KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University)
(Established Under Section 3 of UGC Act 1956)
COIMBATORE-21
(For the candidates admitted from 2019 onwards)

DEPARTMENT OF PHYSICS

SUBJECT: MATHEMATICAL PHYSICS
SEMESTER: I
SUB.CODE:19PHP104 **CLASS: I M.Sc PHYSICS**

Course Objectives

- It is necessary for a physics student to be familiar with different methods in mathematics.
- Give a basic idea about different methods of mathematics, used in Physics.

Course Outcomes (COs)

1. Students will be able to apply integral transform (Fourier and Laplace) to solve mathematical problems of interest in physics, use Fourier transforms as an aid for analyzing experimental data.
2. Students can formulate and express a physical law in terms of tensors, and simplify it by use of coordinate transforms (example: principal axes of inertia).
3. Students will be able to solve some simple classical variation problems.

UNIT I - VECTOR SPACE

Definition of vector space – Linear dependence – Linear independence – Basis – Dimension of a vector space – Representation of Vectors and linear operators with respect to basis – Schmidt orthogonalization process – Inner product.

Tensors : Transformation of coordinates – Summation convention – Contravariant Tensor – Covariant Tensor – Mixed Tensor – Rank of a Tensor – Kronecker delta symbol – symmetric and antisymmetric tensors – Invariant tensors.

UNIT II- COMPLEX VARIABLE

Functions of a complex variable – single and multivalued functions – Cauchy-Riemann differential equation – analytical – line integrals of complex function – Cauchy's integral theorem and integral formula – derivatives of an analytic function – Liouville's theorem - Taylor's series – Laurent's series - Residues and their evaluation - Cauchy's residue theorem – application to the evaluation of definite integrals.

UNIT III- FOURIER TRANSFORM

Properties of Fourier transform – Fourier transform of derivatives – Fourier sine and cosine transforms of derivatives – Fourier transform of functions of two or three variables – Finite Fourier transforms – Simple Applications of FT

Laplace transform – Properties of Laplace transforms – Laplace Transform of derivative of a function – Laplace transform of integral – Laplace transform of periodic functions - Inverse Laplace Transform – Fourier Mellin Theorem - Properties of inverse Laplace Transform – Convolution theorem – Evaluation of Laplace Transform using Convolution theorem.

UNIT IV- FOURIER SERIES

Dirichlet's theorem – change of interval – complex form – Fourier series in the interval (0, T) – Uses of Fourier series - Legendre's polynomials and functions – Differential equations and solutions – Rodrigues formula – Orthogonality – relation between Legendre polynomial and their derivatives – recurrence relations – Lagurae Polynomials – recurrence relations

UNIT V- BESSEL'S FUNCTIONS

Differential equation and solution – generating functions – recurrence relations – Bessel function of second order – Spherical Bessel function -

Hermite differential equation and Hermite polynomials: Generating function of Hermite polynomials – Recurrence formulae for Hermite polynomials – Rodrigue's formula for Hermite Polynomials – Orthogonality of Hermite Polynomials – Dirac's Delta Function.

SUGGESTED READINGS

1. Satya Prakash.,2002. Mathematical Physics , 4th edition, S.Chand & Co, New Delhi.
2. Gupta.B.D., 2002, .Mathematical Physics, 2nd edition, Vikas publishing company, New Delhi.
3. Singaravelu.V., 2008. Numerical methods, 2nd edition, Meenakshi publications, Sirkali.
4. Rajput.B.S., 2003. Mathematical Physics, 16th edition, Pragati Prakashan, Meerut.
5. Gupta. P.P., Yadav., and Malik.,2012. Mathematical Physics, Kedar Nath & Ram Nath, Meerut.
6. Venkataraman.M.K., 2003. Numerical methods in Science & Engineering, 5th edition, The National Publishing Company, Chennai.
7. Butkov, 2007, Mathematical Physics, Addison Wesley, New York
8. A.W. Joshi, 2008, Tensors and Matrices, reprint, Wiley Interscience, New York.

UNIT-I

SYLLABUS

Vector Space - Definition of vector space – Linear dependence – Linear independence – Basis – Dimension of a vector space – Representation of Vectors and linear operators with respect to basis – Schmidt orthogonalization process – Inner product. Tensors : Transformation of coordinates – Summation convention – Contravariant Tensor – Covariant Tensor – Mixed Tensor – Rank of a Tensor – Kronecker delta symbol – symmetric and antisymmetric tensors – Invariant tensors.

Definition of Vector Space

A vector space V is a set that is closed under finite vector addition and scalar multiplication. The basic example is n-dimensional Euclidean space $\mathbb{R}^n$, where every element is represented by a list of n real numbers, scalars are real numbers, addition is component wise, and scalar multiplication is multiplication on each term separately.

For a general vector space, the scalars are members of a field F, in which case V is called a vector space over F.

Euclidean n-space $\mathbb{R}^n$ is called a real vector space, and $\mathbb{C}^n$ is called a complex vector space.

In order for V to be a vector space, the following conditions must hold for all elements $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in V$ and any scalars $r, s \in F$:

1. Commutativity:

$$\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}. \quad (1)$$

2. Associativity of vector addition:

$$(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z}). \quad (2)$$

3. Additive identity: For all $\mathbf{X}$,

$$\mathbf{0} + \mathbf{X} = \mathbf{X} + \mathbf{0} = \mathbf{X}. \quad (3)$$

4. Existence of additive inverse: For any $\mathbf{X}$, there exists a $-\mathbf{X}$ such that

$$\mathbf{X} + (-\mathbf{X}) = \mathbf{0}. \quad (4)$$

5. Associativity of scalar multiplication:

$$r(s\,\mathbf{X}) = (r\,s)\,\mathbf{X}. \quad (5)$$

6. Distributivity of scalar sums:

$$(r + s)\,\mathbf{X} = r\,\mathbf{X} + s\,\mathbf{X}. \quad (6)$$

7. Distributivity of vector sums:

$$r(\mathbf{X} + \mathbf{Y}) = r\,\mathbf{X} + r\,\mathbf{Y}. \quad (7)$$

8. Scalar multiplication identity:

$$1\,\mathbf{X} = \mathbf{X}.$$

Linear Independence and dependence

Let $S=\{v_1,v_2,\ldots,v_k\}$ and $span(S)=W$. Is it possible to find a *smaller (or even smallest)* set, for example, $S^*=\{v_1,v_2,\ldots,v_{k-1}\}$, such that

$$\mathrm{span}(S)=W=\mathrm{span}(S^*)$$

To answer this question, we need to introduce the concept of *linear independence and linear dependence.*

Definition of linear dependence and linear independence:

The vectors $v_1,v_2,\ldots,v_k$ in a vector space V are said to linearly dependent if there exist constants, $c_1,c_2,\ldots,c_k$, not all 0, such that

$$c_1v_1+c_2v_2+\cdots+c_kv_k=0.$$

$v_1,v_2,\ldots,v_k$ are linearly independent if

$$c_1v_1+c_2v_2+\cdots+c_kv_k=0 \Rightarrow c_1=c_2=\cdots=c_k=0.$$

The procedure to determine if $v_1,v_2,\ldots,v_k$ are linearly dependent or linearly independent:

1. Form equation $c_1v_1+c_2v_2+\cdots+c_kv_k=0$, which lead to a homogeneous system.
2. If the homogeneous system has only the trivial solution, then the given vectors are linearly independent; if it has a nontrivial solution, then the vectors are linearly dependent.

Example:

$e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$, and $S=\{e_1,e_2,e_3\}$. Are e_1,e_2 and e_3 linearly independent?

[solution:]

$$c_1e_1+c_2e_2+c_3e_3=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=0$$

$$\Rightarrow\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Therefore, e_1, e_2 *and* e_3 *are linearly independent.*

Example:

$v_1=\begin{bmatrix}1\\2\\3\end{bmatrix},\ v_2=\begin{bmatrix}-2\\1\\1\end{bmatrix},\ v_3=\begin{bmatrix}8\\6\\10\end{bmatrix}$.. Are v_1, v_2 and v_3 linearly independent?

[Solution:]

$$c_1v_1+c_2v_2+c_3v_3=c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}-2\\1\\1\end{bmatrix}+c_3\begin{bmatrix}8\\6\\10\end{bmatrix}=\begin{bmatrix}1&-2&8\\2&1&6\\3&1&10\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=0$$

$$\Rightarrow\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=t\begin{bmatrix}4\\-2\\-1\end{bmatrix},t\in R.$$

Therefore, v_1, v_2 *and* v_3 *are linearly dependent.*

Example:

Determine whether the following set of vectors in the vector space consisting of all 2×2 matrices is linearly independent or linearly dependent.

$$S=\{v_1,v_2,v_3\}=\left\{\begin{bmatrix}2&1\\0&1\end{bmatrix},\begin{bmatrix}3&0\\2&1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix}\right\}.$$

[solution:]

$$c_1v_1 + c_2v_2 + c_3v_3 = c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned} \quad \Leftrightarrow \quad c_1\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The homogeneous system is

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The associated homogeneous system has only the trivial solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, v_1, v_2 *and* v_3 *are linearly independent.*

Example:

Determine whether the following set of vectors in the vector space consisting of all polynomials of degree $\leq n$ is linearly independent or linearly dependent.

$$S = \{v_1, v_2, v_3\} = \{x^2 + x + 2, 2x^2 + x, 3x^2 + 2x + 2\}.$$

[solution:]

$$c_1v_1 + c_2v_2 + c_3v_3 = c_1(x^2 + x + 2) + c_2(2x^2 + x) + c_3(3x^2 + 2x + 2) = 0.$$

Thus,

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + \quad + 2c_3 &= 0 \end{aligned} \quad \Leftrightarrow \quad c_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The associated homogeneous system is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The homogeneous system has infinite number of solutions,

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\ t \in R.$$

Therefore, v_1, v_2 *and* v_3 *are linearly dependent since*

$$tv_1 + tv_2 - tv_3 = 0,\ t \in R.$$

Note:

In the examples with $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix},$ or with $S = \{v_1, v_2, v_3\} = \{x^2 + x + 2, 2x^2 + x, 3x^2 + 2x + 2\},$ v_1, v_2 *and* v_3 *are linearly dependent.* Observe that v_3 in both examples are linear combinations of v_1, v_2,

$$v_3 = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 4v_3 - 2v_2$$

and

$$v_3 = 3x^2 + 2x + 2 = (x^2 + x + 2) + (2x^2 + x) = v_1 + v_2.$$

As a matter of fact, we have the following general result.

Important result:

The nonzero vectors $v_1, v_2, \ldots, v_k$ in a vector space V are linearly dependent if and only if one of the vectors $v_j,\ j \geq 2$, is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{j-1}$.

Note:

Every set of vectors containing the zero vector is linearly dependent. That is, $v_1, v_2, \ldots, v_k$ are *k* vectors in any vector space and v_i is the zero vector, then $v_1, v_2, \ldots, v_k$ are linearly dependent.

Basis and Dimension

Definition of basis:

The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to form a basis of V if

(a) $v_1, v_2, \ldots, v_k$ span V (i.e., $span(v_1, v_2, \ldots, v_k) = V$).

(b) $v_1, v_2, \ldots, v_k$ are linearly independent.

Example:

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $S = \{e_1, e_2, e_3\}$. Are e_1, e_2 and e_3 a basis in R^3?

[solution:]

e_1, e_2 and e_3 form a basis in R^3 since

(a) $span(S) = span(e_1, e_2, e_3) = R^3$ (see the example in the previous section).

(b) e_1, e_2 and e_3 are linearly independent (also see the example in the previous section).

Example:

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Are v_1, v_2 and v_3 a basis in R^2?

[solution:]

v_1, v_2 and v_3 are *not* a basis of R^2 since v_1, v_2 and v_3 are linearly dependent,

$$3v_1 + 4v_2 - v_3 = 0.$$

Note that $span(v_1, v_2, v_3) = R^2$.

Example:

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix}$. Are v_1, v_2 and v_3 a basis in R^3?

[solution:]

v_1, v_2 and v_3 are not a basis in R^3 since v_1, v_2 and v_3 are linearly independent,

$$v_3 = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 4v_3 - 2v_2.$$

Example:

Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } S = \{v_1, v_2, v_3\}.$$

Are S a basis in R^3?

[solution:]

(a) $span(S) = R^3 \Leftrightarrow$ For any vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$, there exist real numbers c_1, c_2, c_3 such that

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = c_1v_1 + c_2v_2 + c_3v_3.$$

$\Leftrightarrow$ we need to solve for the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The solution is

$$c_1 = \frac{-2a+2b+c}{3},\ c_2 = \frac{a-b+c}{3},\ c_3 = \frac{4a-b-2c}{3}.$$

Thus,

$$v = \left(\frac{-2a+2b+c}{3}\right)v_1 + \left(\frac{a-b+c}{3}\right)v_2 + \left(\frac{4a-b-2c}{3}\right)v_3.$$

That is, every vector in R^3 can be a linear combination of v_1, v_2, v_3 and $span(S) = R^3$.

(b) Since

$$c_1v_1 + c_2v_2 + c_3v_3 = \begin{bmatrix} c_1 + c_2 + c_3 \\ 2c_1 + c_3 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow c_1 = c_2 = c_3 = 0,$$

v_1, v_2, v_3 are linearly independent.

By (a) and (b), v_1, v_2, v_3 are a basis of R^3.

Important result:

If $S = \{v_1, v_2, \ldots, v_k\}$ is a basis for a vector space V, then every vector in V can be written in an *unique* way as a linear combination of the vectors in S.

Example:

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $S = \{e_1, e_2, e_3\}$. S is a basis of R^3. Then, for any vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = ae_1 + be_2 + ce_3$$

is uniquely determined.

Important result:

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of nonzero vectors in a vector space V and let $W = span\{v_1, v_2, \ldots, v_k\}$. Then, some subset of S is a basis of W.

How to find a basis (subset of S) of W:

There are two methods:

Method 1:

The procedure based on the proof of the above important result.

Method 2:

Step 1: Form equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0.$$

Step 2: Construct the augmented matrix associated with the equation in step 1 and transform this augmented matrix to the reduced row echelon form.

Step 3: The vectors corresponding to the columns containing the leading 1's form a basis. For example, if $k = 6$ and the reduced row echelon matrix is

$$\left[\begin{array}{cccccc|c} 1 & \times & \times & \times & \times & \times & 0 \\ 0 & 0 & 1 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 1 & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

then the 1'st, the 3'nd, and the 4'th columns contain a leading 1 and thus

v_1, v_3, v_4 are a basis of $W = span\{v_1, v_2, \ldots, v_6\}$.

Example:

Let

$$S = \{e_1, e_2, a_1, e_3, a_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

and $span(S) = R^3$. Please find subsets of S which form a basis of R^3.

[solution:]

Method1:

We first check if e_1 *and* e_2 *are linearly independent.* Since they are linearly independent, we continue to check if e_1, e_2 *and* a_1 *are linearly independent.* Since

$$2e_1 + 3e_2 - a_1 = 0,$$

we delete a_1 from S and form a new set S_1, $S_1 = \{e_1, e_2, e_3, a_2\}$. Then, we continue to check if e_1, e_2 *and* e_3 *are linearly independent.* They are linearly independent. Thus, we finally check if e_1, e_2 e_3 *and* a_2 *are linearly independent.* Since

$$e_1 + 3e_2 + 2e_3 - a_2 = 0,$$

we delete a_1 from S_1 and form a new set S_2, $S_2 = \{e_1, e_2, e_3\}$. Therefore,

$$S_2 = \{e_1, e_2, e_3\}$$

is the subset of S which form a basis of form a basis of R^3.

Method 2:

Step 1:

The equation is

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}2\\3\\0\end{bmatrix}+c_4\begin{bmatrix}0\\0\\1\end{bmatrix}+c_5\begin{bmatrix}1\\2\\3\end{bmatrix}=0.$$

Step 2:

The augmented matrix and its reduced row echelon matrix is

$$\left[\begin{array}{ccccc|c}1 & 0 & 2 & 0 & 1 & 0\\0 & 1 & 3 & 0 & 2 & 0\\0 & 0 & 0 & 1 & 3 & 0\end{array}\right].$$

The 1’st, the 2’nd and 4’th columns contain the leading 1’s. Thus,

$\{e_1, e_2, e_3\}$ forms a basis.

Representation of Vectors and linear operators with respect to basis

Let $S=\{v_1,v_2,\ldots,v_n\}$ be a basis for a vector space V and let $T=\{w_1,w_2,\ldots,w_r\}$ is a linear independent set of vectors in V. Then, $r \le n$.

Corollary:

Let $S=\{v_1,v_2,\ldots,v_n\}$ and $T=\{w_1,w_2,\ldots,w_m\}$ be two bases for a vector space V. Then, $n = m$.

Note:

For a vector space V, there are infinite bases. But the number of vectors in two different bases are the same.

Example:

For the vector space R^3,

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $S = \{v_1, v_2, v_3\}$ is a basis for R^3 (see the previous example). Also,

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $T = \{e_1, e_2, e_3\}$ is basis for R^3.

$\Rightarrow$ There are 3 vectors in both S and T.

Schmidt orthogonalization process

Gram-Schmidt orthogonalization, also called the Gram-Schmidt process, is a procedure which takes a nonorthogonal set of linearly independent functions and constructs an orthogonal basis over an arbitrary interval with respect to an arbitrary weighting function $w(x)$.

Applying the Gram-Schmidt process to the functions 1, x, x^2, ... on the interval $[-1, 1]$ with the usual L^2 inner product gives the Legendre polynomials (up to constant multiples; Reed and Simon 1972, p. 47).

Given an original set of linearly independent functions $\{u_n\}_{n=0}^{\infty}$, let $\{\psi_n\}_{n=0}^{\infty}$ denote the orthogonalized (but not normalized) functions, $\{\phi_n\}_{n=0}^{\infty}$ denote the orthonormalized functions, and define

$$\psi_0(x) \equiv u_0(x) \tag{1}$$

$$\phi_0(x) \equiv \frac{\psi_0(x)}{\sqrt{\int \psi_0^2(x)\, w(x)\, dx}}. \tag{2}$$

Then take

$$\psi_1(x) = u_1(x) + a_{10}\,\phi_0(x), \tag{3}$$

where we require

$$\int \psi_1\, \phi_0\, w\, dx = \int u_1\, \phi_0\, w\, dx + a_{10} \int \phi_0^2\, w\, dx \tag{4}$$

$$= 0. \tag{5}$$

By definition,

$$\int \phi_0{}^2 \, w \, dx = 1, \quad (6)$$

so

$$a_{10} = -\int u_1 \, \phi_0 \, w \, dx. \quad (7)$$

The first orthogonalized function is therefore

$$\psi_1 = u_1(x) - \left[\int u_1 \, \phi_0 \, w \, dx\right] \phi_0, \quad (8)$$

and the corresponding normalized function is

$$\phi_1 = \frac{\psi_1(x)}{\sqrt{\int \psi_1{}^2 \, w \, dx}}. \quad (9)$$

By mathematical induction, it follows that

$$\phi_i(x) = \frac{\psi_i(x)}{\sqrt{\int \psi_i{}^2 \, w \, dx}}, \quad (10)$$

where

$$\psi_i(x) = u_i + a_{i0} \, \phi_0 + a_{i1} \, \phi_1 \ldots + a_{i,i-1} \, \phi_{i-1} \quad (11)$$

and

$$a_{ij} \equiv -\int u_i \, \phi_j \, w \, dx. \quad (12)$$

If the functions are normalized to N_j instead of 1, then

$$\int_a^b [\phi_j(x)]^2 \, w \, dx = N_j^2 \quad (13)$$

$$\phi_i(x) = N_i \frac{\psi_i(x)}{\sqrt{\int \psi_i{}^2 \, w \, dx}} \quad (14)$$

$$a_{ij} = -\frac{\int u_i \phi_j w \, dx}{N_j^2}. \tag{15}$$

Orthogonal polynomials are especially easy to generate using Gram-Schmidt orthonormalization. Use the notation

$$\langle x_i | x_j \rangle \equiv \langle x_i | w | x_j \rangle \tag{16}$$

$$\equiv \int_a^b x_i(x) x_j(x) w(x) \, dx, \tag{17}$$

where $w(x)$ is a weighting function, and define the first few polynomials,

$$p_0(x) \equiv 1 \tag{18}$$

$$p_1(x) = \left[x - \frac{\langle x p_0 | p_0 \rangle}{\langle p_0 | p_0 \rangle}\right] p_0. \tag{19}$$

As defined, p_0 and p_1 are orthogonal polynomials, as can be seen from

$$\langle p_0 | p_1 \rangle = \left\langle \left[x - \frac{\langle x p_0 | p_0 \rangle}{\langle p_0 | p_0 \rangle}\right] p_0 \right\rangle \tag{20}$$

$$= \langle x p_0 \rangle - \frac{\langle x p_0 | p_0 \rangle}{\langle p_0 | p_0 \rangle} \langle p_0 \rangle \tag{21}$$

$$= \langle x p_0 \rangle - \langle x p_0 \rangle \tag{22}$$

$$= 0. \tag{23}$$

Now use the recurrence relation

$$p_{i+1}(x) = \left[x - \frac{\langle x p_i | p_i \rangle}{\langle p_i | p_i \rangle}\right] p_i - \left[\frac{\langle p_i | p_i \rangle}{\langle p_{i-1} | p_{i-1} \rangle}\right] p_{i-1} \tag{24}$$

to construct all higher order polynomials.

To verify that this procedure does indeed produce orthogonal polynomials, examine

$$\langle p_{i+1} | p_i \rangle = \left\langle \left[x - \frac{x p_i | p_i}{p_i | p_i}\right] p_i \middle| p_i \right\rangle - \left\langle \frac{p_i | p_i}{p_{i-1} | p_{i-1}} p_{i-1} \middle| p_i \right\rangle \tag{25}$$

$$= \langle x\, p_i \mid p_i \rangle - \frac{\langle x\, p_i \mid p_i \rangle}{\langle p_i \mid p_i \rangle} \langle p_i \mid p_i \rangle - \frac{\langle p_i \mid p_i \rangle}{\langle p_{i-1} \mid p_{i-1} \rangle} \langle p_{i-1} \mid p_i \rangle \qquad (26)$$

$$= -\frac{\langle p_i \mid p_i \rangle}{\langle p_{i-1} \mid p_{i-1} \rangle} \langle p_{i-1} \mid p_i \rangle \qquad (27)$$

$$= -\frac{\langle p_i \mid p_i \rangle}{\langle p_{i-1} \mid p_{i-1} \rangle} \left[-\frac{\langle p_{i-1} \mid p_{j-1} \rangle}{\langle p_{j-2} \mid p_{j-2} \rangle} \langle p_{j-2} \mid p_{j-1} \rangle \right] \qquad (28)$$

$$= \ldots \qquad (29)$$

$$= (-1)^j \frac{\langle p_j \mid p_j \rangle}{\langle p_0 \mid p_0 \rangle} \langle p_0 \mid p_1 \rangle \qquad (30)$$

$$= 0, \qquad (31)$$

since $\langle p_0 \mid p_1 \rangle = 0$. Therefore, all the polynomials $p_i(x)$ are orthogonal.

Inner product

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.

More precisely, for a real vector space, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following four properties. Let u, v, and w be vectors and α be a scalar, then:

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

2. $\langle \alpha\, v, w \rangle = \alpha \langle v, w \rangle$.

3. $\langle v, w \rangle = \langle w, v \rangle$.

4. $\langle v, v \rangle \geq 0$ and equal if and only if $v = 0$.

The fourth condition in the list above is known as the positive-definite condition. Related thereto, note that some authors define an inner product to be a function $\langle \cdot, \cdot \rangle$ satisfying only the first three of the above conditions with the added (weaker) condition of being (weakly) non-degenerate (i.e., if $\langle v, w \rangle = 0$ for all w, then $v \equiv 0$). In such literature, functions satisfying all four such conditions are typically referred to as positive-definite inner products (Ratcliffe 2006), though inner products which fail to be positive-definite are sometimes called indefinite to avoid confusion. This difference, though subtle, introduces a number of noteworthy phenomena: For example, inner products which fail to be positive-definite may give rise to "norms" which yield an imaginary magnitude for certain vectors (such vectors are called spacelike) and which induce "metrics" which fail to be actual metrics. The Lorentzian inner product is an example of an indefinite inner product.

A vector space together with an inner product on it is called an inner product space. This definition also applies to an abstract vector space over any field.

Examples of inner product spaces include:

1. The real numbers $\mathbb{R}$, where the inner product is given by

$$\langle x, y \rangle = x\, y. \qquad (1)$$

2. The Euclidean space $\mathbb{R}^n$, where the inner product is given by the dot product

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1\, y_1 + x_2\, y_2 + \cdots x_n\, y_n \qquad (2)$$

3. The vector space of real functions whose domain is an closed interval $[a, b]$ with inner product

$$\langle f, g \rangle = \int_a^b f\, g\, d\, x. \qquad (3)$$

When given a complex vector space, the third property above is usually replaced by

$$\langle v, w \rangle = \overline{\langle w, v \rangle}, \qquad (4)$$

where $\bar{z}$ refers to complex conjugation. With this property, the inner product is called a Hermitian inner product and a complex vector space with a Hermitian inner product is called a Hermitian inner product space.

Every inner product space is a metric space. The metric is given by

$$g\,(v, w) = \langle v - w, v - w \rangle. \qquad (5)$$

If this process results in a complete metric space, it is called a Hilbert space. What's more, every inner product naturally induces a norm of the form

$$|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\,, \qquad (6)$$

whereby it follows that every inner product space is also naturally a normed space. As noted above, inner products which fail to be positive-definite yield "metrics" - and hence, "norms" - which are actually something different due to the possibility of failing their respective positivity conditions. For example, n-dimensional Lorentzian Space (i.e., the inner product space consisting of $\mathbb{R}^n$ with the Lorentzian inner product) comes equipped with a metric tensor of the form

$$(ds)^2 = -dx_0^2 + dx_1^2 + \cdots + dx_{n-1}^2 \qquad (7)$$

and a squared norm of the form

$$|\mathbf{v}|^2 = -v_0^2 + v_1^2 + \cdots + v_{n-1}^2 \qquad (8)$$

for all vectors $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$. In particular, one can have negative infinitesimal distances and squared norms, as well as nonzero vectors whose vector norm is always zero. As such, the metric (respectively, the norm) fails to *actually* be a metric (respectively, a norm), though they usually are still called such when no confusion may arise.

Tensor

In *n*-dimensional space V_n (called a "manifold" in mathematics), points are specified by assigning values to a set of *n* continuous real variables $x^1, x^2 \ldots\ldots x^n$ called the *coordinates*. In many cases these will run from -∞ to +∞, but the range of some or all of these can be finite.

Examples: In Euclidean space in three dimensions, we can use cartesian coordinates *x, y* and *z*, each of which runs from -∞ to +∞. For a two dimensional Euclidean plane, Cartesians may again be employed, or we can use plane polar coordinates *r*, □ whose ranges are 0 to ∞ and 0 to 2□ respectively.

Coordinate transformations

The coordinates of points in the manifold may be assigned in a number of different ways. If we select two different sets of coordinates, $x^1, x^2 \ldots\ldots x^n$ and $x'^1, x'^2, \ldots\ldots x'^n$, there will obviously be a connection between them of the form

$$x'^r = f^r(x^1, x^2 \ldots\ldots x^n) \qquad r = 1, 2\ldots\ldots\ldots n. \ (1)$$

where the f's are assumed here to be well behaved functions. Another way of expressing the same relationship is

$$x'^r = x'^r(x^1, x^2 \ldots\ldots x^n) \qquad r = 1, 2\ldots\ldots\ldots n. \ (2)$$

where $x'^r(x^1, x^2 \ldots\ldots x^n)$ denotes the *n* functions $f^r(x^1, x^2 \ldots\ldots x^n)$, *r* = 1, 2……*n*.

Recall that if a variable *z* is a function of two variables *x* and *y*, i.e. *z = f (x, y)*, then the connection between the differentials *dx, dy* and *dz* is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (3)$$

Extending this to several variables therefore, for each one of the new coordinates we have

$$dx'^r = \sum_{s=1}^{n} \frac{\partial x'^r}{\partial x^s} dx^s \qquad r=1, 2........n. \qquad (4)$$

The transformation of the differentials of the coordinates is therefore linear and homogeneous, which is not necessarily the case for the transformation of the coordinates themselves.

Range and Summation Conventions. Equations such as (4) may be simplified by the use of two conventions:

Range Convention: When a suffix is unrepeated in a term, it is understood to take all values in the range 1, 2, 3.....*n*.

Summation Convention: When a suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation being 1, 2, 3.....*n*.

With these two conventions applying, equation (4) may be written as

$$dx'^r = \frac{\partial x'^r}{\partial x^s} dx^s \qquad (5)$$

Note that a repeated suffix is a "dummy" suffix, and can be replaced by any convenient alternative. For example, equation (5) could have been written as

$$dx'^r = \frac{\partial x'^r}{\partial x^m} dx^m \qquad (6)$$

where the summation with respect to *s* has been replaced by the summation with respect to *m*.

Contravariant vectors and tensors. Consider two neighbouring points P and Q in the manifold whose coordinates are x^r and $x^r + dx^r$ respectively. The vector $\vec{PQ}$ is then described by the quantities dx^r which are the *components* of the vector in this coordinate system. In the dashed coordinates, the vector $\vec{PQ}$ is described by the components dx'^r which are related to dx^r by equation (5), the differential coefficients being evaluated at P. The infinitesimal displacement represented by dx^r or dx'^r is an example of a contravariant vector.

Defn. A set of *n* quantities T^r associated with a point P are said to be the components of a contravariant vector if they transform, on change of coordinates, according to the equation

$$T'^r = \frac{\partial x'^r}{\partial x^s} T^s \quad (7)$$

where the partial derivatives are evaluated at the point P. (Note that there is no requirement that the components of a contravariant tensor should be infinitesimal.)

Defn. A set of n^2 quantities T^{rs} associated with a point P are said to be the components of a contravariant tensor of the second order if they transform, on change of coordinates, according to the equation

$$T'^{rs} = \frac{\partial x'^r}{\partial x^m} \frac{\partial x'^s}{\partial x^n} T^{mn} \quad (8)$$

Obviously the definition can be extended to tensors of higher order. A contravariant vector is the same as a contravariant tensor of first order.

Defn. A contravariant tensor of zero order transforms, on change of coordinates, according to the equation

$$T' = T, \quad (9)$$

i.e. it is an *invariant* whose value is independent of the coordinate system used.

Covariant vectors and tensors. Let ϕ be an invariant function of the coordinates, i.e. its value may depend on position P in the manifold but is independent of the coordinate system used. Then the partial derivatives of ϕ transform according to

$$\frac{\partial \phi}{\partial x'^r} = \frac{\partial \phi}{\partial x^s} \frac{\partial x^s}{\partial x'^r} \quad (10)$$

Here the transformation is similar to equation (7) except that the partial derivative involving the two sets of coordinates is the other way up. The partial derivatives of an invariant function provide an example of the components of a covariant vector.

Defn. A set of *n* quantities T_r associated with a point P are said to be the components of a covariant vector if they transform, on change of coordinates, according to the equation

$$T'_r = \frac{\partial x^s}{\partial x'^r} T_s \quad (11)$$

By convention, suffices indicating contravariant character are placed as superscripts, and those indicating covariant character as subscripts. Hence the reason for writing the coordinates as x^r. (Note however that it is only the *differentials* of the coordinates, not the coordinates themselves, that always have tensor character. The latter *may* be tensors, but this is not always the case.)

Extending the definition as before, a covariant tensor of the second order is defined by the transformation

$$T'_{rs} = \frac{\partial x^m}{\partial x'^r}\frac{\partial x^n}{\partial x'^s}T_{mn} \qquad (12)$$

and similarly for higher orders.

Rank of Tensor

The total number of contravariant and covariant indices of a tensor. The rank R of a tensor is independent of the number of dimensions N of the underlyingspace.

An intiative way to think of the rank of a tensor is as follows: First, consider intuitively that a tensor represents a physical entity which may be characterized by magnitude and multiple directions simultaneously (Fleisch 2012). Therefore, the number of simultaneous directions is denoted R and is called the rank of the tensor in question. In N-dimensional space, it follows that a rank-0 tensor (i.e., a scalar) can be represented by $N^0 = 1$ number since scalars represent quantities with magnitude and no direction; similarly, a rank-1 tensor (i.e., a vector) in N-dimensional space can be represented by $N^1 = N$numbers and a general tensor by N^R numbers. From this perspective, a rank-2 tensor (one that requires N^2 numbers to describe) is equivalent, mathematically, to an $N \times N$ matrix.

rank	object
0	scalar
1	vector
2	$N \times N$ matrix
≥ 3	tensor

The above table gives the most common nomenclature associated to tensors of various rank. Some care must be exhibited, however, because the above nomenclature is hardly uniform across the literature. For example, some authors refer to tensors of rank 2 as dyads, a term used completely independently of the related term dyadic used to describe vector direct products (Kolecki 2002). Following such convention, authors also use the terms triad, tetrad, etc., to refer to tensors of rank 3, rank 4, etc.

Some authors refer to the rank of a tensor as its order or its degree. When defining tensors abstractly by way of tensor products, however, some authors exhibit great care to maintain the separation and distinction of these terms.

Mixed tensors and Kroneckar Delta. These are tensors with at least one covariant suffix and one contravariant suffix. An example is the third order tensor T^{r}_{st} which transforms according to

$$T'^{r}_{st} = \frac{\partial x'^{r}}{\partial x^{m}} \frac{\partial x^{n}}{\partial x'^{s}} \frac{\partial x^{p}}{\partial x'^{t}} T^{m}_{np} \qquad (13)$$

Another example is the Kronecker delta defined by

$$\delta^{r}_{s} = 1, \quad r = s$$
$$= 0, \quad r \neq s \qquad (14)$$

It is a tensor of the type indicated because (a) in an expression such as $B^{mn..}_{pq..}\delta^{t}_{m}$, which involves summation with respect to *m*, there is only one non-zero contribution from the Kronecker delta, that for which *m* = *t*, and so $B^{mn..}_{pq..}\delta^{t}_{m} = B^{tn..}_{pq..}$; (b) the coordinates in any coordinate system are necessarily independent of each other, so that $\frac{\partial x^{r}}{\partial x^{s}} = \delta^{r}_{s}$ and $\frac{\partial x'^{r}}{\partial x'^{s}} = \delta'^{r}_{s}$; so these two properties taken together imply that

$$\delta'^{r}_{s} = \frac{\partial x'^{r}}{\partial x^{m}} \frac{\partial x^{n}}{\partial x'^{s}} \delta^{m}_{n}. \qquad (15)$$

Notes. 1. The importance of tensors is that if a tensor equation is true in one set of coordinates it is also true in any other coordinates. e.g. if $T_{mn} = 0$ (which, since *m* and *n* are unrepeated, implies that the equation is true for *all m* and *n*, not just for some particular choice of these suffices), then $T'_{rs} = 0$ also, from the transformation law. This illustrates the fact that any tensor equation is *covariant*, which means that it has the same form in all coordinate systems.

2. A tensor may be defined at a single point P within the manifold, or along a curve, or throughout a subspace, or throughout the manifold itself. In the latter cases we speak of a *tensor field.*

Tensor algebra

Addition of tensors. Two tensors of the same type may be added together to give another tensor of the same type, e.g. if A^r_{st} and B^r_{st} are tensors of the type indicated, then we can define

$$C^r_{st} = A^r_{st} + B^r_{st} . \qquad (16)$$

It is easy to show that the quantities C^r_{st} form the components of a tensor.

Symmetric and antisymmetric tensors. A^{rs} is a *symmetric* contravariant tensor if $A^{rs} = A^{sr}$ and *antisymmetric* if $A^{rs} = -A^{sr}$. Similarly for covariant tensors. Symmetry properties are conserved under transformation of coordinates, e.g. if $A^{rs} = A^{sr}$, then

$$A'^{mn} = \frac{\partial x'^m}{\partial x^r}\frac{\partial x'^n}{\partial x^s}A^{rs} = \frac{\partial x'^m}{\partial x^r}\frac{\partial x'^n}{\partial x^s}A^{sr} = A'^{nm} . \qquad (17)$$

Note however that for a mixed tensor, a relation such as $A^s_r = A^r_s$ does not transform to give the equivalent relation in the dashed coordinates. The concept of symmetry (with respect to a pair of suffices which are either both subscripts or both superscripts) can obviously be extended to tensors of higher order.

Any covariant or contravariant tensor of second order may be expressed as the sum of a symmetric tensor and an antisymmetric tensor, e.g.

$$A^{rs} = \frac{1}{2}(A^{rs} + A^{sr}) + \frac{1}{2}(A^{rs} - A^{sr}) . \qquad (18)$$

Multiplication of tensors. In the addition of tensors we are restricted to tensors of a single type, with the same suffices (though they need not occur in the same order). In the multiplication of tensors there is no such restriction. The only condition is that we never multiply two components with the same suffix at the same level in each. (This would imply summation with respect to the repeated suffix, but the resulting object would not have tensor character - see later.)

To multiply two tensors e.g. A_{rs} and B^m_n we simply write

$$C^m_{rsn} = A_{rs}B^m_n . \qquad (19)$$

It follows immediately from their transformation properties that the quantities C^m_{rsn} form a tensor of the type indicated. This tensor, in which the symbols for the suffices are all different, is called the *outer product* of A_{rs} and B^m_n.

Contraction of tensors. Given a tensor T^m_{np}, then

$$T'^m_{np} = \frac{\partial x'^m}{\partial x^r}\frac{\partial x^s}{\partial x'^n}\frac{\partial x^t}{\partial x'^p}T^r_{st} \qquad (20)$$

Hence replacing *n* by *m* (and therefore implying summation with respect to *m*)

$$\begin{aligned} T'^m_{mp} &= \frac{\partial x'^m}{\partial x^r}\frac{\partial x^s}{\partial x'^m}\frac{\partial x^t}{\partial x'^p}T^r_{st} \\ &= \frac{\partial x^s}{\partial x^r}\frac{\partial x^t}{\partial x'^p}T^r_{st} \\ &= \delta^s_r\frac{\partial x^t}{\partial x'^p}T^r_{st} \\ &= \frac{\partial x^t}{\partial x'^p}T^s_{st} \end{aligned} \qquad (21)$$

so we see that T^m_{mp} behaves like a tensor A_p. The upshot is that *contraction* of a tensor (i.e. writing the same letter as a subscript and a superscript) reduces the order of the tensor by 2 and yields a tensor whose type is indicated by the remaining suffices.

Note that contraction can only be applied successfully to suffices at different levels. We may of course construct, starting with a tensor A^p_{qrs} say, a new set of quantities A^p_{qrr}; but these do not have tensor character (as one can easily check) so are of little interest.

Having constructed the outer product $C^m_{rsn} = A_{rs}B^m_n$ in the example above, we can form the corresponding *inner products* $C^m_{msn} = A_{ms}B^m_n$ and $C^m_{rmn} = A_{rm}B^m_n$. Each of these forms a covariant tensor of second order.

Possible questions –(Part –B- 6 Marks)

1. Explain the properties of Kronecker delta. Prove that Kronecker delta is a mixed tensor of rank 2, and is invariant.
2. Explain Schmidt's orthogonalization method
3. Show that the symmetry properties of a tensor are invariant
4. Describe the operations of outer product and inner product of tensors
5. Show that the set of vectors r_1, r_2, r_3 given by
   r_1= j-2k, r_2= i-j+K, r_3 = i+2j+K is linearly independent
6. Show that vectors (u+v), (u –v) and (u-2v+w) are linearly independent provided (u,v,w) are linearly dependent.
7. Show that Kronecker delta is an invariant mixed tensor of rank 2.
8. Show that in Cartesian coordinate system the contravariant and covariant components of a vector are identical.
9. Explain about the symmetric and antisymmetric tensors.
10. Explain orthogonal and orthonormal vectors. Explain Schmidt's orthogonalization procedure.
11. Explain Einstein's summation convention of tensors

Possible questions –(Part –C- 10 Marks)

1. Explain Schmidt's orthogonalization process and give their properties
2. Show that the symmetry properties of a tensor are invariant
3. Describe the operations of outer product and inner product of tensors
4. Show that the set of vectors r_1, r_2, r_3 given by
   r_1= i+j-5k, r_2= 2i-j+K, r_3 = 8i+2j+K is linearly independent
5. Show that vectors (u+v), (u –v) and (u-2v+w) are linearly independent provided (u,v,w) are linearly dependent.
6. Show that in Cartesian coordinate system the contravariant and covariant components of a vector are identical.
7. Explain about the symmetric and antisymmetric tensors with few examples
8. Explain orthogonal and orthonormal vectors with orthogonolization process.

9. Explain covariant and contravariant tensors and einstein's summation convention of tensors

KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21

DEPARTMENT OF PHYSICS

CLASS:I MSc PHYSICS BATCH:2019-2021

MATHEMATICAL PHYSICS (19PHP104)

MULTIPLE CHOICE QUESTIONS

Questions	**opt1**	**opt2**	**opt3**
UNIT I			
The union of two subspaces of a vector space need not be	a sub space	cyclic	an abelian
If { Vi} is an orthonormal set, then the vectors {Vi} are	linearly dependent	commutativ e	linearly independent
Kronecker delta symbol is	covariant tensor	contravarian t tensor	an invariant
The rank of the tensor A^{ij}_{klm} is	4	5	3
The rank of the outer product of the tensors C^{ij} and D_k is	1	3	2
In an n-dimensional vector space, the number of linearly dependent vectors is	n	2n	n + 1
The rank of the outer product of the tensors C^{ij} and D^k_{lm} is	3	5	2
The dimension of vector space is always	number of linearly	linearly independent	linearly independent
The vectors are said to be orthogonal when the scalar product of	two null vector is one	vector is zero	null vector is zero
The set of all position vectors forms	an abelian group	vector space	sub space
Example of real vector space is	dimensional space	dimensional space	dimensional space
An important example of mixed tensor of rank two is	covariant	Kronecker delts	Invariant
If $x^1, x^2, \ldots.. x^n$ are independent variables, then	$\delta^{\mu}_{\upsilon} = \frac{\partial x^{\mu}}{\partial x_{\nu}}$	$\delta^{\mu}_{\nu} = \frac{\partial x^{\nu}}{\partial x_{\mu}}$	$\delta^{\mu}_{\nu} = \partial x^{\mu\nu}$

If f = f, the function of f is said to be	a scalar	invariant	tensor of rank two
The tensors of rank zero are	scalars	invariant	either (a) or (b)
The tensors of rank one are	scalars	vectors	invariant
A symmetric tensor of rank two is n-dimensional space has independent components	$\frac{n+1}{2}$	$\frac{n(n+1)}{2}$	$\frac{n(n-1)}{2}$
If $A_l^{mns} = - A_l^{msn}$, then tensor A_l^{mns} is antisymmetric with respect to indices	n and s	m and s	m and n
A antisymmetric tensor of rank two is n-dimensional space has independent components	(n+1)/2	n(n+1)/2	n(n-1)/2
If A_{ij} is antisymmetric tensor, then the component A_{11} is	1	0	2
An antisymmetric tensor of rank 'r' is n-dimensional space will have independent components	$nC_r = \frac{n!}{(n-r)!}$	$nC_r = \frac{(n-1)!}{(n-r)!}$	$nC_r = \frac{n!}{r!(n-r)!}$
If a_{ik} is a tensor of rank two, its independent components in 4-dimensional space are	4	2	8
The total number of components a_{ik} tensor of rank two in 4-dimensional space are	4	16	2
The total number of components a_{ik} tensor of rank two in n-dimensional space are	n	n^2	(n-1)
As $a_{ijk}l$ is a tensor of rank 4, the number of components in in 4-dimensional space is	1	0	4
If A_{ij} is antisymmetric tensor, of second order and U^i is a tensor of rank one, then $A_{ij}U^i U^j$ is equal to	1	0	2
The sum of one contravariant and one covariant $A^m B_m$ is	invariant	contravariant	covariant
Kronecker delta is the best example for	covariant	mixed	invariant
A tensor of rank 'r' in n-dimensional space has components	n^r	r_n	n / r
A_l^{mns} are the components of a mixed tensor of rank	1	3	4
In an n-dimensional vector space, the number of linearly dependent vectors is	n	2n	n+1
The rank of the outer product of the tensors Cij and Dklm is	3	5	2
The dimension of vector space is always	than number of	linearly independent	than linearly

The vectors are said to be orthogonal when the scalar product of	two null vector is one	vector is zero	null vector is zero
The set of all position vectors forms	abelian group	vector space	sub space
An important example of mixed tensor of rank two is	covariant	Kronecker delts	Invariant
If f = f, the function of f is said to be	a scalar	invariant	tensor of rank two
The tensors of rank zero are	scalars	invariant	vectors
The tensors of rank one are	scalars	vectors	invariant
Kronecker delta symbol is	covariant tensor	contravariant tensor	an invariant
The rank of the tensor Aijklm is	2	4	3
The rank of the outer product of the tensors Cij and Dk is	1	3	2
In an n-dimensional vector space, the number of linearly dependent vectors is	n	2n	n+1
The rank of the outer product of the tensors Cij and Dklm is	3	5	2
The dimension of vector space is always	than number of	linearly independent	than linearly
The vectors are said to be orthogonal when the scalar product of	two null vector is one	vector is zero	null vector is zero
The set of all position vectors forms	abelian group	vector space	sub space
Example of real vector space is	dimensional space	dimensional space	dimensional space
A symmetrictensor of rank 2 is n-dimensional space independent components	(n+1)/2	n/2	n+1
If A_{ij} is antisymmetric tensor, then the component A_{11} is	1	0	2
If a_{ik} is a tensor of rank two, its independent components are	4	2	8
Kronecker delta is the best example for	covariant	mixed	invariant

The sum of one contravariant and one covariant $A^m B_m$ is	mixed	contravariant	covariant

opt4	**Key**
an invariant	a sub space
distributive	linearly independent
a mixed tensor	a mixed tensor
6	5
0	3
2n + 3	n + 1
6	5
linearly dependent	linearly independent
null vector is one	null vector is zero
cyclic	an abelian group
none of the above	dimensional space
Contravariant	Kronecker delts
$\delta_{\nu}^{\mu} = \partial x_{\nu}^{\mu}$	$\delta_{\nu}^{\mu} = \partial x^{\mu\nu}$

all the above	invariant
vectors	either (a) or (b)
covariant	vectors
$\frac{n-1}{2}$	$\frac{n+1}{2}$
m and l	n and s
(n-1)/2	n(n+1)/2
3	0
$nC_r = \frac{n!}{r!}$	$nC_r = \frac{n!}{r!(n-r)!}$
6	6
8	16
(n+1)	n^2
16	1
4	0
mixed	invariant
contravariant	mixed
r / n	n^r
0	4
2n+3	n+1
6	5
than linearly	linearly independent

null vector is one	null vector is zero
cyclic	abelian group
Contravariant	Kronecker delts
all the above	all the above
none	scalars
covariant	vectors
a mixed tensor	a mixed tensor
6	4
0	3
2n+3	n+1
6	5
linearly independent	linearly independent
null vector is one	null vector is zero
cyclic	abelian group
none	dimensional space
n	(n+1)/2
3	0
6	6
contravariant	mixed

mixed	mixed

UNIT-II

SYLLABUS

Functions of a complex variable – single and multivalued functions – Cauchy-Riemann differential equation – analytical – line integrals of complex function – Cauchy's integral theorem and integral formula – derivatives of an analytic function – Liouville's theorem - Taylor's series – Laurent's series - Residues and their evaluation - Cauchy's residue theorem – application to the evaluation of definite integrals.

Complex Algebra

Formally, the set of complex numbers can be de¯ned as the set of two-dimensional real vectors, $f(x; y)g$, with one extra operation, *complex multiplication*:

$$(x_1; y_1) ¢ (x_2; y_2) = (x_1 x_2 ¡ y_1 y_2; x_1 y_2 + x_2 y_1) : \qquad (1)$$

Together with generic vector addition

$$(x_1; y_1) + (x_2; y_2) = (x_1 + x_2; y_1 + y_2) ; \qquad (2)$$

.With the rules (1)-(2), complex numbers include the real numbers as a subset $f(x; 0)g$ with usual real number algebra. This suggests short-hand notation $(x; 0)$ ´ x; in particular: $(1; 0)$ ´ Complex algebra features commutatively, distributive and associa-tivity.

The two numbers, $1 = (1; 0)$ and $i = (0; 1)$ play a special role. They form a basis in the vector space, so that each complex number can be represented in a unique way as [we start using the notation $(x; 0)$ ´ x]

$$(x; y) = x + iy : \qquad (3)$$

Terminology: The number i is called imaginary unity. For the complex number $z = (x; y)$, the real umbers x and y are called real and imaginary parts, respectively; corresponding notation is: $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$.

The following remarkable property of the number i,

$$i^2 = -1 \qquad (4)$$

renders the representation (3) most convenient for practical algebraic ma-nipulations with complex numbers.|One treats x, y, and i the same way as the real numbers.

Single and Multi valued function

In a **multi-valued function** every input is associated with one or more outputs. Strictly speaking, a "well-defined" function associates one, and only one, output to any particular input. The term "multi-valued function" is, therefore, a misnomer: usually true functions are single-valued.

If only one value of corresponds to each value of z then is of z.**single valued function**

If more than one values of correspond to each value of z then is of z. i.e. A multi-valued function assumes two or more distinct values in its range for at least one point in its domain.

Cauchy Riemann Differential Eqution

Let

$$f(x,y) \equiv u(x,y) + i\,v(x,y), \tag{1}$$

Where

$$z \equiv x + i\,y, \tag{2}$$

So

$$dz = dx + i\,dy. \tag{3}$$

The total derivative of f with respect to z is then

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \tag{4}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right). \tag{5}$$

In terms of u and v, (5) becomes

$$\frac{df}{dz} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] \tag{6}$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\right]. \tag{7}$$

Along the real, or *x*-axis, $\partial f/\partial y = 0$, so

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right). \tag{8}$$

Along the imaginary, or *y*-axis, $\partial f/\partial x = 0$, so

$$\frac{df}{dz} = \frac{1}{2}\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right). \tag{9}$$

If f is complex differentiable, then the value of the derivative must be the same for a given dz, regardless of its orientation. Therefore, (8) must equal (9), which requires that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{10}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{11}$$

These are known as the Cauchy-Riemann equations.

They lead to the conditions

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \tag{12}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}. \quad (13)$$

The Cauchy-Riemann equations may be concisely written as

$$\frac{df}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] \quad (14)$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] \quad (15)$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \quad (16)$$

$$= 0, \quad (17)$$

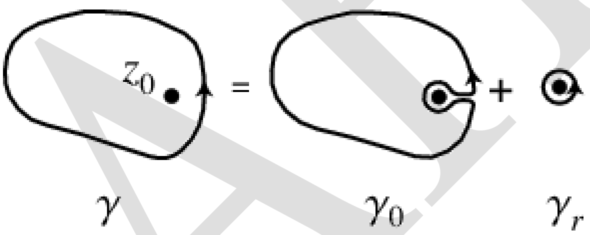
where $\bar{z}$ is the complex conjugate.

If $z = r e^{i\theta}$, then the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad (18)$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad (19)$$

Cauchy Integral Formula



Cauchy's integral formula states that

$$f(z_0) = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)\,dz}{z - z_0}, \quad (1)$$

where the integral is a contour integral along the contour γ enclosing the point z_0.

It can be derived by considering the contour integral

$$\oint_{\gamma}\frac{f(z)\,dz}{z - z_0}, \quad (2)$$

defining a path γ_r as an infinitesimal counterclockwise circle around the point z_0, and defining the path γ_0 as an arbitrary loop with a cut line (on which the forward and reverse contributions cancel each other out) so as to go around z_0. The total path is then

$$\gamma = \gamma_0 + \gamma_r, \quad (3)$$

so

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_0}\frac{f(z)\,dz}{z-z_0}+\oint_{\gamma_r}\frac{f(z)\,dz}{z-z_0}. \tag{4}$$

From the Cauchy integral theorem, the contour integral along any path not enclosing a pole is 0. Therefore, the first term in the above equation is 0 since γ_0 does not enclose the pole, and we are left with

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_r}\frac{f(z)\,dz}{z-z_0}. \tag{5}$$

Now, let $z\equiv z_0+re^{i\theta}$, so $dz=ire^{i\theta}\,d\theta$. Then

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\oint_{\gamma_r}\frac{f\left(z_0+re^{i\theta}\right)}{re^{i\theta}}\,ire^{i\theta}\,d\theta \tag{6}$$

$$=\oint_{\gamma_r}f\left(z_0+re^{i\theta}\right)i\,d\theta. \tag{7}$$

But we are free to allow the radius r to shrink to 0, so

$$\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}=\lim_{r\to 0}\oint_{\gamma_r}f\left(z_0+re^{i\theta}\right)i\,d\theta \tag{8}$$

$$=\oint_{\gamma_r}f(z_0)\,i\,d\theta \tag{9}$$

$$=i\,f(z_0)\oint_{\gamma_r}d\theta \tag{10}$$

$$=2\pi i\,f(z_0), \tag{11}$$

giving (1).

If multiple loops are made around the point z_0, then equation (11) becomes

$$n(\gamma,z_0)\,f(z_0)=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}, \tag{12}$$

where $n(\gamma,z_0)$ is the contour winding number.

A similar formula holds for the derivatives of $f(z)$,

$$f'(z_0)=\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} \tag{13}$$

$$=\lim_{h\to 0}\frac{1}{2\pi i h}\left[\oint_{\gamma}\frac{f(z)\,dz}{z-z_0-h}-\oint_{\gamma}\frac{f(z)\,dz}{z-z_0}\right] \tag{14}$$

$$=\lim_{h\to 0}\frac{1}{2\pi i h}\oint_{\gamma}\frac{f(z)\left[(z-z_0)-(z-z_0-h)\right]dz}{(z-z_0-h)(z-z_0)} \tag{15}$$

$$=\lim_{h\to 0}\frac{1}{2\pi i h}\oint_{\gamma}\frac{h\,f(z)\,dz}{(z-z_0-h)(z-z_0)} \tag{16}$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^2}. \tag{17}$$

Iterating again,

$$f''(z_0) = \frac{2}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^3}. \tag{18}$$

Continuing the process and adding the contour winding number n,

$$n(\gamma, z_0)\, f^{(r)}(z_0) = \frac{r!}{2\pi i}\oint_\gamma \frac{f(z)\,dz}{(z-z_0)^{r+1}}.$$

Cauchy Integral Theorem

If $f(z)$ is analytic in some simply connected region R, then

$$\oint_\gamma f(z)\,dz = 0 \tag{1}$$

for any closed contour γ completely contained in R. Writing z as

$$z \equiv x + iy \tag{2}$$

and $f(z)$ as

$$f(z) \equiv u + iv \tag{3}$$

then gives

$$\oint_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx + i\,dy) \tag{4}$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \tag{5}$$

From Green's theorem,

$$\int_\gamma f(x,y)\,dx - g(x,y)\,dy = -\iint\left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\right)dx\,dy \tag{6}$$

$$\int_\gamma f(x,y)\,dx + g(x,y)\,dy = \iint\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dx\,dy, \tag{7}$$

so (◇) becomes

$$\oint_\gamma f(z)\,dz = -\iint\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy. \tag{8}$$

But the Cauchy-Riemann equations require that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (9)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad (10)$$

so

$$\oint_\gamma f(z)\,dz = 0,$$

Liouville's theorem

Liouville's theorem from complex analysis states that a holomorphic function f(z)f(z) on the plane that is bounded in magnitude is constant. The usual proof uses the Cauchy integral formula

Assume that f(z) is nonconstant. The fact that)f(z) is holomorphic at every point implies that at any given point, there is a direction such that moving in that direction makes |f(z)| larger. But this doesn't prove that |f(z)| is unbounded, because *a priori* its magnitude could behave like 5−1|z| or some such thing.

In the case of f(z)=1P(z) where P(z) is a polynomial, one knows that |f(z)| tends toward 0 as |z|→∞ so that there's some closed disk such that if |f(z)||f(z)| is bounded, then it has a maximum in the interior of the disk, which contradicts the fact that one can always make f(z) larger by moving in a suitable direction. But for general f(z), one doesn't have this argument.

One can try to reason based on the power series expansion of a holomorphic function f(z) that is not a polynomial. Because polynomials are unbounded as |z|→∞ and grow in magnitude in a way that's proportional to their degree, one might think that a power series, which can be regarded as an infinite degree polynomial, would also be unbounded as |z|→∞. This is of course false: take f(z)=sin(z), then as |z|→∞ along the real axis, f(z) remains bounded. The point is that the dominant term in the partial sums of the power series varies with |z|, and that the relevant coefficients change, alternating in sign and tending toward zero rapidly, so that the gain in size corresponding to moving to the next power of z is counterbalanced by the change in coefficient. But there's *some* direction that one can move in for which f(z) is unbounded: in particular, for f(z)=sin(z), f(z) is unbounded along the imaginary axis.

Taylor's Series

A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about a point $x=a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots . \qquad (1)$$

If $a=0$, the expansion is known as a Maclaurin series.

Taylor's theorem (actually discovered first by Gregory) states that any function satisfying certain conditions can be expressed as a Taylor series.

The Taylor (or more general) series of a function $f(x)$ about a point a up to order n may be found using Series[f, {x, a, n}]. The nth term of a Taylor series of a function f can be computed in the Wolfram Language using SeriesCoefficient[f, {x, a, n}] and is given by the inverse Z-transform

$$a_n = \mathcal{Z}^{-1}\left[\frac{1}{z-a}\right](n). \qquad (2)$$

Taylor series of some common functions include

$$\frac{1}{1-x} = \frac{1}{1-a} + \frac{x-a}{(1-a)^2} + \frac{(x-a)^2}{(1-a)^3} + \ldots \qquad (3)$$

$$\cos x = \cos a - \sin a\,(x-a) - \tfrac{1}{2}\cos a\,(x-a)^2 + \tfrac{1}{6}\sin a\,(x-a)^3 + \ldots \qquad (4)$$

$$e^x = e^a\left[1 + (x-a) + \tfrac{1}{2}(x-a)^2 + \tfrac{1}{6}(x-a)^3 + \ldots\right] \qquad (5)$$

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \ldots \qquad (6)$$

$$\sin x = \sin a + \cos a\,(x-a) - \tfrac{1}{2}\sin a\,(x-a)^2 - \tfrac{1}{6}\cos a\,(x-a)^3 + \ldots \qquad (7)$$

$$\tan x = \tan a + \sec^2 a\,(x-a) + \sec^2 a \tan a\,(x-a)^2 + \sec^2 a\left(\sec^2 a - \tfrac{2}{3}\right)(x-a)^3 + \ldots . \qquad (8)$$

To derive the Taylor series of a function $f(x)$, note that the integral of the $(n+1)$ st derivative $f^{(n+1)}$ of $f(x)$ from the point x_0 to an arbitrary point x is given by

$$\int_{x_0}^{x} f^{(n+1)}(x)\,dx = \left[f^{(n)}(x)\right]_{x_0}^{x} = f^{(n)}(x) - f^{(n)}(x_0), \qquad (9)$$

where $f^{(n)}(x_0)$ is the nth derivative of $f(x)$ evaluated at x_0, and is therefore simply a constant. Now integrate a second time to obtain

$$\int_{x_0}^{x}\left[\int_{x_0}^{x} f^{(n+1)}(x)\,dx\right]dx$$
$$= \int_{x_0}^{x}\left[f^{(n)}(x) - f^{(n)}(x_0)\right]dx \tag{10}$$
$$= \left[f^{(n-1)}(x)\right]_{x_0}^{x} - (x-x_0)\,f^{(n)}(x_0)$$
$$= f^{(n-1)}(x) - f^{(n-1)}(x_0) - (x-x_0)\,f^{(n)}(x_0),$$

where $f^{(k)}(x_0)$ is again a constant. Integrating a third time,

$$\int_{x_0}^{x}\int_{x_0}^{x}\int_{x_0}^{x} f^{(n+1)}(x)\,(dx)^3 = f^{(n-2)}(x) - f^{(n-2)}(x_0)$$
$$-(x-x_0)\,f^{(n-1)}(x_0) - \frac{(x-x_0)^2}{2!}\,f^{(n)}(x_0), \tag{11}$$

and continuing up to $n+1$ integrations then gives

$$\underbrace{\int\cdots\int_{x_0}^{x}}_{n+1} f^{(n+1)}(x)\,(dx)^{n+1} = f(x) - f(x_0) - (x-x_0)\,f'(x_0)$$
$$-\frac{(x-x_0)^2}{2!}\,f''(x_0) - \ldots - \frac{(x-x_0)^n}{n!}\,f^{(n)}(x_0). \tag{12}$$

Rearranging then gives the one-dimensional Taylor series

$$f(x) = f(x_0) + (x-x_0)\,f'(x_0) + \frac{(x-x_0)^2}{2!}\,f''(x_0) + \ldots + \frac{(x-x_0)^n}{n!}\,f^{(n)}(x_0) + R_n \tag{13}$$
$$= \sum_{k=0}^{n}\frac{(x-x_0)^k\,f^{(k)}(x_0)}{k!} + R_n. \tag{14}$$

Here, R_n is a remainder term known as the Lagrange remainder, which is given by

$$R_n = \underbrace{\int\cdots\int_{x_0}^{x}}_{n+1} f^{(n+1)}(x)\,(dx)^{n+1}. \tag{15}$$

Rewriting the repeated integral then gives

$$R_n = \int_{x_0}^{x} f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt. \tag{16}$$

Now, from the mean-value theorem for a function $g(x)$, it must be true that

$$\int_{x_0}^{x} g(x)\,dx = (x-x_0)\,g(x^*) \tag{17}$$

for some $x^* \in [x_0, x]$. Therefore, integrating $n+1$ times gives the result

$$R_n = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x^*) \tag{18}$$

so the maximum error after n terms of the Taylor series is the maximum value of (18) running through all $x^* \in [x_0, x]$. Note that the Lagrange remainder R_n is also sometimes taken to refer to the remainder when terms up to the $(n-1)$st power are taken in the Taylor series

Taylor series can also be defined for functions of a complex variable. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{z'-z} \tag{19}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \tag{20}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}. \tag{21}$$

In the interior of C,

$$\frac{|z-z_0|}{|z'-z_0|} < 1 \tag{22}$$

so, using

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \tag{23}$$

it follows that

$$f(z) = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')\,dz'}{(z'-z_0)^{n+1}} \tag{24}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}. \tag{25}$$

Using the Cauchy integral formula for derivatives,

$$f(z) = \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}. \tag{26}$$

An alternative form of the one-dimensional Taylor series may be obtained by letting

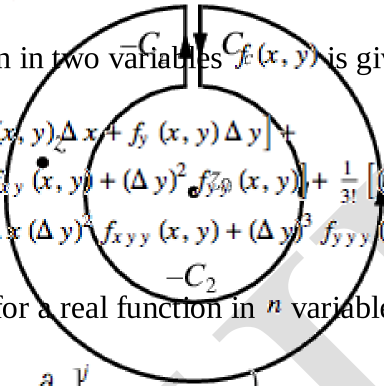
$$x - x_0 \equiv \Delta x \tag{27}$$

so that

$$x \equiv x_0 + \Delta x. \tag{28}$$

Substitute this result into (◇) to give

$$f(x_0 + \Delta x) = f(x_0) + \Delta x\, f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \ldots. \tag{29}$$



A Taylor series of a real function in two variables $f(x, y)$ is given by

$$f(x + \Delta x, y + \Delta y) = f(x, y) + [f_x(x, y)\Delta x + f_y(x, y)\Delta y] + \frac{1}{2!}[(\Delta x)^2 f_{xx}(x, y) + 2\Delta x \Delta y f_{xy}(x, y) + (\Delta y)^2 f_{yy}(x, y)] + \frac{1}{3!}[(\Delta x)^3 f_{xxx}(x, y) + 3(\Delta x)^2 \Delta y f_{xxy}(x, y) + 3\Delta x(\Delta y)^2 f_{xyy}(x, y) + (\Delta y)^3 f_{yyy}(x, y)] + \ldots. \tag{30}$$

This can be further generalized for a real function in n variables,

$$f(x_1, \ldots, x_n) = \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left[\sum_{k=1}^{n}(x_k - a_k)\frac{\partial}{\partial x'_k}\right]^j f(x'_1, \ldots, x'_n)\right\}_{x'_1 = a_1, \ldots, x'_n = a_n}. \tag{31}$$

Rewriting,

$$f(x_1 + a_1, \ldots, x_n + a_n) = \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left(\sum_{k=1}^{n} a_k \frac{\partial}{\partial x'_k}\right)^j f(x'_1, \ldots, x'_n)\right\}_{x'_1 = x_1, \ldots, x'_n = x_n}.$$

Laurent's Series

If $f(z)$ is analytic throughout the annular region between and on the concentric circles K_1 and K_2 centered at $z = a$ and of radii r_1 and $r_2 < r_1$ respectively, then there exists a unique series expansion in terms of positive and negative powers of $(z - a)$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k + \sum_{k=1}^{\infty} b_k (z - a)^{-k}, \tag{1}$$

where

$$a_k = \frac{1}{2\pi i}\oint_{K_1}\frac{f(\zeta)\,d\zeta}{(\zeta - a)^{k+1}} \tag{2}$$

$$b_k = \frac{1}{2\pi i}\oint_{K_2}(\zeta - a)^{k-1} f(\zeta)\,d\zeta \tag{3}$$

Let there be two circular contours C_2 and C_1, with the radius of C_1 larger than that of C_2. Let z_0 be at the center of C_1 and C_2, and z be between C_1 and C_2. Now create a cut line C_c between C_1 and C_2, and integrate around the path $C \equiv C_1 + C_c - C_2 - C_c$, so that the plus and minus contributions of C_c cancel one another, as illustrated above. From the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z')}{z'-z}\,dz' \tag{4}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' + \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_c} \frac{f(z')}{z'-z}\,dz' \tag{5}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{z'-z}\,dz'. \tag{6}$$

Now, since contributions from the cut line in opposite directions cancel out,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' \tag{7}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(\frac{z'-z_0}{z-z_0}-1\right)}\,dz' \tag{8}$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z')}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}\,dz' + \frac{1}{2\pi i}\int_{C_2} \frac{f(z')}{(z-z_0)\left(1-\frac{z'-z_0}{z-z_0}\right)}\,dz'. \tag{9}$$

For the first integral, $|z'-z_0| > |z-z_0|$. For the second, $|z'-z_0| < |z-z_0|$. Now use the Taylor series (valid for $|t| < 1$)

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \tag{10}$$

to obtain

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(z')}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n dz' + \int_{C_2} \frac{f(z')}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n dz'\right] \tag{11}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^{-n-1}\int_{C_2}(z'-z_0)^n f(z')\,dz' \tag{12}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\int_{C_2}(z'-z_0)^{n-1} f(z')\,dz', \tag{13}$$

where the second term has been re-indexed. Re-indexing again,

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n\int_{C_2}\frac{f(z')}{(z'-z_0)^{n+1}}\,dz'. \tag{14}$$

Since the integrands, including the function $f(z)$, are analytic in the annular region defined by C_1 and C_2, the integrals are independent of the path of integration in that region. If we replace paths of integration C_1 and C_2 by a circle C of radius r with $r_1 \le r \le r_2$, then

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' + \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' \quad (15)$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}(z-z_0)^n \int_C \frac{f(z')}{(z'-z_0)^{n+1}}\,dz' \quad (16)$$

$$\equiv \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n. \quad (17)$$

Generally, the path of integration can be any path γ that lies in the annular region and encircles z_0 once in the positive (counterclockwise) direction.

The complex residues a_n are therefore defined by

$$a_n \equiv \frac{1}{2\pi i}\int_\gamma \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'.$$

Cauchy Residue Theorem

An analytic function $f(z)$ whose Laurent series is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad (1)$$

can be integrated term by term using a closed contour γ encircling z_0,

$$\int_\gamma f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz \quad (2)$$

$$= \sum_{n=-\infty}^{-2} a_n \int_\gamma (z-z_0)^n\,dz + a_{-1}\int_\gamma \frac{dz}{z-z_0} + \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz. \quad (3)$$

The Cauchy integral theorem requires that the first and last terms vanish, so we have

$$\int_\gamma f(z)\,dz = a_{-1}\int_\gamma \frac{dz}{z-z_0}, \quad (4)$$

where a_{-1} is the complex residue. Using the contour $z = \gamma(t) = e^{it} + z_0$ gives

$$\int_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\,e^{it}\,dt}{e^{it}} = 2\pi i, \quad (5)$$

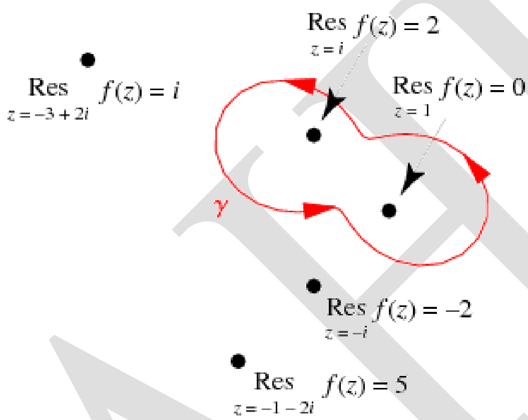
so we have

$$\int_{\gamma} f(z)\,dz = 2\pi i\, a_{-1}. \tag{6}$$

If the contour γ encloses multiple poles, then the theorem gives the general result

$$\int_{\gamma} f(z)\,dz = 2\pi i \sum_{a\in A} \operatorname*{Res}_{z=a_i} f(z), \tag{7}$$

where A is the set of poles contained inside the contour. This amazing theorem therefore says that the value of a contour integral for *any* contour in the complex plane depends *only* on the properties of a few very special points *inside* the contour.



The diagram above shows an example of the residue theorem applied to the illustrated contour γ and the function

$$f(z) = \frac{3}{(z-1)^2} + \frac{2}{z-i} - \frac{2}{z+i} + \frac{i}{z+3-2i} + \frac{5}{z+1+2i}. \tag{8}$$

Only the poles at 1 and i are contained in the contour, which have residues of 0 and 2, respectively. The values of the contour integral is therefore given by

$$\int_{\gamma} f(z)\,dz = 2\pi i\,(0+2) = 4\pi i.$$

Application to evaluation of definite integral

Definite Integrals

We now know how to integrate simple polynomials, but if we want to use this technique to calculate *areas*, we need to know the *limits* of integration. If we specify the limits $x = a$ to $x = b$, we call the integral a *definite integral.*

To solve a definite integral, we first integrate the function as before, then feed in the 2 values of the limits. Subtracting one from the other gives the *area*.

Example

1. What is the area under the curve $y(x) = 2x^2$ between x=1 and x=3? (Note: this is the same problem we did graphically earlier).

Area = $\int_{x=1}^{x=3} 2x^2 . dx$ we write the limits at the top and bottom of the integration sign

= $\left[\frac{2x^3}{3} + k\right]_{x=1}^{x=3}$ we use square brackets to indicate we've calculated the indefinite integral

= (18 + k) - (2/3 + k) feed in the larger value, then the smaller, and subtract the two.

= 18 - 2/3

= <u>17.33 sq. unit</u>

Possible questions (Part B- 6 Marks)

1. State and prove Cauchy Residue theorem.
2. Define and derive Cauchy's integral formula.
3. Derive Cauchy-Riemann equation.
4. Derive and prove Taylor's series.
5. Define and prove Laurent's series
6. Use Cauchy's integral theorem to evaluate

$$\oint_c \frac{dz}{z}.$$

7. Find the Laplace transform of the following functions.

   (i) Sin^2t, (ii) Cos^2t, (iii) $e^{at}\cos\omega t$ and (iv) $e^{at}\sin\omega t$.

8. Explain the complex form of Fourier series
9. State and Explain Dirichlet conditions.
10. Define Laplace Transform. Explain the linearity and change in scale property of Laplace transform.

Possible questions (Part B- 10 Marks)

1. State and prove Cauchy Residue theorem. Explain how it is extended for the case of an isolated first order pole lying on the contour of integration. Using this theorem. show that

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin\pi a} \text{ where } 0 < a < 1.$$

2. Derive Cauchy-Riemann equation and deduce the same in polar form.
3. Derive and prove Taylor's series.
4. Use Cauchy's integral theorem to evaluate

$$\oint_c \frac{dz}{z}.$$

5. Find the Laplace transform of the following functions.

   (i) Sin^2t, (ii) Cos^2t, (iii) $e^{at}\cos\omega t$ and (iv) $e^{at}\sin\omega t$.

8. Explain the complex form of Fourier series
9. Explain how the Dirichlet conditions used to find the functions in physics.

KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21
DEPARTMENT OF PHYSICS
CLASS:I MSc PHYSICS BATCH:2019
MATHEMATICAL PHYSICS (19PHP104)
MULIPLE CHOICE QUESTIONS

Questions	opt1	opt2
The function has	a simple pole at z = 1	a simple pole at z = -1
If a given number is wholly real, it is	a real axis	imaginary axis
A set which entirely consists of inter	an open set	a closed set
If a contour is a unit circle around the	1	0
A connected open set is called	an open set	a closed set
Which is the analytic function of com	|Z|	Re Z
Which is the analytic function of com	| Z |	Sin Z
Which is the analytic function of com	|Z|	e^{sinz}
Which is not the analytic function of	z^{-1}	Z
Which is not the analytic function of	Z^{-1}	$e^{Sin Z}$
Which is not the analytic function of	Z^{-1}	log Z
The function ___________ has	a simple pole at Z= a	a simple pole at Z=_ a
The symbol i with the property i^2=1	Euler	Gauss
arg (Z_1 / Z_2) is equal to	arg Z_1+ arg Z_2	arg Z_1 - arg Z_2
A single valued function f(z) which is	irregular function	analytic function
The function ____________ is analyt	at all points y = x	at all points, except z = 1
In order that the function f(z) = $\|Z\|^2$ /	2	–1
Any function which satisfies the Lapl	harmonic function	analytic function
The value of $\int_C \frac{dz}{z+2}$, C : |Z|	2pi	-2pi
If f(z) is analytic in a closed curve 'C	2pi	2p
The conjugate of 1/1+i is	1–i	1-i/√ 2
The conjugate of (1+i) (3+4i) is	1+7i	1-7i
The Conjugate of 1/i is	–i	i
The value of $i^2 + i^3 + i^4$ is	i	–i
If Z= a+ib, then real part of Z^{-1} is ----	a/ a^2+b^2	–b/ a^2+b^2
If Z= a+ib, then Im(Z^{-1}) is ----	b/ a^2+b^2	b/√ a^2+b^2
The modulus and argument of √ 3 - i	2, ∏/6	2, -∏/6
If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and Z_2= r	$\theta_1 + \theta_2$	$\theta_1 - \theta_2$
The argument of -1 + I is ---	- ∏/4	3∏/4
$(1+ e^{-i\theta}) / (1+ e^{i\theta})$ = -------	cos θ + isin θ	sin θ -icos θ
If X = cos θ + isin θ then the value o	2 cos nθ	2i sin nθ
The value of $(\cos\theta + i\sin\theta)^{-1}$ is -----	cos θ - sin θ	sin θ - icos θ
$(\sin ∏/3+ i \cos ∏/3)^3$ is equal to	-1	1
$(\cos ∏/4+ i \sin ∏/4)^4$ is ------	1/√2 + 1i/√2	b)1
In the Argand diagram, the fourth roo	Straight line	circle

The sum of n^{th} roots of unity are -----	0	1
If $z_1 = 2 + ı$, $z_2 = 1 + 3ı$, then ı Re (z_1	1	i
Polar form of a complex number is	r (tanθ + ıcotθ)	r(secθ + ıcosecθ)
$\lvert z_1 + z_2 \rvert$ =	$\lvert Z_1\rvert + \lvert Z_2$	$\leq\lvert Z_1\rvert + \lvert Z_2\rvert$
The exponential form of a complex n	$z = re^{iq}$	$z = e^{iq}$

opt3	opt4	Answer
a pole at z = 1 of order 2	a simple pole at z =	a simple pole at z = 1 of order 3
x-y plane	space	x-y plane
a banded set	domain	an open set
e^{iq}	e^{iq}	1
a banded set	domain	an open set
Z^{-1}	Log Z	Z^{-1}
Log z	Re Z	Sin Z
log Z	Re Z	e^{sinz}
e^{sinz}	Sin Z	Z
Re Z	SinZ	Re Z
$e^{Sin Z}$	SinZ	log Z
a pole at z=a of order 2	a pole at z=a of ord	a pole at z=a of order 3
Cauchy	Reimann	Euler
real	imaginary	arg Z_1 - arg Z_2
periodic function	all the above	analytic function
at all points, except z = -	at all points, except	at all points, except z = ± 1
0	1	0
periodic function	conjugate function	harmonic function
4pi	0	0
p	ip	2pi
1-i/2	1+i	1-i/√ 2
7-i	-1-7i	-1-7i
1	-1	–i
1	0	–i
a/ $\sqrt{a^2+b^2}$	–b/$\sqrt{a^2+b^2}$	a/ a^2+b^2
-b/ a^2+b^2	–b/$\sqrt{a^2+b^2}$	-b/ a^2+b^2
4, ∏/3	4, -∏/3	2, -∏/6
$\theta_1 \ \theta_2$	θ_1 / θ_2	$\theta_1 + \theta_2$
∏/4	∏/2	3∏/4
cos θ - isin θ	sin θ + icos θ	cos θ - isin θ
2 sin nθ	2i cos nθ	2 cos nθ
cos θ + sin θ	sin θ/2 + icos θ/2	sin θ - icos θ
–i	i	i
-1	i	-1
rectangle	square	square

2	3	0
2i	2	1
r(cosθ + ιsinθ)	r (sinθ + ιcosθ)	r (sinθ + ιcosθ)
$\leq Z_1 + Z_2$	$> Z_1 + Z_2$	$\leq \lvert Z_1 \rvert + \lvert Z_2 \rvert$
z = cos q / r	z = r / cos q	z = reiq

UNIT-III

SYLLABUS

Fourier Transform – Properties of Fourier transform – Fourier transform of derivatives – Fourier sine and cosine transforms of derivatives – Fourier transform of functions of two or three variables – Finite Fourier transforms – Simple Applications of FT Laplace transform – Properties of Laplace transforms – Laplace Transform of derivative of a function – Laplace transform of integral – Laplace transform of periodic functions - Inverse Laplace Transform – Fourier Mellin Theorem - Properties of inverse Laplace Transform – Convolution theorem – Evaluation of Laplace Transform using Convolution theorem.

Fourier Transform

The Fourier transform is a generalization of the complex Fourier series in the limit as $L \to \infty$. Replace the discrete A_n with the continuous $F(k)\,dk$ while letting $n/L \to k$. Then change the sum to an integral, and the equations become

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \quad (1)$$

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx. \quad (2)$$

Here,

$$F(k) = \mathcal{F}_x[f(x)](k) \quad (3)$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \quad (4)$$

is called the *forward* ($-i$) Fourier transform, and

$$f(x) = \mathcal{F}_k^{-1}[F(k)](x) \quad (5)$$

$$= \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \quad (6)$$

is called the *inverse* ($+i$) Fourier transform. The notation $\mathcal{F}_x[f(x)](k)$ is introduced and $\hat{f}(k)$ and $\check{f}(x)$ are sometimes also used to denote the Fourier transform and inverse Fourier transform, respectively.

Properties of Fourier Transform

The properties of the Fourier transform are summarized below. The properties of the Fourier expansion of periodic functions discussed above are special cases of those listed here. In the following,

Linearity

$$\mathcal{F}[ax(t) + by(t)] = a\mathcal{F}[x(t)] + b\mathcal{F}[y(t)]$$

Time shift

$$\mathcal{F}[x(t \pm t_0)] = X(j\omega)e^{\pm j\omega t_0}$$

$$\mathcal{F}[x(t \pm t_0)] \quad : \quad \int_{-\infty}^{\infty} x(t \pm t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t')e^{-j\omega(t' \mp t_0)}dt'$$

$$: \quad e^{\pm j\omega t_0}\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt' = X(j\omega)e^{\pm j\omega t_0}$$

Frequency shift

$$\mathcal{F}^{-1}[X(j\omega \pm \omega_0)] = x(t)e^{\mp j\omega_0 t}$$

$$\mathcal{F}^{-1}[X(j(\omega \pm \omega_0))] \quad : \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega \pm \omega_0))e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')e^{j\omega(\omega' \mp \omega_0)}d\omega'$$

$$: \quad e^{\mp j\omega_0 t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega')e^{j\omega'}d\omega' = x(t)e^{\mp j\omega_0 t}$$

Time reversal

$$\mathcal{F}[x(-t)] = X(-\omega)$$

Proof:

$$\mathcal{F}[x(-t)] = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt$$

Replacing t by , $-t'$ we get

$$\mathcal{F}[x(-t)] = -\int_{\infty}^{-\infty} x(t')e^{j\omega t'}dt' = \int_{-\infty}^{\infty} x(t')e^{j\omega t'}dt' = X(-\omega)$$

and

$$\text{if } x(t) = -x(-t) \quad \text{then} \quad X(j\omega) = -X(-j\omega)$$

Fourier Sine and cosine transform of derivative

The Fourier cosine transform of a real function is the real part of the full complex Fourier transform,

$$\mathcal{F}_x^{(c)}[f(x)](k) = \Re[\mathcal{F}_x[f(x)](k)] \tag{1}$$

$$= \int_{-\infty}^{\infty} \cos(2\pi k x) f(x)\, dx. \qquad (2)$$

The Fourier cosine transform $F_c(k)$ of a function $f(x)$ is implemented as Fourier Cosine Transform[f, x, k], and different choices of a and b can be used by passing the optional Fourier Parameters -> $\{a, b\}$ option. In this work, $a = 0$ and $b = -2\pi$.

Derivative

$$\mathcal{F}c_t\left[f^{(2n)}(t)\right](z) = (-1)^n z^{2n} \mathcal{F}c_t[f(t)](z) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n-1} (-1)^k z^{2k} f^{(2n-2k-1)}(0) \,/;$$
$$\lim_{t\to\infty} f^{(k)}(t) = 0 \bigwedge 0 \le k \le 2n-1 \bigwedge n \in \mathbb{N}^+$$

This formula shows that the Fourier cosine transform of an even-order derivative gives the product of the power function with the Fourier cosine transform plus some even polynomial.

$$\mathcal{F}c_t\left[f^{(2n+1)}(t)\right](z) = (-1)^n z^{2n+1} \mathcal{F}s_t[f(t)](z) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n-1} (-1)^k z^{2k} f^{(2n-2k)}(0) \,/;\, \lim_{t\to\infty} f^{(k)}(t) = 0 \bigwedge 0 \le k \le 2n \bigwedge n \in \mathbb{N}$$

This formula shows that the Fourier cosine transform of an odd-order derivative gives the product of a power function with the Fourier sine transform plus some even polynomial.

The Finite Fourier Transforms

When solving a PDE on a finite interval $0 < x < L$, whether it be the heat equation or wave equation, it can be very helpful to use a *finite Fourier transform*. In particular, we have the *finite sine transform*

$$S_n = S[f] = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx, \quad n = 1, 2, \ldots,$$

with its *inverse sine transform*

$$S^{-1}[S_n] = f(x) = \sum_{n=1}^{\infty} S_n \sin(n\pi x/L).$$

This transform should be used with *Dirichlet* boundary conditions, that specify the value of u at $x = 0$ and $x = L$.

When *Neumann* boundary conditions are used, that specify the value of u_x at $x = 0$ and $x = L$, it is best to use the *finite cosine transform*

$$C_n = C[f] = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx, \quad n = 0, 1, 2, \ldots,$$

with its *inverse sine transform*

$$C^{-1}[C_n] = f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\pi x/L).$$

Both of these transforms can be used to reduce a PDE to an ODE.

Examples of the Sine Transform

Consider the function $f(x) = 1$ on $(0, 1)$. If we apply the finite sine transform to this function, we obtain

$$S_n = 2\int_0^1 \sin(n\pi x)\,dx$$

Solving Problems via Finite Transforms

We illustrate the use of finite Fourier transforms by solving the IBVP

$$u_{tt} = u_{xx} + \sin(\pi x), \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = 1, \quad u_t(x,0) = 0, \quad 0 < x < 1.$$

Because this problem has Dirichlet boundary conditions, we use the finite sine transform. Fron the preceding example, the transform of the initial conditions are

$$S_n(0) = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \quad S_n'(0) = 0.$$

Using the definition and aforementioned properties, we obtain the transform of the PDE,

$$\begin{aligned} S_1''(t) &= -\pi^2 S_1(t) + 1, \\ S_n''(t) &= -(n\pi)^2 S_n(t), \quad n = 2, 3, \ldots. \end{aligned}$$

The ODE for $S_1(t)$ is nonhomogeneous, and can be solved using either the method of undetermine coefficients or variation of parameters. The general solution is

$$S_1(t) = A\cos(\pi t) + B\sin(pit) + C,$$

where A, B and C are constants. Substituting this form of the solution into the ODE and initia conditions yields

$$S_1(t) = \left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t) + \frac{1}{\pi^2}.$$

The ODEs for $S_n(t)$, $n > 1$, are homogeneous and can easily be solved to obtain

$$S_n(t) = \begin{cases} \dfrac{4}{n\pi}\cos(n\pi t) & n = 3, 5, 7, \ldots, \\ 0 & n = 2, 4, 6, \ldots \end{cases}.$$

Applying the inverse sine transform, we conclude that the solution is

$$u(x,t) = \left[\left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t) + \frac{1}{\pi^2}\right]\sin(\pi x) + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos[(2n+1)\pi t]\sin[(2n+1)\pi x].$$

Laplace Transform

The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

The (unilateral) Laplace transform $\mathcal{L}$ (not to be confused with the Lie derivative, also commonly denoted $\mathcal{L}$) is defined by

$$\mathcal{L}_t [f(t)](s) \equiv \int_0^\infty f(t) e^{-st} dt, \qquad (1)$$

where $f(t)$ is defined for $t \geq 0$ (Abramowitz and Stegun 1972). The unilateral Laplace transform is almost always what is meant by "the" Laplace transform, although a bilateral Laplace transform is sometimes also defined as

$$\mathcal{L}_t^{(2)} [f(t)](s) = \int_{-\infty}^\infty f(t) e^{-st} dt$$

Properties of Laplace transform

The properties of Laplace transform are:

Linearity Property

If x(t)⟷L.T X(s)x(t)⟷L.TX(s)

& y(t)⟷L.T Y(s)y(t)⟷L.TY(s)

Then linearity property states that

ax(t)+by(t)⟷L.T aX(s)+bY(s)ax(t)+by(t)⟷L.T aX(s)+bY(s)

Time Shifting Property

If x(t)⟷L.T X(s)x(t)⟷L.TX(s)

Then time shifting property states that

x(t−t0)⟷L.T e−st0X(s)x(t−t0)⟷L.Te−st0X(s)

Frequency Shifting Property

If x(t)⟷L.T X(s)x(t)⟷L.T X(s)

Then frequency shifting property states that

es0t.x(t)⟷L.T X(s−s0)es0t.x(t)⟷L.T X(s−s0)

Time Reversal Property

If x(t)⟷L.TX(s)x(t)⟷L.TX(s)

Then time reversal property states that

x(−t)⟷L.TX(−s)x(−t)⟷L.TX(−s)

Time Scaling Property

If x(t)⟷L.TX(s)x(t)⟷L.TX(s)

Then time scaling property states that

x(at)⟷L.T1|a|X(sa)

Laplace Transform of periodic function

Theorem 1. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is a periodic function of period $T > 0$, i.e. $f(t+T) = f(t)$ for all $t \geq 0$. If the Laplace transform of f exists, then*

$$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}. \tag{1}$$

Proof: We have

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt \\ &= \sum_{n=0}^{\infty} \int_0^T f(u+nT)e^{-su-snT}\,du \qquad u = t - nT \\ &= \sum_{n=0}^{\infty} e^{-snT} \int_0^T f(u)e^{-su}\,du \\ &= \left(\int_0^T f(u)e^{-su}\,du\right) \sum_{n=0}^{\infty} e^{-snT} \\ &= \frac{\int_0^T f(u)e^{-su}\,du}{1 - e^{-sT}}. \end{aligned}$$

The last line follows from the fact that

$$\sum_{n=0}^{\infty} e^{-snT}$$

is a geometric series with common ration $e^{-sT} < 1$ for $s > 0$.

Convolution theorem

Suppose we know that a Laplace transform $H(s)$ can be written as $H(s) = F(s)G(s)$, where $\mathcal{L}\Big(f(t)\Big) = F(s)$ and $\mathcal{L}\Big(g(t)\Big) = G(s)$. We need to know the relation of $h(t) = \mathcal{L}^{-1}\Big(H(s)\Big)$ to $f(t)$ and $g(t)$.

Definition 1. (**Convolution**) *Let f and g be two functions defined in $[0,\infty)$. Then the convolution of f and g, denoted by $f * g$, is defined by*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{2}$$

Note: It can be shown (easily) that $f * g = g * f$. Hence,

$$(f * g)(t) = \int_0^t g(\tau)f(t-\tau)\,d\tau \tag{3}$$

We use either (2) or (3) depending on which is easier to evaluate.

Theorem 2. (**Convolution theorem**) *The convolution $f * g$ has the Laplace transform property*

$$\mathcal{L}\Big((f * g)(t)\Big) = F(s)G(s). \tag{4}$$

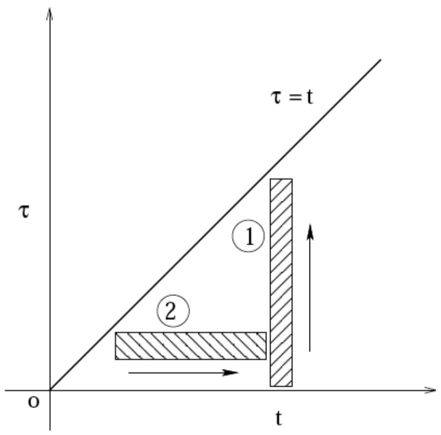
OR conversely

$$\mathcal{L}^{-1}\Big(F(s)G(s)\Big) = (f * g)(t)$$

Proof: Using definition, we find

$$\begin{aligned}\mathcal{L}\Big((f * g)(t)\Big) &= \int_0^\infty (f * g)(t)e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)e^{-st}\,dt\end{aligned}$$

The region of integration is the area in the first quadrant bounded by the t-axis and



the line $\tau = t$. The variable limit of integration is applied on τ which varies from $\tau = 0$ to $\tau = t$.

Let us change the order of integration, thus apply variable limit on t. Then t would vary from $t = \tau$ to $t = \infty$ and τ would vary from $\tau = 0$ to $\tau = \infty$. Hence, we have

$$\begin{aligned}\mathcal{L}\Big((f*g)(t)\Big) &= \int_0^\infty \left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right) f(\tau)\,d\tau \\ &= \int_0^\infty \left(\int_0^\infty e^{-su} g(u)\,du\right) f(\tau)e^{-s\tau}\,d\tau, \qquad t-\tau = u \\ &= \left(\int_0^\infty e^{-su} g(u)\,du\right)\left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right) \\ &= F(s)G(s)\end{aligned}$$

Evaluation of Laplace Transform using Convolution theorem

find inverse Laplace transform of $1/s(s+1)^2$.

Solution: We write $H(s) = F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s+1)^2$. Thus $f(t) = 1$ and $g(t) = te^{-t}$. Hence, using convolution theorem, we find

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t \tau e^{-\tau}\,d\tau = 1 - (t+1)e^{-t}.$$

Note: We have used $f(t-\tau)g(\tau)$ in the convolution formula since $f(t) = 1$. This helps a little bit in the evaluation of the integration.

Laplace transform of integral

The Laplace transform satisfied a number of useful properties. Consider exponentiation. If $\mathcal{L}_t[f(t)](s) = F(s)$ for $s > \alpha$ (i.e., $F(s)$ is the Laplace transform of f), then $\mathcal{L}_t\left[e^{at} f\right](s) = F(s-a)$ for $s > a + \alpha$. This follows from

$$\begin{aligned}F(s-a) &= \int_0^\infty f\, e^{-(s-a)t}\,dt \\ &= \int_0^\infty \left[f(t)\, e^{at}\right] e^{-st}\,dt \\ &= \mathcal{L}_t\left[e^{at} f(t)\right](s).\end{aligned}$$

The Laplace transform also has nice properties when applied to integrals of functions. If $f(t)$ is piecewise continuous.

$$\mathcal{L}_t\left[\int_0^t f(t')\,dt'\right] = \frac{1}{s}\mathcal{L}_t[f(t)](s).$$

Fourier Mellin theorem

Mellin's transformation is closely related to an extended form of Laplace's. The change of variables defined by:

$$t = e^{-x}, \quad dt = -e^{-x}\,dx$$

transforms the integral (11.1) into:

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})e^{-sx}\,dx$$

After the change of function:

$$g(x) \equiv f(e^{-x})$$

one recognizes in (11.13) the *two-sided* Laplace transform of g usually defined by:

$$\mathfrak{L}[g;s] = \int_{-\infty}^{\infty} g(x)\,e^{-sx}\,dx$$

This can be written symbolically as:

$$\mathcal{M}[f(t);s] = \mathfrak{L}[f(e^{-x});s]$$

The occurrence of a strip of holomorphy for Mellin's transform can be deduced directly from this relation. The usual right-sided Laplace transform is analytic in a half-plane $Re(s) > \sigma_1$. In the same way, one can define a left-sided Laplace transform analytic in the region $Re(s) < \sigma_2$. If the two half-planes overlap, the region of holomorphy of the two-sided transform is thus the strip $\sigma_1 < Re(s) < \sigma_2$ obtained as their intersection.

To obtain Fourier's transform, write now $s = a + 2\pi j\beta$ in (11.13):

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\,e^{-ax}\,e^{-j2\pi\beta x}\,dx$$

The result is

$$\mathcal{M}[f(t);a + j2\pi\beta] = \mathfrak{F}[f(e^{-x})\,e^{-ax};\beta]$$

where $\mathfrak{F}$ represents the Fourier transformation defined by:

$$\mathfrak{F}[f;\beta] = \int_{-\infty}^{\infty} f(x)\,e^{-j2\pi\beta x}\,dx$$

Thus, for a given value of $Re(s) = a$ belonging to the definition strip, the Mellin transform of a function can be expressed as a Fourier transform.

Possible questions (Part B- 6 marks)

1. Define Laplace Transform and explain their property.
2. Explain the linearity and change in scale property of Laplace transform.
3. State and explain shifting property of Fourier Transform.
4. Find the Fourier transforms of the following functions, and in each case draw graphs for the function and its transform

   f(x) = 1; │x│ │< a

   f(x) = 0; │x│ │> a
5. Define Inverse Laplace Transform. Find the inverse Laplace transform of $\frac{1-S}{(S+1)(S^2+4S+13)}$
6. Discuss about the change of interval from (-π,π) to (-l, l) in Fourier expansion.
7. Derive any two properties of Fourier transform.
8. State and prove Cauchy's Integral theorem.

   Use Cauchy's integral theorem to evaluate

$$\oint_c \frac{dz}{z}.$$

9. Explain the Taylor's Series with proof.

Possible questions (Part C- 10 marks)

1. Derive four properties of Laplace Transform
2. State and explain shifting property of Fourier Transform.
3. Find the Fourier transforms of the following functions, and in each case draw graphs for the function and its transform

   f(x) = x; │x│ │< a

   f(x) = x^2; │x│ │> a
4. Define Inverse Laplace Transform. Find the inverse Laplace transform of $\frac{1-S}{(S+1)(S^3+4S+13)}$
5. Discuss about the change of interval from (-π,π) to (-k, k) in Fourier expansion.
6. Derive all properties of Fourier transform.
7. Explain the Taylor's Series with proof.

KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21

DEPARTMENT OF PHYSICS

CLASS:I MSc PHYSICS **BATCH:2**

MATHEMATICAL PHYSICS (19PHP104)

MULIPLE CHOICE QUESTIONS

Questions	**opt1**	**opt2**	**opt3**
UNIT III			
Which of the following functions has the period 2p?	cos nx	sin nx	tan nx
If f(x) = -x for -p< x£ 0then its Fourier coefficient a_0 is	π	p/4	p/3
Which of the following is an odd function?	sin x	cos x	x^2
Which of the following is an even function?	x^3	cos x	sin x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	b)f(x) = - f(x)	f(-x) = - f(x)
The function f(x) is said to be an even function of x if	f(-x) = f(x)	b)f(x) = - f(x)	f(-x) = - f(x)
If a periodic function f(x) is odd, it's Fourier expansion contains no ------ terms.	coefficient a_n	sine	coefficient a_0
If a periodic function f(x) is even, it's Fourier expansion contains no ------ terms.	cosine	sine	coefficient a_0
In Fourier series, the function f(x) has only a finite number of maxima and minima. This condition is known as -------	Dirichlet	Kuhn Tucker	Laplace
In dirichlet condition, the function f(x) has only a finite number of finite dis continuities and no ------- discontinuities	semi finite	continuous	infinite
If f(x) is even, then it's Fourier co- efficient -------- is zero.	a_0	a_n	b_n

If the periodic function f(x) is odd, then it's Fourier co-efficient -------- is zero.	a_0	a_n	b_n
The period of cos nx where n is the positive integer is	π/n	π/2n	2π
The Fourier co efficient a_0 in f(x) = x for 0< x£ π is	π	π/2	2π
If the function f (x) = -π in the interval -π x< 0, the coefficient a_0 is	$\pi^2/3$	$2\pi^2/3$	2π/3
If the function f(x) = x sin x, the Fourier coefficient	bn = 0	$a_0 = 1$	$a_0 = \pi^2/3$
For the cosine series, which of the Fourier coefficient variables will be vanish?	a_n	b_n	a_0
For a function f(x) = x^3, the Fourier coefficient	$b_n = 0$	$a_n = 0$	$a_0 = 0$
The function x sin x be a ------- function.	even	odd	continuou s
The function x cos x be a ------- function.	even	odd	continuou s
Lt F(s) = ------ s® ¥	0	1	¥
The Laplace transform of f(t) is denoted by	L { F(s) }	L { f (t) }	L { F(t) }
L (e^{-at}) = ---	1/s+a	1/s-a	1/s * a
L (cos h at) = ----	$a/s^2 - a^2$	$s/s^2 * a^2$	$s/s^2 - a^2$

L (sinh at) = ----	$a/s^2 - a^2$	$s/s^2 - a^2$	$a/s^2 + a^2$
L (cosat) = ----	$s/s^2 - a^2$	$a/s^2 + a^2$	$a/s^2 - a^2$
L (sinat) = ----	$s/s^2 + a^2$	$a/s^2 + a^2$	$a/s^2 - a^2$
L (t^n) = ----	é(n+1)/s^{n+1}	é(n-1)/s^{n+1}	é(n+1)/s^{n-1}
é(n+1) = ----	(n-1)!	n!	(n+1)!
L(1) = ----	1	s	1/s
L(t) = -----	1/s	$1/s^2$	t
L(t^2) = -----	$2/s^3$	$1/t^2$	$2/t^3$
é1/2 = ---	ÖP/2	ÖP/4	ÖP
L (e^{at}) = ----	1/s+a	1/s-a	1/s*a
L (t sinat) = -----	2as/ (s^2-a^2)	2as/ ($s^2$$a^2$)	2as/ (s^2+a^2)
L (tcos at) = -----	s^2-a^2/(s^2+a^2)^2	s^2 +a^2/(s^2+a^2)^2	s^2- a^2/(s^2+a^2)^2
If L^{-1}\{1/$(s+a)^2$ \}= ------	t e^{at}	t e^{-at}	$e^{a t}$

$L^{-1}(1/(s^2+4))$ is equal to	e^{-4t}	cos2t/2	sin2t/2
$L^{-1}(1/s)$ = --------	1	0	t
$L^{-1}[1/(s+a)]$ = ----------	e^{st}	e^{at}	e^{-st}
The function x sin x be a ------- function.	even	odd	continuous
The function x cos x be a ------- function.	even	odd	continuous
Which of the following is an odd function?	sin x	cos x	x^2
Which of the following is an even function	x^3	cos x	sin x
The function f(x) is said to be an odd function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
The function f(x) is said to be an EVEN function of x if	f(-x) = f(x)	f(x) = - f(x)	f(-x) = - f(x)
If a periodic function f(x) is odd, Fourier expansion contains no ------ terms	cosine	sine	coefficient a_0
If a periodic function f(x) is even, Fourier expansion contains no ------ terms	cosine	sine	coefficient a_0
In Fourier series, the function f(x) has only a finite number of maxima and minima	Dirichlet	Kuhn Tucker	Laplace
In dirichlet condition, the function f(x) has no -------- discontinuities	semi finite	continuous	infinite

If f(x) is odd, then it's Fourier co- efficient -------- is zero. a_0 a_n b_n

opt4	Answer
tan x	sin nx
p/2	p/2
$\sin^2 x$	sin x
tan X	cos x
None	f(-x) = - f(x)
None	f(-x) = f(x)
cosine	sine
coefficient a_n	cosine
None	Dirichlet
finite	infinite
none	b_n

none	a_n
$n\pi$	2π
0	$\pi/2$
$(- \pi/2)$	$(- \pi/2)$
$a_0 = -1$	bn = 0
Both a_0 and a_n	b_n
$a_n = b_n = 0$	$a_n = 0$
None	even
None	odd
None	0
L { f(s) }	L { f (t) }
1/s	1/s+a
$a/s^2 + a^2$	$s/s^2 - a^2$

$s/s^2 + a^2$	$a/s^2 - a^2$
$s/s^2 + a^2$	$s/s^2 + a^2$
$s/s^2 - a^2$	$a/s^2 + a^2$
None	é$(n+1)/s^{n+1}$
None	$(n+1)!$
0	$1/s$
$1/t^2$	$1/s^2$
$1/s^2$	$2/s^3$
ÖP/8	ÖP
None	$1/s-a$
None	$2as/ (s^2+a^2)$
None	$s^2-a^2/(s^2+a^2)^2$
None	$t\ e^{-at}$

e^{4t}	sin2t/2
none.	1
e^{-at}	e^{st}
None	even
None	odd
$\sin^2 x$	sin x
$\sin^2 x$	cos x
3	f(-x) = - f(x)
1	f(-x) = f(x)
coefficient a_n	sine
coefficient a_n	cosine
None	Dirichlet
finite	infinite

none	a_n

UNIT-IV

SYLLABUS

Fourier series – Dirichlet's theorem – change of interval – complex form – Fourier series in the interval (0, T) – Uses of Fourier series - Legendre's polynomials and functions – Differential equations and solutions – Rodrigues formula – Orthogonality – relation between Legendre polynomial and their derivatives – recurrence relations – Lagurae Polynomials – recurrence relations

Fourier series

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Examples of successive approximations to common functions using Fourier series are illustrated above.

In particular, since the superposition principle holds for solutions of a linear homogeneous ordinary differential equation, if such an equation can be solved in the case of a single sinusoid, the solution for an arbitrary function is immediately available by expressing the original function as a Fourier series and then plugging in the solution for each sinusoidal component. In some special cases where the Fourier series can be summed in closed form, this technique can even yield analytic solutions.

Any set of functions that form a complete orthogonal system have a corresponding generalized Fourier series analogous to the Fourier series. For example, using orthogonality of the roots of a Bessel function of the first kind gives a so-called Fourier-Bessel series.

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(m x) \sin(n x)\, d x = \pi \delta_{m n} \tag{1}$$

$$\int_{-\pi}^{\pi} \cos(m x) \cos(n x)\, d x = \pi \delta_{m n} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(m x) \cos(n x)\, d x = 0 \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(m x)\, d x = 0 \tag{4}$$

$$\int_{-\pi}^{\pi} \cos(m x)\, d x = 0 \tag{5}$$

for $m, n \neq 0$, where $\delta_{m n}$ is the Kronecker delta.

Using the method for a generalized Fourier series, the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = \cos x$ and $f_2(x) = \sin x$. Since these functions form a complete orthogonal system over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x), \tag{6}$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, d x \tag{7}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n x)\,dx \quad (8)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n x)\,dx \quad (9)$$

and $n = 1$, 2, 3, ... Note that the coefficient of the constant term a_0 has been written in a special form compared to the general form for a generalized Fourier series in order to preserve symmetry with the definitions of a_n and b_n.

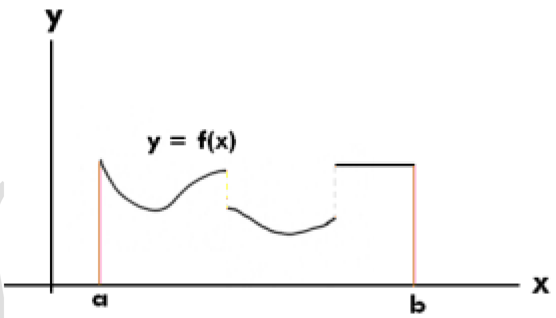
Dirichlet conditions

A piecewise regular function that

1. Has a finite number of finite discontinuities and

2. Has a finite number of extrema

can be expanded in a Fourier series which converges to the function at continuous points and the mean of the positive and negative limits at points of discontinuity.



Def. Sectionally continuous (or piecewise continuous) function. A function $f(x)$ is said to be **sectionally continuous** (or **piecewise continuous**) on an interval a $\leq$ x $\leq$ b if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits. See Figure The requirement that a function be sectionally continuous on some interval [a, b] is equivalent to the requirement that it meet the**Dirichlet conditions** on the interval.

Fourier series. Let $f(x)$ be a sectionally continuous function defined on an interval c < x < c + 2L. It can then be represented by the **Fourier series**

1) $$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + a_3 \cos\frac{3\pi x}{L} + \ldots + b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + b_3 \sin\frac{3\pi x}{L} + \ldots$$

Where

2) $$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$
$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

At a point of discontinuity *f* (x) is given a value equal to its mean value at the discontinuity i.e. if x = a is a point of discontinuity, *f* (x) is given the value

$$f(x) = \frac{\lim_{x\to a_-} f(x) + \lim_{x\to a_+} f(x)}{2}$$

Complex form of Fouries series

We show how a Fourier series can be expressed more concisely if we introduce the complex number i where $i^2 = -1$. By utilising the Euler relation:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

We can replace the trigonometric functions by complex exponential functions. By also combining the

Fourier coefficients a_n and b_n into a complex coefficient c_n through

$C_n = (a_n - ib_n)$

For a given periodic signal, both sets of constants can be found in one operation. We also obtain Parseval's theorem which has important applications in electrical engineering. The complex formulation of a Fourier series is an important precursor of the Fourier transforms which attempts to Fourier analyse non-periodic functions.

So far we have discussed the trigonometric form of a Fourier series i.e. we have represented functions of period T in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{2n\pi t}{T}\right) + b_n\sin\left(\frac{2n\pi t}{T}\right)\right\}$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

We obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals.

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_0 t\ dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_0 t\ dt \qquad n = 1, 2, \ldots$$

An alternative, more concise form, of a Fourier series is available using complex quantities. This form is quite widely used by engineers, for example in Circuit Theory and Control Theory, and leads naturally into the Fourier Transform which is the subject of

Fourier series in the interval (0, T)

We assume that the function f(x) is piecewise continuous on the interval [0,T]. Using the substitution x= Lyπ (−π≤x≤π), we can transform it into the function

F(y)=f(Ly/π)

which is defined and integrable on [−π,π]. Fourier series expansion of this function F(y) can be written as

F(y)=f(Ly/π)=a0/2+ ∑ ($a_n$cosny+$b_n$sinny).

Uses of Fourier series

Fourier series and frequencies

The basic idea of Fourier series is that we try to express the given function as a combination of oscillations, starting with one whose frequency is given by the given function (either its periodicity or the length of the bounded interval on which it is given) and then taking multiples of this frequency, that is, using fractional periods. When we look at coefficients of the resulting "infinite linear combination", we can expect that if some of them are markedly larger then the rest, then this frequency plays an important role in the phenomenon described by the given function. This detection of hidden periodicity can be very useful in analysis, since not every periodicity can be readily seen by looking at a function. In particular, this is true if there are several periods that interact.

Imagine that a function *f* describes temperatures at time *t* over many many years. There is one period that should be easily visible, namely seasonal changes with period one year. We also expect another period going over this basic yearly period, namely 1-day period of cold nights and warm days. Now the interesting question is whether there are also other periods. This is very useful to know, since such knowledge would tell us something important about what is happening with weather and climate. Frequency analysis offers a useful tool for such an investigation, looking over long data sequences it may point out cold ages and other long term changes in climate.

There are areas where decomposition into waves comes naturally, for instance storage of sound. When we are given a sound sample, Fourier transform allows us to decompose it into basic waves and store it in this way. Apart from data compression we also get further memory savings by simply ignoring coefficients that correspond to frequencies that a typical human ear does not hear. This is the basis of the mp3 format (it uses transform that is something like a fourth generation offspring of cosine Fourier series).

Fourier decomposition can be also generalized to more dimensions and then it can be quite powerful in storing visual information - it is for instance the heart of the system used by F.B.I. to store their fingerprint database. Since this decomposition is so useful, one important aspect is the speed at which we can find the coefficients. This inspired further development and today we do not usually use the standard Fourier series but its more powerful offspring, for instance something called Fast Fourier Transform (FFT). Here also hardware helps, there are devices (integrators) that have this wired in, roughly speaking one feeds it a function and the device spits out a Fourier coefficient.

Legendre Polynomial and differential equation

The Legendre differential equation is the second-order ordinary differential equation

$$\left(1-x^2\right)\frac{d^2 y}{d x^2}-2x\frac{d y}{d x}+l(l+1)y=0, \qquad (1)$$

which can be rewritten

$$\frac{d}{d x}\left[\left(1-x^2\right)\frac{d y}{d x}\right]+l(l+1)y=0. \qquad (2)$$

The above form is a special case of the so-called "associated Legendre differential equation" corresponding to the case $m=0$. The Legendre differential equation has regular singular points at -1, 1, and ∞.

If the variable x is replaced by $\cos\theta$, then the Legendre differential equation becomes

$$\frac{d^2 y}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+l(l+1)y=0, \quad (3)$$

Derived below for the associated ($m\neq 0$) case.

Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_l(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_l(x)$ which is singular at ± 1 is called a Legendre function of the second kind. If l is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial.

The Legendre differential equation can be solved using the Frobenius method by making a series expansion with $k=0$,

$$y=\sum_{n=0}^{\infty} a_n x^n \quad (4)$$

$$y'=\sum_{n=0}^{\infty} n a_n x^{n-1} \quad (5)$$

$$y''=\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}. \quad (6)$$

Plugging in,

$$\left(1-x^2\right)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}-2x\sum_{n=0}^{\infty} n a_n x^{n-1}+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0 \quad (7)$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}-\sum_{n=0}^{\infty} n(n-1) a_n x^n \quad (8)$$

$$-2x\sum_{n=0}^{\infty} n a_n x^{n-1}+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0 \quad (9)$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}-\sum_{n=0}^{\infty} n(n-1) a_n x^n \quad (10)$$

$$-2\sum_{n=0}^{\infty} n\,a_n\,x^n + l\,(l+1)\sum_{n=0}^{\infty} a_n\,x^n = 0 \qquad (11)$$

$$\sum_{n=0}^{\infty}(n+2)\,(n+1)\,a_{n+2}\,x^n - \sum_{n=0}^{\infty} n\,(n-1)\,a_n\,x^n \qquad (12)$$

$$-2\sum_{n=0}^{\infty} n\,a_n\,x^n + l\,(l+1)\sum_{n=0}^{\infty} a_n\,x^n = 0 \qquad (13)$$

$$\sum_{n=0}^{\infty}\{(n+1)\,(n+2)\,a_{n+2} + [-n\,(n-1) - 2\,n + l\,(l+1)]\,a_n\} = 0, \qquad (14)$$

so each term must vanish and

$$(n+1)\,(n+2)\,a_{n+2} + [-n\,(n+1) + l\,(l+1)]\,a_n = 0 \qquad (15)$$

$$a_{n+2} = \frac{n\,(n+1) - l\,(l+1)}{(n+1)\,(n+2)}\,a_n \qquad (16)$$

$$= -\frac{[l + (n+1)]\,(l-n)}{(n+1)\,(n+2)}\,a_n. \qquad (17)$$

Therefore,

$$a_2 = -\frac{l\,(l+1)}{1\cdot 2}\,a_0 \qquad (18)$$

$$a_4 = -\frac{(l-2)\,(l+3)}{3\cdot 4}\,a_2 \qquad (19)$$

$$= (-1)^2\,\frac{[(l-2)\,l]\,[(l+1)\,(l+3)]}{1\cdot 2\cdot 3\cdot 4}\,a_0 \qquad (20)$$

$$a_6 = -\frac{(l-4)\,(l+5)}{5\cdot 6}\,a_4 \qquad (21)$$

$$= (-1)^3\,\frac{[(l-4)\,(l-2)\,l]\,[(l+1)\,(l+3)\,(l+5)]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\,a_0, \qquad (22)$$

so the even solution is

$$y_1(x) = 1 + \sum_{n=1}^{\infty}(-1)^n\,\frac{[(l-2\,n+2)\cdots(l-2)\,l]\,[(l+1)\,(l+3)\cdots(l+2\,n-1)]}{(2\,n)!}\,x^{2\,n}. \qquad (23)$$

Similarly, the odd solution is

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+1)\cdots(l-3)(l-1)]\,[(l+2)(l+4)\cdots(l+2n)]}{(2n+1)!} x^{2n+1}. \quad (24)$$

If l is an even integer, the series $y_1(x)$ reduces to a polynomial of degree l with only even powers of x and the series $y_2(x)$ diverges. If l is an odd integer, the series $y_2(x)$ reduces to a polynomial of degree l with only odd powers of x and the series $y_1(x)$ diverges. The general solution for an integer l is then given by the Legendre polynomials

$$P_n(x) = c_n \begin{cases} y_1(x) & \text{for } l \text{ even} \\ y_2(x) & \text{for } l \text{ odd} \end{cases} \quad (25)$$

$$= c_n \begin{cases} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}(l+1); \frac{1}{2}, x^2\right) & \text{for } l \text{ even} \\ x\, {}_2F_1\left(\frac{1}{2}(l+2), \frac{1}{2}(1-l); \frac{3}{2}; x^2\right) & \text{for } l \text{ odd} \end{cases} \quad (26)$$

where c_n is chosen so as to yield the normalization $P_n(1) = 1$ and ${}_2F_1(a, b; c; z)$ is a hypergeometric function.

The *associated* Legendre differential equation is

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0, \quad (27)$$

which can be written

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \quad (28)$$

(Abramowitz and Stegun 1972; Zwillinger 1997, p. 124). The solutions $P_l^m(x)$ to this equation are called the associated Legendre polynomials (if l is an integer), or associated Legendre functions of the first kind (if l is not an integer). The complete solution is

$$y = C_1 P_l^m(x) + C_2 Q_l^m(x), \quad (29)$$

where $Q_l^m(x)$ is a Legendre function of the second kind.

The associated Legendre differential equation is often written in a form obtained by setting $x \equiv \cos\theta$. Plugging the identities

$$\frac{dy}{dx} = \frac{dy}{d(\cos\theta)} \quad (30)$$

$$= -\frac{1}{\sin\theta}\frac{dy}{d\theta} \qquad (31)$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dy}{d\theta}\right) \qquad (32)$$

$$= \frac{1}{\sin^2\theta}\left(\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}\right) \qquad (33)$$

into (◇) then gives

$$\left(\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}\right) + 2\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0 \qquad (34)$$

$$\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]y = 0.$$

Lagurae Polynomials: Definition

Laguerre's Differential Equation is defined as:

$$xy'' + (1-x)y' + ny = 0$$

where n is a real number. When n is a non-negative integer, i.e., $n = 0, 1, 2, 3, \ldots$, the solutions of Laguerre's Differential Equation are often referred to as **Laguerre Polynomials** $L_n(x)$.

Important Properties

Rodrigues' Formula: The Laguerre Polynomials $L_n(x)$ can be expressed by Rodrigues' formula:

$$L_n(x) = \frac{1}{n!e^{-x}}\frac{d^n}{dx^n}(x^n e^{-x})$$ where $n = 0, 1, 2, 3, \ldots$

Generating Function: The generating function of a Laguerre Polynomial is:

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n$$

Orthogonality: Laguerre Polynomials $L_n(x)$, $n = 0, 1, 2, 3, \ldots$, form a *complete orthogonal set* on the interval $0 < x < \infty$ with respect to the weighting function e^{-x}. It can be shown that:

$$\int_0^\infty e^{-x} L_m(x) L_n(x)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

By using this orthogonality, a piecewise continuous function $f(x)$ can be expressed in terms of Laguerre Polynomials:

$$\sum_{n=0}^{\infty} C_n L_n(x) = \begin{cases} f(x) & \text{where } f(x) \text{ is continuous} \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{at discontinuous points} \end{cases}$$

Where:

$$C_n = \int_0^\infty e^{-x} f(x) L_n(x)\,dx$$

This orthogonal series expansion is also known as a **Fourier-Laguerre Series** expansion or a **Generalized Fourier Series** expansion.

Recurrence Relation: A Laguerre Polynomial at one point can be expressed in terms of neighboring Laguerre Polynomials at the same point.

- $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$
- $L_n'(x) = L_{n-1}'(x) - L_{n-1}(x)$
- $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$

Special Rsults

$$L_n(0) = 1$$

$$L_n(x) = \frac{(-1)^n}{n!}\left\{\frac{x^n}{n!} - \frac{nx^{n-1}}{1!(n-1)!} + \frac{n(n-1)x^{n-2}}{2!(n-2)!} - \ldots + (-1)^n\right\}$$

$$\int_0^x L_n(t)\,dt = L_n(x) - L_{n+1}(x)$$

$$\int_0^\infty x^p e^{-x} L_n(x)\,dx = \begin{cases} 0 & \text{if } p < n \\ (-1)^n n! & \text{if } p = n \end{cases}$$

$$\sum_{k=0}^{n} L_k(x) L_k(y) = \frac{L_n(x) L_{n+1}(y) - L_{n+1}(x) L_n(y)}{(n+1)(x-y)}$$

$$\sum_{k=0}^{\infty} \frac{t^k L_k(x)}{k!} = e^t J_0\left(2\sqrt{xt}\right)$$

$$L_n(x) = \frac{1}{n!} \int_0^\infty u^n e^{x-u} J_0\left(2\sqrt{xu}\right) du$$

Where J_0 is 0 order Bessel function of the first kind

Possible questions (Part-B-6 Marks)

1. State and Explain Dirichlet conditions.
2. Show that $\int_{-1}^{+1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$ using Legendre polynomials
3. Show that the Legendre function $P_n(x)$ is the coefficient of z^n in the expansion of $[1 - 2xz + z^2]^{-1/2}$.

   From above, deduce the values of $P_n(1)$. Also, show that $P_n(-x)=(-1)^nP_n(x)$
4. Explain what is Fourier series. Find the Fourier series of the function in the interval $-\pi < x < \pi$
5. Derive Rodrigue's Formula. State and Explain Dirichlet conditions.
6. Explain orthogonal properties of Legendre's polynomials.
7. Explain about the Cauchy Residue theorem
8. Explain orthogonal properties of Legendre's polynomials.
9. Derive recurrence relation for Lagurae formula.

Possible questions (Part-C-10 Marks)

1. Show that $\int_{-1}^{+1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$ using Legendre polynomials
2. Show that the Legendre function $P_n(x)$ is the coefficient of z^n in the expansion of $[1 - 2xz + z^2]^{-1/2}$.

   From above, deduce the values of $P_n(1)$. Also, show that $P_n(-x)=(-1)^nP_n(x)$
3. Explain what is Fourier series. Find the Fourier series of the function in the interval $-2\pi < x < 2\pi$
4. Derive Rodrigue's Formula for legendre polynomial.
5. Explain orthogonal properties of Legendre's polynomials.
6. Derive Rodrigue's Formula for lagurae polynomial
7. Derive the recurrence relation for Legendre formula.

KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21

DEPARTMENT OF PHYSICS

CLASS:I MSc PHYSICS **BATCH:2019**

MATHEMATICAL PHYSICS (19PHP104)

MULIPLE CHOICE QUESTIONS

Questions	**opt1**	**opt2**	**opt3**
UNIT IV			
The value of $J_{-1/2}$ (x) is	$\sqrt{\frac{2}{x}} \sin x$	$\sqrt{\frac{2}{\pi x}} \sin x$	$\sqrt{\frac{2}{\pi x}} \cos x$
The Rodrigue formula for Pn(x), the Legendre polynomial of degree 'n' is	$K = \frac{n!}{2^n}$	$K = \frac{2^n}{n!}$	$K = \frac{1}{2^n n!}$
The value of Jo(x) at the origin is	1	0	–1
The value of P_1(x) is	x	1	$x^2 / 2$
The identical roots of the Legendre's functions are	m = ± n	m = ± 1	m = 0 or m = 1
The value of $J_{1/2}$ (x) is	$\sqrt{\frac{2}{x}} \sin x$	$\sqrt{\frac{2}{\pi x}} \sin x$	$\sqrt{\frac{2}{\pi x}} \cos x$

If Jo and J_1 are Bessel's functions then J_1'(x) is given by	Jo(x) – 1/x J_1(x)	– Jo	Jo(x) + 1/x J_1(x)
integral where Jn(x) is the Bessel function of the first kind of order	0	-2	2
The integral $\int xJo(x)dx$ is equal to	$xJ_1(x) - Jo(x)$	$xJ_1(x)$	$J_1(x)$
– Jo(x) where Jn is the Bessel function of first kind order 'n'. Then	0	2	–1
The value of $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2$ is	$\sqrt{\frac{2}{\pi x}}$	$\frac{2}{\pi x}$	$\frac{2}{\pi}$
The value of Po(x) is	1	x	0
Let Pn(x) be the Legendre polynomial, then Pn(-x) is equal to	$(-1)^{n+1} P_n'(x)$	$(-1)^n P_n'(x)$	$(-1)^n P_n(x)$
Legendre polynomial of order 'n', then $3x^2 + 3x + 1$ can be	$3P_2 + 3P_1$	$4P_2+2P_1 + Po$	$3P_2+3P_1 + Po$
If $\int_{-1}^{+1} P_n(x)dx = 2$, then 'n' is	1	0	–1
is where K is equal to	63/2	63/5	63/10
polynomial of degree n>1, then is equal to	0	1 / (2n+1)	2 / (2n+1)

The value of $\int_{-1}^{+1}(2x+1)P_n(x)dx$ is the third degree Legendre polynomial is	1	–1	2
The Legendre polynomial Pn(x) has	n real zeros between 0 and 1	n zeros of which only one is between –1 and +1	2n-1 real zeros between –1 and 1
The incorrect equation among the following is	$P_o(x) = 1$	$P_1(x) = x$	$P_n(-x) = (-1)^{n+1} P_n(x)$
The value of $P_n(-x)$ is	$- P_n(x)$	$P_n(x)$	$(-1)^n P_n(x)$
The value of $2J_n'$ is	$J_{n-1} – J_{n+1}$	$J_{n-1} + J_{n+1}$	$J_{n+1} – J_{n+1}$
The root of $x^3 – 6x + 4$ lies between	–1 and 0	1 and 2	–2 and 1

opt4	**Answer**
$\sqrt{\frac{2}{x}} \cos \quad x$	$\sqrt{\frac{2}{\pi x}} \sin \quad x$
$K = \frac{1}{2^n (n!)^2}$	$K = \frac{2^n}{n!}$
x	1
½ $(x^2 - 1)$	x
m = 0 or m = -1	m = ± 1
$\sqrt{\frac{2}{x}} \cos \quad x$	$\sqrt{\frac{2}{\pi x}} \sin \quad x$

$Jo(x) - 1/x^2 J_1(x)$	$Jo(x) - 1/x J_1(x)$
1	0
$x^2 J_n(x)$	$xJ_1(x)$
1 $\sqrt{\frac{2}{\pi}}$	1 $\frac{2}{\pi \quad x}$
–1	–1
$P_n''(x)$	$(-1)^n P_n(x)$
$2P_2+3P_1 + 2Po$	$2P_2+3P_1 + 2Po$
μ	0
63/8	63/8
n / (2n+1)	#REF!

0	0
none of these	n real zeros between 0 and 1
$P_n(x) = (-1)^{n+1} P_n(x)$	$P_n(-x) = (-1)^{n+1} P_n(x)$
$(-1)^n P_n(-x)$	$(-1)^n P_n(x)$
$2 J_{n+1}$	$J_{n+1} - J_{n+1}$
0 and 1	–1 and 0

UNIT-V

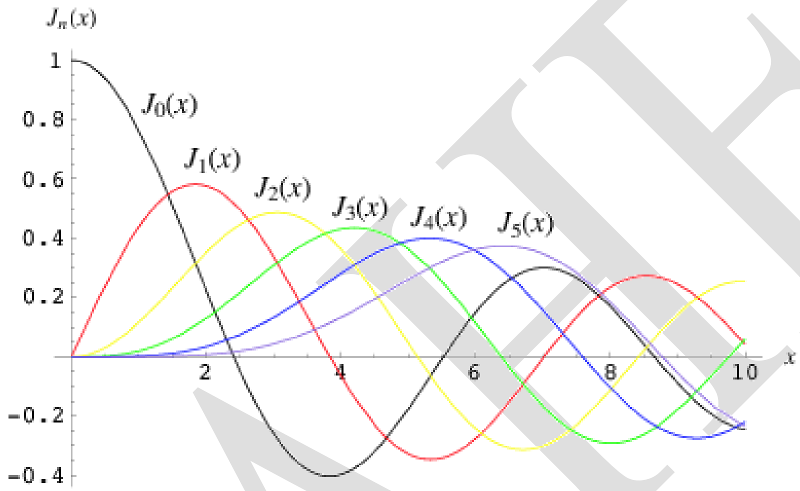
SYLLABUS

Bessel's functions – differential equation and solution – generating functions – recurrence relations – Bessel function of second order – Spherical Bessel function - Hermite differential equation and Hermite polynomials – Generating function of Hermite polynomials – Recurrence formulae for Hermite polynomials – Rodrigue's formula for Hermite Polynomials – Orthogonality of Hermite Polynomials – Dirac's Delta Function

Bessel functions differential equations and solution

The Bessel functions of the first kind $J_n(x)$ are defined as the solutions to the Bessel differential equation

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - n^2) y = 0$$



Which are nonsingular at the origin. They are sometimes also called cylinder functions or cylindrical harmonics. The above plot shows $J_n(x)$ for 1, 2, ..., 5. The notation $J_{z,n}$ was first used by Hansen (1843) and subsequently by Schlömilch (1857) to denote what is now written $J_n(2z)$ (Watson 1966, p. 14). However, Hansen's definition of the function itself in terms of the generating function

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z). \tag{2}$$

is the same as the modern one (Watson 1966, p. 14). Bessel used the notation I_k^h to denote what is now called the Bessel function of the first kind (Cajori 1993, vol. 2, p. 279).

The Bessel function $J_n(z)$ can also be defined by the contour integral

$$J_n(z) = \frac{1}{2\pi i} \oint e^{(z/2)(t-1/t)} t^{-n-1} \, dt, \tag{3}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The Bessel function of the first kind is implemented in the Wolfram Language as BesselJ[nu, z].

To solve the differential equation, apply Frobenius method using a series solution of the form

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}. \tag{4}$$

Plugging into (1) yields

$$x^2 \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n-2} + x \sum_{n=0}^{\infty} (k+n) a_n x^{k+n-1} + x^2 \sum_{n=0}^{\infty} a_n x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0 \tag{5}$$

$$\sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n} + \sum_{n=0}^{\infty} (k+n) a_n x^{k+n} + \sum_{n=2}^{\infty} a_{n-2} x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0. \tag{6}$$

The indicial equation, obtained by setting $n = 0$, is

$$a_0 \left[k(k-1) + k - m^2\right] = a_0 \left(k^2 - m^2\right) = 0. \tag{7}$$

Since a_0 is defined as the first nonzero term, $k^2 - m^2 = 0$, so $k = \pm m$. Now, if $k = m$,

$$\sum_{n=0}^{\infty} \left[(m+n)(m+n-1) + (m+n) - m^2\right] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} \left[(m+n)^2 - m^2\right] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \tag{9}$$

$$\sum_{n=0}^{\infty} n(2m+n) a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \tag{10}$$

$$a_1 (2m+1) x^{m+1} + \sum_{n=2}^{\infty} \left[a_n n(2m+n) + a_{n-2}\right] x^{m+n} = 0. \tag{11}$$

First, look at the special case $m = -1/2$, then (11) becomes

$$\sum_{n=2}^{\infty}[a_n\, n\,(n-1)+a_{n-2}]\, x^{m+n}=0, \tag{12}$$

so

$$a_n=-\frac{1}{n\,(n-1)}\,a_{n-2}. \tag{13}$$

Now let $n\equiv 2\,l$, where $l=1$, 2, ...

$$a_{2\,l}=-\frac{1}{2\,l\,(2\,l-1)}\,a_{2\,l-2} \tag{14}$$

$$=\frac{(-1)^l}{[2\,l\,(2\,l-1)]\,[2\,(l-1)\,(2\,l-3)]\cdots[2\cdot1\cdot1]}\,a_0 \tag{15}$$

$$=\frac{(-1)^l}{2^l\,l!\,(2\,l-1)!!}\,a_0, \tag{16}$$

which, using the identity $2^l\,l!\,(2\,l-1)!!=(2\,l)!$, gives

$$a_{2\,l}=\frac{(-1)^l}{(2\,l)!}\,a_0. \tag{17}$$

Similarly, letting $n\equiv 2\,l+1$,

$$a_{2\,l+1}=-\frac{1}{(2\,l+1)\,(2\,l)}\,a_{2\,l-1}=\frac{(-1)^l}{[2\,l\,(2\,l+1)]\,[2\,(l-1)\,(2\,l-1)]\cdots[2\cdot1\cdot3]\,[1]}\,a_1, \tag{18}$$

which, using the identity $2^l\,l!\,(2\,l+1)!!=(2\,l+1)!$, gives

$$a_{2\,l+1}=\frac{(-1)^l}{2^l\,l!\,(2\,l+1)!!}\,a_1=\frac{(-1)^l}{(2\,l+1)!}\,a_1. \tag{19}$$

Plugging back into (◇) with $k=m=-1/2$ gives

$$y=x^{-1/2}\sum_{n=0}^{\infty}a_n\,x^n \tag{20}$$

$$=x^{-1/2}\left[\sum_{n=1,3,5,\ldots}^{\infty}a_n\,x^n+\sum_{n=0,2,4,\ldots}^{\infty}a_n\,x^n\right] \tag{21}$$

$$=x^{-1/2}\left[\sum_{l=0}^{\infty}a_{2\,l}\,x^{2\,l}+\sum_{l=0}^{\infty}a_{2\,l+1}\,x^{2\,l+1}\right] \tag{22}$$

$$=x^{-1/2}\left[a_0\sum_{l=0}^{\infty}\frac{(-1)^l}{(2\,l)!}\,x^{2\,l}+a_1\sum_{l=0}^{\infty}\frac{(-1)^l}{(2\,l+1)!}\,x^{2\,l+1}\right] \tag{23}$$

$$= x^{-1/2} (a_0 \cos x + a_1 \sin x). \tag{24}$$

The Bessel functions of order $\pm 1/2$ are therefore defined as

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x \tag{25}$$

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \sin x, \tag{26}$$

so the general solution for $m = \pm 1/2$ is

$$y = a_0' J_{-1/2}(x) + a_1' J_{1/2}(x). \tag{27}$$

Now, consider a general $m \neq -1/2$. Equation (◇) requires

$$a_1 (2m+1) = 0 \tag{28}$$

$$[a_n n (2m+n) + a_{n-2}] x^{m+n} = 0 \tag{29}$$

for $n = 2$, 3, ..., so

$$a_1 = 0 \tag{30}$$

$$a_n = -\frac{1}{n(2m+n)} a_{n-2} \tag{31}$$

for $n = 2$, 3, ... Let $n \equiv 2l+1$, where $l = 1$, 2, ..., then

$$a_{2l+1} = -\frac{1}{(2l+1)[2(m+l)+1]} a_{2l-1} \tag{32}$$

$$= \ldots = f(n, m) a_1 = 0, \tag{33}$$

where $f(n, m)$ is the function of l and m obtained by iterating the recursion relationship down to a_1. Now let $n \equiv 2l$, where $l = 1$, 2, ..., so

$$a_{2l} = -\frac{1}{2l(2m+2l)} a_{2l-2} \tag{34}$$

$$-\frac{1}{4l(m+l)} a_{2l-2} \tag{35}$$

$$= \frac{(-1)^l}{[4l(m+l)][4(l-1)(m+l-1)]\cdots[4\cdot(m+1)]} a_0. \tag{36}$$

Plugging back into (◇),

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} = \sum_{n=1,3,5,\ldots}^{\infty} a_n x^{n+m} + \sum_{n=0,2,4,\ldots}^{\infty} a_n x^{n+m} \tag{37}$$

$$= \sum_{l=0}^{\infty} a_{2l+1} x^{2l+m+1} + \sum_{l=0}^{\infty} a_{2l} x^{2l+m} \tag{38}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l}{[4\,l\,(m+l)]\,[4\,(l-1)\,(m+l-1)]\cdots[4\,(m+1)]} x^{2l+m} \tag{39}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{\left[(-1)^l\, m\,(m-1)\cdots 1\right] x^{2l+m}}{[4\,l\,(m+l)]\,[4\,(l-1)\,(m+l-1)]\cdots[4\,(m+1)\,m\cdots 1]} \tag{40}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l\, m!}{2^{2l}\, l!\,(m+l)!} x^{2l+m}. \tag{41}$$

Now define

$$J_m(x) \equiv \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m}\, l!\,(m+l)!} x^{2l+m}, \tag{42}$$

where the factorials can be generalized to gamma functions for nonintegral m. The above equation then becomes

$$y = a_0\, 2^m\, m!\, J_m(x) = a_0'\, J_m(x). \tag{43}$$

Returning to equation (◇) and examining the case $k = -m$,

$$a_1\,(1 - 2\,m) + \sum_{n=2}^{\infty} [a_n\, n\,(n - 2\,m) + a_{n-2}]\, x^{n-m} = 0. \tag{44}$$

However, the sign of m is arbitrary, so the solutions must be the same for $+m$ and $-m$. We are therefore free to replace $-m$ with $-|m|$, so

$$a_1\,(1 + 2\,|m|) + \sum_{n=2}^{\infty} [a_n\, n\,(n + 2\,|m|) + a_{n-2}]\, x^{|m|+n} = 0, \tag{45}$$

and we obtain the same solutions as before, but with m replaced by $|m|$.

$$J_m(x) = \begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+|m|}\, l!\, (|m|+l)!} x^{2l+|m|} & \text{for } |m| \neq \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \cos x & \text{for } m = -\frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \sin x & \text{for } m = \frac{1}{2}. \end{cases} \tag{46}$$

We can relate $J_m(x)$ and $J_{-m}(x)$ (when m is an integer) by writing

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l-m}\, l!\, (l-m)!} x^{2l-m}. \tag{47}$$

Now let $l \equiv l' + m$. Then

$$J_{-m}(x) = \sum_{l'+m=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m}\, (l'+m)!\, l!} x^{2l'+m} \tag{48}$$

$$= \sum_{l'=-m}^{-1} \frac{(-1)^{l'+m}}{2^{2l'+m}\, l'!\, (l'+m)!} x^{2l'+m} + \sum_{l'=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m}\, l'!\, (l'+m)!} x^{2l'+m}. \tag{49}$$

But $l'! = \infty$ for $l' = -m, \ldots, -1$, so the denominator is infinite and the terms on the left are zero. We therefore have

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{2^{2l+m}\, l!\, (l+m)!} x^{2l+m} \tag{50}$$

$$= (-1)^m J_m(x). \tag{51}$$

Note that the Bessel differential equation is second-order, so there must be two linearly independent solutions. We have found both only for $|m| = 1/2$. For a general nonintegral order, the independent solutions are J_m and J_{-m}. When m is an integer, the general (real) solution is of the form

$$Z_m \equiv C_1 J_m(x) + C_2 Y_m(x), \tag{52}$$

where J_m is a Bessel function of the first kind, Y_m (a.k.a. N_m) is the Bessel function of the second kind (a.k.a. Neumann function or Weber function), and C_1 and C_2 are constants. Complex solutions are given by the Hankel functions (a.k.a. Bessel functions of the third kind).

The Bessel functions are orthogonal in $[0, a]$ according to

$$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho = \tfrac{1}{2}a^2\,[J_{\nu+1}(\alpha_{\nu m})]^2\,\delta_{mn}, \tag{53}$$

where $\alpha_{\nu m}$ is the mth zero of $J\,\nu$ and δ_{mn} is the Kronecker delta (Arfken 1985, p. 592).

Except when $2m$ is a negative integer,

$$J_m(z) = \frac{z^{-1/2}}{2^{2m+1/2}\, i^{m+1/2}\,\Gamma(m+1)} M_{0,m}(2iz), \tag{54}$$

where $\Gamma(x)$ is the gamma function and $M_{0,m}$ is a Whittaker function.

In terms of a confluent hypergeometric function of the first kind, the Bessel function is written

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\nu+1; -\tfrac{1}{4}z^2\right). \tag{55}$$

A derivative identity for expressing higher order Bessel functions in terms of $J_0(z)$ is

$$J_n(z) = i^n\, T_n\left(i\frac{d}{dz}\right) J_0(z), \tag{56}$$

where $T_n(z)$ is a Chebyshev polynomial of the first kind. Asymptotic forms for the Bessel functions are

$$J_m(z) \approx \frac{1}{\Gamma(m+1)}\left(\frac{z}{2}\right)^m \tag{57}$$

for $z \ll 1$ and

$$J_m(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{58}$$

for $z \gg |m^2 - 1/4|$ (correcting the condition of Abramowitz and Stegun 1972, p. 364).

A derivative identity is

$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x). \tag{59}$$

An integral identity is

$$\int_0^u u' J_0(u')\, du' = u J_1(u). \tag{60}$$

Some sum identities are

$$\sum_{k=-\infty}^{\infty} J_k(x) = 1 \qquad (61)$$

(which follows from the generating function (◇) with $t = 1$),

$$1 = [J_0(x)]^2 + 2\sum_{k=1}^{\infty}[J_k(x)]^2 \qquad (62)$$

(Abramowitz and Stegun 1972, p. 363),

$$1 = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x) \qquad (63)$$

(Abramowitz and Stegun 1972, p. 361),

$$0 = \sum_{k=0}^{2n}(-1)^k J_k(z) J_{2n-k}(z) + 2\sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \qquad (64)$$

for $n \geq 1$ (Abramowitz and Stegun 1972, p. 361),

$$J_n(2z) = \sum_{k=0}^{n} J_k(z) J_{n-k}(z) + 2\sum_{k=1}^{\infty}(-1)^k J_k(z) J_{n+k}(z) \qquad (65)$$

(Abramowitz and Stegun 1972, p. 361), and the Jacobi-Anger expansion

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}, \qquad (66)$$

which can also be written

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta). \qquad (67)$$

The Bessel function addition theorem states

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z). \qquad (68)$$

Various integrals can be expressed in terms of Bessel functions

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)\,d\theta, \qquad (69)$$

which is Bessel's first integral,

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{iz\cos\theta}\cos(n\theta)\,d\theta \qquad (70)$$

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iz\cos\phi}\,e^{in\phi}\,d\phi \qquad (71)$$

for $n = 1$, 2, ...,

$$J_n(z) = \frac{2}{\pi}\frac{z^n}{(2n-1)!!}\int_0^{\pi/2} \sin^{2n}u\cos(z\cos u)\,du \qquad (72)$$

for $n = 1$, 2, ...,

$$J_n(x) = \frac{1}{2\pi i}\int_\gamma e^{(x/2)(z-1/z)}\,z^{-n-1}\,dz \qquad (73)$$

for $n > -1/2$. The Bessel functions are normalized so that

$$\int_0^\infty J_n(x)\,dx = 1 \qquad (74)$$

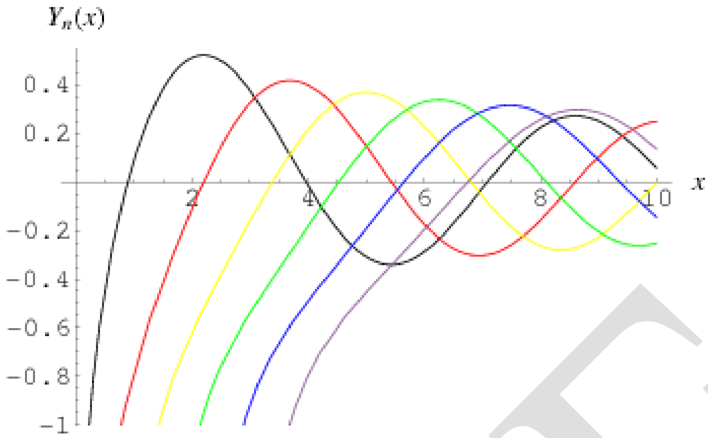
for positive integral (and real) n. Integrals involving $J_1(x)$ include

$$\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 dx = \frac{4}{3\pi} \qquad (75)$$

$$\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 x\,dx = \frac{1}{2}. \qquad (76)$$

Bessel function of second order

A Bessel function of the second kind $Y_n(x)$ (e.g, Gradshteyn and Ryzhik 2000, p. 703, eqn. 6.649.1), sometimes also denoted $N_n(x)$ (e.g, Gradshteyn and Ryzhik 2000, p. 657, eqn. 6.518), is a solution to the Bessel differential equation which is singular at the origin. Bessel functions of the second kind are also called Neumann functions or Weber functions. The above plot shows $Y_n(x)$ for $n = 0$, 1, 2, ..., 5. The Bessel function of the second kind is implemented in the Wolfram Language as BesselY[nu, z].



Let $v \equiv J_m(x)$ be the first solution and u be the other one (since the Bessel differential equation is second-order, there are two linearly independentsolutions). Then

$$x u'' + u' + x u = 0 \qquad (1)$$
$$x v'' + v' + x v = 0. \qquad (2)$$

Take $v\times$ (1) minus $u\times$ (2),

$$x (u'' v - u v'') + u' v - u v' = 0 \qquad (3)$$
$$\frac{d}{dx}[x (u' v - u v')] = 0, \qquad (4)$$

so $x (u' v - u v') = B$, where B is a constant. Divide by $x v^2$,

$$\frac{u' v - u v'}{v^2} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{B}{x v^2} \qquad (5)$$
$$\frac{u}{v} = A + B \int \frac{dx}{x v^2}. \qquad (6)$$

Rearranging and using $v \equiv J_m(x)$ gives

$$u = A J_m(x) + B J_m(x) \int \frac{dx}{x J_m^2(x)} \qquad (7)$$
$$= A' J_m(x) + B' Y_m(x), \qquad (8)$$

where Y_m is the so-called Bessel function of the second kind.

$Y_v(z)$ can be defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{9}$$

(Abramowitz and Stegun 1972, p. 358), where $J_\nu(z)$ is a Bessel function of the first kind and, for ν an integer n by the series

$$Y_n(z) = -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\frac{1}{2}z\right)J_n(z) - \frac{\left(\frac{1}{2}z\right)^n}{\pi}\sum_{k=0}^{\infty}\left[\psi_0(k+1)+\psi_0(n+k+1)\right]\frac{\left(-\frac{1}{4}z^2\right)^k}{k!\,(n+k)!}, \tag{10}$$

where $\psi_0(x)$ is the digamma function (Abramowitz and Stegun 1972, p. 360).

The function has the integral representations

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)\,d\theta - \frac{1}{\pi}\int_0^\infty\left[e^{\nu t} + e^{-\nu t}(-1)^\nu\right]e^{-z\sinh t}\,dt \tag{11}$$

$$= -\frac{2\left(\frac{1}{2}z\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^\infty\frac{\cos(z t)\,dt}{\left(t^2-1\right)^{\nu+1/2}} \tag{12}$$

(Abramowitz and Stegun 1972, p. 360).

Asymptotic series are

$$Y_m(x) \sim \begin{cases}\frac{2}{\pi}\left[\ln\left(\frac{1}{2}x\right)+\gamma\right] & m=0,\ x\ll 1\\ -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m & m\neq 0,\ x\ll 1\end{cases} \tag{13}$$

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\quad x\gg 1,$$

For the special case $n=0$, $Y_0(x)$ is given by the series

$$Y_0(z) = \frac{2}{\pi}\left\{\left[\ln\left(\frac{1}{2}z\right)+\gamma\right]J_0(z) + \sum_{k=1}^{\infty}(-1)^{k+1}H_k\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}\right\},$$

Take the Helmholtz differential equation

$$\nabla^2 F + k^2 F = 0 \tag{1}$$

in spherical coordinates. This is just Laplace's equation in spherical coordinates with an additional term,

$$\frac{d^2 R}{d r^2} \Phi\Theta + \frac{2}{r}\frac{d R}{d r}\Phi\Theta + \frac{1}{r^2 \sin^2\phi}\frac{d^2\Theta}{d\theta^2}\Phi R + \frac{\cos\phi}{r^2\sin\phi}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{d^2\Phi}{d\phi^2}\Theta R + k^2 R\Phi\Theta = 0. \quad (2)$$

Multiply through by $r^2 / R\Phi\Theta$,

$$\frac{r^2}{R}\frac{d^2 R}{d r^2} + \frac{2r}{R}\frac{d R}{d r} + k^2 r^2 + \frac{1}{\Theta\sin^2\phi}\frac{d^2\Theta}{d\theta^2} + \frac{\cos\phi}{\Phi\sin\phi}\frac{d\Phi}{d\phi} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \quad (3)$$

This equation is separable in R. Call the separation constant $n(n+1)$,

$$\frac{r^2}{R}\frac{d^2 R}{d r^2} + \frac{2r}{R}\frac{d R}{d r} + k^2 r^2 = n(n+1). \quad (4)$$

Now multiply through by R,

$$r^2\frac{d^2 R}{d r^2} + 2r\frac{d R}{d r} + \left[k^2 r^2 - n(n+1)\right] R = 0. \quad (5)$$

This is the spherical Bessel differential equation. It can be transformed by letting $x \equiv k r$, then

$$r\frac{d R(r)}{d r} = k r\frac{d R(r)}{k\, d r} = k r\frac{d R(r)}{d(k r)} = x\frac{d R(r)}{d x}. \quad (6)$$

Similarly,

$$r^2\frac{d^2 R(r)}{d r^2} = x^2\frac{d^2 R(r)}{d x^2}, \quad (7)$$

so the equation becomes

$$x^2\frac{d^2 R}{d x^2} + 2x\frac{d R}{d x} + \left[x^2 - n(n+1)\right] R = 0. \quad (8)$$

Now look for a solution of the form $R(r) = Z(x)\, x^{-1/2}$, denoting a derivative with respect to x by a prime,

$$R' = Z' x^{-1/2} - \tfrac{1}{2} Z x^{-3/2} \quad (9)$$

$$R'' = Z''\,x^{-1/2} - \tfrac{1}{2}\,Z'\,x^{-3/2} - \tfrac{1}{2}\,Z'\,x^{-3/2} - \tfrac{1}{2}\left(-\tfrac{3}{2}\right) Z\,x^{-5/2} \quad (10)$$
$$= Z''\,x^{-1/2} - Z'\,x^{-3/2} + \tfrac{3}{4}\,Z\,x^{-5/2}, \quad (11)$$

so

$$x^2\left(Z''\,x^{-1/2} - Z'\,x^{-3/2} + \tfrac{3}{4}\,Z\,x^{-5/2}\right) + 2\,x\left(Z'\,x^{-1/2} - \tfrac{1}{2}\,Z\,x^{-3/2}\right) + \left[x^2 - n\,(n+1)\right] Z\,x^{-1/2} = 0 \quad (12)$$
$$x^2\left(Z'' - Z'\,x^{-1} + \tfrac{3}{4}\,Z\,x^{-2}\right) + 2\,x\left(Z' - \tfrac{1}{2}\,Z\,x^{-1}\right) + \left[x^2 - n\,(n+1)\right] Z = 0 \quad (13)$$
$$x^2\,Z'' + (-x + 2\,x)\,Z' + \left[\tfrac{3}{4} - 1 + x^2 - n\,(n+1)\right] Z = 0 \quad (14)$$
$$x^2\,Z'' + x\,Z' + \left[x^2 - \left(n^2 + n + \tfrac{1}{4}\right)\right] Z = 0 \quad (15)$$
$$x^2\,Z'' + x\,Z' + \left[x^2 - \left(n + \tfrac{1}{2}\right)^2\right] Z = 0. \quad (16)$$

But the solutions to this equation are Bessel functions of half integral order, so the normalized solutions to the original equation are

$$R\,(r) \equiv A\,\frac{J_{n+1/2}\,(k\,r)}{\sqrt{k\,r}} + B\,\frac{Y_{n+1/2}\,(k\,r)}{\sqrt{k\,r}} \quad (17)$$

which are known as spherical Bessel functions. The two types of solutions are denoted $j_n\,(x)$ (spherical Bessel function of the first kind) or $n_n\,(x)$(spherical Bessel function of the second kind), and the general solution is written

$$R\,(r) = A'\,j_n\,(k\,r) + B'\,n_n\,(k\,r), \quad (18)$$

where

$$j_n\,(z) \equiv \sqrt{\frac{\pi}{2}}\,\frac{J_{n+1/2}\,(z)}{\sqrt{z}} \quad (19)$$
$$n_n\,(z) \equiv \sqrt{\frac{\pi}{2}}\,\frac{Y_{n+1/2}\,(z)}{\sqrt{z}}.$$

Spherical Bessel function

The second-order ordinary differential equation

$$\frac{d^2\,y}{d\,x^2} - 2\,x\,\frac{d\,y}{d\,x} + \lambda\,y = 0. \quad (1)$$

This differential equation has an irregular singularity at ∞. It can be solved using the series method

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2\,n\,a_n x^n + \sum_{n=0}^{\infty}\lambda\,a_n x^n = 0 \tag{2}$$

$$(2\,a_2 + \lambda\,a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 2\,n\,a_n + \lambda\,a_n]\,x^n = 0. \tag{3}$$

Therefore,

$$a_2 = -\frac{\lambda\,a_0}{2} \tag{4}$$

and

$$a_{n+2} = \frac{2\,n-\lambda}{(n+2)(n+1)}\,a_n \tag{5}$$

for $n=1$, 2, ... Since (4) is just a special case of (5),

$$a_{n+2} = \frac{2\,n-\lambda}{(n+2)(n+1)}\,a_n \tag{6}$$

for $n=0$, 1, ...

The linearly independent solutions are then

$$y_1 = a_0\left[1 - \frac{\lambda}{2!}x^2 - \frac{(4-\lambda)\lambda}{4!}x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!}x^6 - \ldots\right] \tag{7}$$

$$y_2 = a_1\left[x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \ldots\right]. \tag{8}$$

These can be done in closed form as

$$y = a_0\,{}_1F_1\left(-\tfrac{1}{4}\lambda;\,\tfrac{1}{2};\,x^2\right) + a_1\,x\,{}_1F_1\left(-\tfrac{1}{4}(\lambda-2);\,\tfrac{3}{2};\,x^2\right) \tag{9}$$

$$= a_0\,{}_1F_1\left(-\tfrac{1}{4}\lambda;\,\tfrac{1}{2};\,x^2\right) + a_2\,H_{\lambda/2}(x), \tag{10}$$

where ${}_1F_1(a;b;x)$ is a confluent hypergeometric function of the first kind and $H_n(x)$ is a Hermite polynomial. In particular, for $\lambda=0$, 2, 4, ..., the solutions can be written

$$y_{\lambda=0} = a_0 + \tfrac{1}{2}\sqrt{\pi}\,a_1\,\mathrm{erfi}(x) \tag{11}$$

$$y_{\lambda=2} = a_0\left[e^{x^2} - \sqrt{\pi}\,x\,\mathrm{erfi}(x)\right] + x\,a_1 \tag{12}$$

$$y_{\lambda=4} = \tfrac{1}{4}\left\{2\,e^{x^2}\,x\,a_1 - (2\,x^2-1)\left[4\,a_0 + \sqrt{\pi}\,a_1\,\mathrm{erfi}(x)\right]\right\}, \tag{13}$$

where $\operatorname{erfi}(x)$ is the erfi function.

If $\lambda = 0$, then Hermite's differential equation becomes

$$y'' - 2xy' = 0, \tag{14}$$

which is of the form $P_2(x)y'' + P_1(x)y' = 0$ and so has solution

$$y = c_1 \int \frac{dx}{\exp\left(\int \frac{P_1}{P_2}\,dx\right)} + c_2 \tag{15}$$

$$= c_1 \int \frac{dx}{\exp \int (-2x)\,dx} + c_2 \tag{16}$$

$$= c_1 \int \frac{dx}{e^{-x^2}} + c_2 = c_1 \operatorname{erfi}(x) + c_2. \tag{17}$$

Hermite Polynomial

The Hermite polynomials $H_n(x)$ are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with weighting function e^{-x^2}, illustrated above for $n = 1$, 2, 3, and 4. Hermite polynomials are implemented in the Wolfram Language as HermiteH[*n*, *x*].

The Hermite polynomial $H_n(z)$ can be defined by the contour integral

$$H_n(z) = \frac{n!}{2\pi i} \oint e^{-t^2+2tz} t^{-n-1}\,dt, \tag{1}$$

where the contour encloses the origin and is traversed in a counterclockwise direction (Arfken 1985, p. 416).

The first few Hermite polynomials are

$$H_0(x) = 1 \tag{2}$$
$$H_1(x) = 2x \tag{3}$$
$$H_2(x) = 4x^2 - 2 \tag{4}$$
$$H_3(x) = 8x^3 - 12x \tag{5}$$
$$H_4(x) = 16x^4 - 48x^2 + 12 \tag{6}$$
$$H_5(x) = 32x^5 - 160x^3 + 120x \tag{7}$$
$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120 \tag{8}$$
$$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x \tag{9}$$
$$H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \tag{10}$$
$$H_9(x) = 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x \tag{11}$$

$$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240. \quad (12)$$

When ordered from smallest to largest powers, the triangle of nonzero coefficients is 1; 2; -2, 4; -12, 8; 12, -48, 16; 120, -160, 32; ... (OEIS A059343).

The values $H_n(0)$ may be called Hermite numbers.

The Hermite polynomials are a Sheffer sequence with

$$g(t) = e^{t^2/4} \quad (13)$$
$$f(t) = \frac{1}{2}t \quad (14)$$

(Roman 1984, p. 30), giving the exponential generating function

$$\exp(2xt - t^2) \equiv \sum_{n=0}^{\infty} \frac{H_n(x)\,t^n}{n!}. \quad (15)$$

Using a Taylor series shows that

$$H_n(x) = \left[\left(\frac{\partial}{\partial t}\right)^n \exp(2xt - t^2)\right]_{t=0} \quad (16)$$
$$= \left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}. \quad (17)$$

Since $\partial f(x-t)/\partial t = -\partial f(x-t)/\partial x$,

$$H_n(x) = (-1)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0} \quad (18)$$
$$= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (19)$$

Now define operators

$$\tilde{O}_1 \equiv -e^{x^2} \frac{d}{dx} e^{-x^2} \quad (20)$$
$$\tilde{O}_2 \equiv e^{x^2/2}\left(x - \frac{d}{dx}\right)e^{-x^2/2}. \quad (21)$$

It follows that

$$\tilde{O}_1 f = -e^{x^2} \frac{d}{dx}\left[f e^{-x^2}\right] \quad (22)$$
$$= 2xf - \frac{df}{dx} \quad (23)$$

$$\tilde{O}_2 f = e^{x^2/2}\left(x - \frac{d}{dx}\right)\left[f\, e^{-x^2/2}\right] \tag{24}$$
$$= x\, f + x\, f - \frac{d f}{d x} \tag{25}$$
$$= 2\, x\, f - \frac{d f}{d x}, \tag{26}$$

so

$$\tilde{O}_1 = \tilde{O}_2, \tag{27}$$

and

$$-e^{x^2} \frac{d}{d x} e^{-x^2} = e^{x^2/2}\left(x - \frac{d}{d x}\right) e^{-x^2/2} \tag{28}$$

(Arfken 1985, p. 720), which means the following definitions are equivalent:

$$\exp\left(2\, x\, t - t^2\right) \equiv \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} \tag{29}$$
$$H_n(x) \equiv (-1)^n e^{x^2} \frac{d^n}{d x^n} e^{-x^2} \tag{30}$$
$$H_n(x) \equiv e^{x^2/2}\left(x - \frac{d}{d x}\right)^n e^{-x^2/2} \tag{31}$$

(Arfken 1985, pp. 712-713 and 720).

The Hermite polynomials may be written as

$$H_n(z) = (2\, z)^n\, {}_2F_0\left(-\tfrac{1}{2} n, -\tfrac{1}{2}(n-1); ; -z^{-2}\right) \tag{32}$$
$$= 2^n z^n \left(z^2\right)^{-n/2} U\left(-\tfrac{1}{2} n, \tfrac{1}{2}, z^2\right) \tag{33}$$

(Koekoek and Swarttouw 1998), where $U(a, b, z)$ is a confluent hypergeometric function of the second kind, which can be simplified to

$$H_n(z) = 2^n\, U\left(-\tfrac{1}{2} n, \tfrac{1}{2}, z^2\right) \tag{34}$$

in the right half-plane $\mathbb{R}[z] > 0$.

The Hermite polynomials are related to the derivative of erf by

$$H_n(z) = \tfrac{1}{2} (-1)^n \sqrt{\pi}\, e^{z^2} \frac{d^{n+1}}{d z^{n+1}} \operatorname{erf}(z). \tag{35}$$

They have a contour integral representation

$$H_n(x) = \frac{n!}{2\pi i} \oint e^{-t^2+2tx}\, t^{-n-1}\, dt. \tag{36}$$

They are orthogonal in the range $(-\infty, \infty)$ with respect to the weighting function e^{-x^2}

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = \delta_{mn}\, 2^n\, n!\, \sqrt{\pi}\,. \tag{37}$$

The Hermite polynomials satisfy the symmetry condition

$$H_n(-x) = (-1)^n\, H_n(x). \tag{38}$$

They also obey the recurrence relations

$$H_{n+1}(x) = 2\,x\, H_n(x) - 2\,n\, H_{n-1}(x) \tag{39}$$

$$H_n'(x) = 2\,n\, H_{n-1}(x). \tag{40}$$

By solving the Hermite differential equation, the series

$$H_{2k}(x) = (-1)^k\, 2^k\, (2k-1)!! \left[1 + \sum_{j=1}^{k} \frac{(-4k)(-4k+4)\cdots(-4k+4j-4)}{(2j)!}\, x^{2j}\right] \tag{41}$$

$$= (-2)^k\, (2k-1)!!\; {}_1F_1\left(-k; \tfrac{1}{2}; x^2\right) \tag{42}$$

$$H_{2k+1}(x) = (-1)^k\, 2^{k+1}\, (2k+1)!! \left[x + \sum_{j=1}^{k} \frac{(-4k)(-4k+4)\cdots(-4k+4j-4)}{(2j+1)!}\, x^{2j+1}\right] \tag{43}$$

$$= (-1)^k\, 2^{k+1}\, (2k+1)!!\, x\; {}_1F_1\left(-k; \tfrac{3}{2}; x^2\right) \tag{44}$$

are obtained, where the products in the numerators are equal to

$$(-4k)(-4k+4)\cdots(-4k+4j-4) = 4^j\, (-k)_j, \tag{45}$$

with $(x)_n$ the Pochhammer symbol.

Let a set of associated functions be defined by

$$u_n(x) \equiv \sqrt{\frac{a}{\pi^{1/2}\, n!\, 2^n}}\; H_n(a\,x)\, e^{-a^2 x^2/2}, \tag{46}$$

then the u_n satisfy the orthogonality conditions

$$\int_{-\infty}^{\infty} u_n(x)\frac{d\,u_m}{d\,x}\,d\,x = \begin{cases} a\sqrt{\frac{n+1}{2}} & m=n+1 \\ -a\sqrt{\frac{n}{2}} & m=n-1 \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

$$\int_{-\infty}^{\infty} u_m(x)\,u_n(x)\,d\,x = \delta_{mn} \tag{48}$$

$$\int_{-\infty}^{\infty} u_m(x)\,x\,u_n(x)\,d\,x = \begin{cases} \frac{1}{a}\sqrt{\frac{n+1}{2}} & m=n+1 \\ \frac{1}{a}\sqrt{\frac{n}{2}} & m=n-1 \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

$$\int_{-\infty}^{\infty} u_m(x)\,x^2\,u_n(x)\,d\,x = \begin{cases} \frac{\sqrt{n(n-1)}}{2a^2} & m=n-2 \\ \frac{2n+1}{2a^2} & m=n \\ \frac{\sqrt{(n+1)(n+2)}}{2a^2} & m=n+2 \\ 0 & m\neq n\neq n\pm 2 \end{cases} \tag{50}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_\alpha(x)\,H_\beta(x)\,H_\gamma(x)\,d\,x = \sqrt{\pi}\,\frac{2^s\,\alpha!\,\beta!\,\gamma!}{(s-\alpha)!\,(s-\beta)!\,(s-\gamma)!}, \tag{51}$$

if $\alpha+\beta+\gamma=2s$ is even and $s\geq\alpha$, $s\geq\beta$, and $s\geq\gamma$. Otherwise, the last integral is 0 (Szegö 1975, p. 390). Another integral is

$$\int_{-\infty}^{\infty} u_n(x)\,x^r\,u_m(x)\,d\,x = \begin{cases} 0 & \text{if } r-n-m \text{ is odd} \\ \frac{r!}{(2a)^r}\sqrt{\frac{2^{m+n}}{m!\,n!}}\sum_{p=\max(0,-s)}^{\min(m,n)}\binom{n}{p}\binom{m}{p}\frac{p!}{2^p\,(s+p)!} & \text{otherwise,} \end{cases} \tag{52}$$

where $s=(r-n-m)/2$ and $\binom{n}{k}$ is a binomial coefficient (T. Drane, pers. comm., Feb. 14, 2006).

The polynomial discriminant is

$$D_n = 2^{3n(n-1)/2}\prod_{k=1}^{n} k^k \tag{53}$$

Two interesting identities involving $H_n(x+y)$ are given by

$$\sum_{k=0}^{n}\binom{n}{k} H_k(x) H_{n-k}(y) = 2^{n/2} H_n\left(2^{-1/2}(x+y)\right) \tag{54}$$

and

$$\sum_{k=0}^{n}\binom{n}{k} H_k(x)(2y)^{n-k} = H_n(x+y) \tag{55}$$

(G. Colomer, pers. comm.). A very pretty identity is

$$H_n(x+y) = (H+2y)^n, \tag{56}$$

where $H^k = H_k(x)$ (T. Drane, pers. comm., Feb. 14, 2006).

They also obey the sum

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} H_n(k) = 2^n n!, \tag{57}$$

as well as the more complicated

$$H_n(x) = H_n + \sum_{m=0}^{\lfloor n/2 \rfloor}\left[\sum_{k=1}^{n-2m}(-1)^k S(n-2m,k)(-x)_k\right] \times \frac{(-1)^m 2^{n-2m} n!}{(n-2m)!\, m!}, \tag{58}$$

where $H_n = H_n(0)$ is a Hermite number, $S(n,k)$ is a Stirling number of the second kind, and $(x)_n$ is a Pochhammer symbol. A class of generalized Hermite polynomials $\gamma_n^m(x)$ satisfying

$$e^{mxt-t^m} = \sum_{n=0}^{\infty}\gamma_n^m(x)\, t^n \tag{59}$$

was studied by Subramanyan (1990). A class of related polynomials defined by

$$h_{n,m} = \gamma_n^m\left(\frac{2x}{m}\right) \tag{60}$$

and with generating function

$$e^{2xt-t^m} = \sum_{n=0}^{\infty} h_{n,m}(x)\, t^n \tag{61}$$

was studied by Djordjević (1996). They satisfy

$$H_n(x) = n!\, h_{n,2}(x). \quad (62)$$

defines a generalized Hermite polynomial $H_n^{(\nu)}(x)$ with variance ν.

A modified version of the Hermite polynomial is sometimes (but rarely) defined by

$$\mathrm{He}_n(x) \equiv 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right) \quad (63)$$

The first few of these polynomials are given by

$$\mathrm{He}_1(x) = x \quad (64)$$
$$\mathrm{He}_2(x) = x^2 - 1 \quad (65)$$
$$\mathrm{He}_3(x) = x^3 - 3x \quad (66)$$
$$\mathrm{He}_4(x) = x^4 - 6x^2 + 3 \quad (67)$$
$$\mathrm{He}_5(x) = x^5 - 10x^3 + 15x. \quad (68)$$

The polynomial $\mathrm{He}_n(x)$ is the independence polynomial of the complete graph K_n.

Generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}$$

Recurrence formulas

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$
$$H_n'(x) = 2nH_{n-1}(x)$$

Orthogonality of Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = 0 \qquad m \neq n$$
$$\int_{-\infty}^{\infty} e^{-x^2} \left[H_n(x)\right]^2 dx = 2^n n! \sqrt{\pi}$$

Rodrigue's formula of Hermite polynomial

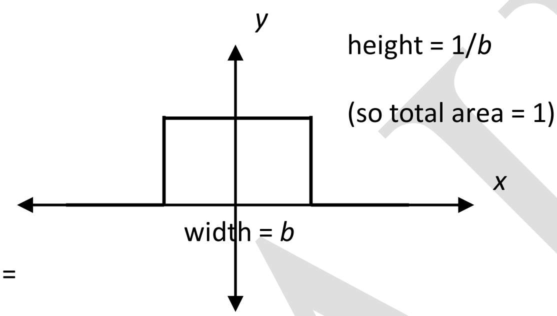
$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n} \qquad n = 0,1,2,\cdots, \ -\infty < t < \infty$$

The first few Hermite polynomials are

$H_0(t) = 1$ $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$ $H_3(t) = 8t - 12t$, $H_4(t) = 16 -- 48t + 12t$,
$H_5(t) = 32t^5 - 160t^3 + 120t$, etc

Dirac delta function

1. Definition as limit. The Dirac delta function can be thought of as a rectangular pulse that grows narrower and narrower while simultaneously growing larger and larger.



rect(x, b) =

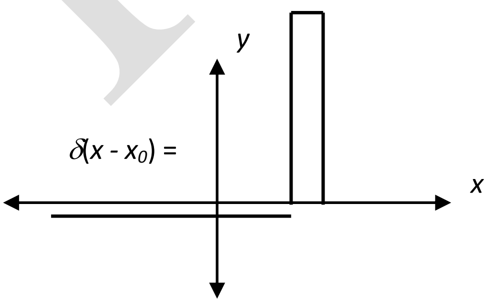
$\delta(x)$ = lim(b→0) rect(x, b)

Note that the integral of the delta function is the area under the curve, and has been held constant at 1 throughout the limit process.

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

Shifting the origin. Just as a parabola can be shifted away from the origin by writing y = $(x - x_0)^2$ instead of just y = x^2, any function can be shifted by plugging in $x - x_0$ in place of its usual argument x.

$\delta(x - x_0)$ =



$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

Shifting the position of the peak doesn't affect the total area if the integral is taken from –∞ to ∞.

Possible questions – (Part B- 6 marks)

1. State and prove the recurrence relations of Bessel's function.
2. Obtain the solution for Hermite Differential equation
3. Write down Hermite differential equation and obtain Hermite polynomial from that.
   (ii) Show that $H_n(-x) = (-1)^n H_n(x)$
4. Derive the Recurrence relations for spherical Bessel functions.
5. Derive Rodrigue's Formula for Hermite polynomial.
6. Discuss about the Dirac – Delta function.
7. Discuss about the Spherical bessal function of zeroth order.
8. Derive the recurrence formula for Hermite polynomial.
9. Discuss about the Bessel's differential equation for Bessel's function of first kind.

Possible questions – (Part C- 10 marks)

1. State and prove the recurrence relations of Bessel's function.
2. Write down the Hermite Differential equation and obtain Hermite polynomial from that.
3. Show that when n is integer,
   a. $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - xsin\theta)\, d\theta$
   b. $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(xcos\varphi)\, d\varphi$
4. Derive the Recurrence relations for spherical Bessel functions.
5. Derive Rodrigue's Formula for Hermite polynomial.
6. Discuss about the Dirac – Delta function.
7. Derive the Rodrique's formula for Hermite polynomial.
8. Discuss about the Bessel's differential equation for Bessel's function of second kind.

KARPAGAM ACADEMY OF HIGHER EDUCATION,COIMBATORE-21

DEPARTMENT OF PHYSICS

CLASS:I MSc PHYSICS **BATCH:2019**

MATHEMATICAL PHYSICS (19PHP104)

MULIPLE CHOICE QUESTIONS

Questions	**opt1**	**opt2**	**opt3**
UNIT V			
The value of $J_{-1/2}(x)$ is	$\sqrt{\frac{2}{x}}\sin x$	$\sqrt{\frac{2}{\pi x}}\sin x$	$\sqrt{\frac{2}{\pi x}}\cos x$
The Rodrigue formula for Pn(x), the Legendre polynomial of degree 'n' is	$K = \frac{n!}{2^n}$	$K = \frac{2^n}{n!}$	$K = \frac{1}{2^n n!}$
The value of Jo(x) at the origin is	1	0	–1
The value of $P_1(x)$ is	x	1	$x^2/2$
The identical roots of the Legendre's functions are	m = ± n	m = ± 1	m = 0 or m = 1
The value of $J_{1/2}(x)$ is	$\sqrt{\frac{2}{x}}\sin x$	$\sqrt{\frac{2}{\pi x}}\sin x$	$\sqrt{\frac{2}{\pi x}}\cos x$
If Jo and J_1 are Bessel's functions then $J_1'(x)$ is given by	Jo(x) – 1/x $J_1(x)$	– Jo	Jo(x) + 1/x $J_1(x)$
The value of the integral where Jn(x) is the Bessel function of the first kind of order n, is equal to	0	-2	2
The integral $\int xJo(x)dx$ is equal to	$xJ_1(x)$ – Jo(x)	$xJ_1(x)$	$J_1(x)$
If $J_{n+1}(x) = (2/x) J_n(x)$ – Jo(x) where Jn is the Bessel function of first kind order 'n'. Then 'n' is	0	2	–1
The value of $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2$ is	$\sqrt{\frac{2}{\pi x}}$	$\frac{2}{\pi x}$	$\frac{2}{\pi}$
The value of Po(x) is	1	x	0
The value of the $\int_{-1}^{1} P_n(x)P_m(x)dx$ if	n = m	n ¹ m	n > m
The polynomial $2x^2+x+3$ interms of Legendre polynomial is	$\frac{1}{3}(4P_2 - 3P_1 + 11P_0)$	$\frac{1}{3}(4P_2+3P_1+11P_0)$	$\frac{1}{3}(4P_2+3P_1+11P_0)$

Let Pn(x) be the Legendre polynomial, then Pn(-x) is equal to	$(-1)^{n+1} P_n'(x)$	$(-1)^n P_n'(x)$	$(-1)^n P_n(x)$
If $P_n(x)$ is the Legendre polynomial of order 'n', then $3x^2 + 3x + 1$ can be expressed as	$3P_2 + 3P_1$	$4P_2+2P_1 + Po$	$3P_2+3P_1 + Po$
If $\int_{-1}^{+1} P_n(x)dx = 2$, then 'n' is	1	0	–1
Legendre polynomial is where K is equal to	63/2	63/5	63/10
Let Pn(x) be Legendre polynomial of degree n>1, then is equal to	0	1 / (2n+1)	2 / (2n+1)
The value of $\int_{-1}^{+1}(2x+1)P_n(x)dx$ is the third degree Legendre polynomial is	1	–1	2
The Legendre polynomial Pn(x) has	n real zeros between 0 and 1	n zeros of which only one is between –1 and +1	2n-1 real zeros between –1 and 1
The incorrect equation among the following is	$P_o(x) = 1$	$P_1(x) = x$	$P_n(-x) = (-1)^{n+1} P_n(x)$
The value of $P_n(-x)$ is	$- P_n(x)$	$P_n(x)$	$(-1)^n P_n(x)$
The value of $2J_n'$ is	$J_{n-1} – J_{n+1}$	$J_{n-1} + J_{n+1}$	$J_{n+1} – J_{n+1}$
The root of $x^3 – 6x + 4$ lies between	–1 and 0	1 and 2	–2 and 1
Bessel's functions also called	cylindrical	circular	square
From Bessel's functions, the value of Jn+1(x) is	$nJ_n(x) + J_n'(x)$	$(n / x) J_n(x) - J_n'(x)$	$nJ_n(x) - J_n'(x)$
The value of $J_{-1/2}(x)$ is	$\sqrt{(2/x)} \sin x$	$\sqrt{(2/\pi x)} \sin x$	$\sqrt{(2/\pi x)} \cos x$
If $J_n(x)$ is the Bessel function of the first kind, then	$x^{-2}J_2(x) + C$	$x^2J_2(x) + C$	$- x^2J_3(x) + C$
When 'n' is an integer, $J_n(x)$ and $J_{-n}(x)$ are	harmonic	linearly independent	orthonormal
Bessel's functions are	indeterminate	simple harmonic	oscillatory functions

If $J_{n+1}(x) = (2/x)\ J_n(x) –$ Jo(x) where Jn is the Bessel function of first kind order 'n'. Then 'n' is	0	2	–1
Let f, g be polynomials of degrees a, b respectively. Let h(x) = f(g(x)). The degree of h is:	ab	a + b	a*b*c
Let f, g, h be nonzero polynomials such that f(x) − g(x) = h(x) and deg f = deg h. Pick the true statement:	deg g ≤ deg f	deg g > deg f	deg g has no relation to deg f
Let f, g, h be polynomials such that f(x) = g(x) + x3 h(x). Then f(j)(0) = g(j)(0) for	j = 0.	j = 1	j = 2
what is the value of d/dx[(x−nJn(x)]	−x-nJn+1(x).	Jn-1 + Jn+1	jn+1
In hermite polynomial what is value for H2(x)	4x2 −2	0	x2

opt4	**Answer**
$\sqrt{\frac{2}{x}}\cos x$	$\sqrt{\frac{2}{\pi x}}\sin x$
$K=\frac{1}{2^n(n!)^2}$	$K=\frac{2^n}{n!}$
x	1
½ $(x^2 - 1)$	x
m = 0 or m = -1	m = ± 1
$\sqrt{\frac{2}{x}}\cos x$	$\sqrt{\frac{2}{\pi x}}\sin x$
Jo(x) – $1/x^2$ $J_1(x)$	Jo(x) – 1/x $J_1(x)$
1	0
$x^2 J_n(x)$	$xJ_1(x)$
1	1
$\sqrt{\frac{2}{\pi}}$	$\frac{2}{\pi x}$
–1	–1
n < m	n ¹ m
$\frac{1}{3}(4P_2 - 3P_1 - 1\,1P_0)$	$\frac{1}{3}(4P_2 - 3P_1 - 1\,1P_0)$

$P_n''(x)$	$(-1)^n P_n(x)$
$2P_2+3P_1 + 2P_o$	$2P_2+3P_1 + 2P_o$
μ	0
63/8	63/8
n / (2n+1)	n / (2n+1)
0	0
none of these	n real zeros between 0 and 1
$P_n(x) = (-1)^{n+1} P_n(x)$	$P_n(-x) = (-1)^{n+1} P_n(x)$
$(-1)^n P_n(-x)$	$(-1)^n P_n(x)$
$2 J_{n+1}$	$J_{n+1} – J_{n+1}$
0 and 1	–1 and 0
linear	cylindrical
$(n / x) J_n(x) + J_n'(x)$	$(n / x) J_n(x) - J_n'(x)$
$\sqrt{(2/x)} \cos x$	$\sqrt{(2/\pi x)} \cos x$
$- x^{-1}J_3(x) + C_2$	$x^{-2}J_2(x) + C$
linearly dependent	harmonic function
critically damped	oscillatory functions

1	1
a/b	ab
deg g = deg f	deg g ≤ deg f
all the above	all of above
jn	−x-nJn+1(x).
x3	4x2 −2