Electrostatics & Magnetostatics

SYLLABUS

Electrostatics: Electric intensity – Electric potential – Gauss Law - Dielectric and its polarization - Electric displacement D – Dielectric constant ε_r – Polarisibiltiy α - Clausius-Mossotti relation (Non-polar molecules) – The Langevin equation (Polar molecules) – Electrostatic energy

Magnetostatics: Current density J – Ampere's law of force – Biot-Savart law – Ampere's circuital law – Magnetic scalar potential ϕ_m (no applications) – Magnetic vector potential A – Magnetisation and magnetization current – Magnetic intensity – Magnetic susceptibility and Permeability.

ELECTRIC INTENSITY

The space surrounding any charge or charge distribution in which the effects of its presence can be felt is called the field of the charge or charge distribution. The strength of the field called electric intensity at a point is defined by a vector function of position in space which determines the force on a unit positive charge at rest at that point of space. The unit of electric intensity in M.K.S system is volts/meter.

If a test charge q_0 is placed at the position r in space at to determine electric field and the test charge experience a force F there, then electric field at the point will be E= F/q0 ------ (1)

Thus in turn implies that the electric field of a point charge at a point r_{j0} will be

$$E_{j0} = F_{j0}/q_0 = (1/4\pi\varepsilon_0)\,(q_j/r^3_{jo})\,r_{jo}$$

$$\text{Or } E = (qr/4\pi\,\varepsilon_0 r^3) \text{ -------- (2)}$$

Where r is the vector distance from q to the point at which E is evaluated.

The total field due to a set of charges i.e., discrete distribution of charges will be

$$E = 1/4\pi\varepsilon_0 \sum_{j=1}^{n} (q_j/r^3_{jo})\,r_{jo} \text{--------- (3)}$$

In continuous distribution of charge, the summation can be changed to integration,

$$E = 1/4\pi\varepsilon_0 \int \frac{r}{r^3}\,dq - - - - - - - - (4)$$

where r represents the vector distance from the element of integration dq to the point at which E is evaluated.

ELECTRIC POTENTIAL ENERGY

Potential energy can be defined as the capacity for doing work which arises from position or configuration. In the electrical case, a charge will exert a force on any other charge and potential energy arises from any collection of charges. For example, if a positive charge Q is fixed at some point in space, any other positive charge which is brought close to it will experience a repulsive force and will therefore have potential energy. The potential energy of a test charge q in the vicinity of this source charge will be:

$$U = kQq/r \text{ -------- (1)}$$

where k is Coulomb's constant.

In electricity, it is usually more convenient to use the electric potential energy per unit charge, just called electric potential.

GAUSS' LAW FOR ELECTRICITY

The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.

The integral form of Gauss' Law finds application in calculating electric fieldsaround charged objects.

In applying Gauss' law to the electric field of a point charge, one can show that it is consistent with Coulomb's law.

While the area integral of the electric field gives a measure of the net charge enclosed, the divergence of the electric field gives a measure of the density of sources. It also has implications for the conservation of charge.

Integral Form

$$\oint \vec{E}.\overrightarrow{dA} = \frac{q}{\varepsilon_0} = 4\pi k q$$

Differential form

$$\nabla.E = \frac{\rho}{\varepsilon_0} = 4\pi k \rho$$

GAUSS' LAW FOR MAGNETISM

The net magnetic flux out of any closed surface is zero. This amounts to a statement about the sources of magnetic field. For a magnetic dipole, any closed surface the magnetic flux directed inward toward the south pole will equal the flux outward from the north pole. The net flux will always be zero for dipole sources. If there were a magnetic monopole source, this would give a non-zero area integral. The divergence of a vector field is proportional to the point source density, so the form of Gauss' law for magnetic fields is then a statement that there are no magnetic monopoles

Integral form, $\oint \overrightarrow{B}.\overrightarrow{dA} = 0$

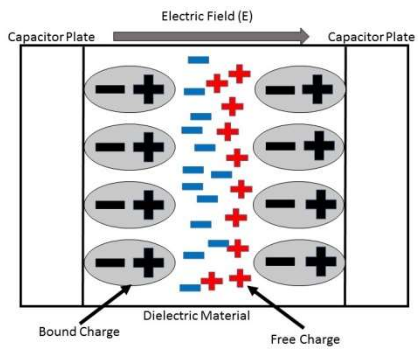
Differential form, Δ.B = 0

POLARIZABILITY

Dielectric polarization occurs when a dipole moment is formed in an insulating material because of an externally applied electric field. When a current interacts with a dielectric (insulating) material, the dielectric material will respond with a shift in charge distribution with the positive charges aligning with the electric field and the negative charges aligning against it. By taking advantage of this response, important circuit elements such as capacitors can be made

Introduction

Dielectric polarization is the term given to describe the behavior of a material when an external electric field is applied on it. A simple picture can be made using a capacitor as an example. The charges in the material will have a response to the electric field caused by the plates.



Using the capacitor model, it is possible to define the relative permittivity or the dielectric constant of the material by setting its relative permittivity equivalent to the ratio of the measured capacitance and the capacitance of a test capacitor, which is also equal to the absolute permittivity of the material divided by the permittivity of a vacuum.

$$\epsilon_r = Q/Q_0 = C/C_o = \epsilon/\epsilon_0 \text{ --------- } (1)$$

The dielectric constant is an important term, because another term known as the electronic polarizability or αeαe can be related to the dielectric constant. The electronic polarizability is a microscopic polarization phenomena that occurs in all materials and is one of the main mechanisms that drives dielectric polarization.

To explain how the dielectric constant relates to the electronic polarizability of a material, the polarization or P of a material should be determined. The polarization of a material is defined as the total dipole moment per unit volume, and its equation is,

$$P=N\alpha eE=\chi_e\epsilon_0E \text{ -..........(2)}$$

where the χ term is known as the electric susceptibility of the material given by the equation $\chi=\epsilon_r-1$. Then, from substituting ϵ_r-1 for χ, an equation relating the relative permittivity and the electronic polarizability is determined.

$$\epsilon_r=1+N\epsilon_e/\epsilon_0$$

Where N is the number of molecules per unit volume.

While this equation does relate the dielectric constant with the electronic polarizability, it only represents the material as a whole, and does not take into effect the local field, or the field experienced by a molecule in a dielectric. This field is known as the Lorentz field, and the equation to define this is given as,

$$E_{loc}=E+1/3\epsilon_0P \text{ .... (3)}$$

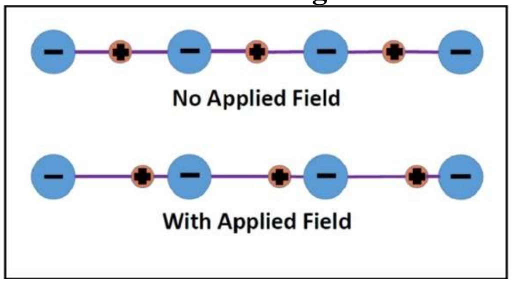
and by substituting this value back for the field used in the previous method, the following equation is determined

$$\epsilon_r-1/\epsilon_r+2=N\alpha_e/3\epsilon_0 \text{ .... (4)}$$

This equation is known as the Clausius-Mossotti equation and is the way to interchange between the microscopic property of electronic permittivity and the dielectric constant. In addition to knowing the electronic polarizability of a material, there are also other sub-factors, such as chemical composition and bond type that determine the total dielectric behavior of a material. However, electronic polarization is always inherent in a dielectric material.

Ionic Polarization

Ionic polarization is a mechanism that contributes to the relative permittivity of a material. This type of polarization typically occurs in ionic crystal elements such as NaCl, KCl, and LiBr. There is no net polarization inside these materials in the absence of an external electric field because the dipole moments of the negative ions are canceled out with the positive ions. However, when an external field is applied, the ions become displaced, which leads to an induced polarization.



The equation to describe this effect is given by,

$$P_av=\alpha_i E_{loc} \text{ ……… } (5)$$

Where P_{av} is the induced average dipole moment per ion pair, αiαi is the ionic polarizability, and E is the local electric field experienced by the pair of ions. Usually the ionic polarizability is greater than the electronic polarizablity by a factor of 10 which leads to ionic substances having high dielectric constants. Similar to electronic polarization, ionic polarization also has a total polarization associated with it. The equation is given by

$$P=N_i P_a v=N_i \alpha_i E_{loc} \text{ ……… } (6)$$

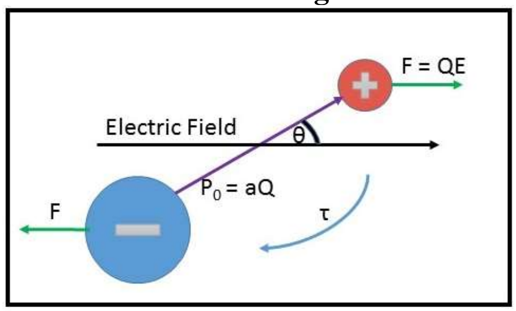
which will also lead to a Clausius-Mossotti equation for ionic polarization,

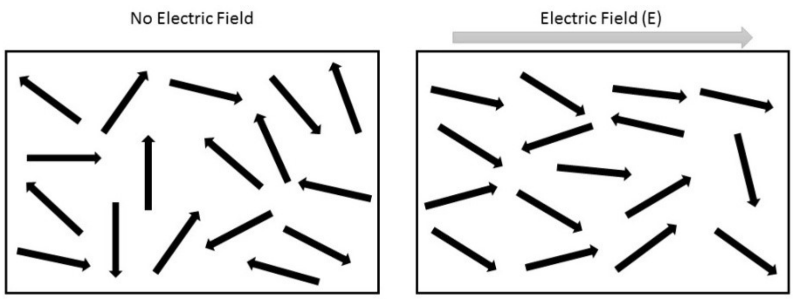
$$\epsilon_r-1/\epsilon_r+2=1/3\epsilon_0 N_i \alpha_i \text{ ……… } (7)$$

Note that these equations assume that there is a charge balance inside the ionic material (eg. NaCl) whereas if a charge imbalance is present, such as in materials like CaF_2, a different set of equations must be used.

Orientational Polarization

Orientational polarization arises when there is a permanent dipole moment in the material. Materials such as HCl and H_2O will have a net permanent dipole moment because the charge distributions of these molecules are skewed. For example, in a HCl molecule, the chlorine atom will be negatively charged and the hydrogen atoms will be positively charged causing the molecule to be dipolar. The dipolar nature of the molecule should cause a dipole moment in the material, however, in the absence of an electric field, the dipole moment is canceled out by thermal agitation resulting in a net zero dipole moment per molecule. When an electric field is applied, however, the molecule will begin to rotate to align the molecule with the field, causing a net average dipole moment per molecule.





To determine if the induced average dipole moment along the electric field, the average potential energy of the dipole must be calculated and compared to the average thermal energy 1/5kT, as determined by thermodynamics for five degrees of freedom. To accomplish this, Force F torque τ acting on the rigid body of the dipolar molecule. Using this model, the equation for the torque is given by

$$\tau=(F\sin\theta)a=EP_0\sin(\theta) \;\ldots\ldots\ldots\; (8)$$

where P_0 is given by $P_0=aQ$. From this equation the maximum potential energy can be calculated by taking the integral at the instant of maximum torque. This maximum potential energy is calculated out to be $2P_0E$ which then means that the average dipole potential energy is $1/2E_{max}$ or P_0E. Once knowing the average dipole potential energy, it is then possible to calculate the average dipole moment P_{av} through Boltzmann's statistics, which would lead to the answer,

$$P_av=P^2_0E/3kT \;\ldots\ldots\ldots (9)$$

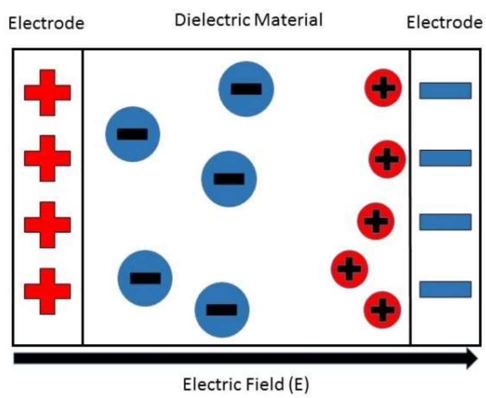
This, leads to the dipolar orientational polarizability, α_d per molecule, which is shown by the equation,

$$\alpha_d=P^2_0/3kt$$

The equation for orientational polarizability shows, that unlike electronic polarizability and ionic polarizability, orientational polarizability is temperature dependent. This is an important factor to consider when choosing a dielectric material for electronic and optical applications.

Interfacial Polarization

Interfacial or space charge polarization occurs when there is an accumulation of charge at an interface between two materials or between two regions within a material because of an external field. This can occur when there is a compound dielectric, or when there are two electrodes connected to a dielectric material. This type of electric polarization is different from orientational and ionic polarization because instead of affecting bound positive and negative charges i.e. ionic and covalent bonded structures, interfacial polarization also affects free charges as well. As a result interfacial polarization is usually observed in amorphous or polycrystalline solids. The electric field will cause a charge imbalance because of the dielectric material's insulating properties. However, the mobile charges in the dielectric will migrate over maintain charge neutrality. This then causes interfacial polarization.



The equation to show the space charge polarizability constant is

$$\alpha_c = \alpha - \alpha_\infty - \alpha_0$$

where α_c is the space charge polarizability and α, α_∞, and α_0 refer to the total, electronic, and orientational polarizations respectively. It is important to note that because the charges are free charges, defects such as grain boundaries or other interfaces can serve as a medium for interfacial polarizability to form.

From the equation of space charge polarization, it is then determined that the total amount of dielectric polarization in a material is the sum of the electronic, orientations, and interracial polarizabilities, or $\alpha = \alpha_c + \alpha_\infty + \alpha_0$.

Clausius-Mossotti Relation

A dielectric medium is made up of identical molecules that develop a dipole moment

$$\mathbf{p} = \alpha\,\epsilon_0\,\mathbf{E} \quad \text{--------- (1)}$$

when placed in an electric field $\mathbf{E}$. The constant α is called the *molecular polarizability*. If N is the number density of such molecules then the polarization of the medium is

$$\mathbf{P} = N\,\mathbf{p} = N\,\alpha\,\epsilon_0\,\mathbf{E}, \quad \text{---------- (2)}$$

Or

$$\mathbf{P} = \frac{N_A\,\rho_m\,\alpha}{M}\,\epsilon_0\,\mathbf{E}, \quad \text{-------- (3)}$$

where ρ_m is the mass density, N_A is Avogadro's number, and M is the molecular weight. But, how does the electric field experienced by an individual molecule relate to the average electric field in the medium? This is not a trivial question because we expect the electric field to vary strongly within the dielectric. Suppose that the dielectric is polarized by a mean electric field $\mathbf{E_0}$ that is uniform, and directed along the **z** -axis. Consider one of the dielectric's constituent molecules. Let us draw a sphere of radius **a** around this particular molecule. The surface of the sphere is intended to represent the boundary between the microscopic and the macroscopic ranges of phenomena affecting the molecule. We shall treat the dielectric outside the sphere as a continuous medium, and the dielectric inside the sphere as a collection of polarized molecules.

$$\sigma_b(\theta) = -P\,\cos\theta \quad \text{------- (4)}$$

on the inside of the sphere's surface, where r , **θ** ,**φ** are conventional spherical coordinates, and $\mathbf{P} = P\,\mathbf{e}_z = \epsilon_0\,(\epsilon - 1)\,E_0\,\mathbf{e}_z$ is the uniform polarization of the uniform dielectric outside the sphere. The magnitude of E_z at the molecule due to this surface charge is

$$E_z = -\frac{1}{4\pi\,\epsilon_0}\int_S \frac{\sigma_b(\theta)\,\cos\theta}{a^2}\,dS, \quad \text{-------- (5)}$$

where $dS = 2\pi a^2 \sin\theta\, d\theta$ is an element of the surface. It follows that

$$E_z = \frac{P}{2\epsilon_0}\int_0^\pi \cos^2\theta \sin\theta\, d\theta = \frac{P}{3\epsilon_0}.$$
-------- (6)

It is easily demonstrated, from symmetry, that $E_\theta = E_\varphi = 0$ at the molecule. Thus, the field at the molecule due to the bound charges distributed on the inside of the sphere's surface is

$$\mathbf{E} = \frac{\mathbf{P}}{3\epsilon_0}.$$
--------------- (7)

The field due to the individual molecules within the sphere is obtained by summing over the dipole fields of these molecules. The electric field a distance $\mathbf{r}$ from a dipole **P** is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}\right].$$
--------- (8)

It is assumed that the dipole moments of the molecules within the sphere are all the same, and also that the molecules are evenly distributed throughout the sphere. This being the case, the value of $\mathbf{E_z}$ at the molecule due to all of the other molecules within in the sphere,

$$E_z = -\frac{1}{4\pi\epsilon_0}\sum_{\text{mols}}\left[\frac{p_z}{r^3} - \frac{3(p_x\, x z + p_y\, y z + p_z\, z^2)}{r^5}\right],$$
---------- (9)

is zero, because, for evenly distributed molecules, $\sum_{\text{mols}} x^2 = \sum_{\text{mols}} y^2 = \sum_{\text{mols}} z^2 = \frac{1}{3}\sum_{\text{mols}} r^2$ ---- (9)

and $\sum_{\text{mols}} x y = \sum_{\text{mols}} y z = \sum_{\text{mols}} z x = 0.$ --------- (10)

It is also easily demonstrated that $E_\theta = E_\varphi = 0$. Hence, the electric field at the molecule due to the other molecules within the sphere averages to zero. It is clear that the net electric field experienced by an individual molecule is $\mathbf{E} = \mathbf{E}_0 + \frac{\mathbf{P}}{3\epsilon_0},$ -------- (11)

(596)

which is larger than the average electric field, $\mathbf{E}_0$, in the dielectric. The above analysis indicates that this effect is ascribable to the long range (rather than the short range) interactions of the molecule with the other molecules in the medium. Making use of Equation as well as the definition $\mathbf{P} = \epsilon_0\,(\epsilon - 1)\,\mathbf{E}_0$, we obtain $\frac{\epsilon - 1}{\epsilon + 2} = \frac{N_A\,\rho_m\,\alpha}{3\,M},$ ------- (12)

which is known as the *Clausius-Mossotti relation*. This expression is found to work very well for a wide class of dielectric liquids and gases. The Clausius-Mossotti relation also yields

$$\frac{d\epsilon}{d\rho_m} = \frac{(\epsilon - 1)\,(\epsilon + 2)}{3\,\rho_m}.$$

DIELECTRIC CONSTANT

Dielectric constant is a property of an electrical insulating material equal to the ratio of the capacitance of a capacitor filled with the given material to the capacitance of an identical capacitor in a vacuum without the dielectric material. The insertion of a dielectric between the plates of, say, a parallel-plate capacitor always increases its capacitance, or ability to store opposite charges on each plate, compared with this ability when the plates are separated by a vacuum. If C is the value of the capacitance of a capacitor filled with a given dielectric and C_0 is the capacitance of an identical capacitor in a vacuum, the dielectric constant, symbolized by the Greek letter kappa, κ, is simply expressed as $\kappa = C/C_0$. The dielectric constant is a number without dimensions. It denotes a large-scale property of dielectrics without specifying the electrical behavior on the atomic scale.

The dielectric constant is sometimes called relative permittivity or specific inductive capacity. In the centimeter–gram–second system the dielectric constant is identical to the permittivity.

LANGEVIN EQUATION

Langevin modified the Clausius –Mossotti relation for polar molecules. The atom has a permanent dipole moment p0 and the only force acting on it due to the field is Em. The couple acting on the dipole whose axis subtends an angle θ with the field.

$$C = q\, 2l \sin\theta E_m = p_0\, E_m \sin\theta$$

So the work done on the dipole in a small rotation d_θ, $dW = Cd\theta = p_0 E_m \sin\theta\, d\theta$

$$W = - p_0 E_m \cos\theta \text{ ------- (1)}$$

This work is store by the dipole as the potential energy.

On the basis of the classical statistical mechanics, the number of molecules per unit volume whose axes makes an angle θ with the field is proportional to $e^{-W/kT}$, where k is the Boltzmann constant and T is the temperature in degree Kelvin. Hence the number of dipoles per unit volume whose axes makes an angle θ and dθ within the solid angle will be

$$dn = Ae^{-W/kT} d\omega$$

$$n = 2\pi A\int_0^{\pi} e^{-W/kT} \sin\theta\, d\theta \text{ ---------- (2)}$$

dn particles contribute a component of electric moment $p_0 \cos\theta$ parrallel to the field while by symmetry the component perpendicular to the field neutralize one another. Hence the polarization of the atoms

$$P = 2\pi A\int_0^{\pi} (p0 \cos\theta)\ e^{-W/kT} \sin\theta\, d\theta \text{ ------- (3)}$$

From equation (2) and (3)

$$p/n = \frac{p0\int_0^{\pi} (\cos\theta)\ e^{-W/kT} \sin\theta\, d\theta}{\int_0^{\pi} e^{-W/kT} \sin\theta\, d\theta} \text{ ------- (4)}$$

if we take, $p_0E_m/kT = u$ and $u \cos\theta = t$

$$\text{or } P = Ps\ [\coth u - 1/u] = PsL(u) \text{ --------- (A)}$$

where $Ps = nP_0$ is the saturation value of polarization. Eqn., (A) is known as Langevin equation.

ELECTROSTATIC ENERGY:

The work done in assembling the charge from infinity to establish the distribution is called the electrostatic energy of the field. Assume that the density of the initial charge distribution is 0 and the charge is brought uniformly from infinity so that at any time $\int$ *the charge density is* $\propto P$ *where* $\propto$

$is\ a\ parameter\ lying\ in\ the\ range\ from\ 0\ to\ 1. When\ the\ density\ from\ the\ \propto$
$P\ the\ potential$

$$V' = \propto V \quad (\text{as } v \propto p) \qquad (1)$$

If we increases the charge density at every point in the distribution from $\propto$ P to $(\propto + d \propto)$P the charge in volume element dτ increases by

$$Dq = d\propto \rho d\tau \qquad \text{……………………..} (2)$$

And the energy supplied to the system in adding charge dq to the particular volume element is

$$du = v' dq \qquad \text{…..}(3)$$

Sub. The values of v′and dq from eqn (1) and (2) respect in (3) if repeat the contribution for all the elements in v_0/τ

$$du = (\propto v)(d \propto \rho d\tau) \qquad \text{………} (4)$$

If repeat the orientation for all the elements in v_o/τ

$$du = \int \propto v \rho d \propto d\tau$$

To increases the charge density to 1 from 0 everywhere

$$U = \int_0^1 \propto d \propto \int_\tau \rho v d\tau$$

$$U = 1/2 \int_\tau \rho v d\tau \qquad \text{………………………..} (A)$$

Or, $$U = 1/2 \int_\tau V Div\ D d\tau$$

As $$\text{div} (SA) = S \text{ div } A + A. \text{ grad } S$$

$$\text{Div} (VD) = V \text{ div } D + D \text{ grad } V$$

Therefore $$u = 1/2 \oint V.D\ ds + 1/2 \int E.D d\tau$$

As τ can be any volume which includes all the charge in the system. So if the volume τ extends to infinity i.e. all space, the surface will vanish and the surface contribution. This in turn means

$$U = 1/2 \int_{all\ space} E.D d\tau$$

1. The interpretation of eqn (B) is certainly possible throughout the field with an energy density

$$U_E = du/d\tau = 1/2\ E.D$$

2. In case of free spaces as

$$U= \quad 1/2\int[E0E.D]d\tau=1/2\varepsilon_0E^2d\tau$$

And electrostatic energy will be,

$$U_E=1/2\varepsilon_0E^2$$

3. If the distribution is discrete

$$\int_{all\ space} E^2 d\tau=\sum_{j=1}^{n} E\ j\hat{}2 \sum_{j=1}^{n} Ej \sum_{j=1}^{n} Ej$$

For the first term the position of other charges is immaterial for any point i.e. this term represents the work done in the creation of charges and is called self-energy u_0

$$U=u_0+E_0/2(\sum_{j=1}^{n}\sum_{j=1}^{n} E_J.E_J) \text{ with } u_0=E_0/2(\sum_{j=1}^{n} E_J2)$$

Second terms represents the work done by the charge in bringing then from infinity to the space to constitute the given distribution of charges

CURRENT DENSITY

In the field of electromagnetism, Current Density is the measurement of electric current (charge flow in amperes) per unit area of cross-section (m^2). This is a vector quantity, with both a magnitude (scalar) and a direction.

$$J = I/A$$

J = current density in amperes/m^2

I = current through a conductor, in amperes

A = cross-sectional area of the conductor, m^2

AMPERE'S LAW OF FORCE

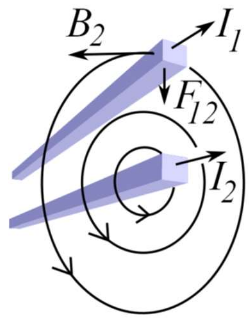
In magnetostatics, the force of attraction or repulsion between two current-carrying wires (see first figure below) is often called **Ampere's force law**. The physical origin of this force is that each wire generates a magnetic field, as defined by the Biot-Savart law, and the other wire experiences a magnetic force as a consequence, as defined by the Lorentz force.

Equation

The best-known and simplest example of Ampère's force law, which underlies the definition of the ampere, the SI unit of current, states that the force per unit length between two straight parallel conductors is

$$F_m = 2k_A\frac{I_1I_2}{r},$$

where k_A is the magnetic force constant, r is the separation of the wires, and I_1, I_2 are the direct currents carried by the wires. This is a good approximation for finite lengths if the distance between the wires is small compared to their lengths, but large compared to their diameters. The value of k_A depends upon the system of units chosen, and the value of k_A decides how large the unit of current will be.



In the SI system,

$$k_A \stackrel{\text{def}}{=} \frac{\mu_0}{4\pi}$$

with μ_0 the magnetic constant, *defined* in SI units as $\mu_0 \stackrel{\text{def}}{=} 4\pi \times 10^{-7}$ newtons / (ampere)^2.

Thus, in vacuum,*the force per meter of length between two parallel conductors – spaced apart by 1 m and* each *carrying a current of 1 A is exactly* 2×10^{-7} N/m.

The general formulation of the magnetic force for arbitrary geometries is based on line integrals and combines the Biot-Savart law and Lorentz force in one equation as shown below.

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} \int_{L_1} \int_{L_2} \frac{I_1 d\vec{\ell}_1 \times (I_2 d\vec{\ell}_2 \times \hat{\mathbf{r}}_{21})}{|r|^2}$$

where

- $\vec{F}_{12}$ is the total force felt by wire 1 due to wire 2 (usually measured in newtons),
- I_1 and I_2 are the currents running through wires 1 and 2, respectively (usually measured in amperes),
- The double line integration sums the force upon each element of wire 1 due to the magnetic field of each element of wire 2,
- $\overrightarrow{dl_1}$ and $\overrightarrow{dl_2}$ are infinitesimal vectors associated with wire 1 and wire 2 respectively (usually measured in metre); see line integral for a detailed definition,

The vector $\hat{\mathbf{r}}_{21}$ is the unit vector pointing from the differential element on wire 2 towards the differential element on wire 1, and $|r|$ is the distance separating these elements.

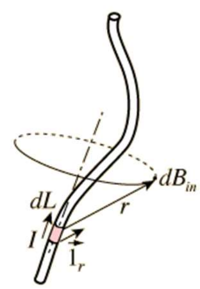
- The multiplication × is a vector cross product,
  - The sign of I_n is relative to the orientation $\overrightarrow{dl_n}$ (for example, if $\overrightarrow{dl_1}$ points in the direction of conventional current, then $I_1>0$).

To determine the force between wires in a material medium, the magnetic constant is replaced by the actual permeability of the medium.

BIOT-SAVART LAW

The Biot-Savart Law relates magnetic fields to the currents which are their sources. In a similar manner, Coulomb's law relates electric fields to the point charges which are their sources. Finding the magnetic field resulting from a current distribution involves the vector product, and is inherently a calculus problem when the distance from the current to the field point is continuously changing.

$$\overrightarrow{dB} = \frac{\mu_0 I \overrightarrow{dL} x \overrightarrow{I_r}}{4\pi r^2}$$



$\overrightarrow{dL}$ −Infinitesimal length of conductor carrying electric current I

$\overrightarrow{I_r}$ −Unit vector tospecify the direction of the vector distance r from the current to the field point.

AMPERE'S CIRCUITAL LAW

Ampere's Circuital Law states the relationship between the current and the magnetic field created by it. This law says, the integral of magnetic field density (B) along an imaginary closed path is equal to the product of current enclosed by the path and permeability of the medium.

$$\oint \overrightarrow{B} \cdot \overrightarrow{dl} = \mu_0 I$$

It alternatively says, the integral of magnetic field intensity (H) along an imaginary closed path is equal to the current enclosed by the path.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

$$\Rightarrow \oint \frac{\vec{B}}{\mu_0} \cdot \vec{dl} = I$$

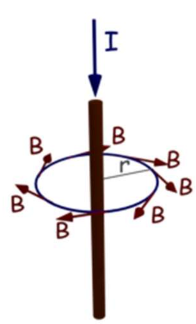
$$\Rightarrow \oint \vec{H} \cdot \vec{dl} = I$$

$$\left[\because \vec{H} = \frac{\vec{B}}{\mu_0}\right]$$

Let us consider an electrical conductor, carrying a current of I ampere. Let us take an imaginary loop around the conductor. This loop is called as Amperian loop, also imagine the radius of the loop is r and the flux density created at any point on the loop due to current through the conductor is B. Consider an infinitesimal length dl of the Amperian loop at the same point. At each point on the Amperian loop, the value of B is constant since the perpendicular distance of that point from the axis of conductor is fixed, but the direction will be along the tangent on the loop at that point. The close integral of the magnetic field density B along the amperian loop, will be,

$$\oint B \cdot dl \qquad [\,dot\ product]$$

$$= B \oint dl = B \cdot (2\pi r)$$



Now, according to **Ampere's Circuital Law**

$$\oint B \cdot dl = \mu_0 \cdot I$$

Therefore,

$$2\pi r\, B = \mu_0\, I$$
$$\Rightarrow \frac{B}{\mu_0} = \frac{I}{2\pi r}$$
$$\Rightarrow H = \frac{I}{2\pi r}$$

Instead of one current carrying conductor, there are N number of conductors carrying same current I, enclosed by the path, then

$$H = \frac{NI}{2\pi r}$$

MAGNETIC SCALAR AND VECTOR POTENTIALS

Let us relate the magnetic field intensity to a **scalar magnetic potential** and write:

$$\vec{H} = -\nabla V_m \quad \text{...................................(1)}$$

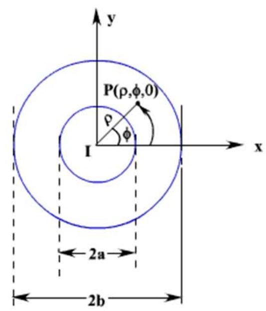
From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J} \quad \text{......................................(2)}$$

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$ ....(3)

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$. Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, V_m in general is not a single valued function of position.

Let us consider the cross section of a coaxial line. In the region a<ρ<b, $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$



If V_m is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

$$\therefore V_m = -\frac{I}{2\pi}\phi + c$$

If we set $V_m = 0$ at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

The lap around the current carrying conductor is complete, reach ϕ_0 again but V_m becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

The value of V_m keeps changing as additional laps to the same point. V_m analogous to electostatic potential V. But for static electric fields, $\nabla\times\vec{E}=0$ and $\oint\vec{E}\cdot d\vec{l}=0$, whereas for steady magnetic field $\nabla\times\vec{H}=0$ wherever $\vec{J}=0$ but $\oint\vec{H}\cdot d\vec{l}=I$ even if $\vec{J}=0$ along the path of integration.

Thee **vector magnetic potential** which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla.\vec{B}=0$and we have the vector identity that for any vector $\vec{A}$, $\nabla.\left(\nabla\times\vec{A}\right)=0$, we can write $\vec{B}=\nabla\times\vec{A}$.

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation. The vector function $\vec{A}$ and related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla.\vec{A}$is made as follows.

$$\nabla\times\nabla\times\vec{A}=\mu\nabla\times\vec{H}=\mu\vec{J} \quad \text{...(4)}$$

By using vector identity, $\nabla\times\nabla\times\vec{A}=\nabla(\nabla.\vec{A})-\nabla^2\vec{A}$ ....(5)

$$\nabla(\nabla.\vec{A})-\nabla^2\vec{A}=\mu\vec{J} \quad \text{..(6)}$$

if we choose $\nabla.\vec{A}=0$.

substituting $\nabla.\vec{A}=0$, we get $\nabla^2\vec{A}=-\mu\vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x \quad \text{.....................................(7a)}$$

$$\nabla^2 A_y = -\mu J_y \quad \text{.....................................(7b)}$$

$$\nabla^2 A_z = -\mu J_z \quad \text{.....................................(7c)}$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \quad \text{..(48)}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho}{R} dv', \qquad R = \left|\vec{r} - \vec{r}'\right| \quad \text{..................(9)}$$

In case of time varying fields we shall see that $\nabla.\vec{A} = \mu\varepsilon\frac{\partial V}{\partial t}$, which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for A_x as

$$A_x = \frac{\mu}{4\pi}\int_{v'} \frac{J_x}{R} dv' \quad \text{...................................(10)}$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{v'} \frac{\vec{J}}{R} dv' \quad \text{.......................................(11)}$$

This equation enables to find the vector potential at a given point because of a volume current density $\vec{J}$. Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_L \frac{I}{R} d\vec{l}' \quad \text{...(12)}$$

$$\vec{A} = \frac{\mu}{4\pi}\int_S \frac{\vec{K}}{R} ds' \quad \text{respectively. ....(13)}$$

The magnetic flux ψ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s} \quad \text{...(14)}$$

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A}.d\vec{s} = \oint_C \vec{A}.d\vec{l} \quad \text{...(15)}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

MAGNETIC VECTOR POTENTIAL

Electric fields generated by stationary charges obey $\nabla \times \mathbf{E} = 0.$ ------ (1)

This immediately allows us to write $\mathbf{E} = -\nabla\phi,$ -------- (2)

since the curl of a gradient is automatically zero. In fact, whenever we come across an irrotational vector field in physics we can always write it as the gradient of some scalar field. This is clearly a useful thing to do, since it enables us to replace a vector field by a much simpler scalar field. The quantity ϕ in the above equation is known as the *electric scalar potential*.

Magnetic fields generated by steady currents (and unsteady currents, for that matter)

satisfy $\nabla \cdot \mathbf{B} = 0.$ ------- (3)

This immediately allows us to write $\mathbf{B} = \nabla \times \mathbf{A}$, ------- (4)

since the divergence of a curl is automatically zero. In fact, whenever we come across a solenoidal vector field in physics we can always write it as the curl of some other vector field. This is not an obviously useful thing to do, however, since it only allows us to replace one vector field by another. Nevertheless, Eq. (4) is one of the most useful equations we shall come across in this lecture course. The quantity $\mathbf{A}$ is known as the *magnetic vector potential*.

The curl of the vector potential gives us the magnetic field via Eq. (4). However, the divergence of $\mathbf{A}$ has no physical significance. the magnetic field is invariant under the transformation $\mathbf{A} \rightarrow \mathbf{A} - \nabla\psi$.--------- (5)

In other words, the vector potential is undetermined to the gradient of a scalar field. This is just another way of saying that we are free to choose $\nabla \cdot \mathbf{A}$. The electric scalar potential is undetermined to an arbitrary additive constant, since the transformation $\phi \rightarrow \phi + c$ ------ (6) leaves the electric field invariant. The transformations are examples call *gauge transformations*.

The choice of a particular function ψ or a particular constant c is referred to as a choice of the gauge. We are free to fix the gauge to be whatever we like. The most sensible choice is the one which makes our equations as simple as possible. The usual gauge for the scalar potential ϕ is such that $\phi \rightarrow 0$ at infinity. The usual gauge for $\mathbf{A}$ is such that $\nabla \cdot \mathbf{A} = 0$.----- (7). This particular choice is known as the *Coulomb gauge*.

It is obvious to add a constant to ϕ so as to make it zero at infinity. Suppose that we have found some vector field $\mathbf{A}$ whose curl gives the magnetic field but whose divergence in non-zero. Let $\nabla \cdot \mathbf{A} = v(\mathbf{r})$. ------ (8). The question is, can we find a scalar field ψ such that after we perform the gauge transformation we are left with $\nabla \cdot \mathbf{A} = 0$. Taking the divergence of Eq. to find a function ψ which satisfies $\nabla^2 \psi = v$. ------- (9)

But this is just Poisson's equation, we can always find a unique solution of this equation. This proves that, in practice, the divergence of $\mathbf{A}$ equal to zero.

Let us again consider an infinite straight wire directed along the z-axis and carrying a current I. The magnetic field generated by such a wire is written

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \left(\frac{-y}{r^2}, \frac{x}{r^2}, 0 \right). \text{ ----- (10)}$$

To find a vector potential $\mathbf{A}$ whose curl is equal to the above magnetic field, and whose divergence is zero. It is not difficult to see that

$$\mathbf{A} = -\frac{\mu_0 I}{4\pi} \left(0,\, 0,\, \ln[x^2 + y^2] \right) \text{ ------- (11)}$$

fits the bill. Note that the vector potential is parallel to the direction of the current. This would seem to suggest that there is a more direct relationship between the vector potential and the current than there is between the magnetic field and the current. The potential is not very well-behaved on the z-axis, but this is just because we are dealing with an infinitely thin current.

Let us take the curl of Eq. (11).

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A}, \text{ ------- (12)}$$

where use has been made of the Coulomb gauge condition. We can combine the above relation with the field equation to give $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}$. ------- (13)

By writing this in component form, we obtain $\nabla^2 A_x = -\mu_0\, j_x,$ ------ (14)

$$\nabla^2 A_y = -\mu_0\, j_y, \quad \text{------- (15)}$$

$$\nabla^2 A_z = -\mu_0\, j_z. \quad \text{------- (16)}$$

This is just Poisson's equation three times over. The unique solutions to the above equations:

$$A_x(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{j_x(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}', \quad \text{……. (17)}$$

$$A_y(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{j_y(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}', \quad \text{-------- (18)}$$

$$A_z(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{j_z(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \quad \text{-------- (19)}$$

These solutions can be recombined to form a single vector solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \quad \text{-------- (20)}$$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\,\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \quad \text{-------- (21)}$$

Equations (20) and (21) are the unique solutions (given the arbitrary choice of gauge) to the field equations. They specify the magnetic vector and electric scalar potentials generated by a set of stationary charges, of charge density ρ(r), and a set of steady currents, of current density j(r). Incidentally, Eq. (20) satisfies the gauge condition $\nabla \cdot \mathbf{A} = 0$ by repeating the analysis of (with W→J and $\mathbf{C} \rightarrow \mu_0 \mathbf{j}$), and using the fact that ∇.j = 0 for steady currents.

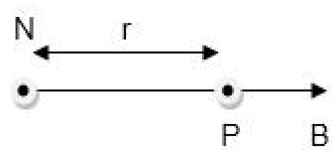
MAGNETISATION AND MAGNETIC INTENSITY

The Magnetic behavior of a magnet is characterized by the alignment of the atoms inside a substance. When a ferromagnetic substance is brought under the application of a strong external magnetic field, then they **experience a torque** wherein the substance **aligns themselves in the direction of the magnetic field applied** and hence gets strongly magnetized in the direction of the magnetic field.

MAGNETIC INTENSITY OR INTENSITY OF MAGNETIC FIELD

The magnetic intensity at a point is defined as the force that unit north - Pole experiences when it is placed in that field.

The intensity of magnetic field at P due to single pole is given by:



$$B = \frac{\mu_0}{4\pi} = \frac{m}{\pi r^2}$$
$$= 10^{-7} \times \frac{m}{\pi r^2}$$

INTENSITY OF MAGNETIZATION

The Magnetic moment of a magnet undergoes a change when it is placed in a magnetic field. This change that is, the magnetic moment change per unit volume is known as in **Intensity of Magnetization**.

Formula of Intensity of Magnetization

$$I = \frac{\text{Magnetic Moment}}{\text{Volume}}$$

$$= \frac{M}{V}$$

$$= \frac{mX2}{AX2} \quad [\text{since } M = mX2| \text{ and } V = AX2]$$

$$= m/A$$

Where, m – Pole strength

A - Area of cross section

Intensity of Magnetisation Unit

The S.I unit of intensity of magnetisation is Ampere/metre or A/m

Magnetic susceptibility and permeability

In a large class of materials, there exists an approximately linear relationship between $\mathbf{M}$ and $\mathbf{H}$. If the material is isotropic then $\mathbf{M} = \chi_m \mathbf{H}$, ----- (1)

where χ_m is called the *magnetic susceptibility*. If χ_m is positive then the material is called *paramagnetic*, and the magnetic field is strengthened by the presence of the material. On the other hand, if χ_m is negative then the material is *diamagnetic*, and the magnetic field is weakened in the presence of the material. The magnetic susceptibilities of paramagnetic and diamagnetic materials are generally extremely small.

A linear relationship between $\mathbf{M}$ and $\mathbf{H}$ also implies a linear relationship between $\mathbf{B}$ and $\mathbf{H}$. In fact, we can write B=μH ------- (1)

where $\mu = \mu_0(1+\chi_m)$ -------- (2)

is termed the *magnetic permeability* of the material in question. (Likewise, μ_0 is termed the *permeability of free space*.) Note that μ has the same units as μ_0. The permeabilities of common diamagnetic and paramagnetic materials do not differ substantially from the permeability of free space.

Possible Questions

2 Marks

1. State Gauss law.
2. What is called electric displacement?
3. What is called current density?
4. State Magnetic vector potential
5. State Magnetic scalar potential
6. What is called dielectric?
7. State Ampere circuital law.
8. State Biot-Savart law.
9. Define Magnetization.
10. Define magnetic intensity.
11. Define Susceptibility.
12. Define permeability and relativity.

6 marks

1. Derive Clausius – Mossotti equation.
2. State Magnetic vector potential. Explain it briefly.
3. Obtain Langevin equation for polar molecules.
4. State Magnetic scalar potential. Explain it briefly.
5. What is called dielectric? Explain dielectric and its polarization.
6. State and derive Ampere circuital law.
7. Explain electrostatic energy and Ampere law of force.
8. State and derive Biot-Savart law.
9. Obtain Lorentz-Lorentz equation for non polar molecules.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 641 021

DEPARTMENT OF PHYSICS

II MSc PHYSICS

ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS (18PHP303)

MULTIPLE CHOICE QUESTIONS

Questions
UNIT-I
The total electric flux linked with a closed surface is ____________ times the charge enclosed by it.
The differential form of Gauss's law is _____________________.

The insulators whose behaviour gets modified in an electric field are called _______________.
dielectric is called _______________.
dielectric is called _______________.
_______________.

The induced dipole moment per unit volume is called _______________.

Dielectric polarization is proportional to ____________________.

The unit of polarization is _________________.

Charge appear on the dielectric surface is _________________.
The charge appear throughout the volume of a dielectric is _______________.

The total bound charge on a dielectric is _______________.

E = _______________.
_______________.
The molecular field is always _____________________ than applied electric field E.
E_m = _________________.

Gauss's law for dielectric is _______________.

In free space i.e., vacuum, P = ______________.

The unit for electric displacement D is ________________.
In case of transparent dielectric, the refractive index n = _______________.

Dielectric materials are called _________________.

For isotrophic dielectrics, the dielectric constant has ________________.

In case of polar molecules, dielectric constant ____________ with increase in frequency of applied field.
Dipole moment acquired by a molecule per unit polarizing field is called ________________.
Clausius-Mossotti relation is _________________.

In case of gases, Clausius-Mossotti relation is _______________.

Langevin equation is applicable to _________________.
polarisability of polar molecules is ________________ proportional to absolute temperature.

Debye relation is used to study the structure of ______________.

In H_2O, the center of +ive and –ive charge _______________ coincide.

_______________ energy.
as the produce of magnitudes of current.
According to Ampere's law of force, the force between current carrying conductors depends on
According to Ampere's law of force, the force is ______________ if the current flows in the same direction.

The vector B is called ________________ vector.

The unit of B is ____________.

The Biot-Savart law for B otherwise called as ________________.
times the total current crossing any surface bounded by the line integral path.

Differential form of Ampere's circuital law is ______________.

Ampere's circuital law signifies that magnetic field is ______________.

B can be expressed in terms of magnetic scalar potential as ______________.

Magnetic scalar potential satisfies _______________ equation.

$\nabla^2 f_m$ = ______________.

Magnetic vector potential satisfies ____________ equation. _______________
the area enclosed by the closed path.
The divergence of magnetic vector potential A is _______________.

B = ________________.

∇X B = ____________.

$\nabla^2 A$ = ______________.

The magnetic field is _______________.
___________ of the two current elements.
According to Ampere's law of force, the force between current carrying conductors varies
The divergence of electric displacement D is equal to ___________.
Dielectric constant of any medium is always ___________ than permittivity.

Clausius-Mossotti relation is applicable to _________________.

Clausius-Mossotti relation is applicable to ____________________.
Clausius-Mossotti relation relates microscopic property polarisability to the macroscopic property

CCl_4 is _______________.

CO_2 is ________________.

Coimbatore.

BATCH : 2018-2020

opt1	**opt2**	**opt3**	**opt4**
ε_0	μ_0	$1/\varepsilon_0$	$1/\mu_0$
div E = ρ/ε_0	-div E = ρ/ε_0	div E = 0	div E = ρ
semiconductor	superconductor	p-type semiconductor	dielectrics
anisotrophic	isotrophic	heterogeneous	homogeneous
anisotrophic	isotrophic	heterogeneous	homogeneous
moment	electric polarization	electronic polarization	moment
orientation polarization	polarization	polarization	above
applied electric field	field	electromagnetic field	electrostatic
coul/m	coul/m^2	coul/m^3	coul^2/m^2
bound charge density	density	free charge density	density
bound charge density	density	free charge density	density
1	-1	0	Same
–grad V	Div V	Curl V	Grad V
electric field	magnetic field	molecular field	field
lesser	greater	equal	nor greater
$-E + P/3\varepsilon_0$	$-E - P/3\varepsilon_0$	$E - P/3\varepsilon_0$	$E + P/3\varepsilon_0$
div D = ρ	div D = -ρ	div D = 0	div D = B
1	-1	2	0
coul^2/m	coul/m^2	coul/m^3	coul^2/m^2
$\sqrt{\varepsilon_\rho}$	ε_r	$-\varepsilon_r$	ε_r^2
ferromagnetic	paramagnetic	ferroelectric	diamagnetic
no value	constant value	infinite value	no dimensions
increases	decreases	is proportional	is same
displacement current	current density	polarizability	charge density
$(\varepsilon_\rho - 1/\varepsilon_\rho + 2) = (n\alpha/3\varepsilon_0)$	$(n\alpha/3\varepsilon_0)$	$(n\alpha/3\varepsilon_0)$	$(n\alpha/3\varepsilon_0)$
$-(\varepsilon_\rho - 1/3) = (n\alpha/3\varepsilon_0)$	$(n\alpha/3\varepsilon_0)$	$-(\varepsilon_\rho - 1/4) = (n\alpha/3\varepsilon_0)$	$(n\alpha/3\varepsilon_0)$
non-polar molecules	polar molecules	complex molecules	molecules
inversely	directly	not	infinitely
atoms	molecules	bond	ionic bond

is supposed to	changed and again	always	do not
electrostatic	electromagnetic	electric	magnetic
inversely	independently	directly	infinitely
colour	nature	property	length
repulsive	infinite	attractive	finite
attractive	infinite	finite	repulsive
magnetic flux	magnetic intensity	magnetic induction	magnetic force
Tesla	Web	Web/m	Web^2/m
laplace formula	poisson formula	Debye formula	formula
ε_0	μ_0	$1/\varepsilon_0$	$1/\mu_0$
$\nabla X\ B = e_0 J$	$\nabla X\ B = -m_0 J$	$\nabla X\ B = m_0 J$	$\nabla X\ B = 0$
finite	infinite	irrotational	rotational
$B = \text{grad}\ \phi_m$	$B = -\text{grad}\ \phi_m$	$B = \text{curl}\ \phi_m$	$B = \text{div}\ \phi_m$
poisson	Debye	Laplace	Langevin
0	1	-1	2
laplace	poisson	debye	Langevin
electrostatic flux	magnetostatic	electric flux	magnetic flux
-1	0	1	2
curl A	–curl A	Div A	–div A
$-\mu_0 J$	$\varepsilon_0 J$	$\mu_0 J$	$-\varepsilon_0 J$
$\mu_0 J$	$-\mu_0 J$	$\varepsilon_0 J$	$-\varepsilon_0 J$
rotational	non-solenoidal	solenoidal	irrotational
length	diameter	cross-section	area
infinitely	finitely	directly	inversely
surface charge density	free charge density	volume charge density	density
greater	lesser	lesser	nor greater
polar liquids	solids	gases	rigid solids
polar liquids	non-polar liquids	solids	rigid solids
dielectric constant	electric displacement	permittivity	permeability
complex molecule	simple molecule	polar	non-polar
non-polar	complex	simple	polar

opt5	opt6	Answer
		$1/\varepsilon_0$
		div $E = \rho/\varepsilon_0$
		dielectrics
		isotrophic
		anisotrophic
		moment
		polarization
		field
		coul/m^2
		density
		density
		0
		–grad V
		molecular field
		greater
		$E + P/3\varepsilon_0$
		div $D = \rho$
		0
		coul/m^2
		$\sqrt{\varepsilon_\rho}$
		ferroelectric
		no dimensions
		decreases
		polarizability
		$= (n\alpha/3\varepsilon_0)$
		$(n\alpha/3\varepsilon_0)$
		polar molecules
		inversely
		molecules

do not

electrostatic

independently

nature

attractive

4

induction

Tesla

laplace formula

μ_0

$\nabla X\ B = m_0 J$

rotational

$B = -\text{grad}\ \phi_m$

Laplace

0

poisson

magnetic flux

0

curl A

$\mu_0 J$

$-\mu_0 J$

solenoidal

length

inversely

density

greater

gases

liquids

dielectric

non-polar

non-polar

FIELD EQUATIONS AND CONSERVATION LAWS

SYLLABUS

Field Equations and Conservation Laws: Equation of continuity - Displacement currents - The Maxwell's equations derivations - physical significance - Poynting vector - Electro magnetic potentials A and ϕ - Maxwell's equations in terms of Electro magnetic potentials - Concept of gauge -Lorentz gauge - Coulomb gauge

EQUATION OF CONTINUITY

Under steady state condition the charge density in any given region will remain constant. It is experimentally verified that the net amount of electric charge in a closed system remains constant. Therefore if the net charge within a certain region decreases with time, this implies that a like charge amount must appear in some other region.

I= - (dq/dt) ---------- (1)

-ive sign indicates that the charge contained in a specified volume decreases with time.

However the definition of current density I = $\oint Jds ------(2)$

So from eqn., (1) and (2) $\oint JdS = -\frac{dq}{dt}$

$$\oint JdS = -\frac{d}{dt}\int \rho d\tau -------- (3)$$

If the surface S fixed in space, the time variation of the volume integral must be solely due to the time variation of ρ. Then $\oint JdS = \int \frac{\partial \rho}{\partial t} d\tau ----- (4)$

But from Gauss Theorem, $\oint JdS = -\int (div\, J) d\tau$ -------- (5)

By comparing (4)and (5) we obtain $\int \left(divJ + \frac{\partial \rho}{\partial t}\right) d\tau = 0 -------- (6)$

The integrand must vanish and the equation is div J +$(\frac{\partial \rho}{\partial t}) = 0 -------- (A)$

Eqn., (A) is called equation of continuity.

DISPLACEMENT CURRENT

Ampere's circuital law in its most general form is given by

$$\oint_C H.d\,I = \int_s J.ds \qquad \text{J – Current density}$$

$$\int_s curl\,H.ds = \int_s J.ds$$

Curl H = J ------------- (1)

Let us now examine the validity of this equation in the event that the fields are allowed to vary with time. If we take the divergence of both sides of equation (1) then,

div (curl) = div J ---------------- (2)

Now as div of curl of any vector is zero, we get from equation (2),

div J = 0 ---------------- (3)

Now the continuity equation in general state

$$\text{div } J = -\frac{\partial \rho}{\partial t} \quad \text{-----------------} \ (4)$$

and will therefore vanish only in the special case that the charge density is static. We must conclude that Ampere's law as stated in equation (1) is valid only for steady state conditions and is insufficient for the case of time-dependent fields. Because of this Maxwell assumed that equation (1) is not complete but should have something be denoted be J_d , then equation (1) can be written as

$$\text{curl } H = J + J_d \quad \text{---------------} \ (5)$$

In order to identify J_d , we calculate the divergence of equation (2) again and get

$$\text{div curl } H = \text{div } (J + J_d) \qquad [\ \text{div curl } H = 0\ ]$$

$$\text{div } (J + J_d) = 0$$

$$\text{div } J + \text{div } J_d = 0$$

$$\text{div} J_d = - \text{div } J$$

$$\text{div} J_d = \frac{\partial \rho}{\partial t} \qquad \{ \text{ from a equation (4) } \}$$

$$\text{div} J_d = \frac{\partial}{\partial t} div\ D$$

$$\text{div} J_d - \frac{\partial}{\partial t} div\ D = 0 \qquad \{ \text{ div } D = \rho \}$$

$$\text{div } (J_d - \frac{\partial D}{\partial t}) = 0 \quad \text{------------} \ (6)$$

As equation (6) is true for any arbitrary volume $J_d = \frac{\partial D}{\partial t}$

And so the modified form of Ampere's circuital law becomes,

$$\text{Curl } H = J + \frac{\partial D}{\partial t}$$

$\partial D / \partial t$ called the displacement current to distinguish it from J, the conduction current. By adding this term to Ampere's law, Maxwell assumed that the time rate of change of displacement produce a magnetic field just as a conduction current.

MAXWELL EQUATIONS

The four fundamental equation of electromagnetism and corresponds to a generalization of certain experimental observations-regarding electricity and magnetism. The following four laws of electricity and magnetism constitutes the so called differential form of Maxwell's equation.

1. Guass law for the electric field of charge yields

FIELD EQUATIONS AND CONSERVATION LAWS

div D = ∇.D = ρ -------------(A)

D – electric displacement in coulombs / m ^{2}

ρ – free charge density in coulombs / m ^{3}

2. Guass law for magnetic field yields

div B = ∇.B =0 ---------------(B)

B – magnetic induction in web / m ^{3}

3. Ampere's Law in circuital form for the magnetic field accompanying a current when modified by Maxwell yields

Curl H = ∇ x H = J + ∂D / ∂t --------------- (C)

H – magnetic field intensity in amperes / m

I – current density amperes / m ^{2}

4. Faradays law in circuital form for the induced electromotive force produced by the rate of change of magnetic flux linked with the path yields.

Curl E =∇ X E = - ∂B / ∂t ------------------(D)

E – electric field intensity in Volts / m

DERIVATIONS

1. div D = ∇.D =ρ

Let us consider a surface S bounding a volume $\tau$with in a dielectric. The volume τcontains no net charge but we allow the dielectric to be polarised say by placing it is an electric field. Some charge on the dielectric body are placed. Thus we have two types charges

a) real charge of density ρ

b) bound charge density ρ' , Guass law then can be written as,

$$\oint_S E.\text{ds} = \frac{1}{\varepsilon\, o}\int(\rho + \rho')\ d\tau$$

$$\varepsilon_0\oint_S E.\text{ds} = \int_\tau \rho d + \ \tau\ \int \rho' d\tau \text{-----------(1)}$$

But as the bound charge density ρ' is defined as

$$\rho' = -\,div\ P$$

$$\oint_S E\ .\text{ds} = \int_\tau div\ E\ d\tau$$

Equation (1) can be written as ,

$$\varepsilon_0\int_\tau div\ E\ d\tau = \int_\tau \rho\ \tau - \int_\tau div\ P\ d\tau$$

$$\int_\tau div\ (\varepsilon0\ \mathrm{E} + P)\ d\tau = \int_\tau \rho\ d\tau \qquad [\varepsilon_0 \mathrm{E} + \mathrm{P} = \mathrm{D}]$$

$$\int_\tau (div\ D - \rho)\ d\tau = 0$$

This equation is true for all volumes, the integration must vanish.

$$\text{div D} = \nabla.\text{D} = \rho$$

2. div B = ∇.B =0

Experiments to data have shown that magnetic poles do not exist. This in turn implies that the magnetic lines of force are either closed group or go off to infinity. Hence the no of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it.

The flux of magnetic induction B across any closed surface is always zero.

$$\oint_s B.ds = 0$$

Transforming this surface integral to volume integral by Guass theorem, we get,

$$\int_\tau div\ B\ d\tau = 0$$

But as the surface bounding the volume is quite arbitrary the above equation will be true only when the integrated vanishes.

$$\text{div B} = \nabla.\text{B} = 0$$

3. Curl H = ∇ x H = J + ∂D / ∂t

From Ampere's circuital law the work done in carrying unit magnetic pole once round a closed arbitrary path linked with the current I is expressed by,

$$\oint_C H.d\ I = I$$

$$\oint_C H.d\ I = \int J.ds \qquad [\ \mathrm{I} = \int I\ .\ \mathrm{ds}\ ]$$

Where S is the surface bounded by the closed path C.

Now changing the line integral into surface integral by Stokes theorem, we get

$$\int_s curl\ H.ds = \int_s J.ds$$

$$\text{Curl H} = \text{J} \qquad \text{------------(2)}$$

But Maxwell found it to be incomplete for changing electric fields and assumed that a quantity,

$J_d = \partial D / \partial t$ is called displacement current must also be included in it so that it may satisfy the continuity equation, J must be replaced by $J + J_d$, so the law becomes,

$$\text{Curl } H = J + J_d$$

$$\text{Curl } H = J + \partial D / \partial t$$

4. Curl E = ∇ X E = - ∂B / ∂t

According to Faradays law of electromagnetic induction, we know that the induced e.m.f is proportional to the rate of change flux

$$\epsilon = -d\,\Phi_B / dt \text{ --------------- (3)}$$

Now if E be the electric intensity at a point the work done in moving a unit charge through a small distance d I is E. d I. So the work done in moving the unit charge once round the circuit is $\oint_c E.dI$. Now as e.m.f is defined as the amount of work done in moving a unit charge once round the electric circuit.

$$\epsilon = \oint_c E.dI \quad \text{--------------- (4)}$$

Comparing equation (3) and (4), we get,

$$\oint_c E.dI = d\ \Phi_B / dt \qquad \text{--------------- (5)}$$

$$\Phi_B = \int_s B.ds$$

$$\text{So ,} \oint_c E.dI = -\frac{d}{dt}\int B.ds$$

Now changing the line integral into surface integral by Stokes theorem, we get

$$\int_s curl\ E.ds = -\frac{d}{dt}\int B.ds$$

The surface S is fixed in space and only B changes with time, above equation yields,

$$\int_s (curl + \frac{\partial B}{\partial T}).ds = 0$$

Integrated vanish is the integral is true for arbitrary,

$$\text{Curl } E = -\partial B / \partial t$$

<u>Special Cases :</u>

1. In a conducting medium of relative permittivity ϵ_r and permeability μ_r as

$$D = \varepsilon E = \epsilon_r \epsilon_0 E$$

$$B = \mu H = \mu_r \mu_0 H \text{ ---------- (A)}$$

And Maxwell equation reduced to

(i) $\nabla.E = \rho / \epsilon_r\epsilon_0$

(ii) $\nabla. H = 0$

(iii) $\nabla \times H = J + \epsilon_r\epsilon_0 \frac{\partial E}{\partial t}$

(iv) $\nabla\times E = - \mu_r\mu_0 \frac{\partial H}{\partial t}$

2. In a non-conducting media of relative permittivity ϵ_r and permeability μ_r as

$$\rho = \sigma = 0$$

$$J = \sigma E = 0 \text{ ---------- (B)}$$

And Maxwell equation reduced to

(i) $\nabla.E = 0$

(ii) $\nabla. H = 0$

(iii) $\nabla \times H = \epsilon_r\epsilon_0 \frac{\partial E}{\partial t}$

(iv) $\nabla\times E = - \mu_r\mu_0 \frac{\partial H}{\partial t}$ ---------- (C)

3. In free space as

$\epsilon_r = \mu_r = 1$

$$\rho = \sigma = 0$$

And Maxwell equation reduced to

(i) $\nabla.E = 0$

(ii) $\nabla. H = 0$

(iii) $\nabla \times H = \epsilon_0 \frac{\partial E}{\partial t}$

(iv) $\nabla\times E = - \mu_0 \frac{\partial H}{\partial t}$

Discussion :

(i) The equation are based on experimental observations the equation (A) and (C) correspond to electricity and (B) and (D) to magnetism.

(ii) These equations are general and apply to all electromagnetic phenomena in media, which are at rest with to respect to the co-ordinate system.

(iii) These equation are not independent of each other as form equation (D) we can derive (B) and from (C), (A). The equation (B) and (D) are called first pair of Maxwell's equation while (A) and (C) are called the second pair.

(iv) The equation A represents coulombs law while C the law of conservation of charge (i.e.) continuity equation.

PHYSICAL SIGNIFICANCE

By means of Gauss and Stokes theorem we can able to write the Maxwell field equations in integral form.

(i) Integrating Maxwell equation div D=ρ over an arbitrary volume τ, $\int_\tau \nabla . D\, d\tau = \int_\tau \rho\, d\tau$, changing the vol integral of LHS into surface integral by Gauss divergence theorem , we obtain $\oint D.ds = q$-------- (A1)

So Maxwell's first equation signifies that the total flux of electric displacement linked with a closed surface is equal to the total charge enclosed by the closed surface.

(ii) Integrating Maxwell's second equation div B = 0 over the arbitrary, $\int_\tau \nabla . B\, d\tau = 0$.

Converting the volume integral into surface integral with the help of Gauss theorem we get$\oint B.ds = 0 ------(B1)$,

Signifies the total flux of magnetic induction linked to a closed surface is zero.

(iii) Integrating III Maxwell equation, curl H= J+ (∂D/∂t) over a surface S bounded by the loop C, we get $\int curl\, H.ds = \int \left(J + \frac{\partial D}{\partial t}\right).ds$

Converting the surface integral into line integral with the help of Stokes theorem we get

$$\oint H.dl = \int \left(J + \frac{\partial D}{\partial t}\right).ds \quad \text{------- (C1)}$$

(iv)Integrating IV Maxwell equation

curl E = − (∂B/ ∂t) over a surface S bounded by the loop C, $\int curl\, E.ds = -\int \left(\frac{\partial B}{\partial t}\right).ds$

Converting the surface integral into line integral with the help of Stokes theorem we get,

$$\oint E.dl = -\frac{\partial}{\partial t}\int B.ds \text{ ------ (D1)}$$

POYNTING THEOREM (OR) ENERGY IN ELECTROMAGNETIC FIELDS :

" The rate of decrease of energy in the electrodynamic fields in a specific region in equal to the sum of rate of work done on charges and rate of escape of energy through the surface in the form of electromagnetic radiation."

According to Lorentz law, the force acting in a electromagnetic field is given by

$$\vec{F} = [\,\vec{E} + (\vec{v} + \vec{B})\,] \qquad \text{------------ (1)}$$

For an elementary volume $d\tau'$, the force experienced in an electromagnetic field is given by

$$\vec{F} = \oint_v [\,\vec{E} + (\vec{v} + \vec{B})\,]\,\rho d\tau [\, q = \oint_v \rho d\tau\,]$$

The work done in causing a displacement dl in the electromagnetic field is given by

$$W = \oint_v [\,\vec{E} + (\vec{v} + \vec{B})\,]\,\rho d\tau . \vec{dl} \qquad \text{-------------- (2)}$$

$$W = \oint_v [\,\vec{E} + (\vec{v} + \vec{B})\,]\,\rho d\tau . \vec{v} dt$$

Rate of work done in an electromagnetic field is given by

$$\frac{dW}{dt} = \oint_v [\,\vec{E} + (\vec{v} + \vec{B})\,]\,\rho d\tau . \vec{v} \qquad \text{------------- (3)}$$

Assuming the rate of work done in the electric field only, we get

$$\frac{dW}{dt} = \oint_v \vec{E} . \rho d\tau . \vec{v} \{\, \vec{v} \times \vec{B} = 0 \,\}$$

$$= \oint_v \vec{E} . \rho\, \vec{v}\, d\tau$$

$$P = \frac{dW}{dt} = \oint_v (\vec{E} . \vec{J}) d\tau \quad \text{---------- (4)}$$

We know that the modified Ampere's law is applicable to electrodynamics.

$$\frac{\nabla \times \vec{B}}{\mu 0} - \epsilon 0 \frac{\partial \vec{E}}{\partial t} = \vec{J} \qquad \text{------------- (5)}$$

Putting the value of J from equation (5) in equation (4) , we get

$$\oint_v (\vec{E} . \vec{J})\, d\tau = \oint_v \vec{E} . [\frac{\nabla \times \vec{B}}{\mu 0} - \epsilon 0 \frac{\partial \vec{E}}{\partial t}]\, d\tau$$

$$= \oint_v \overline{[\vec{E}} . \frac{\nabla \times \vec{B}}{\mu 0}]\, d\tau - \oint_v \epsilon 0\, \vec{E} . \frac{d\vec{E}}{dt} d\tau \qquad \text{--------------- (6)}$$

We know that $\nabla . [\,\vec{E} \times \frac{\vec{B}}{\mu 0}\,] = [\frac{\vec{B}}{\mu 0} . (\nabla \times \vec{E}) - \vec{E} . \frac{\nabla \times \vec{B}}{\mu 0}]$

Now $\oint_v \vec{E} . \vec{J} . d\tau = \oint_v \frac{\vec{B}}{\mu 0} (\nabla \times \vec{E})\, d\tau - \oint_V \nabla . [\,\vec{E} \times \frac{\vec{B}}{\mu 0}\,]\, d\tau - \oint_V \varepsilon 0\, \vec{E} . \frac{d\vec{E}}{dt} d\tau$

$$= \oint_V \frac{\vec{B}}{\mu 0} . (\nabla \times \vec{E})\, d\tau - \oint_V \nabla . [\, \vec{E} \times \frac{\vec{B}}{\mu 0} \,]\, d\tau - \frac{\varepsilon 0}{2} \oint \frac{dE\,2}{dt}\, d\tau$$

According to Maxwell's third equation in the differential form,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint_V \vec{E}.\vec{J}\, .\, d\tau = \frac{1}{\mu 0} \oint_V [\vec{B} . (-\frac{\partial \vec{B}}{\partial t})]\, d\tau - \oint_V \nabla . (\, \vec{E} . \vec{H}\,)\, d\tau - \frac{\varepsilon 0}{2} \oint_V \frac{dE\,2}{dt}\, d\tau$$

$$\oint_V \vec{E} . \vec{J} . d\tau = -\frac{1}{2\mu 0} \oint \frac{dB\,2}{dt}\, d\tau - \frac{\varepsilon 0}{2} \oint_V \frac{dE\,2}{dt}\, d\tau - \oint_V \nabla . (\, \vec{E} . \vec{H}\,)\, d\tau$$

$$= -\frac{\partial}{\partial t} \oint_V [\, \frac{B2}{2\mu 0} + \frac{1}{2}\varepsilon_0 E^2\,] d\tau - \oint_V \nabla . (\, \vec{E} . \vec{H}\,)\, d\tau$$

$$= -\frac{\partial}{\partial t} \oint_V [\, \frac{B2}{2\mu 0} + \frac{1}{2}\varepsilon_0 E^2\,] d\tau - \oint_S (\, \vec{E} \times \vec{H}\,) . d\vec{a} \qquad \text{------------- (8)}$$

$$-\frac{\partial}{\partial t} \oint_V [\, \frac{B2}{2\mu 0} + \frac{1}{2}\varepsilon_0 E^2\,] d\tau = \oint_V (\, \vec{E} . \vec{j}\,)\, d\tau + \oint_S (\, \vec{E} \times \vec{H}\,) . d\vec{a} \qquad \text{------------- (9)}$$

$\vec{E} \times \vec{H}$ is called the pointing vector or power density. It is denoted by symbol S.

$$\vec{S} = \vec{E} \times \vec{H}$$

The unit of pointing vector is Watts/m^2. The pointing theorem,

$-\frac{\partial}{\partial t} \oint_V [\, \frac{B2}{2\mu 0} + \frac{1}{2}\varepsilon_0 E^2\,] d\tau = \oint_V (\, \vec{E} . \vec{j}\,)\, d\tau + \oint_S (\, \vec{E} \times \vec{H}\,) . d\vec{a}$ is integral form.

POYNTING VECTOR (OR) POWER DENSITY :

According to the law of conservation of energy is an electromagnetic fields,

$$S = E \times H$$

E – the electric field H – magnetic field

The amount of the field energy passing through unit area of the surface in a direction perpendicular to the plane containing E and H per unit time.

ELECTROMAGNETIC POTENTIALS A AND ϕ

The analysis of an electromagnetic field is often facilitated by the use of auxiliary functions know as electromagnetic potentials. At every point of space the field vectors satisfy the equations,

$$\textbf{div D } \rho = \rho \qquad \text{------------------ (A)}$$

$$\textbf{div B} = 0 \qquad \text{------------------- (B)}$$

$$\textbf{Curl H} = \textbf{J} + \partial \textbf{D} / \partial t \qquad \text{-------------------- (C)}$$

Curl = - ∂B / ∂t ------------------- (D)

According to equation (B) , the field of vector B is always solenoidal, B can be represented as the curl of another vector say A .

B = curl A --------------- (1)

Where A is a vector which is function of space (x, y, z) and time (t) both. Now sub the value B in equation (1) we get,

$$\text{Curl E} = -\frac{\partial}{\partial t} curl\ A$$

$$\text{Curl} \left(E + \frac{\partial A}{\partial t} \right) = 0 \quad \text{-------------- (2)}$$

(i.e.) $E + \frac{\partial A}{\partial t}$ is a irrotational and must be equal to the gradient of some scalar function.

$$E + \frac{\partial A}{\partial t} = -\text{ grad } \varphi$$

$$E = -\text{ grad } \varphi - \frac{\partial A}{\partial t} \text{---------------- (3)}$$

Thus we have introduced a vector A and a scalar φ both being functions of position and time. These are called electromagnetic potentials. The scalar φ is called the scalar potential and vector A , vector potential.

Properties of scalar and vector potential :

(i) These are mathematical function, which are not physically measurable.

(ii) They are not independent of each other.

(iii) They play an important role in relativistic electrodynamics.

MAXWELL EQUATION IN TERMS OF ELECTROMAGNETIC POTENTIALS

Consider Maxwell equation, μ curl H = μJ + μ (∂D/∂t) ------ (1)

Or curl B = μJ+με (∂E/∂t) -------- (2)

Substituting the equation for E and B in above equation, we obtain

Curl (curl A) = μJ +με ∂/∂t (-grad ϕ -∂A/∂t)

i.e. $\nabla^2 A - \mu\varepsilon\, \partial^2 A/\partial t^2$ – grad (div A + με ∂ϕ/∂t) = -μJ ------ (3)

similarly we get div D = ρ

ε div E = ρ

div (-grad ϕ - ∂A/∂t) = ρ/ε

$\nabla^2\phi + \partial/\partial t$ (div A) = -ρ/ε

By adding and subtracting, we get $\nabla^2\phi - \mu\varepsilon\, \partial^2\phi/\partial t^2 + \partial/\partial t\,(\text{div } A + \mu\varepsilon\, \partial\phi/\partial t) = -\rho/\varepsilon$ ---- (4)

Eqn., (3) and (4) are field vectors in terms of electromagnetic potentials.

NON-UNIQUENESS OF ELECTROMAGNETIC POTENTIALS AND CONCEPT OF GAUGE

In terms of electromagnetic potentials field vectors are given by,

$$B = \text{curl } A \qquad \text{------------------ (1)}$$

And
$$E = -\text{ grad } \varphi - \frac{\partial A}{\partial t} \qquad \text{------------------ (2)}$$

From equations (1) and (2) it is clear that for a given A and φ, each of the field vectors B and E has only value i.e. A and φ determine B and E uniquely. However the converse is not true i.e. field vectors do not determine the potentials A and φ completely. This in turn implies that for a given A and φ there will be only one E and B while for a given E and B there can be infinite number of A' S and φ' S. This is because the curl of the gradient of any scalar vanishes identically and hence we may add to A the gradient of a scalar Λ without affecting B. That is A may be replaced by,

$$A' = A + \text{grad } \Lambda \text{ ------------------ (3)}$$

But if this is done equation (2) becomes,

$$E = -\text{ grad } \varphi - \frac{\partial}{\partial t}(A' - \text{grad } \Lambda)$$

$$E = -\text{ grad } (\varphi - \frac{\partial \Lambda'}{\partial t}) - \frac{\partial \Lambda'}{\partial t}$$

So if we make the transformation given by equation (3). We must also replace φ by

$$\Phi' = \varphi - \frac{\partial \Lambda}{\partial t} \text{-------------- (4)}$$

The expressions for field vectors E and B remain unchanged under transformations equations (3) and (4).

$$B = \text{curl } A = \text{curl } (A' - \text{grad } \Lambda) = \text{curl } A'$$

And
$$E = -\text{ grad } \varphi - \frac{\partial A}{\partial t} = -\text{ grad } (\varphi' + \frac{\partial \Lambda}{\partial t}) - \frac{\partial}{\partial t} (A' - \text{grad } \Lambda)$$

$$E = \text{grad } \varphi' - \frac{\partial A'}{\partial t}$$

We get the same field vectors whether we use the set (A, φ) or (A',φ'). So electromagnetic potentials define the field vectors uniquely through they themselves are non-unique.

The transformations given by equations (3) and (4) are called gauge transformations and the arbitrary scalar Λ gauge function. From the above it is also clear that even though we add the gradient of a scalar function, the field vectors remain unchanged. Now it is the field quantities and not the potentials that possess physical meaningfulness. We therefore Say that the field vectors are invariant to gauge transformations i.e. they are gauge invariant.

LORENTZ GAUGE

The Maxwell's field equations in terms of electromagnetic potentials are,

$$\nabla^2 A - \mu\varepsilon \frac{\partial 2A}{\partial t2} - \text{grad}\left(\text{div } A + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu J \quad \text{--------------- (1)}$$

$$\nabla^2 \varphi - \mu\varepsilon \frac{\partial 2\phi}{\partial t2} + \frac{\partial}{\partial t}\left(\text{div } A + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon} \quad \text{--------------- (2)}$$

A casual glance at equations (1) and (2) reveals that these equations will be much more simplified (i.e. will become identical and uncoupled) if we choose

$$\text{div } A + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0 \quad \text{--------------- (3)}$$

This requirement is called the Lorentz condition when the vector and scalar potential satisfy it, the gauge is called is known as Lorentz gauge.

So with Lorentz condition field equation reduce to

$$\nabla^2 A - \mu\varepsilon \frac{\partial 2\phi}{\partial t} = -\mu J \quad \text{-------------- (4)}$$

$$\nabla^2 \varphi - \mu\varepsilon \frac{\partial 2\phi}{\partial t2} = -\frac{\rho}{\varepsilon} \quad \text{-------------- (5)}$$

But as $\mu\varepsilon = 1/v^2$

So equations (4) and (5) can be written as

$$\boxdot^2 \; A = -\mu J \quad \text{---------------- (6)}$$

$$\boxdot^2 \; \varphi = -\frac{\rho}{\varepsilon} \quad \text{---------------- (7)}$$

With $\boxdot^2 = \nabla^2 - \frac{1}{v\,2}\frac{\partial 2}{\partial t2}$

Equations (6) and (7) are inhomogeneous wave equations and are known as D' Alembertian equations and can be solved in general by a method similar to that we use to solve Poisson's equation. The potentials obtained by solving these equations are called retarted potentials.

In order to determine the requirement that Lorentz condition places on Λ, we substitute A' and φ' from equations (3) and (4).

$$div\,(A' - grad\,\Lambda) + \mu\varepsilon \frac{\partial}{\partial t}(\varphi' + \frac{\partial \Lambda}{\partial t}) = 0$$

$$div\,A' + \mu\epsilon \frac{\partial \phi'}{\partial t} = \nabla^2\Lambda - \mu\varepsilon \frac{\partial 2\Lambda}{\partial t2}$$

So A' and φ' will also satisfy equation (3) i.e. Lorentz condition provided that

$$\nabla^2\Lambda - \mu\varepsilon \frac{\partial 2\Lambda}{\partial t2} = 0$$

i.e. $\boxdot^2 \Lambda = 0$ ------------- (8)

Lorentz condition is invariant under those gauge transformations for which the gauge functions are solutions of the homogeneous wave equations.

<u>The advantages of this particular gauge are :</u>

(i) It makes the equations for A and φ independent of each other.

(ii) It leads to the wave equations which treat φ and A on equivalent footings.

(iii) It is a concept which is independent of the co-ordinate system chosen and so fits naturally into the considerations of special theory of relativity.

COULOMB GAUGE

An inspection of field equations in terms of electromagnetic potentials,

$$\nabla^2 A - \mu\varepsilon \frac{\partial 2A}{\partial t} - grad\,(div\,A + \mu\varepsilon \frac{\partial \phi}{\partial t}) = - \mu J \quad \text{--------------- (1)}$$

$$\nabla^2\varphi - \mu\varepsilon \frac{\partial 2\phi}{\partial t2} + \frac{\partial}{\partial t}(div\,A + \mu\varepsilon \frac{\partial \phi}{\partial t}) = - \frac{\rho}{\varepsilon}$$

i.e. $$\nabla^2\varphi + \frac{\partial}{\partial t}(div\,A) = - \frac{\rho}{\varepsilon} \quad \text{------------- (2)}$$

Shows that if we assume ,

$$div\,A = o \quad \text{------------ (3)}$$

equation (2) reduces to Poisson's equation

$$\nabla^2 \varphi_{(r,t)} = - \frac{\rho}{\varepsilon}(r', t) \quad \text{-------------- (4)}$$

Whose solution is ,

$$\varphi_{(r,t)} = \frac{1}{4\pi\varepsilon} \int \frac{\rho(r',t)}{R}\, d\tau' \quad \text{-------------- (5)}$$

i.e. the scalar potential is just the instantaneous Coulombian potential due to charge ρ (x', y', z', t). This is the origin of the name Coulomb gauge. Equation (1) in the light of (3) reduced to

$$\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = - \mu J + \mu\varepsilon\nabla \frac{\partial \phi}{\partial t} \quad \text{----------------- (6)}$$

Now to express equation (6) in more convenient way we use Poisson's equation (4) which with the help of (5) can be written as

$$\nabla^2 \left\{ \frac{1}{4\pi\varepsilon} \int \frac{\rho(r',t)}{R} \, d\tau' \right\} = -\frac{\rho}{\varepsilon}(r',t) \qquad \text{--------------- (7)}$$

Now as Poisson's equation holds good for scalar and vectors both, replacing ρ(r' , t) by J' we get,

$$\nabla^2 \left\{ \frac{1}{4\pi\varepsilon} \int \frac{J'}{R} \, d\tau' \right\} = -\frac{J'}{\varepsilon} \qquad \text{-------------- (8)}$$

Now from the vector identity

$$\nabla \times \nabla \times G = \nabla(\nabla . G) - \nabla^2 G$$

$$\nabla^2 G = \nabla(\nabla . G) - \nabla \times \nabla \times G$$

Taking $G = \int \left(\frac{J}{R}\right) d\tau'$, we get

$$\nabla^2 \int \left(\frac{J}{R}\right) d\tau' = \nabla\left(\nabla . \int \frac{J'}{R} d\tau'\right) - \nabla \times \nabla \times \int \frac{J'}{R} \, d\tau'$$

Which in the light of equation (8) reduces to

$$-4\pi J' = \nabla\left(\nabla . \int \frac{J'}{R} d\tau'\right) - \nabla \times \nabla \times \int \frac{J'}{R} \, d\tau'$$

i.e. $J' = -\frac{1}{4\pi} \nabla\left(\nabla . \int \frac{J'}{R} d\tau'\right) + \frac{1}{4\pi} \nabla \times \nabla \times \int \frac{J'}{R} \, d\tau'$ ------------- *(9)*

Now as $\nabla . \int \frac{J'}{R} d\tau'$

$= \int \left[\frac{1}{R} \nabla . J' + J' . \nabla\left(\frac{1}{R}\right)\right] \{ as \nabla(sv) = s\nabla . v + v . \nabla s \}$

$= \int J' . \nabla \frac{1}{R} d\tau'$ { as J' is not a function x, y and z }

$= -\int J' . \nabla' \frac{1}{R} d\tau'$ $\{ as \nabla\left(\frac{1}{R}\right) = -\nabla'\left(\frac{1}{R}\right) \}$

$= \int \left[\nabla' . \frac{J'}{R} - \nabla' . \left(\frac{J'}{R}\right) \right] d\tau'$ $\{ as \nabla' . \frac{J'}{R} = \left(\frac{1}{R}\right) \nabla' . J' + J' . \nabla'\left(\frac{1}{R}\right) \}$

$= \int \nabla' . \frac{J'}{R} \, d\tau' - \oint_s \left(\frac{J'}{R}\right) . ds$ $\{ as \int \nabla'\left(\frac{J'}{R}\right) d\tau' = \oint_s \left(\frac{J'}{R}\right) . ds \}$

As J' is confined to the volτ', the surface contribution will vanish so

$$\nabla . \int \left(\frac{J'}{R}\right) d\tau' = \nabla' . \frac{J'}{R} \, d\tau' \qquad \text{----------------- (10)}$$

And $\nabla \times \int \left(\frac{J'}{R}\right) d\tau'$

$= \int \left[\nabla \times \frac{J'}{R} - J' \times \nabla \left(\frac{1}{R}\right) \right] d\tau'$ { as curl SV = S curl V – V X grad S }

$= - \int J' X \nabla \left(\frac{1}{R}\right) d\tau'$ { as J' is not a function of x, y, and z }

$= \int J' X \nabla' \left(\frac{1}{R}\right) d\tau'$ { as $\nabla \left(\frac{1}{R}\right) = -\nabla' \left(\frac{1}{R}\right)$ }

$= \int [\nabla' X \frac{J'}{R} - \nabla' X \left(\frac{J'}{R}\right)] d\tau'$ { as ∇ x (J'/R) = (1/R) ∇' X J' − J' X ∇' (1/R) }

$= \int \nabla' X \frac{J'}{R} d\tau' + \oint \frac{J'}{R} X ds$ { as $\int \nabla X V d\tau' = -\oint_s V X ds$ }

As J' is confined to vol′, surface contribution will vanish so

$$\nabla x \int \left(\frac{J'}{R}\right) d\tau' = \int \nabla' X \frac{J'}{R} d\tau' \qquad \text{------------------ (11)}$$

So equation (9) becomes

$$J' = -\frac{1}{4\pi} \nabla \int \nabla' . \frac{J'}{R} d\tau' + \frac{1}{4\pi} \nabla X \int \nabla' X \frac{J'}{R} d\tau'$$

i.e. $J' = J'_l + J'_T$ ---------------- (12)

With $J'_l = -\frac{1}{4\pi} \nabla \int \nabla' . \frac{J'}{R} d\tau'$ and $J'_T = \frac{1}{4\pi} \nabla X \int \nabla' X \frac{J'}{R} d\tau'$ ----------- (13)

Now as

$$\nabla x \; J'_l = \nabla x [-\frac{1}{4\pi} \nabla \int \nabla' . \frac{J'}{R} d\tau']$$

∇' x J'_l = 0 { as curl grad φ = 0 } ------------- (14)

And $\nabla . J'_T = \nabla . [\nabla X \int \nabla' X \frac{J'}{R} d\tau']$

$\nabla . J'_T$ = 0 { as div curl V =0 } -------------- (15)

The first term on R.H.S of equation (12) is irrotational and second is solenoid. The first term is called longitudinal current and the other transverse current.

So in the light of equation (12),(6) can be written as

$$\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = -\mu (J_l + J_T) + \mu\varepsilon \nabla \frac{\partial \phi}{\partial t}$$

$\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = -\mu J_T - \mu J_l + \mu\varepsilon \nabla \frac{\partial}{\partial t} [\frac{1}{4\pi\varepsilon} \int \frac{\rho(r',t)}{R} d\tau']$ { Substituting φ from equation (5) }

$\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = -\mu J_T - \mu J_l + \mu \frac{1}{4\pi} \nabla \int - \frac{-\nabla . J}{R} d\tau'$ { as from continuity equation $\partial \frac{\rho(r',t)}{R} = - - \nabla . J$ }

0r $\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = -\mu J_T - \mu J_l + \mu J_l$ { from equation (13) }

$$\nabla^2 A - \frac{1}{V2} \frac{\partial 2A}{\partial t2} = -\mu J_T$$

$$\Box^2 \ A = - \mu J_T \qquad \text{------------- (16)}$$

The equation for A can be expressed entirely in terms of the transverse current. So this gauge sometimes is also called as transverse gauge.

The Coulomb gauge has a entire advantage. In it the scalar potential is exactly the electrostatic potential and electric field,

$$E = - \text{grad } \varphi - \frac{\partial A}{\partial t}$$

Is separable into an electrostatic field V = φ and a wave field given by – (∂A/∂t).

This gauge is often used when no sources are present. Then according to equation 5, φ =0 and A satisfies the homogeneous wave equation 16. The fields are given by,

$$E = - \frac{\partial A}{\partial t} \text{ and } B = \nabla X A$$

Possible Questions

2 marks

1. What do mean by gauge?
2. What is called Lorentz gauge?
3. What is the concept of gauge?
4. State Poynting theorem.
5. What is called displacement current?
6. Write down four Maxwell's equations.
7. What are the significance of Maxwell's equations?

6 marks

1. Obtain Maxwell equations.
2. State and explain Coulomb Gauge.
3. Derive Poynting vector.
4. Explain the non uniqueness of electromagnetic potential and Lorentz Gauge.
5. Obtain an equation for electromagnetic potential (A and ϕ) and Maxwell equation in terms of electromagnetic potential.
6. Discuss about displacement current.
7. What is the concept of gauge? Explain Lorentz gauge.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 641 021
DEPARTMENT OF PHYSICS
II MSc PHYSICS
ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS (18PHP303)

MULTIPLE CHOICE QUESTIONS

Questions
UNIT-II
The net amount of electric charge in a closed system is ________________.
The continuity equation is ___________________.
The equation I = - (dq/dt) implies that charge contained in a specified volume ___________ with time.
The div of curl of any vector is _______________.
The magnitude of displacement current is equal to the time rate of change of _______________.
The displacement current in a good conductor is _______________.
Displacement current have a _____________ value in perfect vacuum.
J_d = _______________.
Displacement current density J_d makes conduction current ______________ across a discontinuity.
The unit for free charge density is _____________.
The gauss's law for magnetic field is ______________.
The unit for current density is ______________.
The unit for magnetic induction B is ________________.
enclosed by the closed surface.
Gauss's law for electric field of charges is ______________________.
Gauss's law for magnetic field is ______________________.
Unit of electric field intensity is ________________.
The flux of magnetic induction B across any closed surface is always _______________________.
Electric and magnetic phenomena are ____________________.
Electric and magnetic phenomena are asymmetry arises due to the non-existense of _________________.
Maxwell's first equation signifies that the total flux of electric displacement linked with a closed surface is _______________ the total charge enclosed by the closed surface.
Maxwell's second equation signifies that the total flux of magnetic induction linked with a closed surface is ________________.
conducting current plus displacement current linked with the path.
Maxwell's fourth equation signifies that __________ is equal to the negative rate of change of magnetic
perpendicular to the plane containing E and H per unit time.

from the point source of radiation.
The definition of a poynting vector is not a ___________________.
If the poynting vector is ________________ then no electromagnetic energy flows across a closed surface.
In case of time varying fields S= E X H gives the _____________ value of the poynting vector.
The time rate of change of the sum of particle momentum and field momentum is equal to the ______________________ which would be exerted by the Maxwell stresses on the region considered.
The striking phenomena resulting directly from the effects of radiation pressure is found in
The field vector B is always _________________.
The field vectors E and B are ________________.
Electromagnetic potentials are ______________.
___________ plays an important role in relativistic electrodynamics.
Electromagnetic potentials define the field vectors _______________.
Field vectors are _______________ to gauge transformation.
Lorentz condition is div A + me ∂f/∂t = _______________.
Gauge functions are solutions of the __________________ wave equation.
Lorentz gauge is _______________ of the co-ordinate system.
_______________ gauge is often used when no sources are present.
The dipole moment of an electric dipole is p= _____________.
_________________________.
Accelerated charges radiates _________________.
dipole.
The magnetic induction which varies as $1/r^2$ is __________________.
The magnetic induction which varies as 1/r is _______________.
frequency.
current I_0.
The radiation resistance of half wave antenna is ___________ ohms.
If charge is placed at infinity, it's potential is
If we move a positive charge to a positive plate, then potential energy of charge is
In an electric field energy per unit positive charge is
A charge in electric field always
If a charge is moved from lower power potential to higher potential, then energy should be
____________________ at a point may be defined as equal to lines of force passing normally through a uni
The electrostatic force between two charges of one coulomb each and placed at a distance of 0.5 m will be

BATCH: 2018-2020						
opt1	**opt2**	**opt3**	**opt4**	**opt5**	**opt6**	**Answer**

constant	decreased	increased	varying			constant
div J - ∂ρ/ ∂t = 0	∂ρ/ ∂t = 0	∂ρ/ ∂t =0	∂ρ/ ∂t = 0			∂ρ/ ∂t =0
increases	decreases	varies	equal			decreases
1	-1	Grad	0			0
displacement	density J	density ρ	the above			displace
constant	to time	density	negligible			negligible
finite	infinite	nal value	no			finite
-∂D/ ∂t	J + ∂D/ ∂t	∂D/ ∂t	J - ∂r/ ∂t			∂D/ ∂t
discontinuous	continuous	finite	infinite			us
$coul/m^2$	$coul^2/m$	$coul/m^3$	$coul^2/m^2$			$coul/m^3$
div B = 0	B = curl H	Div B = r	$H + J_d$			div B = 0
amp/m	amp/m^2	amp/m^3	amp^2/m^2			amp/m^2
web/m	web^2/m	web^2/m^2	web/m^2			web/m^2
proportional	proportional	equal	e_0 times			equal
div D = 0	div D = r	ρ	div D = μ			div D = ρ
div B = -H	div B = H	div B = μ	div B = 0			div B = 0
$volts/m^2$	$volts^2/m$	volts/m	$volts.m^2$			volts/m
0	unity	varying	constant			0
symmetric	asymmetric	same	converse			asymmetric
dipoles	electric field	magnetic field	monopole			monopole
equal to	lesser than	than	proportio			equal to
unity	same	zero	constant			zero
magnetomotive	e	restoring	the above			motive
magnetomotive	electromotiv	electric	magnetic			electrom
electric energy	energy	vector	al energy			vector

inversely	directly	lly	ly			inversely
vector	scalar	mandator	none of			mandator
unity	finite	infinite	zero			zero
instantaneous	total	random	half the			eous
magnetic force	electric force	otive	force			force
stars	comets	sun	solar			comets
rotational	irrotational	l	the above			l
unique	non-unique	scalar	tensor			unique
unique	non-unique	scalar	tensor			non- unique
B	energy	energy	gnetic			gnetic
uniquely	non-uniquely	sinusoida lly	randomly			uniquely
variant	proportional	equal	invariant			invariant
1	–1	2	0			0
non-homogeneous	us	Linear	c			eous
Dependent	Independent	function	ial			ent
lorentz	screw	coloumb	the above			coloumb
q. dl	–q.dl	q^2. dl	–q^2. dl			q. dl
charge element	density	charge	element			element
charges	current	energy	the above			energy
zero	minimum	m	same			minimum
imaginary part	real part	part	part			part
imaginary part	real part	part	part			part
cube	fourth	square	the above			fourth
independent	dependent	nal	that			ent
73	72.1	73.1	70			73.1
zero	infinite	1	-1			zero
increase	increase	constant	d			increase
A. voltage	current	frequency	resistance			voltage
lower potential to	from	higher	lower			from
released	same	supplied	converted			supplied
Electric intensity	Magnetic flux	Electric fl	density			Electric intensity
36 × 10^6 N.	36 × 10^7 N	36 × 10^8 N	36 × 10^9 N			36 × 10^9 N

Propagation of Electromagnetic Waves

SYLLABUS

Propagation of Electromagnetic Waves: Electromagnetic waves in Free space - Isotropic dielectric - Anisotropic dielectric – Conducting media - Ionized gases.

Radiating systems: Oscillating electric dipole – Radiation from an oscillating dipole – Radiation from small current element.

ELECTROMAGNETIC WAVES IN FREE SPACE

Maxwell's equations possess propagating wave-like solutions. Let us start from Maxwell's equations in free space (*i.e.*, with no charges and no currents):

$\nabla.E=0$ --------(1)

$\nabla.B = 0$ --------(2)

$\nabla X E= -\partial B/\partial t$ ------------ (3)

$\nabla X B = \varepsilon_0\mu_0 \, \partial E/\partial t$ ------------ (4)

These equations exhibit a nice symmetry between the electric and magnetic fields. There is an easy way to show that the above equations possess wave-like solutions, and a hard way. The easy way is to assume that the solutions are going to be wave-like beforehand.

let us search for plane-wave solutions of the form:

$E(r,t) = E_0 \cos (k.r-\omega t)$ ----------- (5)

$B(r,t) = B_0 \cos (k.r-\omega t+\phi)$ ----------- (6)

Here, E_0 and B_0 are constant vectors, $\mathbf{k}$ is called the wave-vector, and ω is the angular frequency. The frequency in hertz, f, is related to the angular frequency via $\omega=2\pi f$. The frequency is conventionally defined to be positive. The quantity ϕ is a phase difference between the electric and magnetic fields. Actually, it is more convenient to write

$E=E_0 \, e^{ik.r-\omega t}$ ------ (7)

$B= B_0 \, e^{ik.r-\omega t}$ ------ (8)

where, by convention, the physical solution is the *real part* of the above equations. The phase difference ϕ is absorbed into the constant vector B by allowing it to become complex. Thus, $\mathbf{B}_0 \rightarrow \mathbf{B}_0 \, e^{i\phi}$. In general, the vector E_0 is also complex.

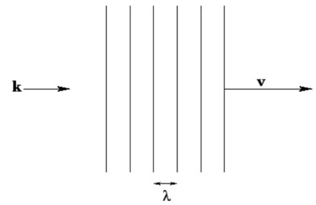
A wave maximum of the electric field satisfies $k.r=\omega t+n2\pi+\phi$ ------- (9)

where n is an integer and ϕ is some phase angle. The solution to this equation is a set of equally spaced parallel planes (one plane for each possible value of n), whose normal lie in the direction of the wave-vector $\mathbf{k}$, and which propagate in this direction with phase-velocity

$$v=\omega/k \text{ -------- (10)}$$

Propagation of Electromagnetic Waves

The spacing between adjacent planes (*i.e.*, the wave-length) is given by $\lambda=2\pi/k$ ----- (11)



Consider a general plane-wave vector field $A=A_0\, e^{ik.r-\omega t}$ ------(12)

We have $\nabla.A= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=(A_{0x}\,i\,k_x+A_{0y}\,i\,k_y+A_{0z}\,i\,k_z)\,e^{i\,(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$= ik.A$ ------ (13)

$(\nabla \times A)_x = \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=(i\,k_y A_z - i\,k_z A_y) = i(k \times A)_x$ ------ (14)

This is easily generalized to $\nabla \times A = ik \times A$ ------- (15)

The vector field operations on a plane-wave simplify to replacing the ∇ operator with ik. The first Maxwell equation reduces to $ik.E_0 = 0$ ----- (16)

using the assumed electric and magnetic fields. Thus, the electric field is perpendicular to the direction of propagation of the wave. Likewise, the second Maxwell equation gives

$ik.B_0 = 0$ ----- (17)

implying that the magnetic field is also perpendicular to the direction of propagation. Clearly, the wave-like solutions of Maxwell's equation are a type of *transverse wave*. The third Maxwell equation gives

$ik \times E_0 = i\omega\, B_0$ ----- (18)

where use has been made of Eq. 15. Dotting this equation with E_0 yields

$$\mathbf{E}_0\cdot\mathbf{B}_0=\frac{\mathbf{E}_0\cdot\mathbf{k}\times\mathbf{E}_0}{\omega}=0. \text{ ------ (19)}$$

Thus, the electric and magnetic fields are mutually perpendicular. Dotting equation 18 with B_0 yields $\mathbf{B}_0\cdot\mathbf{k}\times\mathbf{E}_0=\omega\,{B_0}^2>0.$ ------ (20)

Thus, the vector E_0,B_0 and k are mutually perpendicular, and form a right-handed set. The final Maxwell equation gives

$$i\,\mathbf{k}\times\mathbf{B}_0=-i\,\epsilon_0\mu_0\,\omega\,\mathbf{E}_0. \text{ ------- (21)}$$

Combining this with Eq. (18) yields

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0)=(\mathbf{k}\cdot\mathbf{E}_0)\,\mathbf{k}-k^2\,\mathbf{E}_0=k^2\,\mathbf{E}_0=-\epsilon_0\mu_0\,\omega^2\,\mathbf{E}_0, \text{ ------ (22)}$$

or $k^2 = \epsilon_0\mu_0\omega^2$, ----- (23)

where use has been made of Eq. (15). However, we know from Eq. that the phase-velocity c is related to the magnitude of the wave-vector and the angular wave frequency via c=ω/k. Thus, we obtain

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}. \text{ ----- (24)}$$

So, transverse wave solutions of the free-space Maxwell equations, propagating at some phase-velocity c, which is given by a combination of ε_0 and μ_0. The constants ε_0 and μ_0 are easily measurable. The former is related to the force acting between stationary electric charges, and the latter to the force acting between steady electric currents. Both of these constants were fairly well-known in Maxwell's time. Maxwell, incidentally, was the first person to look for wave-like solutions of his equations, and, thus, to derive Eq. (24). The modern values of ε_0 and μ_0 are

$\varepsilon_0 = 8.8542 \times 10^{-12} C^2 N^{-1} m^{-2}$--------(25)

$\mu_0 = 4\pi \times 10^{-7} NA^{-2}$--------(26)

Let us use these values to find the phase-velocity of ``electromagnetic waves.'' We obtain

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.998 \times 10^8\, \mathrm{m\,s^{-1}}. \text{ -------- (27)}$$

Maxwell was able to make another remarkable prediction. The wave-length of light was well-known in the late nineteenth century from studies of diffraction through slits, *etc.* Visible light actually occupies a surprisingly narrow wave-length range. The shortest wave-length blue light which is visible has λ=0.54 microns (one micron is 10^{-6} meters). The longest wave-length red light which is visible has λ=0.76 microns. However, there is nothing in our analysis which suggests that this particular range of wave-lengths is special. Electromagnetic waves can have any wave-length. Maxwell concluded that visible light was a small part of a vast spectrum of previously undiscovered types of electromagnetic radiation. Since Maxwell's time, virtually all of the non-visible parts of the electromagnetic spectrum have been observed. Electromagnetic waves are of particular importance because they are our only source of information regarding the universe around us. Radio waves and microwaves (which are comparatively hard to scatter) have provided much of our knowledge about the centre of our own galaxy. This is completely unobservable in visible light, which is strongly scattered by interstellar gas and dust lying in the galactic plane. For the same reason, the spiral arms of our galaxy can only be mapped out using radio waves. Infrared radiation is useful for detecting proto-stars, which are not yet hot enough to emit visible radiation. Of course, visible radiation is still the mainstay of astronomy. Satellite based ultraviolet observations have yielded invaluable insights into the structure and distribution of distant galaxies. Finally, X-ray and γ-ray astronomy usually concentrates on exotic objects in the Galaxy, such as pulsars and supernova remnants and the relation c=ω/k, imply that

$B_0=E_0/c$------ (28)

Thus, the magnetic field associated with an electromagnetic wave is smaller in magnitude than the electric field by a factor c. Consider a free charge interacting with an electromagnetic wave. The force exerted on the charge is given by the Lorentz formula f=q(E +v x B) ---- (29)

The ratio of the electric and magnetic forces is $\frac{f_{\rm magnetic}}{f_{\rm electric}} \sim \frac{v\,B_0}{E_0} \sim \frac{v}{c}.$ ----- (30)

So, unless the charge is relativistic, the electric force greatly exceeds the magnetic force. Clearly, in most terrestrial situations electromagnetic waves are an essentially *electric* phenomenon. For this reason, electromagnetic waves are usually characterized by their wave-vector (which specifies the direction of propagation and the wave-length) and the plane of polarization (*i.e.*, the plane of oscillation) of the associated electric field. For a given wave-vector $\mathbf{k}$, the electric field can have any direction in the plane normal to $\mathbf{k}$. However, there are only two *independent* directions in a plane (*i.e.*, we can only define two linearly independent vectors in a plane). This implies that there are only two independent polarizations of an electromagnetic wave, once its direction of propagation is specified.

Let us now derive the velocity of light from Maxwell's equation the hard way. Suppose that we take the curl of the fourth Maxwell equation, Eq. We obtain

$$\nabla\times\nabla\times\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = -\nabla^2\mathbf{B} = \epsilon_0\mu_0\,\frac{\partial\,\nabla\times\mathbf{E}}{\partial t}. \quad \ldots\ldots (31)$$

Here, we have used the fact that ∇.B=0. The third Maxwell equation, yields

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \mathbf{0}, \quad \ldots\ldots (32)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \mathbf{0}, \quad \ldots\ldots. (33)$$

We have found that electric and magnetic fields both satisfy equations of the form $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \mathbf{0}$ ------ (34)

in free space. As is easily verified, the most general solution to this equation (with a positive frequency) is

$$A_x = F_x\,[k.r-ct] \;\ldots\ldots\; (35)$$

$$A_y = F_y\,[k.r-ct] \;\ldots\ldots\; (36)$$

$$A_z = F_z\,[k.r-ct] \;\ldots\ldots\; (37)$$

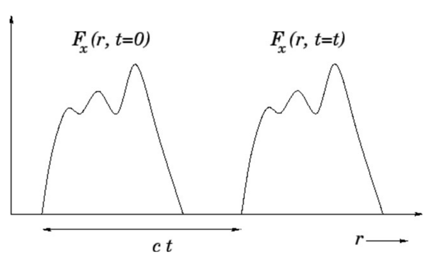
where $F_x\,(\phi)$,$F_y\,(\phi)$ and $F_z\,(\phi)$ are one-dimensional scalar functions. Looking along the direction of the wave-vector, so that r=(k/k)r, we find that

$$A_x = F_x\,[k(r-ct)] \;\ldots\ldots\; (38)$$

$$A_y = F_y\,[k(r-ct)] \;\ldots\ldots\; (39)$$

$$A_z = F_z\,[k(r-ct)] \;\ldots\ldots\; (40)$$

The x-component of this solution is shown schematically in Fig. It clearly propagates in r with velocity c. If we look along a direction which is perpendicular to k then k.r = 0, and there is no propagation. Thus, the components of A are arbitrarily shaped pulses which propagate, without changing shape, along the direction of k with velocity c. These pulses can be related to the sinusoidal plane-wave solutions which we found earlier by Fourier transformation. Thus, any arbitrary shaped pulse propagating in the direction of k with velocity c can be broken down into lots of sinusoidal oscillations propagating in the same direction with the same velocity.



The operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ ------ (41) is called the *d'Alembertian*. It is the four-dimensional equivalent of the Laplacian. Recall that the Laplacian is invariant under rotational transformation. The d'Alembertian goes one better than this, since it is both rotationally invariant and *Lorentz invariant*. The d'Alembertian is conventionally denoted $\Box^2$. Thus, electromagnetic waves in free space satisfy the wave equations $\Box^2 E=0$ & $\Box^2 B=0$

When written in terms of the vector and scalar potentials, Maxwell's equations reduce to

$$\Box^2\phi = -\rho/\varepsilon_0$$

$\Box^2 A = -\mu_0 j$

ELECTRO-MAGNETIC WAVES IN ISOTROPIC DIELECTRIC MEDIUM

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \text{ ------ (1)}$$

$$\nabla \cdot \mathbf{B} = 0 \text{ ------ (2)}$$

$$\nabla \times \mathbf{H} = J + \frac{\partial D}{\partial t} \text{ ------ (3)}$$

$$\nabla \times \mathbf{E} = \frac{-\partial B}{\partial t} \text{ ------ (4)}$$

for isotropic medium,

$$J = \sigma E$$

$$B = \mu H$$

$$D = \varepsilon E$$

here,

$$\sigma = 0$$

$$\rho = 0$$

then the Maxwell's equation reduces to,

$$\nabla . E = 0 \text{ ------ (5)}$$

$$\nabla . H = 0 \text{ ------ (6)}$$

$$\nabla \times \mathrm{H} = \varepsilon \frac{\partial \mathrm{E}}{\partial \mathrm{t}} \text{ ------ (7)}$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \text{ ------ (8)}$$

taking curl for third and fourth equation,

for third equation,

$$\nabla^2 H - \frac{1}{v^2}\frac{\partial^2 H}{\partial t^2} = 0 \text{ ------ (9)}$$

for fourth equation,

$$\nabla^2 E - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0 \text{ ------ (10)}$$

these two waves satisfies the wave equation,

$$\nabla^2 \Psi - \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = 0 \text{ ------ (11)}$$

the solution for the wave equation is,

$$\Psi = \Psi_0 e^{-i(\omega t - k.r)} \text{ ------ (12)}$$

the solution of equations are will be of the given in the form,

$$E = E_0 e^{-i(\omega t - kr)} \text{------ (13)}$$

$$H = H_0 e^{-i(wt - kr)} \text{------ (14)}$$

where k is the wave vector

$$k = k_n = \frac{2\pi}{\lambda} n = \frac{2\pi f_n}{c} = \frac{\omega_n}{c}$$

with n as the unit vector in the direction of wave propagation. The equation can be written as

$$k.E = 0 \text{------ (a)}$$

$$k.H = 0 \text{------ (b)}$$

$$-k \times H = \omega \varepsilon_0 E \text{------ (c)}$$

$$k \times E = \mu_0 H \text{------ (d)}$$

Propagation of electromagnetic waves in dielectric

1. The waves E and H are orthogonal.
2. The electromagnetic wave is transverse in nature.
3. The electric and magnetic vectors are also mutually orthogonal.

PROPAGATION OF ELECTROMAGNETIC WAVES IN ANISOTROPIC MEDIUM

An anisotropic medium is one in which electromagnetic field properties depend on direction. Let us consider a non-magnetic homogeneous isotropic medium in which

$$J=0,\ \rho=0 \text{ and } \mu=\mu_0 \ldots(1)$$

Permittivity is tensor not scalar so, that the components of electric displacement D are in general related to components of E by the equations

$$D_x=\varepsilon_{xx}E_x+\varepsilon_{xy}E_y+\varepsilon_{xz}E_z$$

$$D_y=\varepsilon_{yx}E_x+\varepsilon_{yy}E_y+\varepsilon_{yz}E_z$$

$$D_z=\varepsilon_{zx}E_x+\varepsilon_{zy}E_y+\varepsilon_{zz}E_z \ldots(2)$$

where the coefficients are scalar constants for a homogeneous medium. If we choose the co-ordinate axes as the principle axes of Freshnel ellipsoid for the medium, then D and E are related rather simple equations

$$D_x=\varepsilon_x E_x=K_x\varepsilon_0 E_x$$

$$D_y=\varepsilon_y E_y=K_y\varepsilon_0 E_y$$

$$D_z=\varepsilon_z E_z=K_z\varepsilon_0 E_z \ldots(3)$$

So Maxwell's equations in an anisotropic dielectric medium take the form

$$\text{Div } D = 0 \ldots (a)$$

$$\text{Div } H = 0 \ldots (b)$$

$$\text{Curl } E = -\mu_0 \partial H/\partial t \ldots (c)$$

$$\text{Curl } E = \partial D/\partial t \ldots (d) \ldots (4)$$

So, Maxwell's equations in an isotropic dielectric medium take form shown in equation (4). D and E do not possess the same direction so; div E is not the symmetrical as of equation (4a).

Now consider a plane wave advancing with angular frequency ω and phase velocity ν along the direction of propagation vector K. Then

$$E = E_0 e^{ik.r-\omega t} \ldots (5)$$

where r is the radius vector from origin and

$k = kn = (2\pi/\lambda)\, n = (\omega/\nu) n$ Here n is unit vector along k. from equation(4c) $\partial H/\partial t = (1/\mu_0)$ curlE

$$= (1/\mu_0)\, \text{curl}(E_0 e^{ik.r-\omega t}) \ldots (6)$$

Using vector identity $\text{curl}(\phi A) = \phi \text{curl} A - A \times \text{grad}\phi \ldots (7)$

Here Curl $E_0=0$ is spatially constant. So, equation (6) gives

$$\partial H/\partial t = (E_0/\mu_0) \times \text{grad}(E_0 e^{ik.r-i\omega t})$$

$$\text{grad}(E_0 e^{ik.r-i\omega t}) = ike^{ik.r-i\omega t}$$

$$\partial H/\partial t = (E_0/\mu_0) \times (ike^{ik.r-i\omega t})$$

$$= (ie^{ik.r-i\omega t}/\mu_0) E_0 \times (K)$$

$$\text{Integrating}, H = -(i/\mu_0)(e^{ik.r-i\omega t}/i\omega) E_0 \times k$$

$$= -(1/\mu_0 \omega) e^{ik.r-i\omega t} E_0 \times k$$

$$= -(1/\mu_0 \omega) k \times E [\text{using equation}(5)]$$

$$H = -(1/\mu_0 \omega) E \times k$$

This shows that H is normal to the plane of E and k. Now from equation (4d)

$$\partial D/\partial t = \text{curl } H$$

$$= \text{curl}(H_0 e^{(ik.r-i\omega t)}) [\because H = H_0 e^{(ik.r-i\omega t)}]$$

Using vector identity in equation (7) and nothing that curl H=0, we obtain

$$\partial D/\partial t = -H_0 \times \text{grad}(e^{(ik.r-i\omega t)})$$

$$= -H_0 \times ike^{(ik.r-i\omega t)}$$

$$D = -H_0 \times ike^{(ik.r-i\omega t)}/(-i\omega)$$

$$= (1/\omega) H \times k \ldots (10)$$

This equation shows that D is normal to k and H and both H and D are normal to the direction if propagation vector k. Therefore the electromagnetic wave in anisotropic non conducting medium is transverse with respect to H and D.

Now substituting values of H from (9) to(10), we get

$$D=1/\omega(1\mu_0\omega k\times E)\times k$$

$$=(1/\mu_0\omega^2)k\times(k\times E)=1/\mu_0\omega^2[(k.k)E-(E.k)E]$$

$$=-1/\mu_0\omega^2[k^2E-(k.E)k]\ldots(11)$$

This equation shows D, E and k all lie in the same plane.

Now Poynting vectors S=E×H is normal to the plane containing E and H. This implies that vector D,E, k and S are coplanar. Also, since k is normal to D; and S is normal to E; therefore the angle between S and k is equal to the angle between E and D. In other word S is in the direction of k. As a result an anisotropic medium energy is not propagated in general, in the direction of wave propagation: since poynting vectors k represents the direction of energy flow.

since poynting vectors k represents the direction of energy flow.

Fresnel's law for phase velocity ν.

Equation (11) can be written as

$$D=K^2/\mu_0\omega^2[(E-n.E)n][\text{since}K=kn]$$

$$=1/\mu_0 v^2[E-(n.E)E][\text{since}k=\omega v]$$

If cos α, cos β and cos γ are direction cosines of unit wave vector n; then

$$n=\hat{i}\cos\alpha+\hat{j}\cos\beta+\hat{k}\cos\gamma\ldots(12)$$

Now the components of D are $D_x=(1/\mu_0 v^2)[E_x-(n.E)\cos\alpha]$

$$D_y=(1/\mu_0 v^2)[E_y-(n.E)\cos\beta]$$

$$D_z=(1/\mu_0 v^2)[E_z-(n.E)\cos\gamma]\ \ldots(12)$$

Now from equation (3) $E_x=D_x/K_x\varepsilon_0$,

$$E_y=D_y/K_y\varepsilon_0$$

$$\text{And } E_z=D_z/K_z\varepsilon_0$$

From definition of refractive index

$$n_x=\sqrt{K_x}=c/v_x,\ n_y=\sqrt{K_y}=c/v_y \text{ and } n_z=\sqrt{K_z}=c/v_z$$

$$c=1/\sqrt{(\mu_0\varepsilon_0)}$$

Using these equations, we get

$$E_x=\mu_0 v^2{}_x D_x;$$

$$E_y=\mu_0 v^2_y D_y$$

And $E_z=\mu_0 v^2_z D_z$

So that equation (12), (13) and (14) becomes

$$D_x=(1/\mu_0)(n.E)\cos\alpha/v^2_x-v^2$$

$$D_y=(1/\mu_0)(n.E)\cos\beta/v^2_y-v^2$$

$$D_z=(1/\mu_0)(n.E)\cos\gamma/v^2_z-v^2$$

Since vector D is normal to n, we get

$$D.n=D_x\cos\alpha+D_y\cos\beta+D_z\cos\gamma=0$$

using equation(13),we obtain

$$\cos^2\alpha/(v^2_x-v^2)+\cos^2\beta/(v^2_y-v^2)+\cos^2\gamma/(v^2_z-v^2)=0\ldots(14)$$

This is well known Freshnel's law for phase velocity. This equation indicates that the phase velocity v, in general, can have two values in any given direction. However for most anisotropic media, it is found that there are two directions for which equation (14) has only one solution i.e. it gives only one velocity phase. These directions are called optic axes and the medium is said to be biaxial.

PROPAGATION OF ELECTROMAGNETIC WAVES IN CONDUCTING MEDIA

In a conducting medium, such that the conductivity σ≠0 Maxwell's equations become

$$\nabla\cdot\vec{D}=0,\ \nabla\cdot\vec{B}=0,\ \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}\ \text{and}\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}.$$

By neglecting resonant or other effects we may use the linear approximations $\vec{J}=\sigma\vec{E},\ \vec{D}=\epsilon\vec{E}$ and $\vec{B}=\mu\vec{H}$ where ϵ,μ and σ are independent of time.

Maxwell's equations become $\nabla\cdot\vec{E}=0,\ \nabla\cdot\vec{H}=0,\ \nabla\times\vec{H}=\sigma\vec{E}+\epsilon\frac{\partial\vec{E}}{\partial t}$ and $\nabla\times\vec{E}=-\mu\frac{\partial\vec{B}}{\partial t}.$

Taking the curl of the last of these gives

$$\nabla\times(\nabla\times\vec{E})=-\nabla\times(\mu\frac{\partial\vec{B}}{\partial t})=-\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})=-\mu\frac{\partial}{\partial t}(\sigma\vec{E}+\epsilon\frac{\partial\vec{E}}{\partial t})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}.$$

We have now the wave equation in a conducting medium: $\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\epsilon\frac{\partial^2\vec{E}}{\partial^2 t}.$

Similarly $\nabla^2\vec{B}=\mu\sigma\frac{\partial\vec{B}}{\partial t}+\mu\epsilon\frac{\partial^2\vec{B}}{\partial^2 t}.$

The last two equations are called the telegraph equations and incorporate damping terms $\mu\sigma\frac{\partial \vec{E}}{\partial t}$ and $\mu\sigma\frac{\partial \vec{B}}{\partial t}$ so electromagnetic waves travelling in a conducting medium experience attenuation proportional to the conductance.

By assuming $\vec{B}$ and $\vec{E}$ are of complex exponential form $Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$ the last two of Maxwells equations above become $\nabla\times\vec{H}=-i\omega\mu\vec{H}$ and $\nabla\times\vec{H}=\sigma\vec{E}=i\omega(\epsilon-\frac{i\sigma}{\omega})\vec{E}$.

The first telegraph equation then becomes $\nabla^2\vec{E}+\omega^2\mu(\epsilon-\frac{i\omega}{\omega})\vec{E}=0$ which has the form of the Helmhotz equation $(\nabla^2+k^2)\vec{E}=0$ with $k=k_1=\omega\sqrt{\mu(\epsilon-\frac{i\sigma}{\omega})}$.

We may use the identity $k=\frac{n\omega}{c}=\omega\sqrt{\mu\epsilon}$ to demonstrate that the equations for conducting and non conducting media are the same if the dielectric constant ϵ is replaced by a complex dielectric constant $\hat{\epsilon}=\epsilon-\frac{i\sigma}{\omega}$.

Since we have replaced k by a complex equivalent, we must obtain a complex equivalent for the refractive index. This is done by writing

$N=n(1-i\kappa)$ where k is a constant called the extinction coefficient.

We replace the propagation constant k by $N\frac{\omega}{c}=\frac{\omega n}{c}(1-i\kappa)$.

Assuming that $\vec{k}$ is parallel to the x – axis, then

$$\vec{E}=\vec{E}_0 e^{i\omega t}e^{-\frac{i\omega n}{c}(1-i\kappa)z}.$$

This wave is attenuated by the factor $e^{-\frac{\omega}{c}n\kappa z}$.

PROPAGATION OF ELECTROMAGNETIC WAVES IN IONIZED GASES:-

In certain situations such as the ionosphere or tenuous plasma there is so little air that the electrons may vibrate without colliding with the molecules. So the force on a charged particle is an electromagnetic field, neglecting the earth's magnetic field will be

$$\boldsymbol{F}=\boldsymbol{e}[\boldsymbol{E}+(\boldsymbol{v}\times\boldsymbol{B})] \text{ -------(1)}$$

now as in a plane wave

$$B=\frac{n\times E}{c}$$

$$|v\times B|=vB=\frac{v}{c}E$$

and also,

$$E = E_0 e^{-i(\omega t - k.r)}$$

$$E = E_0 e^{-i(\omega t \ \ (2\pi \backslash \lambda) n.r)}$$

$$E = E_0 e^{-i(\omega t)}$$

so equation reduces to,

$$F = eE_0 e^{-i\omega t}$$

$$m\frac{d^2r}{dt^2} = eE_0 e^{-i\omega t}$$

$$\frac{d^2r}{dt^2} = \frac{e}{m}E_0 e^{-i\omega t}$$

$$\frac{dr}{dt} = \frac{eE_0 e^{-i\omega t}}{m(-i\omega)}$$

$$v = i\frac{e}{m\omega}E \text{-------}\mathbf{(2)}$$

now if there are N electrons per unit volume then as

$$J = Nev$$

substituting the value of v from equation we get,

$$J = i\frac{Ne^2}{m\omega}E \text{-------}\mathbf{(3)}$$

$$J = \sigma E$$

we find that the conductivity is purely imaginary,

$$\sigma = i\frac{Ne^2}{m\omega} \text{-------}\mathbf{(4)}$$

various shortcuts are possible in deriving equations of wave propagation in an ionized medium but it worthwhile to go all the way back to Maxwell's equation.

$$\nabla . D = \rho$$

$$\nabla . B = 0$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \text{-------}\mathbf{(5)}$$

which for the present situation reduces to

$$\nabla . E = 0$$

$$\nabla . H = 0$$

$$\nabla \times H = \sigma E + \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}$$ **-------(6)**

in case of ionized medium $\rho = 0, \varepsilon_r = 1 \; and \; \mu_r = 1$.Now taking curl for fourth equation,

$$\nabla \times (\nabla \times E) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times H)$$

solving this we get,

$$\nabla^2 E - \sigma\mu_0 \frac{\partial E}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0$$ **-------(7)**

similarly taking curl for third equation,

$$\nabla^2 H - \sigma\mu_0 \frac{\partial H}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0$$ **-------(8)**

the solution of these two equations be,

$$\Psi = \Psi_0 e^{-i(\omega t - k.r)}$$

then,

$$\binom{E}{H} = \binom{E_0}{H_0} e^{-i(\omega t - k.r)}$$ **-------(A)**

so that field equation reduces,

$$\nabla(K^2 - i\mu_0\omega\sigma - \mu_0\varepsilon_0\omega^2)\binom{E}{H} = 0$$

as vector E or H is not zero,

$$K^2 = \mu_0\varepsilon_0\omega^2\left[1 + \frac{i\sigma}{\varepsilon_0\omega}\right]$$

$$K^2 = \frac{\omega^2}{c^2}\left[1 - \frac{{\omega_p}^2}{c^2}\right]$$ **-------(9)**

$$m = \frac{c}{v} = \frac{c}{\omega/k} = \frac{kc}{\omega}$$

so the index of refraction in this case will be given by,

$$n = \sqrt{\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)}$$ **-------(10)**

from this equation it is clear that for frequencies $\omega^2 > {\omega_p}^2$

In region of vanishing small ionization and high frequency range index of refraction is real and so waves propagate freely as in dielectric, however if the plasma frequency increases with distance, the index of refractive will decreases according. This is turn means that the beam will bends in a direction away from the normal as it moving from a region of higher index of refraction to that of lower index of refraction. This bending of high frequency or short wavelength electromagnetic wave by earth's ionosphere is used in long distance radio transmission.

In the limit $\omega^2 \gg {\omega_p}^2$ as $n \to 1 = constant$, the transmission is unaffected by the presences of ionosphere this is why the radar signals that have been received after reflection from the moon had to be rather higher frequency waves to pass through the ionized part of earth's atmosphere. For frequency $\omega^2 < {\omega_p}^2$ in heavily ionized region and for low frequencies ranges the index of refraction is purely imaginary. so if we write n →in then from equation

$$k = \frac{\omega}{c}(in) = i\frac{\omega}{c}\sqrt{\left(\frac{{\omega_p}^2}{\omega^2} - 1\right)}$$

so that

$$\binom{E}{H} = \binom{E_0}{H_0} e^{-\beta(n.r)} e^{-i(\omega.t)}$$

with $\beta = \frac{\omega_n}{c}$

Oscillating Electric Dipole

An electric dipole is formed by a pair of charge of equal magnitude and of opposite sign separated by a small distance. The dipole moment of a dipole is defined as

P=qd1

Where d1 has direction from –ive to +ive charge along the line joining the charges.

However if the charge q varies sinusoid ally the dipole moment will vary accordingly

i.e if $q=q_o e^{-iwt}$

Then $p=(q_o e^{-iwt})\, d1$ or $p=p_o e^{-iwt}$ with $p_o=q_o d1$

A dipole whose dipole moment varies sinusoid ally is called an oscillating dipole.

Regarding an oscillating dipole it is worthy to note that

(i) As due to the oscillation of charges current I=dq/dt flows

So $p = \frac{dp}{dt} = \frac{d}{dt}(q\ d1) = \frac{dq}{dt}d1 = I\ d1$

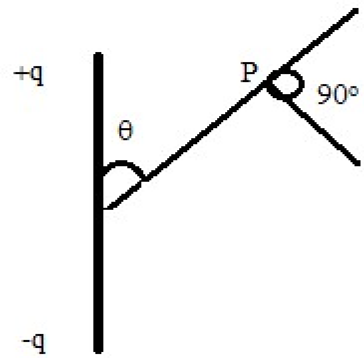
i.e the time rate of change of dipole moment of an oscillating dipole is equal to a current element.

(ii) As, in oscillation , charge are always undergoing acceleration an electric dipole radiates energy(because a charge undergoing acceleration radiates energy).

Propagation of Electromagnetic Waves

Radiation from an Oscillatin dipole

In order to calculate the various quantities of ineterst for an oscillating dipole assume that the length of the dipole dl is small aas compared to the wavelength λ corresponding to any periodic function so that all source points have the same retarted time and no phase difference occurs between waves from different parts of the source.



(i) Vector potential

Vector potential is given by $A = \frac{\mu_0}{4\pi} \int \left[\frac{J}{r}\right] dl$

As in case of an oscillating dipole Idl = dp/dt = p

$$\text{So } A = \frac{\mu_0 [p]}{4\pi r} \text{ ------ (1)}$$

From eqn., (1) it is clear that

(i) Vector potential is independent of θ and ϕ and varies as r^{-1} with distance r.

(ii) Vector potential is every where parallel to polar axis dl.

(iii)If n is a unit vector along polar axis

$$n = n_r \cos\theta - n_\theta \sin\theta$$

then $A = \frac{\mu_0 [p]}{4\pi} [n_r \cos\theta - n_\theta \sin\theta]$

the components will be $A_r = \frac{\mu_0 [p]}{4\pi} \cos\theta$ $\quad A_\theta = -\frac{\mu_0 [p]}{4\pi r} \sin\theta$ and $A_\phi = 0$

(ii) Scalar potential

As Lorentz condition is div A + με ∂φ/∂t = 0

$\partial\Phi/\partial t = -1/\mu_0\varepsilon_0$ div A = $-c^2$ div A

i.e. $\partial\Phi/\partial t = \frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\left\{\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right\}\cos\theta\right]$

i.e. $\Phi = 1/4\pi\epsilon_0 \left\{\frac{[p]\cos\theta}{r^2} + \frac{[\dot{p}]\cos\theta}{rc}\right\}$ ……. (3)

From eqn., (3) it is clear

(i) scalar potential varies as cos θ and is zer in the equatorial plane where the fields of the two changes cancel exactly.

(ii) The scalar potential varies as 1/r2 for small values of r and as 1/r for large value of r.

$$\Phi \approx 1/4\pi\epsilon_0 \left\{\frac{p\cos\theta}{r^2}\right\} = \frac{1}{4\pi\epsilon_0}\, p.r/r^3$$

(iii) Magnetic Induction B

B = curl A

Substituting the value of A0 and Ar in the eqn., B and expanding

$$B = \{(\mu_0[\dot{p}]\ \sin\ \theta)/(4\pi r^{\wedge}2) + (\mu_0[\ddot{p}]\ \sin\ \theta)/4\pi rc\ \} \text{------ (4)}$$

From eqn., (4) it is observed that

(i) The magnetic induction vector B varies as sin θ and is minimum along the axis of the dipole and maximum along the equatorial plane.

(ii) The magnetic induction varies as $1/r^2$ for small values of r while as 1/r for large values of r. The term varies as r^{-2} is called inductive part and term varies as r^{-1} is called radiative part. Actually for small r

$$B = \frac{\mu_0[\dot{p}]\cos\theta}{4\pi r^2}\, n_\phi = \frac{\mu_0 Idl\sin\theta}{4\pi r^2}\, n_\phi = \frac{\mu_0 Idl\ x\ r}{4\pi r^3}$$

(iii) Electric Intensity E

Electric intensity is given by E = -grad Φ - ∂A/∂t

$$E = -\left[\frac{\partial\Phi}{\partial t}\, n_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta} n_\theta + \frac{1}{rsin\theta}\frac{\partial\Phi}{\partial\phi}\, n_\phi\right] - \frac{\partial}{\partial t}\left[\frac{\mu_0[\dot{p}]\cos\theta}{4\pi r} n_r - \frac{\mu_0[\dot{p}]\sin\theta}{4\pi r}\, n_\theta\right]$$

$$\text{Or } E = \left[\frac{2[\dot{p}]\cos\theta}{4\pi\epsilon_0 r^3} n_r - \frac{[\dot{p}]\sin\theta}{4\pi r^3}\, n_\theta\right] + \frac{2[\dot{p}]\cos\theta}{4\pi\epsilon_0 r^3 c} n_r + \frac{[\dot{p}]\sin\theta}{4\pi r^2 c}\, n_\theta + \frac{\sin\theta}{4\pi r^3}\, n_\theta[\ddot{p}]$$

This is the required result.

(i) The first bracketed terms in E is the only static field of an electric dipole and because it falls off as $1/r^3$ it is dominant only at small distances.

(ii) The second bracketed term E depends on $[\dot{p}]$ or the current I and gives fields which would arise from steady currents. They fall off as $1/r^2$ and are called induced fields or inductive part.

(iii) The third term falls off as 1/r and predominates at large distance and represents radiation field because the energy given by integrating S over a sphere round the dipole does not decrease as the radius gets larger. This term is called radiative part.

RADIATION FROM A SMALL CURRENT ELEMENT

As the dipole moment of an oscillating dipole is given by p = q dl

So $\dot{p}$ = dq/dt dl = Idl

i.e. $\ddot{p}$ = dI / dt dl

Now if the current is sinusoidal, $I = I_0 \cos \omega t = R\,e\,(I_0 e^{-i\omega t})$

$$\ddot{p} = i\omega I_0\, dl\, e^{-i\omega(t-r/c)} \text{ ----- (1)}$$

In the light of equation (1) compute quantities of for a small current element.

(i) Electric field: The electric field is given by $E = \frac{\ddot{p}\sin\theta}{4\pi\epsilon_0 r c^2} n_\theta$

$$\text{In this case } E = -I\, i\, \frac{\omega I0 \text{ dl} \sin\theta}{4\pi\epsilon_0 r c^2}\, e^{-i\omega(t-r/c)} n_\theta \text{ --------- (2)}$$

(ii) Magnetic field: The magnetic field is given by $B = \frac{\mu_0 \ddot{p}\sin\theta}{4\pi\epsilon_0 r c} n_\phi$

$$\text{In this case } B = -i\, \frac{\mu_0 \omega I0 \text{ dl} \sin\theta}{4\pi\epsilon_0 r c}\, e^{-i\omega(t-r/c)} n_\phi \text{ ------- (3)}$$

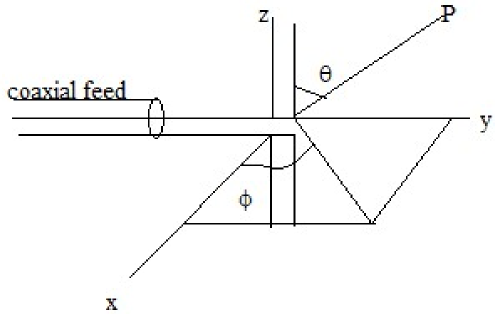
(iii) Poynting Vector: Poynting Vector is defined by S = ½ Re(E x H)

In this case $\langle S\rangle = \frac{1}{2}\, \frac{\omega 2 I02 \text{ dl2} \sin 2\,\theta}{16\pi 2\epsilon_0 2 r c 3} n_\theta \times n_\phi$

$$\langle S\rangle = \frac{\omega 2 I02 \text{ dl2} \sin 2\,\theta}{32\pi 2\epsilon_0 2 r c 3} n_r \text{ .... (4)}$$

(iv) Radiation Resistance: The ordinary ohmic power loss is $P = RI^2_{rms}$ is found that

$$R = \left\{ \begin{array}{c} \frac{2\pi}{3}\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\right)\left(\frac{\omega dl}{2\pi c}\right)^2 \\ \frac{2\pi}{3}\, z_0 \left(\frac{dl}{\lambda}\right)^2 \\ 80\,\pi^2\left(\frac{dl}{\lambda}\right)^2 = 787\,\left(\frac{dl}{\lambda}\right)^2 \end{array} \right\} \text{ohms --------- (5)}$$



The resistance Rr is called radiation resistance. It is the resistance which would dissipate the same power than the current element. A simple example of current element radiator is a centre fed short linear antenna whose length dl is small compared to wavelength λ.

It must be noted that for all dl<< λ the radiation resistane is much less than the ohmic resistance. For example if dl = 0.01λ then the radiation resistance is only 0.08Ω. While the ohmic resistance of an antenna could have appreciably larger than this value.A short linear antenna is therefore in general only an inefficient radiator. dl must be comparable with λ in order that electromagnetic energy can be efficiently radiated, but the dipole approximation is not valid.

Propagation of Electromagnetic Waves

Possible Questions

2 marks

1. What is called free space?
2. What is called ionized gas?
3. What is called conducting medium?
4. What is called an isotropic medium?
5. Define radiation.

6 marks

1. How the electromagnetic waves propagate in a free space? Explain it.
2. Explain the propagation of electromagnetic waves in an ionized gas.
3. How the electromagnetic waves propagate in an isotropic dielectric medium? Explain it.
4. How the EMW waves propagate in an anisotropic medium? Explain it.
5. Discuss the problem of radiation from oscillating dipole.
6. How the electromagnetic vector propagates in the conducting media.
7. Find the total power radiated from the current element.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 641 021

DEPARTMENT OF PHYSICS

II MSc PHYSICS

ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS (18PHP303)

MULTIPLE CHOICE QUESTIONS

Questions
UNIT-III
Electromagnetic waves propagates in free space with the velocity of _______________.
The velocity of electromagnetic waves in free space is ___________________.
The field vector operator Ñ is equivalent to ________________.
In an electromagnetic wave the amplitude of electric vector E is ___________ times that of the magnetic vector H.
The flow of energy in a electromagnetic wave in free space is in the direction of _______________.
The electromagnetic energy density is equal to the ___________ energy density.
The electromagnetic field vectors E and H are in _____________.
Z_0 is ________________.
In plane electromagnetic wave, the wave vectors E, H and K are ______________.
The field vector operator ∂/∂t is equivalent to ______________.
electromagnetic waves in free space.
_______________ radiation is emitted due to the interaction of uniformly moving charged particles with the dielectric medium.
When high energy particles having velocities greater than c passes through a dielectric a __________ light known as cerenkov radiation is emitted.
The poynting vector in case of propagation of electromagnetic waves in iotrophic dielectric is _____________ times of the poynting vector if the same wave propagates through free space.
The total energy density in case of electromagnetic waves in isotrophic dielectrics is ___________ times of the energy density if the wave propagates through free space.
In case of propagation of electromagnetic waves in isotropic dielectrics the electromagnetic energy density is _______________ the magnetostatic energy density.
In an anisotrophic medium, the energy is _________________ in the wave propagation.
In case of propagation of EMW in conducting medium the wave gets _____________ with penetration.
and H.
In case of propagation of EMW in conducting medium, magnetic energy is ___________________ electric energy density.
When electromagnetic waves crosses a boundary surface, then the normal component of the electric displacement is _________________ by an amount equal to the free density of charge.
The normal component of the magnetic induction is ___________ across a surface of discontinuity.

The tangential component of magnetic intensity is ________________ by an amount equal to the surface current density.
The tangential component of E is _________________ across a surface of discontinuity.
refraction.
According to law of reflection, the angle of reflection is __________angle of incidence.
________________ ratio of the refractive index of the two media.
vibrations are perpendicular to the plane of incidence.
_______________ angle is also called as polarizing angle.
If light is incident on a glass plate at 56°, the reflected light will _____________.
are incident on it at 84°.
All the light is reflected as the angle of incidence approaches 90°, the angle is called ________________.
_____________ glasses transmit only one direction of vibration.
The value of angle of incidence for which q_T becomes 90° is called ________________.
The ______________ velocity is a function of angle of incidence.
The waves which do not have energy are called _____________ waves.
angle, the reflected wave will be __________________.
The phenomena of total internal reflection is used to produce ________________ polarized lights.
All good conductors are good ______________ and good _______________.
Good conductor of electricity are _____________ to light.
If white light is incident on thin gold foils, then the transmitted light appears ______________.
The reflection coefficient of substance of high conductivity at low frequency will ________________.
________________.
If the cross-section of the waveguide is rectangular, it is called ____________ waveguide.
If the cross-section of waveguide is circular, it is called ____________ waveguide.
The walls of the waveguide are perfectly _______________.
walls of the wave guide.
____________________.
The phase velocity becomes __________ exactly at cutoff frequency.
In the waveguide the Maxwell's first equation for the propagation of EMW is _______________.
If the EMW propagates in a waveguide, the Maxwell's second equation is ________________.
______________ waves cannot be propagated along the axis of a waveguide.
longitudinal component of magnetic vector B_z.
The ___________ mode is called the principal or dominant mode.
longitudinal component of the electric field E-$_z$.
In ____________ waveguide TE and TM have the same set of cutoff frequencies.
Waveguides are used in ______________ region.
Waveguides are used in ___________ region.

The reflection of the electromagnetic waves at the conducting plane involves no change in ____________.
If the magnetic field H of electromagnetic wave has a component along the assumed axis of propagation then the wave is called __________________.

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BATCH :	2018-2020					

opt1	**opt2**	**opt3**	**opt4**	**opt5**	**opt6**	**Answer**
light	sound	electron	proton			light
30 X 10^8	356 X 10^8	330 m/s	3 X 10^8 m/s			3 X 10^8 m/s
–ik	ik	ik^2	i^2k^2			ik
e_0	m_0	Z_0	E/H			Z_0
electric	magnetic	electrons	wave			wave
tatic	electrostatic	flux	electric flux			magnetostatic
out of	phase	proportional	none of the			phase
reactance	reactance	of free space	above			free space
parallel	rotational	irrotational	orthogonal			orthogonal
-iω	iω	$i\omega^2$	$i^2 \omega^2$			-iω
greater	lesser	absolute	above			lesser
X-rays	Gamma	Alpha	Cerenkov			Cerenkov
greenish	bluish	reddish	greenish-blue			bluish
ν/ε_r	n/μ_r	$-n/\varepsilon_r$	$-n/\mu_r$			n/μ_r
μ_r	$-\mu_r$	ε_r	$-\varepsilon_r$			ε_r
less than	greater than	equal to	none of the above			equal to
not propagate	propagated	orthogonal	parallel			not propagated
reflected	refracted	attenuated	scattered			attenuated
nal	parallel	transverse	above			transverse
greater than	lesser than	equal to	inversely proportional			greater than
equals	continuous	proportional	discontinuous			discontinuous
equals	continuous	proportional	s			continuous

uous	continuous	proportional	equal			discontinuous
uous	l	continuous	equals			continuous
d	changed	decreased	unchanged			unchanged
the	greater than	lesser than	above			equal to the
than	equal to the	lesser than	above			equal to the
2π	π/2	π	π/3			π
s angle	angel	angle	above			angle
polarized	polarized	polarized	polarized			polarized
cannot	can	relfection	above			cannot
angle	angle	angle	angle			Grazing angle
light	dark	colour	crown			dark
angle	angle	angle	Snell’s angle			critical angle
group	angular	phase	linear			phase
cerenkov	Radio	Microwaves	Evanescent			Evanescent
circularly	elliptically	spherically	plane			elliptically
y	spherically	plane	above			elliptically
and	and	refractor	above			reflectors
opaque	absorbers	reflectors	scatterers			opaque
or	or greenish	bluish	bluish			bluish
unity	be infinite	will be unity	be finite			will be unity
ion tube	wave guide	tube	reflection			wave guide
l	circular	square	rectangular			rectangular
l	circular	elliptical	rectangular			cylindrical
conductin	conducting	conducting	insulating			conducting
multipies	vanishes	coincides	above			vanishes
c	0.9 c	0.3 c	0.2 c			c
finite	zero	infinite	unity			infinite
div E = ρ	ρ/ε_0	div E = -ρ	div E = 0			div E = 0
H = 0	Div B = 0	H = 1	Div B = H			Div B = 0
stationary	TE	TEM	TM			TEM
TM	TE	TEM	Transverse			TE
TE_{11}	TE_{10}	TE-$_{01}$	TE_{00}			TE-$_{01}$
TM	TE	TEM	Longitudinal			TM
circular	all	square	rectangular			rectangular
MUF	VHF	OHF	OWF			OHF
ve	ultra-violet	infra-red	radio-wave			microwave

frequency	amplitude	phase	energy			amplitude
H-wave	E-wave	EH-wave	Transverse wave			H-wave

Interaction of E.M.Waves with matter

SYLLABUS

Interaction of E.M.Waves with matter (Macroscopic): Boundary conditions at interfaces - Reflection and refraction – Frenel's laws-Brewster's law and degree of polarization - Total internal reflection and critical angle.

Interaction of E.M.Waves with matter (Microscopic): Scattering and Scattering parameters - Scattering by a free electron (Thomson Scattering) - Scattering by a Bound electron (Rayleigh scattering) – Dispersion Normal and Anomalous – Dispersion in gases (Lorentz theory) – Dispersion in liquids and solids.

BOUNDARY CONDITION

The electromagnetic field vectors changes as one moves across the boundary.

(i) The normal component of the electric displacement is discontinuous by an amount equal to the free surface density of charge at the boundary D_{1n}-D_{2n}=σ

(ii) The normal componentof the magnetic induction is continuous across s surface of discontinuity B_{1n}-B_{2n} = 0

(iii) The tangential component of magnetic intensity is discontinuous by an amount equal to the free surface current density H_{1t}-H_{2t}=J_s

(iv) The tangential component of E is continuous across surface of discontinuity E_{1t}-E_{2t}=0

REFLECTION AND REFRACTION OF ELECTROMAGNETIC WAVE

Consider that when plane electromagnetic waves which are travelling in one medium are incident upon an infinite plane surface separation this medium from another, with different electromagnetic properties. When an electric wave is travelling through space there is an exact balance between the electric and magnetic field. Half of the energy of wave as a matter of fact is the electric field and half in the magnetic. If the wave enters some different medium, there must be a new distribution of energy, whether the new medium is a dielectric a magnetic a conducting or an ionised region, there will have to be a readjustment of energy related as the wave reaches its surface. Since no energy can be added to the wave as it only way that a new balance can be achieved is for some of the incident energy to be reflected.

The transmitted energy constitutes the reflected wave and the reflected one the reflected wave. The reflection and refraction of light at a plane surface between two media of different dielectric properties is a familiar, example of reflection and refraction of electromagnetic waves. The various aspects of the phenomenon divide themselves into two classes.

Kinematic properties :

Following are the kinematic properties of reflection and refraction.

(i) Law of frequency :

The frequency of the wave remains unchanged by reflection or refraction.

(ii) The reflected and refracted waves are in the same plane as the incident wave and the normal to the boundary surface.

(iii) Law of reflection :

In case of reflection the angle of reflection is equal to the angel of incident.

$\theta_I = \theta_R$

(IV) Snell's Law :

In case of reflection the ration of the sin of the angle of refraction to the sin of angle of incident is equal to the ratio of the refractive indices of two media.

$n_1 \sin \theta_i = n_2 \sin \theta_R$

Dynamic properties

These properties are concerned with the

(i) intensities of reflected and refracted waves

(ii) Phase changes and polarisation of waves

The kinematic properties follow immediately from the wave nature of phenomenon and the fact that these are boundary condition to be satisfied. But they do not depends on the nature of the wave or the boundary conditions.

FRESNEL FORMULAE

The formulae relating the amplitude of the reflected and transmitted electromagnetic waves with that of incident one when the boundary is between two dielectrics are called Fresnel formulae. These are contained in the boundary condition.

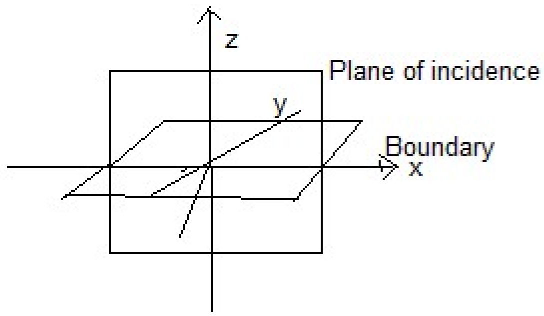
$$(D_i)_n + (D_R)_n = (D_T)_n \quad \text{-------------------- (1)}$$

$$(B_i)_n + (B_R)_n = (B_T)_n \quad \text{-------------------- (2)}$$

$$(E_i)_t + (E_R)_t = (E_T)_t \quad \text{-------------------- (3)}$$

$$(H_i)_t + (H_R)_t = (H_T)_t \quad \text{-------------------- (4)}$$

The condition (1) and (2) when coupled with Snell's law yield no information not included in the (3) and (4) conditions. So it is necessary to consider only condition (3) and (4). Now to derive the desired formulae we consider a plane EMW in x-z plane incident on a plane boundary and consider it as a superposition of two waves one with the electric vector perpendicular to the plane of incidence. Therefore it is sufficient to consider these two cases separately. The general result may be obtained from the appropriate linear combination of the two cases.



CASE I : E parallel to the plane of incidence

The situation is shown in figure. The electric and propagation vectors in two media are indicated. The directions of H vector are chosen as to give a positive flow of energy in the direction of wave vectors. In this situation the magnetic vectors are all parallel to the boundary surface.

$(H_i)_t = H_i$

$$(H_R)_t = H_R$$

$$(H_T)_t = H_T$$

And $(E_I)_t = E_i \cos\theta_i$

$$(E_R)_t = -E_R \cos\theta_R$$

$$(E_T)_t = E_T \cos\theta_T$$

So the boundary condition (3) and (4) reduce to,

$$E_i \cos\theta_i - E_R \cos\theta_R = E_T \cos\theta_T \quad \text{---------------- (5)}$$

$$H_i - H_p = H_T \quad \text{---------------- (6)}$$

In equation (5) and (6) we have emitted the zero subcript on E and H, it being understood that the phases now cancel and equation are relations between amplitudes.

$\theta_i = \theta_R$ and $H = (E/Z) = (n/\mu_r Z_0) = E$

$$H = (n/Z_0)E$$

So equation (5) and (6) reduce to

$$E_i \cos\theta_i - E_R \cos\theta_R = E_r \cos\theta_r \quad \text{--------------- (7)}$$

$$n_1 E_i + n_2 E_R = n_2 E_r \quad \text{---------------- (8)}$$

The interest lies in the fraction of incident amplitudes which are reflected and transmitted. So eliminating E_r from equation (7) with the help of (8) we get

$(E_i - E_R) \cos \theta_I = \frac{n1}{n2}(E_i + E_R) \cos \theta_r$

$$\left(\frac{E_R}{E_i}\right) = \frac{\frac{n2}{n1}\cos\theta i - \cos\theta}{\frac{n2}{n1}\cos\theta i + \cos\theta}$$

$$\left(\frac{E_R}{E_i}\right) = \frac{\left(\frac{\sin\theta i}{\sin\theta T}\right)\cos\theta i - \cos\theta}{\left(\frac{\sin\theta i}{\sin\theta T}\right)\cos\theta i + \cos\theta}$$

$$\left(\frac{E_R}{E_i}\right) = \frac{\sin\theta\ i\ \cos\theta i - \sin\theta\ r\cos\theta\ r}{\sin\quad i\ \cos\theta i + \sin\theta\ r\cos\theta\ r} = \frac{\sin 2\theta i - \sin 2\theta T}{\sin 2\theta i + \sin 2\theta T} \qquad \text{-----------------}\ (A)$$

Similarly eliminating E_R from equation (7) with the help of (8)

$$E_i \cos\theta_I - \left(\frac{n2}{n1} E_T - E_I\right) \cos\theta = E_T \cos\theta_T$$

$$\left(\frac{E_T}{E_i}\right)_{II} = \frac{2\ COS\ \theta\ i}{\frac{n2}{n1}\cos\theta\ i + \cos\quad T}$$

$$\left(\frac{E_T}{E_i}\right)_{II} = \frac{2\ COS\ \theta\ i\ \sin\theta\ T}{Sin\ \theta\ i\cos\theta\ i + \sin\theta\ r\cos\theta\ r}$$

$$\left(\frac{E_T}{E_i}\right)_{II} = \frac{2\ COS\ \theta\ i\ \sin\theta\ T}{\sin(\theta\ i + \theta\ T)Cos\ (\theta\ i - \theta\ T)} \qquad \text{------------------}\ (B)$$

CASE II : E perpendicular to the plane of incidence

The situation is shown the magnetic field vectors and the propagation vectors are indicated. The electric vectors all directed into the plane of the figure.

Since the electric vectors are all parallel to the boundary surface,

$$(E_I)_t = E_i$$

$$(E_R)_t = E_R$$

$$(E_T)_t = E_T$$

And

$$(H_i)_t = - H_I \cos\theta_i$$

$$(H_R)_t = H_R \cos\theta_p$$

$$(H_T)_t = - H_T \cos\theta_T$$

So boundary condition (3) and (4) reduce to

$$E_I - E_R = E_T$$

$$H_i \cos\theta_i - H_R \cos\theta_R = H_T \cos\theta_T \qquad \text{----------------}\ (9)$$

$$\theta_i = \theta_R \text{ and } H = (E/Z) = (n\varepsilon / Z_0) \qquad \text{----------------}\ (10)$$

So equation (10) reduce to

$$n_1 E_1 \cos\theta_1 - n_2 E_R \cos\theta_R = n_2 E_T \cos\theta_T \qquad \text{---------------}\ (11)$$

Now eliminating E_R from equation (11) with help of (9) we get,

$$(E_i - E_R)\, n_1 \cos\theta_I = n_2 \cos\theta_T (E_i + E_R)$$

$$\left(\frac{E_R}{E_i}\right)_\perp = \frac{\cos\ i - \frac{n2}{n1}\cos\theta T}{\cos\theta\, i + \frac{n2}{n1}\cos\theta T}$$

$$\left(\frac{E_R}{E_i}\right)_\perp = \frac{\cos\theta\, i - \frac{\sin\theta\, i}{\sin\theta\, T}\cos\theta T}{\cos\ i + \frac{\sin\theta\, i}{\sin\theta\, T}\cos\theta T}$$

$$\left(\frac{E_R}{E_i}\right)_\perp = \frac{\sin\theta\, r\cos\theta\, i - \cos\ T\sin\theta\, i}{\sin\ r\cos\ i + \cos\theta\, T\, \mathrm{si}\ \ i}$$

$$\left(\frac{E_R}{E_i}\right)_\perp = \frac{\sin(\theta\, i - \theta\, T)}{\sin(\theta\, i - \theta\, T)} \qquad \text{---------- (C)}$$

Similarly eliminating E_R from equation (11) with the help of (9) we get,

$$n_1 E_1 \cos\theta_1 - n_1 (E_T - E_I)\cos\theta_I = n_2 E_T \cos\theta_T$$

$$\left(\frac{E_T}{E_i}\right)_\perp = \frac{2\cos\theta\, i}{\cos\theta\, i + \frac{n2}{n1}\cos\theta T}$$

$$\left(\frac{E_T}{E_i}\right)_\perp = \frac{2\cos\theta\, i}{\cos\theta\, i + \frac{\sin\ i}{\sin\theta\, T}\cos\theta T}$$

$$\left(\frac{E_T}{E_i}\right)_\perp = \frac{2\cos\theta\, i\sin\theta\, T}{\cos\theta\, i\sin\theta\, T + \mathrm{si}\ \ i\cos\theta\, T}$$

$$\left(\frac{E_T}{E_i}\right)_\perp = \frac{2\cos\theta\, i\, \mathrm{si}\ \ T}{\sin(\theta i + \theta\, T)} \qquad \text{----------- (D)}$$

Equation (A), (B), (C), (D) are the desired result known as Fresnel formulae.

BREWSTER'S LAW AND POLARIZATION OF E.M.W.

From Fresnel's formula (A) i.e

$$\left(\frac{E_R}{E_i}\right)_I = \frac{\tan(\theta I - \theta r)}{\tan(\theta I + \theta r)}$$

It is evident that $(E_R/E_i)_I=0$ for

$$\tan(\theta_I - \theta_r) = 0 \quad \text{i.e } \theta_I - \theta_r = 0$$

$$\tan(\theta_I + \theta_r) = \infty \quad \text{i.e } \theta_I + \theta_r = \pi/2$$

Or

The first result is trivial since it implies that the twwo media are optically identical*. But the secound result shows that when the reflected and refracted rays are perpndicular there is no energy carried by the reflected ray. The angle of incidence for which this occurs is called brewster's angleθ_B.

Now as from snell's law

$$n_1 \sin\theta_i = n_2 \sin\theta_r$$

$$\frac{\sin\theta I}{\sin\theta r} = \frac{n_2}{n_1}$$

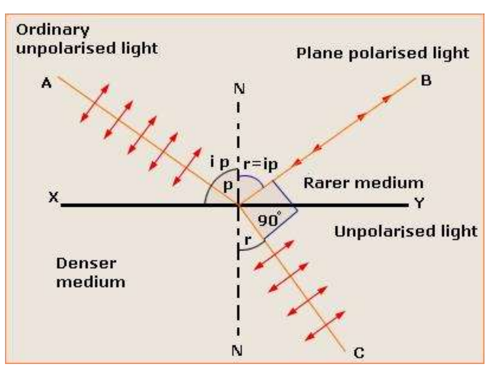
i.e so $\frac{\sin\theta B}{\sin[\frac{\pi}{2} - \theta B]} = \frac{n_2}{n_1}$ [as $\theta_B + \theta_r = \frac{\pi}{2}$

i.e $\tan\theta =_B \frac{n2}{n1}$

$$\theta_B = \tan^{-1}[\frac{n_2}{n_1}]$$

$\theta_B = \tan^{-1}({}_1n_2)$ ……..(a)

Thus if an unpolarised wave is incident on the boundary surface with $\theta_i = \theta_B$only the portion of the wave with electric vector perpendicular to thr plan of incidence will be reflected. That is the reflected wave is linearly polarised with vibration to the plan of incidence. Brewster's angle is therefore some times called the polarising angle. For light incident on glass with n_2=1.5 θ_B=56° and so θ_r=34°. So if light is incident on a glass plate at 56° , the reflected light will be plane polarized with vibrations to the plane of incidence and transmitted light will also be plane polarized with vibration parallel to the plane incidence. For radio wave incident on water as n_2=9, θ_B=84° and so θ_r=6° so water cannot reflect radio wave which are polarised with vibration in the plane of incident and are incident on it at 84°,



Even if the unpolarised incident wave is reflected at angles other than the Brewster's angle, there is a tendency for the reflected wave to be predominantly polarized with vibration perpendicular to the plane of incident. The success of dark glasses which selectively transmits only one direction of vibration depends on this fact. In such cases the degree of polarisation is defined as

$$P(\theta_i)=\frac{R1-R1}{R1+R1} \quad \text{.....(b)}$$

Where $R_{\underline{1}}$and R_1 are the reflection coefficient for the I and II components of reflected light and are given by

$$R_1 = \frac{tan^2(\theta i-\theta r)}{tan^2(\theta i+\theta r)} \quad \text{and } R_1 = \frac{sin^2(\theta i-\theta r)}{sin^2(\theta i-\theta r)}$$

The curves for R_I and $R_{\underline{I}}$ for glass are shown in fig 6.12

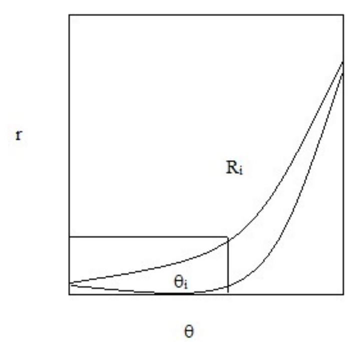
From these it is clear that

For θ_I=0 R_i= $R_{\underline{I}}$ = ·04 so $P_{(0)}$=0

For θ_I=θ_B R_I = 0 and $R_{\underline{I}}$ = ·16 so $P_{(\theta B)}$ = 1

For θ_I = 90 R_I = $R_{\underline{I}}$= 1 so $P_{(90^o)}$ =0

i.e the reflection wave is partially polarized with vibrations perpendicular to the plane of incidence except near 0° and 90°.



Further fro the curves it is clear that the reflecting power at normal incidence is 4% only and it falls to zero at polarising angle for the light with vibration in the plane of incidence and for light with its vibration Ito the plane of incidence it is 15% . All the light is reflected as the angle of incidence approaches 90°*i.e* grazing angle.

TOTAL INTERNAL REFLECTION AND CRITICAL ANGLE

If n_1 is greater than n_2 then in the light of snell's law we

Sin $\theta_i = \frac{n_2}{n_1}$ sin θ_r …..(1)

$\theta_i < \theta_r$ (as $\frac{n_2}{n_1}$) < 1)

So for a particular value of angle of incidence, θ_r will become 90º . The value of angle of incidence for which θ_r become 90º is called critical angle and is represented by θ_c. so for

Sin $\theta_i = \frac{n_2}{n_1}$sin 90º = $\frac{n_2}{n_1}$ ………..(2)

The refracted wave is propagated parallel to the boundary surface

Now it is certainly possible to have θ_i>θ_c.in this situation from equation (1) in the light of (2) we get

Sin $\theta_i = \frac{\text{Sin}\,\theta_i}{(\frac{n_2}{n_1})} = \frac{\text{Sin}\,\theta_i}{\text{Sin}\,\theta_c}$> 1 [as $\theta_i > \theta_c$] …..(3)

Equation (3) means that θ_r is imaginary (as the sine of any real angle can never be greater than 1) . To find the meaning of imaginary θ_r we consider the transmitted electromagnetic wave

E_r=$E_c{}^{e^{-i(\omega 1-K.r)}}$

i.e E_r=$E_c{}^{e^{-i[\omega t-Kr(x\sin\theta_r+z\cos c)]}}$ ……………………(4)

but from equation (3)

cos θ_r= √(1-sin² θ_r)=I √{($\frac{\sin\theta_r}{\sin\theta_c}$)²-1 }= ib ………………(5)

So equation (4) reduces to

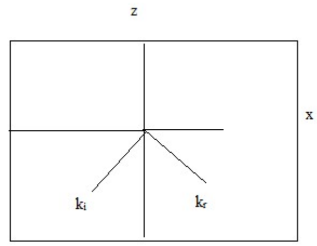
E_r=$E_c{}^{e^{-i[\omega t-\{Kr(\frac{\text{Sin}\,\theta_i}{\text{Sin}\,\theta_c})\}x-ikr\ \ ]}}$ ……………….(6)

Equation (6) shows that $\theta_i > \theta_c$ the transmitted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface.* regarding the surface wave it is worthy to note that

(i) The phase velocity of the wave is a function of angle of incidence i.e

$$\gamma_r = \frac{\omega}{Kr\left(\frac{\text{Sin}\,\theta_i}{\text{Sin}\,\theta_c}\right)} = \frac{\omega}{Kr}\frac{\text{Sin}\,\theta_i}{\text{Sin}\,\theta_c} = \frac{\omega}{Kr}\frac{n_2}{n_1\text{Sin}\,\theta_i}$$

Or $\gamma_r = \frac{\gamma x c/\gamma}{n_1\text{Sin}\,\theta_i} = \frac{c}{n_1\text{Sin}\,\theta_i}$ [as $K_{r=\omega}$/γ and n=c/γ



And in the event the first medium is air it become equal to c for grazing incidence as expected

(ii) These waves do not carry any energy inyo the second medium This is because the time average normal component of the pointing vector just inside the surface

$<S_r>$*n=$<S_r>$ cosθ_r

$<S_r>$*n=$\frac{1}{2}$ Re $(E_r * 11 *)$cosθ_r

$<S_r>$*n=$\frac{1}{2}$ Re $[E_r * \frac{n_1(\text{u}*E_r*)}{Z_o}]cos\theta_r$

i.e $<S_r>$*n=$\frac{1}{2}$ Re$\frac{n_1}{Z_o}[E_r.E_r *]cos\theta_r$n (as n and E_r $are\ perpendicular$)

i.e $< S_r>$*n=$\frac{n_1}{2Z_o}$ Re$[E_o{}^{e-bkrZ}]^2$ (ib) n

i.e $<S_r>$*n=0 (as it has no real part)

such waves are called evanescent waves and are set by the transient effect of the incident waves which strike the surface first.

It is also interesting to see what Fresnel's equations tells us about the field when we use the value of sinθ_r and cos$\theta_r$given by equation (3) and (5) we consider first the case in which E is perpendicular to the plane of incidence i.e

$\frac{E_R}{E_r}=\frac{\cos\theta_i-\frac{n_2}{n_1}\cos\theta_r}{\cos\theta_i+\frac{n_2}{n_1}\cos\theta_r}=\frac{\text{a}-\text{ib}}{\text{a}+\text{ib}}$ with {a=cosθ_r b= $\frac{n_2}{n_1}$ cosθ_r}

` $(\frac{E_R}{E_r})$=e$^{-\emptyset 1}$with tan$\frac{\emptyset 1}{2}=\frac{b*}{\text{a}}$ …………..(7)

Similarly when E is parallel to the plane of incidence

$(\frac{E_R}{E_r})=\frac{\cos\theta_i-\frac{n_2}{n_1}\cos\theta_r}{\cos\theta_i+\frac{n_2}{n_1}\cos\theta_r}=\frac{\text{a}-\text{ib}}{\text{a}+\text{ib}}$ with {a= cosθ_r b= $\frac{n_2}{n_1}$ cosθ_r}

Or $(\frac{E_R}{E_r})$=e$^{-\emptyset 1}$with tan$\frac{\emptyset 1}{2}=\frac{b*}{\text{a}}$ ………(8)

So from equation 7 and 8 it is clear that the amplitudes and the intensity of the reflected wave is equal to that of incident wave i.e the wave is totally reflected and that the phases of the perpendicular and parallel reflected wave are $\emptyset 1$ and $\emptyset 1$ and depend on the angle of incidence. The general consequence of this is that if a linearly polarised wave is reflected from the boundary at an incident angle greater than the critical angle the reflected wave will be elliptically polarised.

Assume that the incident wave is linearly polarized by this we mean that the oscillating electric field always lies along the same direction. If two write the field in their real forms. The component of the linearly polarised field can be written.

$(E_R)_I = (E_o)_I \cos \omega t$ and $(E_R)_I = (E_o)_I \cos \omega t$

After reflection each of these component will have its phase changed by different amount and can be written as

$(E_R)_I = (E_o)_I \cos (\omega t-\emptyset_1)$ and $(E_R)_I = (E_R)_I = (E_o)_I \cos (\omega t-\emptyset_1)$

So now the components no longer vanish nor reach their maxima simultaneously and when we add the two we find the resultant E trace an ellipse.

The phenomenon of total internal reflection is used to produce elliptically or circularly polarised light. It is also exploited in many applications where it is required to transmit light without loss in intensity. In nuclear physics Lucite or ‘light pipes’ are used to carry light emitted from a scintillation crystal because of the passage of an ionizing particle to a photo-multiplier . where it is converted into a useful electric signal.

SCATTERING AND SCATTERING PARAMETERS

If an electromagnetic wave is incident on a system of charged particles, the electric and magnetic components of the wave will Exert Lorentz force on the charge and they will be set into motion. Since the electromagnetic wave is periodic in time, so will be the motion of the particles Thus there will be changes in the directions of motion and hence there will be acceleration. The system will therefore radiate ; that is energy will be absorbed from the incident wave by the particles and will be re-emitted into space in all directions. We describe such a process as scattering of the electromagnetic wave by the system of charge particles. If the energies of the incident and scattered radiations are equal the scattering is called elastic otherwise inelastic.

Scattering is most conveniently characterized by following parameters-

(A) **Differential Scattering Cross-Section :** It is defined as the ratio of the amount of energy scattered by the system per unit time per unit solid angle to the energy flux density or intensity (i.e energy per unit area per unit time in a normal direction) of the incident radiations. So if a solid angle dΩ is substandard at the system of an area ds. The mean power (i.e energy per unit time) scattered by the system will be given by $dP_{sr}=S_{sr}ds$

Where S_{sr} is the intensity of scattered radiation. The mean energy scattered per unit time per unit solid angle will therefore be

$$\frac{dPsr}{d\Omega} = \frac{Ssrds}{d\Omega}$$

Or $$\frac{dPsr}{d\Omega} = S_{sr}r^2 \qquad (\text{as } d\Omega=\frac{ds}{r2}) \quad ......1$$

If the incident energy flux density i.e intensity (pointing vector is S_{tr} , the differential scattering cross-section will be

$$\frac{do}{d\Omega} = \frac{dPsr}{d\Omega}/S_{tr} \qquad ....(A)$$

$$\frac{do}{d\Omega} = \frac{Ssr.r2}{SIr} \quad ....(1A)$$

(Substituting the value of $\frac{dPsr}{d\Omega}$ from eqn .1)

From the above it is clear that the differential scattering cross section has the dimension of the area.

(B) Total Scattering Cross-section : we have defined the differential scattering cross-section as

$$\frac{do}{d\Omega} = \frac{Ssr.r2}{SIr}$$

$$d\sigma = \frac{Ssr.r2d\Omega}{SIr}$$

$$\sigma = \int \frac{Ssr}{SIr} ds \qquad (\text{as } d\Omega = \frac{ds}{r2})$$

$$\sigma = \frac{Psr}{SIr} \qquad [\text{ as } P_{sr} = \int Ssrds] \ldots(B)$$

σ is called the total scattering cross-section and is defined as the ratio of the power scattered (total energy scattered per sec) to the intensity (energy per unit area per unit time) of the incident radiations.

SCATTERING BY A FREE ELECTRON (THOMSON SCATTERING)

Let there be an electron of mass m and change q in the path of a plane polarized monochromatic wave in vacuum. Both the electric and magnetic wave vector E and B will exert a force on a q given by the Lorentz formula

$$F = q(E + v \times B) \ldots\ldots\ldots\ldots(1)$$

Where v is the velocity of the particle its self-produced by wave, and assumed though out this treatment to be non-relativistic so that v <<c. because in a plane wave.

$$B = n \times \frac{E}{c}$$

So expression (1) become

$$F = q[E + v/c \times (n \times E)] \text{ --------------(2)}$$

$$F = qE$$

Thus only action of the electricity field on the charge needs to be considered. And so the equation of motion.

$$F = ms = m\frac{d2r}{dt2} \text{------------------>(3)}$$

In the light of (2) becomes

$$m\frac{d2r}{dt2} = qE \text{ ------------------(4)}$$

And becomes in a plane wave

$$E = E_0 e^{-i(\omega t - k.r)}$$

Equation (4) becomes

$$\frac{d2r}{dt} = \frac{qEo}{m} e^{-i\omega t - k.r)} \ldots\ldots\ldots\ldots(5)$$

Equation (5)implies that the acceleration, velocity and displacement of the particle are all in the same direction as E_0 which itself is constant and that the charge is oscillating sinusoid ally.

Now if the incident electromagnetic wave in which the electric vector is along x-axis is moving along z-axis as shown in fig 7.2 (a) then the acceleration in the x- direction will be given by

$$\frac{d2x}{dt2} = \frac{qE0}{m}\, e^{-i\,(\omega t-kz)} \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

So that the displacement x at time t will be given by

$$x = \frac{qE0}{m\omega 2} e\ {-i(}^{\omega t-kz}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Now as an oscillating charge behaves like an oscillating dipole with dipole moment

P=qx.

It follows from equation (7) that

$$p0 = q2.\frac{E0}{m\omega 2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

But as the average energy radiated per sec. per unit area in a normal direction by an oscillating dipole is given by

$$Ssr = \frac{1}{4\pi\varepsilon 0}\frac{q4E02}{8\pi m2c3r4} sin2\theta \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Further as for a plane wave

S1r=E×H

So the average value of S_1r will be given by

$$S1r = \frac{1}{2}\varepsilon 0cE02 \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

So the differential scattering cross-section $\frac{d\boldsymbol{\sigma}}{d\boldsymbol{\Omega}}$ = ro2 sθ = rc2 cos2Φ **……………..(A)**

the incident radiation here has been taken to be plane polarized. For unpolarised an average must be taken over all orientation of the plane of AB is the direction of E in another wave of incident on the particle of fig (a) containing field point is ψ. It is now preferable to express the scattering in terms of the angle***Φ*** which common to all azimuth . in fig (b) the plane POB is drawn perpendicular to the plane containing AB so that the length BQ is given both by r cosθ and by r sin***Φ*** cosψ .

$$\cos\theta = \sin\boldsymbol{\Phi}\cos\psi$$

$$\sin^2\theta=\sin^2\boldsymbol{\Phi}\cos^2\psi$$

$$\sin^2\theta=1-\cos^2\psi(1-\cos^2\boldsymbol{\Phi})^* \ldots\ldots\ldots\ldots\ldots(11)$$

averaging equation (11) over all ψ, we get

$$\sin^2\theta=1/2(1+\cos^2\boldsymbol{\Phi}) \text{---------------------}(12)$$

substituting the value of $\sin^2\theta$ from eqn (12) in (A), we get

$$\frac{d\sigma}{d\Omega} = r02\frac{1}{2}(1 + cos2\Phi) \text{ ....(B)}$$

This is called Thomson formula for scattering of radiation and is appropriate for the scattering of x-ray by electrons or gamma rays by photons. In it angle is called the scattering angle and the factor 1/2(1+cos² ***Φ***) is called degree of depolarization. From expression (B) it is clear that:

i) Scattering of electromagnetic waves is independent of the nature of incident wave,

(ii) Scattering occurs in all direction and is maximum when φ=0 or π,while when is minimum

$$\varphi = \frac{\pi}{2} or 3\pi/2,$$

(iii) Scattering depends on the nature of the charge particle i.e. scattered and is symmetrical about the line given by $\varphi = \pi/2$.

For s plane polarized light as ψ=0

$\sin^2\theta = 1 - (1 - \cos^2\varphi) = \cos^2\varphi$ which is also evident from fig (a)in which $\theta = (\frac{\pi}{2} - \varphi)$

The total scattering cross-section will be

$$\sigma = \frac{\int d\,\sigma}{d\Omega}\, d\Omega$$

$$= \int r02\frac{1}{2}(1 + cos2\varphi)d\Omega$$

$$\Omega = 2\pi(1 - cos\varphi)$$

$$\sigma r = \frac{8\pi}{3}\, r02 \text{ ....(c)}$$

Result (c) was first of all derived by Thomson and so after his name it is called Thomson scattering cross-section.

A quantum mechanical calculation carried out by Klein and Mishna shows that derivations from Thomson result become significant for incident photon energy hγ whoch is comperable with or larger than the rest energies of the scattering electron mc^2.according to them

$$\sigma KN = r02\langle\frac{8\pi}{3}\,(1 - 2h\frac{\gamma}{mc2} + \cdots)\rangle \qquad \text{for } h\gamma \ll mc2.$$

$$\sigma KN = r02\langle\frac{\pi mc2}{h\gamma}\,(loge(2h\frac{\gamma}{mc2} + 1/2\,))\rangle \text{for } h\gamma \gg mc2$$

From these curves it is clear that;

(i) The scattering depends on the nature of incident radiations.

Quantum mechanical result approaches the classical one on the long wavelength side as the frequency $\gamma = \frac{\omega}{2\pi}$goes to zero.

(ii) The scattering is not symmetrical. In general the scattered radiation is more concentrated in the forward direction.

Apart from these is another feature to Thomson scattering which is modified by quantum considerations. The relation between the wavelength of the scattered radiation at an angle φ and the incident radiation is

$$\lambda s = \lambda 1 + \frac{h}{mc}(1 - cos\varphi)$$

Scattering by a bound electron (Rayleigh Scattering)

Considering a charge whose restoring force is $m\omega_0^2x$ is displaced. Assume that there is a small amount of damping proportional to dx/dt which may be produced in particle by collisions or radiation. The quation of motion becomes,

$$m(d^2x/dt^2) = qE -(m\gamma)\, dx/dt - m\omega_0^2x$$

whereγ is the damping constant per unit mass. Thus

$$(d^2x/dt^2) + (\gamma)\, dx/dt + \omega_0^2x = (q/m)Eoe^{-i(\omega t-kz)} \text{ ----- } (1)$$

The solution of this differential equation consists of two parts:

(a) The complementary function: It is obtained by solving equation

$$(d^2x/dt^2) + (\gamma)\, dx/dt + \omega_0^2x = 0 \text{ -------- } (2)$$

Let the solution of equation (2) be

$$X = A^{\alpha t} \text{ ---------- } (3)$$

So that $dx/dt = A\alpha e^{\alpha t}$

Substituting the values of x, dx/dt and d^2x/dt^2 in equation (2) we get

$$\alpha^2 + \gamma\alpha + \omega^2{}_0 = 0$$

$$\text{i.e. } \alpha = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

so that $x = A_1 e^{[-(\gamma/2)+i(\sqrt{(\omega_0^2-\gamma^2/4)}t)} + A_2 e^{[-\left(\frac{\gamma}{2}\right)-i(\sqrt{(\omega_0^2-\gamma^2/4)}t)}$

i.e. $x = e^{-(\gamma/2)t}\,[A_1 e^{i\beta t} + A_2 e^{-i\beta t}]$ ------ (a)

The constant A_1 and A_2 can be determined by applying initial conditions.

(b) Perpendicular integral:

It is obtained by solving the equation

$$(d^2x/dt^2) + (\gamma)\,dx/dt + \omega_0^2 x = (q/m)Eoe^{-i(\omega t-kz)} \text{ ------ (4)}$$

So that $dx/dt = -i\omega Be^{-i(\omega t-kz)}$

$d^2x/dt^2 = \omega^2 Be^{-i(\omega t-kz)}$

substituting these values of x, dx/dt and d^2x/dt^2 in eqn., (4) we obtain

$$-\omega^2 B + \gamma(-2\omega B) + \omega^2{}_0 B = q/m\,(E_0)$$

$$\text{Or } B = \frac{qE_0}{m[\omega_0^2-\omega^2-i\gamma\ \ ]}$$

$$\text{So that } x = \frac{qE_0}{m[\omega_0^2-\omega^2-i\gamma\omega]}\,e^{-i(\omega t-kz)}$$

$$x = \frac{qE_0 e^{-i(\omega t-kz-\delta)}}{m[(\omega_0^2-\omega^2)+\gamma^2\omega^2]^{\frac{1}{2}}} \text{ --------- (b)}$$

with $\delta = \tan^{-1}(\gamma\omega/[\omega_0^2-\omega^2)]$

So from expression (a) and (b) conclude that the general solution of equation (1) will be

$$x = e\text{-}(\gamma/2)t[A1ei\beta t + A2e\text{-}i\beta t] + \frac{qE_0 e^{-i(\omega t-kz-\delta)}}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}} \text{ ------- (c)}$$

In this solution first term on RHS represents free damped vibrations of the charge. These vibrations die out soon on account of the factor of the charge e-(γ/2)t and the first term can be neglected in considering the final motion. So eqn., © reduces to

$$x = \frac{qE_0 e^{-i(\omega t-kz-\delta)}}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}} \text{ ------- (d)}$$

An oscillating charge is equivalent to an induced electric dipole of moment p=qx

It follows from eqn., (d)

$$p = \frac{q^2 E_0 e^{-i(\omega t - kz - \delta)}}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}}$$

$$\text{or } q= \frac{qE_0}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}} \quad \text{------- (6)}$$

But as average energy radiated per sec per unit area in a normal direction by an oscillating dipole is given by

$$S=(1/4\pi\varepsilon_0)\,(\omega^4 p^2_0/8\pi c^3 r^2)\,\sin^2\theta$$

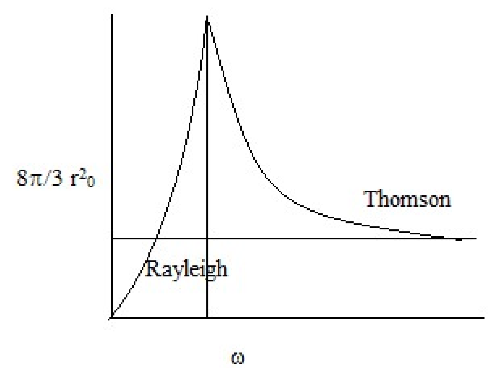
For the present situation $d\sigma/d\Omega = r^2{}_0 \dfrac{\omega^4 \sin^2\theta}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}}$ ----- (A)

Now if ϕ is the angle of scattering and incident radiations are unpolarised then $\sin^2\theta = ½(1+\cos^2\phi)$ --------- (7)

So eqn. (A) in the light of (7) becomes, $d\sigma/d\Omega = \dfrac{\frac{1}{2}(1+\cos^2\phi)r_0^2\omega^4}{m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}}$ ----- (B)

This is the required result. It is clear that

(i) Scattering depends on the nature of the incident radiations, ω
(ii) Scattering depends on the angle of scattering ϕ
(iii) Scattering depends on the nature of the scatter ω_0 and γ.



The total scattering cross-section will be $\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$

$$\sigma = 8\pi/3 \frac{r_0^2\omega^4}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{\frac{1}{2}}} \quad \text{-------- (C)}$$

The expression c gives the total scattering cross-section for an elastically bound electron. The total scattering cross-section is an evident of a function of the frequency of incident radiations.

(i) If $\omega >> \omega_0$ then $\sigma \rightarrow \sigma_T$

(ii) If $\omega \sim \omega_0$, then $\sigma = (8\pi/3)\ r^2{}_0\ (\omega_0/\gamma)^2$ which is very large compared to Thomson scattering cross-section. This is known as resonance scattering.

(iii) If$\omega << \omega_0$ the $\sigma \rightarrow K/\lambda^4$ i.e., amount of scattered light is proportional to $1/\lambda^4$ where λ is the wavelength of the incident radiation. This scattering is known as Rayleigh scattering. This will occur when ω corresponds to the frequencies of visible light and ωo to ultraviolet.

Dispersion normal and anomalous

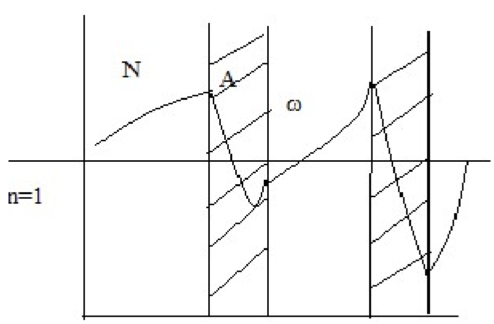
If in the medium the index of refraction various with frequency then the medium is said to be dispersive the phenomena its self is called dispersion and the rate of change refractive index with wavelength i.e. $dn/d\lambda$ is known as dispersive power

Generally the variation of η is such that

(i) The index of refraction increase as the frequency increase

(ii) The rate of increases $dn/d\omega$i.e. the slop of the $\eta - \omega$ curve is greater at the high frequency

However it is also found that over small frequency range there is often decrease of index of refraction with increase in frequency.



In these is narrow spectral region due to it is abnormal behavior the dispersion is called anomalous

DISPERSION IN GASES (LORENTZ THEORY)

In order to investigate the frequency dependence of refractive index n or dielectric constant ϵr and to discuss dispersive Lorentz assumed that in cases of gases

(i) There is no appreciable interaction between the atom cases of atomic gases or between the molecules in case of molecules gases.

(ii) As a electromagnetic wave passes through a gas electric field induces dipole moment in the gas molecule.

(iii) In polarization the position of the electrons are altered from their equilibrium value while nuclei remain stationary.

(iv) The electrons are bound to the nucleus is an atom by linear restoring force.

(v) There is a damping proportional to the velocity of the electron.

(vi) Over an atom or a molecule E is constant in space i.e

$$E=E_0e^{-i(\omega t - k.r)} \approx E0e - i\omega t$$

In the light of above assumption the equation of motion of an electron will be

$$m\frac{d2r}{dt2} + m\gamma 0\frac{dr}{dt} + m\omega o2r = eE$$

$$\frac{d2r}{dt2} + \gamma 0\frac{dr}{dt} + \omega o2r = \frac{e}{M}E\text{.....................(1)}$$

This equation is fully discussed in 7.3 and its solutions was found to be

$$r = \left(\frac{e}{m}\right)E0e - \frac{i\omega t}{\omega o2-\omega} - i\gamma o\omega\text{................(2)}$$

So that dipole moment which results from the displacement of the electron under consideration

$$p = er = \left(\frac{e2}{m}\right)EOe - i\omega t \,/\, \omega o2 - \omega 2 - i\gamma o\omega \quad\text{...................(3)}$$

Now if there are N electrons per unit volume in the gas, the polarization vector

$$P=Np$$

In the light of equation (3) is given by

$$P = N\left(\frac{e2}{m}\right)E/\omega o2 - \omega 2 - i\gamma o\omega \quad\text{..............(4)}$$

In deriving the above equation we have assumed that there is only one type of charge which is characterized by the constants ωg and γg. It is quite reasonable to expect that the electrons are not all in identical situations within molecules and that there should be different pair of characteristic frequency ω_{oj} and associated damping factors γ_{oj}, pair reflecting that particular environment in which the given type of electron is found. So if we define f_j as the probability that an electron has characteristics frequency $\gamma 0 f$ and damping coefficient then the generalization of equation yields This is the required result which expresses the frequency dependence of ϵr or n. to study what this equation implies we consider the following situations;

(A) **Static case** : if $\omega \rightarrow 0$ i.e the frequency of the incident wave is very small in comparision to the natural frequency of the electrons, equation (A) reduces to

$$n2 = 1 + \frac{1}{4\pi\epsilon o}\frac{4\pi Ne2}{m}\frac{\Sigma f1}{\omega o}\text{...............(a)}$$

Equation (a) clearly shows that the index of refraction is a constant greater than unity and depends on the nature of the medium

(B) **normal dispersion:** If γ_{0j} and $\omega < \omega oj$ i.e the region is remote from the natural frequency is of the electrons equation a reduce to

$$n2 = [1 + \frac{1}{4\pi\epsilon o}\frac{4\pi Ne}{m}\frac{\Sigma f1}{\omega oj2-}\text{..............(b)}$$

From equation B is clear that the refractive index in real and increases with frequency of the incident wave

(c) Anomalous dispersion If $\omega \simeq \omega oj$ i.e in the extremely narrow spectral region in which the impressed frequencies include one of the so many natural frequencies of the electron .for simplicity we assume that there is one natural frequency i.e., $\omega 0 = \omega oj$ so that equation(A)becomes Now as the index of refraction of gases under normal conditions is a approximately unity so that expression $(1+y)^{1/2}=1+1/2y$ for $y< 1$ may be employed to obtain Multiplying numerator and denominator of the second term on R.H.S by its complex conjugated equation shows that the index of refraction is a complex function of frequency of the electromagnetic waves propagating through the gas.

The real part of equation i.e n is plotted as a function of frequency ω. At very low frequency n is slightly greater than unity, n increase with increasing ω, reaching a maximum at $\omega 1 \simeq \left(\omega 0 - \frac{\gamma 0}{2}\right)$ falling to unity $(\omega 0 - \frac{\gamma 0}{2})$ where apart it increase again and approache unity asymptotically for large values of ω

The imaginary part of $n2$ is corresponding to absorption of the electromagnetic waves propagating through the gas. The imaginary of n* gas a typical resonance shape. X is maximum at $\omega = \omega 0$ where n is unity and a width at half maximum approximately equal to $\gamma 0$ .therefore in region where n changes rapidly, the gas is relatively high absorbing.

For any real gas exist many imaginary resonant frequency $\omega 0j$ and corresponding damping coefficient $\gamma 0j$ so that

$$n * 1 + \frac{1}{4\pi\varepsilon o} . \frac{2\pi Ne2}{m} \boldsymbol{\Sigma fr}/(\ \omega 0j2 - \ \omega 2 + \ \gamma 0j2\ \omega 4)\text{.........................}(d)$$

the behavior of the real and imaginary parts of the equation (c) is illustrated .

It is worthy to note here that this classical theory cannot of course predict the values of the resonance $\omega 0j$; only a correlation of observation relating to optical properties of matter can be attempted. Quantum theory must be used for complete description and even though a quantum calculation can in principle yield value for the resonance frequency, the computation can be carried out exactly only for the most simple cases.

DISPERSION IN LIQUID AND SOLIDS

In these materials the molecules are sufficiently close to each other do the effect of interactions among the molecules can longer be neglected . Since the material is polarised we expect that the actual electric field on a given charge will have a contribution from the polarisation and hence it will be different from the applied field. The usual way of approximating this is imagine a small sphere centered at the position of the electron question to be cut out of the material.The sphere is to be large enough microscopically and small enough macroscopically o that the material outside it can be described in terms of the continuous polarisation vector P. Then, because of the discontinuity in P there are bound surface changes on the surface of a spherical volume with density

$$\sigma' = P.n = p \cos\theta$$

So the contribution to the local field E' by the change on area ds

$$dE' = \frac{\sigma' ds}{4\pi\varepsilon 0R2}$$

The components normal to the direction of P clearly cancel so integrating over all the surface we find that local field E' parallel to P is equal to

Interaction of E.M.Waves with matter

$$E' = \int dE' \cos\theta = \frac{P}{4\pi\varepsilon 0}\int_0^{2x}\int_0^{x}\frac{\cos\ \theta R^2 \sin\theta\, d\theta\, d\emptyset}{R^2} = \frac{P}{3\varepsilon_0} \quad ....(1)$$

The result is the contribution to the local field from the material outside the sphere. We still have to calculate the contribution due to the molecules within the small sphere. It can be shown that this contribution averages to zero in an isotropic material such as a liquid or when the molecules are arranged in a cubic lattice. Therefore if E is applied electric field the total electric field acting on the average is

$$E_1 = E + E' = E + \frac{P}{3\varepsilon_0}\text{(using equation 1)} \quad ....(2)$$

The equation of motion of an electron in this case will be

$$m\frac{d^2r}{dt^2} + m\gamma_0\frac{dr}{dt} + m\gamma\omega_0 2_{r=}e\ [E + \frac{P}{3\varepsilon_0}]$$

$$m\frac{d^2r}{dt^2} + m\gamma_0\frac{dr}{dt} + m\gamma\omega_0 2_{r=}e[E + \frac{Ner}{3\varepsilon_0}] \quad \text{(as P= N}er)$$

$$\frac{d^2r}{dt^2} + \gamma_0\frac{dr}{dt} + (\omega_0{}^2\frac{Ne^2}{3\varepsilon_0 m})\ r = \frac{e}{m}.\ E \quad ....(3)$$

The solution of equation (3)

$$R = \frac{\left(\frac{e}{m}\right)E}{[\omega_0{}^2\frac{Ne^2}{3m\varepsilon_0} - \omega^2 - i\gamma_{0\ \omega}]} \quad ....(4)$$

However as electron are not all in identical situation with the molecules there should be different pairs of characteristic frequency ω_0 and associated damping factors γ_0 each pair reflecting the particular environment in which the given type of electron is found .So if we define f_j as the probability that an electron has characteristic frequency ω_{0j} and damping coefficient $\gamma_0 j$ then the generalisation of equation (4) yields

$$P = \frac{Ne^2}{m}\sum\frac{f_j}{[\omega_0^2\frac{Ne^2 f_j}{3m\varepsilon_0} - \omega^2 - i\gamma_{0\ \omega}]}$$

$$\alpha =^{p} E = \frac{Np}{NE} = \frac{P}{NE} = \frac{e^2}{m}\sum_j\frac{f_j}{[\omega_0^2\frac{Ne^2 f_j}{3m\varepsilon_0} - \omega^2 - i\gamma_{0\ \omega}]}$$

Using the result of Clausius-Mossotti relation obtain an equation called Lorentz-Lorentz formula

$$(n^2-1)/(n^2+1)\ (M/\rho) = \text{constant}$$

Possible Questions

2 marks

1. State Fresnel law
2. State Brewster's law
3. What is called polarizing angle?
4. What is called critical angle?
5. What is called scattering cross-sections?
6. What is called scattering angle?

6 marks

1.Explain briefly about boundary conditions.

2.Briefly discuss about Rayleigh scattering.

3.Explain Frenel's law.

4.Briefly discuss about Lorentz theory.

5.Discuss the phenomenon of total internal reflection and critical angle.

6.Explain the dispersion in liquids and solids.

7.Discuss about dynamic properties.

8.Explain Thomson scattering.

9.Explain scattering and scattering parameters.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 641 021

DEPARTMENT OF PHYSICS

II MSc PHYSICS

ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS (18PHP303)

MULTIPLE CHOICE QUESTIONS

Questions

UNIT-IV

If the EMW incident on a system of charged particles, the electric and magnetic fields of the wave exert a ________________ force on the charges.

If the energies of the incident and scattered radiations are equal the scattering is called ________________.

If the energies of the incident and scattered radiations are not equal, the scattering is called ________________.

________________ is defined as the ratio of the energy scattered by the system per unit time per unit solid angle to the energy flux density of the incident radiation.

________________ is defined as the ratio of the power scattered to the intensity of the incident radiation.

Lorentz formula is F= ______________________.

The factor ____________ is called degree of polarization.

Scattering depends on the nature of the ______________ particles.

Thomson formula for scattering is appropriate for the scattering for ________________.

Scattering occurs in all directions and is maximum when f = ________________.

Scattering occurs in all directions and is minimum when f = ________________.

The total scattering cross-section according to Thomson's scattering is s_T = ___________________.

In general the scattered radiation is more concentrated in the _____________ direction.

Scattering of electromagnetic waves is ______________ of the nature of the incident wave.

An oscillating charge behave like an oscillating dipole with dipole moment p = _______________.

_______________.

If the amount of scattered light is proportional to 1/l where l is the wavelength of the incident radiation, then scattering is known as ___________ scattering.

The blue color of the sky is due to _____________ scattering.

________________ light has longest wavelength in the visible region.

________________.

The rate of change of refractive index with wavelength is known as ___________________.

The index of refraction _______________ as the frequency increases.

____________ molecules.

The electrons are bound to the nucleus in an atom by ________________.

_____________ remains stationary.

The classical radius of the electron r_0 = ________________.

In a plane wave B = _______________.

If the index of refraction decreases with the increase in frequency over small frequency range, then it is called _______________ dispersion.

In dispersion in gases, there is a damping proportional to the velocity of the ________________.

The dipole moment results from the displacement of the electron is p= ______________.

In case of gases, $\varepsilon_r \rightarrow$ ________________.

____

of the electron.

Thomson scattering is known as _____________ scattering.

Oscillating charge is equivalent to an induced _______________ of moment p = qx.

than rest energies of the scattering electron.

____

that of the incoming waves.

_______________ of the scattering.

The restoring force is __________________.

Example of resonance scattering is _______________.

The color of the sky during sunset or sunrise is ________________.

_________________ light has shorter wavelength in visible region.

According to normal dispersion, the refractive index is _______________.

According to normal dispersion, the refractive index _______________ with frequency of the incident waves.

frequencies.

frequency.

For anomalous dispersion, there is _____________ natural frequency.

The index of refraction is a ________________ function of frequency of the electromagnetic waves propagating through the gas.

At very low frequencies, the index of refraction is slightly _____________ unity.

The imaginary part of index of refraction corresponds to the __________________ of electromagnetic waves propagating through gases.

For any real gas, there exists ______________ resonant frequencies.

BATCH : 2018-2020

opt1	opt2	opt3	opt4	opt5	opt6	Answer
Lorentz	Mechanical	Electrical	No			Lorentz
inelastic	coherent	elastic	incoherent			elastic
inelastic	coherent	elastic	incoherent			inelastic
cross-	section	section	scattering			scattering
section	section	scattering	scattering			scattering
v X B)	B)	X B)	B)			B)
$\cos^2 \phi$)	ϕ)	$\cos^2 \phi$)	ϕ)			$\cos^2 \phi$)
charged	uncharged	y	above			charged
alpha	neutrons	cosmic	electrons			electrons
0°	270°	90°	45°			0°
180°	0	270°	360°			90° or 270°
ρ_0^2	$8\pi/3\ \rho_0^2$	$8\pi/3\ \rho_0$	$-8\pi/3\ \rho_0$			$8\pi/3\ \rho_0^2$
forward	backward	random	above			forward
t	independent	infinite	finite			t
$-qx$	qx^2	qx^2	$-qx^2$			qx^2
(h/mc)	(h/mc) (1-	(h/mc)	(h/mc) (1+			(h/mc) (1-
Thomson	Compton	Inelastic	Rayleigh			Rayleigh
Rayleigh	Compton	Thomson	above			Rayleigh
blue	violet	red	green			red
rarer	dispersive	denser	above			dispersive
n	index	e power	above			power
equals	decreases	varies	increases			increases
tic energy	moment	tatic	above			moment
bond	ionic bond	restoring	above			restoring

nuclei	neutron	proton	muon	nuclei
$-q^2/4\pi\varepsilon_0$	$q^2/4\pi\varepsilon_0 mc$	$q^2/4\pi\varepsilon_0 m$	$q/4\pi\varepsilon_0 mc^2$	$q^2/4\pi\varepsilon_0 mc^2$
E)	(n X E)	E)/c	(n X E)/c	(n X E)/c
normal	abnormal	finite	anomalous	anomalous
proton	electron	neutron	muon	electron
εr	$-\varepsilon r$	εr^2	$-\varepsilon r^2$	εr
–1	0	1	2	1
very large	very small	zero	above	very small
resonance	abnormal	normal	anomalous	resonance
dipole	dipole	gnetic	above	dipole
mc	mv	mc^2	mv^2	mc^2
than	lesser than	equal to	above	lesser than
angle	nature	energy	momentum	angle
$m\omega_0^2 x$	$-m\omega_0^2 x$	$-m\omega_0 x$	$-m^2\omega_0 x$	$-m\omega_0^2 x$
neon lamp	mercury vapour lamp	fluorescent lamp	sodium vapour lamp	sodium vapour
red	blue	yellow	yellowish red	red
red	blue	violet	green	violet
Real	imaginary	complex	rational	Real
proportional	equals	increases	decreases	increases
blue	green	yellow	red	red
red	violet	blue	green	violet
one	two	four	three	one
linear	logarithmic	complex	exponential	complex
lesser than	greater than	equal to	proportional to	greater than
emission	reflection	absorption	refraction	absorption
many	two	three	four	many

RELATIVISTIC ELECTRODYNAMICS

Relativistic Electrodynamics: Purview of special theory of relativity – 4-vectors and Tensors - Transformation equations for charge and current densities J and ρ – For electromagnetic potentials A and ϕ - Electromagnetic field tensor $F_{\mu\nu}$ - Transformation equations for the field vectors E and B - Covariance of field equations in terms of 4-vectors - Covariance of Maxwell equations in 4-tensor forms – Covariance and transformation law of Lorentz force.

RELATIVISTIC ELECTRODYNAMICS

Purview of special theory of relativity

This theory is based on the postulates that:

(i) All inertial frames are equivalent for the description of nature i.e., physical laws are covariant under change of inertial frames.

(ii) The velocity of light is a universal constant, independent of the motion of source and observer.

According to this theory the relations linking the event as (x, y, z, t) in inertial frame S and as (x',y',z't') in inertial frame S' which is moving along z axis with a uniform velocity v relative to S are

$$x' = x \quad \text{------- (A)}$$

$$y' = y \quad \text{-------- (B)}$$

$$z' = (z - vt)/\sqrt{(1-\beta^2)} \quad \text{------(C)}$$

$$t' = \frac{t - \frac{v}{c^2} z}{\sqrt{(1-\beta^2)}} \quad \text{---------- (D)}$$

From frame S to S' and are called Lorentz transformations. Following are the important consequences of these transformation:

(a) **Length :** The length of an object viewed by a moving observation in the direction of motion is motion is $\sqrt{(1-\beta^2)}$ times the length observed by a stationary observer in the same direction. The length of the objection is same for both the observers in directions perpendicular to the direction of motion. So if the dimension of in object W.r.t a stationary observer are d_x d_y and d_z then the dimensions of the same object with respect to an observer moving with a velocity v along z-axis are given by

$$dx' = dx$$

$$dy' = dy$$

$$dz' = dz\sqrt{(1-\beta^2)} = dz/\gamma$$

(b) Time : The time interval between two event occurring at the same place as observed bya moving observer is $\frac{1}{\sqrt{(1-\beta 2)}}$

*i.e*γ times the time interval between the same event as observed by a stationary observer if dt' and dt are time intervals between any two event occurring at the same place as observed by moving and stationary observes respectively then

$$dt' = \frac{dt}{\sqrt{(1-\beta 2)}} = \gamma\ dt$$

(c) Mass : The mass of an object viewed by a moving observer is $\frac{1}{\sqrt{(1-\beta 2)}}$ times the mass of the same object as viewed by a stationary observer. So if m' and m are the masses of an object as viewed by a moving and stationary observer then

$$m' = \frac{m}{\sqrt{(1-\beta)2}} = \gamma m.$$

(d) Energy and Momentum : The energy and momentum of a body whose apparent mass is m' according to this theory are given by

$$E == m'\ c^2$$

CURRENT DENSITY 4-VECTOR

Let us now consider the laws of electromagnetism. These laws are compatible with the relativity principle. In order to achieve this, it is necessary for us to make an *assumption* about the transformation properties of electric charge. The assumption is well substantiated experimentally, is that charge, unlike mass, is *invariant*. That is, the charge carried by a given particle has the same measure in all inertial frames. In particular, the charge carried by a particle does not vary with the particle's velocity.

Let us suppose, following Lorentz, that all charge is made up of elementary particles, each carrying the invariant amount e. Suppose that n is the number density of such charges at some given point and time, moving with velocity u, as observed in a frame S. Let n_0 be the number density of charges in the frame S_0 in which the charges are momentarily at rest. As is well-known, a volume of measure V in S has measure γ(u)V in S_0 (because of length contraction). Since observers in both frames must agree on how many particles are contained in the volume, and, hence, on how much charge it contains, it follows that $n = \gamma(u)n_0$. If ρ=en and $\rho_0 = en_0$ are the charge densities in S and S_0, respectively, then $\rho = \gamma(u)\rho_0$ ------- (1)

The quantity ρ_0 is called the *proper density*, and is obviously Lorentz invariant.

Suppose that x^μ are the coordinates of the moving charge in S. The *current density 4-vector* is constructed as follows: $$J^\mu = \rho_0 \frac{dx^\mu}{d\tau} = \rho_0 U^\mu.$$ ------- (2)

$$J^\mu = \rho_0 \gamma(u)\,(\mathbf{u}, c) = (\mathbf{j}, \rho\, c),$$ ------- (3)

where j=ρu is the current density 3-vector. Clearly, charge density and current density transform as the time-like and space-like components of the same 4-vector.

Consider the invariant 4-divergence of J^μ

$$\partial_\mu J^\mu = \nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t}.$$ ---------- (4)

one of the caveats of Maxwell's equations is the charge conservation law

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0.$$ --- (5)

It is clear that this expression can be rewritten in the manifestly Lorentz invariant form

$$\partial_\mu J^\mu = 0.$$ -----(6)

This equation tells us that there are no net sources or sinks of electric charge in nature: *i.e.*, electric charge is neither created nor destroyed.

4 VECTORS AND TENSORS

The covariance of space-time interval and D'Alembertian are

$$x^2+y^2+z^2-c^2t^2 \text{ and } \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}$$

suggested that the fourth dimension may be taken as ict. The vector with components x,y,x and ict is called 4 dimensional radius vector. Denote this components by x_μ where μ=1,2,3 and 4.

$$x_1 = x \text{ ---------- } (1)$$

$$x_2 = y \text{ -------- } (2)$$

$$x_3 = z \text{ -------- } (3)$$

$$x_4 = ict \text{ ---------- } (4)$$

Under a transformation from one inertial reference system to another i.e., under a Lorentz transformation the components of the fourth dimensional radius vector transform according to

$$x_1' = x_1 \text{ ------------ } (5)$$

$$x_2' = x_2 \text{ ------------ } (6)$$

$$x_3' = (x_3 - vt) / \sqrt{(1-\beta^2)} = \gamma(x_3 + i\beta x_4) \text{ ------------ } (7)$$

$$x_4' = ic\,(t - vx^3/c^2) / \sqrt{(1-\beta^2)} = \gamma(x_4 - i\beta x_3) \text{ -------- } (8)$$

We can write the above transformation as $x_1' = 1.x_1+0.x_2+0.x_3+0.x_4$

$$x_2' = 0.x_1+1.x_2+0.x_3+0.x_4$$

$$x_3' = 0.x_1+0.x_2+\gamma.x_3+i\gamma\beta.x_4$$

$$x_4' = 1.x_1+0.x_2+i\gamma\beta.x_3+\gamma.x_4$$

All the above results can be written as $x_\mu' = \alpha_{\mu\nu}x_\nu$ --------- (A) in matrix form

For example if $\mu=3$

$$x_3' = \alpha_3\nu_{x\nu} = \alpha_{31}x_1+\alpha_{32}x_2+\alpha_{33}x_3+\alpha_{34}x_4 = \gamma[x_3+i\beta x_4] \text{ ----- (B)}$$

So the law of transformation of any 4 vector is $A_\mu' = \alpha_{\mu\nu}A_\nu$ ---------- (C)

Four vectors have properties which are very similar to those of ordinary vectors. Actually 4 vectors are tensors of the first rank in a four dimensional space tensor of higher ranks are defined in a analogous way. A second rank tensor $T\mu\nu$ is a set of sixteen quantities which transform according to the law$T_{\mu\nu}=\alpha_{\nu\mu}\alpha_{\nu\sigma}T\lambda_\alpha$ -------- (D)

TRANSFORMATION EQUATIONS FOR CHARGE AND CURRENT DENSITIES J AND ρ

The law of conservation of charge is mathematically expressed by the continuity equation

$$div\, J + \frac{\partial \rho}{\partial t} = 0$$

Current density and charge density cannot be distinct and completely separable entities since charge distribution that is static in one reference frame will appear as a current distribution in a moving reference frame. The current density J and the charge density ρ according to

$$J_\mu = (J, ic\rho) \text{ -------- (1)}$$

To justify this consider the charge contained in a small volume dτ i.e., $dq=\rho d\tau$ ------- (2)

Multiply both sides by dx_μ, we get $dq\, dx_\mu = \rho(dx_\mu/dt)d\tau dt$ ----------(3)

As dq is a scalar and dx_μ is a vector, LHS of eq., (3) is a four vector. So RHS must also be a 4 vector. But as $d\tau dt = dx_1dx_2dx_3dt$

$$= (1/ic)\, dx_1dx_2dx_3dx_4 \text{ is a Lorentz}$$

scalar i.e., invariant. So $\rho(dx_\mu/dt)$ must be a four vector.

$$J_\mu = \rho(dx_\mu/dt)$$

$$J_1 = \rho(dx_1/dt) = \rho v_1$$

$$J_2 = \rho(dx_2/dt) = \rho v_2$$

$$J_3 = \rho(dx_3/dt) = \rho v_3$$

$$J_4 = \rho(dx_4/dt) = ic\rho$$

The components of the 4 vector J_μ are given by $J_\mu=(J,ic\rho)$

Transform reference frame S to S' under Lorentz transformation as $J_{\mu\nu}' = \alpha_{\mu\nu}J_\gamma$

$$\text{So that } J_1' = J_1 \text{ ------- (A)}$$

$J_2' = J_2$ ------- (B)

$J_3' = J_3 - v\rho/\sqrt{(1-v^2/c^2)}$ ------- (C)

$J_4' = \gamma[J_4 - i\beta J_3]$

Gives, $\rho' = \frac{\rho - \frac{\vartheta}{c^2}J_3}{\sqrt{(1-\frac{\vartheta^2}{c^2})}}$ ------ (D)

Eqn., (A), (B), (C) and (D) are the required laws for the transformation of charge and current from one system S to other S'.

Discussion of the results

(i) In the above results, the charge and current densities transformed from reference frame S to S'. The inverse transformations are obtained by replacing ν to –ν and changing the primed and unprimed quantities.

$J_1 = J_1'$; $J_2 = J_2'$; $J_3 = \frac{J_3' + \vartheta\rho}{\sqrt{(1-\frac{\vartheta^2}{c^2}}}$ *and* $\rho = \frac{(\rho' + \frac{\vartheta}{c^2})\, J_3}{\sqrt{(1-\vartheta^2/c^2)}}$

(ii)The continuity equation, div J + $\partial\rho/\partial t = 0$.

can be written as $\Delta J + \partial(ic\rho)/\partial(ict) = 0$

i.e., $\partial J_\mu/\partial x_\mu = J_\mu = 0$. --------- (E)

(iii) If the charge is at rest in frame S than J=0. So the transformation (A), (B), (C) and (D) yield

$J_1' = 0$------- (a)

$J_2' = 0$------- (b)

$J_3' = -v\rho'$ ------- (c)

$\rho' = \gamma\rho$ --------- (d)

This in turn implies that an observer in frame S' observes a charge at rest in frame S at a charge density $\rho' = \gamma\rho$ together with a convection current.

TRANSFORMATION EQUATIONS FOR ELECTROMAGNETIC POTENTIALS A AND ϕ

The magnetic field vector B is represented as curl of vector potential A. Since A is not completely determined by the specification of its curl alone we have choose the div of A.

In Lorentz gauge: div $A+(1/c^2)(\partial\phi/\partial t) = 0$ -------- (1)

This equ., suggest A_μ as $A_\mu=(A, i/c\ \phi)$ --------- (2)

Consider the field equations, $\nabla^2 A-(1/c^2)(\partial^2 A/\partial t^2) = -\mu_0 J$ -------- (3)

$\nabla^2\phi-(1/c^2)(\partial^2\phi/\partial t^2) = -\rho/\varepsilon_0$ -------- (4)

These eqn., can be written as $\nabla^2 A_1 - \frac{1}{c^2}\frac{\partial^2 A_1}{\partial t^2} = \mu_0 J_1$ -------- (5a)

$\nabla^2 A_2 - \frac{1}{c^2}\frac{\partial^2 A2}{\partial t^2} = \mu_0 J_2$ -------- (5b)

$\nabla^2 A_3 - \frac{1}{c^2}\frac{\partial^2 A_3}{\partial t^2} = \mu_0 J_3$ -------- (5c)

$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon_0$ -------- (5d)

The eqn., 5d can be written as $\nabla^2\left(\frac{i\phi}{c}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{i\phi}{c}\right) = -i\rho/c\varepsilon 0$

i.e., $\nabla^2 A_4 - \frac{1}{c^2}\frac{\partial^2 A4}{\partial t^2} = \mu_0 J_4$ -------- (6)

So that eqn., (5) can be written in general as $\Box^2 A_\mu = -\mu_0 J_\mu$

$\Box^2$ is the Lorentz invariant and Jμ is a 4 vector, so that A_μ must be a 4 vector. If A_μ is a 4 vector, the transformation will be $A'_\mu = \alpha_{\mu\nu} A_\nu$

So that $A'_1 = \alpha_{1\nu} A_\nu$ i.e., $A'_1=1'A_1$ -------- (A)

$A'_2 = \alpha_{2\nu} A_\nu$ i.e., $A'_1=1'A_2$ -------- (B)

$$A'_3 = \alpha_{3\nu}A_\nu = \frac{A_3-\frac{\vartheta}{c^2}\phi}{\sqrt{(1-\frac{\vartheta^2}{c^2}}} \text{ ------ (C)}$$

$A'_4 = \alpha_{4\nu}A_\nu = \gamma[A_4 - i\beta A_3]$

i.e., $\phi' = \frac{\phi-\nu A_3}{\sqrt{(1-\frac{\vartheta^2}{c^2})}}$ ------- (D)

Eqn., (A), (B), (C) and (D) are the desired transformation laws.

In the light of above, Lorentz condition div A+ $(1/c^2)$ $(\partial\phi/\partial t) = 0$ can be written as

$$\text{div A} + \frac{\partial}{\partial}\left(\frac{\frac{i\phi}{c}}{ict}\right) = 0$$

$$\text{i.e., } \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0$$

$$\text{i.e., } \frac{\partial A_\mu}{\partial x_\mu} = \Box A_\mu = 0. -----(E)$$

where $= \frac{\partial}{\partial x_\mu}$ *is called* four dimensional divergence operator.

ELECTROMAGNETIC FIELD TENSOR

Let us now write the components of the electric and magnetic fields as the components of some *proper* 4-tensor. There is an obvious problem here. The former components transform differently under parity inversion than the latter components. Consider a proper-3-tensor whose covariant components are written B_{ik}, and which is antisymmetric: $B_{ij} = -B_{ji}$. ------ (1)

This immediately implies that all of the diagonal components of the tensor are zero. In fact, there are only three independent non-zero components of such a tensor.

Let us write $B^i = \frac{1}{2}\epsilon^{ijk}B_{jk}$. ------- (2)

It is clear that B^i transforms as a contravariant pseudo-3-vector. It is easily seen that

$$B^{ij} = B_{ij} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}, \qquad \text{-------- (3)}$$

where $B^1 = B_1 = B_x$, *etc*. In this manner, we can actually write the components of a pseudo-3-vector as the components of an antisymmetric proper-3-tensor. In particular, we can write the components of the magnetic field B in terms of an antisymmetric proper magnetic field 3-tensor which we shall denote B_{ij}.

Let us now examine Eqs. 1 and 2. It follows that we can write Eq. (1) in the form

$$E_i = -\partial_i \Phi_4 + \partial_4 \Phi_i . \qquad \text{------ (4)}$$

Likewise, Eq. (2) can be written $$c\, B^i = \frac{1}{2}\, \epsilon^{ijk}\, c\, B_{jk} = -\epsilon^{ijk}\, \partial_j \Phi_k . \qquad \text{------ (5)}$$

Let us multiply this expression by ε_{iab}, making use of the identity

$$\epsilon_{iab}\, \epsilon^{ijk} = \delta_a^j\, \delta_b^k - \delta_b^j\, \delta_a^k . \qquad \text{------- (6)}$$

We obtain $$\frac{c}{2}\,(B_{ab} - B_{ba}) = -\partial_a \Phi_b + \partial_b \Phi_a , \qquad \text{------ (7)}$$

or $$c\, B_{ij} = -\partial_i \Phi_j + \partial_j \Phi_i , \qquad \text{------ (8)}$$

since $B_{ij} = -B_{ji}$.

Let us define a proper-4-tensor whose covariant components are given by

$$F_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu . \qquad \text{------- (9)}$$

It is clear that this tensor is antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$. ------ (10)

This implies that the tensor only possesses six independent non-zero components. So that equations (6 and 7) becomes, $F_{4i} = \partial_4 \Phi_i - \partial_i \Phi_4 = E_i$. ------ (11)

Likewise, Eqs. (8) and (9) imply that $F_{ij} = \partial_i \Phi_j - \partial_j \Phi_i = -c\, B_{ij}$. ------ (12)

Thus, $F_{i4}=-F_{4i} = -E_i$ ------- (13)

$F_{ij} = -F_{ji} = -cB_{ij}$ ------ (14)

In other words, the completely space-like components of the tensor specify the components of the magnetic field, whereas the hybrid space and time-like components specify the components of the electric field. The covariant components of the tensor can be written

$$F_{\mu\nu} = \begin{pmatrix} 0 & -c\,B_z & +c\,B_y & -E_x \\ +c\,B_z & 0 & -c\,B_x & -E_y \\ -c\,B_y & +c\,B_x & 0 & -E_z \\ +E_x & +E_y & +E_z & 0 \end{pmatrix}.$$ ------- (15)

Fμν is usually called the *electromagnetic field tensor*. The above expression, which appears in all standard textbooks, is very misleading. We cannot form a proper-4-tensor from the components of a proper-3-vector and a pseudo-3-vector. The expression only makes sense if we interpret B_x as representing the component B_{23} of the proper magnetic field 3-tensor B_{ij}

The contravariant components of the electromagnetic field tensor are given by

$F^{i4}=-F^{4i} = -E^i$ ------- (16)

$F^{ij} = -F^{ji} = -cB^{ij}$ ------- (17)

Or $$F^{\mu\nu} = \begin{pmatrix} 0 & -c\,B_z & +c\,B_y & +E_x \\ +c\,B_z & 0 & -c\,B_x & +E_y \\ -c\,B_y & +c\,B_x & 0 & +E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}.$$ -------- (18)

Let us now consider two of Maxwell's equations: $\nabla.E=\rho/\varepsilon_0$ ------- (19)

$$\nabla \times B = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$ ------- (20)

The 4-current is defined $J^\mu = (j, c\rho)$. The first of these equations can be written

$$\partial_i E^i = \partial_i F^{i4} + \partial_4 F^{44} = \frac{J^4}{c\,\epsilon_0}. \qquad \text{--------- (21)}$$

since $F^{44} = 0$. The second of these equations takes the form

$$\epsilon^{ijk}\,\partial_j(c\,B_k) - \partial_4 E^i = \epsilon^{ijk}\,\partial_j(1/2\,\epsilon_{kab}\,c\,B^{ab}) + \partial_4 F^{4i} = \frac{J^i}{c\,\epsilon_0}. \qquad \text{------ (22)}$$

The above expression reduces to

$$\frac{1}{2}\,\partial_j(c\,B^{ij} - c\,B^{ji}) + \partial_4 F^{4i} = \partial_j F^{ji} + \partial_4 F^{4i} = \frac{J^i}{c\,\epsilon_0}. \qquad \text{------ (23)}$$

The above equation can be combined to give $\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c\,\epsilon_0}.$ ------ (24)

This equation is consistent with the equation of charge continuity, $\partial_\mu J^\mu = 0$, because of the antisymmetry of the electromagnetic field tensor.

Lorentz Transformation of the Fields

Let us consider the Lorentz transformation of the fields. A_μ just transforms like a vector. We could derive the transformed E and B fields using the derivatives of A_μ.

Maxwell's equations indicate that if we transform a static electric field to a moving frame, a magnetic field will be generated, because there is a current in that frame. It was clear from E&M that $\vec{E}$ and $\vec{B}$ were not simply parts of 4-vectors.

The Electric and Magnetic fields are part of a rank 2 tensor and so they transform accordingly.

$$F'_{\mu\nu} = B_{\mu\rho} F_{\rho\sigma} B^{-1}_{\sigma\nu}$$

$$F'_{\mu\nu} = \begin{pmatrix} \gamma & i\beta\gamma & 0 & 0 \\ -i\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & iE_x & iE_y & iE_z \\ -iE_x & 0 & B_z & -B_y \\ -iE_y & -B_z & 0 & B_x \\ -iE_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -i\beta\gamma & 0 & 0 \\ i\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'_{\mu\nu} = \begin{pmatrix} \gamma & i\beta\gamma & 0 & 0 \\ -i\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta\gamma E_x & i\gamma E_x & iE_y & iE_z \\ -i\gamma E_x & -\beta\gamma E_x & B_z & -B_y \\ -i\gamma E_y - i\beta\gamma B_z & -\beta\gamma E_y - \gamma B_z & 0 & B_x \\ -i\gamma E_z + i\beta\gamma B_y & -\beta\gamma E_z + \gamma B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\beta\gamma^2 E_x + \beta\gamma^2 E_x & i\gamma^2 E_x - i\beta^2\gamma^2 E_x & i\gamma E_y + i\beta\gamma B_z & i\gamma E_z - i\beta\gamma B_y \\ i\beta^2\gamma^2 E_x - i\gamma^2 E_x & \beta\gamma^2 E_x - \beta\gamma^2 E_x & \beta\gamma E_y + \gamma B_z & \beta\gamma E_z - \gamma B_y \\ -i\gamma E_y - i\beta\gamma B_z & -\beta\gamma E_y - \gamma B_z & 0 & B_x \\ -i\gamma E_z + i\beta\gamma B_y & -\beta\gamma E_z + \gamma B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & iE_x & i\gamma(E_y + \beta B_z) & i\gamma(E_z - \beta B_y) \\ -iE_x & 0 & \gamma(B_z + \beta E_y) & -\gamma(B_y - \beta E_z) \\ -i\gamma(E_y + \beta B_z) & -\gamma(B_z + \beta E_y) & 0 & B_x \\ -i\gamma(E_z - \beta B_y) & \gamma(B_y - \beta E_z) & -B_x & 0 \end{pmatrix}$$

$$E'_\perp = \gamma(E_\perp - \vec{\beta} \times \vec{B})$$

$$B'_\perp = \gamma(B_\perp + \vec{\beta} \times \vec{E})$$

Since we could choose any direction for the x axis that we boosted along, these results for the field transformation are correct for all boosts.

COVARIANCE OF FIELD EQUATIONS IN TERMS OF 4-VECTORS

An equation does not change its form by the change of frame of reference. All frames moving with constant velocities are equivalent for the description of nature, any physical law must be put in a form which is unaltered by the change of frame of reference.

Maxwell's equation in terms of scalar and vector potentials are given by $\Box^2 A = -\mu_{0J}$

$$\Box^2 \phi = -\frac{\rho}{\mu_0} \text{ ------- (1)}$$

The four vector for potential and charge are defined as $A_\mu = \left(A; \frac{i}{c}\phi\right)$ & $J_\mu = (J, ic\rho) - - - (2)$

and the law of their transformation under Lorentz invariance is $\begin{Bmatrix} A'_\mu \\ J'_\mu \end{Bmatrix} = \alpha_{\mu\vartheta} \begin{Bmatrix} A_\vartheta \\ J_\vartheta \end{Bmatrix} - - - - (3)$

where $\alpha_{\mu\vartheta}$ are the coefficients of the transformation matrix

$$\alpha_{\mu\vartheta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \end{pmatrix} - - - - (4)$$

So the components of the 4 vectors transform according to the relations

$$J'_1 = J_1 \qquad A'_1 = A_1$$

$$J'_2 = J_2 \qquad A'_2 = A_2$$

$$J_3' = \gamma(J_3 - v\rho) \qquad A'_3 = \gamma(A_3 - v/c^2\ \phi)$$

$$J'_4 = \gamma[J_4 - i\beta J_3] \quad A'_4 = \gamma[A_4 - i\beta A_3] \quad \text{-------} (5)$$

And in terms of 4 vectors Maxwell's equation given by expression (1) becomes

$$\Box^2 A_\mu = -\mu_0 J_\mu \text{ with } \partial A_\mu/\partial x_\mu = 0 \text{ ------ } (6)$$

The invariance of these equations require that in any other inertial frame S' the form of these equations must be retained i.e. $\Box'^2 A'_\mu = -\mu_0 J'_\mu$ with $\partial A'_\mu/\partial x'_\mu = 0$ ------ (7)

So Maxwell's equations are invariant under Lorentz transformations.

COVARIANCE OF MAXWELL EQUATIONS IN 4-TENSOR FORMS

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho \text{ ------ } (1)$$

$$\nabla \cdot \mathbf{B} = 0 \text{ ------- } (2)$$

$$\nabla \times \mathbf{H} = J + \frac{\partial D}{\partial t} \text{ ------- } (3)$$

$$\nabla \times \mathbf{E} = \frac{-\partial B}{\partial t} \text{ -------- } (4)$$

If $\mu_r = 1 = \varepsilon_r$, the above equations reduces to

$$\nabla . E = \frac{\rho}{\epsilon_0}$$

$$\nabla . B = 0$$

$$\nabla \times \mathrm{B} - \frac{1}{\mathrm{c}^2}\frac{\partial \mathrm{E}}{\partial \mathrm{t}} = \mu_0 \mathrm{J}$$

$$\nabla \times B + \frac{\partial B}{\partial t} = 0$$

By introducing co-ordinates, $x=x_1$; $y=x_2$; $z=x_3$; $ict= x_4$

Above equation can be written as

$$\nabla . E = \frac{\rho}{\epsilon_0} \text{ -------- } (5)$$

$$\nabla . B = 0 \text{ -------- } (6)$$

$$\nabla \times \mathrm{B} - \frac{\mathrm{i}}{\mathrm{c}^2}\frac{\partial \mathrm{E}}{\partial \mathrm{x}_4} = \mu_0 \mathrm{J} \text{ -------- } (7)$$

$$\nabla \times E + ic\frac{\partial B}{\partial \mathrm{x}_4} = 0 - - - - - - (8)$$

Considering the non homogenous pair of equations (7) and (5)

$$0+\frac{\partial B_3}{\partial x} - \frac{\partial B_2}{\partial x} - \frac{i}{c}\frac{\partial E_1}{\partial x_4} = \mu_0 J_1 - - - - - - (9)$$

$$-\frac{\partial B_3}{\partial x1} + 0 + \frac{\partial B_1}{\partial x3} - \frac{i}{c}\frac{\partial E_2}{\partial x_4} = \mu_0 J_2 - - - - - - (10)$$

$$\frac{\partial B_2}{\partial x1} - \frac{\partial B_1}{\partial x2} + 0 - \frac{i}{c}\frac{\partial E_3}{\partial x_4} = \mu_0 J_3 - - - - - - (11)$$

$$\frac{i}{c}\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\frac{\partial E_3}{\partial x_3} = \mu_0 J_4 - - - - - - (12)$$

By introducing four current density in LHS,

$F_{11}=0$ $F_{12}=B_3$ $F_{13}=-B_2$ $F_{14}=-(i/c)E_1$

$F_{21}=-B_3$ $F_{22}=0$ $F_{23}=-B_1$ $F_{24}=-(i/c)E_2$

$F_{31}=B_2$ $F_{32}= - B_1$ $F_{33}=-0$ $F_{34}=-(i/c)E_3$

$F_{41}=(i/c) E_1$ $F_{42}=(i/c)E_2$ $F_{43}=-(i/c) E_3$ $F_{44}=0$

(a) If $\mu=1$, then in general,

$$\frac{\partial E_{1v}}{\partial x_v} = \mu_0 J_1$$

(b) If $\mu=4$, then in general,

$$\frac{\partial E_{4v}}{\partial x_v} = \mu_0 J_4$$

$$Now\ consider\ eqn.,(8) and\ (6)$$

$$0+\frac{\partial E_3}{\partial x2} + \frac{\partial E_2}{\partial x3} + \frac{i}{c}\frac{\partial B1}{\partial x_4} = 0 ------(13)$$

$$\frac{\partial E_3}{\partial x1} + 0 + \frac{\partial E_1}{\partial x3} + \frac{i}{c}\frac{\partial B_2}{\partial x_4} = 0 ------(14)$$

$$\frac{\partial E_2}{\partial x1} - \frac{\partial E_1}{\partial x2} + 0 + \frac{i}{c}\frac{\partial B_3}{\partial x_4} = 0 ------(15)$$

$$\frac{i}{c}\frac{\partial B_1}{\partial x_1} + \frac{i}{c}\frac{\partial B_2}{\partial x_2} + \frac{i}{c}\frac{\partial B_3}{\partial x_3} + 0 = 0 ------(16)$$

Dividing these eqn., with ic

$$0-i/c\,\frac{\partial E_3}{\partial x2} + \frac{i}{c}\frac{\partial E_2}{\partial x3} + \frac{\partial E_1}{\partial x_4} = 0$$

$$\frac{i}{c}\frac{\partial E_3}{\partial x1} + 0 - \frac{i}{c}\frac{\partial E_1}{\partial x3} + \frac{\partial B_2}{\partial x_4} = 0$$

$$-\frac{i}{c}\frac{\partial E_2}{\partial x1} + \frac{i}{c}\frac{\partial E_1}{\partial x2} + 0 + \frac{\partial B_3}{\partial x_4} = 0$$

$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} + 0 = 0$$

Using Fμν these eqn., can be written as

$$0+\frac{\partial F_{34}}{\partial x} + \frac{\partial F_{42}}{\partial x} + \frac{\partial 23}{\partial x_4} = 0 \quad -------(17)$$

$$\frac{\partial F_{43}}{\partial x} + 0 + \frac{\partial F_{14}}{\partial x} + \frac{\partial F_{31}}{\partial x_4} = 0-------(18)$$

$$\frac{\partial F_{24}}{\partial x1} + \frac{\partial F_{41}}{\partial x2} + 0 + \frac{\partial F_{21}}{\partial x_4} = 0-------(19)$$

$$\frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_3} + 0 = 0-------(20)$$

All these eqn., can be written as

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} = 0$$

This equation express Maxwell's equation in tensor form.

COVARIANCE AND TRANSFORMATION LAW OF LORENTZ FORCE

The Lorentz force equation which gives the force experienced by single partivle having charge e and moving with velocity v in an electric field E and magnetic field B is

$$F=e(E+V\times B)$$

So if in volume τ there are n charge carriers then the force experienced in the volume τ will be

$$F=ne(E+V\times B)$$

$$\text{Or } F=q(E+V\times B)$$

Therefore the force experienced by a small volume element $d\tau$ containing charge dq will be

$$d\,F=dq(E+v\times B)$$

$$d\,F=dqE+dq\,dl/dt\times B$$

$$d\,F=dqE+IdI\times B$$

so the force experienced per unit volume

$$f=\frac{dF}{d\tau}=\frac{dq}{d\tau}\;E+dl\frac{I}{d\tau}\times B$$

$$F=\rho E+\frac{dl}{dsdl}\,(l\times B)$$

$$F=\;\rho E+(J\times B)\qquad \text{as } J=In/ds.....(2)$$

In terms of components equation (2) can be written as

$$f1=\rho E1+J2B3-J3B2............(3)$$

$$f2=\rho E2+J3B1-J1B3..........(4)$$

$$f3=\rho E3+J1B2-J2B1...........(5)$$

And if we use electromagnetic field tensor F$\mu\gamma$these equations can be written as

$$f1=F11J1+F12J2+F13J3+F14J4...........(6)$$

$$f1=F21J1+F22J2+F23J3+F24J4\;...\;...\;...\;...\;...\,.\,.\,(7)$$

$$f1=F31J1+F32J2+F33J3+F34J4....................(8)$$

Equation 6,7and 8 can be written more compactly as $f\alpha=F\alpha\gamma J\gamma$

The right hand side ofthe above equation is evidently the space component of a 4- vector; so f must be space part of a 4- vector fμ with

$$f\mu=F\mu\gamma J\gamma$$

To see the meaning of the fourth component of the 4- vector force density we write

$$f4=F4\gamma\,J\gamma$$

$$= F41J1 + F42J2 + F43J3 + F44J4$$

$$= \frac{1}{C}(E1J1 + E2J2 + E3J3)$$

f 1=1/c (E.J) ………………………….(C)

equation (c) shows that f_4 is imaginary and within the factor i/c represents the work done per unit volume per unit time .the equation (B)can also be written as

$$f\rho = \frac{1}{\mu 0} F\mu\gamma(\frac{\partial F}{\partial X})\text{……………….(D)}$$

Equation (D) is a tensor equation of rank one so it is invariant under Lorentz transformation i.e equation (d) or (b) is the covariant from of Lorentz force equation and gives the rate of change of mechanical moment per unit volume as it is time space part and mechanical energy per unit volume as its time part. Alternatively it may be viewed as given the space and time derivatives of something of the dimensions of work per unit volume

As in the rest system of charges no works done on moving charges

$$f_4=0$$

so the law of transformation of 4- vector given by

$$f\mu' = \alpha\mu\gamma f\gamma\text{…..(9)}$$

$$f1' = f1\text{……..(10)}$$

$$f2' = f2\text{…….(11)}$$

$$f3 = \gamma f3\text{…….(12)}$$

The component of total force which will be exert on a particular charge distribution or on a particular volume of space containing charge will therefore be .

$$\mathrm{F}\alpha' = \int\tau f\alpha' d\tau'$$

So that $\mathrm{F}1' = \int\tau f1' d\tau$

$$=\int \surd\,(1-\beta 2) f1 d\tau$$

$$\mathrm{F1}=\surd\,(1-\beta 2)F1$$

$$\mathrm{F2}=\surd\,(1-\beta 2)F2 \quad \text{…………..(E)}$$

$$\mathrm{F1}=F2$$

expression (E) is the require transformation for the Lorentz force and can be written more confidently as $\mathrm{F}_\perp == \surd\,(1-\beta 2)\, F_\perp$ With inverse $\mathrm{F}_\perp == \surd\,(1-\beta 2)\, F_\perp$…….(F)

Where $F_{\perp}$ is component of Lorentz force in a plane transfer to the direction of motion while f1 is the component of Lorentz force in the direction of motion. The transform=motion given byexpression (E) or (F) are in agreement with the mechanical force transformation, provingthereby that the nature of course (i.e electrical or mechanical) does not effect its transformation properties.

RELATIVISTIC ELECTRODYNAMICS

Possible Questions

2 marks

1. Give a note on 4 vectors.
2. What is called tensor?
3. What is called vector?
4. Give a note on special theory of relativity?
5. Define current density.

6 marks

1. Discuss in detail about 4 tensors and 4 vectors.
2. Explain about the covariance of Maxwell equation in terms of 4 vectors.
3. Discuss in detail about transformation equation for charge and current density.
4. Explain about the covariance of Maxwell equation in terms of 4 tensor.
5. Obtain transformation equation for electromagnetic potential A and ϕ.
6. Briefly discuss about electromagnetic field tensor $F_{\mu\upsilon}$.
7. Explain Covariance and transformation law of Lorentz force.
8. Explain the transformation equation of field vector E and B.

KARPAGAM ACADEMY OF HIGHER EDUCATION, COIMBATORE – 641 021

DEPARTMENT OF PHYSICS

II MSc PHYSICS

ELECTROMAGNETIC THEORY AND ELECTRODYNAMICS (18PHP303)

MULTIPLE CHOICE QUESTIONS
Questions

UNIT-V

The three dimensional formulae are __________________ under Lorentz transformation.
The fourth component of four vector is ___________________.
________________ have properties similar to those of ordinary vectors.
4 – vectors are tensors of rank __________________.

The current and charge density cannot be ________________.
The fourth component of current density is ______________.
The charge distribution that is static in one reference frame will appear as a ___________ distribution in a moving reference frame.

The law of conservation of charge is mathematically expressed by the equation _____________.
If the charge is at rest, then J= _________.
The fourth component of A is _______________.
The electromagnetic field vector E in terms of scalar and vector potential is E = ___________.
The electromagnetic field vector B in terms of scalar and vector potential is B = _____________.
The electromagnetic field tensor is ______________ in nature.
Electromagnetic field tensor is a tensor of rank ____________.
$\partial / \partial x_m$ is a _______________ dimensional divergence operator.
We group together the current density J and current density r as _____________.
The inverse transformation are obtained by replacing v by ___________.
Maxwell's equation are _____________ under Lorentz transformation.
$\nabla . B$ = ______________.
$\nabla X E - \partial B/\partial t$ = ______________.
$\nabla . D$ = ________________.
Tensor equations are ________________ under co-ordinate transformation.
E and magnetic field B is given by ________________.
[illegible] ________________ another co-ordinate frame.
The electromagnetic field tensor has ___________ form in all Lorentz reference frames.
When we move along a perpendicular direction to a static field, then E gets _________ and transverse B gets ___________.

E.B = ______________.

E.B = 0 in all inertial frames only if E and B are mutually ______________ to each other.

The nature of force _____________ effect its transformation properties.

The velocity of light is a universal _______________.

Physics laws are ______________ under change of inertial frames.

D'Alembertian operator is ______________.

E.B is ________________ under Lorentz transformation.

Maxwell's equation _____________ change its form by the change of frame of reference.

As the electromagnetic field tensor is antisymmetric, then F_{12} = _______________.

D'Alembertian operator is defined as the ______________ of a 4 – vector.

The fourth component of 4 - vector force density is f_4 = ___________________.

In the fourth component of 4 – vector force density, the factor (i/c) represents the ________________ per unit volume per unit time.

In the rest system of charges, the fourth component of 4 – vector f_4 = _____________.

BATCH : 2018-2020

opt1	opt2	opt3	opt4	opt5	opt6	Answer
not covariant	covariant	not possible	possible			not covariant
icu	icv	ict	icx_4			ict
five	four	three	tensors			four
two	three	four	one			one
distinct	different	proportio nal	inverse			distinct
ict	icu	icr	icv			icr
volume	surface	current	charge			current
div J + $\partial r/\partial t$	div J - $\partial r/\partial t$	-div J + $\partial r/\partial t$	-div J - $\partial r/\partial t$			div J + $\partial r/\partial t$
1	-1	-2	0			0
(i/c) ρ	(i/c)ϕ	-(i/c) ϕ	icϕ			(i/c)ϕ
+ $\partial r/\partial t$	$\partial r/\partial t$	+ $\partial A/\partial t$	$\partial A/\partial t$			$\partial A/\partial t$
div A	–div A	Curl A	–Curl A			Curl A
constant	antisymm	symmetri	nal			antisymm
one	three	four	two			two
four	three	two	one			four
J, ict	J, icr	–J, ict	–J, icr			–J, icr
v^2	$-v^2$	$-v^3$	$-v$			$-v^3$
variant	nal	invariant	the above			invariant
A	Curl A	0	1			A
1	-1	0	J			0
r	J	A	–J			r
variant	invariant	nal	the above			invariant
X B)	X B)	X B)	B)			X B)
of	electric	magnetic	electric			of
different	same	nal	the above			same
increased	reduced,	decreased	increased			reduced,
,	added/	,	,			added/

1	-1	0	the above	0
parallel	nal	inverse	ular	ular
will	does not	proportio	the above	does not
constant	value	variant	the above	constant
variant	covariant	nal	proportio	covariant
variant	nal to	proportio	under	under
invariant	variant	nal	proportio	invariant
will	proportio	does not	inverse	does not
$-F_{12}$	$-F_{21}$	F_{22}	F_{21}	$-F_{21}$
curl	grad	div	the above	div
(i/c) (E.J)	(E.J)	$(E.J)^2$	$(E.J)^{-1}$	(i/c) (E.J)
energy	work done	power	force	work done
1	-1	E	0	0