dy/dx)^2)^3/2)/(d^2x/dx^2)	((1+(dy/dx)^2))/(d^2 x/dx^2)	((dy/dx)^2)^3/2)/(d^ 2x/dx^2)	((1+(dy/dx)^2)^3/2) (dx/dx)	((1+(dy/dx)^2)^3/2)/(d^2x/dx^2)
The value of d/dx(c),where c is constant then-------------	1	0	-1	infinity	1
The value of d/dx(x) is-------------	0	1	-1	infinity	1
The value of d/dx(logx) is-------------	x	1/x	x^2	0	1/x
The value of d/dx(e^x) is-------------	1	e^x	e^-x	-e^x	e^x
The value of d/dx(a^x) is-------------	a^xloga	a^x	loga	0	a^xloga
The value of d/dx(sinx) is-------------	cosx	sinx	tanx	-cosx	cosx
The value of d/dx(cosx) is-------------	cosx	-sinx	sinx	-cosx	-sinx
The value of d/dx(tanx) is-------------	cosecx	tanx	(secx)^2	(cosecx)^2	(secx)^2
The value of d/dx(secx) is-------------	secxtanx	tanx	secx	cosx	secxtanx
The value of d/dx(cosecx) is-----------	cosecx	-cosecxcotx	cotx	sinx	-cosecxcotx
The value of d/dx(cotx) is-------------	sec^2x	tanx	cosx	-cosce^2x	-cosce^2x
The value of d/dx(sinhx) is-------------	coshx	sinhx	tanhx	-coshx	coshx
The value of d/dx(coshx) is-------------	coshx	sinhx	tanhx	-coshx	sinhx
The value of d/dx(tanhx) is-------------	sec^2hx	tanhx	coshx	cosce^2hx	sec^2hx
The value of d/dx(cothx) is-------------	sec^2hx	tanhx	coshx	-cosce^2hx	-cosce^2hx
The value of d/dx(sechx) is-------------	tanhx	-sechxtanhx	sechx	coshx	-sechxtanhx

UNIT-II

Integration of $\frac{f^{'}(x)}{f(x)}, f^{'}\sqrt{f(x)}, (px+q)\Big/\sqrt{(ax^2+bx+c)}, \left(\sqrt{(x-a)(b-x)}\right), 1\Big/\left(\sqrt{(x-a)(b-x)}\right),$ $(1/a\cos x + b\sin x + c), 1\Big/\left(a\cos^2 x + b\sin^2 x + c\right),$ *Integration by parts.*

INTEGRAL CALCULUS

INVERSE PROCESS OF DIFFERENTIATION

Integration is the reverse or inverse process of differentiation.

Problem:

1. Integrate the following with respective to x

(a) $\int x^5 dx$

Solution:

$$= \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{5+1}}{5+1} + c$$

$$= \frac{x^6}{6} + c$$

(b) $\int x^{-4} dx$

Solution:

$$= \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{-4+1}}{-4+1} + c$$

$$= \frac{x^{-3}}{-3} + c$$

$$= -\frac{1}{3x^3} + c$$

(c) $\int \frac{ax^2 + bx + c}{x^2} dx$

Solution:

$$= \int \frac{ax^2}{x^2} dx + \int \frac{bx}{x^2} dx + \int \frac{c}{x^2} dx$$

$$= a\int dx + b\int \frac{1}{x} dx + c\int \frac{1}{x^2} dx$$

$$= a\int dx + b\int \frac{1}{x} dx + c\int x^{-2} dx$$

$$= ax + b(\log x) + c\left(\frac{x^{-1}}{-1}\right) + k$$

$$= ax + b(\log x) + \frac{-c}{k} + k$$

(d) $\int \frac{ax^{-2} + bx^{-1} + c}{x^{-4}} dx$

Solution:

$$= \int \frac{ax^{-2}}{x^{-4}} dx + \int \frac{bx^{-1}}{x^{-4}} dx + \int \frac{c}{x^{-4}} dx$$

$$= a\int x^2 dx + b\int x^3 dx + c\int x^4 dx$$

$$= a\frac{x^3}{3} + b\frac{x^4}{4} + c\frac{x^5}{5} + k$$

(e) $\int \tan^2 x dx$

Solution:

$$=\int(\sec^2 x-1)dx$$

$$=\int\sec^2 dx-\int dx$$

$$=\tan x-x+c$$

(f) $\int\cot^2 xdx$

Solution:

$$=\int(\cos ec^2 x-1)dx$$

$$=\int\cos ec^2 dx-\int dx$$

$$=-\cot x-x+c$$

$$=-(\cot x+x-c)$$

(g) $\int\frac{1}{\sin^2 x\cos^2 x}dx$

Solution:

$$=\int\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\frac{\sin^2 x}{\sin^2 x\cos^2 x}dx+\int\frac{\cos^2 x}{\sin^2 x\cos^2 x}dx$$

$$=\int\frac{1}{\cos^2 x}dx+\int\frac{1}{\sin^2 x}dx$$

$$=\int\sec^2 xdx+\int\cos ec^2 xdx$$

$$= \tan x - \cot x + c$$

(h) $\int \frac{1}{1-\sin x} dx$

Solution:

$$= \int \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} dx$$

$$= \int \frac{1+\sin x}{1-\sin^2 x} dx$$

$$= \int \frac{1+\sin x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \tan x \frac{1}{\cos x} dx$$

$$= \int \sec^2 x dx + \int \tan x \sec x dx$$

$$= \tan x + \sec x + c$$

DEFINITE INTEGRAL:

Let $\int f(x) dx = f(x)$ then the definite integral is given by,

$$\int_a^b f(x) dx = f(b) - f(a)$$

Problems:

1. Evaluate $\int_1^2 (x^2 + x) dx$

Solution:

$$=\int_1^2 x^2 dx + \int_1^2 x dx$$

$$=\left[\frac{x^3}{3}\right]_1^2 + \left[\frac{x^2}{2}\right]_1^2$$

$$=\left[\frac{(2)^3}{3}-\frac{(1)^3}{3}\right]+\left[\frac{(2)^2}{2}-\frac{(1)^2}{2}\right]$$

$$=\left[\frac{8}{3}-\frac{1}{3}\right]+\left[\frac{4}{2}-\frac{1}{2}\right]$$

$$=\frac{7}{3}+\frac{3}{2}$$

$$=\frac{14+9}{6}$$

$$=\frac{23}{6}$$

2. Evaluate $\int_0^{\pi/2} \cos^2\left(\frac{x}{2}\right) dx$

Solution:

$$=\int_0^{\pi/2} \frac{1+\cos x}{2} dx$$

$$= \int_0^{\pi/2} \frac{1}{2} dx + \int_0^{\pi/2} \frac{\cos x}{2} dx$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} dx + \int_0^{\pi/2} \cos x dx \right]$$

$$= \frac{1}{2}\left[[x]_0^{\pi/2} + [\sin x]_0^{\pi/2} \right]$$

$$= \frac{1}{2}\left[\pi/2 - 0 + \sin \pi/2 - \sin 0 \right]$$

$$= \frac{1}{2}\left[\pi/2 + 1 - 0 \right]$$

$$= \frac{1}{2}\left[\pi/2 + 1 \right]$$

$$= \pi/4 + \frac{1}{2}$$

TYPE 1

$$\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$$

Proof:

Consider,

L.H.S let $t = f(x)$

Differentiating,

$$dt = f'(x) dx$$

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log f(x) + c$$

=R.H.S

Hence the proof

PROBLEMS:

1. Evaluate $\int \frac{e^x}{e^x + 10} dx$

Solution:

Let $t = e^x + 10$

$$dt = e^x dx$$

$$\int \frac{e^x}{e^x + 10} dx = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(e^x + 10) + c$$

2. Evaluate $\int \frac{\log x}{x} dx$

Solution:

Let $t = \log x$

$$dt = \frac{1}{x} dx$$

$$\int \frac{\log x}{x} dx = \int t dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{(\log x)^2}{2} + c$$

$$= \frac{2 \log x}{2} + c$$

$$= \log x + c$$

3. Evaluate $\int \frac{1}{x \log x} dx$

Solution:

Let $t = \log x$

$$dt = \frac{1}{x} dx$$

$$\int \frac{1}{x \log x} dx = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(\log x) + c$$

4. Evaluate $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx$

Solution:

Let $t = \sin x + \cos x$

$$dt = (\cos x + \sin x)dx$$

$$\int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(\sin x + \cos x) + c$$

5. Evaluate $\int \cot x dx$

Solution:

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx$$

Let $t = \sin x$

$$dt = \cos x dx$$

$$\int \cot x = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(\sin x) + c$$

6. Evaluate $\int \sec x dx$

Solution:

Multiple and divided by $(\sec x + \tan x)$

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

Let $t = (\sec x + \tan x)$

$$dt = \sec x(\tan x + \sec x)dx$$

$$\int \sec x = \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(\sec x + \tan x) + c$$

7. Evaluate $\int \cos ecxdx$

Solution:

Multiple and divided by $(\cos ecx + \cot x)$

$$\int \cos ecxdx = \int \frac{\cos ecx(\cos ecx + \cot x)}{(\cos ecx + \cot x)} dx$$

Let $t = (\cos ecx + \cot x)$

$$dt = -\cos ecx(\cos ecx + \cot x)dx$$

$$-dt = \cos ecx(\cos ecx + \cot x)dx$$

$$\int \cos ecx = \int -\frac{dt}{t}$$

$$= -\log t + c$$

$$= -\log(\cos ecx + \cot x) + c$$

TYPE 2:

$$\int f(x) f'(x)dx = \frac{[f(x)]^2}{2} + c$$

Proof:

Let $t = f(x)$

$dt = f'(x)dx$

$\int f(x) f'(x)dx = \int tdt$

$= \frac{t^2}{2} + c$

$= \frac{[f(x)]^2}{2} + c$

$\int f(x) f'(x)dx = \frac{[f(x)]^2}{2} + c$

Hence the proof

PROBLEM:

1. Evaluate $\int x\sqrt{x^2+1}dx$

Solution:

Let $t = x^2 + 1$

$dt = 2xdx$

$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{t}dt$

$= \frac{1}{2}\int t^{1/2}dt$

$$=\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]+c$$

$$=\frac{1}{2}\left[\frac{(x^2+1)^{3/2}}{3/2}\right]+c$$

$$=\left[\frac{(x^2+1)^{3/2}}{3}\right]+c$$

2. Evaluate $\int xe^{x^2}dx$

Solution:

Let $t=x^2$

$$dt=2xdx$$

$$\frac{dt}{2}=xdx$$

$$\int xe^{x^2}dx=\int e^t.\frac{dt}{2}$$

$$=\frac{1}{2}\int e^t dt$$

$$=\frac{1}{2}e^t+c$$

$$=\frac{e^{x^2}}{2}+c$$

3. Evaluate $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution:

Let $t = \sin^{-1} x$

$$dt = \frac{1}{\sqrt{1-x^2}}dx$$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t.dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{(\sin^{-1} x)^2}{2} + c$$

4. Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2}dx$

Solution:

Let $t = \tan^{-1} x$

$$dt = \frac{1}{1+x^2}dx$$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2}dx = \int e^t .dt$$

$$= e^t + c$$

$$= e^{\tan^{-1} x} + c$$

TYPE 3:

$$\int f(x) f'(x)dx = 2\sqrt{f(x)} + c$$

Proof:

Let $t = f(x)$

$dt = f'(x)dx$

$$\int f(x) f'(x) dx = \int \frac{dt}{\sqrt{t}}$$

$$= \int t^{-1/2} dt$$

$$= \frac{t^{1/2}}{1/2} + c$$

$$= 2\sqrt{f(x)} + c$$

$$\int f(x) f'(x) dx = 2\sqrt{f(x)} + c$$

Hence the proof

PROBLEMS:

1. Evaluate $\int \frac{x^2 + 2x}{\sqrt{x^3 + 3x^2 + 2}} dx$

Solution:

Let $t = x^3 + 3x^2 + 2$

$dt = (3x^2 + 6x)dx$

$dt = 3(x^2 + 2x)dx$

$$\frac{dt}{3} = (x^2 + 2x)dx$$

$$\int \frac{x^2 + 2x}{\sqrt{x^3 + 3x^2 + 2}} dx = \int \frac{dt/3}{\sqrt{t}}$$

$$= \frac{1}{3} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{3} \int t^{-1/2} dt$$

$$= \frac{1}{3} \frac{t^{1/2}}{1/2} + c$$

$$= \frac{1}{3} 2\sqrt{t} + c$$

$$= \frac{1}{3} 2\sqrt{x^3 + 3x^2 + 2} + c$$

2. Evaluate $\int \frac{x}{\sqrt{3+5x^2}} dx$

Solution:

Let $t = 3 + 5x^2$

$$dt = 10x dx$$

$$\frac{dt}{10} = x dx$$

$$\int \frac{x}{\sqrt{3+5x^2}} dx = \int \frac{dt/10}{\sqrt{t}}$$

$$=\frac{1}{10}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{1}{10}\int t^{-1/2}dt$$

$$=\frac{1}{10}\frac{t^{1/2}}{1/2}+c$$

$$=\frac{1}{10}2\sqrt{t}+c$$

$$=\frac{1}{10}2\sqrt{3+5x^2}+c$$

$$=\frac{1}{5}\sqrt{3+5x^2}+c$$

3. Evaluate $\int\frac{x+1}{\sqrt{x^2+2x+1}}dx$

Solution:

Let $t=x^2+2x+1$

$dt=(2x+2)dx$

$dt=2(x+1)dx$

$\frac{dt}{2}=(x+1)dx$

$$\int\frac{x+1}{\sqrt{x^2+2x+1}}dx=\int\frac{dt/2}{\sqrt{t}}$$

$$=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{1}{2}\int t^{-1/2}dt$$

$$=\frac{1}{2}\frac{t^{1/2}}{1/2}+c$$

$$=\frac{1}{2}2\sqrt{t}+c$$

$$=\frac{1}{2}2\sqrt{x^2+2x+1}+c$$

$$=\sqrt{x^2+2x+1}+c$$

TYPE 4:

1. Evaluate $\int\frac{dx}{x^2+2x+5}$

Solution:

$$x^2+2x+5=x^2+2x+1-1+5$$

$$=(x+1)^2+4$$

$$=(x+1)^2+2^2$$

$$\int\frac{dx}{x^2+2x+5}=\int\frac{dx}{(x+1)^2+2^2}$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$$

2. Evaluate $\int\frac{dx}{x^2+8x-7}$

Solution:

$$x^2+8x-7=x^2+8x+16-16-7$$

$$=(x+4)^2-23$$

$$=(x+4)^2+(\sqrt{23})^2$$

$$\int\frac{dx}{x^2+8x-7}=\int\frac{dx}{(x+4)^2+(\sqrt{23})^2}$$

$$=\frac{1}{2\sqrt{23}}\log\left(\frac{(x+4)+\sqrt{23}}{(x+4)+\sqrt{23}}\right)+c$$

3. Evaluate $\int\frac{dx}{x^2+2x+3}$

Solution:

$$x^2+2x+3=x^2+2x+1-1+3$$

$$=(x+1)^2+2$$

$$=(x+1)^2+\sqrt{2}$$

$$\int\frac{dx}{x^2+2x+5}=\int\frac{dx}{(x+1)^2+\sqrt{2}^2}$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$$

TYPE 6:

1. Evaluate $\int\frac{dx}{\sqrt{x^2-x+2}}$

Solution:

$$x^2 - 3x + 2 = x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$\int \frac{dx}{\sqrt{x^2 - x + 2}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \log\left[\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right] + c$$

Or

$$= \cosh^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right) + c$$

$$= \cosh^{-1}\left(\frac{\frac{2x-3}{2}}{\frac{1}{2}}\right) + c$$

$$= \cosh^{-1}(2x - 3) + c$$

2. Evaluate $\int \frac{dx}{\sqrt{-2x^2 + 3x}}$

Solution:

$$= -\left(2x^2 - \frac{3}{2}x\right)$$

$$= -2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)$$

$$= -2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}\right]$$

$$= -2\left[\left(x - \frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]$$

$$= 2\left[\left(\frac{3}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2\right]$$

$$\int \frac{dx}{\sqrt{-2x^2 + 3x}} = \int \frac{dx}{2\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2}}$$

$$= 2\sin^{-1}\left(\frac{x - \frac{3}{4}}{\frac{3}{4}}\right) + c$$

$$= 2\sin^{-1}\left(\frac{\frac{4x-3}{4}}{\frac{3}{4}}\right) + c$$

$$= 2\sin^{-1}\left(\frac{4x-3}{3}\right) + c$$

3. Evaluate $\int \frac{dx}{\sqrt{-x^2 + 3x - 2}}$

Solution:

$$-x^2+3x-2=-(x^2-3x+2)$$

$$=-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}+2\right)$$

$$=-\left[\left(x-\frac{3}{2}\right)^2-\frac{1}{4}\right]$$

$$=-\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]$$

$$\int\frac{dx}{\sqrt{-x^2+3x-2}}=\int\frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2+\left(x-\frac{3}{2}\right)^2}}$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right)+c$$

$$=\sin^{-1}\left(\frac{\frac{2x-3}{2}}{\frac{1}{2}}\right)+c$$

$$=\sin^{-1}(2x-3)+c$$

TYPE 18:

FORMULA:

Put $t=\tan\frac{x}{2}$

$$\cos x=\frac{1+t^2}{1+t^2}$$

$\sin x = \frac{2t}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

1. Evaluate $\int \frac{dx}{5-4\sin x}$

Solution:

Put $t = \tan\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$$\int \frac{dx}{5-4\sin x} = \int \frac{\frac{2dt}{1+t^2}}{5-4\left(\frac{2t}{1+t^2}\right)}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{5(1+t^2)-8t}{1+t^2}}$$

$$= \int \frac{2dt}{5(1+t^2)-8t}$$

$$= \int \frac{2dt}{5t^2-8t+5}$$

$$\int \frac{dx}{5-4\sin x} = 2\int \frac{dt}{5t^2-8t+5}$$

Consider,

$$5t^2 - 8t + 5 = 5\left(t^2 - \frac{8}{5}t + 1\right)$$

$$= 5\left(t^2 - \frac{8}{5}t + \frac{16}{25} - \frac{16}{25} + 1\right)$$

$$= 5\left(\left[t - \frac{4}{5}\right]^2 + \frac{9}{25}\right)$$

$$= 5\left(\left[t - \frac{4}{5}\right]^2 + \left(\frac{3}{5}\right)^2\right)$$

$$\int \frac{dx}{5 - 4\sin x} = 2\int \frac{dt}{5\left(\left[t - \frac{4}{5}\right]^2 + \left(\frac{3}{5}\right)^2\right)}$$

$$= \frac{2}{5}\left[\frac{1}{3/5}\tan^{-1}\left(\frac{t - \frac{4}{5}}{3/5}\right)\right] + c$$

$$= \frac{2}{5}\left[5/3 \tan^{-1}\left(\frac{5t - 4}{3}\right)\right] + c$$

$$= \frac{2}{3}\left[\tan^{-1}\left(\frac{5t - 4}{3}\right)\right] + c$$

$$\int \frac{dx}{5 - 4\sin x} = \frac{2}{3}\left[\tan^{-1}\left(\frac{5\left(x/2\right) - 4}{3}\right)\right] + c$$

2. Evaluate $\int \frac{dx}{4 + 5\cos x}$

Solution:

Put $t = \tan\frac{x}{2}$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\int \frac{dx}{4+5\cos x} = \int \frac{\frac{2dt}{1+t^2}}{4+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\frac{2dt}{1+t^2}}{\frac{4(1+t^2)+5(1-t^2)}{1+t^2}}$$

$$= \int \frac{2dt}{4(1+t^2)+5(1-t^2)}$$

$$= \int \frac{2dt}{4+4t^2+5-5t^2}$$

$$= \int \frac{2dt}{-t^2+9}$$

$$= 2\int \frac{dt}{3^2-t^2}$$

$$\int \frac{dx}{4+5csx} = 2\int \frac{dt}{3^2 - t^2}$$

$$= 2 \times \frac{1}{6} \log\left(\frac{3+t}{3-t}\right) + c$$

$$= \frac{1}{3} \log\left(\frac{3+t}{3-t}\right) + c$$

$$\int \frac{dx}{4+5\cos x} = \frac{1}{3} \log\left(\frac{3+t}{3-t}\right) + c$$

$$\int \frac{dx}{4+5\cos x} = \frac{1}{3} \log\left(\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right) + c$$

INTEGRATION BY PARTS:

1. $\int x\cos axdx$

Solution:

$u = x \qquad dv = \cos axdx$

$du = dx \qquad v = \frac{\sin ax}{a}$

$$\int udv = uv - \int vdu$$

$$= x\frac{\sin ax}{a} - \int \frac{\sin ax}{a}dx$$

$$= \frac{x}{a}\sin ax - \frac{1}{a}\int \sin axdx$$

$$= \frac{x}{a}\sin ax - \frac{1}{a}\left[\frac{-\cos ax}{a}\right]$$

$$= \frac{x}{a}\sin ax + \frac{\cos ax}{a^2}$$

$$\int x\cos axdx = \frac{x}{a}\sin ax + \frac{\cos ax}{a^2} + c$$

2. $\int x\sin axdx$

Solution:

$u = x \qquad dv = \sin axdx$

$du = dx \qquad v = \frac{-\cos ax}{a}$

$$\int udv = uv - \int vdu$$

$$= -x\frac{\cos ax}{a} + \int \frac{\cos ax}{a}dx$$

$$= -\frac{x}{a}\cos ax + \frac{1}{a}\int \cos axdx$$

$$= \frac{x}{a}\sin ax + \frac{1}{a}\left[\frac{\sin ax}{a}\right]$$

$$= \frac{x}{a}\sin ax + \frac{\sin ax}{a^2}$$

$$\int x\cos axdx = \frac{x}{a}\sin ax + \frac{\sin ax}{a^2} + c$$

3. $\int xe^x dx$

Solution:

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$\int u dv = uv - \int v du$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$

$$= e^x (x-1) + c$$

$$\int xe^x dx = e^x (x-1) + c$$

4. $\int \log x dx$

Solution:

$u = \log x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int u dv = uv - \int v du$$

$$= \log x x - \int x \frac{1}{x} dx$$

$$= x \log x - x + c$$

$$= x(\log x - 1) + c$$

$\int \log x dx == x(\log x - 1) + c$

Possible Questions

2 Mark questions

1. Evaluate: $\int \frac{2x-1}{\sqrt{x^2+5x+6}}$ dx.

2. Evaluate: (i) $\int_0^\infty e^{-x^2} x^3$ dx

   (ii)$\int x log(x+1)dx$.

3. Evaluate:$\int \frac{ex}{e^{\frac{x}{2}}-1}$ dx.

4. Evaluate:$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$.

5. Integrate $\int (\log x)dx$.

6. Integrate $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$.

7. Evaluate $\int \frac{\cos x}{a+bsinx} dx$.

8. Evaluate $\int \frac{2x+1}{(x-3)^2} dx$.

9. Evaluate $\int \sqrt{(x-3)(7-x)}dx$.

10. Evaluate $\int \frac{dx}{5+4cosx}$.

11. Evaluate $\int \frac{(2x+1)}{x^2+3x+1} dx$.

12. Evaluate $\int \frac{dx}{3\sin x+4\cos x}$.

13. Evaluate (i) $\int \sqrt{\frac{x-2}{5-x}}dx$ (ii)$\int x^2 \cos x \, dx$.

14. Evaluate $\int sin^n$ x dx.

15. Evaluate : $\int_0^3 \int_1^2 xy(x+y)dydx$.

6 Mark questions

1. Evaluate : $\int (2x-3)\sqrt{4x+1}\, dx$.

2. Integrate : $\int x^3 e^{x^2}\, dx$.

3. Evaluate : $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\, dx$.

4. Evaluate : $\int \frac{x\sin x}{1+\cos x}\, dx$.

5. Evaluate :

   (i) $\int \frac{dx}{3\cos x+4\sin x+6}$.

   (ii) $\int \frac{(x+1)}{x^2+6x+25}\, dx$.

6. Evaluate :

   (i) $\int \sqrt{\frac{x-2}{5-x}}\, dx$.

   (ii) $\int x^2 \cos x\, dx$.

7. Evaluate : $\int \frac{2x-4}{\sqrt{3x^2-4x-7}}\, dx$.

Evaluate : $\int \frac{\log(1+x)}{(2x-1)^2}\, dx$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
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Subject: Mathematics-I | **Subject Code: 18PHU304**

Class : II - B.Sc. Physics | **Semester : III**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The value of ∫dx/(x^2+a^2)is-----------------	tan^(-1)x	1/a tan^(-1)x	1/a tan^(-1)(x/a)	tanx	tan^(-1)x
The ∫(sinθ)dθ value is--------------	sinθ	cosθ	tanθ	secθ	cosθ
The ∫(cosθ)dθ value is--------------	sinθ	-cosθ	tanθ	secθ	sinθ
The ∫(tanθ)dθ value is--------------	0	–logcosθ	logsinθ	secθ	–logcosθ
Integral a to b value of f(x) = --------------	F(a)-F(b)	F(b)-F(a)	F(a)	F(b)	F(b)-F(a)
The value of the integral 0 to pi/6 for the function (cos^2 (x/2))is---------------	π	π/12+1/4	0	π/2	π/12+1/4
The ∫(cotθ)dθ value is--------------	0	logsinθ	sinθ	secθ	logsinθ
The ∫(cosecθ)dθ value is--------------	logtan(x/2)	tan(x/2)	logsin(x/2)	sin(x/2)	logtan(x/2)
The ∫dx/x value is--------------	x	x^2	logx	1/x^2	logx
Which is following rational algebric function-----	∫1/xdx	∫xdx	∫x^2dx	∫(x+1)dx	∫1/xdx
The ∫dx/(x^2+a^2) value is--------------	1/a tan^(-1)(x/a)	1/a tan^(-1)x	tan^(-1)x	tanx	1/a tan^(-1)(x/a)
The ∫dx/(x^2-a^2) value is--------------	1/a log(x-a)	1/a log((x-a)/(x+a))	log((x-a)/(x+a))	log((x+a)	1/a log((x-a)/(x+a))
The ∫sinθdθ value is--------------	-sinθ	-cosθ	tanθ	secθ	-cosθ
The value ∫(f'(x))/(f(x)) dx is---------------	logx	logf(x)	0	f(x)	logf(x)
The ∫(secθtanθ)dθ value is--------------	secθ	-cosθ	logsinθ	cosθ	secθ
The ∫(cosecθcotθ)dθ value is--------------	secθ	-cosecθ	logsinθ	tanθ	-cosecθ
The ∫(coshx)dx value is--------------	sinhx	coshx	sechx	tanhx	sinhx

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The value of ∫sinhxdx value is--------------	sinhx	coshx	-coshx	tanhx	coshx
The ∫(x^n)dx,if n=-1then value is--------------	sinx	cosx	logx	tanx	logx
The value of ∫cf(x)dx is equal to--------------	∫f(x)dx	c∫f(x)dx	c^2∫f(x)dx	c∫dx	c∫f(x)dx
The value of ∫(u+v)dx is equal to--------------	∫udx-∫vdx	∫udx*∫vdx	∫udx+∫vdx	∫uvdx	∫udx+∫vdx
The value of ∫(u-v)dx is equal to--------------	∫udx-∫vdx	∫udx*∫vdx	∫udx+∫vdx	∫uvdx	∫udx-∫vdx
The value of ∫f(ax+b)dx is equal to--------------	∫f(x)dx	1/a(∫f(x)dx)	∫af(x)dx	∫dx	1/a(∫f(x)dx)
The value of ∫e^(ax+b)dx is equal to----------	1/a(e^(ax+b))	a(e^(ax+b))	1/a(e^(ax))	1/a(e^(b))	1/a(e^(ax+b))
The integral ∫(f(x)^n)(x^n-1)dx is equal to-----	(n+1)∫f(x)dx	n(∫f(x)dx)	∫f(x)dx	1/n(∫f(x)dx)	1/n(∫f(x)dx)

UNIT-III

Reduction formulae- problems- evaluation of double and triple integrals- applications to calculations of areas and volumes-areas in polar coordinates.

REDUCTION FORMULAE:

Reduction formula is the formula which connects a given integral with another integral which is of the same type but of a lower degree or is otherwise easier to evaluate.

Problem:

1. Find the reduction formula for $\int x^n e^{ax} dx, n$ being a positive integer.

Solution:

$$I_n = \int x^n e^{ax} dx$$

$$u = x^n \qquad dv = e^{ax} dx$$

$$du = nx^{n-1} dx \qquad v = \frac{e^{ax}}{a}$$

$$\int u dv = uv - \int v du$$

$$I_n = x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} nx^{n-1} dx$$

$$= x^n \frac{e^{ax}}{a} - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

The auxiliary integral is of the same type as the given integral but with index n reduction by 1. Such a formula is called reduction formula and by successing approximation we can evaluate I_n.

Problem:

1. Evaluate $\int x^2 e^{-x} dx$ by reduction formula.

Solution:

Let $I_n = \int x^2 e^{-x} dx$..................(1)

Reduction formula is given by

$$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1} \text{.....................(2)}$$

Here a=-1, n=2.

$$I_2 = x^2 \frac{e^{-x}}{-1} - \frac{2}{-1} I_{2-1}$$

$$I_2 = -x^2 e^{-x} + 2I_1 \text{.....................(3)}$$

$$I_1 = -xe^{-x} + I_0 \text{.....................(4)}$$

$$I_0 = -e^{-x}$$

$$I_2 = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x})$$

$$I_2 = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

2. Evaluate $\int e^x (x-1)^2 dx$ by reduction formula.

Solution:

Let $I_n = \int e^x (x-1)^2 dx$

$$I_n = \int e^x (x^2 - 2x + 1)dx$$

$$I_n = \int (e^x x^2 - 2e^x x + e^x)dx$$

$$I_n = \int e^x x^2 dx - 2\int xe^x dx + \int e^x dx ....(1)$$

Consider,

$$\int e^x x^2 dx$$

Reduction formula is given by

$$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1} ....(2)$$

Here a=1, n=2.

$$I_2 = x^2 e^x - 2I_{2-1}$$

$$I_2 = x^2 e^x + 2I_1$$

$$I_1 = xe^x - I_0$$

$$I_0 = e^x$$

$$I_2 = x^2 e^x - 2(xe^x + e^x)$$

$$I_2 = x^2 e^x - 2xe^x + 2e^x$$

$$\int e^x x^2 dx = x^2 e^x - 2xe^x + 2e^x + c ....(1)$$

Consider,

$$\int xe^x dx$$

Here a=1, n=1

$I_1 = xe^x - I_0$

$I_0 = e^x$

$I_1 = xe^x - e^x$

$\int xe^x dx = xe^x - e^x + c \Lambda\ \Lambda\ \Lambda\ (2)$

Eqn (1) becomes,

$I_n = x^2e^x - 2xe^x + 2e^x - 2(xe^x - e^x) + e^x + c$

$I_n = x^2e^x - 2xe^x + 2e^x - 2xe^x + 2e^x + e^x + c$

$I_n = x^2e^x - 4xe^x + 5e^x + c$

$\int e^x (x-1)^2 dx = x^2e^x - 4xe^x + 5e^x + c$

3. Evaluate $\int x^3 e^{2x} dx$ by reduction formula.

Solution:

Let $I_n = \int x^3 e^{2x} dx$...................(1)

Reduction formula is given by

$I_n = x^n \dfrac{e^{ax}}{a} - \dfrac{n}{a} I_{n-1}$.....................(2)

Here a=2, n=3.

$I_3 = x^3 \dfrac{e^{2x}}{2} - \dfrac{3}{2} I_{3-1}$

$$I_3 = x^3 \frac{e^{2x}}{2} - \frac{3}{2} I_2 \text{.........................(3)}$$

$$I_2 = x^2 \frac{e^{2x}}{2} - \frac{3}{2} I_{2-1}$$

$$I_2 = x^2 \frac{e^{2x}}{2} - \frac{3}{2} I_1 \text{.............................(4)}$$

$$I_1 = x \frac{e^{2x}}{2} - \frac{1}{2} I_0 \text{.....................(5)}$$

$$I_0 = \frac{e^{2x}}{2}$$

$$I_n = x^3 \frac{e^{2x}}{2} - \frac{3}{2}\left(x^2 \frac{e^{2x}}{2} - \frac{3}{2}\left(x \frac{e^{2x}}{2} - \frac{1}{2}\frac{e^{2x}}{2} \right)\right)$$

$$I_n = x^3 \frac{e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3xe^{2x}}{4} + \frac{3e^{2x}}{8} + c$$

$$\int x^3 e^{2x} dx = x^3 \frac{e^{2x}}{2} - \frac{3}{4}\left[x^2 e^{2x} - xe^{2x} - \frac{e^{2x}}{2} \right] + c$$

4. Find the reduction formula for $\int x^n \cos axdx$, x being positive integer.

Solution:

Let $I_n = \int x^n \cos axdx$..................(1)

$u = x^n \qquad dv = \cos axdx$

$du = nx^{n-1}dx \quad v = \frac{\sin ax}{a}$

$$\int udv = uv - \int vdu$$

$$I_n = x^n \frac{\sin ax}{a} - \int \frac{\sin ax}{a} nx^{n-1} dx$$

$$= \frac{x^n}{a} \sin ax - \frac{n}{a} \int \sin axx^{n-1} dx$$

$$I_n = \frac{x^n}{a} \sin ax - \frac{n}{a} I_{n-1} ....(2)$$

Now,

$$I_{n-1} = \int x^{n-1} \frac{\sin ax}{a} dx$$

$u = x^{n-1}$ $\quad dv = \sin axdx$

$du = (n-1)x^{n-2}dx$ $\quad v = \frac{-\cos ax}{a}$

$\int udv = uv - \int vdu$

$$I_{n-1} = -x^{n-1} \frac{\cos ax}{a} - \int \frac{-\cos ax}{a} (n-1) x^{n-2} dx$$

$$= -\frac{x^{n-1}}{a} \cos ax + \frac{n-1}{a} \int \cos axx^{n-2} dx$$

$$I_n = \frac{x^n}{a} \sin ax - \frac{n}{a} \left(-\frac{x^{n-1}}{a} \cos ax + \frac{n-1}{a} \int \cos axx^{n-2} dx \right)$$

$$I_n = \frac{x^n}{a} \sin ax + \frac{nx^{n-1}}{a^2} \cos ax - \frac{n-1}{a^2} I_{n-2} ....(3)$$

Which is the reduction formula

The ultimate integral is

$\int x\cos axdx$ When n is odd

$\int \cos axdx$ When n is even

Case 1:

When n is odd

$$\int x\cos axdx$$

$u = x \qquad dv = \cos axdx$

$du = dx \qquad v = \frac{\sin ax}{a}$

$$\int udv = uv - \int vdu$$

$$= x\frac{\sin ax}{a} - \int \frac{\sin ax}{a}dx$$

$$= \frac{x}{a}\sin ax - \frac{1}{a}\int \sin axdx$$

$$= \frac{x}{a}\sin ax - \frac{1}{a}\left[\frac{-\cos ax}{a}\right]$$

$$= \frac{x}{a}\sin ax + \frac{\cos ax}{a^2}$$

$$\int x\cos axdx = \frac{x}{a}\sin ax + \frac{\cos ax}{a^2} + c$$

Case 2:

When n is even

$\int \cos ax dx = \frac{\sin ax}{a} + c$

5. Find the reduction formula for $\int x^n \sin ax dx$, x being positive integer.

Solution:

Let $I_n = \int x^n \sin ax dx$..................(1)

$u = x^n \qquad dv = \sin ax dx$

$du = nx^{n-1}dx \quad v = \frac{-\cos ax}{a}$

$$\int udv = uv - \int vdu$$

$$I_n = x^n \frac{-\cos ax}{a} - \int \frac{-\cos ax}{a} nx^{n-1} dx$$

$$= -\frac{x^n}{a}\cos ax + \frac{n}{a}\int \cos ax x^{n-1} dx$$

$$I_n = -\frac{x^n}{a}\cos ax + \frac{n}{a} I_{n-1} \text{...............(2)}$$

Now,

$$I_{n-1} = \int x^{n-1} \cos ax dx$$

$u = x^{n-1} \qquad dv = \sin ax dx$

$du = (n-1)x^{n-2}dx \qquad v = \frac{\sin ax}{a}$

$\int udv = uv - \int vdu$

$$I_{n-1} = x^{n-1}\frac{\sin ax}{a} - \int \frac{\sin ax}{a}(n-1)x^{n-2}dx$$

$$= \frac{x^{n-1}}{a}\frac{\sin ax}{a} - \frac{n-1}{a}\int \sin axx^{n-2}dx$$

$$I_n = -\frac{x^n}{a}\cos ax - \frac{n}{a}\left(\frac{x^{n-1}}{a}\sin ax - \frac{n-1}{a}\int \sin axx^{n-2}dx\right)$$

$$I_n = -\frac{x^n}{a}\cos ax + \frac{nx^{n-1}}{a^2}\sin ax - \frac{n-1}{a^2}I_{n-2}\text{........................}(3)$$

Which is the reduction formula

The ultimate integral is

$\int x\sin axdx$ When n is odd

$\int \sin axdx$ When n is even

Case 1:

When n is odd

$$\int x\sin axdx$$

$u = x$ $\quad dv = \sin axdx$

$du = dx$ $\quad v = \frac{-\cos ax}{a}$

$$\int udv = uv - \int vdu$$

$$= -x\frac{\cos ax}{a} + \int \frac{\cos ax}{a}dx$$

$$= -\frac{x}{a}\cos ax + \frac{1}{a}\int \cos ax dx$$

$$= \frac{x}{a}\sin ax + \frac{1}{a}\left[\frac{\sin ax}{a}\right]$$

$$= \frac{x}{a}\sin ax + \frac{\sin ax}{a^2}$$

$$\int x\cos ax dx = \frac{x}{a}\sin ax + \frac{\sin ax}{a^2} + c$$

Case 2:

When n is even

$$\int \cos ax dx = -\frac{\cos ax}{a} + c$$

6. Shoe that if $I_n = \int_0^{\pi/2} x^n \cos x dx$,show that $I_n + n(n-1)I_{n-2} = \left(\frac{\pi}{2}\right)^n$

Solution:

Let $I_n = \int x^n \cos x dx$...................(1)

$u = x^n$ $dv = \cos x dx$

$du = nx^{n-1}dx$ $v = \sin x$

$$\int udv = uv - \int vdu$$

$$I_n = \left(x^n \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} \sin x nx^{n-1} dx$$

$$= \left(x^n \sin x\right)_0^{\pi/2} - n\int_0^{\pi/2} \sin x x^{n-1} dx$$

$$I_n = \left(x^n \sin x\right)_0^{\pi/2} - nI_{n-1} \ldots\ldots\ldots\ldots..(2)$$

Now,

$$I_{n-1} = \int_0^{\pi/2} \sin x x^{n-1} dx$$

$u = x^{n-1}$ $dv = \sin x dx$

$du = (n-1)x^{n-2}dx$ $v = -\cos x$

$\int udv = uv - \int vdu$

$$I_{n-1} = \left(-x^{n-1}\cos x\right)_0^{\pi/2} - \int_0^{\pi/2} -\cos x(n-1)x^{n-2}dx$$

$$= \left(-x^{n-1}\cos x\right)_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos x x^{n-2}dx$$

$$I_n = \left(x^n \sin x\right)_0^{\pi/2} - n\left(\left(-x^{n-1}\cos x\right)_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos x x^{n-2}dx\right)$$

$$I_n = \left(\left(\pi/2\right)^n \sin x\, \pi/2 - 0\right) - n\left(\left(-\left(\pi/2\right)^{n-1}\cos \pi/2\right) + (n-1)\int_0^{\pi/2} \cos x x^{n-2}dx\right)$$

$$I_n = \left(\left(\pi/2\right)^n 1 - 0\right) - n\left(0 + (n-1)I_{n-2}\right)$$

$$I_n = \left(\pi/2\right)^n - n(n-1)I_{n-2}$$

$$I_n + n(n-1)I_{n-2} = \left(\pi/2\right)^n$$

Hence the proof

7. Find the reduction formula for $I_n = \int_0^{\pi/2} x^3 \sin x dx$

Solution:

Let $I_n = \int_0^{\pi/2} x^3 \sin x dx$...................(1)

$u = x^3$ $\quad dv = \sin x dx$

$du = 3x^2 dx$ $\quad v = -\cos x$

$$\int u dv = uv - \int v du$$

$$I_n = \left(-x^3 \cos x\right)_0^{\pi/2} - 3\int_0^{\pi/2} -\cos x x^2 dx$$

$$= \left(-x^3 \cos x\right)_0^{\pi/2} + 3\int_0^{\pi/2} \cos x x^2 dx$$

$$I_n = \left(-x^3 \cos x\right)_0^{\pi/2} + 3I_{n-1}\text{...............(2)}$$

Now,

$$I_{n-1} = \int_0^{\pi/2} x^2 \cos x dx$$

$u = x^2$ $\quad dv = \cos x dx$

$du = 2x dx$ $\quad v = \sin x$

$\int u dv = uv - \int v du$

$$I_{n-1} = \left(x^2 \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} \sin x 2x dx$$

$$= \left(-x^2 \cos x\right)_0^{\pi/2} - 2\int_0^{\pi/2} x \sin x dx$$

$$I_n = \left(-x^3 \cos x\right)_0^{\pi/2} + 3\left(\left(x^2 \sin x\right)_0^{\pi/2} - 2\int_0^{\pi/2} x \sin x dx\right)$$

$$I_n = \left(\left(\pi/2\right)^3 \cos \pi/2 + 0\right) - 3\left(\left(\left(\pi/2\right)^2 \sin \pi/2\right) + (n-1)\int_0^{\pi/2} \cos x x^{n-2} dx\right)$$

$$I_n = 0 + 3\left(\left(\pi/2\right)^2 - 2I_1\right)$$

$$I_n = 3\left(\left(\pi/2\right)^2 - 2I_1\right)$$

$$I_n = \frac{3\pi^2}{4} - 6I_1$$

$$I_n + 6I_1 = \frac{3\pi^2}{4}$$

8. $\int \sin^6 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_6 = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6} I_4$$

$$I_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$I_2 = \frac{-\sin x \cos x}{2} + \frac{1}{2} I_0$$

$$I_0 = 0$$

$$I_0 = x$$

$$I_4 = \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left(\frac{-\sin x \cos x}{2} + \frac{x}{2}\right)$$

$$I_4 = \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3x}{8}$$

$$I_6 = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6}\left(\frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3x}{8}\right)$$

$$I_6 = \frac{-\sin^5 x \cos x}{6} - \frac{5\sin^3 x \cos x}{24} - \frac{15}{48}\sin x \cos x + \frac{15x}{48} + c$$

9. $\int \sin^5 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_5 = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5} I_3$$

$$I_3 = \frac{-\sin^2 x \cos x}{3} + \frac{2}{3} I_2$$

$$I_1 = -\cos x$$

$$I_3 = \frac{-\sin^2 x \cos x}{3} + \frac{2}{3}(-\cos x)$$

$$I_3 = \frac{-\sin^2 x \cos x}{3} - \frac{2}{3}\cos x$$

$$I_5 = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5}\left(\frac{-\sin^2 x \cos x}{3} - \frac{2}{3}\cos x\right)$$

$$I_5 = \frac{-\sin^4 x \cos x}{5} - \frac{4\sin^2 x \cos x}{15} - \frac{8}{15}\cos x + c$$

10. Find the reduction formula for $\int_0^{\pi/2} \sin^n x dx$

Solution:

Let $I_n = \int_0^{\pi/2} \sin^n x dx$

$$I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x dx$$

Let $u = \sin^{n-1} x$ $\quad dv = \sin x$

$du = n-1 \sin^{n-2} x \cos x dx$ $\quad v = -\cos x$

$$I_n = \left(-\sin^{n-1} x \cos x\right)_0^{\pi/2} - \int_0^{\pi/2} -\cos x (n-1) \sin^{n-2} x \cos x dx$$

$$= \left(-\sin^{n-1} x \cos x\right)_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx$$

$$= \left(-\sin^{n-1} x \cos x\right)_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x (1-\sin^2 x) dx$$

$$= \left(-\sin^{n-1} x \cos x\right)_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x dx - (n-1) \int_0^{\pi/2} \sin^n x dx$$

$$I_n = 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x dx - (n-1) \int_0^{\pi/2} \sin^n x dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$nI_n = (n-1) I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

This is the reduction formula

Replacing n by (n-2)

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_n = \frac{(n-1)}{n} \frac{n-3}{n-2} I_{n-4}$$

Generalizing (1) and (2) we get the following 2 cases

Case 1:

When n is even

$$I_n = \frac{(n-1)(n-3)........1}{n(n-2)..........2} I_0$$

Here $I_0 = x$

$$I_0 = (x)_0^{\pi/2}$$

$$I_0 = \left(\pi/2\right)$$

$$I_n = \frac{(n-1)(n-3)........1}{n(n-2)..........2} \left(\pi/2\right)$$

Case 2:

When n is odd

$$I_n = \frac{(n-1)(n-3)........2}{n(n-2)...............3} I_1$$

Here $I_1 = -\cos x$

$$I_1 = (-\cos x)_0^{\pi/2}$$

$$I_1 = \left(-\cos \pi/2 - (-\cos 0)\right)$$

$$I_1 = 0 - (-1)$$

$$I_1 = 1$$

$$I_n = \frac{(n-1)(n-3)........2}{n(n-2)...............3} (1)$$

$$I_n = \frac{(n-1)(n-3)........2}{n(n-2)...............3}$$

11. $\int_0^{\pi/2} \sin^4 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_4 = \left[\frac{-\sin^3 x \cos x}{4}\right]_0^{\pi/2} + \frac{3}{4} I_2$$

$$I_4 = 0 + \frac{3}{4} I_2$$

$$I_4 = \frac{3}{4} I_2$$

$$I_2 = \left[\frac{-\sin x \cos x}{2}\right]_0^{\pi/2} + \frac{1}{2} I_0$$

$$I_2 = 0 + \frac{1}{2} I_0$$

$$I_2 = \frac{1}{2} I_0$$

$$I_0 = (x)_0^{\pi/2}$$

$$I_0 = \pi/2$$

$$I_2 = \pi/4$$

$$I_4 = \frac{3}{4} \times \pi/4$$

$$I_4 = 3\pi/16$$

$$\int_0^{\pi/2} \sin^4 x dx = 3\pi/16$$

12. $\int_0^{\pi/2} \sin^7 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_7 = \left[\frac{-\sin^6 x \cos x}{7}\right]_0^{\pi/2} + \frac{6}{7} I_5$$

$$I_7 = 0 + \frac{6}{7} I_5$$

$$I_7 = \frac{6}{7} I_5$$

$$I_5 = \left[\frac{-\sin^4 x \cos x}{5}\right]_0^{\pi/2} + \frac{4}{5} I_3$$

$$I_5 = 0 + \frac{4}{5} I_3$$

$$I_5 = \frac{4}{5} I_3$$

$$I_3 = \left[\frac{-\sin x^2 \cos x}{3}\right]_0^{\pi/2} + \frac{2}{3} I_1$$

$$I_3 = 0 + \frac{2}{3} I_1$$

$$I_3 = \frac{2}{3} I_1$$

$$I_5 = \frac{4}{5} . \frac{2}{3}$$

$$I_5 = \frac{8}{15}$$

$$I_7 = \frac{6}{7} . \frac{8}{15}$$

$$I_7 = \frac{16}{35}$$

$$\int_0^{\pi/2} \sin^7 x dx = \frac{16}{35}$$

13. $\int \cos^6 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_6 = \frac{\sin x \cos^5 x}{6} + \frac{5}{6} I_4$$

$$I_4 = \frac{\sin x \cos^3 x}{4} + \frac{3}{4} I_2$$

$$I_2 = \frac{\sin x \cos x}{2} + \frac{1}{2} I_0$$

$$I_0 = x$$

$$I_4 = \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\left(\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

$$I_4 = \frac{\sin x \cos^3 x}{4} + \frac{3}{8}\sin x \cos x + \frac{3x}{8}$$

$$I_6 = \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\left(\frac{\sin x \cos^3 x}{4} + \frac{3}{8}\sin x \cos x + \frac{3x}{8}\right)$$

$$I_6 = \frac{\sin x \cos^5 x}{6} + \frac{5 \sin x \cos^3 x}{24} + \frac{15}{48}\sin x \cos x + \frac{15x}{48} + c$$

14. $\int \cos^7 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_7 = \frac{\sin x \cos^6 x}{7} + \frac{6}{7} I_5$$

$$I_5 = \frac{\sin x \cos^4 x}{5} + \frac{4}{5} I_3$$

$$I_3 = \frac{\sin x \cos^2 x}{3} + \frac{2}{3} I_1$$

$$I_1 = -\sin x$$

$$I_5 = \frac{\sin x \cos^4 x}{5} + \frac{4}{5}\left(\frac{\sin x \cos^2 x}{3} + \frac{2}{3}(\sin x)\right)$$

$$I_5 = \frac{\sin x \cos^4 x}{5} + \frac{4}{15}\sin x \cos^2 x + \frac{8}{15}\sin x$$

$$I_7 = \frac{\sin x \cos^6 x}{7} + \frac{6}{7}\left(\frac{\sin x \cos^4 x}{5} + \frac{4}{15}\sin x \cos^2 x + \frac{8}{15}\sin x\right)$$

$$I_7 = \frac{\sin x \cos^6 x}{7} + \frac{6 \sin x \cos^3 x}{35} + \frac{24}{105}\sin x \cos x + \frac{48}{105}\sin x + c$$

15. $\int \tan^6 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_6 = \frac{\tan^5 x}{5} - I_4$$

$$I_4 = \frac{\tan^3 x}{3} - I_2$$

$$I_2 = \frac{\tan x}{1} - I_0$$

$$I_0 = x$$

$$I_4 = \frac{\tan^3 x}{3} - (\tan x - x)$$

$$I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

16. $\int \tan^5 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_5 = \frac{\tan^4 x}{4} - I_3$$

$$I_3 = \frac{\tan^2 x}{2} - I_1$$

$$I_1 = \log(\sec x)$$

$$I_3 = \frac{\tan^2 x}{2} - \log(\sec x)$$

$$I_5 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log(\sec x) + c$$

17. $\int \cot^6 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

$$I_6 = \frac{-\cot^5 x}{5} - I_4$$

$$I_4 = \frac{-\cot^3 x}{3} - I_2$$

$$I_2 = \frac{-\cot x}{1} - I_0$$

$$I_0 = x$$

$$I_4 = \frac{-\cot^3 x}{3} + \cot x + x$$

$$I_6 = \frac{-\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$$

18. $\int \cot^5 x dx$

Solution:

We know the reduction formula is

$$I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$$

$$I_5 = \frac{-\cot^4 x}{4} - I_3$$

$$I_3 = \frac{-\cot^2 x}{2} - I_1$$

$$I_1 = \log(\sin x)$$

$$I_3 = \frac{-\cot^2 x}{2} - \log(\sin x)$$

$$I_5 = \frac{-\cot^4 x}{4} + \frac{-\cot^2 x}{2} + \log(\sin x) + c$$

19. Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x dx$

Solution:

Here m=6 and n=4 m and n are even

The reduction formula

$$I_{m,n} = \frac{(n-1)(n-3)(m-1)(m-3)}{(m+n)(m+n-2)m(m-2)2}\left(\frac{\pi}{2}\right)$$

$$I_{6,4} = \frac{(3)(1)(5)(3)}{(10)(8)6(4)2}\left(\frac{\pi}{2}\right)$$

$$I_{6,4} = \frac{3\pi}{512}$$

20. Evaluate $\int_0^{\pi/2} \sin^6 x \cos^5 x dx$

Solution:

Here m=6 and n=4 m and n are even

The reduction formula

$$I_{m,n} = \frac{(n-1)(n-3)......1}{(m+n)(m+n-2).......m+1}$$

$$I_{6,5} = \frac{4.2.......1}{11.9......7}$$

$$I_{6,5} = \frac{8}{693}$$

MULTIPLE INTEGRALS:

1. Evaluate $\int_0^a \int_0^b (x^2 + y^2) dy dx$

Solution:

$$\int_0^a \int_0^b (x^2 + y^2) dy dx = \int_0^a \left(x^2 y + \frac{y^3}{3} \right)_0^b dx$$

$$= \int_0^a \left(x^2 b + \frac{b^3}{3} \right) dx$$

$$= \left(\frac{x^3}{3} b + x \frac{b^3}{3} \right)_0^a + c$$

$$= \left(\frac{x^3}{3} b + x \frac{b^3}{3} \right)_0^a + c$$

$$= \frac{a^3 b}{3} + \frac{ab^3}{3} + c$$

$$\int_0^a \int_0^b (x^2 + y^2) dy dx = \frac{a^3 b}{3} + \frac{ab^3}{3} + c$$

2. Evaluate $\int_0^3 \int_1^2 xy(x + y) dy dx$

Solution:

$$\int_0^3\int_1^2 (x^2y + xy^2)dydx = \int_0^3 \left(x^2\frac{y^2}{2} + x\frac{y^3}{3} \right)_1^2 dx$$

$$= \int_0^3 \left(\frac{4x^2}{2} + \frac{8x}{3} - \frac{x^2}{2} - \frac{x}{3} \right) dx$$

$$= \int_0^3 \left(\frac{3x^2}{2} + \frac{7x}{3} \right) dx$$

$$= \left(\frac{3x^3}{6} + \frac{7x^2}{6} \right)_0^3 + c$$

$$= \left(\frac{x^3}{2} + \frac{7x^2}{6} \right)_0^3 + c$$

$$= \frac{27}{2} + \frac{63}{6}$$

$$= \frac{81+63}{6} + c$$

$$= \frac{144}{6} + c$$

$$= 24$$

$$\int_0^3\int_1^2 xy(x+y)dydx = 24$$

3. Evaluate $\int_0^a\int_0^b xy(x-y)dydx$

Solution:

$$\int_0^a \int_0^b (x^2 y - xy^2) dydx = \int_0^a \left(x^2 \frac{y^2}{2} - x\frac{y^3}{3} \right)_0^b dx$$

$$= \int_0^a \left(\frac{x^2 b^2}{2} - \frac{xb^3}{3} \right) dx$$

$$= \int_0^a \left(\frac{x^2 b^2}{2} - \frac{xb^3}{3} \right) dx$$

$$= \left(\frac{x^3 b^2}{6} - \frac{x^2 b^3}{6} \right)_0^a + c$$

$$= \frac{a^3 b^2}{6} - \frac{a^2 b^3}{6}$$

$$= \frac{a^3 b^2}{6}(a-b)$$

$$\int_0^a \int_0^b xy(x-y) dydx = \frac{a^3 b^2}{6}(a-b)$$

4. Evaluate $\int_0^a \int_0^x (x^2 + y^2) dydx$

Solution:

$$\int_0^a \int_0^x (x^2 + y^2) dydx = \int_0^a \left(x^2 y + \frac{y^3}{3} \right)_0^x dx$$

$$= \int_0^a \left(x^2 x + \frac{x^3}{3} \right) dx$$

$$= \int_0^a \left(x^3 + \frac{x^3}{3} \right) dx$$

$$=\left(\frac{x^4}{4}+\frac{x^4}{12}\right)_0^a+c$$

$$=\left(\frac{a^4}{4}+\frac{a^4}{12}\right)+c$$

$$=\frac{3a^4+a^4}{12}$$

$$=\frac{4a^4}{12}$$

$$=\frac{a^4}{3}$$

$$\int_0^a\int_0^x (x^2+y^2)dydx=\frac{a^4}{3}$$

Unit-III

Possible Questions

2 Mark questions

1. Evaluate : $\int_0^3 \int_1^2 xy(x^2+y^2)dydx$.

2. Evaluate: $\int_0^{\frac{\pi}{2}} sin^5x \cos^7 x\, dx$.

3. If $I_n = \int cos^n x\, dx$ (n, being a positive integer) then prove that $nI_n = cos^{n-1}xsinx + (n-I)I_{n-2}$.

4. Evaluate $\int\int xy\, dx\, dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.

5. If $I_n = \int_0^{\frac{\pi}{2}} x^n cosxdx$, show that $I_n + n(n-1)I_{n-2} = (\frac{\pi}{2})^n$.

6. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

7. If $u_n = \int (\log x)^n\, dx$, show that $u_n + nu_{n-1} = (xlogx)^n$.

8. Evaluate $_S\int\int \sqrt{4x^2-y^2dx\, dy}$, where S is the region bounded by x=0,y=x,x=1.

9. Evaluate $\int x^3 \cos 2x\, dx$ using reduction formula.

10. Evaluate $\int_0^{\pi}\int_o^{1-cos\theta} r\, dr\, d\theta$.

11. If $I_n = \int_0^a \mathrm{x^n e^{-x}}\, \mathrm{dx}$ prove that $I_n - \mathrm{(n+a)I_{n-1} + a(n-1)I_{n-2}} = 0$.

12. Find reduction formula for $\int sin^n \mathrm{x}\, dx$.

13. Find the reduction formula for $\int x^m(\log x)^n\ dx$. hence find $\int x^4(\log x)^3\ dx$.

14. Find $\int \frac{x^m}{(\log x)^n} dx$.

15. Evaluate $\int \frac{dx}{x+1\sqrt{x^2}+x+1}$.

6 Mark questions

1. If $\int_0^{\frac{\pi}{2}} cos^m x\, cosnx\, dx = f(m,n)$, prove that $f(m,n) = \frac{m}{m+n} f(m-1, n-1)$. hence prove

   that $f(n,n) = \frac{\pi}{2^{n+1}}$.

2. Find the area of the cardiod $r = a(1 + \cos\theta)$.

3. If $U_n = \int_0^a x^n e^{-x}\, dx$ prove that $u_n - (n + a)u_{n-1} + a(n-1)u_{n-2} = 0$.

4. Evaluate $\iiint xyz\, dx\, dy\, dz$ take through the positive octant of the sphere $x^2 + y^2 +$

   $z^2 = a^2$.

5. Evaluate :$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$.

6. Evaluate :$\int_0^1 \int_0^2 \int_0^3 z^2\, yx\, dx\, dy\, dz$.

7. Evaluate $\int sin^n x\, dx$.

8. Obtain the reduction formula for $\int cos^n x\, dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} cos^n x\, dx$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Mathematics-I **Subject Code: 18PHU304**

Class : II - B.Sc. Physics **Semester : III**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Integration of 0 to π/2 with the function (sinx)^n--------------	((n-1)/n)x(n-3)/(n-2)…	(n+1)/n (n+3)/(n+2)…	(n-1)/(n-2) (n-3)/(n-4)…	n/(n+1) (n+3)/(n+2)…	((n-1)/n)x(n-3)/(n-2)…
Doble Integration of 0 to a and 0 to b with the function x^2+y^2 --------------	(ab)^3/9	(b)^3/9	(a)^3/9	(ab)^2/9	(ab)^3/9
Integration of 0 to π/2 with the function (sinx)^6--------------	0	π/2	3π/2	5π/6	5π/6
Doble Integration of 0 to 1 and 0 to 1 with the function f(x,y)=1--------------	0	-1	2	1	1
Integration of 0 to π/2 with the function (cosx)^n then if n is even --------------	1	π/2	3π/2	π/6	π/2
Integration of 0 to π/2 with the function (cosx)^n then if n is odd --------------	1	π/2	3π/2	π/6	1
Integration of 0 to π/2 with the function (sinx)^n then if n is even --------------	1	π/2	π/3	π/6	π/2
Integration of 0 to π/2 with the function (sinx)^n then if n is odd--------------	1	π/2	π/3	π/6	1
The value of ∫(sinx)^m(cosx) dx =------------------	(sinx)^(m+1) /(m+1)	(sinx)^(m-1) /(m-1)	(cosx)^(m+1) /(m+1)	(cosx)^m /m	(sinx)^(m+1) /(m+1)

The value of ∫(sinx)(cosx)^n dx =------------------	(sinx)^(m+1) /(m+1)	(sinx)^(m-1) /(m-1)	-(cosx)^(n+1) /(n+1)	(cosx)^n /n	-(cosx)^(n+1) /(n+1)
The value of ∫(sinx)^m(cosx)^n dx with limit of 0 to π/2 if n=1then ------------------	1/m	1/m+1	1/m-1	1/m+2	1/m+1
Integration of 0 to π/2 with the function (cosx)^5 then --------------	1.6	0.533333333	0.32	0	0.533333333
The value of ∫(sinx)^6 (cosx)^5 dx with limit of 0 to π/2 then ------------------	8/693	9/693	10/693	11/693	8/693
The value of ∫(sinx)^6 (cosx)^4 dx with limit of 0 to π/2 then ------------------	3π/511	3π/512	3π/510	3π/501	3π/512
If ∫(x)^m(logx)^n dx then if n=0 =------------------	x^m+1/m+1	x^m-1/m-1	x^m/m+1	x^m/m	x^m+1/m+1
If ∫(cosx)^m(cosnx) dx=f(m,n) then f(n,n)= ------------------	π/(2^n+1)	π/(2^n)	π/(2^n-1)	π/(2^n+2)	π/(2^n+1)
If ∫(cosx)^m(cosnx) dx=f(m,n) then f(m,n)= ------------------	m/(m+n)f(m+1,n+1)	m/(m+n)f(m-1,n+1)	m/(m+n)f(m-1,n-1)	m/(m+n)f(m,n)	m/(m+n)f(m-1,n-1)
If ∫(cosx)^m(sinnx) dx=f(m,n) then f(m,n)= ---------	(1/m+1)+m/(m+n) f(m-1,n-1)	(1/m-1)+m/(m+n)f(m+1,n+1)	(1/m)+m/(m+n)f(m+1,n+1)	(1/m+2)+m/(m+n) f(m+1,n+1)	(1/m+1)+m/(m+n) f(m-1,n-1)
The area of the cardioid r=a(1+cosθ) is---------	3πa^2/2	3πa/2	3πa^3/2	3πa^2/4	3πa^2/2
The area of the lemniscase bernoulli r^2=a^2 cos2θ---------	a	2a	a^2	a^3	a^2
The perimeter of the cardioid r=a(1+cosθ) is---------	a	4a	8a	2a	8a
if the f(x,y) be a continuous and single real valued function of x and y within the region R bounded by a closed curve C and upon the boundary C is called--------	integration	doble integration	triple inegration	differentiation	doble integration
The value of ∬xy dx dy taken over the positive quadrant of the circle x^2+y^2=a^2---------	a/2	(a^3)/2	(a^4)/8	a^2	(a^4)/8
In triple integral when integrationg with respect to x in the integral y and z are treated as---------	variables	function	mapping	constants	constants

In triple integral when integrationg with respect to x in the integral ---------are treated as constants	y only	y and z	x only	y and x	y and z

UNIT-IV

Change of order of integration in double integral- change of variables in double and triple integrals.

CHANGE OF VARIABLES

JACOBIAN:

If $u = f(x,y)$ & $v = \phi(x,y)$ be continuous function of the independent variables x and y such that,

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

Are also continuous in x and y then

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Is called the Jacobian of u,v with respect to x,y and denoted by

$$J\left(\frac{u,v}{x,y}\right)(or)\frac{\partial(u,v)}{\partial(x,y)}$$

In case of three variables u,v,w which are function of x,y,z the Jacobian is respected by

$$J\left(\frac{u,v,w}{x,y,z}\right)(or)\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

TWO IMPORTANT RESULTS REGARDING JACOBIAN:

1. If u,v are functions of x,y and x,y are themselves function of ξ,η then

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{\partial(u,v)}{\partial(\xi,\eta)}$$

PROOF:

L.H.S,

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(\xi,\eta)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial x}\cdot\frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}\\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial x}\cdot\frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \eta}\end{vmatrix}\text{....................(1)}$$

Since $u=f(x,y)\ \&\ v=\phi(x,y)$

Also $x=f_1(\xi,\eta)\ \&\ y=f_2(\xi,\eta)$

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial x}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

$$\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial x}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}.\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}.\frac{\partial y}{\partial \eta}$$

Equation (1) becomes,

$$\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(\xi,\eta)}$$

=R.H.S

$$\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}$$

Hence the proof

2. $\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(\xi,\eta)} = 1$

PROOF:

L.H.S

We know that

$$\frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial(u,v)}{\partial(\xi,\eta)}$$

Put $\xi = u$ and $\eta = v$

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial(u,v)}{\partial(u,v)}$$

$$\frac{\partial(u,v)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial u}{\partial u} & \frac{\partial u}{\partial v}\\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}$$

$$=1$$

=R.H.S

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(\xi,\eta)}=1$$

Hence the proof

CHANGE OF VARIABLE IN CASE OF TWO VARIABLES:

Suppose we have to change the variable in integral $\iint_R F(x,y)dxdy$ then $dxdy$ changes to

$\frac{\partial(x,y)}{\partial(u,v)}dudv$

$$\iint_R F(x,y)dxdy=\iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv$$

Similarly in the case of three variables

$$\iiint_R F(x,y,z)dxdydz=\int\iint F(u,v,w)\frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw$$

Transformation from Cartesian to polar co-ordinate:

Let the polar co-ordinates of a point p(x,y).

Put

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Hence $dxdy$ has to be changed to $r.dr.d\theta$

Transformation from Cartesian to spherical polar co-ordinates:

Let the co-ordinates $p(x,y,z)$ in the Cartesian and (r,θ,ϕ) in spherical co-ordinate. Put

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \pm r^2 \sin\theta$$

Hence $dxdydz$ has to be changed to $+r^2 \sin\theta dr d\theta d\phi$

NOTE:

$dxdydz$ can also be changed to $-r^2 \sin\theta dr d\theta d\phi$

PROBLEM:

1. Given that $x+y=u; y=uv$ change the variables to u,v in the integral $\iint [xy(1-x-y]^{1/2} dxdy$ taken over the area of triangle with sides $x=0; y=0; x+y=1$ and evaluate it.

SOLUTION:

Given,

$x+y=u; y=uv$

$x+uv=u$

$x=u-uv$

$x = u(1-v); y = uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - uv + uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = u$$

The area of the triangle with sides $x = 0; y = 0; x + y = 1$ changes to

When $x = 0$

$u(1-v) = 0$
$u = 0 \;\&\; v = 1$

When $y = 0$

$0 = uv$
$u = 0 \;\&\; v = 0$

When $x + y = 1$

$u = 1$

Limits

u varies from 0 to 1

v varies from 0 to 1

we know that

$$\iint_R F(x,y)dxdy = \iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv$$

$$\iint [xy(1-x-y)]^{1/2}dxdy = \int_0^1\int_0^1 [u(1-v)uv(1-u)]^{1/2}ududv$$

$$= \int_0^1\int_0^1 [u^2v(1-v)(1-u)]^{1/2}ududv$$

$$= \int_0^1\int_0^1 u^2v^{1/2}(1-u)^{1/2}(1-v)^{1/2}dudv$$

$$= \int_0^1\left[\int_0^1 u^2(1-u)^{1/2}du\right]v^{1/2}(1-v)^{1/2}dv$$

Now, $\int_0^1 u^2(1-u)^{1/2}du$

Let $t^2 = 1-u$

$2tdt = -du$

$$-2tdt = du$$

When $u = 0, t = 1$

$$u = 1, t = 0$$

$$\int_0^1 u^2 (1-u)^{1/2} du = \int_1^0 (1-t^2)^2 t(-2tdt)$$

$$= -\int_0^1 (1-t^2)^2 t(-2tdt)$$

$$= 2\int_0^1 (1-t^2)^2 t^2 dt$$

$$= 2\int_0^1 (1-2t^2+t^4)t^2 dt$$

$$= 2\int_0^1 (t^2-2t^4+t^6)dt$$

$$= 2\left[\frac{t^3}{3} - \frac{2t^5}{5} + \frac{t^7}{7}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right]$$

$$= \frac{16}{105}$$

$$\int_0^1 \left[\int_0^1 u^2 (1-u)^{1/2} du \right] v^{1/2} (1-v)^{1/2} dv = \int_0^1 \frac{16}{105} v^{1/2} (1-v)^{1/2} dv$$

$$= \int_0^1 \frac{16}{105} \sqrt{v(1-v)} dv$$

$$= \frac{16}{105} \int_0^1 \sqrt{v(1-v)} dv$$

$$= \frac{16}{105} \int_0^1 \sqrt{(v-v^2)} dv$$

Now,

$$v - v^2 = -(v^2 - v)$$

$$= -\left(v^2 - v + \frac{1}{4} - \frac{1}{4} \right)$$

$$= -\left(\left(v - \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right)$$

$$v - v^2 = \left(\frac{1}{2} \right)^2 - \left(v - \frac{1}{2} \right)^2$$

$$\int_0^1 \left[\int_0^1 u^2 (1-u)^{1/2} du \right] v^{1/2} (1-v)^{1/2} dv = \frac{16}{105} \int_0^1 \sqrt{(v-v^2)} dv$$

$$\int_0^1 \left[\int_0^1 u^2 (1-u)^{1/2} du \right] v^{1/2} (1-v)^{1/2} dv = \frac{16}{105} \int_0^1 \sqrt{\left(\frac{1}{2} \right)^2 - \left(v - \frac{1}{2} \right)^2} dv$$

$$=\frac{16}{105}\left[\frac{1}{2}\left(v-\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2-\left(v-\frac{1}{2}\right)^2}+\frac{1/4}{2}\sin^{-1}\left(\frac{v-1/2}{1/2}\right)\right]_o^1$$

$$=\frac{16}{105}\left[\frac{1}{2}\left(1-\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2-\left(1-\frac{1}{2}\right)^2}+\frac{1/4}{2}\sin^{-1}\left(\frac{1-1/2}{1/2}\right)\right]$$
$$-\left[\frac{1}{2}\left(0-\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2-\left(0-\frac{1}{2}\right)^2}+\frac{1/4}{2}\sin^{-1}\left(\frac{0-1/2}{1/2}\right)\right]$$

$$=\frac{16}{105}\left(\begin{array}{l}\left[\frac{1-1/2}{2}\sqrt{\left(\frac{1}{2}\right)^2-\left(\frac{2-1}{2}\right)^2}+\frac{1}{8}\sin^{-1}\left(\frac{2-1/2}{1/2}\right)\right]\\ +\frac{1}{4}\sqrt{\left(\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)^2}-\frac{1}{8}\sin^{-1}(-1)+c\end{array}\right)$$

$$=\frac{16}{105}\left[0+\frac{1}{8}\sin^{-1}(1)+\frac{1}{4}(0)-\frac{1}{8}\frac{3\pi}{2}+c\right]$$

$$=\frac{16}{105}\left[\frac{1}{8}\left(\frac{\pi}{2}\right)-\frac{1}{8}\frac{3\pi}{2}+c\right]$$

$$=\frac{16}{105}\times\frac{1}{8}\left[\left(\frac{\pi}{2}\right)-\frac{3\pi}{2}+c\right]$$

$$=\frac{2}{105}\left[-\frac{2\pi}{2}\right]+c$$

$$=-\frac{2\pi}{105}+c$$

$$= \frac{2\pi}{105} + c$$

2. Evaluate $\iint_R (x-y)^4 e^{x+y} dxdy$ where R is the square with verticies (0,1) (2,1) (1,2) (0,1)

Solution:

The sides of the square for (1,0) and (2,1)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-1}{2-1}$$

$$\frac{y}{1} = \frac{x-1}{1}$$

$$y = x-1$$

$$x-y-1=0$$

$$x-y=1$$

(2,1) and (1,2)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-1}{2-1}=\frac{x-2}{1-2}$$

$$\frac{y-1}{1}=\frac{x-2}{-1}$$

$$x+y=3$$

(1,2) and (0,1)

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2}{1-2}=\frac{x-1}{0-1}$$

$$\frac{y-2}{-1}=\frac{x-1}{-1}$$

$$x-y=-1$$

(0,1) and (0,1)

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-1}{0-1}=\frac{x-0}{1-0}$$

$$\frac{y-1}{-1}=\frac{x}{1}$$

$$x+y=1$$

Taking $u = x + y$, $v = x - y$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$
$$= -1 - (1)$$
$$= -2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

u varies from 1 to 3

v varies from -1 to 1

we know that,

$$\iint_R F(x,y)\,dxdy = \iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}\,dudv$$

$$\iint_R (x-y)^4 e^{x+y}\,dxdy = \int_{-1}^{1}\int_{1}^{3} v^4 e^u \left(\frac{-1}{2}\right) dudv$$

$$=\frac{-1}{2}\int_{-1}^{1}\left(e^{u}\right)_{1}^{3}v^{4}dv$$

$$=\frac{-1}{2}\int_{-1}^{1}\left(e^{3}-e^{1}\right)v^{4}dv$$

$$=\frac{-1}{2}\left(e^{3}-e^{1}\right)\int_{-1}^{1}v^{4}dv$$

$$=\frac{-1}{2}\left(e^{3}-e^{1}\right)\left(\frac{v^{5}}{5}\right)_{-1}^{1}$$

$$=\frac{-1}{2}\left(e^{3}-e^{1}\right)\frac{1}{5}(1-(-1))$$

$$=\frac{-1}{2}\left(e^{3}-e^{1}\right)\frac{1}{5}(2)$$

$$=\frac{-\left(e^{3}-e^{1}\right)}{5}$$

$$\iint_R (x-y)^4 e^{x+y}dxdy=\frac{-\left(e^{3}-e\right)}{5}$$

$$\iint_R (x-y)^4 e^{x+y}dxdy=\frac{\left(e^{3}-e\right)}{5}$$

3. Evaluate $\iint_R xydxdy$ where R is the region in the first quadrant bounded by the hyperbola $x^2-y^2=a^2; x^2-y^2=b^2$ and the circles $x^2+y^2=c^2; x^2+y^2=d^2$

Solution:

Let $u = x^2 - y^2$, $v = x^2 + y^2$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}$$
$$= 4xy - (-4xy)$$
$$= 4xy + 4xy$$
$$= 8xy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 8xy$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{8xy}$$

u varies from a^2 to b^2

v varies from c^2 to d^2

we know that,

$$\iint_R F(x,y)dxdy = \iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv$$

$$\iint_R xydxdy = \int_{c^2}^{d^2}\int_{a^2}^{b^2} xy\left(\frac{1}{8xy}\right)dudv$$

$$= \int_{c^2}^{d^2}\int_{a^2}^{b^2}\left(\frac{1}{8}\right)dudv$$

$$= \frac{1}{8}\int_{c^2}^{d^2}\int_{a^2}^{b^2} dudv$$

$$= \frac{1}{8}\int_{c^2}^{d^2}(u)_{a^2}^{b^2}\, dv$$

$$= \frac{1}{8}\int_{c^2}^{d^2}\left(b^2 - a^2\right)dv$$

$$= \frac{b^2 - a^2}{8}\int_{c^2}^{d^2} dv$$

$$= \frac{b^2 - a^2}{8}(v)_{c^2}^{d^2}$$

$$= \frac{b^2 - a^2}{8}\left(d^2 - c^2\right)$$

$$\iint_R xydxdy = \frac{\left(b^2 - a^2\right)\left(d^2 - c^2\right)}{8}$$

4. Evaluate $\iint_R \left(x^2 + y^2\right)dxdy$ where R is the region in the first quadrant bounded by the hyperbola $x^2 - y^2 = a; x^2 - y^2 = b$ and the circles $2xy = c; 2xy = d$

Solution:

Let $u = x^2 - y^2$, $v = 2xy$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \\ &= 4x^2 - (-4y^2) \\ &= 4x^2 + 4y^2 \\ &= 4(x^2 + y^2) \end{aligned}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 4(x^2 + y^2)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4(x^2 + y^2)}$$

u varies from a to b

v varies from c to d

we know that,

$$\iint_R F(x,y)dxdy = \iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv$$

$$\iint_R (x^2+y^2)dxdy = \int_c^d \int_a^b 4(x^2+y^2)\left(\frac{1}{4(x^2+y^2)}\right)dudv$$

$$= \int_c^d \int_a^b \left(\frac{1}{4}\right)dudv$$

$$= \frac{1}{4}\int_c^d \int_a^b dudv$$

$$= \frac{1}{4}\int_c^d (u)_a^b dv$$

$$= \frac{1}{4}\int_c^d (b-a)dv$$

$$= \frac{b-a}{4}\int_c^d dv$$

$$= \frac{b-a}{4}(v)_c^d$$

$$= \frac{b-a}{4}(d-c)$$

$$\iint_R (x^2+y^2)dxdy = \frac{(b-a)(d-c)}{4}$$

5. Evaluate $\iint_R (x+y)^2\, dxdy$ where R is the region in the first quadrant bounded by the hyperbola $x+y=0; x+y=2$; $3x-2y=0; 3x-2y=3$

Solution:

Let $u = x + y$, $v = 3x - 2y$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$$
$$= -2 - 3$$
$$= -5)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -5$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-5}$$

u varies from 0 to 2

v varies from 0 to 3

we know that,

$$\iint_R F(x,y)\,dxdy = \iint F(u,v)\frac{\partial(x,y)}{\partial(u,v)}\,dudv$$

$$\iint_R (x+y)^2\,dxdy = \int_0^3\int_0^2 u^2\left(\frac{1}{-5}\right)dudv$$

$$= \frac{-1}{5}\int_0^3\left(\frac{u^3}{3}\right)_0^2 dv$$

$$= \frac{-1}{5}\int_0^3 \frac{8}{3}dv$$

$$= \frac{-8}{15}\int_0^3 dv$$

$$= \frac{-8}{15}(v)_0^3$$

$$= \frac{-8}{15}(3)$$

$$= \frac{-8}{5}$$

$$\iint_R (x+y)^2\,dxdy = \frac{8}{5}$$

TRIPLE INTEGRAL:

1. Evaluate $\iiint xyz\,dxdydz$ over the positive quadrant of the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical co-ordinates.

Solution:

Put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dxdydz = r^2\sin\theta dr d\theta d\phi$

Limits:

r Varies from 0 to a

θ Varies from 0 to $\frac{\pi}{2}$

ϕ Varies from 0 to $\frac{\pi}{2}$

$$\iiint xyz dxdydz = \int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r\sin\theta\cos\phi r\sin\theta\sin\phi r\cos\theta r^2\sin\theta dr d\theta d\phi$$

$$= \int_0^a\int_0^{\pi/2}\int_0^{\pi/2}\left(r^5\sin^3\theta\cos\phi\sin\phi\cos\theta\right) dr d\theta d\phi$$

$$= \int_0^a\int_0^{\pi/2} r^5\sin^3\theta\cos\theta\left(\frac{1}{2}\right) dr d\theta$$

$$= \frac{1}{2}\int_0^a\int_0^{\pi/2}\left(r^5\sin^2\theta\sin\theta\cos\theta\right) dr d\theta$$

$$=\frac{1}{2}\int_0^a r^5\left(\frac{1}{4}\right)dr$$

$$=\frac{1}{8}\left[\frac{r^6}{6}\right]_0^a$$

$$=\frac{1}{8}\left[\frac{a^6}{6}\right]$$

$$=\frac{a^6}{48}$$

$$\iiint xyzdxdydz=\frac{a^6}{48}$$

CHANGE OF ORDER OF INTEGRATION:

dydx= vertical strip

dxdy= Horizontal strip

PROBLEMS:

1. By changing the order of integration. Evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dxdy$

Solution:

Limits:

$y = x$ to $y = \infty$

$x = 0$ to $x = \infty$

New limits:

$x = 0$ to $x = y$

$y = 0$ to $y = \infty$

By changing the order of integration

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \int_0^y dxdy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} ydy$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= \left(\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1} \right)$$

$$= 1$$

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy = 1$$

2. Change of order of integration the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy$ and evaluate it.

Solution:

Given,

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy$$

Limits:

$y = \frac{x^2}{a}$ to $y = 2a - x$

$x^2 = ay$ to $x + y = 2a$

$x = 0$ to $x = a$

New limits:

Since the region is divide into two points.

I_1

$x = 0$ to $x = \sqrt{ay}$

$y = 0$ to $y = a$

I_2

$x = 0$ to $x = 2a - y$

$y = a$ to $y = 2a$

By changing the order of integration

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydxdy = \int_0^a \int_0^{\sqrt{ay}} xydxdy + \int_a^{2a} \int_0^{2a-y} xydxdy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy + \int_0^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{1}{2}\int_0^a y(ay)dy + \frac{1}{2}\int_0^{2a} y(2a-y)^2 dy$$

$$= \frac{a}{2}\int_0^a y^2 dy + \frac{1}{2}\int_0^{2a} y(4a^2 + y^2 - 4ay)dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a + \frac{1}{2}\left(\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3}\right)_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(2a^2(4a^2 - a^2) + \frac{16a^4 - a^4}{4} - \frac{4a}{3}(8a^3 - a^3)\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(2a^2(3a^2) + \frac{15a^4}{4} - \frac{4a}{3}(7a^3)\right)$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{72a^4 + 45a^4 - 112a^4}{12}\right)$$

$$= \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{4a^4 + 5a^4}{24}$$

$$= \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy = \frac{3a^4}{8}$$

Possible Questions

2 Mark questions

1. Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy$ and evaluate it.

2. If $x+y+z=u, y+z=uv, z=uvw$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

3. Show that $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.

4. Given the $x+y=u, y=uv$, change the variables to u, v in the integral $\int\int[xy(1-x-y)]^{1/2}\, dx\, dy$ takes over the area of the triangle with sides $x=0, y=0, x+y=1$ and evaluate it.

5. Change the order of integration in $\int_0^a \int_x^a (x^2+y^2)\, dy\, dx$ and evaluate it.

6. If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$, then find $J\left(\frac{y_1, y_2, y_3}{x_{1,} x_2, x_3}\right)$.

7. Evaluate $\int_0^3 \int_o^{\sqrt{4-y}} (x+y\,) dx\, dy$ by changing the order of integration.

8. Evaluate $_V\iiint xyz\, dxdy\, dz$ over the positive octant of the sphere $x^2+y^2+\ z^2=a^2$.

9. Evaluate $_R\iint (x-y)^4 e^{x+y} dx\, dy$ where R is the square with vertices (0,1),(2,1),(1,2) and (0,1).

10. If $u=\frac{y^2}{x}, \mathrm{v}=\frac{\mathrm{x}^2}{\mathrm{y}}$,find $\frac{\partial(u,v)}{\partial(x,y)}$.

11. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate it.

12. Find the value of $\iint xydx\, dy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

13. Calculate $\int_1^6 x^2 dx$ by trapezoidal rule and find the error by actual integration.

14. Calculate $\int_0^5 \sqrt{125-x^3}$. by Simpsons rule.

15. Evaluate $\int \frac{dx}{(x+1)\sqrt{1-x^2}}$.

6 Mark questions

1. If $u = xyz, v = xy + yz + zx$ and $w = x + y + z$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

2. Change the order of integration $\int_0^1 \int_x^1 \frac{x\,dx\,dy}{(x^2+y^2)}$.

3. If $x = \frac{u}{1+v^2}$ and $y = \frac{uv}{1+v^2}$ prove that $\frac{\partial(x,y)}{\partial(u,v)} . \frac{\partial(u,v)}{\partial(x,y)} = 1$.

4. Change the order of integration and evaluate $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$.

5. Compute $\int_0^6 \frac{dx}{1+x^2}$ by Simpon's rule, taking six intervals.

6. Evaluate $\iiint xyz\,dx\,dy\,dz$ take through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$ by transforming into spherical co-ordinates.

7. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^{\frac{x^2+y^2}{4}} dxdydz$.

8. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Mathematics-I **Subject Code: 18PHU304**

Class : II - B.Sc. Physics **Semester : III**

Unit IV

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
(∂(u,v,w))/(∂(x,y,z))×(∂(x,y,z))/(∂(u,v,w))=----------	0	1	2	3	1
(∂(u,v,))/(∂(r,s))×(∂(r,s))/(∂(x,y))=----------	r	1	0	-1	1
If u=x+y,v=x-y then (∂(u,v))/(∂(x,y))=----------	-2	1	0	-1	-2
If x=rcosθ,y=rsinθ, then (∂(x,y))/(∂(r,θ))=---------	1	r	1/r	-1	1/r
(∂(u,v,))/(∂(x,y)).(∂(x,y))/(∂(u,v))=----------	1	r	1/r	-1	1
If u and v are functions of x and y then (∂(u,v,))/(∂(x,y))×(∂(x,y))/(∂(u,v))=----------	0	1	2	3	1
In Polar coordinates,(∂(x,y))/(∂(r,θ))=----------	1	r	1/r	-1	r
if x=rsinθcosφ,y=rsinθsinφ,z=rcosθ then (∂(x,y,z))/(∂(r,φ,θ))=---------	-r(sinθ)^2	r(sinθ)^2	-r(cosθ)^2	r(cosθ)^2	-r(sinθ)^2
Change the order of integration ∬f(x,y)dxdy=--------	∫f(x,y)dxdy	∬f(x,y)dxdy	∬f(x,y)dydx	∬f(x,y)dxdy	∬f(x,y)dydx
∭dxdydz -------------------	x	xy	xyz	yz	xyz
∭xdxdydz -------------------	(x^2)yz	(x^2)/2 yx	(x^2)/2 x	(x^2)z	(x^2)/2 yx
If u=x-y,v=x+y then (∂(u,v))/(∂(x,y))=--------------	1	2	3	0	2
The positive quadrant of circle x^2+y^2=a^2 is --------------	0 to √(a^2-x^2)	-√(a^2-x^2) to √(a^2-x^2)	0 to √(a^2+x^2)	√(a^2-x^2) to 0	0 to √(a^2-x^2)
∬dydz -------------------	y	z	yz	0	yz

∭abcdxdydz -------------------	0	abc	xyz	abcxyz	abcxyz
In cartesian plane all (x,y) value positive in -------------quaderent	first	second	third	fourth	first
In cartesian plane x positive and y negative value positive in -------------quaderent	first	second	third	fourth	fourth
In cartesian plane x negative and y negative value positive in -------------quaderent	first	second	third	fourth	third
In cartesian plane x negative and y positive value positive in -------------quaderent	first	second	third	fourth	second
In polar coordinates is rdrdθ and hence the area of region R is------------	∬ drdθ	∬ rdθ	∬ rdr	∬ rdrdθ	∬ rdrdθ
The cordinates of the center of gravity and the moment inertia in the case of polar coordinates can be obtained by changing the corresponding formula in cartesian coordianates into-----------coordinates	cartesian	polar	sperical polar	circle	polar
The equation of the cardiod is ----------	r=acosθ	r=a(1+cosθ)	r=a(1-cosθ)	r=asinθ	r=a(1+cosθ)
∬dxdz -------------------	xy	xz	yz	x	xz
If x+y=u,y=uv then ∂(x,y)/∂(u,v)=-------------------	u	v	uv	xy	u
Change the order of integration ∭f(x,y)dxdydz=--------	∭f(x,y)dxdydz	∭f(x,y)dzdydx	∬f(x,y)dxdy	∬f(x,y)dydx	∭f(x,y)dzdydx

UNIT-V

Beta and Gamma integrals-their properties, relation between them- evaluation of multiple integrals using Beta and Gamma functions.

IMPROPER INTEGRAL

INFINITE INTEGRAL:

In $\int_a^b f(x)dx$ the integral $f(x)$ to be bounded and the integrable of integration from a to b. now we shall consider the cases where,

Case 1:

The integrant to $\infty \int_a^\infty f(x)dx$ let $f(x)$ be bounded and integrable in the interval (a,x) where a is fixed and x is any number>a.

We defined $\int_a^\infty f(x)dx$ to be $\lim_{x\to\infty}\int_a^x f(x)dx$ provided this limit exist. Then $\int_a^\infty f(x)dx$ has an infinite integral and say that it convergence.

If a to x $\int_a^x f(x)dx \to \infty as X \to \infty$ the infinite integral $\int_a^\infty f(x)dx$ is said to divergent to ∞ or does not exist.

Case 2:

The integrant to $-\infty \int_{-\infty}^b f(x)dx$ let $f(x)$ be bounded and integrable in the interval (x,b) where b is fixed and x is any number<b

We defined $\int_{-\infty}^b f(x)dx$ to be $\lim_{x\to-\infty}\int_x^b f(x)dx$ provided this limit exist. Then $\int_{-\infty}^b f(x)dx$ has an infinite integral and say that it convergence.

If $\int_{-\infty}^b f(x)dx$ may diverges to ∞ to $-\infty$ or oscillate finitely or infinitely.

Case 3:

Integrals from ∞ to $-\infty$. $\int_{-\infty}^{\infty} f(x)dx$ if the infinite integrals $\int_{-\infty}^{x} f(x)dx$ and $\int_{x}^{\infty} f(x)dx$ both convergence then $\int_{-\infty}^{\infty} f(x)dx$ is convergence and is equal to their sum.

Note:

The value of integral is independent of the x is used.

PROBLEMS:

1. Discuss the convergence of $\int_{1}^{\infty} \frac{dx}{x^2}$

Solution:

$$\int_{1}^{\infty} \frac{dx}{x^2} = \lim_{X\to\infty} \int_{0}^{X} \frac{dx}{x^2}$$

$$= \lim_{X\to\infty} \left(\frac{-1}{x}\right)_0^X$$

$$= \lim_{X\to\infty} \left(\frac{-1}{X} + 1\right)$$

$$= \frac{-1}{\infty} + 1$$

$$\int_{1}^{\infty} \frac{dx}{x^2} = 1$$

The given integral converges to 1.

2. Discuss the convergence of $\int_1^\infty e^{-x}dx$

Solution:

$$\int_1^\infty e^{-x}dx = \lim_{X\to\infty}\int_0^X e^{-x}dx$$

$$= \lim_{X\to\infty}\left(-e^{-x}\right)_0^X$$

$$= \lim_{X\to\infty}\left(-e^{-X} + e^{-0}\right)$$

$$= -e^{-\infty} + 1$$

$$= 0 + 1$$

$$\int_1^\infty e^{-x}dx = 1$$

The given integral converges to 1.

3. Discuss the convergence of $\int_1^\infty \frac{dx}{a^2+x^2}$

Solution:

$$\int_1^\infty \frac{dx}{a^2+x^2} = \lim_{X\to\infty}\int_0^X \frac{dx}{a^2+x^2}$$

$$= \lim_{X\to\infty}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^X$$

$$= \lim_{X\to\infty}\left(\frac{1}{a}\tan^{-1}\frac{X}{a} - \frac{1}{a}\tan^{-1}\frac{0}{a}\right)$$

$$= \lim_{X\to\infty}\left(\frac{1}{a}\tan^{-1}\frac{X}{a} - 0\right)$$

$$= \lim_{X\to\infty}\left(\frac{1}{a}\tan^{-1}\frac{X}{a}\right)$$

$$= \left(\frac{1}{a}\tan^{-1}\infty\right)$$

$$= \frac{1}{a}\frac{\pi}{2} - 0$$

$$= \frac{\pi}{2a}$$

$$\int_1^\infty \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}$$

The given integral converges to $\frac{\pi}{2a}$

4. Discuss the convergence of $\int_1^\infty \frac{dx}{\sqrt{x}}$

Solution:

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{X\to\infty}\int_1^X x^{-1/2}$$

$$= \lim_{X\to\infty}\left(\frac{x^{1/2}}{1/2}\right)_1^X$$

$$= \lim_{X\to\infty}\left(\frac{X^{1/2}}{1/2} - \frac{1^{1/2}}{1/2}\right)$$

$$= \lim_{X \to \infty}\left(2X^{1/2} - 2(1)\right)$$

$$= 2\sqrt{\infty} - 2$$

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \infty$$

The given integral does not exist or diverges to ∞.

5. If $a > 0, \int_a^\infty \sin x dx$

Solution:

$$\int_a^\infty \sin x dx = \lim_{X \to \infty} \int_a^X \sin x dx$$

$$= \lim_{X \to \infty}\left(-\cos x\right)_a^X$$

$$= \lim_{X \to \infty}\left(-\cos X + \cos a\right)$$

$$= \left(-\cos\infty + \cos a\right)$$

$\cos x$ is bounded and lies between ± 1

The integral oscillates finitely.

i.e, neither converges nor diverges.

6. $\int_{-\infty}^0 \frac{dx}{(1-3x)^2}$

Solution:

$$\int_{-\infty}^{0} \frac{dx}{(1-3x)^2} = \lim_{X \to -\infty} \int_{X}^{0} \frac{dx}{(1-3x)^2}$$

Put $t = 1 - 3x$

$$dt = -3dx$$

$$dx = \frac{-dt}{3}$$

$$\int_{-\infty}^{0} \frac{dx}{(1-3x)^2} = \lim_{X \to -\infty} \int_{X}^{0} \frac{\frac{-dt}{3}}{t^2}$$

$$= \lim_{X \to -\infty} \int_{X}^{0} \frac{-dt}{3t^2}$$

$$= \frac{-1}{3} \lim_{X \to -\infty} \int_{X}^{0} \frac{dt}{t^2}$$

$$= \frac{-1}{3} \lim_{X \to -\infty} \left(\frac{-1}{t} \right)_{X}^{0}$$

$$= \frac{1}{3} \lim_{X \to -\infty} \left(\frac{1}{1-3x} \right)_{X}^{0}$$

$$= \frac{1}{3} \lim_{X \to -\infty} \left(1 - \frac{1}{1-3X} \right)$$

$$= \frac{1}{3} \left(1 - \frac{1}{1-3(-\infty)} \right)$$

$$= \frac{1}{3}(1-0)$$

$$= \frac{1}{3}$$

The given integral converges to $\frac{1}{3}$.

7. $\int_{-\infty}^{0} \cosh x dx$

Solution:

$$\int_{-\infty}^{0} \cosh x dx = \lim_{X \to -\infty} \int_{X}^{0} \cosh x dx$$

$$= \lim_{X \to -\infty} (\sinh x)_X^0$$

$$= \lim_{X \to -\infty} \left[(\sinh(0) - \sinh(X)\right]$$

$$= \lim_{X \to -\infty} \left[0 - \frac{e^X - e^{-X}}{2}\right]$$

$$= \lim_{X \to -\infty} \left[0 - \frac{e^X - 1/e^X}{2}\right]$$

$$= \lim_{X \to -\infty} \left[\frac{e^X - 1/e^X}{2}\right]$$

$$= \frac{1}{2}\left(e^{-\infty} - 1/e^{-\infty}\right)$$

$$= \infty$$

The given integral does not exist or divergent to ∞.

8. $\int_{-\infty}^{0} x\sin x dx$

Solution:

$$\int_{-\infty}^{0} x\sin x dx = \lim_{X\to-\infty}\int_{X}^{0} x\sin x dx$$

Let $u = x$ $\quad dv = \sin x dx$

$du = dx$ $\quad v = -\cos x$

$$= (-x\cos x)_X^0 - \int_X^0 -\cos x dx$$

$$= (-x\cos x)_X^0 - (\sin x)_X^0$$

$$= \lim_{X\to-\infty}[-x\cos x + \sin x]_X^0$$

$$= \lim_{X\to-\infty}[-X\cos X + \sin X]$$

$$= [-\infty\cos\infty - \sin\infty]$$

$\cos x$ is bounded and lies between ± 1

The integral oscillates finitely.

i.e, neither converges nor diverges.

9. Show that converges of $\int_{-\infty}^{\infty}\frac{dx}{4+x^2}$

Solution:

$$\int_{-\infty}^{\infty}\frac{dx}{4+x^2}=\int_{-\infty}^{\infty}\frac{dx}{2^2+x^2}$$

$$\int_{-\infty}^{\infty}\frac{dx}{4+x^2}=\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-\infty}^{\infty}$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-\frac{-\pi}{2}\right]$$

$$=\frac{1}{2}[\pi]$$

$$=\frac{\pi}{2}$$

The given integral is converges to $\frac{\pi}{2}$.

10. Show that converges of $\int_{1}^{\infty}\log x dx$

Solution:

$$\int_{1}^{\infty}\log x dx=\lim_{x\to\infty}\int_{1}^{X}\log x dx$$

$$=\lim_{x\to\infty}(x\log x-x)_{1}^{X}$$

$$=\lim_{x\to\infty}(X\log X-X-\log 1+1)$$

$$=(\infty\log\infty-\infty-\log 1+1)$$

$$=\infty+1$$

$$=\infty$$

The given integral does not exist or divergent to ∞ .

11. Show that converges of $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

Solution:

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{x\to\infty}\int_0^X \frac{dx}{(1+x)\sqrt{x}}$$

Put $t^2 = x$

$$2tdt = dx$$

$$= \lim_{x\to\infty}\int_0^X \frac{2tdt}{(1+t^2)t}$$

$$= \lim_{x\to\infty}\int_0^X \frac{2dt}{(1+t^2)}$$

$$= 2\lim_{x\to\infty}\int_0^X \frac{dt}{(1+t^2)}$$

$$= 2\lim_{x\to\infty}\left[\tan^{-1}\left(\frac{t}{1}\right)\right]_0^X$$

$$= 2\lim_{x\to\infty}\left[\tan^{-1}\sqrt{x}\right]_0^X$$

$$= 2\lim_{x\to\infty}\left[\tan^{-1}\sqrt{x} - \tan^{-1}0\right]$$

$$= 2\left[\tan^{-1}\sqrt{\infty}\right]$$

$$= 2\left[\frac{\pi}{2}\right]$$

$= \pi$

The given integral converges to π.

12. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$f(x) = \frac{dx}{\sqrt{1-x^2}}$$

$$f(0) = 1$$

$$f(1) = \infty$$

By definition

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to 0} \int_1^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{\varepsilon \to 0} \left(\sin^{-1}(x)\right)_1^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left(\sin^{-1}(1-\varepsilon) - \sin^{-1}(1)\right)$$

$$= \lim_{\varepsilon \to 0} \left(\sin^{-1}(1-\varepsilon) - \frac{\pi}{2}\right)$$

$$= \left(\sin^{-1}(1-0) - \frac{\pi}{2}\right)$$

$$= \left(\sin^{-1}(1) - \frac{\pi}{2}\right)$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} \right)$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = 0$$

13. Evaluate $\int_{-1}^{1} \frac{dx}{x}$

Solution:

$$\int_{-1}^{1} \frac{dx}{x}$$

$$f(x) = \frac{dx}{x}$$

$$f(0) = \infty$$

$$f(-1) = 1$$

$$f(1) = 1$$

By definition

$$\int_{-1}^{1} \frac{dx}{x} = \lim_{\varepsilon \to 0} \left[\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} \right]$$

$$= \lim_{\varepsilon \to 0} \left[(\log x)_{-1}^{-\varepsilon} + (\log x)_{\varepsilon}^{1} \right]$$

$$= \lim_{\varepsilon \to 0} \left[\log(-\varepsilon) - \log(-1) + \log x(1) - \log(\varepsilon) \right]$$

$$= \lim_{\varepsilon \to 0} \left[\log(0) - 0 + 0 - \log(0) \right]$$

$$\int_0^1 \frac{dx}{x} = 0$$

14. Evaluate $\int_{-1}^{1} \frac{dx}{x^{1/2}}$

Solution:

$$\int_{-1}^{1} \frac{dx}{x^{1/2}}$$

$$f(x) = \frac{1}{x^{1/2}}$$

$$f(0) = \infty$$

$$f(-1) = 1$$

$$f(1) = 1$$

By definition

$$\int_{-1}^{1} \frac{dx}{x} = \lim_{\varepsilon \to 0}\left[\int_{-1}^{-\varepsilon} \frac{dx}{x^{1/2}} + \int_{\varepsilon}^{1} \frac{dx}{x^{1/2}}\right]$$

$$= \lim_{\varepsilon \to 0}\left[(2\sqrt{x})_{-1}^{-\varepsilon} + (2\sqrt{x})_{\varepsilon}^{1}\right]$$

$$= \lim_{\varepsilon \to 0}[(2\sqrt{-\varepsilon} - (2\sqrt{-1})(2\sqrt{1} - (2\sqrt{\varepsilon})$$

$$= [2 + 2]$$

$$= 4$$

$$\int_{-1}^{1} \frac{dx}{x^{1/2}} = 4$$

BETA AND GAMMA FUNCTION:

The bita function denoted $\beta(m,n)$ which is given by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m,n > 0$$

The gamma function is denoted by Γn which is given by

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx, n > 0$$

Convergence of Γn

We know that $\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx$

$$= \int_0^1 x^{n-1}e^{-x}dx + = \int_1^\infty x^{n-1}e^{-x}dx$$

The first integral is

$\int_0^1 x^{n-1}e^{-x}dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{n-1}e^{-x}dx$ if this limit exist when x is small. The integral x^{n-1} and the limit exist, if n>0.

The second integral certainly exist for $e^x > \frac{x^r}{r!} > \frac{x^{n+1}}{r!}$ also $r > n+1$. Hence $x^{n-1}e^{-1}$ is $< \frac{r!}{x^2}$

$\int_1^\infty x^{n-1}e^{-x}dx$ is does not exist constant multiple of the 2nd integral which diverges.

Γn converges for n>0.

RECURRENCE FORMULA FOR GAMMA FUNCTION:

We have to prove that $\Gamma n+1 = n\Gamma n, n > 0$

Proof:

We know that

$$\Gamma n = \int_1^\infty x^{n-1} e^{-x} dx$$

$$\Gamma n+1 = \int_1^\infty x^n e^{-x} dx$$

$u = x^n \qquad dv = e^{-x} dx$

$u = nx^{n-1} dx \qquad v = -e^{-x}$

$$= (x^n(-e^{-x}))_0^\infty - \int_1^\infty -e^{-x} nx^{n-1} dx$$

$$= (-x^n e^{-x})_0^\infty - \int_1^\infty -e^{-x} nx^{n-1} dx$$

$$= 0 - \int_1^\infty -e^{-x} nx^{n-1} dx$$

$$= -\int_1^\infty -e^{-x} nx^{n-1} dx$$

$$= n\int_1^\infty e^{-x} x^{n-1} dx$$

$$= n\Gamma n$$

$$\Gamma n+1 = n\Gamma n$$

1. $\int_1^\infty \frac{dx}{x^3}$

Solution:

$$\int_1^\infty \frac{dx}{x^3} = \lim_{X\to\infty}\int_0^X x^{-3}$$

$$= \lim_{X\to\infty}\left(\frac{x^{-2}}{-2}\right)_0^X$$

$$= \frac{-1}{2}\lim_{X\to\infty}\left(\frac{1}{x^2}\right)_0^X$$

$$= \frac{-1}{2}\lim_{X\to\infty}\left(\frac{1}{X^2}-\frac{1}{1}\right)$$

$$= \frac{-1}{2}\lim_{X\to\infty}\left(\frac{1}{X^2}-1\right)$$

$$= \frac{-1}{2}\left(\frac{1}{\infty}+1\right)$$

$$= \frac{-1}{2}(0-1)$$

$$= \frac{-1}{2}(-1)$$

$$= \frac{1}{2}$$

$$\int_1^\infty \frac{dx}{x^3} = \frac{1}{2}$$

The given integral converges to $\frac{1}{2}$.

2. $\int_0^\infty \frac{dx}{x^{1/2}}$

Solution:

$$\int_a^\infty \frac{dx}{x^{1/2}} = \lim_{X\to\infty}\int_a^X x^{-1/2}$$

$$= \lim_{X\to\infty}\left(\frac{x^{1/2}}{1/2}\right)_1^X$$

$$= \lim_{X\to\infty}\left(2X^{1/2}\right)_a^X$$

$$= \lim_{X\to\infty}\left(2X^{1/2} - 2(a)\right)$$

$$= 2\sqrt{\infty} - 2\sqrt{a}$$

The given integral is divergent.

3. $\int_{-\infty}^0 e^{2x}dx$

Solution:

$$\int_{-\infty}^0 e^{2x}dx = \lim_{X\to\infty}\int_X^0 e^{2x}dx$$

$$= \lim_{X\to\infty}\left(\frac{e^{2x}}{2}\right)_X^0$$

$$= \frac{1}{2} \lim_{X\to\infty}\left(e^{2(0)} + e^{2X}\right)$$

$$= \frac{1}{2} \lim_{X\to\infty}\left(e^{0} + e^{2X}\right)$$

$$= \frac{1}{2}\left(1 + e^{2(-\infty)}\right)$$

$$= \frac{1}{2}\left(1 + e^{\infty}\right)$$

$$= \frac{1}{2}(1)$$

$$\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}$$

The given integral does not exist or diverges to $-\infty$.

4. $\displaystyle\int_{a}^{\infty} \frac{dx}{b^2 + x^2}$

Solution:

$$\int_{a}^{\infty} \frac{dx}{b^2 + x^2} = \lim_{X\to\infty} \int_{a}^{X} \frac{dx}{b^2 + x^2}$$

$$= \lim_{X\to\infty}\left(\frac{1}{b}\tan^{-1}\frac{x}{b}\right)_a^X$$

$$= \lim_{X\to\infty}\left(\frac{1}{b}\tan^{-1}\frac{X}{b} - \frac{1}{b}\tan^{-1}\frac{a}{b}\right)$$

$$= \frac{1}{b}\lim_{X\to\infty}\left(\tan^{-1}\frac{X}{b} - \tan^{-1}\frac{a}{b}\right)$$

$$= \frac{1}{b}\left(\frac{\pi}{2} - \tan^{-1}\frac{a}{b}\right)$$

The given integral converges to $\frac{1}{b}\left(\frac{\pi}{2} - \tan^{-1}\frac{a}{b}\right)$

PROPERTIES OF BETA FUNCTION:

(i) $\beta(m,n) = \beta(n,m)$

Proof:

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = 1-y \quad y = 1-x$

$dx = -dy$

When $x = 0$, $y = 1$

$x = 1, y = 0$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_1^0 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

$\beta(m,n)=\beta(n,m)$

Hence the proof

(ii) $\beta(m,n)$ can be expressed as a definite integral with $(0,\infty)$ as limit.

Proof:

we know that,

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x=\dfrac{y}{1+y}$

$$dx=\frac{(1+y)dy-ydy}{(1+y)^2}$$

$$dx=\frac{(1+y-y)dy}{(1+y)^2}$$

$$dx=\frac{1dy}{(1+y)^2}$$

When $x=0, y=0$

$x=1, y=\infty$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=\int_0^\infty\left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\,dy$$

$$=\int_0^\infty\left(\frac{y}{1+y}\right)^{m-1}\left(\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\,dy$$

$$=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}}\,dy$$

$$=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$\beta(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

(iii) $\beta(m,n)=2\int_0^{\pi/0} \sin^{2m-1} x\cos^{2n-1} x\,dx$

Proof:

We know that

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x=\sin^2\theta$

$dx=2\sin\theta\cos\theta d\theta$

When $x=0,\theta=0$

$x=1,\theta=\pi/2$

$$\beta(m,n)=\int_0^{\pi/2}\left(\sin^2\theta\right)^{m-1}\left(1-\sin^2\theta\right)^{n-1}2\sin\theta\cos\theta d\theta$$

$$=2\int_0^{\pi/2}\sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \cos^{2n-2+1}\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x dx.................(1)$$

We know that

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$$

$$m = 2m-1$$

$$n = 2n-1$$

$$I_{2m-1,2n-1} = \int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x dx$$

(1) Becomes $\beta(m,n) = 2I_{2m-1,2n-1}$

$$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = 2I_{m,n}$$

$$I_{m,n} = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

COROLLARY:

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

Proof:

$$\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma m+n}$$

Put $m=n=\frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}}$$

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\pi/2}\sin^{2\left(\frac{1}{2}\right)-1}\theta\cos^{2\left(\frac{1}{2}\right)-1}\theta d\theta$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=2\int_0^{\pi/2}d\theta$$

$$=2(\theta)_0^{\pi/2}$$

$$=2\left(\pi/2-0\right)$$

$$=2\left(\pi/2\right)$$

$$=\pi$$

Eqn (1) becomes,

$$\beta\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\left(\Gamma\frac{1}{2}\right)^2}{\sqrt{1}}$$

$$\pi = \left(\Gamma\frac{1}{2}\right)^2$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

Hence proved

PROBLEMS:

1. $\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx$

Solution:

Put $\log\left(\frac{1}{x}\right) = t$

$$\frac{1}{x} = e^t$$

$$x = e^{-t}$$

$$dx = -e^{-t}dt$$

When $x = 0, t = \infty$

$$x = 1, t = 0$$

$$\int_0^1 x^m \log\left(\frac{1}{x}\right)^n dx = \int_\infty^0 e^{-tm} t^n (-e^{-t})dt$$

$$= -\int_0^\infty e^{-tm} t^n (-e^{-t})dt$$

$$= \int_0^\infty e^{-t(m+1)} t^n \, dt$$

Put $y = t(m+1) \Rightarrow t = \frac{y}{m+1}$

$$dy = (m+1)dt$$

$$\frac{dy}{m+1} = dt$$

$$\int_0^\infty e^{-t(m+1)} t^n \, dt = \int_0^\infty e^{-y} \left(\frac{y}{m+1} \right)^n \frac{dy}{m+1}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n \, dy$$

$$= \frac{1}{(m+1)^{n+1}} \Gamma n+1$$

2. $\int_0^\infty e^{-x^2} dx$

Solution:

$$t = x^2$$

$$dt = 2x dx$$

$$\frac{dt}{2x} = dx$$

$$\frac{dt}{2\sqrt{t}} = dx$$

When $x = 0, t = 0$

$x=\infty, t=\infty$

$$\int_0^\infty e^{-x^2}\,dx=\int_0^\infty e^{-t}\,\frac{dt}{2\sqrt{t}}$$

$$=\frac{1}{2}\int_0^\infty e^{-t}t^{-1/2}\,dt$$

$$=\frac{1}{2}\Gamma\frac{1}{2}$$

$$=\frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

3. $\int_0^1 x^7(1-x)^8\,dx$

Solution:

We know that,

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$m-1=7, n-1=8$

$m=8, n=9$

$$\int_0^1 x^7(1-x)^8\,dx=\beta(8,9)$$

$$=\frac{\Gamma 8\Gamma 9}{\Gamma 8+9}$$

$$= \frac{\Gamma 8 \Gamma 9}{\Gamma 17}$$

$$= \frac{7!8!}{16!}$$

$$\int_0^1 x^7 (1-x)^8 \, dx = \frac{1}{102960}$$

4. Evaluate $\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta d\theta$

Solution:

We know that,

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Here m=7 and n=5

$$\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta d\theta = \frac{1}{2} \beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{8}{2}, \frac{6}{2}\right)$$

$$= \frac{1}{2} \beta(4,3)$$

$$= \frac{1}{2} \frac{\Gamma 4 \Gamma 3}{\Gamma 4+3}$$

$$= \frac{1}{2} \frac{\Gamma 4 \Gamma 3}{\Gamma 7}$$

$$=\frac{1}{2}\frac{3!2!}{6!}$$

$$=\frac{1}{120}$$

$$\int_0^{\pi/2}\sin^7\theta\cos^5\theta d\theta=\frac{1}{120}$$

5. $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Solution:

$$\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\int_0^{\pi/2}(\tan\theta)^{1/2}d\theta$$

$$=\int_0^{\pi/2}\left(\frac{\sin\theta}{\cos\theta}\right)^{1/2}d\theta$$

$$=\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta d\theta$$

$$m=\frac{1}{2},n=\frac{-1}{2}$$

We know that,

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta=\frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

$$\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{\frac{-1}{2}}\theta d\theta=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$$

$$\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta d\theta = \frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma\frac{3+1}{4}}$$

$$\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta d\theta = \frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma\frac{4}{4}}$$

$$\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta d\theta = \frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{1}$$

$$\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta d\theta = \frac{1}{2}\Gamma\frac{3}{4}\Gamma\frac{1}{4}$$

APPLICATION OF GAMMA FUNCTION TO MULTIPLE INTEGRAL:

1. Evaluate $\iiint x^p y^q z^r \, dzdydx$ then taken over the volume of the tetrahedron given by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$

Solution:

Limits

x varies from 0 to 1

y varies from 0 to 1-x

z varies from 0 to 1-x-y

$$\iiint x^p y^q z^r \, dzdydx = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^p y^q z^r \, dzdydx$$

$$= \int_0^1 \int_0^{1-x} x^p y^q \left(\frac{z^{r+1}}{r+1}\right)_0^{1-x-y} dydx$$

$$= \frac{1}{r+1}\int_0^1\int_0^{1-x} x^p y^q (1-x-y)^{r+1}\,dydx$$

$$\iiint x^p y^q z^r\,dzdydx = \frac{1}{r+1}\int_0^1\int_0^{1-x} x^p y^q (1-x-y)^{r+1}\,dydx$$

The area over which the integrant to be a triangle OAB

Let $x + y = u$...............(1)

$y = uv$...............(2)

Solve (1) & (2)

$x + uv = u$

$x = u(1-v)$

When x=0

u=0,v=1

When y=0

u=0 ,v=0

When x+y=1

u=1,v=0

Limits:

U varies from 0 to 1

V varies from 0 to 1

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - uv + uv$$

$$= u$$

$$\frac{1}{r+1}\int_0^1\int_0^{1-x} x^p y^q (1-x-y)^{r+1}\,dydx = \frac{1}{r+1}\int_0^1\int_0^1 (u(1-v))^p (uv)^q (1-u)^{r+1}\,ududv$$

$$= \frac{1}{r+1}\int_0^1\int_0^1 u^p (1-v)^p u^q v^q (1-u)^{r+1}\,ududv$$

$$= \frac{1}{r+1}\left[\int_0^1 u^{p+q+1}(1-u)^{r+1}\,du\int_0^1 v^q (1-v)^p\,dv\right]$$

Now,

$$\int_0^1 u^{p+q+1}(1-u)^{r+1}\,du$$

Here m-1=p+q+1

m=p+q+2

n-1=r+1

n=r+2

$$\int_0^1 u^{p+q+1}(1-u)^{r+1}\,du = \beta(p+q+2, r+2)$$

$$= \frac{\Gamma p+q+2\Gamma r+2}{\Gamma p+q+2+r+2}$$

$$= \frac{\Gamma p+q+2 \times r+1 \times \Gamma r+1}{\Gamma p+q+r+4}$$

Consider,

$$\int_0^1 v^q (1-v)^p\,dv$$

Here m-1=q

m=q+1

n-1=p

n=p+1

$$\int_0^1 v^q (1-v)^p\,dv = \beta(q+1, p+1)$$

$$= \frac{\Gamma q+1\Gamma p+1}{\Gamma p+q+1+1}$$

$$= \frac{\Gamma q+1\Gamma p+1}{\Gamma p+q+2}$$

$$\frac{1}{r+1}\int_0^1\int_0^{1-x} x^p y^q (1-x-y)^{r+1}\,dydx = \frac{1}{r+1}\left[\frac{\Gamma p+q+2 \times r+1 \times \Gamma r+1}{\Gamma p+q+r+4} \times \frac{\Gamma q+1\Gamma p+1}{\Gamma p+q+2}\right]$$

$$\frac{1}{r+1}\int_0^1\int_0^{1-x} x^p y^q (1-x-y)^{r+1}\,dydx = \frac{\Gamma p+1\Gamma q+1\Gamma r+1}{\Gamma p+q+4}$$

Possible Questions

2 Mark questions

1. Discuss the convergence of the following :

   (i) $\int_a^\infty \frac{dx}{x^{1/2}}, a > 0$

   (ii) $\int_{-\infty}^0 \frac{dx}{(1-3x)^2}$.

2. Prove that $\sqrt{(\frac{1}{2})} = \sqrt{\pi}$.

3. Evaluate : $\int_0^{\frac{\pi}{2}} sin^{10}\theta \, d\theta$.

4. Show that $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}x cos^{2n-1}x dx$.

5. Prove that $\beta(m,n) = \beta(n,m)$.

6. Prove that $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx * \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x}} = \pi$.

7. Evaluate $\int \frac{e^{-5x}}{\sqrt{x}} dx$ using Gamma function.

8. Discuss the convergence of $\int_{-\infty}^{\infty} \frac{dx}{4+x^2}$.

9. Prove that $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos\theta}} = \pi$.

10. Prove that $\sqrt{\frac{p}{2}}\sqrt{\frac{p+1}{2}} = \frac{\sqrt{\pi}}{2^{p-1}}\sqrt{p}$

11. Discuss the convergence of $\int_a^\infty \frac{x}{(1+x)^3} dx$.

12. Prove that $\sqrt{n} = (n-1)\sqrt{(n-1)}$ where n>1 and $\sqrt{n} = (n-1)!$ where n is a positive integer.

13. If $f(m,n) = \int_0^{\frac{\pi}{2}} cos^m x$ cosnx dx, prove that $f(m,n) = \frac{m}{m+n}$f(m − 1, n − 1).

14. Prove that $\int_2^3 \sqrt{(x-2)(3-x)}\,dx = \frac{\pi}{8}$.

15. Evaluate (i) $\int \sqrt{(x-3)(7-x)}dx$. (ii) $\int_1^\infty \frac{dx}{x^2}$

6 Mark questions

1. Prove that (i) $\sqrt{n+1} = n!, n > 0$ (ii) Evaluate $\int_0^\infty e^{-x^2}\, dx$.

2. Prove that $\beta(m,n) = \frac{\sqrt{m}\sqrt{n}}{\sqrt{(m+n)}}$.

3. Show that: $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$.

4. Show that: $\int_0^\infty \frac{t^2dt}{1+t^4} = \frac{\pi}{2\sqrt{2}}$.

5. Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$.

6. Prove that $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4}$.

7. Prove that $\int_0^{\frac{\pi}{2}} \frac{cos^{2m-1}\theta sin^{2n-1}\theta\, d\theta}{(acos^2\theta+bsin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$.

8. Show that $\int_0^\infty \frac{x^2}{(1+x^4)^3}\, dx = \frac{5\pi\sqrt{2}}{128}$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Subject: Mathematics-I **Subject Code: 18PHU304**

Class : I - B.Sc. Physics **Semester : III**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
Γ(n+1)= -------------------	n!	n+1	n-1	n	n!
Γ(1/2)=-------------------	π	1	0	√π	√π
β(m,n)=-------------------	Γ(m)Γ(n)/Γ(m-n)	Γ(m)Γ(n)/Γ(m+n)	(Γ(m)-Γ(n))/Γ(m-n)	Γ(m+n)/Γ(m-n)	Γ(m)Γ(n)/Γ(m+n)
β(1/2,1/2)=------------------	0	1	π	-1	π
Γ(n+1)= -----------------	nΓn	(n+1)Γn	Γn	n	nΓn
Γ(n) converges for------------------	n=0	n>1	n>0	n<1	n>0
Γ(n+1)=------------------	integral 0 to infinity (x^n) e^(-x) dx,n>-1	integral 0 to infinity (x^n e^(-x) dx,n>1	integral 0 to infinity (x^n) e^(-x) dx,n=-1	integral 0 to infinity (x^n)e^(-x) dx,n<-1	integral 0 to infinity (x^n) e^(-x) dx,n>-1
∫e^(-x) dx with the limit of 0 to infinity then =-------	√π	√(π/2)	√2π	0	√π
β(m,n) exist if -----------------	m>0,n>0	m>0,n<0	m<0,n<0	m=0,n<0	m>0,n>0
Γ(m+n)β(m,n)=-------------------	Γ(m)-Γ(n)	Γ(m)Γ(n)	Γ(m)+Γ(n)	Γ(m)/Γ(n)	Γ(m)Γ(n)
β(1,1)=-------------------	0	1	2	-1	1
Γ(n)=∫e^(-x) dx with the limit of 0 to infinity then n possible value is =------------------	n=0	n>0	n<0	n≠0	n>0
The value of Γ(1) is ------------------	1	0	2	-1	1

The value of ∫e^(-x^2) dx with the limit of 0 to infinity is---------------	π	π/2	π^2	√π/2	π/2
β(m,n)=------------------	β(n,n)	β(m)	β(n)	β(n,m)	β(n,m)
Γ(n+1)=∫(x^n)(e^(-x))dx with the limit of 0 to infinity then n possible value is =-------------	n=0	n>-1	n<0	n≠0	n>-1
Γ(1/4)Γ(3/4) equal to -----------------	2	π√2	π	0	π√2
Γ(n+2/3) is equal to -----------------	(n-1/3)Γ(n-1/3)	(n+1/3)Γ(n-1/3)	(n-1/3)Γ(n+1/3)	(n-1/3)Γ(1/3)	(n-1/3)Γ(n-1/3)
Γ(n)Γ(n-1)= -----------------	β(n,1-n)	β(n,n)	β(n-1,n)	β(n-1,n-1)	β(n,1-n)
Γ(1)= -----------------	1	2	0	-1	1
Γ(1/2)Γ(1/2)= -----------------	β(1/2,1/2)	β(1,1/2)	β(1/2,1)	β(1,1)	β(1/2,1/2)
Γ(n+1)=nΓnthe recurrence formula is true only when -----	n=0	n>0	n<0	n≠0	n>0
Γ(n)Γ(n-1)=-----------------	π/sinnπ	π/cosnπ	2π/sinnπ	3π/sinnπ	π/sinnπ
Γ(3/2)= -----------------	1	π√2	π	0	1