Semester-II

19CHU611 MATHEMATICS-II - PRACTICAL 4H -2C

Instruction Hours / week: L: 0 T: 0 P: 4 **Marks:** Internal: **40** External: **60** Total: **100**

End Semester Exam: 3 Hours

Course Objectives

This course enables the students to learn

- To solve simultaneous linear algebraic equations using various methods.
- To evaluate definite integrals using numerical techniques.
- Problem-solving through (computer language) programming.

Course Outcomes (COs)

On successful completion of this course, the student will be able to

- Familiarize with the programming environment for numerical methods.
- Develop proficiency in skills to solve the algebraic equations.
- Evaluate the definite integrals using computer programming techniques

List of Practical

1. Compute Fourier Coefficients.
2. Solution of simultaneous linear algebraic equations – Gauss Elimination method
3. Solution of simultaneous linear algebraic equations – Gauss Jordan method
4. Solution of simultaneous linear algebraic equations – Gauss Jacobi method
5. Solution of simultaneous linear algebraic equations – Gauss Seidal method
6. Numerical Integration – Simpson's one third rule
7. Numerical Integration – Simpson's three eighth rule
8. Numerical Integration – Trapezoidal rule

UNIT - 2

FOURIER SERIES

PART - A

1. State the Sufficient Conditions for a function $f(x)$ to be expressed as a fourier Series [M/J 2017, N/D 2016, N/D 2014

Ans: A function $f(x)$ defined in $c < x < c+2l$ can be expanded as an infinite trigonmetric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

statisfied conditions are

(i) $f(x)$ is periodic, Single Valued and finite in $(c, c+2l)$

(ii) $f(x)$ is continuous with finite number of discontinuities in $(c, c+2l)$

(iii) $f(x)$ has atmost a finite number of maxima or minima in $(c, c+2l)$

2. If the fourier series of the function $f(x) = x+x^2$, in the interval $(-\pi, \pi)$ is
$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx \right]$$

[M/J 2017

Ans: Given $f(x) = x+x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{4}{n^2} \cos nx - \frac{2}{n} \sin nx \right\}$ — ①

Put $x = \pi$, which is an end point of the interval

$\therefore$ The sum of the series

$$= \frac{1}{2}\left[f(-\pi) + f(\pi-) \right]$$

$$= \frac{1}{2}\left[-\cancel{\pi} + \pi^2 + \cancel{\pi} + \pi^2 \right] = \pi^2$$

$$① \Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{4}{n^2} \cos n\pi - \cancel{\frac{2}{n} \sin n\pi} \right\}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{4}{n^2} (-1)^n \right\}$$

$$\Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} (-1)^{2n} \times \frac{4}{n^2}$$

$$\Rightarrow \frac{\cancel{2}\pi^2}{3} = \cancel{4}^{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\because (-1)^{2n} = 1)$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \infty$$

7. If the fourier series of the function $f(x) = x$, $-\pi < x < \pi$ with period 2π is given by

$$f(x) = 2\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \right)$$

Then find sum of the series

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ [A/M 2015

Ans: Given: $f(x) = x$ in $(-\pi, \pi)$

& $f(x) = 2\left\{\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right\}$ — ①

To find sum of the series

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Put $x = \pi/2$ in ① ($\because$ It is continuous)

$$① \Rightarrow \frac{\pi}{2} = 2\left\{\sin\frac{\pi}{2} - 0 + \frac{\sin\frac{3\pi}{2}}{3} - 0 + \cdots\right\}$$

$$\Rightarrow +\frac{\pi}{2} = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \right\}$$

$$\Rightarrow \frac{\pi}{2} \times \frac{1}{2} = \left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right\}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

5. Find the value of b_n in the fourier series expansion of $f(x) = \begin{cases} x+\pi & \text{in } (-\pi, 0) \\ -x+\pi & \text{in } (0, \pi) \end{cases}$

[M|J-2016

Ans:-

Given $f(x) = \begin{cases} x+\pi & \text{in } (-\pi, 0) \\ -x+\pi & \text{in } (0, \pi) \end{cases}$

$$f(x) = \begin{cases} \phi_1(x) & \text{in } (-\pi, 0) \\ \phi_2(x) & \text{in } (0, \pi) \end{cases}$$

Where $\phi_1(x) = x+\pi$, $\phi_2(x) = -x+\pi$

$$\phi_1(-x) = -x+\pi = \phi_2(x)$$

$$\therefore \phi_1(-x) = \phi_2(x)$$

The given function $f(x)$ is an even function. $\therefore b_n = 0$

6. Find the root mean square value of $f(x) = x(l-x)$ in $0 \le x \le l$ [N|D 2015

Ans:- Given $f(x) = x(l-x)$ in $(0, l)$

$$= xl - x^2$$

$$(f(x))^2 = (xl-x^2)^2 = x^2l^2 + x^4 - 2x^3l$$

To find : $\text{RMS} = \sqrt{\frac{1}{b-a}\int_a^b (f(x))^2\,dx}$

$$= \sqrt{\frac{1}{l}\int_0^l (x^2l^2 + x^4 - 2x^3l)\,dx} = \frac{l^2}{\sqrt{30}}$$

3. Expand $f(x) = 1$ in $(0, \pi)$ as a half range sine series. [N/D 2016, N/D 2015

Ans:- Given: $f(x) = 1$ in $(0, \pi)$

To expand half range sine series in $(0, \pi)$ of given $f(x)$

(ie) $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{\cos 0}{n}\right]$$

$$= \frac{2}{n\pi}\left\{-(-1)^n + 1\right\}$$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is an odd} \\ 0 & \text{if } n \text{ is an even} \end{cases}$$

$$\therefore f(x) = \sum_{n=\text{odd}}^{\infty} \frac{4}{n\pi} \sin nx$$

$$= \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n} \sin nx$$

4. Find the Value of the fourier series of $f(x) = \begin{cases} 0 & \text{in } (-c, 0) \\ 1 & \text{in } (0, c) \end{cases}$ at the point of discontinuity $x = 0$ [M/J 2016]

Ans:- Given $f(x) = \begin{cases} 0 & \text{in } (-c, 0) \\ 1 & \text{in } (0, c) \end{cases}$

To find: The value of the fourier series at the point of discontinuity $x = 0$

But the fourier series of given

$$f(x) = \frac{1}{2} + \frac{2}{c}\left\{ \sin\left(\frac{\pi x}{c}\right) + \frac{\sin(3\pi x/c)}{3} + \frac{\sin(5\pi x/c)}{5} + \cdots \right\}$$

Put $x = 0$

$$f(x) = \frac{1}{2} + 0 + 0 = \frac{1}{2}$$

$$\boxed{f(x) = 1/2}$$

8. If $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, $0 < x < 2\pi$, then deduce that the value of $\sum \frac{1}{n^2}$

[N/D 2014

Ans: Given: $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

Put $x = 0$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos 0}{n^2}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{3\pi^2 - \pi^2}{3 \times 4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

UNIT - 2

FOURIER SERIES

PART - B

1. Find the fourier series of period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$

[M/J - 2017]

Solution :-

Given: $f(x) = x\cos x$ in $(0, 2\pi)$

The fourier series of $f(x)$ in $(0, 2\pi)$ defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x}\,\underset{v}{\cos x\,dx}$$

$u = x \quad , u' = 1$

$V_1 = \sin x \, , \, V_2 = -\cos x$

$$= \frac{1}{\pi}\left\{ \cancel{(x\sin x)_0^{2\pi}} + (\cos x)_0^{2\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \cos 2\pi - \cos 0 \right\} = \frac{1}{\pi}(1-1) = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\cos x\,dx$$

$$= \frac{1}{\pi}\times\frac{1}{2}\left\{\int_0^{2\pi} x\left\{\cos(n+1)x + \cos(n-1)x\right\}dx\right\}$$

$$\left[\cos(A+B) + \cos(A-B) = 2\cos A\cos B\right]$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\cos(n+1)x\,dx + \int_0^{2\pi} x\cos(n-1)x\,dx\right\}$$

$u = x$, $u' = 1$, $V_1 = \frac{\sin(n+1)x}{n+1}$, $V_2 = -\frac{\cos(n+1)x}{(n+1)^2}$ | $u = x$, $u' = 1$, $V_1 = \frac{\sin(n-1)x}{n-1}$, $V_2 = -\frac{\cos(n-1)x}{(n-1)^2}$

$$= \frac{1}{2\pi}\left\{\left[\cancel{\left(x\frac{\sin(n+1)x}{n+1}\right)_0^{2\pi}} + \left(\frac{\cos(n+1)x}{(n+1)^2}\right)_0^{2\pi}\right] + \left[\cancel{\left(x\frac{\sin(n-1)x}{(n-1)}\right)_0^{2\pi}} + \left(\frac{\cos(n-1)x}{(n-1)^2}\right)_0^{2\pi}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\cos 2\pi(n+1)}{(n+1)^2} - \frac{\cos 0}{(n+1)^2} + \frac{\cos 2\pi(n-1)}{(n-1)^2} - \frac{\cos 0}{(n-1)^2}\right\}$$

6 ②

$$= \frac{1}{2\pi}\left\{\frac{1}{\cancel{(n+1)^2}} - \frac{1}{\cancel{(n+1)^2}} \; \frac{-1}{(n-1)^2} - \frac{1}{(n-1)^2}\right\}$$

$\cos(n+1)(2\pi)$

$(\because \cos(2\pi+2n\pi)$

$= \cos 2n\pi = 1$

$\cos[(n-1)2\pi]$

$= +\cos((1-n)2\pi)$

$= \cos(1-n)2\pi$

$= \cos(2\pi - 2n\pi)$

$= -\cos 2n\pi)$

$= -1]$

$$= \frac{1}{\cancel{2}\pi}\left\{\frac{-\cancel{2}}{(n-1)^2}\right\}.$$

$$= \frac{-1}{(n-1)^2}$$

$$a_n = \frac{-1}{(n-1)^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cos x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin nx \cos x \, dx$$

$$= \frac{1}{\pi} \times \frac{1}{2}\int_0^{2\pi} x\left[\sin(n+1)x + \sin(n-1)x\right] dx$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x \sin(n+1)x \, dx + \int_0^{2\pi} x \sin(n-1)x \, dx\right\}$$

$u = x, \; u' = 1 \qquad\qquad \Big| \qquad u = x \quad , \; u' = 1$

$V_1 = -\dfrac{\cos(n+1)x}{n+1}, \; V_2 = -\dfrac{\sin(n+1)x}{(n+1)^2} \qquad \Big| \qquad V_1 = -\dfrac{\cos(n-1)x}{n-1}, \; V_2 = -\dfrac{\sin(n-1)x}{(n-1)^2}$

$$= \frac{1}{2\pi}\left\{\left[\left(-\frac{x\cos(n+1)x}{n+1}\right)_0^{2\pi} + \cancel{\left(+\frac{\sin(n+1)x}{(n+1)^2}\right)_0^{2\pi}}\right]\right.$$

$$+ \left[\left(-\frac{x\cos(n-1)x}{(n-1)}\right)_0^{2\pi} + \cancel{\left(\frac{\sin(n-1)x}{(n-1)^2}\right)_0^{2\pi}}\right]$$

$$= \frac{1}{2\pi}\left\{\left(-\frac{2\pi\cos(n+1)(2\pi)}{n+1}\right) + 0 + \left(-\frac{2\pi\cos(n-1)2\pi}{n-1}\right) + 0\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right\}$$

$$= \frac{1}{\cancel{2\pi}} \times \cancel{2\pi}\left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\} = \left\{\frac{\cancel{-n}+1+\cancel{n}+1}{n^2-1}\right\}$$

$$= \frac{2}{n^2-1} \quad \text{if } n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \left[\sin 2x \right] dx$$

$u = x, \quad u' = 1$

$V_1 = -\frac{\cos 2x}{2}, \quad V_2 = -\frac{\sin 2x}{4}$

$$= \frac{1}{2\pi} \left\{ \left(-\frac{x \cos 2x}{2} \right)_0^{2\pi} + \left(\cancel{\frac{\sin 2x}{4}} \right)_0^{2\pi} \right\}$$

$$= \frac{1}{\cancel{2\pi}} \left\{ -\frac{\cancel{2\pi} \cos 4\pi}{\cancel{2}} \right\}$$

$$\boxed{b_1 = -\frac{1}{2}}$$

$\therefore$ The fourier series of $f(x)$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} \frac{1}{(n-1)^2} \cos nx + \left[\frac{-1}{2}\right] \sin x$$

$$+ \sum_{n=2}^{\infty} \frac{2}{n^2-1} \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)^2} \cos nx - \frac{1}{2} \sin x + 2 \sum_{n=2}^{\infty} \frac{\sin nx}{n^2-1}$$

Find the complex form of the fourier series of $f(x) = e^{-ax}$ in $-l < x < l$ [A/M - 2017

Solns: Given: $f(x) = e^{-ax}$, $-l < x < l$

The complex form of $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where $C_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} \cdot dx$

$$= \frac{1}{2l} \int_{-l}^{l} e^{-ax} e^{-\frac{in\pi x}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^{l} \left[\frac{e^{-\frac{(a+in\pi)x}{l}}}{-\frac{(a+in\pi)}{l}} \right]_{-l}^{l}$$

$$= \frac{1}{-2(a+in\pi)} \left\{ e^{-a} \cdot e^{-in\pi} - e^{a} \cdot e^{in\pi} \right\}$$

$$= \frac{a-in\pi}{-2(a^2+n^2\pi^2)} \left\{ e^{-a} \{\cos n\pi - i \sin n\pi\} - e^{a} (\cos n\pi + \sin n\pi) \right\}$$

$$= \frac{a-in\pi}{a^2+n^2\pi^2} (-1)^n \sinh a$$

The complex fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(a-in\pi)(-1)^n}{a^2+n^2\pi^2} \sinh a \cdot e^{in\pi x}.$$

FOURIER SERIES IN THE INTERVAL $(-l, l)$

(4)

2.2.1 Find the fourier series of $f(x) = x$ in $-\pi < x < \pi$

Solns: Given: $f(x) = x$ in $(-\pi, \pi)$ [M|J 2016]

Put $x \to -x$

$$f(-x) = -x = -f(x)$$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$

Here $a_0, a_n = 0$

$\therefore$ The fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\, \underset{v}{\sin nx}\, \underset{dv}{dx}$$

$u = x$, $u' = 1$

$V_1 = -\frac{\cos nx}{n}$, $V_2 = -\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\left\{\left(\frac{-x\cos nx}{n}\right)_0^{\pi} + \left(\frac{\cancel{\sin nx}}{n^2}\right)_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + 0\right\} = \frac{2}{\cancel{\pi}}\left\{\frac{-\cancel{\pi}(-1)^n}{n}\right\}$$

$$\boxed{b_n = -\frac{2(-1)^n}{n}} \Rightarrow b_n = \frac{(-1)\,2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$\therefore$ The required fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

2.2.2

Find the fourier series of x^2 in $-\pi < x < \pi$. Hence deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$ [(N/D 2014), (M/J 2013) N/D 2016]

Solns:

Given: $f(x) = x^2$ in $(-\pi, \pi)$

Put $x \to -x$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$\Longrightarrow f(-x) = f(x)$$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$

The required the fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad (\text{Here } b_n = 0)$$

Where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}} \left[\frac{\pi^{\cancel{3}\,2}}{3} \right] = \frac{2\pi^2}{3}$$

$$\therefore \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underset{u}{x^2} \; \underset{v}{\cos nx} \; \underset{dv}{dx}$$

$u = x^2, \quad u' = 2x, \quad u'' = 2$

$V_1 = \frac{\sin nx}{n}, \quad V_2 = -\frac{\cos nx}{n^2}, \quad V_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi} \left\{ \cancel{\left(x^2 \frac{\sin nx}{n}\right)_0^{\pi}} + \left(2x \frac{\cos nx}{n^2}\right)_0^{\pi} - 2\cancel{\left(\frac{\sin nx}{n^3}\right)_0^{\pi}} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{2\pi \cos n\pi}{n^2} - 0 \right\}$$

$$= \frac{2}{\cancel{\pi}} \times \frac{2\cancel{\pi}(-1)^n}{n^2} = \frac{4}{n^2}(-1)^n \quad (\because \cos n\pi = (-1)^n)$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$\therefore$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Put $x = \pi$, $\Rightarrow \underbrace{f(-\pi) + f(\pi)} = \frac{\pi^2}{3} + 4\left[-\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} + \dots + \infty\right]$

$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

By parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \frac{\left(\frac{2\pi^2}{3}\right)^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{4}{n^2}(-1)^n\right)^2+0\right]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx = \frac{\frac{\cancel{4}\pi^4}{9}}{\cancel{4}}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n}$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{9}+\frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad (\because (-1)^{2n}=1)$$

$$\Rightarrow \frac{1}{2\pi}\left\{\frac{\pi^5}{5}-\frac{(-\pi)^5}{5}\right\} = \frac{\pi^4}{9}+8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{1}{\cancel{2\pi}}\left[\frac{\cancel{2}\pi^{\cancel{5}\,4}}{5}\right] = \frac{\pi^4}{9}+8\left\{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\infty\right\}$$

$$\Rightarrow \frac{\pi^4}{5}-\frac{\pi^4}{9} = 8\left\{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\infty\right\}$$

$$\Rightarrow \frac{\cancel{4}\pi^4}{45}\times\frac{1}{\cancel{8}\,2} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\infty$$

$$\Rightarrow \boxed{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \frac{\pi^4}{90}}$$

2.52 Find the fourier series as far as the second harmonic to represent the function $f(x)$ with the period 6. given the following table. (1)

[(NID 2009), (NID 2010), (M/J 2012), (NID 2012)] [A/M - 2017] [N/D 2016]

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Soln: Given:

Here six values of x are given, which are left end values of the subintervals of the interval $(0, 6)$

Here $2l = 6 \Rightarrow l = 3$ and $m = 6$, and $h = 1$

Then the fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l}$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3}$$

Put $\theta = \frac{\pi x}{3}$, Then values of θ corresponding to value of x are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, $\therefore h = \frac{\pi}{3}$

$\therefore$ The fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + b_1 \sin\theta + b_2 \sin 2\theta$$

Where $a_0 = \frac{2}{m}\sum y$, $a_n = \frac{2}{m}\sum y \cos n\theta$, $b_n = \frac{2}{m}\sum y \sin n\theta$

x	$\theta = \frac{\pi x}{3}$	$\cos\theta$	$\sin\theta$	$\cos 2\theta$	$\sin 2\theta$	y	$y\cos\theta$	$y\sin\theta$	$y\cos 2\theta$	$y\sin 2\theta$
0	0	1	0	1	0	9	9	0	9	0
1	$\frac{\pi}{3}$	0.5	0.866	-0.5	0.866	18	9	15.588	-9	15.588
2	$\frac{2\pi}{3}$	-0.5	0.866	-0.5	-0.866	24	-12	20.784	-12	-20.784
3	$\frac{3\pi}{3}$	-1	0	1	0	28	-28	0	28	0
4	$\frac{4\pi}{3}$	-0.5	-0.866	-0.5	0.866	26	-13	-22.516	-13	22.516
5	$\frac{5\pi}{3}$	0.5	-0.866	-0.5	-0.866	20	10	-17.32	-10	-17.32
					Total	125	-25	-3.464	-7	0

$$a_0 = \frac{2}{m}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{m}\sum y\cos\theta = \frac{2}{6}(-25) = -8.33$$

$$a_2 = \frac{2}{3}\sum y\cos 2\theta = \frac{2}{6}(-7) = -2.33$$

$$b_1 = \frac{2}{m}\sum y\sin\theta = \frac{2}{6}(-3.464) = -1.15$$

$$b_2 = \frac{2}{m}\sum y\sin 2\theta = \frac{2}{6}(0) = 0$$

$$y = \frac{41.67}{2} - 8.33\cos\theta - 2.33\cos 2\theta - 1.15\sin\theta$$

$$= 20.835 - 8.33\cos\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 1.15\sin\frac{\pi x}{3}$$

2.3.5

Find the half range sine Series of $f(x)$ (12)

$$= \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$. Hence deduce the sum

[A/M 2015]

of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Solns:

Given: $$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$

To find: Half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx \right\}$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right\}$$

$$= \frac{2}{\pi} \left\{ \left(-x \frac{\cos nx}{n} \right)_0^{\pi/2} + \left(\frac{\sin nx}{n^2} \right)_0^{\pi/2} \right.$$

$$\left. + \left(-(\pi - x) \frac{\cos nx}{n} \right)_{\pi/2}^{\pi} + \left(-\frac{\sin nx}{n^2} \right)_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi}\left\{\left[\left(-\frac{\pi}{2}\,\frac{\cancel{\cos n\pi/2}}{n} - 0\right) + \left(\frac{\sin n\pi/2}{n^2}\right) - 0\right]\right.$$

$$\left. + \left[0 + \frac{\pi}{2}\,\frac{\cancel{\cos n\pi/2}}{n} - \cancel{\frac{\sin n\pi}{n^2}} + \frac{\sin n\pi/2}{n^2}\right]\right\}$$

$$= \frac{2}{\pi}\left\{ 2\,\frac{\sin n\pi/2}{n^2}\right\}$$

$b_n = \dfrac{4}{\pi n^2}\sin n\pi/2$ if n is an odd

$b_n = 0$ if n is an even

$$f(x) = \sum_{n = odd}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx.$$

$$f(x) = \frac{4}{\pi}\sum_{n = odd}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$$

2.36 Obtain the fourier series of cosine expansion of $x \sin x$ in $(0,\pi)$ and hence find the value of $1 + \frac{2}{1\cdot3} - \frac{2}{3\cdot5} + \frac{2}{5\cdot7} - \frac{2}{7\cdot9} + \cdots$ (13)

Solns:

[N/D 2013]

Given: $f(x) = x \sin x$ in $(0,\pi)$

Half range cosine series expansion of

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) + \sin x\Big]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left[-\cancel{\pi}\cos\pi\right]$$

$$= 2\left[-(-1)\right] = 2$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx \sin x\, dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\} dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x\, dx - \int_0^{\pi} x\sin(n-1)x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{x\cos(n+1)x}{n+1} + \frac{\sin(\cancel{n+1})x}{(\cancel{n}+1)^2}\right)_0^{\pi} - \left(-\frac{x\cos(n-1)x}{n-1} + \frac{\sin(\cancel{n-1})x}{(\cancel{n}-1)^2}\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n+1} - \frac{\pi\cos n\pi}{n-1}\right\}$$

$$= \frac{1}{\pi}\times\pi(-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= (-1)^n\left\{\frac{\cancel{n}-1-\cancel{n}-1}{n^2-1}\right\} = \frac{(-1)^n(-2)}{n^2-1}$$

$$a_n = \frac{(-1)^n(-2)}{n^2-1} \quad \text{if } n \neq 1$$

(17) 13

(14)

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left\{-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi\cos 2\pi}{2} + 0\right\}$$

$$\boxed{a_1 = \frac{-1}{2}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{(-1)^n(-2)}{n^2-1}\cos nx$$

Put $x = 0$ proved Deduce part

2.3.7 Find the half range sine series of $f(x) = 4x - x^2$ in the interval $(0, 4)$. hence deduce the value of the series $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \infty$ [M|J 2014] (13)

Solns:

Given: $f(x) = 4x - x^2$ in $(0, 4)$

Half range sine series expansion

$$\text{of } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{Where } b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$(0, l) = (0, 4)$$

$$\Rightarrow l = 4$$

$$= \frac{2}{4}\int_0^4 (4x - x^2) \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2}\left\{\left[(4x - x^2)\left(-\frac{\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right)\right]_0^4 + \left[\frac{(4-2x)\sin\left(\frac{n\pi x}{4}\right)}{\frac{n^2\pi^2}{16}}\right]_0^4\right.$$

$$\left. + \left[\frac{-2\cos\left(\frac{n\pi x}{4}\right)}{\frac{n^3\pi^3}{64}}\right]_0^4\right\}$$

$$= \frac{1}{2}\left\{0 + 0 - \left[2\cos n\pi\left(\frac{64}{n^3\pi^3}\right) - 2\cos 0\left(\frac{64}{n^3\pi^3}\right)\right]\right.$$

$$= \frac{2}{\cancel{2}} \times \frac{\overset{32}{\cancel{64}}}{n^3\pi^3} \left\{ -\cos n\pi + \cos 0 \right\}$$

$$= \frac{64}{n^3\pi^3} \left\{ (-1)^n + 1 \right\}$$

$$b_n = \frac{128}{n^3\pi^3} \quad \text{if } n \text{ is an even}$$

$$= 0 \quad \text{if } n \text{ is an odd}$$

$\therefore$ The fourier series is

$$f(x) = \frac{64}{\pi^3} \sum_{n=\text{even}}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi x}{4}\right)$$

$$f(x) \times \frac{\pi^3}{128} = \frac{1}{2^3} \sin\left(\frac{2\pi x}{4}\right) + \frac{1}{4^3} \sin\left(\frac{4\pi x}{4}\right) + \cdots$$

Q.38

Obtain the fourier cosine series expansion of $f(x) = x$ in $0 < x < 4$ Hence deduce the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \infty$ (16) [N/D 2014]

Soln:

Given $f(x) = x$ in $(0, 4)$

Half range cosine series expansion of $f(x)$ in $(0, l)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Here $(0, l) \rightarrow (0, 4)$

$\Rightarrow l = 4$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{4}\int_0^4 x\,dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^4 = \frac{1}{2}\left(\frac{16}{2}\right) = 4$$

$$\boxed{a_0 = 4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^4 x\cos\left(\frac{n\pi x}{4}\right)dx = \frac{1}{2}\int_0^4 x\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left\{ \left(\frac{x \sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)} \right)_0^4 + \left(\frac{\cos\left(\frac{n\pi x}{4}\right)}{\frac{n^2\pi^2}{16}} \right)_0^4 \right\}$$

$$= \frac{1}{2}\left\{ (\cos n\pi) \times \frac{16}{n^2\pi^2} - \cos 0 \left(\frac{n^2\pi^2}{16}\right) \right\}$$

$$= \frac{1}{2} \times \frac{n^2\pi^2}{16} \left\{ (-1)^n - 1 \right\}$$

$$a_n = \frac{n^2\pi^2}{32}(-2) = -\frac{\cancel{2}n^2\pi^2}{3\cancel{2}16} \quad \text{if } n \text{ is an odd}$$

$$= 0 \quad \text{if } n \text{ is an even}$$

$$\therefore f(x) = \frac{4}{2} + \sum_{n=1}^{\infty} \frac{n^2\pi^2}{16} \cos\left(\frac{n\pi x}{4}\right)$$

2.3.9 Find the half range sine series of $f(x) = lx - x^2$ in $(0, l)$ [N|D 2013] (17)

Solns:

Given: $f(x) = lx - x^2$ in $(0, l)$

Half range sine Series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{l} \int_0^l \underbrace{(lx - x^2)}_{u} \underbrace{\sin\left(\frac{n\pi x}{l}\right)}_{v} \underbrace{dx}_{dv}$$

here $u = lx - x^2 \longrightarrow V_1 = -\cos\left(\frac{n\pi x}{l}\right) \times \frac{l}{n\pi}$

$u' = l - 2x \longrightarrow V_2 = -\sin\left(\frac{n\pi x}{l}\right) \times \frac{l^2}{n^2\pi^2}$

$u'' = -2 \longrightarrow V_3 = +\cos\left(\frac{n\pi x}{l}\right) \times \frac{l^3}{n^3\pi^3}$

$$= \frac{2}{l} \left\{ \left[(lx - x^2)\left(-\cos\left(\frac{n\pi x}{l}\right) \times \frac{l}{n\pi}\right) \right]_0^l + \left[(l - 2x) \cancel{\sin\left(\frac{n\pi x}{l}\right)} \times \frac{l^2}{n^2\pi^2} \right]_0^l - \left(2\cos\left(\frac{n\pi x}{l}\right) \times \frac{l^3}{n^3\pi^3} \right)_0^l \right\}$$

$$= \frac{2}{l}\left\{ -2\cos\left(\frac{n\pi \cancel{l}}{\cancel{l}}\right) \times \frac{l^3}{n^3\pi^3} + 2\cos 0 \times \frac{l^3}{n^3\pi^3} \right\}$$

$$= \frac{2}{l}\left\{ -2\cos n\pi \times \frac{l^3}{n^3\pi^3} + 2\cos 0 \times \frac{l^3}{n^3\pi^3} \right\}$$

$$= \frac{2}{\cancel{l}} \times \frac{l^{\cancel{3}\,2}}{n^3\pi^3}\left\{ -2\cos n\pi + 2\cos 0 \right\}$$

$$= \frac{-4l^2}{n^3\pi^3}\left\{ (-1)^n - 1 \right\}$$

$$b_n = \frac{8l^2}{n^3\pi^3} \quad \text{if } n \text{ is an odd}$$

$$= 0 \quad \text{if } n \text{ is an even.}$$

$$\therefore f(x) = \sum_{n=\text{odd}} \frac{8l^2}{n^3\pi^3} \sin\left(\frac{n\pi x}{l}\right)$$

(18)

2.7.2. Find the complex form of the fourier Series of $f(x) = e^{-x}$ in $-1 < x < 1$

[(N/D 2009) (A/M 2015)]

Solns: Given $f(x) = e^{-x}$, $-1 < x < 1$

Here $l = 1$

The complex form of $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

where $C_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$

$$= \frac{1}{2} \int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$\Rightarrow C_n = \frac{1}{2} \int_{-1}^{1} e^{-x} e^{-in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{-2(1+in\pi)}\left\{e^{-(1+in\pi)} - e^{(1+in\pi)}\right\}$$

$$= \frac{1}{-2(1+in\pi)}\left\{e^{-1}\cdot e^{-in\pi} - e^{1}\cdot e^{in\pi}\right\}$$

$$= \frac{(1-in\pi)}{-2(1+n^2\pi^2)}\left\{e^{-1}\left\{\cos n\pi - i\cancel{\sin n\pi}\right\} - e\left(\cos n\pi + i\cancel{\sin n\pi}\right)\right\}$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)}\left[(e^{-1} - e^{1})\cos n\pi\right]$$

$$= \frac{(1-in\pi)(-1)^n}{1+n^2\pi^2}\sinh(1) \quad \left\{\because \frac{e^{1}-e^{-1}}{2} = \sinh(1)\right\}$$

The complex fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(1-in\pi)(-1)^n}{1+n^2\pi^2}\sinh(1)\cdot e^{in\pi x}$$

$$\Rightarrow e^{-x} = \sinh 1 \sum_{n=-\infty}^{\infty} \frac{(1-in\pi)(-1)^n}{1+n^2\pi^2} e^{in\pi x}$$

(19)

$$\Rightarrow C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin x \cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-inx}}{i^2n^2+a^2}\left[-in\sin x \pm \cos x\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left\{ e^{-in\pi}\left[-in\cancel{\sin\pi} \pm \cos\pi\right] - e^{in\pi}\left[-in\cancel{\sin(-\pi)} \pm \cos(-\pi)\right]\right\}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left\{ (\cos n\pi - i\cancel{\sin n\pi})(\cos\pi) + (\cos n\pi + i\cancel{\sin n\pi})\cos\pi \right\}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left\{ \cos n\pi \cos\pi + \cos n\pi \cos\pi \right\}$$

$$= \frac{\cancel{2}\cos n\pi \cos\pi}{\cancel{2}\pi(a^2-n^2)} = \frac{(-1)^n(-1)}{(a^2-n^2)\pi} = \frac{(-1)^{n+1}}{\pi(a^2-n^2)}$$

$\therefore$ The complex form of fourier series

is $$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{\pi(a^2-n^2)} e^{inx}$$

$$= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{a^2-n^2} e^{inx}.$$

Find the fourier series expansion of $f(x) = |\cos x|$ in $-\pi < x < \pi$ [M|J 2016]

Solns:

Given: $f(x) = |\cos x| \quad -\pi < x < \pi$

$$f(-x) = |\cos(-x)|$$
$$= |\cos x|$$
$$= f(x)$$

$f(x)$ is an even function

The required fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$b_n = 0$$

Where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx.$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} |\cos x|\,dx$$

(19)

19 (20)

$$= \frac{2}{\pi} \int_0^{\pi} |\cos x| \, dx$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} -\cos x \, dx \right\} \quad \left[\because |\cos x| = \begin{cases} \cos x & \text{if } 0 \le x \le \pi/2 \\ -\cos x & \text{if } \pi/2 < x < \pi \end{cases} \right.$$

$$= \frac{2}{\pi} \left\{ (\sin x)_0^{\pi/2} + (-\sin x)_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \sin \pi/2 - 0 + - \cancel{\sin \pi} + \sin \pi/2 \right\}$$

$$= \frac{2}{\pi} \left\{ 1 + 1 \right\}$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} f(x) \cos nx \, dx + \int_{\pi/2}^{\pi} f(x) \cos nx \, dx \right\}$$

$$= \frac{2}{\pi} \left\{ \int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} -\cos x \cos nx \, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi/2} 2\cos nx \cos x\, dx - \int_{\pi/2}^{\pi} 2\cos nx \cos x\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi/2} \left(\cos(n+1)x + \cos(n-1)x\right) dx - \int_{\pi/2}^{\pi} \left(\cos(n+1)x + \cos(n-1)x\right) dx \right\}$$

$$= \frac{1}{\pi}\left\{ \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \left(\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0\right) - \left(0 - \left(\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{(n-1)}\right)\right) \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\sin\left(\frac{n\pi}{2} + \pi/2\right)}{n+1} + \frac{\sin\left(n\pi/2 - \pi/2\right)}{n-1} \right\}$$

$$= \frac{2}{\pi}\left\{\frac{\cos\frac{m\pi}{2}}{m+1} - \frac{\cos\frac{m\pi}{2}}{m-1}\right\}$$

$$= \frac{2}{\pi}\cos\frac{m\pi}{2}\left\{\frac{1}{m+1} - \frac{1}{m-1}\right\} \quad \left(\because \sin\left(\frac{m\pi}{2} - \frac{\pi}{2}\right) = \sin\left(-\left(\pi/2 - m\pi/2\right)\right) = -\sin\left(\pi/2 - m\pi/2\right) = -\cos m\pi/2\right)$$

$$= \frac{2}{\pi}\cos\frac{m\pi}{2}\left\{\frac{\cancel{m}-1-\cancel{m}-1}{m^2-1}\right\}$$

$$= \frac{-4}{\pi(m^2-1)}\cos m\pi/2$$

if $a_m = \begin{cases} 0 & \text{if } m \text{ is an odd} \\ \frac{-4}{\pi(m^2-1)} & \text{if } m \text{ is an even} \end{cases}$

$$\therefore f(x) = \frac{2}{\pi} + \sum_{m=\text{even}}^{\infty} \frac{-4}{\pi(m^2-1)}\cos m\pi/2 \cos mx. \quad -(1)$$

But $a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos x\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\cos x\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos^2 x\,dx - \int_{\pi/2}^{\pi}\cos^2 x\,dx\right\}$$

$$= \frac{2}{\pi}\times\frac{1}{2}\left\{\int_0^{\pi/2}(1+\cos 2x)\,dx - \int_{\pi/2}^{\pi}(1+\cos 2x)\,dx\right.$$

$$= \frac{1}{\pi}\left\{\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi}{2} + \frac{\sin \cancel{2}\times\pi/\cancel{2}}{2}\right) - 0 - \left(\left(\pi + \cancel{\frac{\sin 2\pi}{2}}\right) - \left(\pi/2 + \frac{\sin 2\times\pi/2}{2}\right)\right)\right\}$$

$$= \frac{1}{\pi}\left\{\cancel{\frac{\pi}{2}} - \cancel{\pi} + \cancel{\frac{\pi}{2}}\right\} = 0$$

$$a_1 = 0$$

$$(1) \Rightarrow |\cos x| = \frac{2}{\pi} + \left(\frac{-4}{\pi}\right)\sum_{n=\text{even}} \frac{\cos\frac{n\pi}{2}}{n^2-1}\cos nx.$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=\text{even}} \frac{(-1)^{n/2}}{n^2-1}\cos nx.$$

$$\left(\because \cos n\pi/2 = (\cos n\pi)^{1/2} = \left((-1)^n\right)^{1/2} = (-1)^{n/2}\right)$$

— x —

* Complex form of fourier Series:

2.4.1 Find the Complex form of the fourier Series of $f(x) = e^{ax}$, $-\pi < x < \pi$ [A/M 2016] [N/D 2016]

Solns: Given $f(x) = e^{ax}$, $-\pi < x < \pi$ [N/D 2015]

The complex form of fourier series in $(-\pi, \pi)$ is $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$

Where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{a-in} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)} \left\{ e^{(a-in)\pi} - e^{-(a-in)\pi} \right\}$$

$$= \frac{a+in}{2\pi(a+in)(a-in)} \left\{ e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi} \right\}$$

$$= \frac{a+in}{2\pi(a^2+n^2)}\left[e^{a\pi}\left\{\cos n\pi - i\cancel{\sin n\pi}\right\} - e^{-a\pi}\left\{\cos n\pi + i\cancel{\sin n\pi}\right\}\right]$$

$$= \frac{a+in}{2\pi(a^2+n^2)}\left\{\left(e^{a\pi} - e^{-a\pi}\right)\cos n\pi\right\}$$

$$= \frac{a+in}{\cancel{2}\pi(a^2+n^2)}\left\{\cancel{2}\sinh \pi a \cdot \cos n\pi\right\}$$

$$= \frac{(a+in)(-1)^n \sinh \pi a}{\pi(a^2+n^2)}$$

$\therefore$ The complex form of fourier Series is

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(a+in)(-1)^n \sinh \pi a}{\pi(a^2+n^2)} e^{inx}$$

$$e^{ax} = \frac{1}{\pi}\sinh \pi a \sum_{n=-\infty}^{\infty} \frac{(a+in)(-1)^n}{a^2+n^2} e^{inx}$$

2.4.2 Find the complex form of the fourier Series of $f(x) = e^{-x}$ in $-1 < x < 1$

[(N/D 2009) (A/M 2015)]

• Half range Fourier Series :

(5)

2.3.1 Find the half range Cosine Series of the function $f(x) = x(\pi - x)$ in the interval $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

Soln:

[A/M 2015]

Given: $f(x) = x(\pi - x)$ in $(0, \pi)$

To find : Half range Cosine Series in $(0, \pi)$

(ie) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} x(\pi - x)\, dx = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2)\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2 \pi}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right] = \frac{2}{\pi} \left[\frac{3\pi^3 - 2\pi^3}{6} \right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}} \times \frac{\pi^{\cancel{3}2}}{\cancel{6}3} = \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underbrace{x(\pi-x)}_{u}\ \underbrace{\cos nx}_{v}\ \underbrace{dx}_{dv}.$$

$U = x(\pi - x)$, $U' = \pi - 2x$ $\quad U'' = -2$

$= x\pi - x^2$

$V_1 = \frac{\sin nx}{n}$, $V_2 = -\frac{\cos nx}{n^2}$, $V_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left\{ \cancel{\left((x\pi - x^2)\frac{\sin nx}{n}\right)_0^{\pi}} + \left((\pi-2x)\left(+\frac{\cos nx}{n^2}\right)\right)_0^{\pi} + \cancel{\left(\frac{2\sin nx}{n^3}\right)_0^{\pi}} \right\}$$

$$= \frac{2}{\pi}\left\{ -\pi\frac{\cos n\pi}{n^2} - \pi\frac{\cos 0}{n^2} \right\}$$

$$= \frac{2}{\cancel{\pi}}\left(\frac{-\cancel{\pi}}{n^2}\right)\left\{\cos n\pi + \cos 0\right\}$$

$$= -\frac{2}{n^2}\left[(-1)^n + 1\right] \qquad a_n = \begin{cases} \frac{-4}{n^2} & \text{if } n \text{ is an even} \\ 0 & \text{if } n \text{ is an odd.} \end{cases}$$

The fourier series be

$$x(\pi - x) = \frac{\pi^2}{6} + \sum_{n=\text{even}}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx.$$

(8)

Deduce part:

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}$$

Using parseval's theorem:

$$\frac{1}{\pi}\int_0^{\pi}(f(x))^2\,dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2 \quad -(1)$$

Now

$$f(x)=x(\pi-x)=x\pi-x^2$$

$$(f(x))^2=(x\pi-x^2)^2=x^2\pi^2+x^4-2x^3\pi$$

$$\int_0^{\pi}(x^2\pi^2+x^4-2x^3\pi)\,dx$$

$$=\left(\frac{x^3\pi^2}{3}+\frac{x^5}{5}-\frac{2x^4\pi}{4}\right)_0^{\pi}$$

$$=\left(\frac{\pi^5}{3}+\frac{\pi^5}{5}-\frac{2\pi^5}{4}\right)-0=\frac{\pi^3}{30}$$

$$(1)\Rightarrow\frac{1}{\cancel{\pi}}\left\{\frac{\pi^{\cancel{5}\,4}}{30}\right]=\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=\text{even}}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow\frac{\pi^4}{30}-\frac{\pi^4}{36}=8\sum_{n=\text{even}}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow\frac{\pi^4}{180}=8\sum_{n=1}^{\infty}\frac{1}{(2n)^4}$$

$$\Rightarrow \frac{\pi^4}{180} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

2.3.2

Find the half range fourier cosine series of $f(x) = (\pi - x)^2$ in the interval $(0, \pi)$ hence find the sum of the series

$\frac{1}{1^4} + \frac{1}{2^4} + \cdots + \infty$ [M/J 2012]

N/D 2013

Solns:

Given: $f(x) = (\pi - x)^2$ in $(0, \pi)$

To find: Half range cosine series of $f(x)$ in $(0, \pi)$

(ie) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$= \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2\, dx$

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$$= \frac{2}{\pi}\left[\frac{(\pi-x)^3}{3(-1)}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{0 + \frac{\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi-x)^2 \cos nx \, dx$$

$u = (\pi - x)^2$, $u' = 2(\pi - x)(-1)$ $\quad u'' = 2$

$= -2(\pi - x)$

$= -2\pi + 2x$

$V_1 = \frac{\sin nx}{n}$ $\quad V_2 = -\frac{\cos nx}{n^2}$ $\quad V_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left\{\left(\cancel{(\pi-x)^2 \frac{\sin nx}{n}}\right)_0^{\pi} + \left((-2\pi+2x)\frac{\cos nx}{n^2}\right)_0^{\pi} - \left(\cancel{2\frac{\sin nx}{n^3}}\right)_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{0 - (-2\pi + 0)\frac{\cos 0}{n^2}\right] = \frac{2}{\cancel{\pi}} \times 2\cancel{\pi} \times \frac{1}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

$\therefore$ The fourier cosine series be $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx$

Deduce part: $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\infty$

Using parseval's theorem

$$\frac{1}{\pi}\int_0^{\pi}(f(x))^2\,dx = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}a_n^2$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi}\left((\pi-x)^2\right)^2dx = \frac{\left(\frac{\cancel{4}\pi^4}{9}\right)}{\cancel{4}}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)^2$$

$$\Rightarrow \frac{1}{\pi}\left\{\int_0^{\pi}(\pi-x)^4\,dx = \frac{\pi^4}{9}+\frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}\right.$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{\pi} = \frac{\pi^4}{9}+8\left\{\frac{1}{1^4}+\frac{1}{2^4}+\cdots+\infty\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{0+\frac{\pi^5}{5}\right]-\frac{\pi^4}{9} = 8\left\{\frac{1}{1^4}+\frac{1}{2^4}+\cdots+\infty\right\}$$

$$\Rightarrow \frac{\pi^4}{5}-\frac{\pi^4}{9} = 8\left\{\frac{1}{1^4}+\frac{1}{2^4}+\cdots+\infty\right\}$$

$$\Rightarrow \frac{\cancel{4}\pi^4}{45}\times\frac{1}{\cancel{8}_2} = \frac{1}{1^4}+\frac{1}{2^4}+\cdots+\infty$$

$$\Rightarrow \boxed{\frac{1}{1^4}+\frac{1}{2^4}+\cdots+\infty = \frac{\pi^4}{90}}$$

Q.33 Obtain the fourier Cosine Series of $(x-1)^2$, $0<x<1$ and hence show that (16)

$\frac{1}{1^2}+\frac{1}{2^2}+ \quad = \frac{\pi^2}{6}$ [(M|J 2013)(N|D 2014)]

Soln:

Given: $f(x)=(x-1)^2$, $(0,1)$

$\therefore$ half range cosine Series in $(0,l)$ is defined as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

here $(0,l) = (0,1)$

$\Rightarrow l=1$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$= \frac{2}{1}\int_0^1 (x-1)^2\,dx$$

$$= \left(2\,\frac{(x-1)^3}{3}\right)_0^1$$

$$= \frac{2}{3}\left(0-(-1)\right) = \frac{2}{3}$$

$$\boxed{a_0 = \frac{2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$= \frac{2}{1}\int_0^1 \underset{u}{(x-1)^2}\ \underset{v}{\cos\left(\frac{n\pi x}{1}\right)}\ \underset{dv}{dx}$$

$$u = (x-1)^2 \qquad u' = 2(x-1)(+1) \qquad u'' = 2$$

$$= 2(x-1)$$

$$V_1 = \frac{\sin\left(\frac{n\pi x}{1}\right)}{n\pi},\quad V_2 = -\frac{\cos\left(\frac{n\pi x}{1}\right)}{n^2\pi^2},\quad V_3 = \frac{-\sin\left(\frac{n\pi x}{1}\right)}{n^3\pi^3}$$

$$= 2\left\{ \left((x-1)^2 \cancel{\frac{\sin n\pi x}{n\pi}} \right)_0^1 + \left((2(x-1))\left(\frac{\cos(n\pi x)}{n^2\pi^2}\right)\right)_0^1 + \left(2\left(-\cancel{\frac{\sin(n\pi x)}{n^3\pi^3}}\right)\right)_0^1 \right\}$$

$$= 2\left\{ (0)\frac{\cos n\pi}{n^2\pi^2} + 2\frac{\cos 0}{n^2\pi^2} \right\}$$

$$= \frac{2}{n^2\pi^2}(+2)$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}},$$ The fourier cosine series of $f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi x)$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x \quad (1)$$

Put $x=0$, here $x=0$ is a finite point of continuity in the middle

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0$$

$$1 - \frac{1}{3} = \frac{4}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \infty\right\}$$

$$\Rightarrow \frac{\cancel{2}}{3} \times \frac{\pi^2}{\cancel{4}_2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \infty$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \infty$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \infty = \frac{\pi^2}{6}}$$

2.3.4

Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$ [(N/D 2010)(N/D 2012)] N/D 2013

Soln: Given $f(x) = x$ in $(0, \pi)$

∴ half range

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{\cancel{2}}{\pi}\left[\frac{x^2}{\cancel{2}}\right]_0^{\pi}$$

$$= \frac{\pi^2}{\pi} - 0$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(x\,\cancel{\frac{\sin nx}{n}}\right)_0^{\pi} + \left(\frac{\cos nx}{n^2}\right)_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right\}$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

if n is an odd $a_n = \frac{-4}{n^2\pi^2}$

if n is an even $a_n = 0$.

$\therefore$ The fourier series be : $f(x) = \frac{\pi}{2} + \sum_{n=odd} \frac{-4}{n^2\pi^2}\cos nx$

2^{*}

2.2.3

Obtain the fourier series of $f(x) = x \sin x$ in $(-\pi, \pi)$

Solns: Given: $f(x) = x \sin x \quad (-\pi, \pi)$ [N/D 2013]

put $x \to -x$

$$f(-x) = (-x)\sin(-x)$$
$$= -x(-\sin x)$$
$$= x \sin x$$
$$= f(x)$$

$\Rightarrow f(-x) = f(x)$ is an even function

The required fourier series be

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (\text{Here } b_n = 0)$$

Where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \underset{u}{x} \; \underset{v}{\sin x} \; \underset{dv}{dx}$$

$u = x \quad , \quad u' = 1$

$V_1 = -\cos x \quad , \quad V_2 = -\sin x$

$$= \frac{2}{\pi} \left\{ \left(-x \cos x\right)_0^{\pi} + \left(\cancel{\sin x}\right)_0^{\pi} \right] = \frac{2}{\cancel{\pi}} \left\{ -\cancel{\pi} \cos \pi \right\}$$

$$= -2\cos\pi = -2(-1) = 2$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \sin x\, dx$$

$\left(\because \sin(A+B) = \sin A\cos B + \cos A \sin B \right.$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] = \cos A \sin B$

Here $A = nx$, $\left. B = x \right)$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} x \times \frac{1}{\cancel{2}} \left\{ \sin(nx+x) - \sin(nx-x)\right\} dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \underset{u}{x} \underset{v}{\sin(n+1)x}\, \underset{dv}{dx} - \frac{1}{\pi}\int_0^{\pi} \underset{u}{x} \underset{v}{\sin(n-1)x}\, \underset{dv}{dx}$$

$u = x$, $u' = 1$

$V_1 = -\frac{\cos(n+1)x}{n+1}$, $V_2 = -\frac{\sin(n+1)x}{n+1}$

$u = x$, $u' = 1$

$V_1 = -\frac{\cos(n-1)x}{n-1}$, $V_2 = -\frac{\sin(n-1)x}{n-1}$

$$= \frac{1}{\pi}\left\{\left(-\frac{x\cos(n+1)x}{n+1}\right)_0^{\pi} + \cancel{\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi}}\right\} - \frac{1}{\pi}\left\{\left(-\frac{x\cos(n-1)x}{n-1}\right)_0^{\pi} + \cancel{\left(\frac{\sin(n-1)x}{n-1}\right)_0^{\pi}}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi\cos(n+1)\pi}{n+1} - 0\right\} - \frac{1}{\pi}\left\{-\frac{\pi\cos(n-1)\pi}{n-1} - 0\right\}$$

$$= \frac{1}{\pi}\left\{+\frac{\pi\cos n\pi}{n+1} - \frac{\pi\cos n\pi}{n-1}\right\}$$

$$\left(\because \cos(n+1)\pi = \cos(n\pi+\pi) = \cos(\pi+n\pi) = -\cos n\pi\right.$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n+1} - \frac{\pi(-1)^n}{n-1}\right]$$

$$\cos(n-1)\pi = \cos(n\pi-\pi) = \cos(-(\pi-n\pi)) = \cos(\pi-n\pi) = -\cos n\pi)$$

$$= \frac{1}{\cancel{\pi}} \times \cancel{\pi}(-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= (-1)^n\left\{\frac{n-1-n-1}{n^2-1}\right\}$$

$$= \frac{-2}{n^2-1}(-1)^n$$

$$a_1 = \frac{2}{\pi}\int_0^\pi f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{\cancel{2}}{\pi}\int_0^\pi x\,\frac{\sin 2x}{\cancel{2}}\,dx$$

$$u = x, \longrightarrow V_1 = -\frac{\cos 2x}{2}$$

$$u' = 1 \longrightarrow V_2 = -\frac{\sin 2x}{4}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{x\cos 2x}{2}\right)_0^\pi + \cancel{\left(\frac{\sin 2x}{4}\right)_0^\pi}\right\}$$

$$= \frac{1}{\cancel{\pi}}\left[-\frac{\cancel{\pi}\cos 2\pi}{2} - 0\right] = \frac{-1}{2}, \quad \boxed{a_1 = \frac{-1}{2}}$$

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}\cos nx$$

2.2.4

Obtain the fourier series to represent the function $f(x) = |x|$, $-\pi < x < \pi$ and deduce $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ [M/J 2012]

N/D 2015

Soln:

Given: $f(x) = |x| \quad -\pi < x < \pi$

$$\therefore f(-x) = |-x| = |x| = f(x)$$

$\therefore f(x)$ is an even function.

The required fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Here $b_n = 0$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \, dx \quad \left(\because |x| = x \text{ if } x \geq 0 \right)$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] = \frac{\cancel{2}}{\cancel{\pi}} \left[\frac{\pi^{\cancel{2}}}{\cancel{2}} \right] = \pi$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underset{u}{x} \; \underset{v}{\cos nx} \; \underset{dv}{dx}$$

$$u = x, \quad u' = 1$$

$$V_1 = + \frac{\sin nx}{n}, \quad V_2 = \frac{\cos nx}{n^2}$$

$$= \frac{2}{\pi} \left\{ \left(-x \frac{\sin nx}{n} \right)_0^{\pi} + \left(\frac{\cos nx}{n^2} \right)_0^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \left(-\pi \frac{\cancel{\sin n\pi}}{\cancel{n}} - 0 \right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right) \right\}$$

$$= \frac{2}{\pi} \left\{ + \frac{(-1)^n}{n^2} - \frac{1}{n^2} \right\} = \frac{2}{\pi n^2} \left\{ + (-1)^n - 1 \right\}$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is an odd} \\ 0 & \text{if } n \text{ is an even} \end{cases}$$

The fourier series is

$$f(x) = |x| = \frac{\pi}{2} + \sum_{n=odd} \frac{-4}{n^2\pi} \cos nx$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=odd} \frac{\cos nx}{n^2}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots + \infty \right\}$$

Put $x = 0$

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \cdots \infty \right\}$$

$$\Rightarrow \cancel{-}\frac{\pi}{2} = \cancel{-}\frac{4}{\pi} \left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty \right\}$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}$$

• Harmonic analysis

2.5.1 Compute upto Second harmonics of the fourier Series of $f(x)$ given by the following table

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x)$	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

[NID 2013]

Solution:

Given the length of the interval is T.

$\therefore \quad 2l = T \Rightarrow l = T/2$

number of subintervals $m = 6$

$$\therefore \frac{n\pi x}{l} = \frac{n\pi t}{T/2} = \frac{2n\pi t}{T} \; ; \; n = 0, 1, 2 \ldots$$

(Here $x = t$)

$$A = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + b_2 \sin\frac{4\pi t}{T}$$

Put $\theta = \dfrac{2\pi t}{T}$

The Values of θ are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

$$A = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + b_1 \sin\theta + b_2 \sin 2\theta$$

where $a_0 = \frac{2}{m} \sum y$, $a_n = \frac{2}{m} \sum y \cos n\theta$

$b_n = \frac{2}{m} \sum y \sin n\theta$

Here $y = A$

t	$\theta=\frac{2\pi t}{T}$	$\cos\theta$	$\sin\theta$	$\cos2\theta$	$\sin2\theta$	$A=y$	$y\cos\theta$	$y\sin\theta$	$y\cos2\theta$	$y\sin2\theta$
0	0	1	0	1	0	1.98	1.98	0	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	0.5	0.866	−0.5	0.866	1.30	0.65	1.126	−0.65	1.126
$\frac{T}{3}$	$\frac{2\pi}{3}$	−0.5	0.866	−0.5	−0.866	1.05	−0.525	0.909	−0.525	−0.909
$\frac{T}{2}$	π	−1	0	1	0	1.30	−1.30	0	1.30	0
$\frac{2T}{3}$	$\frac{4\pi}{3}$	−0.5	−0.866	−0.5	0.866	−0.88	0.44	0.762	0.44	−0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	0.5	−0.866	−0.5	−0.866	−0.25	−0.125	0.217	0.125	0.217
					Total	4.5	1.12	3.014	2.67	−0.328

$\therefore \quad a_0 = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{6}(1.12) = 0.373, \quad a_2 = \frac{2}{6}(2.67) = 0.89$

$b_1 = \frac{2}{6}(3.014) = 1.005, \quad b_2 = \frac{2}{6}(-0.328) = -0.109$

$\therefore \quad A = \frac{1.5}{2} + 0.373\cos\theta + 0.89\cos2\theta + 1.005\sin\theta - 0.109\sin2\theta$

$= 0.75 + 0.373\cos\theta + 0.89\cos2\theta + 1.005\sin\theta - 0.109\sin2\theta$

2.53 Find the fourier cosine series up to third harmonic to represent the function given by the following data [M/J 2016]

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

Soln: Given

Here six values of x are given, which are left end values of the subintervals of the interval $(0, 6)$

Here $2l = 6 \Rightarrow l = 3$ and $n = 6$, and $h = 1$

Then the fourier cosine series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + a_3 \cos\frac{3\pi x}{l}$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + a_3 \cos\frac{\cancel{3}\pi x}{\cancel{3}}$$

$$= \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + a_3 \cos\pi x$$

Put $\theta = \frac{\pi x}{3}$, Then values of θ corresponding to value of x are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, $\therefore h = \frac{\pi}{3}$

$\therefore$ The fourier Series is given by

$$f(x) = \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta$$

Where $a_0 = \frac{2}{m}\sum y$, $a_n = \frac{2}{m}\sum y\cos n\theta$

X	$\theta = \frac{\pi x}{3}$	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	Y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	1	1	+1	9	9	9	9
1	$\frac{\pi}{3}$	0.5	-0.5	-1	18	9	-9	-18
2	$\frac{2\pi}{3}$	-0.5	-0.5	1	24	-12	-12	24
3	$\frac{3\pi}{3} = \pi$	-1	1	-1	28	-28	28	-28
4	$\frac{4\pi}{3}$	-0.5	-0.5	1	26	-13	-13	26
5	$\frac{5\pi}{3}$	0.5	-0.5	-1	20	10	-10	-20
				Total	125	-25	-7	-7

$$a_0 = \frac{2}{m}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{m}\sum y\cos\theta = \frac{2}{6}(-25) = -8.33$$

$$a_2 = \frac{2}{m}\sum y\cos 2\theta = \frac{2}{6}(-7) = -2.33$$

$$a_3 = \frac{2}{m}\sum y\cos 3\theta = \frac{2}{6}(-7) = -2.33$$

$$y = \frac{41.67}{2} - 8.33\cos\theta - 2.33\cos 2\theta - 2.33\cos 3\theta$$

$$- 8.33\cos\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 2.33\cos\frac{3\pi x}{3}$$

2.54 Find the fourier Series up to second harmonic for $y = f(x)$ from the following Values

x:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

[(A/M 2011)(N/D 2012), (M/J 2014), (A/M 2015)]

Soln:

The Values are given in the interval $(0, 2\pi)$ with number of intervals $m = 6$ and width $h = \frac{\pi}{3}$

The fourier series upto Second harmonic

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

where $a_0 = \frac{2}{m} \Sigma y$, $a_n = \frac{2}{m} \Sigma y \cos nx$,

$$b_n = \frac{2}{m} \Sigma y \sin nx$$

To find the co-efficients we shall use the left end Values. So we shall take the first six values and display the computations in a table.

x	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	y	$y\cos x$	$y \sin x$	$y \cos 2x$	$y \sin 2x$
0	1	0	1	0	1	1	0	1	0
$\frac{\pi}{3}$	0.5	0.866	-0.5	0.866	1.4	0.7	1.212	-0.7	1.212
$\frac{2\pi}{3}$	-0.5	0.866	-0.5	-0.866	1.9	-0.95	1.65	-0.95	-1.65
$\frac{3\pi}{3}$	-1	0	1	0	1.7	-1.7	0	1.7	0
$\frac{4\pi}{3}$	-0.5	-0.866	-0.5	0.866	1.5	-0.75	-1.299	-0.75	1.299
$\frac{5\pi}{3}$	0.5	-0.866	-0.5	-0.866	1.2	0.6	-1.039	-0.6	-1.039
				Total	8.7	-1.1	0.524	-0.3	-0.178

$a_0 = \frac{2}{6} \times 8.7$

$= 2.9$

$a_1 = \frac{2}{6} \sum y \cos x$

$= \frac{1}{3}(-1.1)$

$= -0.367$

$a_2 = \frac{2}{m} \sum y \cos 2x$

$= \frac{2}{6}(-0.3)$

$= -0.1$

$a_3 = \frac{2}{m} \sum y \cos 3x$

$= \frac{2}{6}(0.1)$

$= 0.033$

$b_1 = \frac{2}{m} \sum y \sin x = \frac{2}{6}(0.524) = 0.175$

$b_2 = \frac{2}{m} \sum y \sin 2x = \frac{2}{6}(-0.178) = -0.0593$

$\therefore$ The fourier series upto third harmonics is

$y = 1.45 - 0.367 \cos x - 0.1 \cos 2x + 0.033 \cos 3x$
$+ 0.175 \sin x - 0.0593 \sin 2x$

FOURIER TRANSFORM

PART-A.

① Find the Fourier sine transform of $1/x$

[A/M 2015] [N/D 2016] [A/M 2017]

solution:

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} \qquad \int_0^\infty \frac{\sin ax}{x} dx = \pi/2$$

$$= \sqrt{\pi/2}$$

② If $F(s)$ is the Fourier transform of $f(x)$ then $F[f(x-a)] = e^{isa}F(s)$ [A/M 2017]

solution:

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

$$t = x - a \Rightarrow dt = dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt.$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt.$$

$$= e^{isa} F(s)$$

(3) State change of scale property of Fourier transforms. [N/D 2016] [N/D 2015] [N/D 2014]

Solution:

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

put $t = ax \Rightarrow dt = a\,dx$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)} \frac{dt}{a}.$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t} dt$$

$$= \frac{1}{a} F(s/a).$$

(4) State Fourier Integral theorem [M/J 2016]

Solution:

A function $f(x)$ satisfies the Dirichlet's conditions in $(-l, l)$ then

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda (t - x)\, dt\, d\lambda$$

(5) If the Fourier transform of $f(x)$ is $F[f(x)] = F(s)$ then show that $F[f(x-a)] = e^{iax} F(s)$ [A/M 2015]

Solution.

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx} dx$$

put $t = x - a \Rightarrow dt = dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)} dt.$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt.$$

$$= e^{isa} F(s)$$

(6) State and prove modulation theorem on Fourier transform [N/D 2014] 2

Solution:

If $F[f(x)] = F(s)$ then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s-a) + F(s+a)]$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}\,dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$

(7) Write Fourier transform pair [M/J 2013]

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\,e^{-isx}\,ds.$$

PART-B:

(1) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < 2 \\ 0, & |x| > 2 \end{cases}$

hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$ [A/M 2017]

solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} [\cos sx + i \sin sx]\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^2 \cos sx\,dx = \sqrt{\frac{2}{\pi}} \int_0^2 \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^2 = \sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s}\right]$$

using Inverse Fourier transform of $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{\sin 2s}{s}\right][\cos sx - i \sin sx]\,ds.$$

$$f(x) = \frac{2}{\pi} \int_{-\cancel{0}}^{\infty} \left[\frac{\sin 2s}{s}\right] \cos sx\,ds.$$

put $x = 0$

$$1 = \frac{2}{\pi} \int_0^\infty \left[\frac{\sin 2s}{s}\right] ds.$$

put $2s = t, \Rightarrow \frac{dt}{ds} = 2 \Rightarrow \frac{dt}{2} = ds \Rightarrow s = t/2$

$$1 = \frac{2}{\pi} \int_0^\infty \frac{\sin t}{(t/2)} \cdot \frac{dt}{2}$$

$$1 = \frac{2}{\pi} \int_0^\infty \frac{\cancel{2} \sin t}{t} \cdot \frac{dt}{\cancel{2}}$$

$$\int_0^{\infty} \frac{\sin t}{t}\, dt = \pi/2$$

put $t = x$

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

using parseval's Identity.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$$

$$\int_{-2}^{2} (1)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left(\frac{\sin 2s}{s}\right)\right]^2 ds.$$

$$2\int_0^{2} dx = 2 \times \frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin 2s}{s}\right]^2 ds.$$

$$2(x)_0^2 = \frac{4}{\pi}\int_0^{\infty}\left[\frac{\sin 2s}{s}\right]^2 ds.$$

$$\cancel{4} \times \frac{\pi}{\cancel{4}} = \int_0^{\infty}\left[\frac{\sin 2s}{s}\right]^2 ds.$$

put $2s = t \Rightarrow$

$$\pi = \int_0^{\infty}\left[\frac{\sin t}{(t/2)}\right]^2 \frac{dt}{2}$$

$$\pi = \int_0^{\infty} \cancel{4}\left[\frac{\sin t}{t}\right]^2 \frac{dt}{\cancel{2}}$$

$$\int_0^{\infty}\left[\frac{\sin t}{t}\right]^2 dt = \pi/2$$

$$\int_0^{\infty}\left[\frac{\sin x}{x}\right]^2 dx = \pi/2.$$

(3) Find the Fourier co-sine transform of $f(x) = e^{-a^2x^2}, a>0$.

[A/M 2017]

solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \, R.P \, e^{isx} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \int_0^\infty e^{-a^2x^2+isx} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \int_0^\infty e^{-[(ax)^2 - isx]} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \frac{1}{2} \int_{-\infty}^\infty e^{-[(ax)^2 - \frac{2a}{2a} isx]} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \frac{1}{2} \int_{-\infty}^\infty e^{-[(ax) - 2(ax)(\frac{is}{2a})]} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \frac{1}{2} \int_{-\infty}^\infty e^{-[(ax) - 2(ax)(\frac{is}{2a}) + (\frac{is}{2a})^2 - (\frac{is}{2a})^2]} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \frac{1}{2} \int_{-\infty}^\infty e^{-[ax - \frac{is}{2a}]^2 - \frac{s^2}{4a^2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \, R.P \, e^{-\frac{s^2}{4a^2}} \frac{1}{2} \int_{-\infty}^\infty e^{-(ax - is/2a)^2} dx$$

put $u = ax - \frac{is}{2a} \Rightarrow \frac{du}{dx} = a$

$\frac{du}{a} = dx$

$$= \frac{R.P \, e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-u^2} \frac{du}{a}.$$

$$= \frac{R.P \, e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^\infty e^{-u^2} du.$$

$$= R.P \frac{e^{-s^2/4a^2}}{a\sqrt{2}\cancel{\sqrt{\pi}}} (\cancel{\sqrt{\pi}})$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$ // Ans.

(*) Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ [A/M 2017] [N/D 2015] [N/D 2014].

Solution:

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \cdot G_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right)$$

$$\int_0^\infty F_s(s)\, G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right]\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right)\right] ds = \int_0^\infty \left(e^{-ax} e^{-bx}\right) dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \int_0^\infty e^{-(a+b)x}\, dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-(a+b)}\right]$$

$$= \frac{1}{(a+b)}$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}\, ds = \frac{\pi}{2(a+b)}.$$

put $a=1$ and $b=2$

$$\int_0^\infty \frac{s^2}{(s^2+1)(s^2+4)}\, ds = \frac{\pi}{2(3)}$$

put $s = x$

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{6} \text{ // Ans.}$$

(*) Find Infinite Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ 5

Hence deduce the infinite Fourier sine transform of $1/x$ [N/D 2016]

Solution:

Let $f(x) = \frac{e^{-ax}}{x}$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Diff w.r.t "s" we get.

$$\frac{d}{ds} F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial x} (\sin sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} (\cos sx) (\cancel{x}) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right)$$

Integrate w.r.t "x"

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int \left(\frac{a}{s^2+a^2} \right) ds.$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1} \left(\frac{s}{a} \right).$$

(7) Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$

and Hence deduce that $\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \pi/3$.

[M/J 2016] [N/D - 2016]
[N/D 2015] [M/J 2013]

Solution

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x) [\cos sx + i \sin sx] dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \left(\frac{\sin sx}{s}\right) - (-1) \left(-\frac{\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2(s/2)}{s^2} \right]$$

using parseval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^4(s/2)}{s^4} ds = 2 \int_0^1 (1-x)^2 dx$$

$$\frac{4}{\pi} \int_0^{\infty} \frac{\sin^4(s/2)}{s^4} ds = 2 \left[\frac{(1-x)^3}{-3} \right]_0^1$$

$$\int_0^{\infty} \left[\frac{\sin^4(s/2)}{s^4} \right] ds = \pi/24$$

$$t = s/2 \Rightarrow 2dt = ds.$$

$$\int_0^{\infty} \frac{\sin^4 t}{t^4} dt = \pi/3.$$

(*) Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}, \lambda > 0$

[N/D 2016]

Solution:

Given $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}$

Instead of variable s the letter λ is used

Multiply $\sqrt{\frac{2}{\pi}}$ on both side

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\,e^{-\lambda}$$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\,e^{-\lambda}$$

Inverse Fourier transform:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\lambda)\cos x\lambda\,d\lambda$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,e^{-\lambda}\cos x\lambda\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos x\lambda\,d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos x\lambda + x\sin x\lambda)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{e^0}{1+x^2}(-\cos 0 + x\sin 0)\right]$$

$$= \frac{2}{\pi}\,\frac{1}{1+x^2}$$

$$f(x) = \frac{2}{\pi}\,\frac{1}{1+x^2} \text{ // Ans.}$$

(9) Find Fourier transform of $f(x) = e^{-a^2x^2}$, $a>0$ Hence show that the function $e^{-x^2/2}$ is self reciprocal

[A/M 2015] [M/J 2016] [N/D 2016] [N/D 2014]

Solution.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - isx]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - \frac{2a\, isx}{a}]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - 2(ax)(\frac{is}{a})]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - 2(ax)(\frac{is}{a}) + (\frac{is}{a})^2 - (\frac{is}{2a})^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax - is/a)^2] - \frac{s^2}{4a^2}}\, dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax - is/a)]^2}\, dx$$

put $u = ax - is/a \Rightarrow \frac{du}{dx} = a$.

$$F[e^{-a^2x^2}] = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} (\sqrt{\pi})$$

put $a = 1/\sqrt{2}$

$$F[e^{-x^2/2}] = e^{-\frac{s^2}{2}}$$

// Ans.

(7) Find the Fourier sine and co-sine transform of $f(x) = e^{-ax}$, $x \geq 0$, $a > 0$. Hence deduce that $\int_0^\infty \frac{\cos sx}{s^2+a^2} ds$ and $\int_0^\infty \frac{s \sin sx}{s^2+a^2} ds$. [M/J 2016]

Solution:

$$f(x) = e^{-ax}$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \text{ and } F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Inverse Fourier co-sine transform.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[e^{-ax}] \cos sx \, ds.$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right) \cos sx \, ds.$$

$$\int_0^\infty \frac{\cos sx}{s^2+a^2} ds = \pi/2a \, e^{-ax}$$

changing variable x by m and s by x

$$\int_0^\infty \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{2a} e^{-am}$$

Inverse Fourie sine transform.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[e^{-ax}] \sin sx \, ds.$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \sin sx \, ds.$$

$$\int_0^\infty \frac{s \sin sx}{s^2+a^2} ds = \pi/2 \, e^{-ax}$$

changing variable x by m and s by x

$$\int_0^\infty \frac{x \sin mx}{x^2+a^2} dx = \pi/2 \, e^{-ma}, \quad a > 0.$$

(*) Find the Fourier co-sine transform of the function

$$f(x) = \frac{e^{-ax} - e^{-bx}}{x}, \quad x>0 \quad [N/D\ 2015]$$

Solution:

Let $f(x) = \dfrac{e^{-ax}}{x}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x}\right) \cos sx\, dx$$

Diff w.r.t "s" we get.

$$\frac{d}{ds} F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{e^{-ax}}{x}\right) \frac{\partial}{\partial x} (\cos sx)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)$$

Integrate w.r.t "s" we get

$$F_c[f(x)] = -\sqrt{\frac{2}{\pi}} \int \left(\frac{s}{s^2+a^2}\right) ds.$$

$$= -\sqrt{\frac{2}{\pi}} \times \frac{1}{2} \log(s^2+a^2)$$

$$F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2) + \frac{1}{\sqrt{2\pi}} \log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

(*) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ hence evaluate $\int_0^\infty \frac{\sin x}{x}\,dx$, $\int_0^t \frac{\sin^2 t}{t^2}\,dt = \pi/2$

[M/J 2016]

solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}[\cos sx + i\sin sx]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \cos sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin sa}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sa}{s}\right] = F(s).$$

using Inverse Transform.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}ds.$$

$$1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa}{s}\right][\cos sx - i\sin sx]ds.$$

$$1 = \frac{2}{\pi}\int_0^\infty\left(\frac{\sin sa}{s}\right)\cos sx$$

put $x=0$, $a=1$, $s=t$.

$$\int_0^\infty\left(\frac{\sin t}{t}\right)dt = \pi/2$$

using parseval's Identity.

$$\int_{-\infty}^{\infty}|F[f(x)]|^2 ds = \int_{-\infty}^{\infty}|f(x)|^2 dx.$$

$$\int_{-\infty}^{\infty}\frac{2}{\pi}\left(\frac{\sin sa}{s}\right)^2 ds = \int_{-a}^{a}(1)dx \Rightarrow \frac{4}{\pi}\int_0^\infty\left(\frac{\sin sa}{s}\right)^2 ds = 2a.$$

put $a=1$ $s=t$.

$$\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt = \pi/2.$$

(*) Find the Fourier co-sine transform of x^{n-1}

[A/M 2015]

solution:

W.K.T. $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$.

put $a = is$.

$$\int_0^\infty e^{-isx} x^{n-1} dx = \frac{\Gamma n}{(is)^n}$$

$$\int_0^\infty x^{n-1} [\cos sx - i \sin sx] dx = \frac{\Gamma n}{s^n} (-i)^n$$

$$= \frac{\Gamma n}{s^n} [\cos \pi/2 - i \sin \pi/2]^n.$$

$$= \frac{\Gamma n}{s^n} [\cos n\pi/2 - i \sin n\pi/2]$$

Equating real part.

$$\int_0^\infty x^{n-1} \cos sx \, dx = \frac{\Gamma n}{s^n} \cos n\pi/2$$

Multiply $\sqrt{2/\pi}$ on both side

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{n-1} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos n\pi/2$$

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \left(\frac{n\pi}{2}\right).$$

(*) Find the Fourier transform of $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & \text{otherwise}\end{cases}$

Hence deduce that $\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt$. [N/D 2014]
[M/J 2013]

Solution:

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-|x|)[\cos sx+i\sin sx]\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(1-x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2(s/2)}{s^2}\right]$$

using Inverse formula.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2(s/2)}{s^2}\right][\cos sx-i\sin sx]\,ds.$$

$$1-|x|=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2(s/2)}{s^2}\cos sx\,dx$$

put $x=0$. and $t=s/2 \Rightarrow \frac{dt}{2}=ds$.

$$\int_0^{\infty}\frac{\sin^2 t}{4t^2}\,2\,dt=\pi/4$$

$$\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt=\pi/2$$

* Find $f(x)$ if its sine transform is $\frac{e^{-sa}}{s}$

Hence Find $F_s^{-1}(1/s)$ [N/D 2013]

Solution:

$$F_s[f(x)] = \frac{e^{-sa}}{s}.$$

By Inverse Formula we get

$$f(x) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx \, ds$$

$$f(x) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \frac{e^{-sa}}{s} \sin sx \, ds$$

$$\frac{d}{dx} f(x) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \frac{d}{dx} \frac{e^{-sa}}{s} \sin sx \, ds$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty \left[\frac{e^{-sa}}{s} s \cos sx\right] ds$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty e^{-sa} \cos sx \, ds$$

$$= \sqrt{\tfrac{2}{\pi}} \left(\frac{a}{a^2+x^2}\right)$$

Int w.r.t "x" we get

$$f(x) = \sqrt{\tfrac{2}{\pi}} \int \frac{a}{x^2+a^2} dx$$

$$= \sqrt{\tfrac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)$$

put $a=0$

$$F_s^{-1}(1/s) = \sqrt{\tfrac{2}{\pi}} \frac{\pi}{2} = \sqrt{\pi/2}$$

(*) using transform Methods evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

[N/D 2013]

Solution.

Let $f(x) = e^{-ax}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

using parseval's Identity

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c[f(x)]|^2 ds.$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left[\sqrt{2/\pi}\left(\frac{a}{s^2+a^2}\right)\right]^2 ds.$$

$$\int_0^\infty e^{-2ax} dx = \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(s^2+a^2)}$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(s^2+a^2)}$$

$$\left[\frac{0-1}{-2a}\right] = \frac{2a^2}{\pi} \int_0^\infty \frac{ds}{(s^2+a^2)}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)} = \left(\frac{\pi}{2a^2}\right)\left(1/2a\right) = \frac{\pi}{4a^2}$$

put $s = x$

$$\int_0^\infty \frac{dx}{(x^2+a^2)} = \frac{\pi}{4a^2}$$

UNIT –III

LAPLACE TRANSFORM

Def. Exponential order

A function $f(t)$ is said to be of exponential order if

$$\underset{t\to\infty}{\mathrm{Lt}}\ e^{-st} f(t) = 0$$

Example 1 Show that x^n is of exponential order as $x \to \infty$, $n > 0$.

Solution :

$$\underset{x\to\infty}{\mathrm{Lt}}\ e^{-ax} x^n = \underset{x\to\infty}{\mathrm{Lt}}\ \frac{x^n}{e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

$$= \underset{x\to\infty}{\mathrm{Lt}}\ \frac{n x^{n-1}}{a e^{ax}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

[Apply L' Hospital Rule]

$$= \underset{x\to\infty}{\mathrm{Lt}}\ \frac{n(n-1)\dots 1}{a^n e^{ax}} \text{ [Repeating this process we get]}$$

$$= \underset{x\to\infty}{\mathrm{Lt}}\ \frac{n!}{a^n e^{ax}} \text{ [Applying L'Hospital's rule]}$$

$$= \frac{n!}{\infty} = 0$$

Hence x^n is of exponential order.

Example Show that t^2 is of exponential order.

Solution : $$\underset{t\to\infty}{\mathrm{Lt}}\ e^{-st} t^2 = \underset{t\to\infty}{\mathrm{Lt}}\ \frac{t^2}{e^{st}} \left[\frac{\infty}{\infty} \text{ i.e., Indeterminant form}\right]$$

[Apply L'Hospital's rule]

$$= \underset{t\to\infty}{\mathrm{Lt}}\ \frac{2t}{se^{st}} \left[\frac{\infty}{\infty} \text{ form}\right] \text{[Apply L'Hospital's Rule]}$$

$$= \underset{t\to\infty}{\mathrm{Lt}}\ \frac{2}{s^2 e^{st}} = \frac{2}{\infty}$$

$$= 0$$

Hence t^2 is of exponential order.

Example Show that the function

$f(t) = e^{t^2}$ is not of exponential order.

Solution : $$\underset{t\to\infty}{\mathrm{Lt}}\ e^{-st} e^{t^2} = \underset{t\to\infty}{\mathrm{Lt}}\ e^{-st + t^2}$$

$$= e^{\infty} = \infty$$

So $f(t) = e^{t^2}$ is not of exponential order.

Define function of class A.

Solution : A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A.

♦ Important Result

(1) $L[1] = \frac{1}{s}$ where $s > 0$

(2) $L[t^n] = \frac{n!}{s^{n+1}}$ where $n = 0, 1, 2, \ldots$

(3) $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$ where n is not a integer.

(4) $L[e^{at}] = \frac{1}{s-a}$ where $s > a$ or $s - a > 0$

(5) $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$

(6) $L[\sin at] = \frac{a}{s^2+a^2}$ where $s > 0$

(7) $L[\cos at] = \frac{s}{s^2+a^2}$ where $s > 0$

(8) $L[\sinh at] = \frac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

(9) $L[\cosh at] = \frac{s}{s^2-a^2}$ where $s^2 > a^2$

(10) $L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]$ [Linearity property]

Note : (1) $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots$

$$e^{\infty} = 1 + \frac{\infty}{\lfloor 1} + \frac{\infty^2}{\lfloor 2} + \ldots$$

(2) $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

(3) $\Gamma_{n+1} = n!$

(4) $\Gamma_{n+1} = \int_0^\infty x^n e^{-x} dx$

(5) $\Gamma_{n+1} = n \Gamma_n$

(6) $\Gamma_{1/2} = \sqrt{\pi}$

(7) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$

(8) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$

(9) $\sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3\theta]$

(10) $\cos^3 \theta = \frac{1}{4} [\cos 3\theta + 3 \cos \theta]$

(11) $\sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$

(12) $\cos A \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)]$

(13) $\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$

(14) $\sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$

5.2 TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Result (1) : Prove that $L[1] = \frac{1}{s}$ where $s > 0$

Proof : We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt$

Here $f(t) = 1$

$$\therefore L[1] = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}[0 - 1] \text{ by note (2)}$$

$$= \frac{1}{s}, \; s > 0$$

Result (2) : Prove that $L[t^n] = \frac{n!}{s^{n+1}}$ $[n = 0, 1, 2, ...]$

Proof : We know that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt = \int_0^\infty t^n\, d\left[\frac{e^{-st}}{-s}\right]$$

$$= t^n \left(\frac{e^{-st}}{-s}\right)\Bigg]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n\, t^{n-1}\, dt$$

$$= (0 - 0) + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1}\, dt$$

i.e., $$L[t^n] = \frac{n}{s} L[t^{n-1}]$$

Similarly $$L[t^{n-1}] = \frac{n-1}{s} L[t^{n-2}]$$

$$L[t^{n-2}] = \frac{n-2}{s} L[t^{n-3}]$$

....

....

$$L[t^{n-(n-1)}] = \frac{n-(n-1)}{s} L\left[t^{[n-(n-1)]-1}\right]$$

$$= \frac{1}{s} L\left[t^o\right] = \frac{1}{s} L[1] = \frac{1}{s}\frac{1}{s}$$

$$\therefore L[t^n] = \frac{n}{s}\frac{n-1}{s} \dots \frac{2}{s}\frac{1}{s}\frac{1}{s} = \frac{n!}{s^n}\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}} \text{ where } [n = 0, 1, 2, ...]$$

Result (3) Prove that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$ where n is not a integer.

Proof : We know that $$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt$$

Put $st = x$ as $t \to 0 \Rightarrow x \to 0$

$s\,dt = dx$ as $t \to \infty \Rightarrow x \to \infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$$

i.e., $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$ $\left[\because \int_0^\infty x^n e^{-x}\, dx = \Gamma_{n+1}\right]$

when n is a positive integer.

we get $\Gamma_{n+1} = n!$

$$L\left[t^n\right] = \frac{n!}{s^{n+1}}$$

II. PROBLEMS BASED ON TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Example 1 Find L[t]

Solution : $L[t^n] = \dfrac{n!}{s^{n+1}}$ [we know that]

$$L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

Example 2 Find L $[t^3]$

Solution : We know that $L[t^n] = \dfrac{n!}{s^{n+1}}$

$$L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Example 3 Find L$[\sqrt{t}]$

Solution : We know that $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$

$$L[\sqrt{t}] = L[t^{1/2}] = \frac{\Gamma_{1/2+1}}{s^{1/2+1}}$$

$$= \frac{\frac{1}{2}\Gamma_{1/2}}{s^{3/2}} \qquad [\because \Gamma_{n+1} = n\,\Gamma_n \;;\; \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{\Gamma_{1/2}}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Example 4. Find L $[t^{3/2}]$

Solution :

We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[t^{3/2}\right] = \frac{\Gamma_{3/2+1}}{s^{3/2+1}} = \frac{\frac{3}{2}\Gamma_{3/2}}{s^{5/2}}$$

$$= \frac{\frac{3}{2}\Gamma_{1/2+1}}{s^{5/2}} = \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma_{1/2}}{s^{5/2}}$$

$$= \frac{\left(\frac{3}{4}\right)\sqrt{\pi}}{s^{5/2}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}}$$

Example 5.2.5. Find L $\left[\frac{1}{\sqrt{t}}\right]$

Solution : We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right] = \frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}$$

$$= \frac{\Gamma_{1/2}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} \qquad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

Result 4. Prove that L $[e^{at}] = \frac{1}{s-a}$ where $s > a$.

Proof : We know that

$$L[f(t)] = \int_C^{\infty} e^{-st} f(t)\, dt$$

$$L\left[e^{at}\right] = \int_0^\infty e^{-st}\ e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}[0-1] = \frac{1}{s-a} \text{ where } s-a>0$$

Example 6. Find the value $L\left[e^{3t}\right]$

Solution : We know that

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

Example 7 Find $L[e^{3t+5}]$

Solution :

W.K.T $L[e^{at}] = \dfrac{1}{s-a}$

$$L[e^{3t+5}] = L[e^{3t}\ e^{5}]$$

$$= e^5\ L\ [e^{3t}] = e^5\left[\frac{1}{s-3}\right] = \frac{e^5}{s-3}$$

Example 8 Find $L\left[\dfrac{e^{at}}{a}\right]$

Solution : W.K.T $L[e^{at}] = \dfrac{1}{s-a}$

$$L\left[\frac{e^{at}}{a}\right] = \frac{1}{a}L\,[e^{at}] = \frac{1}{a}\left[\frac{1}{s-a}\right]$$

Example 9 Find $L\ [2^t]$

Solution W.K.T. $L[e^{at}] = \dfrac{1}{s-a}$

$$L[2^t] = L\left[e^{\log 2^t}\right]$$

$$= L\left[e^{t\log 2}\right]$$

$$= L\left[e^{(\log 2)\,t}\right]$$

$$= \frac{1}{s-\log 2}$$

Result 5. Prove that $L[e^{-at}] = \frac{1}{s+a}$, $(s+a) > 0$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= -\frac{1}{s+a}[0 - 1]$$

$$= \frac{1}{s+a} \text{ where } (s+a) > 0$$

Example 10. Find $L[e^{-bt}]$

Solution : W.K.T $L[e^{-at}] = \frac{1}{s+a}$

$$L[e^{-bt}] = \frac{1}{s+b}$$

Example 11. Find $L[2e^{-3t}]$

Solution : W.K.T. $L[e^{-at}] = \frac{1}{s+a}$

$$L[2e^{-3t}] = 2L[e^{-3t}]$$

$$= 2\left[\frac{1}{s+3}\right] = \left[\frac{2}{s+3}\right]$$

Result 6. Prove that $L[\sin at] = \frac{a}{s^2+a^2}$ $(s > 0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$$

$$\boxed{\int e^{ax} b(x)\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]}$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s \sin at - a \cos at]\right]_0^\infty \text{ by Note 7.}$$

$$= 0 - \left[\frac{(-a)}{s^2 + a^2}\right] = \frac{a}{s^2 + a^2} \text{ where } s > 0.$$

Example 5.2.12. Find L [sin 2t]

Solution : W.K.T $L[\sin at] = \frac{a}{a^2 + a^2}$

$$L[\sin 2t] = \frac{2}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 4}$$

Example 5.2.13. Find L [sin π t]

Solution : W.K.T $L[\sin at] = \frac{a}{s^2 + a^2}$

$$L[\sin \pi t] = \frac{\pi}{s^2 + \pi^2}$$

Result : 7. Prove that $L[\cos at] = \frac{s}{s^2 + a^2}$ $(s > 0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2}[-s\cos at + a\sin at]\right]_0^\infty$$

$$= 0 - \left[\frac{1}{s^2 + a^2}(-s)\right]$$

$$= \frac{s}{s^2 + a^2} \quad (s > 0)$$

Example 5.2.14. Find L [cos 2t]

Solution : W.K.T. $L[\cos at] = \frac{s}{s^2 + a^2}$

$$L[\cos 2t] = \frac{s}{s^2 + 4}$$

Example 15 **Prove that L [cos at] = $\frac{s}{s^2 + a^2}$ and L [sin at] = $\frac{a}{s^2 + a^2}$**

Solution : By Euler's theorem

$$e^{ix} = \cos x + i \sin x$$

$$e^{iat} = \cos at + i \sin at$$

$$L[e^{iat}] = L[\cos at + i \sin at]$$

$$= L[\cos at] + i\, L[\sin at]$$

$$L[\cos at] + i\, L[\sin at] = L[e^{iat}]$$

$$= \frac{1}{s - ia}$$

$$= \left[\frac{1}{s - ia}\right] \left[\frac{s + ia}{s + ia}\right]$$

$$= \frac{s + ia}{s^2 + a^2}$$

Equating real & Imaginary parts we get

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

$$L[\sin at] = \frac{a}{s^2 + a^2}$$

Example 16 **Find L [cos (at + b)]**

Solution : $L[\cos (at + b)]$

$$= L[\cos at \cos b - \sin at \sin b]$$

$$= \cos b\, L[\cos at] - \sin b\, L[\sin at]$$

$$= \cos b \left[\frac{s}{s^2 + a^2}\right] - \sin b \left[\frac{a}{s^2 + a^2}\right]$$

$$= \frac{s \cos b - a \sin b}{s^2 + a^2}$$

Example 17 **Find L [$\sin^2 2t$]**

Solution : $L[\sin^2 2t] = L\left[\frac{1 - \cos 4t}{2}\right] = \frac{1}{2} L[1 - \cos 4t]$

$$= \frac{1}{2}\left[L[1] - L[\cos 4t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 16}\right]$$

Example 18 Find L [sin 5t cos 2t]

Solution : $L[\sin 5t \cos 2t] = \frac{1}{2} L[\sin 7t + \sin 3t]$ by Note 11.

$$= \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

Example 19 Find $L[(\sin t - \cos t)^2]$

Solution : $L[(\sin t - \cos t)^2] = L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$

$$= L[1 - \sin 2t] = L[1] - L[\sin 2t]$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

Result 8. Prove that $L[\sinh at] = \frac{a}{s^2 - a^2}$ where $s > |a|$

Proof : $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} - e^{-at}] = \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}, \; s > |a|$$

Result 9. Prove that $L[\cosh at] = \frac{s}{s^2 - a^2}$, $s > |a|$

Proof : $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} L[e^{at} + e^{-at}] = \frac{1}{2}\left[L[e^{at}] + L[e^{-at}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}, \; s > |a|$$

Result 10. Linearity property.

Prove that $L[a f(t) \pm bg(t)] = a L[f(t)] \pm b L[g(t)]$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[a f(t) \pm bg(t)] = \int_0^\infty e^{-st} [af(t) \pm bg(t)]\, dt$$

$$= \int_0^\infty e^{-st} af(t)\, dt \pm \int_0^\infty e^{-st} bg(t)\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt \pm b\int_0^\infty e^{-st} g(t)\, dt$$

$$= a L[f(t)] \pm b\, L[g(t)]$$

Example $L[e^{4t} + t^4 + 7]$

Solution : $L[e^{4t} + t^4 + 7]$

$$= L[e^{4t}] + L[t^4] + L[7]$$

$$= \frac{1}{s-4} + \frac{4!}{s^5} + 7\, L[1]$$

$$\frac{1}{s-4} + \frac{24}{s^5} + 7\left[\frac{1}{s}\right]$$

Example 5.2.26. Find $L[f(t)]$ **if** $f(t) = \begin{cases} e^{-t}, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

Solution : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^4 e^{-st} e^{-t}\, dt + \int_4^\infty e^{-st}\, 0\, dt$$

$$= \int_0^4 e^{-(s+1)t}\, dt + 0$$

$$= \left.\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4 = \frac{-1}{(s+1)}\left[e^{-(s+1)t}\right]_0^4$$

$$= \frac{-1}{s+1}\left[e^{-4(s+1)} - 1\right] = \frac{1}{s+1}[1 - e^{-4(s+1)}]$$

Result 11. Prove that L [f′ (t)] = s L [f (t)] − f (0)

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \int_0^\infty e^{-st}\, d\,[f(t)]$$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty f(t)\,(-s)\, e^{-st}\, dt$$

$$= [0 - f(0)] + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\, L\, [f(t)]$$

$$= s\, L\, [f(t)] - f(0)$$

Result 12. Prove that L [f′′ (t)] = s^2 L [f (t)] − s f (0) − f ′ (0)

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st}\, f(t)\, dt$

$$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\, dt$$

$$= \int_0^\infty e^{-st}\, d\,[f'(t)]$$

$$= e^{-st} f'(t)\Big]_0^\infty - \int_0^\infty f'(t)\,(-s)\, e^{-st}\, dt$$

$$= [0 - f'(0)] + s\int_0^\infty e^{-st} f'(t)\, dt$$

$$= -f'(0) + s\,L[f'(t)]$$

$$= -f'(0) + s\,[sL[f(t)] - f(0)] \text{ by result } (11)$$

$$= s^2\, L\,[f(t)] - s\,f(0) - f'(0)$$

Note : (15)

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

Result : 13. FIRST SHIFTING THEOREM

- If $L\,[f(t)] = \varphi(s)$ then $L[e^{at} f(t)] = \varphi(s - a)$
- If $L\,[f(t)] = \varphi(s)$ then $L[e^{-at} f(t)] = \varphi(s + a)$

Proof : W.K.T $L[f(t)] = \varphi(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \varphi(s - a)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \varphi(s+a)$$

III. PROBLEMS BASED ON FIRST SHIFTING THEOREM AND SECOND SHIFTING THEOREM

Example **Find L $[t^n e^{-at}]$**

Solution : $L[t^n e^{-at}] = \left[L(t^n)\right]_{s \to (s+a)}$

$$= \left[\frac{n!}{s^{n+1}}\right]_{s \to (s+a)}$$

$$= \frac{n!}{(s+a)^{n+1}}$$

Example **Find L $[e^{-at} \cos bt]$**

Solution : $L[e^{-at} \cos bt] = \left[L[\cos bt]\right]_{s \to (s+a)}$

$$= \left[\frac{s}{s^2+b^2}\right]_{s \to (s+a)}$$

$$= \frac{s+a}{(s+a)^2+b^2}$$

Example **Find L $[e^{at} \sinh bt]$**

Solution : $L[e^{at} \sinh bt] = \left[L[\sinh bt]\right]_{s \to (s-a)}$

$$= \left[\frac{b}{s^2-b^2}\right]_{s \to (s-a)} = \frac{b}{(s-a)^2-b^2}$$

Example **Find L $\left[e^t t^{-1/2}\right]$**

Solution : $L\left[e^t t^{-1/2}\right] = \left[L[t^{-1/2}]\right]_{s \to (s-1)}$

$$= \left[\frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}\right]_{s \to (s-1)} = \left[\frac{\Gamma_{1/2}}{s^{1/2}}\right]_{s \to (s-1)}$$

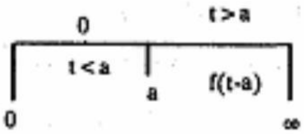
$$= \left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]_{s \to (s-1)} = \left[\sqrt{\frac{\pi}{s}}\right]_{s \to (s-1)}$$

$$\sqrt{\pi}$$

Result 14. Second shifting theorem.

- If $L[f(t)] = \varphi(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

  then $L[G(t)] = e^{-as}\,\varphi(s)$



Proof : $L[G(t)] = \int_0^{\infty} e^{-st}\, G(t)\, dt$

$$= \int_0^{a} e^{-st}\, 0\, dt + \int_a^{\infty} e^{-st} f(t-a)\, dt$$

$$= \int_a^{\infty} e^{-st} f(t-a)\, dt$$

Put $t - a = u$ $\quad t \to a \Rightarrow u \to 0$

$\quad dt = du$ $\quad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^{\infty} e^{-s(u+a)} f(u)\, du$$

$$= e^{-sa} \int_0^{\infty} e^{-su} f(u)\, du$$

$$= e^{-sa} \int_0^{\infty} e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$= e^{-sa}\, L[f(t)]$$

$$= e^{-sa}\, \varphi(s)$$

Result : 15. If $L[F(t)] = \varphi(s)$ and $C > 0$ then

$L[F(t-c)\, H(t-c)] = e^{-cs}\varphi(s)$ where $H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

Proof : $L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$

$$L[F(t-c)\, H(t-c) = \int_0^{\infty} e^{-st}\, F(t-c)\, H(t-c)\, dt$$

DERIVATIVES AND INTEGRALS OF TRANSFORMS ·
TRANSFORMS OF DERIVATIVES AND INTEGRALS

Result : 17. Transforms of Derivatives

If $L[f(t)] = \varphi(s)$ **then** $L[t f(t)] = -\frac{d}{ds}\varphi(s) = -\varphi'(s)$

Proof : $\varphi(s) = L[f(t)]$

$$\frac{d}{ds}\varphi(s) = \frac{d}{ds}L[f(t)]$$

$$\varphi'(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right] = \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)\,dt = -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -L[t f(t)]$$

Put $t - c = u \qquad t \to 0 \Rightarrow u \to -c$

$dt = du \qquad t \to \infty \Rightarrow u \to \infty$

$$= \int_{-c}^{\infty} e^{-s(u+c)} F(u)\, H(u)\, du$$

$$= e^{-sc} \int_{-c}^{\infty} e^{-su} F(u)\, H(u)\, du$$

$$= e^{-sc}\left[\int_{-c}^{0} e^{-su} F(u)\, 0\, du + \int_0^\infty e^{-su} F(u)\, du\right]$$

$$= e^{-sc} \int_0^\infty e^{-su} F(u)\, du$$

$$= e^{-sc} \int_0^\infty e^{-st} F(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$= e^{-sc} L[F(t)] = e^{-sc} \varphi(s)$$

$$L[t f(t)] = -\varphi'(s)$$

Corollary :- If $L[f(t)] = \varphi(s)$ then $L[t^n f(t)] = (-1)^n \varphi^n(s)$.

Proof : W.K.T. $L[t f(t)] = -\varphi'(s)$

$$L[t^2 f(t)] = L[t \cdot t f(t)]$$

$$= -\frac{d}{ds} L[t f(t)]$$

$$= -\frac{d}{ds}\left[\frac{-d}{ds} L f(t)\right]$$

$$= (-1)^2 \frac{d^2}{ds^2}[L f(t)]$$

$$= (-1)^2 \frac{d^2}{ds^2} \varphi(s)$$

....

....

....

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \varphi(s) = (-1)^n \varphi^n(s)$$

PROBLEMS BASED ON TRANSFORMS OF DERIVATIVES

Example 1. Find L [t sin 2t]

Solution : W.K.T. $L[t^n f(t)] = (-1)^n \varphi^n(s)$

$$L(t \sin 2t) = -\frac{d}{ds}[L(\sin 2t)] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left[\frac{-4s}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$$

Example 2. Find L $[t^2 e^{-3t}]$

Solution : W.K.T $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\varphi(s)]$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}] = \frac{d^2}{ds^2}\left[\frac{1}{s+3}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{2}{(s+3)^3}$$

Example 3. **Find $L[te^{-2t}\sin t]$**

Solution : $L[t e^{-2t} \sin t] = -\frac{d}{ds}[L(e^{-2t}\sin t)]$

$$= -\frac{d}{ds}\left[[L[\sin t]]_{s \to (s+2)}\right] = -\frac{d}{ds}\left[\left[\frac{1}{s^2+1}\right]_{s \to (s+2)}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{(s+2)^2+1}\right] = \frac{2(s+2)}{[(s+2)^2+1]^2}$$

Example **Find $L[t \sin 3t \cos 2t]$**

Solution : $L[t\sin 3t\cos 2t] = -\frac{d}{ds}[L(\sin 3t\cos 2t)]$

$$= -\frac{d}{ds}\left[\frac{1}{2}[L(\sin 5t) + L(\sin t)]\right] = -\frac{1}{2}\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

Example 5. **Given that $L[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$ find L.T. of $\frac{1}{\sqrt{t}}\cos\sqrt{t}$**

Solution : Let $f(t) = \sin\sqrt{t}$

$$f'(t) = \frac{1}{2\sqrt{t}}\cos\sqrt{t}$$

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L\left[\frac{1}{2\sqrt{t}}\cos\sqrt{t}\right] = L[f'(t)]$$

$$= s\frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s} - 0 \quad [\because f(0) = 0]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$\frac{1}{2} L\left[\frac{1}{\sqrt{t}}\cos\sqrt{t}\right] = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$L\left[\frac{1}{\sqrt{t}}\cos\sqrt{t}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Example 6 show that $\int_0^{\infty} e^{-t}\, t\cos t\, dt = 0$

Solution $\quad \underset{0}{}\ t\,dt = \left[L[t\cos t]\right]_{s=1}$

$$= \left[-\frac{d}{ds}L(\cos t)\right]_{s=1} = \left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=1} = \left[-\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]\right]_{s=1}$$

$$= \left[-\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]_{s=1}\right] = \left[-\left[\frac{1-s^2}{(s^2+1)^2}\right]\right]_{s=1} = [-(0)] = 0$$

Example 7 Find $L[t e^{-t}\cosh t]$

Solution : $\quad {}^{-t}\cosh t] = -\frac{d}{ds}L[e^{-t}\cosh t]$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2-1}\right] = -\left[\frac{[(s+1)^2-1]-(s+1)\,2\,(s+1)]}{[(s+1)^2-1]^2}\right]$$

$$= -\left[\frac{(s+1)^2-1-2(s+1)^2}{[(s+1)^2-1]^2}\right] = \frac{(s+1)^2+1}{(s^2+2s)^2} = \frac{s^2+2s+2}{s^4+4s^2+4s^3}$$

Result 18. Integrals of transform

If $L[f(t)] = \varphi(s)$ and $\frac{1}{t}f(t)$ has a limit as $t \to 0$ then

$$L\left[\frac{1}{t}f(t)\right] = \int_s^{\infty} \varphi(s)\, ds$$

Proof : $\varphi(s) = L[f(t)]$

$$\int_s^{\infty} \varphi(s)\, ds = \int_s^{\infty} L[f(t)]\, ds$$

$$= \int_s^{\infty}\int_0^{\infty} e^{-st} f(t)\, dt\, ds = \int_0^{\infty}\int_s^{\infty} e^{-st} f(t)\, ds\, dt$$

[since s and t are independent variables and hence the order of integration in the double integral can be interchanged]

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}\, ds\right] dt = \int_0^{\infty} f(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt$$

$$= \int_0^{\infty} f(t)\left[0+\frac{e^{-st}}{t}\right] dt = \int_0^{\infty} f(t)\frac{e^{-st}}{t}\, dt$$

$$= \int_0^{\infty} e^{-st}\,\frac{f(t)}{t}\, dt = L\left[\frac{1}{t}f(t)\right]$$

i.e., $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds$

PROBLEMS BASED ON INTEGRALS OF TRANSFORM

Example 8 . **Find** $L\left[\frac{1-e^t}{t}\right]$

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L[1-e^t]\,ds = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-1}\right] ds$$

$$= \left[\log s - \log(s-1)\right]_s^\infty = \left[\log\frac{s}{s-1}\right]_s^\infty$$

$$= \left[\log\frac{s}{s(1-1/s)}\right]_s^\infty = \left[\log\frac{1}{1-1/s}\right]_s^\infty$$

$$= 0 - \log\frac{s}{s-1} = \log\left(\frac{s-1}{s}\right)$$

Example 9 **Find** $L\left[\frac{\sin at}{t}\right]$ **[A.U., March 1996]**

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left[\frac{s}{a}\right] = \tan^{-1}\left[\frac{a}{s}\right]$$

Note : $\cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$$\frac{s}{a} = \cot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{s}{a}$$

PROBLEMS BASED ON INITIAL VALUE AND FINAL VALUE THEOREM

Example 5.4.1. If $L[f(t)] = \dfrac{1}{s(s+a)}$, find $\underset{t \to \infty}{\text{Lt}} f(t)$ and $\underset{t \to 0}{\text{Lt}} f(t)$

Solution : $\underset{t \to 0}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}}\, s \,.\, F(s)$

$$= \underset{s \to \infty}{\text{Lt}}\, s \frac{1}{s(s+a)} = \underset{s \to \infty}{\text{Lt}} \frac{1}{s+a} = \frac{1}{\infty} = 0$$

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}}\, s\,F(s) = \underset{s \to 0}{\text{Lt}}\, s \frac{1}{s(s+a)}$$

$$= \underset{s \to 0}{\text{Lt}} \frac{1}{s+a} = \frac{1}{a}$$

Example 2. Verify the initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution : Initial value theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

$$\text{L.H.S} = \lim_{t \to 0} f(t) = 1 + 1 = 2$$

$$\text{R.H.S} = \lim_{s \to \infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s \to \infty}\left[1 + \frac{s(s+2)}{(s+1)^2+1}\right] = \lim_{s \to \infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right]$$

$$= \lim_{s \to \infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1 + 1 = 2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Initial value theorem verified.

Final value theorem states that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, F(s)$$

$$\text{L.H.S.} = \lim_{t \to \infty} [1 + e^{-t} (\sin t + \cos t)]$$

$$= 1 + 0 = 1$$

$$\text{R.H.S.} = \lim_{s \to 0} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1}\right]$$

$$= 1 + 0 = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Final value theorem verified.

Example 3. Verify the initial and final value theorems for $f(t) = 3e^{-2t}$

Solution : $f(t) = 3e^{-2t}$

$$F(s) = L[f(t)] = L[3e^{-2t}] = \frac{3}{s+2}$$

Initial value theorem : $\underset{t \to 0}{\text{Lt}}\, f(t) = \underset{s \to \infty}{\text{Lt}}\, s\, F(s)$

$$\text{L.H.S.} = \underset{t \to 0}{\text{Lt}}\, f(t) = \underset{t \to 0}{\text{Lt}}\, 3e^{-2t} = 3$$

$$\text{R.H.S} = \underset{s \to \infty}{\text{Lt}}\, s\, F(s) = \underset{s \to \infty}{\text{Lt}}\, s\left(\frac{3}{s+2}\right) = \underset{s \to \infty}{\text{Lt}} \frac{3s}{s+2}$$

$$= \underset{s \to \infty}{\text{Lt}} \frac{3s}{s\left(1 + \frac{2}{s}\right)}$$

$$= \underset{s \to \infty}{\text{Lt}} \frac{3}{1 + \left(\frac{2}{s}\right)} = 3$$

$$\text{L.H.S} = \text{R.H.S.}$$

Hence Initial value theorem verified.

Final value theorem $\underset{t \to \infty}{\text{Lt}} \; f(t) = \underset{s \to 0}{\text{Lt}} \; s\,F(s)$

$$\text{L.H.S.} = \underset{t \to \infty}{\text{Lt}} f(t) = \underset{t \to \infty}{\text{Lt}} \; 3e^{-2t} = 0 \qquad [\because e^{-\infty} = 0]$$

$$\text{R.H.S.} = \underset{s \to 0}{\text{Lt}} \; s\,F(s) = \underset{s \to 0}{\text{Lt}} \; s\left(\frac{3}{s+2}\right) = 0$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Hence Final value theorem verified.

TRANSFORMS OF UNIT STEP FUNCTION AND IMPULSE FUNCTION

♦ UNIT STEP FUNCTION (OR) HEAVISIDE'S UNIT STEP FUNCTION

PROBLEMS BASED ON UNIT STEP FUNCTION

Example 1. Define the unit step function.

Solution :

The unit step function, also called Heavi side's unit function is defined as

$$U(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t > a \end{cases}$$

This is the unit step functions at $t = a$

It can also be denoted by $H(t-a)$.

Example 2. Give the L.T. of the unit step function. **[M.U. Oct., 96]**

Solution :

The L.T. of the unit step function is given by

$$L[U(t-a)] = \int_0^\infty e^{-st} \, U(t-a)\, dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st}\,(1)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 - \left(\frac{e^{-sa}}{-s}\right) = \frac{e^{-as}}{s}$$

TRANSFORM OF PERIODIC FUNCTIONS

Define periodic function and state its Laplace transform formula.

◆ Def. Periodic

A function $f(x)$ is said to be "periodic" if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function $f(t)$ with period p given by $\dfrac{1}{1-e^{-ps}} \displaystyle\int_0^p e^{-st} f(t)\, dt$

Proof :
$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

Put $t = u + p$ in the second integral

i.e., $u = t - p$ $\qquad t \to p \Rightarrow u \to 0$

i.e., $du = dt$ $\qquad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^p e^{-st} f(t)\, dt + \int_0^\infty e^{-(u+p)s} f(u+p)\, du$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-su} f(u)\, du \quad [\because f(u+p) = f(u)]$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$L[f(t)] = \int_0^p e^{-st} f(t)\, dt + e^{-sp}\, L[f(t)]$$

$$[1 - e^{-sp}]\, L[f(t)] = \int_0^p e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

Example 1 Find the Laplace transform of the Half wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0 \quad , & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

Solution : This function is a periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right]$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2 + \omega^2} \left[-s \sin \omega t - \omega \cos \omega t \right] \right]_0^{\pi/\omega}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-s\pi/\omega}\, \omega + \omega}{s^2 + \omega^2} \right]$$

$$= \frac{\omega \left[1 + e^{\frac{-s\pi}{\omega}}\right]}{\left[1 - e^{-s\pi/\omega}\right] \left[1 + e^{-s\pi/\omega}\right] (s^2 + \omega^2)}$$

$$= \frac{\omega}{(s^2 + \omega^2) \left(1 - e^{-s\pi/\omega}\right)}$$

Example 2 **Find the Laplace Transform of**

$$f(t) = \begin{cases} 1 & , \ 0 < t < a \\ 2a-t, & a < t < 2a \end{cases} \text{ with } f(t+2a) = f(t)$$

Solution : $$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st}\, t\, dt + \int_a^{2a} e^{-st}\, (2a-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left[t\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left[-t\, \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(-\frac{1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right]$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{[1-e^{-as}]^2}{s^2(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{1}{s^2}\tanh\left[\frac{as}{2}\right]$$

INVERSE LAPLACE TRANSFORM

Now we obtain $f(t)$ when $\phi(s)$ is given, then we say that inverse Laplace transform of $\phi(s)$ is $f(t)$.

(1) If $L[f(t)] = \phi(s)$, then $L^{-1}[\phi(s)] = f(t)$
where L^{-1} is called the inverse Laplace transform operator.

(2) If $\varphi_1(s)$ and $\varphi_2(s)$ are L.T. of $f(t)$ and $g(t)$ respectively then
$L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)] = C_1L^{-1}[\varphi_1(s)] + C_2L^{-1}[\varphi_2(s)]$

Proof : Given : $L[f(t)] = \varphi_1(s)$

$$f(t) = L^{-1}[\varphi_1(s)]$$

$$L[g(t)] = \varphi_2(s)$$

$$g(t) = L^{-1}[\varphi_2(s)]$$

W.K.T. $L[C_1f(t) + C_2g(t)] = C_1L[f(t)] + C_2L[g(t)]$

$$= C_1\varphi_1(s) + C_2\varphi_2(s)$$

$$C_1f(t) + C_2g(t) = L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)]$$

i.e., $L^{-1}[C_1\varphi_1(s) + C_2\varphi_2(s)] = C_1f(t) + C_2g(t)$

$$= C_1L^{-1}\varphi(s) + C_2L^{-1}\varphi_2(s)$$

Note : (1) If $L[f(t)] = \varphi(s)$ then $L[e^{at}f(t)] = \varphi(s-a)$

i.e., If $L^{-1}[\varphi(s)] = f(t)$ then

$$L^{-1}[\varphi(s-a)] = e^{at}f(t) = e^{at}L^{-1}[\varphi(s)]$$

Note : (2) If $L[f(t)] = \varphi(s)$ then $L[e^{-at}f(t) = \varphi(s+a)$

i.e., If $L^{-1}[\varphi(s)] = f(t)$ then

$$L^{-1}[\varphi(s+a)] = e^{-at}f(t) = e^{-at}[L^{-1}\varphi(s)]$$

UNIT-V

SOLUTIONS OF SYSTEM OF EQUATIONS

Introduction

We come across, very often simultaneously linear algebraic equations for its solutions, especially, in the fields of science and engineering. In lower classes, we have solved such equations by Cramer's rule (determinant methods) or by matrix methods. These methods become tedious when the number of unknown in the system is large. After the availability of computers, we go to numerical methods which are suited for computer operations. These numerical methods are of two types namely: (*i*) direct and (*ii*) iterative.

We will study a few methods below deals with the solution of simultaneous Linear Algebraic Equations

Gauss Elimination Method (Direct Method)

This is a direct method based on the elimination of the unknowns by combining equations such that the n unknowns are reduced to an equation upper triangular system which could be solved by back substitution.

Consider the n linear equations in n unknowns, viz.

$$a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=b_n \quad \dots(1)$$

Where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to $AX=B$ .....(2)

Where $A=\begin{pmatrix} a_{11} & a_{12} \dots\dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots\dots a_{2n} \\ \dots\dots\dots & \dots\dots\dots \; \dots\dots\dots \\ a_{n1} & a_{n2} \dots\dots a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix (A,B) to upper triangular matrix.

$$(A,B) = \left(\begin{array}{ccc|c} a_{11} & a_{12} \ldots\ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots\ldots\ldots & a_{2n} & b_2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \vdots \\ a_{n1} & a_{n2} \ldots\ldots\ldots & a_{nn} & b_n \end{array}\right) \quad ..(3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\dfrac{a_{i1}}{a_{11}}$ and add to the ith row of (A,B), where i=2,3,…,n. By thia, all elements in the first column of (A,B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} \ldots\ldots\ldots & a_{1n} & b_1 \\ 0 & b_{22} \ldots\ldots\ldots & b_{2n} & c_2 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \vdots \\ 0 & b_{n2} \ldots\ldots\ldots & b_{nn} & c_n \end{array}\right) \quad \ldots\ldots.(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second

row of (4) by $-\dfrac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row (i=3,4,…,n). Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} \ldots\ldots\ldots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} \ldots\ldots\ldots & b_{2n} & c_2 \\ 0 & 0 & c_{23} \ldots\ldots\ldots & c_{3n} & d_3 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \vdots \\ 0 & 0 & c_{n3} \ldots\ldots\ldots & c_{nn} & d_n \end{array}\right) \quad \ldots\ldots (5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A,B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots\ \ldots & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \ldots\ \ldots\ldots\ \ldots & & b_{2n} & c_2 \\ 0 & 0 & c_{23} & c_{34}\ldots\ldots & & c_{3n} & d_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots\ldots & c_{nn} & d_n \end{array}\right) \quad \ldots(6)$$

From, (6) the given system of linear equations is equivalent to

$$\left.\begin{array}{r} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots.+a_{1n}x_n=b_1 \\ b_{22}x_2+b_{23}x_3+\ldots.+b_{2n}x_n=c_2 \\ c_{33}x_3+\ldots.+c_{3n}x_n=d_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{nn}x_n=k_n \end{array}\right\}$$

Going from the bottom of these equation, we solve for $x_n=\frac{k_n}{\alpha_{nn}}$. Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method for we solve x_n , x_{n-1}, x_{n-2}, ……x_2, x_1.

Gauss – Jordan Elimination Method (Direct Method)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX=B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making the matrix A not above the leading diagonal of A also as zeros. By this way, the system $AX=B$ will reduce to the form.

$$\left(\begin{array}{cccccc|c} a_{11} & 0 & 0 & 0 & \ldots\ldots\ldots\ \ldots & a_{1n} & b_1 \\ 0 & b_{22} & 0 & 0 & \ldots\ \ldots\ldots\ \ldots & b_{2n} & c_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & d_3 \\ 0 & 0 & 0 & 0 & \ldots\ \ldots\ldots & \alpha_{nn} & k_n \end{array}\right) \ldots(7)$$

From (7)

$$x_n = \frac{k_n}{\alpha_{nn}}, \ \ldots\ldots, \ x_2 = \frac{c_2}{b_{22}}, \ x_n = \frac{b_1}{a_{11}}$$

Note: By this method, the values of $x_1, x_2, \ldots x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss – Jordan method.*

x+2y+z=3, 2x+3y+3z=10, 3x-y+2z=13.

Solution. (By Gauss method)

This given system is equivalent to

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

$$A\,X = B$$

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now, we will make the matrix A upper triangluar.

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad R_2+(-2)R_1,\ R_3+(-3)R_1$$

Now, take b_{22}=-1 as the pivot and make b_{32} as zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_{32}(-7) \ldots\ldots\ldots\ldots(2)$$

From this, we get

$$x+2y+z = 3, \qquad -y+z = 4 \qquad, \quad -8z = -24$$

$\therefore$ $z = 3$, $y = -1$, $x = 2$ by back substitution.

$$x = 2,\ y = -1,\ z = 3$$

Solution. (Gauss – Jordan method)

In stage 2, make the element, in the position (1,2), also zero.

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_{12}(2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_{13}(3),\ R_{23}(1)$$

i.e., $x = 2,\ y = -1,\ z = 3$

Example 2 *Solve the system by Gauss- Elimination method*

$2x+3y-z = 5$; $4x+4y-3z = 3$ and $2x-3y+2z = 2$.

Solution. The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \qquad X = B$$

$$(A,B) = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right]$$

Step 1. Taking $a_{11} = 2$ as the pivot, reduce all elements below that to zero.

$$(A,B) = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right] \quad R_{21}(-2),\ R_{31}(-1)$$

Step 2. Taking the element -2 in the position (2,2) as pivot, reduce all elements all elements below that to zero.

$$(A,B) = \left[\begin{array}{cccc} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right] \qquad R_{32}(-3)$$

Hence $2x+3y-z = 5$

$$-2y-z = -7$$

$$6z = 18$$

$\therefore$ $z = 3$, $y = 2$, $x = 1$. By back substitution

Example 3 Solve the following system by Gauss - Jordan method

$$5x_1 + x_2 + x_3 + x_4 = 4; \quad x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5; \quad x_1 + x_2 + x_3 + 4x_4 = -6$$

Solution. Interchange the first and the last equation, so that coefficient of x_1 in the first equation is 1. Then we have

$$(A,B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{6} & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) R_{21}(-1),\ R_{31}(-1),\ R_{41}(-5)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{1} & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) R_2\left(\frac{1}{6}\right) \text{ to make the pivot as 1}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{5} & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_{12}(-1),\ R_{42}(4)$$

$$\sim \left(\begin{array}{ccccc} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{1} & -0.6 & 0.2 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_3\left(\frac{1}{5}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right) \quad R_{12}(-1),\ R_{43}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \quad R_4\left(\frac{1}{23.4}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) R_{34}\left(-\frac{3}{5}\right),\ R_{24}\left(-\frac{1}{2}\right),\ R_{14}(5.1)$$

$x_1 = 1,\ \ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

Example 4. *Solve the system of equations by Gauss – Jordan method:*

$$x + y + z + w = 2$$

$$2x - y + 2z - w = -5$$

$$3x + 2y + 3z + 4w = 7$$

$$x - 2y - 3z + 2w = 5$$

Solution.

$$(A,B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \\ \ \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad R_2\left(-\frac{1}{3}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right) \begin{array}{l} R_1+(-1)R_2 \\ \\ R_3+R_2 \\ R_4+3R_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 & -3 \end{array}\right) \begin{array}{l} \\ \\ R_3\left(\frac{1}{2}\right) \\ R_4\left(-\frac{1}{4}\right) \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \text{Interchanging} \\ R_3 \text{ and } R_4 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \\ \\ R_1+(-1)R_3 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \begin{array}{l} \\ R_1+(-1)R_4 \\ R_2+(-1)R_4 \\ R_3+R_4 \end{array}$$

$\therefore \quad x = 0, \;\; y = 1, z = -1, w = 2$

Example 5. *Apply Gauss – Jordan method to find the solution of the following system:*

$$10x + y + z = 12; \;\; 2x + 10y + z = 13; \qquad x + y + 5z = 7.$$

Solution. since the coefficient of x in the last equation is unity, we rewrite the equations interchanging the first and the last. Hence the augmented matrix is

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad \begin{array}{l} R_2+(-2)R_1 \\ R_3+(-10)R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{473}{8} & -\frac{473}{8} \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_3\left(-\frac{8}{473}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad R_1+(-1)R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{l} R_2+\left(\frac{9}{8}\right)R_3 \\ R_1+\left(-\frac{49}{8}\right)R_3 \end{array}$$

$$\therefore \quad x = 1, \quad y = 1, z = 1$$

Method Of Triangularization (Or Method Of Factorization) (*Direct Method*)

This method is also called as *decomposition* method. In this method, the coefficient matrix A of the system $AX = B$, decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. we will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_{11}x_1+ a_{12}x_2+ a_{13}x_3 = b_1$$

$$a_{21}x_1+ a_{22}x_2+ a_{23}x_3 = b_2$$

$$a_{31}x_1+ a_{32}x_2+ a_{33}x_3 = b_3$$

This system is equivalent to $AX = B$

Where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

And an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \text{ so that}$$

$LUX = B$ Let $UX = Y$ And hence $LY = B$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\therefore \quad y_1 = b, \; l_{21}y_1+y_2 = b_2, \; l_{31}y_1+l_{32}y_2+y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4), $\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$, $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.Now L and U can be found from $LU = A$

i.e., $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

i.e.,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3 l's and 6 u's.

That is, L and U re known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first rows of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$ $\quad l_{21}u_{12}+u_{22} = a_{22}$ $\quad l_{21}u_{13}+u_{23} = a_{23}$

$$l_{21} = \frac{a_{21}}{a_{11}}, \; u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.\, a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.\, a_{13}$$

again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12}+l_{32}u_{22} = a_{32}$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = a_{33}$

solving, $\quad l_{31} = \dfrac{a_{31}}{a_{11}}$, $l_{32} = \dfrac{a_{32} - \frac{a_{31}}{a_{11}} . a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} . a_{12}}$

$$u_{33} = a_{33} - \left[\frac{a_{31}}{a_{11}} . a_{13}\right] - \left[\frac{a_{32} - \frac{a_{31}}{a_{11}} . a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} . a_{12}}\right] a_{33} - \frac{a_{31}}{a_{11}} . a_{13}$$

Therefore L and U are known.

Example 1: *By the method of triangularization, solve the following system.*

$5x - 2y + z = 4, \quad 7x + y - 5z = 8, \quad 3x + 7y + 4z = 10.$

Solution. The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$$A \qquad X = B$$

Now, let $\quad LU = A$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$

Multiplying and equating coefficients,

$$u_{11} = 5, \quad u_{12} = -2, \quad u_{13} = 1$$

$$l_{21}u_{11} = 7 \qquad l_{21}u_{12} + u_{22} = 1 \qquad l_{21}u_{13} + u_{23} = -5$$

$$l_{21} = \frac{7}{5}, \quad u_{22} = 1 - \frac{7}{5}.(-2) = \frac{19}{5} \text{ and}$$

$$u_{23} = -5 - \frac{7}{5}.(1) = -\frac{32}{5}$$

Again equating elements in the third row,

$l_{31}u_{11} = 3, \quad l_{31}u_{12} + l_{32}u_{22} = 7$ and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$

$$\therefore \qquad l_{31} = \frac{3}{5}, \quad l_{32} = \frac{7 - \frac{3}{5}.(-2)}{\frac{19}{5}} = \frac{41}{19}$$

$u_{33} = 4 - \frac{3}{5}.\left(1\right) - \frac{41}{19}\left(-\frac{32}{5}\right) = 4 - \frac{3}{5} + \frac{1312}{95}$

$= \frac{1635}{95} = \frac{327}{19}$

Now L and U are known.Since $LUX = B$, $LY = B$ where $UX = Y$.

From $LY = B$,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$y_1 = 4,\ \frac{7}{5}y_1 + y_2 = 8,\ \frac{3}{5}y_1 + \frac{41}{19}y_2 + y_3 = 10$

$y_2 = 8 - \frac{28}{5} = \frac{12}{5}$

$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \times \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$

$UX = Y$ gives $\begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$

$5x - 2y + z = 4$

$\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$

$\frac{327}{19}z = \frac{46}{19}$

$z = \frac{46}{327}$

$$\frac{19}{5}y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$$

$$y = \frac{284}{327}$$

$$5x = 4+2y - z = 4 + 2\left(\frac{568}{327}\right) - \frac{46}{327}$$

$$\therefore \qquad x = \frac{366}{327}$$

$$\therefore \qquad x = \frac{366}{327}, \quad y = \frac{284}{327}, \quad z = \frac{46}{327}$$

Example 2: *Solve, by triangularization method, the following system:*

x + 5y + z = 14, 2x + y + 3z = 13, 3x + y + 4z = 17.

Solution. this is equivalent to

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$$A \qquad X = B$$

Now, let $LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$

By seeing, we can write $u_{11} = 1,\quad u_{12} = 5 \quad u_{13} = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} 1 & 5 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Hence, $l_{21} = 2,\quad 5l_{21}+u_{22} = 1 \qquad l_{21}+u_{23} = 3$

$l_{21} = 2,\quad u_{22} = -9,\qquad u_{23} = 1$

again, $l_{31} = 3,\quad 5l_{31}+l_{32}u_{22} = 1$ and $l_{31}+l_{32}u_{23}+u_{33} = 4$

$$l_{32} = \frac{1-15}{-9} = \frac{14}{9};\quad u_{33} = 4 - 3 - \frac{14}{9} = -\frac{5}{9}$$

$LUX = B$ *implies* $LY = B$ where $UX = Y$.

$LY = B$, *gives*,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$y_1 = 14$, $2\ y_1 + y_2 = 13$, $3\ y_1 + \frac{14}{9}\ y_2 + y_3 = 17$

$$y_1 = 14, \quad y_2 = -15, \quad y_3 = -\frac{5}{3}$$

$UX = Y$ gives $\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{pmatrix}$

$$x + 5y + z = 14$$

$$-9y + z = -15$$

$$-\frac{5}{9}z = -\frac{5}{3}$$

$\therefore \quad x = 1, \ y = 2, \ z = 3$

Jacobi Method Of Iteration or Gauss – Jacobi Method

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3 \quad \ldots\ldots\ldots\ldots (1)$$

Let us assume $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Then, iterative method can be used for the system (1). Solve for *x, y, z* (whose coefficients are the larger values) in terms of the other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \;\ldots\ldots\ldots.\; (2)$$

If x^0, y^0, z^0 are the initial values of *x, y, z* respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(0)} - c_2 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(0)} - b_3 y^{(0)}) \;\ldots\ldots\ldots\; (3)$$

Again using these values $x^{(2)}, y^{(2)}, z^{(2)}$ in (2), we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)}) \;\ldots.(4)$$

Proceeding in the same way, if the rth iterates are $x^{(r)}, y^{(r)}, z^{(r)}$, the iteration scheme reduces to

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r)} - b_3 y^{(r)}) \;\ldots..(5)$$

The procedure is continued till the convergence is assured (correct to required decimals).

Note 1: To get the (r+1)th iterates, we use the values of the rth iterates in the scheme (5).

Note 2: In the absence of the initial values of x, y, z we take, usually, (0, 0, 0) as the initial estimate.

Gauss – Seidel Method of Iteration:

This is only a refinement of Guass – Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values y^0, z^0 for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of x^0 as in Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)} and\ y^{(1)}$, use $x^{(1)}$ for x $and\ y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss – seidel method is very fast when compared to Gauss – Jacobi method. The rate of convergence in Gauss – Seidel method is roughly two times than that of Gauss – Jacobi method. As we saw the sufficient condition already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest* coefficient is greater than the sum of the absolute values of all the remaining coefficients. (The largest coefficients must be the coefficients for different unknowns).

Note 1: For all systems of equations, this method will not work (since convergence is not assured). It converges only for special systems equations.

Note 2: Iteration method is self – correcting method. That is, any error made in computation, is corrected in the subsequent iterations.

Note 3: The iteration is stopped when the values of *x, y, z* start repeating with the required degree of accuracy.

Example 1. Solve the following system by Gauss – Jacobi and Gauss – Seidel methods:

10x-5y-2z = 3 ; 4x-10y+3z = -3 ; x+6y+10z = -3.

Solution: Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$ is diagonally dominant, since $|10| > |-5| + |-2|$,

$|-10| > |4| + |3|$,

$|10| > |1| + |6|$

Gauss – Jacobi method, solving for *x, y, z* we have

$x = \frac{1}{10}(3+5y+2z)$ ………………. (1)

$y = \frac{1}{10}(3+4x+3z)$ ………………. (2)

$z = \frac{1}{10}(-3-x-6y)$ ………………. (3)

First iteration: Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}(3 + 5(0) + 2(0)) = 0.3$$

$$y^{(1)} = \frac{1}{10}(3 + 4(0) + 3(0)) = 0.3$$

$$z^{(1)} = \frac{1}{10}(-3 - (0) - 6(0)) = -0.3$$

Second iteration: using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}(3 + 5(0.3) + 2(-0.3)) = 0.39$$

$$y^{(2)} = \frac{1}{10}(3 + 4(0.3) + 3(-0.3)) = 0.33$$ $$z^{(2)} = \frac{1}{10}(-3 - (0.3) - 6(0.3)) = -0.51$$

Third iteration: using these values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3), we get,

$$x^{(3)} = \frac{1}{10}(3 + 5(0.33) + 2(-0.51)) = 0.363$$

$$y^{(3)} = \frac{1}{10}(3 + 4(0.39) + 3(-0.51)) = 0.303$$

$$z^{(3)} = \frac{1}{10}(-3 - (0.39) - 6(0.33)) = -0.537$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10}(3 + 5(0.303) + 2(-0.537)) = 0.3441$$

$$y^{(4)} = \frac{1}{10}(3 + 4(0.363) + 3(-0.537)) = 0.2841$$

$$z^{(4)} = \frac{1}{10}(-3 - (0.363) - 6(0.303)) = -0.5181$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10}(3 + 5(0.2841) + 2(-0.5181)) = 0.33843$$

$$y^{(5)} = \frac{1}{10}(3 + 4(0.3441) + 3(-0.5181)) = 0.2822$$

$z^{(5)} = \frac{1}{10}$ (-3- (0.3441) – 6(0.2841)) = - 0.50487

Sixth iteration:

$x^{(6)} = \frac{1}{10}$ (3 + 5(0.2822) + 2 (-0.50487)) = 0.340126

$y^{(6)} = \frac{1}{10}$ (3 + 4(0.33843) + 3(-0.50487)) = 0.283911

$z^{(6)} = \frac{1}{10}$ (-3- (0.33843) – 6(0.2822)) = - 0.503163

Seventh iteration:

$x^{(7)} = \frac{1}{10}$ (3 + 5(0.283911) +2(-0.503163)) =0.3413229

$y^{(7)} = \frac{1}{10}$ (3 + 4(0.340126) +3(-0.503163)) = 0.2851015

$z^{(7)} = \frac{1}{10}$ (-3- (0.340126) – 6(0.283911)) = - 0.5043592

Eighth iteration:

$x^{(8)} = \frac{1}{10}$ (3 + 5(0.2851015) +2 (-0.5043592))

=0.34167891

$y^{(8)} = \frac{1}{10}$ (3 + 4(0.3413229) + 3(-0.5043592))

= 0.2852214

$z^{(8)} = \frac{1}{10}$(-3-(0.3413229)– 6(0.2851015))

= - 0.50519319

Ninth iteration:

$x^{(9)} = \frac{1}{10}$ (3 + 5(0.2852214) + 2 (-0.50519319))

= 0.341572062

$y^{(9)} = \frac{1}{10}(3 + 4(0.34167891) + 3(-0.50519319))$

$= 0.285113607$

$z^{(9)} = \frac{1}{10}(-3 - (0.34167891) - 6(0.2852214)) = -0.505300731$

Hence, correct to 3 decimal places, the values are

$x = 0.342, \quad y = 0.285, \quad z = -0.505$

Gauss – Seidel method: Initial values : $y = 0$, $z = 0$.

First iteration: $x^{(1)} = \frac{1}{10}(3 + 5(0) + 2(0)) = 0.3$

$y^{(1)} = \frac{1}{10}(3 + 4(0.3) + 3(0)) = 0.42$

$z^{(1)} = \frac{1}{10}(-3 - (0.3) - 6(0.42)) = -0.582$

Second iteration:

$x^{(2)} = \frac{1}{10}(3 + 5(0.42) + 2(-0.582)) = 0.3936$

$y^{(2)} = \frac{1}{10}(3 + 4(0.3936) + 3(-0.582)) = 0.28284$

$z^{(2)} = \frac{1}{10}(-3 - (0.3936) - 6(0.28284)) = -0.509064$

Third iteration:

$x^{(3)} = \frac{1}{10}(3 + 5(0.28284) + 2(-0.509064)) = 0.3396072$ $y^{(3)} = \frac{1}{10}(3 + 4(0.3396072) + 3(-0.509064)) = 0.28312368$

$z^{(3)} = \frac{1}{10}(-3 - (0.3396072) - 6(0.28312368))$

$= -0.503834928$

Fourth iteration:

$x^{(4)} = \frac{1}{10}(3 + 5(0.28312368) + 2(-0.503834928))$

$= 0.34079485$

$y^{(4)} = \frac{1}{10}$ (3 + 4(0.34079485) + 3(-0.503834928))

= 0.285167464

$z^{(4)} = \frac{1}{10}$ (-3- (0.34079485) – 6(0.285167464))

= - 0.50517996

Fifth iteration:

$x^{(5)} = \frac{1}{10}$ (3 + 5(0.285167464) + 2(-0.50517996))

= 0.34155477

$y^{(5)} = \frac{1}{10}$ (3 + 4(0.34155477) + 3(-0.50517996))

= 0.28506792

$z^{(5)} = \frac{1}{10}$ (-3- (0.34155477) – 6(0.28506792))

= - 0.505196229

Sixth iteration:

$x^{(6)} = \frac{1}{10}$ (3 + 5(0.28506792) + 2(-0.505196229))

= 0.341494714

$y^{(6)} = \frac{1}{10}$ (3 + 4(0.341494714) + 3(-0.505196229))

= 0.285039017

$z^{(6)} = \frac{1}{10}$ (-3- (0.341494714) – 6(0.28506792))

= - 0.5051728

Seventh iteration:

$x^{(7)} = \frac{1}{10}$ (3 + 5(0.285039017) + 2(-0.5051728))

= 0.3414849

$y^{(7)} = \frac{1}{10}$ (3 + 4(0.3414849) + 3(- 0.5051728))

= 0.28504212

$z^{(7)} = \frac{1}{10}$ (-3- (0.3414849) – 6(0.28504212))

= - 0.5051737

The values at each iteration by both methods are tabulated below:

Iteration	Gauss - jacobi method			Gauss – seidel method		
	x	*y*	*z*	*x*	*y*	*z*
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-0.51	0.3936	0.2828	-0.5090
3	0.363	0.303	-0.537	0.3396	0.2831	-0.5038
4	0.3441	0.2841	-0.5181	0.3407	0.2851	-0.5051
5	0.3384	0.2822	-0.5048	0.3415	0.2850	-0.5051
6	0.3401	0.2839	-0.5031	0.3414	0.2850	-0.5051
7	0.3413	0.2851	-0.5043	0.3414	0.2850	-0.5051
8	0.3416	0.2852	-0.5051			
9	0.3411	0.2851	-0.5053			

The values correct to 3 decimal places are

$$x = 0.342, \; y = 0.285, \; z = -0.505$$

Example 2. Solve the following system of equations by using Gauss – jacobi and Gauss – Seidel methods (correct to 3 decimal places):

$$8x - 3y + 3z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35.$$

Solution: since the diagonal elements are dominant in the coefficient matrix, we write *x*, *y*, *z* as follows

$$x = \frac{1}{8}(20 + 3y - 2z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$y = \frac{1}{11}(33 + 4x + z)$ ………………. (2)

$z = \frac{1}{12}(35 - 6x - 3y)$ ………………. (3)

Gauss – Jacobi method:

First iteration: Let the initial values be $x = 0$, $y = 0$, $z = 0$

Using the values $x = 0$, $y = 0$, $z = 0$ in (1), (2), (3) we get,

$$x^{(1)} = \frac{1}{8}(20 + 3(0) - 2(0)) = 2.5$$

$$y^{(1)} = \frac{1}{11}(33 + 4(0) + (0)) = 3.0$$

$$z^{(1)} = \frac{1}{12}(35 - 6(0) - 3(0)) = 2.916666$$

Second iteration: using these values of $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ in (1), (2), (3), we get,

$x^{(2)} = \frac{1}{8}(20 + 3(3.0) - 2(2.916666)) = 2.895833$

$y^{(2)} = \frac{1}{11}(33 + 4(2.5) + (2.916666)) = 2.356060$

$z^{(2)} = \frac{1}{12}(35 - 6(2.5) - 3(3.0)) = 0.916666$

Third iteration:

$x^{(3)} = \frac{1}{8}(20 + 3(2.356060) - 2(0.916666)) = 3.154356$

$y^{(3)} = \frac{1}{11}(33 + 4(2.895833) + (0.916666)) = 2.030303$ $z^{(3)} = \frac{1}{12}(35 - 6(2.895833) - 3(2.356060)) = 0.879735$

Fourth iteration:

$x^{(4)} = \frac{1}{8}(20 + 3(2.030303) - 2(0.879735)) = 3.041430$

$y^{(4)} = \frac{1}{11}(33 + 4(3.154356) + (0.879735)) = 2.932937$

$z^{(4)} = \frac{1}{12}(35 - 6(3.154356) - 3(2.030303)) = 0.831913$

Fifth iteration:

$x^{(5)} = \frac{1}{8}(20 + 3(2.932937) - 2(0.831913)) = 3.016873$

$y^{(5)} = \frac{1}{11}(33 + 4(3.041430) + (0.831913)) = 1.969654$

$z^{(5)} = \frac{1}{12}(35 - 6(3.041430) - 3(2.932937)) = 0.912717$

Sixth iteration:

$x^{(6)} = \frac{1}{8}(20 + 3(1.969654) - 2(0.912717)) = 3.010441$

$y^{(6)} = \frac{1}{11}(33 + 4(3.016873) + (0.912717)) = 1.985930$

$z^{(6)} = \frac{1}{12}(35 - 6(3.016873) - 3(1.969654)) = 0.915817$

Seventh iteration:

$x^{(7)} = \frac{1}{8}(20 + 3(1.985930) - 2(0.915817)) = 3.015770$

$y^{(7)} = \frac{1}{11}(33 + 4(3.010441) + (0.915817)) = 1.988550$

$z^{(7)} = \frac{1}{12}(35 - 6(3.010441) - 3(1.985930)) = 0.914964$

Eighth iteration:

$x^{(8)} = \frac{1}{8}(20 + 3(1.988550) - 2(0.914964)) = 3.016946$

$y^{(8)} = \frac{1}{11}(33 + 4(3.015770) + (0.914964)) = 1.986535$

$z^{(8)} = \frac{1}{12}(35 - 6(3.015770) - 3(1.988550)) = 0.911644$

Ninth iteration:

$x^{(9)} = \frac{1}{8}(20 + 3(1.986535) - 2(0.911696)) = 3.017039$

$y^{(9)} = \frac{1}{11}$ (33 + 4(3.016946) + (0.911696)) = 1.985805

$z^{(9)} = \frac{1}{12}$ (35 – 6(3.016946) – 3(1.986535)) = 0.911560

Tenth iteration:

$x^{(9)} = \frac{1}{8}$ (20 + 3(1.985805) – 2(0.911560)) = 3.016786

$y^{(9)} = \frac{1}{11}$ (33 + 4(3.017039) + (0.911560)) = 1.985764

$z^{(9)} = \frac{1}{12}$ (35 – 6(3.017039) – 3(1.985805)) = 0.911696

In 8^{th} , 9^{th} and 10^{th} iterations the values of x, y, z are same correct to 3 decimal places. Hence, we stop at this level.

Gauss – Seidel method:

We take the initial values are $y = 0$, $z = 0$ and use equations (1)

First iteration:

$x^{(1)} = \frac{1}{8}$ (20 + 3(0) – 2(0)) = 2.5

$y^{(1)} = \frac{1}{11}$ (33 + 4(2.5) + (0)) = 2.090909

$z^{(1)} = \frac{1}{12}$ (35 – 6(2.5) – 3(2.090909)) = 1.143939

Second iteration:

$x^{(2)} = \frac{1}{8}$ (20 + 3(2.090909) – 2(1.143939)) = 2.998106

$y^{(2)} = \frac{1}{11}$ (33 + 4(2.998106) + (1.143939)) = 2.013774$z^{(2)} = \frac{1}{12}$ (35 – 6(2.998106) – 3(2.013774)) = 0.914170

Third iteration:

$x^{(3)} = \frac{1}{8}$ (20 + 3(2.013774) – 2(0.914170)) = 3.026623

$y^{(3)} = \frac{1}{11}$ (33 + 4(3.026623) + (0.914170)) = 1.982516 $z^{(3)} = \frac{1}{12}$ (35 – 6(3.026623) – 3(1.982516)) = 0.907726

Fourth iteration:

$x^{(4)} = \frac{1}{8}$ (20 + 3(1.982516) – 2(0.907726)) = 3.016512

$y^{(4)} = \frac{1}{11}$ (33 + 4(3.026623) + (0.907726)) = 1.985607

$z^{(4)} = \frac{1}{12}$ (35 – 6(3.016512) – 3(1.985607)) = 0.912009

Fifth iteration:

$x^{(5)} = \frac{1}{8}$ (20 + 3(1.985607) – 2(0.912009)) = 3.016600

$y^{(5)} = \frac{1}{11}$ (33 + 4(3.016600) + (0.912009)) = 1.985964

$z^{(5)} = \frac{1}{12}$ (35 – 6(3.016600) – 3(1.985964)) = 0.911876

Sixth iteration:

$x^{(6)} = \frac{1}{8}$ (20 + 3(1.985964) – 2(0.911876)) = 3.016767

$y^{(6)} = \frac{1}{11}$ (33 + 4(3.016767) + (0.911876)) = 1.985892

$z^{(6)} = \frac{1}{12}$ (35 – 6(3.016767) – 3(1.985892)) = 0.911810

(The values of *x*, *y*, *z* got by jacobi method correct to 3 decimal places are got even in the 6^{th} iteration by Gauss – seidel method.)

Seventh iteration:

$x^{(7)} = \frac{1}{8}$ (20 + 3(1.985892) – 2(0.911810)) = 3.016757

$y^{(7)} = \frac{1}{11}$ (33 + 4(3.016757) + (0.911810)) = 1.985889

$z^{(7)} = \frac{1}{12}$ (*35* – 6(3.016757) – *3*(1.985889)) = 0.911816

Since the seventh and eighth iterations give the same values for *x, y, z* correct to 4 decimal places, we stop here.

∴ x = 3.0168, y =1.9859, z = 0.9118

The values of *x, y, z* by both methods at each iteration are tabulated below:

Iteration	Gauss – jacobi method			Gauss – seidel method		
	x	*y*	*z*	*x*	*y*	*z*
1	2.5	3.0	2.9166	2.5	2.0909	1.1439
2	2.8958	2.3560	0.9166	2.9981	2.0137	0.9141
3	3.1543	2.0303	0.8797	3.0266	1.9825	0.9077
4	3.0414	1.9329	0.8319	3.0165	1.9856	0.9120
5	3.0168	1.9696	0.9127	3.0166	1.9859	0.9118
6	3.0104	1.9859	0.9158	3.0167	1.9858	0.9118
7	3.0157	1.9885	0.9149	3.0167	1.9858	0.9118
8	3.0169	1.9865	0.9116			
9	3.0170	1.9858	0.9115			
10	3.0167	1.9857	0.9116			

This shows that the convergence is rapid in Gauss – seidel method when compared to Gauss – Jacobi method. We see that 10 iterations are necessary in jacobi method to get the same accuracy as got by 7 iterations in Gauss – Seidel method.

Example 3. Since the diagonal elements in the coefficient matrix are not dominant, we arrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

28x + 4y – z = 32, x + 3y + 10z = 24, 2x +17y + 4z = 35

Solution: Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

Hence, $x = \frac{1}{28}(32 - 4y + z)$ ………………. (1)

$y = \frac{1}{17}(35 - 2x - 4z)$ ………………. (2)

$z = \frac{1}{10}(24 - x - 3y)$ ………………. (3)

setting $y = 0$, $z = 0$, we get

First iteration:

$$x^{(1)} = \frac{1}{28}(32 - 4(0) + 0) = 1.1429$$

$$y^{(1)} = \frac{1}{17}(35 - 2(1.1429) - 4(0)) = 1.9244$$

$$z^{(1)} = \frac{1}{10}(24 - 1.1429 - 3(1.9244)) = 1.8084$$

Second iteration:

$$x^{(2)} = \frac{1}{28}(32 - 4(1.9244) + 1.8084) = 0.9325$$

$$y^{(2)} = \frac{1}{17}(35 - 2(0.9325) - 4(1.8084)) = 1.5236$$

$$z^{(2)} = \frac{1}{10}(24 - 0.9325 - 3(1.5236)) = 1.8497$$

Third iteration:

$$x^{(3)} = \frac{1}{28}(32 - 4(1.5236) + 1.8497) = 0.9913$$

$$y^{(3)} = \frac{1}{17}(35 - 2(0.9913) - 4(1.8497)) = 1.5070$$

$$z^{(3)} = \frac{1}{10}(24 - 0.9913 - 3(1.5070)) = 1.8488$$

Fourth iteration:

$$x^{(4)} = \frac{1}{28}(32 - 4(1.5070) + 1.8488) = 0.9936$$

$$y^{(4)} = \frac{1}{17}(35 - 2(0.9936) - 4(1.8488)) = 1.5069$$

$$z^{(4)} = \frac{1}{10}(24 - 0.9936 - 3(1.5069)) = 1.8486$$

Fifth iteration:

$$x^{(5)} = \frac{1}{28}(32 - 4(1.5069) + 1.8486) = 0.9936$$

$$y^{(5)} = \frac{1}{17}(35 - 2(0.9936) - 4(1.8486)) = 1.5069$$

$$z^{(5)} = \frac{1}{10}(24 - 0.9936 - 3(1.5069)) = 1.8486$$

Since the values of x, y, z in the 4th and 5th iterations are same, we stop the process here.

Hence, $x = 0.9936$, $y = 0.5069$ and $z = 1.8486$

Numerical Integration

We know that $\int_a^b f(x)dx$ represents the area between y = f(x), x – axis and the ordinates x = a and x = b. This integration is possible only if the f(x) is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows: Given as set of (n+1) paired values (x_i,y_i), i = 0,1,2,…,n of the function y=f(x), where f(x) is not known explicitly, it is required to compute $\int_{x_0}^{x_n} y\,dx$.

As we did in the case of interpolation or numerical differentiation, we replace f(x) by an interpolating polynomial $P_n(x)$ and obtain $\int_{x_0}^{x_n} P_n(x)dx$ which is approximately taken as the value for $\int_{x_0}^{x_n} f(x)dx$.

A general quadrature formula for equidistant ordinates (or Newton – cote's formula)

For equally spaced intervals, we have Newton's forward difference formula as

$y(x)=y(x_0+uh)=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\ldots$ ……(1)

Now, instead of $f(x)$, we will replace it by this interpolating formula of Newton.

Here, $u = \frac{x-x_0}{h}$ where h is interval of differencing.

Since $x_n = x_0 + nh$, and $u = \frac{x-x_0}{h}$ we have $\frac{x-x_0}{h} = n = u$.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$$= \int_{x_0}^{x_n+nh} P_n(x)\, dx$$ where $P_n(x)$ is interpolating polynomial

$$=\int_0^n \left(y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right)(hdu)$$

Since $dx = hdu$, and when $x = x_0$, $u = 0$ and when $x = x_0+nh$, $u = n$.

$$= h\left[y_0(u)+ \frac{u^2}{2}\Delta y_0 + \frac{\left(\frac{u^3}{3}-\frac{u^2}{2}\right)}{2}\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$\int_{x_0}^{x_n} f(x)dx = h\left[ny_0+ \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right] \quad \ldots(2)$$

The equation (2), called Newton-cote's quadrature formula is a general quadrature formula. Giving various values for n, we get a number of special formula.

Trapezoidal rule

By putting n = 1, in the quadrature formula (i.e there are only two paired values and interpolating polynomial is linear).

$\int_{x_0}^{x_n+nh} f(x)dx = h\left[1.y_0 + \frac{1}{2}\Delta y_0\right]$ since other differences do not exist if n = 1.

$$=\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$= \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_n+2h} f(x)dx + \ldots\ldots\ldots +$

$$\int_{x_0+(n-1)h}^{x_n+nh} f(x)\,dx$$

$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots.+y_{n-1})\right]$

$= \frac{h}{2}$ [(sum of the first and the last ordinates) + 2(sum of the remaining ordinates)]

This is known as Trapezoidal Rule and the error in the trapezoidal rule is of the order h^2.

Note

Though this method is very simple for calculation purposes of numerical integration; the error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.

Truncating error on Trapezoidal rule

In the neighborhood of $x = x_0$, we can expand $y = f(x_0)$ by Taylor series in power of $x - x_0$. That is,

$y(x) = y_0 + (x-x_0)\, y'_0 + (x-x_0)2y''_0 + \ldots\ldots +$

where $y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots\ \ldots(1)$ where $y_0' = [y'(x)]\ x = x_0$

$\int_{x_0}^{x_1} y\,dx = \int_{x0}^{x1}\left[y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots..\ dx\right]$

$= \left[y_0 x + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0'' + \cdots\right]_{x_0}^{x_1}$

$= y_0(x_1 - x_0) + \frac{(x-x_0)^2}{2!} y_0' + \frac{(x-x_0)^3}{3!} y_0'' + \ldots..$

$= h\,y_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \ldots. \qquad \ldots\ldots\ (2)$

If h is the equal interval length.

Also $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)$ = area of the first trapezium = A_0…..(3)

Putting $x=x_1$ in (1)

$$y(x_1) = y_1 = y_0 + \frac{(x_1-x_0)}{1!}y_0' + \frac{(x_2-x_0)^2}{2!}y_0'' + \ldots$$

i.e., $$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \ldots \qquad \ldots\ldots\ldots \; (4)$$

$$A_0 = \frac{h}{2}\left[y_0 + y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots\right]$$

Using (4) in (3).

$$= h\,y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2*2!}y_0'' + \ldots\ldots\ldots\ldots$$

Subtracting A_0 value from (2),

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left(\frac{1}{3!} - \frac{1}{2*2!}\right) + \ldots\ldots\ldots$$

$$= -\frac{1}{12}h^3 y_0'' + \ldots\ldots$$

Therefore the error in the first interval (x_0, x_1) is $-\frac{1}{12}h^3 y_0''$ (neglecting other terms)

Similarly the error in the ith interval = $-\frac{1}{12}h^3 y_{i-1}''$

Therefore, the total cumulative error (approx.),

$$E = -\frac{1}{12}h^3(y_0'' + y_1'' + y_2'' + \ldots\ldots + y_{n-1}'')$$

$|E| < \frac{nh^3}{12}(M)$ where M is the maximum value of $|y_0''|$, $|y_1''|$, $|y_2''|$, …..

$< \frac{(b-a)h^2}{12}(M)$ if the interval is (a,b) and

$$h = \frac{b-a}{n}$$

Hence, the error in the trapezoidal rule is of the order h^2.

Simpson's one-third rule

Setting n = 2 in Newton- cote's quadrature formula, we have $\int_{x_0}^{x_n} f(x)dx = h \left(2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right)$ (since other terms vanish)

$$= \frac{h}{3}(y_2 + y_1 + y_0)$$

Similarly, $\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$

If n is an even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all the integrals, if n is an even positive integer, that is, the number of ordinates y_0, y_1, $y_2 \ldots . y_n$ is odd, we have

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots\ldots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \ldots .) + \ldots\ldots. + 4(y_1 + y_3 + \ldots ..)\right]$$

$= \frac{h}{3}$ [(sum of the first and the last ordinates) + 2(sum of remaining odd ordinates) + 2(sum of even ordinates)]

Note. Though y_2 has suffix even, it is third ordinate (odd).

Simpson's three-eighths rule

Putting n = 3 in Newton – cotes formula

$$= \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+ y_2+y_4+y_5+\ldots+y_{n-1})+2(y_3+ y_6+y_9+\ldots+y_n)\right] \quad \ldots..(2)$$

Equation (2) is called *Simpson's three – eighths rule* which is applicable only when n is a multiple of 3.Truncation error in simpson's rule is of the order h

Note 1: In trapezoidal rule , y(x) is a linear function of x. The rule is the simplest one but it is least accurate.

Note 2: In simpson's one – third rule, y(x) is a polynomial of degree two. To apply this rule n, the number of intervals must be even. That is, the number of ordinates must be odd.

Note 3: In weddle's rule, y(x) is a polynomial of degree six and this rule is applicable only if n, the number of intervals, is a multiple of six. A minimum number of 7 ordinates is necessary.

Truncation error in simpson's rule

By taylor expansion of y=f(x) in the neighborhood of $x = x_0$ we get,

$$y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \ldots \quad \ldots\ldots..(1)$$

$$\int_{x_0}^{x_2} y\,dx = \int_{x_0}^{x_2}\left[y_0 + \frac{(x-x_0)}{1!}\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \ldots\right]dx$$

$$= \left[y_0\,x + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0'' + \cdots\right]_{x_0}^{x_2}$$

$$= y_0\,(x_2 - x_0) + \frac{(x_2-x_0)^2}{2!}y_0' + \frac{(x_2-x_0)^3}{3!}y_0'' + \ldots..$$

$$=2h\,y_0 + \frac{4h^2}{2!}\,y_0' + \frac{8h^3}{3!}\,y_0''+\frac{16h^4}{4!}y_0'''+\ldots$$

$$=2h\,y_0 + 2h^2\,y_0' + \frac{4}{3}\,h^3y_0''+\frac{2h^4}{3}y_0'''+\frac{4h^5}{15}\,y_0''''+\ldots \quad \ldots..(2)$$

$$A_1 = \text{area} = \int_{x_0}^{x_2} y\,dx = \frac{h}{3}\,(y_2+4y_1+y_0)$$

by simpson's rule ……….(3)

Putting $x = x_1$ in (1)

$$y_1 = y_0 + \frac{(x_1-x_0)}{1!}y_0' + \frac{(x_2-x_0)^2}{2!}y_0'' + \ldots$$

$$= y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \ldots \qquad \ldots\ldots\ldots (4)$$

Putting $x = x_2$ in (1)

$$y_1 = y_0 + \frac{2h}{1!}y_0' + \frac{4h^2}{2!}y_0'' + \ldots \qquad \ldots\ldots\ldots (5)$$

substituting (4) in (5) , in (3),

$A_1 = 2hy_0 + 2h^2 y_0' + \frac{4}{3}h^3y_0'' + \frac{2h^4}{3}y_0''' + \frac{5h^5}{18}y_0'''' + \ldots$ …..(6) equations (2) – (6) give

$$\int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0'''' + \ldots$$

$$= -\frac{h^5}{90} y_0'''' + \ldots$$

Leaving the remaining terms involving h^6 and higher powers of h, principal part of the error in (x_0,x_2) is

$$= -\frac{h^5}{90} y_0'''' + \ldots$$

Similarly the principal part of the error in (x_2,x_4) is

$$= -\frac{h^5}{90} y_2'''' \text{ and so far each interval.}$$

Hence the total error in all the intervals is given by

$$E = = -\frac{h^5}{90}(y_0'''' + y_2'''' + \ldots)$$

$|E| < \frac{nh^5}{90}$ (M) where M is the numerically greater value of y_0'''', y_2'''',…y_{2n-2}

since (x_{2n},x_{2n}) is the last paired value because we require odd number of ordinates to apply simpson's one – third rule. (i.e., 2n intervals).

If the interval is(a,b) then b – a = h(2n). using this, $|E| < \frac{(b-1)h^4}{180}$ (M).

Hence, the error in simpson's one – third rule is of the order h

Example 1

Evaluate $\int_{-3}^{3} x^4$dx by using (1) trapezoidal rule (2)simpson's rule. Verify your results by actual integration.

Solution

Here $y(x) = x^4$. Interval length($b - a$) = 6. So, we divide 6 equal intervals with $h = \frac{6}{6} =$ 1.

We form below the table

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

(i) By trapezoidal rule:

$\int_{-3}^{3} y\,dx = \frac{h}{2}$ [(sum of the first and the last ordinates) +

2(sum of the remaining ordinates)]

$= \frac{1}{2}$ [(81+81)+2(16+1+0+1+16)]

=115

(ii) By Simpson's one - third rule (since number of ordinates is odd)**:**

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 2(1+1) + 4(16+0+16)]

= 98.

(iii) Since n = 6, (multiple of three), we can also use **simpson's three - eighths rule.** By this rule,

$\int_{-3}^{3} y\,dx = \frac{1}{3}$ [(81+81) + 3(16+1+1+16) + 2(0)]

= 99

(iv) By actual integration,

$\int_{-3}^{3} x^4 dx = 2* \left[\frac{x^5}{5} \right]_0^3 = \frac{2*243}{5} = 97.2$

From the results obtained by various methods, we see that simpson's rule gives better result than trapezoidal rule

x	0	0.2	0.4	0.6	0.8	1.0
$y=1/1+x^2$	1	0.96154	0.86207	0.73529	0.60976	0.50

Example 2

Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with h = 0.2. hence obtain an approximate value of π. Can you use other formulae in this case.

Solution.

Let $y(x) = \frac{1}{1+x^2}$

Interval is (1-0) = 1. Since the value of y are calculated as points taking h =0.2

(i) By Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2} \left[(y_0+y_n) + 2(y_1+y_2+ y_3 + \cdots .+y_{n-1}) \right]$$

$=\frac{0.2}{2}[(1+0.5)+2(0.96154+0.8620+0.73529+0.60976)]$

= (0.1)[1.5+6.33732]

= 0.783732

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4}$$

$\frac{\pi}{4} \approx 0.783732$

$\pi \approx 3.13493$ (approximately).

In this case, we cannot use simpson's rule (both) and weddle's rule. (since number of intervals is 5).

Example 3

From the following table, find the areas bounded by the curve and the x-axis from x = 7.47 to x = 7.52.

x	7.47	7.48	7.49	7.50	7.51	7.52
y=f(x)	1.93	1.95	1.98	2.01	2.03	2.06

Solution

Since only 6 ordinates (n = 5) are given, we cannot use Simpson's rule. So, we will use trapezoidal rule.

Area=$\int_{7.47}^{7.52} f(x)\, dx$

$=\frac{0.01}{2}$[(1.93+2.06)+2(1.95+1.98+2.01+2.3)]

= 0.09965.

Example 4

Evaluate $\int_0^6 \frac{dx}{1+x}$, using (i) Trapezoidal rule (ii) Simpson's rule (both) . Also, check up by direct integration.

Solution

Take the number of intervals as 6.

$h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
y	1	0.5	1/3	1/4	1/5	1/6	1/7

i) By Trapezoidal rule

$$\left(\frac{1}{7}\right) = \frac{1}{2}\left[\left(1+\frac{1}{7}\right)+2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\right]$$

$= 2.02142857$

ii) By Simpsons's one – third rule,

$$I = \frac{1}{3}\left[\left(1+\frac{1}{7}\right)+2\left(\frac{1}{3}+\frac{1}{5}\right)+4\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)\right]$$

$$= \frac{1}{3}\left(1+\frac{1}{7}+\frac{16}{15}+\frac{22}{6}\right) = 1.95873016$$

iii) By Simpsons's three - eighths rule,

$$I = \left[\frac{3*1}{8}\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right]$$

$= 1.96607143$

iv) By actual integration,

$$\int_0^6 \frac{1}{1+x} = [\log(1+x)]_0^6 = \log_e 7 = 1.94591015$$

Example 5

By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x dx$ by trapezoidal and Simpson's rule. Verify your answer with integration.

x	0	π/10	2π/10	3π/10	4π/10	5π/10
y=sinx	0	0.3090	0.58878	0.8090	0.9511	1.0
x	6π/10	7π/10	8π/10	9π/10	π	
y=sinx	0.9511	0.8090	0.578	0.3090	0	

Solution

Range = π - 0 = π

Hence h = $\frac{\pi}{10}$

We tabulate below the values of y at different x's

Note that the values are symmetrical about x = $\frac{\pi}{2}$

(i) By Trapezoidal rule,

I = $\frac{\pi}{20}$ [(0 + 0) + 2(0.3090+0.5878+0.8090+

0.9511+1.0+0.9511+0.8090+0.5878+0.3090)]

= 1.9843 nearly.

(ii) By Simpsons's one – third rule,

I = $\frac{1}{3}\left(\frac{\pi}{10}\right)$ [(0+0)+2(0.5878+0.9511+0.5878+0.9511) +

4(0.3090+0.8090+1+0.3090+0.8090)]

= 2.00091

Note: We cannot use Simpson's three eighth's rule.

(iii) By actual integration, I = $(-\cos x)_0^{\pi}$ = 2.

Hence, Simpson's rule is more accurate than the trapezoidal rule.

Example 6

Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Romberg's method. Hence obtain an approximate value of π.

Solution

To use the method, we shall give various values of h and evaluate the integral.

By taking h=0.5, tabulate the values of y=1/1+x^2

x: 0 0.5 1

y: 1 0.80 0.50

By Trapezoidal rule

Therefore I = 0.5/2[1.5+2(0.8)] =0.775

By taking h=0.25, we have the table

x:	0	0.25	0.5	0.75	1
y:	1	0.9412	0.8	0.64	0.5

Therefore I = 0.25/2[1.5+2(0.9412+0.8+0.64)]=0.78280

By taking h=0.125, we have the table

x:	0	0.125	0.250	0.375	0.5	0.625	0.75	0.875	1
y:	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

Therefore

I = 0.125/2[1.5+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)]= 0.78475

The different values got by Trapezoidal rule for various h's are

0.775 0.78280 0.78280

Applying the formula I = I_2+ 1/3[I_2- I_1], we will get two important values, namely 0.7828+1/3[0.7828-0.7750]=0.7854

0.78475+1/3[0.78475-0.7750] =0.7854

As these two values happen to be equal, we finalise the result.

Hence I = 0.7854.

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1$$

$$= \frac{\pi}{4} = 0.7854$$

$$= \pi = 3.1416$$

POSSIBLE QUESTIONS:

1. Applying Gauss Elimination method to find the solution of the following system
   10x+y+z = 12;2x+10y+z = 13;x+y+5z = 7
2. By the Gauss Jordan method solve the following equations.
   5x-2y+z=4; 7x+y-5z=8; 3x+7y+4z=10
3. By the Method of Triangularization solve the following system
   5x-2y+z =4; 7x+y-5z = 8; 3x+7y+4z =10
4. Solve the system of equation by Gauss Jacobi method.
   5x-2y+z= -4; x+6y-2z= -1; 3x+y+5z=13
5. Solve the system of equation by Gauss Seidel method
   10x-5y-2z =3; 4x-10y+3z =-3; x+6y+10z =-3
6. Solve the system of equations by Gauss Seidel method correct to 3 decimal places.
   8x-3y+2z=20; 4x+11y-z=33; 6x+3y-12z=35
7. Applying Gauss Jacobi method to find the solution of the following system
   10x+2y+z=9; 2x+20y-2z= - 44; -2x+3y+10z=22
8. Solve the following system by Relaxation method.
   10x-2y-2z =6; -x+10y+-2z =7; -x-y+10z =8
9. Solve the following system of equations by relaxation method
   10x-2y+z=12, x+9y-z = 10, 2z-y+11z=20.
10. By dividing the range into 10 equal parts evaluate $\int_0^{\pi} sinx dx$ by Trapezoidal & Simpson's rule. Verify your answer with integration.
11. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule.
12. Evaluate $\int_{-3}^3 x^4 \, dx$ using Simpson's rule.