KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

		Semester – VI
		L T P C
17PHU602	**MATHEMATICS-II**	**4 - - 4**

Scope: Mathematics is an important tool for the study of physics. Basic mathematical tools like Fourier analysis, Fourier transform, Laplace Transform, differential equations etc. are explained in this paper.

Objective: This paper gives the clear idea about Fourier series & transforms, Laplace transforms, differential equations and numerical methods. The integration involved is very useful in the application of mathematics.

UNIT I

Ordinary Differential Equations: Equations of First Order and of Degree Higher than one – Solvable for p, x, y – Clairaut's Equation – Simultaneous Differential Equations with constant coefficients of the form i)$f_1D(x) + g_1D(y) = \phi_1(t)$ ii) $f_2D(x) + g_2D(y) = \phi_2(t)$, where$f_1, g_1, f_2$ and g_2 are rational functions $D = \frac{d}{dt}$with constant coefficients ϕ_1 and ϕ_2 explicit functions of t.

UNIT II

Finding the solution of Second and Higher Order with constant coefficients with Right Hand Side is of the form Ve^{ax}, where V is a function of x – Euler's Homogeneous Linear Differential Equations– System of simultaneous linear differential equations with constant coefficients.

UNIT III

Partial Differential Equations: Formation of Partial Differential Equation by eliminating arbitrary constants and arbitrary functions – Solutions of Partial Differential Equations by direct integration – Solution of standard types of first order partial differential equations.

UNIT IV

Laplace transforms: Definition – Laplace Transforms of standard functions – First Shifting Theorem – Transform of $tf(t), \frac{f(t)}{t}, f'(t), f''(t)$- Inverse Laplace Transforms – Applications to solutions of First Order and Second Order Differential Equations with constant coefficients.

UNIT V

Interpolation with unequal intervals – Lagrange's interpolation – Newton's divided difference interpolation – Interpolation with equal intervals – Newton's forward and backward difference formulae.

SUGGESTED READINGS

1. Treatment as in Kandasamy. P, Thilagavathi. K "Mathematics for B.Sc – Branch – I Volume III", S. Chand and Company Ltd, New Delhi, 2004.
2. S. Narayanan and T.K. ManickavasagamPillai, Calculus, S. Viswanathan (Printers and Publishers) Pvt. Ltd, Chennai 1991
3. N.P. Bali, Differential Equations, Laxmi Publication Ltd, New Delhi, 2004
4. Dr. J. K. Goyal and K.P. Gupta, Laplace and Fourier Transforms, PragaliPrakashan Publishers, Meerut, 2000.
5. SankaraRao K., Numerical methods for scientists and Engineers, Prentice Hall of India Private, 3rd Edition, New Delhi, 2007.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

Staff name: R.Gayathri

Subject Name: Mathematics-II — **Sub.Code:17PHU602**

Semester: VI — **Class: III B.Sc.Physics**

S.No	Lecture Duration Period	Topics to be Covered	Support Material/ Page Nos
UNIT-I			
1	1	Equations of First Order and Degree Higher than one – Solvable for p.	S1:Chapter-1, Pg.No : 1-16
2	1	Problems on Equations of First Order and of Degree Higher than one – Solvable for x.	S1:Chapter-1, Pg.No : 1-16
3	1	Problems on Equations of First Order and of Degree Higher than one – Solvable for y .	S1:Chapter-1, Pg.No : 1-16
4	1	Clairaut's Equation problems.	S1:Chapter-1, Pg.No : 17-28
5	1	Continuation onClairaut's Equation problems.	S1:Chapter-1, Pg.No : 17-28
6	1	Simultaneous Differential Equations with constant coefficients of the form i)$f_1D(x)+g_1D(y)=\phi_1(t)$ where f_1, g_1, f_2 and g_2 are rational functions $D=\frac{d}{dt}$with constant coefficients ϕ_1 and ϕ_2 explicit functions of t.	S1:Chapter-1, Pg.No : 28-41
7	1	Simultaneous Differential Equations with constant coefficients of the form ii) $f_2D(x)+g_2D(y)=\phi_2(t)$, where f_1, g_1, f_2 and g_2 are rational functions $D=\frac{d}{dt}$with constant coefficients ϕ_1 and ϕ_2 explicit functions of t.	S1:Chapter-1, Pg.No : 28-41
8	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-8Hours			
UNIT-II			
1	1	Finding the solution of Second and Higher Order with constant coefficients with Right Hand Side is of the form $V\,e^{ax}$, where V is a function of x	S3:Chapter-3, Pg.No : 222-235
2	1	Continuation onFinding the solution of Second and Higher Order with constant coefficients with Right Hand Side is of the form $V\,e^{ax}$, where V is a function	S3:Chapter-3, Pg.No : 222-235

		of x	
3	1	Euler's Homogeneous Linear Differential Equations	S3:Chapter-5, Pg.No : 286-313
4	1	Continuation onEuler's Homogeneous Linear Differential Equations	S3:Chapter-5, Pg.No : 286-313
5	1	Continuation onEuler's Homogeneous Linear Differential Equations	S3:Chapter-5, Pg.No : 286-313
6	1	Problems on System of simultaneous linear differential equations with constant coefficients.	S3:Chapter-9, Pg.No : 417-428
7	1	Problems on System of simultaneous linear differential equations with constant coefficients.	S3:Chapter-9, Pg.No : 417-428
8	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-8 Hours			
		UNIT-III	
1	1	Formation of Partial Differential Equation by eliminating arbitrary constants and arbitrary functions.	S1 : Chapter 5, Pg.No :117-126
2	1	Continuation on Formation of Partial Differential Equation by eliminating arbitrary constants and arbitrary functions.	S1 : Chapter 5, Pg.No :117-126
3	1	Solutions of Partial Differential Equations by direct integration	S2 : Chapter 8, Pg.No :179-185
4	1	Continuation onSolutions of Partial Differential Equations by direct integration	S2 : Chapter 8, Pg.No :179-185
5	1	Solution of standard types of first order partial differential equations.	S1 : Chapter 5, Pg.No :133-150
6	1	Continuation onSolution of standard types of first order partial differential equations.	S1 : Chapter 5, Pg.No :133-150
7	1	Continuation on Solution of standard types of first order partial differential equations.	S1 : Chapter 5, Pg.No :133-150
8	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-8 Hours			
		UNIT-IV	
1	1	Laplace Transforms: Definition and Problems	S4 : Chapter 1, Pg.No: 9-10
2	1	Problems on Laplace Transforms of standard functions	S4 : Chapter 1, Pg.No: 9-10
3	1	Linearity property	S4 : Chapter 1, Pg.No: 10-11
4	1	First Shifting Theorem	S4 : Chapter 1, Pg.No: 11-12
5	1	Transform of $tf(t), \frac{f(t)}{t}, f'(t), f''(t)$ Problems	S4 : Chapter 1, Pg.No: 12-23
6	1	Inverse Laplace Transforms: Definitions and Problems	S4 : Chapter 1, Pg.No: 99-110
7	1	Applications to solutions of First Order and Second Order Differential Equations with constant coefficients.	S4 : Chapter 1, Pg.No: 114-140
8	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned-8 Hours			

UNIT-V			
1	1	Interpolation with unequal intervals problems	S5 : Chapter 6, Pg.No :94-96
2	1	Lagrange's interpolation problems	S5 : Chapter 6, Pg.No :96-112
3	1	Newton's divided difference interpolation problems	S5 : Chapter 6, Pg.No :113-116
4	1	Newton's forward and backward difference problems	S5 : Chapter 6, Pg.No :116-125
5	1	Recapitulation and discussion of possible questions	
6	1	Discussion of previous year ESE question papers	
7	1	Discussion of previous year ESE question papers	
8	1	Discussion of previous year ESE question papers	
Total No. of Lecture hours planned-8 Hours			
Total Planned Hours			**40**

SUGGESTED READINGS

1. Treatment as in Kandasamy. P, Thilagavathi. K "Mathematics for B.Sc – Branch – I Volume III", S. Chand and Company Ltd, New Delhi, 2004.
2. S. Narayanan and T.K. Manickavasagam Pillai, Calculus, S. Viswanathan (Printers and Publishers) Pvt. Ltd, Chennai 1991
3. N.P. Bali, Differential Equations, Laxmi Publication Ltd, New Delhi, 2004
4. Dr. J. K. Goyal and K.P. Gupta, Laplace and Fourier Transforms, PragaliPrakashan Publishers, Meerut, 2000.
5. Sankara Rao K.,Numerical Methods for scientists and Engineers,Prentice Hall of India Private,3rd Edition ,New Delhi,2007.

Signature student Representative   Signature of the Course Faculty

Signature of the Class Tutor   Signature of Coordinator

Head of the Department

Ordinary Differential Equations: Equations of First Order and of Degree Higher than one – Solvable for p, x, y – Clairaut's Equation – Simultaneous Differential Equations with constant coefficients of the form i) $f_1 D(x) + g_1 D(y) = \phi_1(t)$ ii) $f_2 D(x) + g_2 D(y) = \phi_2(t)$, where f_1, g_1, f_2 and g_2 are rational functions $D = \frac{d}{dt}$ with constant coefficients ϕ_1 and ϕ_2 explicit functions of t.

DEFINITION:

Differential equations which involve only one independent variable are called **O**rdinary **D**ifferential **E**quations.

1.1 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

1.1(a) General form of a linear differential equation of the nth order with constant coefficients is

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots . + k_n y = X \text{ ....(1)}$$

Where $k_1, k_2, \ldots., k_n$ are constants. Such equations are most important in the study of electro –mechanical vibrations and other engineering problems.

1.1(b). General form of the linear differential equation of second order is

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ Or $D^2y + PDy + Qy = R$ Where $D' = \frac{d}{dx}$

Where P and Q are constants and R is a function of x or Constants.

Complete Solution = Complementary Function + Particular Integral

To Find the Complementary Functions:

S.No.	**Roots of Auxillary Equation**	**Complementary Functions**
1	Roots are Real and Different $m_1, m_2 \ (m_1 \neq m_2)$	$y = Ae^{m_1 x} + Be^{m_2 x}$
2	Roots are real and equal $m_1 = m_2 = m$(say)	$y = (Ax + B)e^{mx}$ Or $y = (A + Bx)e^{mx}$
3	Roots are imaginary $\alpha \pm i\beta$	$y = e^{\propto x}(A\cos\beta x + B\sin\beta x)$

To Find the Particular Integral: $\boldsymbol{P.I = \frac{1}{f(D)}X}$

S.No	**X**	**P.I**
1.	e^{ax}	**P.I=** $\frac{1}{f(D)}e^{ax} = e^{ax}\frac{1}{f(D)}$, $\boldsymbol{D = a, D' = b}$
2.	x^n	**P.I=** $\frac{1}{f(D)}x^n$, Expand $[\boldsymbol{f(D)}]^{-1}$ and then operate.

3.	Sinax or Cosax	**P.I=** $\frac{1}{f(D)}$ Sinax or Cosax
4.	$e^{ax}\varphi(x)$	**P.I=** $\frac{1}{f(D+a)}e^{ax}\varphi(x)$ Replace $D^2\ by - a^2$

Result:

1. $\frac{1}{D-a}\varphi(x) = e^{ax}\int e^{-ax}\varphi(x)\,dx$
2. $\frac{1}{D+a}\varphi(x) = e^{-ax}\int e^{ax}\varphi(x)\,dx$

1.1.1 Problems based on R.H.S of the given differential equation is Zero.

1. **Solve $(D^2+1)y=0$ given $y(0)=0$ $and\ y'(0)=1$ [AU, April 1996]**

Solution:

Given $(D^2+1)y=0$

Auxillary Equation is $m^2+1=0$

$$m^2 = \pm i$$
$$y(x) = Acosx + Bsinx$$
$$y(0) = Acos0 + Bsin0$$
$$y'(0) = B = 1$$

A=0, B= 1

$$y'(x) = -A\,Sinx + B\,Cosx$$

i.e., $y = (0)\,(cosx) + sinx$

$$\boldsymbol{y = sinx}$$

2. **Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$**

Solution:

$$(D^3 - 3D^2 + 3D - 1)y = 0$$

The Auxillary Equation is $m^3 - 3m^2 + 3m - 1 = 0$

$$(m-1)^3 = 0$$
$$m = 1,1,1$$

Hence the Solution is $\boldsymbol{y = e^x(A + Bx + Cx^2)}$

3. **Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$**

Solution:

Given $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$$(D^3 - 6D^2 + 11D - 6)y = 0$$

The Auxillary Equation is $m^3 - 6m^2 + 11 - 6 = 0$

If $m = 1$, $1 - 6 + 11 - 6 = 0$

$m = 1$ is a root

$$(m-1)(m-2)(m-3) = 0$$

$m = 1,2,3$

Hence the Solution is $\boldsymbol{y = Ae^{x} + Be^{2x} + Ce^{3x}}$

1.1.2 Problems based on P.I =$\frac{1}{f(D)}e^{ax}$ ⟹ Replace D by a

1. **Solve** $(4D^2 - 4D + 1)y = 4$ [AU, March 1996]

Solution:

Given $(4D^2 + 4D + 1)y = 4$

$$4m^2 - 4m + 1 = 0$$
$$4m^2 - 2m - 2m + 1 = 0$$
$$2m(2m-1) - 1(2m-1) = 0$$
$$(2m-1)^2 = 0$$
$$m = 1/2, 1/2$$

Complementary function = $\boldsymbol{(Ax + B)\, e^{1/2x}}$

Particular Integral =$\frac{1}{(4D^2-4D+1)}4e^{0x}$

=$\frac{4}{1}e^{0x}$

= 4

Y= C.F + P.I

$$\boldsymbol{y = (Ax + B)e^{1/2x} + 4}$$

2. **Solve** $\boldsymbol{\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\, coshx}$ [AU, April 2002]

Solution:

Given $(D^2 + 4D + 5)y = -2coshx$

The Auxillary Equations is $m^2 + 4m + 5 = 0$

$\boldsymbol{m = \frac{-4 \pm \sqrt{16-20}}{2}}$

$$\boldsymbol{m = \frac{-4 \pm 2i}{2}}$$
$$\boldsymbol{m = -2 \pm i}$$

Complimentary Function =$\boldsymbol{e^{-2x}[Acosx + B\, sinx]}$

Particular Integral

P.I = $\frac{1}{D^2+4D+5}(-2\, coshx)$

$$= \frac{-2}{D^2+4D+5}\frac{[e^x+e^{-x}]}{2}$$

$$= \frac{-1}{D^2+4D+5}[e^x+e^{-x}]$$

$$= \frac{-1}{D^2+4D+5}e^x + \frac{-1}{D^2+4D+5}e^{-x}$$

$$= \frac{-1}{1+4+5}e^x + \frac{1}{1-4+5}e^{-x}$$

$$= \frac{-1}{10}e^x - \frac{1}{2}e^{-x}$$

Y= Complementary Function + Particular value

Y= $e^{-2x}[A\cos x + B\sin x] + \frac{-1}{10}e^x - \frac{1}{2}e^{-x}$

3. Solve $(D^2-1)y = \sinh x$**.**

Solution:

The given ODE is $(D^2-1)y = \sinh x = \frac{e^x - e^{-x}}{2}$ ------(1)

The A.E of (1) is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

$\therefore\ C.F = Ae^x x + Be^{-x}$ P.I= $\frac{1}{D^2-1}\left(\frac{e^x-e^{-x}}{2}\right)$

$$= \frac{1}{2}\left(\frac{1}{D^2-1}e^x - \frac{1}{D^2-1}e^{-x}\right) \text{\{replace } D \text{ by } a\text{\}}$$

$$= \frac{1}{2}\left(\frac{1}{1^2-1}e^x - \frac{1}{(-1)^2-1}e^{-x}\right) \quad \text{(orf in I \& II terms)}$$

$$= \frac{1}{2}\left(x\frac{1}{2D}e^x - x\frac{1}{2D}e^{-x}\right) \quad \text{\{replace } D \text{ by } a\text{\}}$$

$$= \frac{1}{2}\left(x\frac{1}{2(1)}e^x - x\frac{1}{2(-1)}e^{-x}\right)$$

$$. = \frac{x}{2}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{x}{2}\cosh x$$

$\therefore$ The general solution is $y = C.F + P.I = Ae^x x + Be^{-x} + \frac{x}{2}\cosh x$.

1.1.3 Problems based on P.I = $\frac{1}{f(D)}$ sinax or $\frac{1}{f(D)}$ cosax . Replace D^2 by $-a^2$

1. **Find the Particular Integral of $(D^2+4)y = cos\ 2x$** [AU, May 2001]

Solution:

Given $(D^2+4)y = cos\ 2x$

P. I $= \frac{1}{D^2+4}\ cos\ 2x$

$= \frac{1}{-4+4}\ cos\ 2x$ Replace $D^2\ by - (2)^2$ [Ordinary Rule fail]

$= x\frac{1}{2D}cos2x$

$= \frac{x}{2}\frac{1}{D}$ cos2x

$= \frac{x}{2}\int cos2x\ dx$

$= \frac{x}{2}\left(\frac{sin2x}{2}\right)$

P.I $= \frac{x}{4}sin2x$

2. **Find the particular integral of $(D^2+1)y = sinx\ sin2x$**

Solution: [AU, May 1997]

Given $(D^2+1)y = sinx\ sin2x$

$= -\frac{1}{2}\ [cos3x - cosx]$

$= -\frac{1}{2}cos3x + \frac{1}{2}\ cosx$

P. I $= \frac{1}{D^2+1}\left[-\frac{1}{2}cos3x\right] + \frac{1}{D^2+1}\left[\frac{1}{2}cosx\right]$

$= -\frac{1}{2}\left[\frac{1}{-9+1}\right]cos3x\ + \frac{1}{2}\frac{1}{-1+1}\ cosx$ [Ordinary Rule Fail]

$= \frac{1}{16}cos3x\ + \frac{1}{2}x\ \frac{1}{2D}cosx$

$= \frac{1}{16}cos3x\ + \frac{x}{4}\int cosx\ dx$

P.I $= \frac{1}{16}cos3x\ + \frac{x}{4}sinx$

3. Solve $(D^2+16)y=\cos^3 x$ **.(AU Dec 2010)**

Solution:

The given ODE is $(D^2+16)y=\cos^3 x=\frac{1}{4}(\cos 3x+3\cos x)$ -----(1)

The A.E of (1) is $m^2+16=0 \Rightarrow m^2=-16 \Rightarrow m=\pm 4i$

$\therefore\ C.F=e^{0z}[A\cos 4x+B\sin 4x]=A\cos 4x+B\sin 4x$

$$\text{P.I}=\frac{1}{D^2+16}\left[\frac{1}{4}(\cos 3x+3\cos x)\right]$$

$$=\frac{1}{4}\left[\frac{1}{D^2+16}\right]\cos 3x+\frac{3}{4}\left[\frac{1}{D^2+16}\right]\cos x$$

$$=\frac{1}{4}\left[\frac{1}{-3^2+16}\right]\cos 3x+\frac{3}{4}\left[\frac{1}{-1^2+16}\right]\cos x \quad \{\text{ Replace } D^2 \text{ by } -a^2\}$$

$$=\frac{1}{4}\left[\frac{1}{7}\right]\cos 3x+\frac{3}{4}\left[\frac{1}{15}\right]\cos x$$

$$=\frac{1}{28}\cos 3x+\frac{1}{20}\cos x$$

$\therefore$ The general solution is

$$y=C.F+P.I=A\cos 4x+B\sin 4x+\frac{1}{28}\cos 3x+\frac{1}{20}\cos x.$$

1.1.4 Problems based on R.H.S = $e^{ax}+\ sinax\ (or)e^{ax}+\ cosax$

1. Solve $(D^2-4D+4)y=e^{2x}+\ cos\ 2x$

Solution:

Given $(D^2-4D+4)y=e^{2x}+\ cos\ 2x$

The Auxillary Equation is $m^2-4m+4=0$

$$(m-2)^2=0$$

$$m = 2, 2$$

Complementary Function $= (\boldsymbol{Ax + B})\boldsymbol{e^{2x}}$

Particular Integral = $\frac{1}{D^2-4D+4} e^{2x} + \frac{1}{D^2-4D+4} \cos 2x$

$$= \frac{1}{4-8+4} e^{2x} \quad + \frac{1}{-2^2-4D+4} \cos 2x$$

$$= \frac{1}{0} e^{2x} \quad + \frac{1}{-4D} \cos 2x$$

$$= \text{x}\frac{1}{2D-4} e^{2x} - \frac{1}{4D} \cos 2x$$

$$= x\frac{1}{0} e^{2x} - \frac{1}{4}\left[\frac{1}{D} \cos 2x\right]$$

$$= x^2 \frac{1}{2} e^{2x} - \frac{1}{4}\left[\frac{\sin 2x}{2}\right]$$

$$= x^2 \frac{1}{2} e^{2x} - \left[\frac{\sin 2x}{8}\right]$$

Y= C.F + P.I

$$\textbf{Y}= (\boldsymbol{Ax + B})\boldsymbol{e^{2x}} + \boldsymbol{x^2 \frac{1}{2} e^{2x}} - \left[\boldsymbol{\frac{\sin 2x}{8}}\right]$$

2. **Solve** $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$ ***.(Jan. 2005, Nov/ Dec.2009*****Solution:**

The given ODE is $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$ **-----(1)**

The A.E of (1) is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$
$$m = 1, m = 2$$

C.F= $Ae^x + Be^{2x}$.

$$\text{P.I}= 2\frac{1}{f(D)}\cos(2x+3) + 2\frac{1}{f(D)}e^x = 2P.I_1 + 2P.I_2$$

$$\text{Now } P.I_1 = \frac{1}{D^2 - 3D + 2}\cos(2x+3) = \frac{1}{-2^2 - 3D + 2}\cos(2x+3)$$

$$=\frac{1}{-(3D+2)}\cos(2x+3)=-\frac{(3D-2)}{(3D)^2-2^2}\cos(2x+3)$$

$$=-\frac{(3D-2)}{9D^2-2^2}\cos(2x+3)=-\frac{(3D-2)}{-40}\cos(2x+3)$$

$$=\frac{[-6\sin(2x+3)-2\cos(2x+3)]}{40}=-\frac{[3\sin(2x+3)+\cos(2x+3)]}{20}$$

$$P.I_2=\frac{1}{D^2-3D+2}e^x=\frac{1}{1-3+2}e^x \text{(Ordinary rule fails)}$$

$$=x\frac{1}{2D-3}e^x=-xe^x$$

$$\therefore P.I=2P.I_1+2P.I_2=-\frac{[3\sin(2x+3)+\cos(2x+3)]}{10}-2xe^x$$

The general solution of (1) is $y(x)=\text{C.F+P.I}$

$$=Ae^x+Be^{2x}-\frac{[3\sin(2x+3)+\cos(2x+3)]}{10}-2xe^x.$$

1.1.5 Problems based on R.H.S = x

The following formulae are important.

$(1+x)^{-1}=1-x+x^2-x^3+\ldots,$ $(1-x)^{-1}=1+x+x^2+x^3+\ldots,$

$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots,$ $(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots,$

1. Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x^2+3$

Solution: Given $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x^2+3$

i.e., $(D^2-5D+6)y=x^2+3$

Auxillary Equation is $m^2-5m+6=0$

$$(m-2)(m-3)=0$$

$m=2,\ m=3$

Complimentary function is $Ae^{2x}+Be^{3x}$

Particular Integral $=\dfrac{1}{D^2-5D+6}(x^2+3)$

$$=\frac{1}{6\left[1+\dfrac{D^2-5D}{6}\right]}(x^2+3)$$

$$=\frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left(\frac{D^2-5D}{6}\right)^2-....,\right](x^2+3)$$

$$=\frac{1}{6}\left[1-\frac{D^2}{6}+\frac{5D}{6}+\frac{D^4}{36}+\frac{25D^2}{36}-\frac{10D^3}{36}-...,\right](x^2+3)$$

$$=\frac{1}{6}\left[(x^2+3)-\frac{2}{6}+\frac{5(2x)}{6}+0+\frac{25(2)}{36}\right]$$

$$=\frac{1}{6}\left[(x^2+3)-\frac{1}{3}+\frac{5x}{3}+0+\frac{25}{18}\right]$$

$$=\frac{1}{108}\left[18x^2+30x+73\right]$$

The Complete Solution is y = C.F + P.I

$$y=Ae^{2x}+Be^{3x}+\frac{1}{108}\left[18x^2+30x+73\right]\qquad .$$

2. **Find the Particular Integral of** $(\boldsymbol{D^2-1})\boldsymbol{y}=\boldsymbol{x}$

Solution:

Given $(\boldsymbol{D^2-1})\boldsymbol{y}=\boldsymbol{x}$

Auxillary Equation $m^2-1=0$

$m^2=1$

$m=\pm 1$

Complementary function is $Ae^{-x}+Be^{x}$

Particular Integral $=\dfrac{1}{D^2-1}x$

$$=\frac{-1}{1-D^2}x$$

$$=-1(1-D^2)^{-1}x$$

$$= -\left[1 + D^2 + (D^2)^2 + ...\right]x$$
$$= -\left[x + 0 + 0 + ...\right]$$
$$= -x$$

The Complete Solution is $y = Ae^{-x} + Be^{x} - x$.

1.1.6 Problems based on R.H.S = $e^{ax}x$. Particular Integral = $\frac{1}{f(D)}e^{ax}x = e^{ax}\frac{1}{f(D+a)}x$

1. **Solve:** $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$

Solution:

Given $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$

A.E is $m^2 + 4m + 3 = 0$

$(m+1)(m+3) = 0$

$m = -1, m = -3$

$C.F = Ae^{-x} + Be^{-3x}$

$$P.I_1 = \frac{1}{D^2 + 4D + 3}e^{-x}\sin x$$
$$= e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 3}\sin x$$
$$= e^{-x}\frac{1}{D^2 - 2D + 1 + 4D - 4 + 3}\sin x$$
$$= e^{-x}\frac{1}{D^2 + 2D}\sin x$$
$$= e^{-x}\frac{1}{-1 + 2D}\sin x$$

Take Conjugate we get,

$$= e^{-x}\frac{2D+1}{(2D)^2 - 1}\sin x$$

$$= e^{-x}\,\frac{2D+1}{4D^2-1}\sin x$$

$$= e^{-x}\,\frac{2D+1}{-4-1}\sin x$$

$$= \frac{e^{-x}}{-5}(2D+1)\sin x$$

$$P.I_1 = \frac{e^{-x}}{-5}(2\cos x+\sin x)$$

$$P.I_2 = \frac{1}{D^2+4D+3}xe^{3x}$$

$$= e^{3x}\,\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$= e^{3x}\,\frac{1}{D^2+6D+9+4D+12+3}x$$

$$= e^{3x}\,\frac{1}{D^2+10D+24}x$$

$$= \frac{e^{3x}}{24}\frac{1}{\left[\frac{D^2}{24}+\frac{10}{24}D+1\right]}x$$

$$= \frac{e^{3x}}{24}\left[1+\frac{5}{12}D+\frac{D^2}{24}\right]^{-1}x$$

$$= \frac{e^{3x}}{24}\left[1-\left(\frac{5}{12}D+\frac{D^2}{24}\right)+....,\right]x$$

$$= \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

$$y = C.F + P.I$$

$$y = Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}(2\cos x+\sin x)+\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

2. Solve $(D^2-2D+2)y = e^x x^2+5+e^{-2x}$

Solution:

Given $(D^2-2D+2)y = e^x x^2 + 5 + e^{-2x}$

A.E is $m^2-2m+2=0$

$m = 1 \pm i$

$C.F = e^x(A\cos x + B\sin x)$

$$P.I_1 = \frac{1}{D^2-2D+2} e^x x^2$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+2} x^2$$

$$= e^x \frac{1}{D^2+2D+1-2D-2+2} x^2$$

$$= e^x \frac{1}{D^2+1} x^2$$

$$= e^x (D^2+1)^{-1} x^2$$

$$= e^x (1-D^2+\ldots)x^2$$

$$P.I_1 = e^x(x^2-2)$$

$$P.I_2 = \frac{1}{D^2-2D+2} 5e^{0x}$$

$$P.I_2 = \frac{5}{2}$$

$$P.I_3 = \frac{1}{D^2-2D+2} e^{-2x}$$

$$= \frac{1}{4+4+2} e^{-2x}$$

$$P.I_3 = \frac{1}{10} e^{-2x}$$

y = C.F +P.I

$$y = e^x(A\cos x + B\sin x) + e^x(x^2-2) + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

3. Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$. **(Nov./Dec. 2002)**

Solution:

The given ODE is $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$ **----(1)**

The A.E of (1) is $m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1, m=-3$

C.F= $Ae^{-x}+Be^{-3x}$.

$$\text{P.I}=\frac{1}{f(D)}e^{-x}\sin x+\frac{1}{f(D)}xe^{3x}=P.I_1+P.I_2$$

$$\text{Now } P.I_1=\frac{1}{D^2+4D+3}e^{-x}\sin x=e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$=e^{-x}\frac{1}{D^2+2D}\sin x=e^{-x}\frac{1}{-1+2D}\sin x=e^{-x}\frac{(2D+1)}{(2D)^2-1^2}\sin x$$

$$=e^{-x}\frac{(2D+1)}{-4-1}\sin x=-\frac{e^{-x}}{5}(2\cos x+\sin x)$$

$$P.I_2=\frac{1}{f(D)}xe^{3x}=\frac{1}{D^2+4D+3}e^{3x}x=e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$=e^{3x}\frac{1}{D^2+10D+24}x=\frac{e^{3x}}{24}\left[1+\left(\frac{D^2+10D}{24}\right)\right]^{-1}x$$

$$=\frac{e^{3x}}{24}\left[1-\left(\frac{D^2+10D}{24}\right)+\left(\frac{D^2+10D}{24}\right)^2-\dots\right]x$$

$$=\frac{e^{3x}}{24}\left[1-\frac{5}{12}D\right]x \quad \textit{omitting Higher order derivatives}$$

$$=\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]\therefore P.I=P.I_1+P.I_2=-\frac{e^{-x}}{5}(2\cos x+\sin x)+\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

The general solution of (1) is $y(x)=$C.F+P.I

$$= Ae^{-x} + Be^{-3x} - \frac{e^{-x}}{5}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left[x - \frac{5}{12}\right].$$

4.Solve $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$. **(April/May 2003)**

Solution:

The given ODE is $(D^2 - 2D + 2)y = e^x x^2 + 5 + e^{-2x}$ **----(1)**

The A.E of (1) is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = 1 \pm i$$

C.F= $e^x(A\cos x + B\sin x)$.

$$\text{P.I}= \frac{1}{f(D)}e^x x^2 + \frac{1}{f(D)}5 + \frac{1}{f(D)}e^{-2x} = P.I_1 + P.I_2 + P.I_3$$

Now $P.I_1 = \frac{1}{D^2 - 2D + 2}e^x x^2 = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}x^2$

$$= e^x \frac{1}{D^2 + 1}x^2 = e^x(1 + D^2)^{-1}x^2 = e^x(1 - D^2 + (D^2)^2 - ...)x^2$$

$$= e^x(1 - D^2)x^2 = e^x(x^2 - 2)$$

$$P.I_2 = 4\frac{1}{D^2 - 2D + 2}e^{0x} = 4\frac{1}{2} = 2$$

$$P.I_3 = \frac{1}{D^2 - 2D + 2}e^{-2x} = \frac{1}{(-2)^2 - 2(-2) + 2}e^{-2x} = \frac{e^{-2x}}{10}$$

P.I = $P.I_1 + P.I_2 + P.I_3$

$$= e^x(x^2 - 2) + 2 + \frac{e^{-2x}}{10}$$

The general solution of (1) is $y(x) =$ C.F+P.I

$$= e^x(A\cos x + B\sin x) + e^x(x^2 - 2) + 2 + \frac{e^{-2x}}{10}$$

1.1.7 Problems based on $f(x) = x^n \sin ax \ \ or \ x^n \cos ax$

To find Particular Integral when

$$f(x) = x^n \sin ax \ \ or \ x^n \cos ax \ P.I = \frac{1}{f(D)} x^n \sin ax \ \ (or) \ x^n \cos ax$$

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left[\frac{d}{dD}\frac{1}{f(D)}\right]V$$

$$\frac{1}{f(D)}(x.V) = x\frac{1}{f(D)}V - \left[\frac{f^{'}(D)}{f(D)}\frac{1}{f(D)}\right]V$$

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \left[\frac{f^{'}(D)}{\left[f(D)\right]^2}\right]V$$

1. Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Solution:

Given $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

The roots are $m = 2, 2$.

Complementary Function is $(c_1 x + c_2)e^{2x}$

Particular Integral $= \dfrac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x$

$$= 8e^{2x}\frac{1}{(D-2)^2}x^2 \sin 2x$$

$$= 8e^{2x}\frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - 2x\left(\frac{-\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right)\right\}$$

$$= e^{2x}\left\{\frac{1}{D}\left(-4x^2 \cos 2x\right) + \frac{1}{D}(4x \sin 2x) + \frac{1}{D}(2\cos 2x)\right\}$$

$$= e^{2x}\left[\left\{\left(-4x^2\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{4}\right)\right\} + 4\left\{x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right) + \sin 2x\right\}\right]$$

$$= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]$$

The general Solution is y = C.F + P.I.

$$y = (c_1 x + c_2)e^{2x} + e^{2x}(3 - 2x^2)\sin 2x - 4x\cos 2x$$

2. Solve the differential equation $(D^2 + 4)y = x^2 \cos 2x$ **(May/ June 2009)**

Solution:

The given ODE is $(D^2 + 4)y = x^2 \cos 2x$ **----(1)**

The A.E of (1) is $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$

C.F= $A\cos 2x + B\sin 2x$

$$\text{P.I}=\left[\frac{1}{f(D)}\right]x^2 \cos 2x = \left[\frac{1}{D^2+4}\right]x^2 R.P\ of\ e^{i2x}$$

$$= R.P\ \ of\ e^{i2x}\left[\frac{1}{(D+2i)^2+4}\right]x^2 = R.P\ of\ e^{i2x}\left[\frac{1}{D^2+4Di}\right]x^2$$

$$= R.P\ of\left[\frac{e^{i2x}}{4Di\left(1+\frac{D^2}{4Di}\right)}\right]x^2$$

$$= R.P\ of\ \frac{-i^2 e^{i2x}}{4Di}\left(1+\frac{D}{4i}\right)^{-1} x^2$$

$$= R.P\ of\ \frac{-ie^{i2x}}{4D}\left(1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2-\left(\frac{D}{4i}\right)^3+\ldots\right)x^2$$

$$= R.P\ of\ -\frac{ie^{i2x}}{4}\left(\frac{1}{D}-\frac{1}{4i}+\left(-\frac{D}{16}\right)-\left(-\frac{D^2}{64i}\right)\right)x^2$$

$$= R.P\ of\ \frac{e^{i2x}}{4}\left(-\frac{i}{D}+\frac{1}{4}+\left(\frac{Di}{16}\right)-\left(\frac{D^2}{64}\right)\right)x^2$$

$$= R.P\ of\ \frac{e^{i2x}}{4}\left(-i\left(\frac{x^3}{3}\right)+\frac{x^2}{4}+\left(\frac{2xi}{16}\right)-\left(\frac{2}{64}\right)\right)$$

$$= R.P\ of\ \frac{(\cos 2x + i\sin 2x)}{4}\left(\left(\frac{x^2}{4}-\frac{1}{32}\right)-i\left(\frac{x^3}{3}-\frac{x}{8}\right)\right)$$

$$= \frac{1}{4}\left[\left(\frac{x^2}{4}-\frac{1}{32}\right)\cos 2x + \left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x\right]$$

The general solution of (1) is $y(x) = \text{C.F+P.I}$

$$= A\cos 2x + B\sin 2x + \frac{1}{4}\left[\left(\frac{x^2}{4}-\frac{1}{32}\right)\cos 2x + \left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x\right].$$

1.1.8 Problems based on $\frac{1}{D-a} f(x) = e^{ax}\int e^{-ax} f(x)dx$ Type.

[General Method of finding the Particular Integral of any function f(x)]

1. Solve $(D^2 + a^2)y = \sec ax$.

Solution:

Given $(D^2 + a^2)y = \sec ax$

A. E. is $m^2 + a^2 = 0$

The Roots are $m = \pm\, ia$

Complementary function = $A\cos ax + B\sin ax$.

$$P.I = \frac{1}{(D^2 + a^2)}\sec ax$$

$$P.I = \frac{1}{(D-ia)(D+ia)}\sec ax$$

$$= \left(\frac{\frac{1}{2ia}}{D-ia} - \frac{\frac{1}{2ia}}{D+ia}\right)\sec ax$$

$$= \frac{1}{2ia}e^{iax}\int e^{-iax}\sec ax dx - \frac{1}{2ia}e^{-iax}\int e^{iax}\sec ax dx \quad [\frac{1}{D-m}X = e^{mx}\int Xe^{-mx}dx]$$

$$= \frac{1}{2ia}e^{iax}\int (1 - i\tan ax)dx - \frac{1}{2ia}e^{-iax}\int (1 + i\tan ax)dx$$

$$= \frac{1}{2ia}e^{iax}(x - \frac{i}{a}\log\sec ax) - \frac{1}{2ia}e^{-iax}(x + \frac{i}{a}\log\sec ax)$$

$$= \frac{x}{a}\left(\frac{e^{iax}-e^{-iax}}{2i}\right)-\frac{1}{a^2}\log\sec ax\left(\frac{e^{iax}+e^{-iax}}{2i}\right)$$

$$= \frac{x}{a}\sin ax-\frac{1}{a^2}\log\sec ax\cos ax$$

General Solution is y = C.F + P.I

$$y = A\cos ax + B\sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\log\sec ax\cos ax$$

Simultaneous First Order Linear Equations With Constant Coefficients

Linear differential equation in which there are two or more dependent variables and a single independent variable. Such equations are known as simultaneous linear equations. Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the depend variables is obtained in a similar manner**.**

Problems Based on Simultaneous First Order Linear Equations With Constant coefficients

1. Solve the simultaneous equations $\frac{dx}{dt}+2x+3y=2e^{2t}$, $\frac{dy}{dt}+3x+2y=0$

Solution:

Given $\frac{dx}{dt}+2x+3y=2e^{2t}$ $\frac{dy}{dt}+3x+2y=0$

(i.e.,)$Dx+2x+3y=2e^{2t}$ $Dy+3x+2y=0$

$(D+2)x+3y=2e^{2t}$ …. (1) $(D+2)y+3x=0$ …..(2)

$(1)X(D+2) => (D+2)^2x+3(D+2)y=2(D+2)e^{2t}$ …….(3)

$(2)X3$ => $9x+3(D+2)y=0$ …….(4)

(3) – (4) => $[(D+2)^2-9]x=2(D+2)\,e^{2t}$

$$(D^2+4D+4-9]x=2(2+2)\,e^{2t}$$

$$(D^2+4D-5]x=8\,e^{2t}$$

It's Auxillary equation is $m^2+4m-5=0$

$$(m+5)(m-1)=0$$

$$m=-5, m=1$$

Complementary Function = $Ae^{-5t}+Be^{t}$

Particular Integral = $\frac{1}{(D^2+4D-5]}8\,e^{2t}$

$$=8\,\frac{1}{(D^2+4D-5]}e^{2t}$$

$$=8\,\frac{1}{((2)^2+4(2)-5]}e^{2t}$$

$$=\frac{8}{7}e^{2t}$$

$$x = Ae^{-5t}+Be^{t}+\frac{8}{7}e^{2t}$$

Differentiate with respect to 't'

$$\frac{dx}{dt}=-5Ae^{-5t}+Be^{t}+\frac{16}{7}e^{2t}$$

Substitute above values in (1) we get,

$[-5Ae^{-5t}+Be^{t}+\frac{16}{7}e^{2t}]+2[5Ae^{-5t}+Be^{t}+\frac{8}{7}e^{2t}]+3y=2e^{2t}$

$-5Ae^{-5t}+Be^{t}+\frac{16}{7}e^{2t}+2Ae^{-5t}+2Be^{t}+\frac{16}{7}e^{2t}+3y=2e^{2t}$

$-3Ae^{-5t}+3Be^{t}+\frac{32}{7}e^{2t}+3y=2e^{2t}$

$-Ae^{-5t}+Be^{t}+\frac{6}{7}e^{2t}+y=0$

$y = Ae^{-5t}+Be^{t}-\frac{6}{7}e^{2t}$

Hence the desired solutions are

$x = Ae^{-5t}+Be^{t}+\frac{8}{7}e^{2t}$, $y= Ae^{-5t}+Be^{t}-\frac{6}{7}e^{2t}$.

1. **Solve the simultaneous equations** $\frac{dx}{dt}+2y=5e^{t};\frac{dy}{dt}-2x=5e^{t}$ **given that** $x=-1,\ y=3\ at\ \ t=0$**.**

Solution:

The given simultaneous equations are $\frac{dx}{dt}+2y=5e^t$; $\frac{dy}{dt}-2x=5e^t$

i.e. $Dx+2y=5e^t$ ----------(1) $Dy-2x=5e^t$ ----------------(2)

Eliminate x from (1) and (2)

$(1)\times 2 \Rightarrow \quad 2Dx+4y=10e^t$ --------------(3)

$(2)\times D \Rightarrow D^2y-2Dx=5e^t$ --------------- (4)

---

(3)+(4)$\Rightarrow$ $(D^2+4)y=15e^t$ ----------------(5)

The A.E of (5) is $m^2+4=0$

$\Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

$\therefore$ C.F $=A\cos 2t+B\sin 2t$

$$\text{P.I}=\left[\frac{1}{D^2+4}\right](15e^t)$$

$$=15\left[\frac{1}{1^2+4}\right]e^t=15\frac{1}{5}e^t=3e^t$$

The general solution of (5) is $y(t)=C.F+P.I=A\cos 2t+B\sin 2t+3e^t$

$(2)\Rightarrow 2x(t)=y'(t)-5e^t$

$=-2A\sin 2t+2B\cos 2t+3e^t-5e^t$

$=-2A\sin 2t+2B\cos 2t-2e^t$

$\therefore x(t)=-A\sin 2t+B\cos 2t-e^t$

$\therefore$ The solutions of (1) and (2) are $x(t)=-A\sin 2t+B\cos 2t-e^t$ and

$y(t)=A\cos 2t+B\sin 2t+3e^t$.

Given $x(0)=-1=-A(0)+B(1)-e^0 \Rightarrow B=0$

Given $y(0)=3=A(1)+B(0)\Rightarrow A=0$

$\therefore$ The solutions of (1) and (2) are $x(t)=-e^{t}$ and $y(t)=3e^{t}$.

1. **Solve** $(D+5)x+y=e^{t};(D+3)y-x=e^{2t}$.

Solution

The given simultaneous equations

are $(D+5)x+y=e^{t}---(1);\ -x+(D+3)y=e^{2t}----(2)$

Eliminate x from (1) and (2)

$(1)\times 1\Rightarrow (D+5)x+y=e^{t}$ ------------------------**(3)**

(2) $\times(D+5)\Rightarrow -(D+5)x+(D+5)(D+3)y=(D+5)e^{2t}$ -------------**(4)**

----

(3)+(4) $\Rightarrow (1+(D+3)(D+5))y=e^{t}+(D+5)e^{2t}$

$(D^{2}+8D+16)y=e^{t}+2e^{2t}+5e^{2t}$

$(D^{2}+8D+16)y=e^{t}+7e^{2t}$ ------------------------------ **(5)**

The A.E of (5) is $m^{2}+8m+16=0\Rightarrow m=-4,-4$

C.F= $(At+B)e^{-4t}$

$$\therefore P.I=\left[\frac{1}{(D+4)^{2}}\right](e^{t}+7e^{2t})$$

$$=\left[\frac{1}{(1+4)^{2}}\right]e^{t}+7\left[\frac{1}{(2+4)^{2}}\right]e^{2t}$$

$$=\frac{1}{25}e^{t}+\frac{7}{36}e^{2t}$$

The general solution of (5) is $y(t)=(At+B)e^{-4t}+\frac{1}{25}e^{t}+\frac{7}{36}e^{2t}$

(2) $\Rightarrow x(t)=Dy+3y-e^{2t}$

$$= Ae^{-4t} - 4Ate^{-4t} - 4Be^{-4t} + \frac{e^{t}}{25} + \frac{14e^{2t}}{36} + 3Ate^{-4t} + 3Be^{-4t}$$

$$+ \frac{3}{25}e^{t} + \frac{21}{36}e^{2t} - e^{2t}$$

$$x(t) = Ae^{-4t}(1-t) - Be^{-4t} + \frac{4e^{t}}{25} - \frac{e^{2t}}{36}$$

Thesolutions of (1) and (2) are $y(t) = (At+B)e^{-4t} + \frac{1}{25}e^{t} + \frac{7}{36}e^{2t}$ and

$$x(t) = Ae^{-4t}(1-t) - Be^{-4t} + \frac{4e^{t}}{25} - \frac{e^{2t}}{36}$$

2. Solve the simultaneous equations $\frac{dx}{dt} + 2x - 3y = t; \frac{dy}{dt} - 3x + 2y = e^{2t}$**.** **(Jan. 2006)**

Solution:

The given simultaneous equations are $\frac{dx}{dt} + 2x - 3y = t;\quad \frac{dy}{dt} - 3x + 2y = e^{2t}$

i.e $Dx + 2x - 3y = t;\qquad Dy - 3x + 2y = e^{2t}$

$(D+2)x - 3y = t ---(1)\qquad (D+2)y - 3x = e^{2t} ---(2)$

Eliminate x from (1) and (2)

$(1)\times 3 \Rightarrow\quad 3(D+2)x - 9y = 3t$ -------------------------(3)

$(2)\times(D+2) \Rightarrow (D+2)^2 y - 3(D+2)x = (D+2)e^{2t}$ -------(4)

$(3)+(4) \Rightarrow -9y + (D+2)^2 y = 3t + (D+2)e^{2t}$

$\Rightarrow -9y + (D^2 + 4D + 4)y = 3t + 2e^{2t} + 2e^{2t}$

$\Rightarrow (D^2 + 4D - 5)y = 3t + 4e^{2t}$ **-----------------------(5)**

The A.E of (5) is $m^2 + 4m - 5 = 0$

$\Rightarrow (m+5)(m-1) = 0 \Rightarrow m = 1,\ -5$

$\therefore$ C.F= $Ae^{t} + Be^{-5t}$

$$\text{P.I}= 3\left[\frac{1}{D^2+4D-5}\right]t+4\left[\frac{1}{(D+5)(D-1)}\right]e^{2t}$$

$$=\frac{3}{-5}\left[\frac{1}{1-\left(\frac{D^2+4D}{5}\right)}\right]t+4\left[\frac{1}{(2+5)(2-1)}\right]e^{2t}$$

$$=-\frac{3}{5}\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1}t+\frac{4}{7}e^{2t}$$

$$=-\frac{3}{5}\left[1+\left(\frac{D^2+4D}{5}\right)+\left(\frac{D^2+4D}{5}\right)^2+\ldots\right]t+\frac{4}{7}e^{2t}$$

$$=-\frac{3}{5}\left[1+\frac{4D}{5}\right]t+\frac{4}{7}e^{2t}$$

$$=-\frac{3}{5}\left[t+\frac{4}{5}\right]+\frac{4}{7}e^{2t}$$

The general solution of (5) is $y(t)=C.F+P.I=Ae^{t}+Be^{-5t}-\frac{3}{5}\left[t+\frac{4}{5}\right]+\frac{4}{7}e^{2t}$

$(2)\Rightarrow 3x(t)=(D+2)y(t)-e^{t}$

$$=(D+2)\left(Ae^{t}+Be^{-5t}-\frac{3}{5}t-\frac{12}{25}+\frac{4e^{2t}}{7}\right)-e^{2t}$$

$$=\left(Ae^{t}-5Be^{-5t}-\frac{3}{5}+\frac{8e^{2t}}{7}\right)+\left(2Ae^{t}+2Be^{-5t}-\frac{6}{5}t-\frac{24}{25}+\frac{8e^{2t}}{7}\right)-e^{2t}$$

$$=3Ae^{t}-3Be^{-5t}+\frac{9e^{2t}}{7}-\frac{6}{5}t-\frac{39}{25}$$

$$\Rightarrow x(t)=Ae^{t}-Be^{-5t}+\frac{3e^{2t}}{7}-\frac{2}{5}t-\frac{13}{25}$$

$\therefore$ The solutions of (1) and (2) are

$$x(t) = Ae^{t} - Be^{-5t} + \frac{3e^{2t}}{7} - \frac{2}{5}t - \frac{13}{25} \text{ and } y(t) = Ae^{t} + Be^{-5t} - \frac{3}{5}\left[t + \frac{4}{5}\right] + \frac{4}{7}e^{2t}.$$

3. Solve $\frac{dx}{dt} - y = t;\ \frac{dy}{dt} + x = t^2$ **. (Nov./Dec. 2003) (Nov./Dec. 2006).**

Solution:

The given simultaneous equations are $\frac{dx}{dt} - y = t;\quad \frac{dy}{dt} + x = t^2$

i.e $Dx - y = t ---(1) \quad Dy + x = t^2 ---(2)$

Eliminate y from (1) and (2)

$(1)\times D \Rightarrow \qquad D^2x - Dy = Dt$ --------------------(3)

$(2)\times 1 \Rightarrow \qquad Dy + x = t^2$ ----------------------(4)

$(3)+(4) \Rightarrow \qquad D^2x + x = Dt + t^2$

i.e $(D^2+1)x = 1 + t^2$ ----------------(5)

The A.E of (5) is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$ Here $\alpha = 0; \beta = 1$

$\therefore$ C.F$= e^{0t}(A\cos t + B\sin t) = A\cos t + B\sin t$

$$\text{P.I} = \left[\frac{1}{D^2+1}\right](1+t^2) = (1+D^2)^{-1}(1+t^2)$$

$$=[1 - D^2 + (D^2)^2 - ...](1+t^2)$$

$= [1 - D^2](1+t^2)$ omitting Hr,. derivatives

$$=[1 + t^2 - 2] = t^2 - 1$$

The general solution of (5) is $x(t) = C.F + P.I = A\cos t + B\sin t + t^2 - 1$

$(1) \Rightarrow y(t) = Dx(t) - t = -A\sin t + B\cos t + 2t - t$

$$= -A\sin t + B\cos t + t$$

$\therefore$ The solutions of (1) and (2) are $x(t) = A\cos t + B\sin t + t^2 - 1$ and

$y(t) = -A\sin t + B\cos t + t$.

4. Solve the simultaneous equations $\frac{dx}{dt} + 2x - 3y = 5t;\ \frac{dy}{dt} - 3x + 2y = 0$ given that $x(0) = 0, y(0) = -1$.

Solution:

The given simultaneous equations are

$$\frac{dx}{dt} + 2x - 3y = 5t;\ \frac{dy}{dt} - 3x + 2y = 0$$

i.e $(D+2)x - 3y = 5t$ ---(1) $(D+2)y - 3x = 0$ ---------(2)

Eliminate x from (1) and (2)

$(1)\times 3 \Rightarrow$ $3(D+2)x - 9y = 15t$ --------------------(3)

$(2)\times(D+2) \Rightarrow (D+2)^2 y - 3(D+2)x = 0$ -----------------(4)

(3)+(4) $\Rightarrow (D+2)^2 y - 9y = 15t$

i.e $(D^2 + 4D - 5)y = 15t$ ----------------(5)

The A.E of (5) is $m^2 + 4m - 5 = 0$

$\Rightarrow (m+5)(m-1) = 0$

$\Rightarrow m = -5 \text{ or } m = 1$

$\therefore$ C.F= $Ae^t + Be^{-5t}$

$$\text{P.I} = \frac{1}{D^2 + 4D - 5}(15t)$$

$$= 15\left[\frac{1}{-5\left(1 - \left(\frac{D^2+4D}{5}\right)\right)}\right]t$$

$$= -3\left[1 - \left(\frac{D^2+4D}{5}\right)\right]^{-1} t$$

$$= -3\left[1 + \left(\frac{D^2+4D}{5}\right) + \left(\frac{D^2+4D}{5}\right)^2 + \dots\right]t$$

$$= -3\left[1+\frac{4D}{5}\right]t$$

$$= -3\left[t+\frac{4}{5}\right]$$

The general solution of (5) is $y(t) = C.F + P.I = Ae^{t} + Be^{-5t} - 3t - \frac{12}{5}$

$$(2) \Rightarrow (D+2)y - 3x = 0$$

$$\Rightarrow 3x(t) = (D+2)(Ae^{t} + Be^{-5t} - 3t - \frac{12}{5})$$

$$= Ae^{t} - 5Be^{-5t} - 3 + 2Ae^{t} + 2Be^{-5t} - 6t - \frac{24}{5}$$

$$= 3Ae^{t} - 3Be^{-5t} - 6t - \frac{39}{5}$$

$$\Rightarrow x(t) = Ae^{t} - Be^{-5t} - 2t - \frac{13}{5}.$$

$\therefore$ The solutions of (1) and (2) are $x(t) = Ae^{t} - Be^{-5t} - 2t - \frac{13}{5}$ and

$y(t) = Ae^{t} + Be^{-5t} - 3t - \frac{12}{5}$

Given $x(0) = 0 = A(1) - B(1) - 0 - \frac{13}{5} \Rightarrow A - B = \frac{13}{5}$ ------(6)

Given $y(0) = -1 = A(1) + B(1) - 0 - \frac{12}{5} \Rightarrow A + B = \frac{7}{5}$ ------(7)

(6)+(7)$\Rightarrow 2A = 4 \Rightarrow A = 2$

(7)$\Rightarrow B = \frac{7}{5} - 2 = -\frac{3}{5}$

$\therefore$ The solutions of (1) and (2) are $x(t) = 2e^{t} + \frac{3}{5}e^{-5t} - 2t - \frac{13}{5}$ and

$$y(t) = 2e^t - \frac{3}{5}Be^{-5t} - 3t - \frac{12}{5}$$

5. **Solve** $(2D-3)x + Dy = e^t$, $Dx + (D+2)y = \cos 2t$. **(April/May 2005)**

Solution

The given simultaneous equations are $(2D-3)x + Dy = e^t$ **------(1)**

and $Dx + (D+2)y = \cos 2t$ **-------**(2)

Eliminate x from (1) and(2)

$(1)\times D \Rightarrow D(2D-3)x + D^2 y = D(e^t)$ **------------------------------------** (3)

(2)$\times(2D-3) \Rightarrow (2D-3)Dx + (2D-3)(D+2)y = (2D-3)\cos 2t$ **-----** (4)

---

(3)-**(4)**$\Rightarrow D^2 y - (2D-3)(D+2)y = D(e^t) - (2D-3)\cos 2t$

$\Rightarrow D^2 y - (2D^2 + 4D - 3D - 6)y = e^t - (2(-2\sin 2t) - 3\cos 2t)$

$\Rightarrow -(D^2 + D - 6)y = e^t + 4\sin 2t + 3\cos 2t$

$\Rightarrow (D^2 + D - 6)y = -e^t - 4\sin 2t - 3\cos 2t$

The A.E of (5) is $m^2 + m - 6 = 0 \Rightarrow m = -3, 2$

C.F= $Ae^{-3t} + Be^{2t}$

$$\therefore P.I = \left[\frac{1}{D^2 + D - 6}\right]\left[-(e^t + 4\sin 2t + 3\cos 2t)\right]$$

$$= -\left[\frac{1}{D^2+D-6}\right]e^t - 4\left[\frac{1}{D^2+D-6}\right]\sin 2t - 3\left[\frac{1}{D^2+D-6}\right]\cos 2t$$

$$= -\left[\frac{1}{1^2+1-6}\right]e^t - 4\left[\frac{1}{-2^2+D-6}\right]\sin 2t - 3\left[\frac{1}{-2^2+D-6}\right]\cos 2t$$

$$= \frac{1}{4}e^t - 4\left[\frac{1}{D-10}\right]\sin 2t - 3\left[\frac{1}{D-10}\right]\cos 2t$$

$$=\frac{1}{4}e^{t}-4\left[\frac{(D+10)}{D^{2}-100}\right]\sin 2t-3\left[\frac{(D+10)}{D^{2}-100}\right]\cos 2t$$

$$=\frac{1}{4}e^{t}-4\left[\frac{(D+10)}{-4-100}\right]\sin 2t-3\left[\frac{(D+10)}{-4-100}\right]\cos 2t$$

$$=\frac{1}{4}e^{t}+4\frac{(2\cos 2t+10\sin 2t)}{104}+3\frac{(-2\sin 2t+10\cos 2t)}{104}$$

$$=\frac{1}{4}e^{t}+\frac{(8\cos 2t+40\sin 2t)}{104}+\frac{(-6\sin 2t+30\cos 2t)}{104}$$

$$=\frac{1}{4}e^{t}+\frac{(38\cos 2t+34\sin 2t)}{104}$$

The general solution of (5) is $y(t)= Ae^{-3t}+Be^{2t}+\frac{1}{4}e^{t}+\frac{1}{104}(38\cos 2t+34\sin 2t)$

$$(1)\times 1-(2)\times 2 \Rightarrow -3x-Dy-4y=e^{t}-2\cos 2t$$

$$\Rightarrow -3x-\left[-3Ae^{-3t}+2Be^{2t}+\frac{1}{4}e^{t}+\frac{1}{104}(-76\sin 2t+68\cos 2t)\right]$$

$$-4\left[Ae^{-3t}+Be^{2t}+\frac{1}{4}e^{t}+\frac{1}{104}(38\cos 2t+34\sin 2t)\right]=e^{t}-2\cos 2t$$

$$\Rightarrow -3x-Ae^{-3t}-6Be^{2t}-\frac{5}{4}e^{t}-\left[\frac{1}{104}(60\sin 2t+228\cos 2t)\right]$$

$$x(t)=\frac{1}{3}Ae^{-3t}-2Be^{2t}-\frac{5e^{t}}{12}-\left(\frac{20\cos 2t+76\sin 2t}{104}\right)$$

Thesolutions of (1) and (2) are

$$y(t)= Ae^{-3t}+Be^{2t}+\frac{1}{4}e^{t}+\frac{1}{104}(38\cos 2t+34\sin 2t)$$

and $x(t)=\frac{1}{3}Ae^{-3t}-2Be^{2t}-\frac{5e^{t}}{12}-\left(\frac{20\cos 2t+76\sin 2t}{104}\right)$

6. **Solve** $Dx+y=\sin t,\ x+Dy=\cos t$ **given that** $x=2, y=0\quad at\quad t=0$.
(April/May 2006, May/June 2009)

Solution:

The given simultaneous equations are $Dx + y = \sin t ---(1)$ $x + Dy = \cos t ---(2)$

$(1)\times 1 \Rightarrow Dx + y = \sin t ------------(3)$

$(2)\times D \Rightarrow Dx + D^2 y = D(\cos t) --------(4)$

(3)-(4)$\Rightarrow (1-D^2)y = \sin t + \sin t = 2\sin t$

$(1-D^2)y = 2\sin t$ **----------------------(5)**

The A.E of (5) is $1-m^2 = 0 \Rightarrow m = \pm 1$

C.F= $Ae^t + Be^{-t}$

$$\therefore P.I = \left[\frac{1}{1-D^2}\right]2\sin t = 2\left[\frac{1}{1-(-1^2)}\right]\sin t = \sin t$$

The general solution of (5) is $y(t) = Ae^t + Be^{-t} + \sin t$

$\Rightarrow Dy = Ae^t - Be^{-t} + \cos t$

(2)$\Rightarrow x = \cos t - Dy = -Ae^t + Be^{-t}$

Thesolutions of (1) and (2) are $y(t) = Ae^t + Be^{-t} + \sin t$ and
$x(t) = -Ae^t + Be^{-t}$

Given $x(0) = 2 = -Ae^0 + Be^{-0} \Rightarrow -A + B = 2$ **-------------------(6)**

$y(0) = 0 = Ae^0 + Be^{-0} + \sin 0 \Rightarrow A + B = 0$ **-------------(7)**

(6)+(7)$\Rightarrow 2B = 2 \Rightarrow B = 1$ and **(7)**$\Rightarrow A = -1$

Thesolutions of (1) and (2) are $x(t) = e^t + e^{-t}$ and

$y(t) = -e^t + e^{-t} + \sin t$

7. **Solve** $\frac{dx}{dt} + y = \sin t + 1; \frac{dy}{dt} + x = \cos t$ **given that** $x = 1, y = 2 \quad at \ t = 0$ **.**

Solution:

The given simultaneous equations are

$$\frac{dx}{dt}+y=\sin t+1; \quad \frac{dy}{dt}+x=\cos t$$

i.e $Dx+y=\sin t+1---(1) \quad Dy+x=\cos t---(2)$

Eliminate x from (1) and (2)

$(1)\times 1 \Rightarrow \qquad Dx+y=\sin t+1$ -----------------------(3)

$(2)\times D \Rightarrow \qquad D^2y+Dx=D(\cos t)$ ------------------- (4)

$(3)\text{-}(4) \Rightarrow \qquad y-D^2y=\sin t+1-D(\cos t)$

i.e $(1-D^2)y=\sin t+1+\sin t$

$(1-D^2)y=2\sin t+1$ ------------------ (5)

The A.E of (5) is $1-m^2=0$

$\Rightarrow m^2=1 \Rightarrow m=\pm 1$

$\therefore$ C.F$= Ae^t+Be^{-t}$

$$\text{P.I}=\left[\frac{1}{1-D^2}\right](2\sin t+1)$$

$$=2\left[\frac{1}{1-D^2}\right]\sin t+\left[\frac{1}{1-D^2}\right]e^{0t}$$

$$=2\left[\frac{1}{1-(-1^2)}\right]\sin t+\left[\frac{1}{1-0^2}\right]e^{0t}$$

$$=2\left[\frac{1}{2}\right]\sin t+1$$

$$=\sin t+1$$

The general solution of (5) is $\quad y(t)=C.F+P.I=Ae^t+Be^{-t}+\sin t+1$

$(2)\Rightarrow x(t)=\cos t-D[Ae^t+Be^{-t}+\sin t+1]$

$= \cos t - [Ae^{t} - Be^{-t} + \cos t]$

$\Rightarrow x(t) = -Ae^{t} + Be^{-t}$

$\therefore$ The solutions of (1) and (2) are $x(t) = -Ae^{t} + Be^{-t}$ and

$y(t) = Ae^{t} + Be^{-t} + \sin t + 1$

Givcn $x(0) = 1 = -Ae^{0} + Be^{-0} \Rightarrow 1 = -A + B$ **---------------- (6)**

$y(0) = 2 = Ae^{0} + Be^{-0} + \sin 0 + 1 \Rightarrow 2 = A + B + 1 \Rightarrow A + B = 1$ **---(7)**

(6)+(7)$\Rightarrow 2B = 2 \Rightarrow B = 1$

(6)-(7) $\Rightarrow -2A = 0 \Rightarrow A = 0$

$\therefore$ The solutions of (1) and (2) are $x(t) = e^{-t}$ and $y(t) = e^{-t} + \sin t + 1$

8. Solve $Dx + y = \sin 2t;\ -x + Dy = \cos 2t$. **(June 2003)**

Solution:

The given simultaneous equations are

$Dx + y = \sin 2t$ **---------(1)** $-x + Dy = \cos 2t$ **-----------(2)**

$(1)\times 1 \Rightarrow Dx + y = \sin 2t$ **------------------------------(3)**

$(2)\times D \Rightarrow -Dx + D^{2}y = D(\cos 2t)$ **-------------------------(4)**

(3)+(4)$\Rightarrow (1 + D^{2})y = \sin 2t + D(\cos 2t)$

$= \sin 2t - 2\sin 2t = -\sin 2t$

$(D^{2} + 1)y = -\sin 2t$ **----------------------(5)**

The A.E of (5) is $m^{2} + 1 = 0 \Rightarrow m = \pm i$

C.F= $A\cos t + B\sin t$

$$\therefore P.I = \frac{1}{D^{2} + 1}(-\sin 2t)$$

$$= -\left[\frac{1}{-2^2+1}\right]\sin 2t$$

$$= \frac{1}{3}\sin 2t$$

The general solution of (5) is $y(t) = A\cos t + B\sin t + \frac{1}{3}\sin 2t$

$$\textbf{(2)} \Rightarrow x = Dy - \cos 2t$$

$$= -A\sin t + B\cos t + \frac{2}{3}\cos 2t - \cos 2t$$

$$= -A\sin t + B\cos t - \frac{1}{3}\cos 2t$$

The solutions of (1) and (2) are $y(t) = A\cos t + B\sin t + \frac{1}{3}\sin 2t$ and

$$x(t) = -A\sin t + B\cos t - \frac{1}{3}\cos 2t\,\textbf{.}$$

9. Solve $\frac{dx}{dt} + 2y = -\sin t;\ \ \frac{dy}{dt} - 2x = \cos t$ **given** $x = 1$ *and* $y = 0$ *at* $t = 0$ ***(Jan. 2005, Dec 2010)***

Solution:

The given simultaneous equations are

$$\frac{dx}{dt} + 2y = -\sin t;\ \ \frac{dy}{dt} - 2x = \cos t$$

i.e $Dx + 2y = -\sin t ---(1)\quad Dy - 2x = \cos t ---(2)$

Eliminate x from (1) and (2)

$(1)\times 2 \Rightarrow \quad 2Dx + 4y = -2\sin t$ ---------------------(3)

$(2)\times D \Rightarrow \quad D^2y - 2Dx = D(\cos t)$ ------------------(4)

---

(3)+(4)$\Rightarrow \quad 4y + D^2y = -2\sin t - \sin t$

i.e $(D^2+4)y = -3\sin t$ **--------------(5)**

The A.E of (5) is $m^2 + 4 = 0$

$\Rightarrow m = \pm i2$ Here $\alpha = 0;\ \beta = 2$

$\therefore$ C.F= $e^{0t}(A\cos 2\,t + B\sin 2\,t)$

$$= A\cos 2\,t + B\sin 2\,t$$

$$\text{P.I} = \left[\frac{1}{D^2+4}\right](-3\sin t)$$

$= -3\left[\dfrac{1}{-1^2+4}\right]\sin t$

$= -\sin t$

The general solution of (5) is $y(t) = C.F + P.I = A\cos 2t + B\sin 2\,t - \sin t$

$(2) \Rightarrow D[A\cos 2\,t + B\sin 2\,t] - 2x = \cos t$

$\Rightarrow\ 2x = 2[-A\sin 2\,t + B\cos 2\,t] - \cos t$

$\Rightarrow\ x = -A\sin 2\,t + B\cos 2\,t - \dfrac{\cos t}{2}$

$\therefore$ The solutions of (1) and (2) are

$$x(t) = -A\sin 2\,t + B\cos 2\,t - \frac{\cos t}{2} \text{ and}$$

$y(t) = A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t - \sin t$.

Given $x(0) = 1 = -A(0) + B(1) - \dfrac{(1)}{2} \Rightarrow B = \dfrac{3}{2}$

Given $y(0) = 0 = A(1) + B(0) - 0 \Rightarrow A = 0$.

$\therefore$ The solutions of (1) and (2) are $x(t) = \dfrac{3}{2}\cos 2\,t - \dfrac{\cos t}{2}$

and

$y(t) = \dfrac{3}{2}\sin 2\,t - \sin t$.

10. Solve $\frac{dx}{dt}-\frac{dy}{dt}+2y=\cos 2t;\quad \frac{dx}{dt}+\frac{dy}{dt}-2x=\sin 2t$ **. (Nov./Dec. 2005)**

Solution:

The given simultaneous equations are

$$\frac{dx}{dt}-\frac{dy}{dt}+2y=\cos 2t;\quad \frac{dx}{dt}+\frac{dy}{dt}-2x=\sin 2t$$

i.e $Dx-(D-2)y=\cos 2t---(1)\quad Dy+(D-2)x=\sin 2t---(2)$

Eliminate y from (1) and (2)

$(1)\times D \Rightarrow \quad D^2x-D(D-2)y=D(\cos 2t)$--------------------------(3)

$(2)\times D-2 \Rightarrow \quad D(D-2)y+(D-2)^2x=(D-2)\sin 2t$-----------------(4)

(3)+(4)$\Rightarrow \quad (D-2)^2x+D^2x=-2\sin 2t+2\cos 2t-2\sin 2t$

i.e $\quad (2D^2-4D+4)x=-4\sin 2t+2\cos 2t$

i.e $\quad (D^2-2D+2)x=-2\sin 2t+\cos 2t$ **--------------------------(5)**

The A.E of (5) is $m^2-2m+2=0$

$$\Rightarrow m=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}$$

$$=\frac{2\pm\sqrt{-4}}{2(1)}$$

$$=\frac{2\pm 2i}{2(1)}$$

$\Rightarrow m=1\pm i$ Here $\alpha=1;\beta=1$

$\therefore$ C.F$=e^t(A\cos t+B\sin t)$

$$\text{P.I}=\left[\frac{1}{D^2-2D+2}\right]\cos 2t-2\left[\frac{1}{D^2-2D+2}\right]\sin 2t$$

$$=\left[\frac{1}{-2^2-2D+2}\right]\cos 2t-2\left[\frac{1}{-2^2-2D+2}\right]\sin 2t$$

$$=\frac{1}{-2D-2}\cos 2t-\frac{2}{-2D-2}\sin 2t$$

$$= -\frac{1}{2}\left[\frac{1}{D+1}\right]\cos 2t + \left[\frac{1}{D+1}\right]\sin 2t$$

$$= -\frac{1}{2}\left[\frac{D-1}{D^2-1^2}\right]\cos 2t + \left[\frac{D-1}{D^2-1}\right]\sin 2t$$

$$= -\frac{1}{2}\left[\frac{D-1}{-2^2-1^2}\right]\cos 2t + \left[\frac{D-1}{-2^2-1}\right]\sin 2t$$

$$= -\frac{1}{2}\left[\frac{-2\sin 2t - \cos 2t}{-5}\right] + \left[\frac{2\cos 2t - \sin 2t}{-5}\right]$$

$$= -\frac{1}{10}\left[2\sin 2t + \cos 2t\right] - \frac{1}{5}\left[2\cos 2t - \sin 2t\right]$$

$$= -\frac{1}{10}\left[2\sin 2t + \cos 2t + 4\cos 2t - 2\sin 2t\right]$$

$$= -\frac{1}{10}\left[5\cos 2t\right]$$

$$= -\frac{1}{2}\cos 2t$$

The general solution of (5) is $x(t) = C.F + P.I \quad = e^t(A\cos t + B\sin t) - \frac{1}{2}\cos 2t$

$$(1)+(2) \Rightarrow 2\frac{dx}{dt} + 2y - 2x = \cos 2t + \sin 2t$$

$$\Rightarrow 2y = \cos 2t + \sin 2t + 2x - 2\frac{dx}{dt}$$

$$\Rightarrow 2y = \cos 2t + \sin 2t + 2e^t(A\cos t + B\sin t) - \cos 2t$$

$$-2\left[(A\cos t + B\sin t)e^t + e^t(-A\sin t + B\cos t)\right] - 2\frac{1}{2}(-2\sin 2t)$$

$$\Rightarrow 2y = \sin 2t + 2e^t(A\cos t + B\sin t)$$

$$-2\left[(A\cos t + B\sin t)e^t + e^t(-A\sin t + B\cos t)\right] - 2\frac{1}{2}(-2\sin 2t)$$

$$\Rightarrow 2y = \sin 2t + 2e^t(A\cos t + B\sin t)$$

$$-2\left[(A\cos t + B\sin t)e^t + e^t(-A\sin t + B\cos t)\right] - 2\frac{1}{2}(-2\sin 2t)$$

$$y = e^t(A\cos t - B\sin t) - \frac{\sin 2t}{2}$$

$\therefore$ The solutions of (1) and (2) are

$$x(t) = e^t(A\cos t + B\sin t) + \frac{(-2\sin 2t - \cos 2t)}{10} - \frac{(2\cos 2t - \sin 2t)}{5}$$

and $y(t) = e^t(A\cos t - B\sin t) - \frac{\sin 2t}{2}$.

11. Solve $\frac{dx}{dt} + 2y = \sin 2t;\ \frac{dy}{dt} - 2x = \cos 2t$ ***(Nov. Dec/2009)***

Solution:

The given simultaneous equations are

$$\frac{dx}{dt} + 2y = \sin 2t;\ \frac{dy}{dt} - 2x = \cos 2t$$

i.e $Dx + 2y = \sin 2t ---(1) \quad Dy - 2x = \cos 2t ---(2)$

Eliminate x from (1) and (2)

$(1)\times 2 \Rightarrow \quad 2Dx + 4y = 2\sin 2t$ --------------------(3)

$(2)\times D \Rightarrow \quad D^2y - 2Dx = D(\cos 2t)$ ----------------- (4)

$(3)+(4) \Rightarrow \quad 4y + D^2y = 2\sin 2t - 2\sin 2t$

i.e $(D^2 + 4)y = 0$ ---------------- (5)

The A.E of (5) is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$ Here $\alpha = 0; \beta = 2$

$\therefore$ C.F$= e^{0t}(A\cos 2t + B\sin 2t) = A\cos 2t + B\sin 2t$

The general solution of (5) is $y(t) = C.F = A\cos 2t + B\sin 2t$

$$(2) \Rightarrow 2x = Dy - \cos 2t - - - (2)$$

$$= (-2A\sin 2t + 2B\cos 2t) - \cos 2t$$

$$x(t) = -A\sin 2t + B\sin 2t - \frac{1}{2}\cos 2t$$

$\therefore$ The solutions of (1) and (2) are $x(t) = -A\sin 2t + B\sin 2t - \frac{1}{2}\cos 2t$ and

$y(t) = A\cos 2t + B\sin 2t$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: MATHEMATICS-II **Subject Code: 17PHU602**

Class : III - B.Sc. Physics **Semester : VI**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
An equation involving one or more dependent variables with respect to one or more independent variables is called……………………....	differential equations	intergral equation	constant equation	Eulers equation	differential equations
An equation involving one or more ……….. variables with respect to one or more independent variables is called differential equations	single	dependent	independent	constant	dependent
An equation involving one or more dependent variables with respect to one or more……….variables is called differential equations	dependent	independent	single	different	independent
A differential equation involving ordinary derivatives of one or moredependentvariables with respect to single independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	ordinary differential equations

A differential equation involving ordinary derivatives of one or more dependentvariables with respect to………. independent variables is called ordinary differential equations	zero	single	different	one or more	single
A differential equation involving ………… derivatives of one or more dependentvariables with respect to single independent variables is called ordinary differential equations	partial	different	total	ordinary	ordinary
A differential equation involving partial derivatives of one or more dependent variables with respect to oneor more independent variables is called ……………	differential equations	partial differential equations	ordinary differential equations	total differential equations	partial differential equations
A differential equation involving partial derivatives of one or more dependentvariables with respect to………. independent variables is called partial differential equations	zero	single	different	one or more	oneormore
A differential equation involving ………… derivatives of one or more dependentvariables with respect to one or moreindependent variables is called partial differential equations	partial	different	total	ordinary	partial
The order of ……...derivatives involvedin the differential equations is called order of the differential equation	zero	lowest	highest	infinite	highest
The order of highest derivatives involvedin the differential equations is called …………….. of the differential equation	order	power	value	root	order
The order of highest ……………….. involvedin the differential equations is called order of the differential equation	derivatives	intergral	power	value	derivatives
The order of the differential equations is (d^2 y)/〖dx〗^2 +xy(dy/dx)^2=1	0	1	2	4	2
A non linear ordinary differential equation is an ordinary differential equation that is not …………	linear	non linear	differential	intergral	linear

A ……………..ordinary differential equation is an ordinary differential equation that is not linear	linear	non linear	differential	intergral	non linear
A non linear ordinary differential equation is an ……………. differential equation that is not linear	ordinary	partial	single	constant	ordinary
………….. ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	linear	non linear	differential	intergral	linear
Linear ………………. differential equations are further classified according to the nature of the coefficients of the dependent variables and its derivatives	ordinary	partial	single	constant	ordinary
Linear ordinary differential equations are further classified according to the nature of the coefficients of the ………………variables and its derivatives	single	dependent	independent	constant	dependent
Linear ordinary differential equations are further classified according to the nature of the coefficients of the dependent variables and its ……………………..	integrals	constant	derivatives	roots	derivatives
Both explicit and implicit solutions will usually be called simply ……………	solutions	constant	equations	values	solutions
Both ……………….. solutions will usually be called simply solutions.	general and particular	singular and non singular	ordinary and partial	explicit and implicit	explicit and implicit
Let f be a real function defined for all x in a real interval I and having nth order derivatives then the function f is called …………solution of the differential equations	constant	implicit	explicit	general	explicit
Let f be a real function defined for all x in a real interval I and having …………..order derivatives then the function f is called explicit solution of the differential equations	1st	2nd	nth	(n+1)th	nth

The relation g(x,y)=0 is called the ………solution of F[x,y,(dy/dx)………..(dy/dx)^n]=0	constant	implicit	explicit	general	implicit

UNIT-II

Finding the solution of Second and Higher Order with constant coefficients with Right Hand Side is of the form $V\,e^{ax}$, where V is a function of x – Euler's Homogeneous Linear Differential Equations– System of simultaneous linear differential equations with constant coefficients.

1.1.6 Problems based on R.H.S = $e^{ax}x$. Particular Integral = $\frac{1}{f(D)}e^{ax}x = e^{ax}\frac{1}{f(D+a)}x$

1. **Solve:** $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

   Solution:

   Given $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

   A.E is $m^2+4m+3=0$

   $(m+1)(m+3)=0$

   $m=-1, m=-3$

   $C.F = Ae^{-x}+Be^{-3x}$

$$P.I_1 = \frac{1}{D^2+4D+3}e^{-x}\sin x$$

$$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$= e^{-x}\frac{1}{D^2-2D+1+4D-4+3}\sin x$$

$$= e^{-x}\frac{1}{D^2+2D}\sin x$$

$$= e^{-x}\frac{1}{-1+2D}\sin x$$

Take Conjugate we get,

$$= e^{-x}\frac{2D+1}{(2D)^2-1}\sin x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}\sin x$$

$$= e^{-x}\frac{2D+1}{-4-1}\sin x$$

$$= \frac{e^{-x}}{-5}(2D+1)\sin x$$

$$P.I_1 = \frac{e^{-x}}{-5}(2\cos x + \sin x)$$

$$P.I_2 = \frac{1}{D^2+4D+3} xe^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2+4(D+3)+3} x$$

$$= e^{3x} \frac{1}{D^2+6D+9+4D+12+3} x$$

$$= e^{3x} \frac{1}{D^2+10D+24} x$$

$$= \frac{e^{3x}}{24} \frac{1}{\left[\frac{D^2}{24}+\frac{10}{24}D+1\right]} x$$

$$= \frac{e^{3x}}{24}\left[1+\frac{5}{12}D+\frac{D^2}{24}\right]^{-1} x$$

$$= \frac{e^{3x}}{24}\left[1-\left(\frac{5}{12}D+\frac{D^2}{24}\right)+....,\right]x$$

$$= \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

$$y = C.F + P.I$$

$$y = Ae^{-x} + Be^{-3x} - \frac{e^{-x}}{5}(2\cos x + \sin x) + \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

2. Solve $(D^2-2D+2)y = e^x x^2 + 5 + e^{-2x}$

Solution:

Given $(D^2-2D+2)y = e^x x^2 + 5 + e^{-2x}$

A.E is $m^2-2m+2=0$

$m = 1 \pm i$

$C.F = e^{x}(A\cos x + B\sin x)$

$P.I_1 = \frac{1}{D^2 - 2D + 2} e^{x} x^2$

$= e^{x} \frac{1}{(D+1)^2 - 2(D+1) + 2} x^2$

$= e^{x} \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2} x^2$

$= e^{x} \frac{1}{D^2 + 1} x^2$

$= e^{x}(D^2 + 1)^{-1} x^2$

$= e^{x}(1 - D^2 + ...)x^2$

$P.I_1 = e^{x}(x^2 - 2)$

$P.I_2 = \frac{1}{D^2 - 2D + 2} 5e^{0x}$

$P.I_2 = \frac{5}{2}$

$P.I_3 = \frac{1}{D^2 - 2D + 2} e^{-2x}$

$= \frac{1}{4 + 4 + 2} e^{-2x}$

$P.I_3 = \frac{1}{10} e^{-2x}$

y = C.F +P.I

$y = e^{x}(A\cos x + B\sin x) + e^{x}(x^2 - 2) + \frac{5}{2} + \frac{1}{10} e^{-2x}$

3. Solve $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. **(Nov./Dec. 2002)**

Solution:

The given ODE is $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$ **----(1)**

The A.E of (1) is $m^2 + 4m + 3 = 0$

$(m+1)(m+3) = 0$

$m = -1, m = -3$

C.F= $Ae^{-x}+Be^{-3x}$.

$$\text{P.I}=\frac{1}{f(D)}e^{-x}\sin x+\frac{1}{f(D)}xe^{3x}=P.I_1+P.I_2$$

Now $P.I_1=\frac{1}{D^2+4D+3}e^{-x}\sin x=e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$

$=$

$$e^{-x}\frac{1}{D^2+2D}\sin x=e^{-x}\frac{1}{-1+2D}\sin x=e^{-x}\frac{(2D+1)}{(2D)^2-1^2}\sin x$$

$$=e^{-x}\frac{(2D+1)}{-4-1}\sin x=-\frac{e^{-x}}{5}(2\cos x+\sin x)$$

$$P.I_2=\frac{1}{f(D)}xe^{3x}=\frac{1}{D^2+4D+3}e^{3x}x=e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$=e^{3x}\frac{1}{D^2+10D+24}x=\frac{e^{3x}}{24}\left[1+\left(\frac{D^2+10D}{24}\right)\right]^{-1}x$$

$$=\frac{e^{3x}}{24}\left[1-\left(\frac{D^2+10D}{24}\right)+\left(\frac{D^2+10D}{24}\right)^2-\ldots\right]x$$

$$=\frac{e^{3x}}{24}\left[1-\frac{5}{12}D\right]x \quad \textit{omitting Higher order derivatives}$$

$$=\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]\therefore P.I=P.I_1+P.I_2=-\frac{e^{-x}}{5}(2\cos x+\sin x)+\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

The general solution of (1) is $y(x)=$ C.F+P.I

$$=Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}(2\cos x+\sin x)+\frac{e^{3x}}{24}\left[x-\frac{5}{12}\right].$$

4.Solve $(D^2-2D+2)y=e^x x^2+5+e^{-2x}$**. (April/May 2003)**

Solution:

The given ODE is $(D^2-2D+2)y = e^x x^2 + 5 + e^{-2x}$ **-----(1)**

The A.E of (1) is $m^2-2m+2=0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = 1 \pm i$$

C.F= $e^x(A\cos x + B\sin x)$.

$$\text{P.I}= \frac{1}{f(D)}e^x x^2 + \frac{1}{f(D)}5 + \frac{1}{f(D)}e^{-2x} = P.I_1 + P.I_2 + P.I_3$$

$$\text{Now } P.I_1 = \frac{1}{D^2-2D+2}e^x x^2 = e^x \frac{1}{(D+1)^2 - 2(D+1)+2}x^2$$

$$= e^x \frac{1}{D^2+1}x^2 = e^x(1+D^2)^{-1}x^2 = e^x(1-D^2+(D^2)^2 - \dots)x^2$$

$$= e^x(1-D^2)x^2 = e^x(x^2-2)$$

$$P.I_2 = 4\frac{1}{D^2-2D+2}e^{0x} = 4\frac{1}{2} = 2$$

$$P.I_3 = \frac{1}{D^2-2D+2}e^{-2x} = \frac{1}{(-2)^2-2(-2)+2}e^{-2x} = \frac{e^{-2x}}{10}$$

$$\text{P.I} = P.I_1 + P.I_2 + P.I_3$$

$$= e^x(x^2-2) + 2 + \frac{e^{-2x}}{10}$$

The general solution of (1) is $y(x) = \text{C.F+P.I}$

$$= e^x(A\cos x + B\sin x) + e^x(x^2-2) + 2 + \frac{e^{-2x}}{10}$$

1.1.7 Problems based on $f(x) = x^n \sin ax$ or $x^n \cos ax$

To find Particular Integral when $f(x) = x^n \sin ax \;\; or \;\; x^n \cos ax$

$$P.I = \frac{1}{f(D)} x^n \sin ax \quad (or) \;\; x^n \cos ax$$

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left[\frac{d}{dD}\frac{1}{f(D)}\right]V$$

$$\frac{1}{f(D)}(x.V) = x\frac{1}{f(D)}V - \left[\frac{f'(D)}{f(D)}\frac{1}{f(D)}\right]V$$

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \left[\frac{f'(D)}{\left[f(D)\right]^2}\right]V$$

1. Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Solution:

Given $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

The roots are $m = 2, 2$.

Complementary Function is $(c_1 x + c_2)e^{2x}$

Particular Integral $= \dfrac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x$

$$= 8e^{2x}\frac{1}{(D-2)^2}x^2 \sin 2x$$

$$= 8e^{2x}\frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - 2x\left(\frac{-\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right)\right\}$$

$$= e^{2x}\left\{\frac{1}{D}\left(-4x^2 \cos 2x\right) + \frac{1}{D}(4x \sin 2x) + \frac{1}{D}(2\cos 2x)\right\}$$

$$= e^{2x}\left[\left\{\left(-4x^2\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{4}\right)\right\} + 4\left\{x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right) + \sin 2x\right\}\right]$$

$$= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]$$

The general Solution is y = C.F + P.I.

$$y=(c_1x+c_2)e^{2x}+e^{2x}(3-2x^2)\sin 2x-4x\cos 2x$$

2. Solve the differential equation $(D^2+4)y=x^2\cos 2x$ **(May/ June 2009)**

Solution:

The given ODE is $(D^2+4)y=x^2\cos 2x$ **----(1)**

The A.E of (1) is $m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm 2i$

C.F= $A\cos 2x+B\sin 2x$

$$\text{P.I}=\left[\frac{1}{f(D)}\right]x^2\cos 2x = \left[\frac{1}{D^2+4}\right]x^2 R.P\ of\ e^{i2x}$$

$$= R.P\ \ of\ e^{i2x}\left[\frac{1}{(D+2i)^2+4}\right]x^2 = R.P\ of\ e^{i2x}\left[\frac{1}{D^2+4Di}\right]x^2$$

$$= R.P\ of\left[\frac{e^{i2x}}{4Di\left(1+\frac{D^2}{4Di}\right)}\right]x^2$$

$$= R.P\ of\ \frac{-i^2e^{i2x}}{4Di}\left(1+\frac{D}{4i}\right)^{-1}x^2$$

$$= R.P\ of\ \frac{-ie^{i2x}}{4D}\left(1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2-\left(\frac{D}{4i}\right)^3+\ldots\right)x^2$$

$$= R.P\ of\ -\frac{ie^{i2x}}{4}\left(\frac{1}{D}-\frac{1}{4i}+\left(-\frac{D}{16}\right)-\left(-\frac{D^2}{64i}\right)\right)x^2$$

$$= R.P\ of\ \frac{e^{i2x}}{4}\left(-\frac{i}{D}+\frac{1}{4}+\left(\frac{Di}{16}\right)-\left(\frac{D^2}{64}\right)\right)x^2$$

$$= R.P\ of\ \frac{e^{i2x}}{4}\left(-i\left(\frac{x^3}{3}\right)+\frac{x^2}{4}+\left(\frac{2xi}{16}\right)-\left(\frac{2}{64}\right)\right)$$

$$= R.P\ of\ \frac{(\cos 2x + i\sin 2x)}{4}\left(\left(\frac{x^2}{4}-\frac{1}{32}\right)-i\left(\frac{x^3}{3}-\frac{x}{8}\right)\right)$$

$$= \frac{1}{4}\left[\left(\frac{x^2}{4}-\frac{1}{32}\right)\cos 2x + \left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x\right]$$

The general solution of (1) is $y(x) = \text{C.F+P.I}$ =

$$A\cos 2x + B\sin 2x + \frac{1}{4}\left[\left(\frac{x^2}{4}-\frac{1}{32}\right)\cos 2x + \left(\frac{x^3}{3}-\frac{x}{8}\right)\sin 2x\right].$$

1.1.8 Problems based on $\frac{1}{D-a}f(x) = e^{ax}\int e^{-ax} f(x)dx$ Type.
[General Method of finding the Particular Integral of any function f(x)]

1. Solve $(D^2 + a^2)y = \sec ax$.

Solution:

Given $(D^2 + a^2)y = \sec ax$

A. E. is $m^2 + a^2 = 0$

The Roots are $m = \pm\, ia$

Complementary function = $A\cos ax + B\sin ax$ **.**

$$P.I = \frac{1}{(D^2+a^2)}\sec ax$$

$$P.I = \frac{1}{(D-ia)(D+ia)}\sec ax$$

$$= \left(\frac{\frac{1}{2ia}}{D-ia} - \frac{\frac{1}{2ia}}{D+ia}\right)\sec ax$$

$$= \frac{1}{2ia}e^{iax}\int e^{-iax}\sec ax dx - \frac{1}{2ia}e^{-iax}\int e^{iax}\sec ax dx \quad [\frac{1}{D-m}X = e^{mx}\int X e^{-mx}dx]$$

$$= \frac{1}{2ia}e^{iax}\int (1 - i\tan ax)dx - \frac{1}{2ia}e^{-iax}\int (1 + i\tan ax)dx$$

$$= \frac{1}{2ia}e^{iax}(x - \frac{i}{a}\log\sec ax) - \frac{1}{2ia}e^{-iax}(x + \frac{i}{a}\log\sec ax)$$

$$= \frac{x}{a}\left(\frac{e^{iax}-e^{-iax}}{2i}\right)-\frac{1}{a^2}\log\sec ax\left(\frac{e^{iax}+e^{-iax}}{2i}\right)$$

$$= \frac{x}{a}\sin ax-\frac{1}{a^2}\log\sec ax\cos ax$$

General Solution is y = C.F + P.I

$$y = A\cos ax + B\sin ax+\frac{x}{a}\sin ax-\frac{1}{a^2}\log\sec ax\cos ax$$

Homogeneous Equations of Euler Type [Cauchy's Type]

Linear Differential Equations with Variable Co-efficient

An Equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^n}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_ny=f(x) \qquad \ldots\ldots\ldots\ldots(1)$$

Where$a_1, a_2, \ldots a_n$re constants and f(x) is a function of x.

Equation(1) can be reduced to linear differential equation with constant

Co - efficient by putting the substitution.

$$x=e^z(or)z=logx$$

$$x\frac{dy}{dx}=\frac{dy}{dz}=D'y \qquad (2)$$

$$D'=\frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2}=x^2\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$= (D'^2-D')y$ where $D'=\frac{d}{dz}$ (3)

Similarly,

$$x^2\frac{d^2y}{dx^2}=D'(D'-1)(D'-2)y \qquad (4)$$

and so on, substituting (2), (3), (4) and so on in (1) we get a differential equation with constant coefficients and can be solved by any one of the known methods.

PROBLEMS BASED ON CAUCHY'S TYPE

1. **Solve** $x^2y'' + 2xy' + 2y = 0$.

Solution:

The given ODE is $x^2y'' + 2xy' + 2y = 0$. i.e $(x^2D^2 + 2xD + 2)y = 0$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) + 2D'+2)y = 0$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 + D'+2)y = 0$ **----------------(2)**

The A.E of (2) is $m^2 + m + 2 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$\text{C.F} = e^{-\frac{1}{2}z}\left(A\cos\frac{\sqrt{7}}{2}z + B\sin\frac{\sqrt{7}}{2}z\right)$$

$\therefore$ The general solution of (1) is $y(x) = \text{C.F}$

$$= e^{-\frac{1}{2}\log x}\left[A\cos\left(\frac{\sqrt{7}}{2}\log x\right) + B\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right]$$

$$= \frac{1}{\sqrt{x}}\left[A\cos\left(\frac{\sqrt{7}}{2}\log x\right) + B\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right].$$

2. **Solve** $x^2y'' - xy' + y = x$. **(June 2004)**

Solution:

The given ODE is $x^2y'' - xy' + y = x$. i.e $(x^2D^2 - xD + 1)y = x$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) - D'+1)y = e^z$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 - 2D' + 1)y = e^z$ **-----------------(2)**

The A.E of (2) is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1,1$

$\therefore$ C.F= $(Az + B)e^z$

Now P.I= $\frac{1}{f(D')}e^z = \frac{1}{(D'-1)^2}e^z = \frac{1}{0}e^z$ (Ordinary rule fails)

$= z\frac{1}{2(D'-1)}e^z$ (Ordinary rule fails)

$= z^2\frac{1}{2}e^z$

The general solution of (2) is $y(z) =$ C.F+P.I= $(Az + B)e^z + z^2\frac{1}{2}e^z$ $\therefore$ $y(x) =$

$(A\log x + B)e^{\log x} + (\log x)^2 \frac{e^{\log x}}{2}$

$= (A\log x + B)x + (\log x)^2\frac{x}{2}$ is the required general Solution of (1)

3. Solve $(x^2D^2 - 7xD + 12)y = x^2$.

Solution:

The given ODE is $(x^2D^2 - 7xD + 12)y = x^2$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x$, $xD = D'$; $x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) - 7D' + 12)y = e^{2z}$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 - 8D' + 12)y = e^{2z}$ **-----------------(2)**

The A.E of (2) is $m^2 - 8m + 12 = 0 \Rightarrow (m-6)(m-2) = 0 \Rightarrow m = 2,6$

$\therefore$ C.F= $Ae^{2z} + Be^{6z}$

Now P.I= $\frac{1}{f(D')}e^{2z} = \frac{1}{D'^2 - 8D' + 12}e^{2z} = \frac{1}{0}e^{2z}$ (Ordinary rule fails)

$$= z\frac{1}{2D'-8}e^{2z} = -\frac{ze^{2z}}{4}$$

The general solution of (2) is $y(z) = \text{C.F+P.I} = Ae^{2z} + Be^{6z} - \frac{ze^{2z}}{4}$ $\therefore$ $y(x) =$ $Ax^2 + Bx^6 - \frac{x^2\log x}{4}$ is the required general Solution of (1)

4. Solve $(x^2D^2 + 4xD + 2)y = \log x$ **given that when** $x = 1, y = 0, \frac{dy}{dx} = 0$**.**

Solution:

The given ODE is $(x^2D^2 + 4xD + 2)y = \log x$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) + 4D'+2)y = z$ $\quad D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2+3D'+2)y = e^z z$ **----------------(2)**

The A.E of (2) is $m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1,\ -2$

$\therefore$ C.F= $Ae^{-z} + Be^{-2z}$

Now P.I= $\frac{1}{f(D')}z = \frac{1}{D'^2+3D'+2}z = \frac{1}{2\left(1+\frac{D'^2+3D}{2}\right)}z$

$$= \frac{1}{2}\left[1+\left(\frac{D'^2+3D}{2}\right)\right]^{-1} z = \frac{1}{2}\left[1-\left(\frac{D'^2+3D}{2}\right)+\left(\frac{D'^2+3D}{2}\right)^2 - \ldots\right]z$$

$= \frac{1}{2}\left[1-\frac{3D}{2}\right]z$ omitting second and Higher derivatives

$$= \frac{1}{2}\left[z-\frac{3}{2}\right]$$

The general solution of (2) is $y(z) = \text{C.F+P.I} = Ae^{-z} + Be^{-2z} + \frac{1}{2}\left[z - \frac{3}{2}\right] \therefore\ y(x) =$

$Ax^{-1} + Bx^{-2} + \frac{1}{4}\left[2\log x - 3\right]$ is the required general solution of (1).

Given that $y(1) = 0;\ y'(1) = 0$

$y(1) = 0 = A + B + \frac{1}{4}\left[0 - 3\right] \Rightarrow A + B = \frac{3}{4}$ **-------(3)**

$y'(x) = -Ax^{-2} - 2Bx^{-3} + \frac{1}{2x}$

$y'(1) = -A - 2B + \frac{1}{2} = 0\ \ A + 2B =$

5. Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x\log x$ **. (Nov./Dec. 2006)**

Solution:

The given ODE is $(x^2D^2 + 4xD + 2)y = x\log x$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ \ xD = D';\ \ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) + 4D'+2)y = ze^z$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 + 3D' + 2)y = e^z z$ **-----------------(2)**

The A.E of (2) is $m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1,\ -2$

$\therefore$ C.F= $Ae^{-z} + Be^{-2z}$

Now P.I= $\frac{1}{f(D')}e^z z = \frac{1}{D'^2 + 3D' + 2}e^z z = e^z \frac{1}{(D'+1)^2 + 3(D'+1) + 2} z$

$= e^z \frac{1}{D'^2 + 5D' + 6} z$

$$= e^z \frac{1}{6} \frac{1}{\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]} z = e^z \frac{1}{6}\left[1+\left(\frac{D'^2+5D'}{6}\right)\right]^{-1} z$$

$$= e^z \frac{1}{6}\left[1-\left(\frac{D'^2+5D'}{6}\right)+\left(\frac{D'^2+5D'}{6}\right)^2 - \ldots\right] z$$

$$= e^z \frac{1}{6}\left[1-\frac{5D'}{6}\right] z$$ omitting second and Hr. order derivatives

$$= e^z \frac{1}{6}\left[z-\frac{5}{6}\right]$$

The general solution of (2) is $y(z) =$ C.F+P.I$= Ae^{-z} + Be^{-2z} + e^z \frac{1}{6}\left[z-\frac{5}{6}\right]$ $\therefore$ $y(x) = Ax^{-1} + Bx^{-2} + \frac{x}{6}\left[\log x - \frac{5}{6}\right]$ is the required general solution of (1).

6. **Solve** $(x^2D^2 - 2xD - 4)y = 32(\log x)^2$. **(April/May 2005)**

Solution:

The given ODE is $(x^2D^2 - 2xD - 4)y = 32(\log x)^2$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) - 2D'-4)y = 32z^2$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 - 3D' - 4)y = 32z^2$ **----------------(2)**

The A.E of (2) is $m^2 - 3m - 4 = 0 \Rightarrow (m-4)(m+3) = 0 \Rightarrow m = -3,\ 4$

$\therefore$ C.F= $Ae^{-3z} + Be^{4z}$

Now P.I$= \frac{1}{f(D')} 32z^2 = \frac{1}{D'^2 - 3D' - 4} 32z^2 = \frac{1}{-4\left[1-\frac{(D'^2-3D')}{4}\right]} 32z^2$

$$= \frac{1}{-4}\left[1-\left(\frac{D'^2-3D'}{4}\right)\right]^{-1} 32z^2$$

$$= \frac{32}{-4}\left[1+\left(\frac{D'^2-3D'}{4}\right)+\left(\frac{D'^2-3D'}{4}\right)^2 \ldots\right]z^2$$

$$= -8\left[1+\left(\frac{D'^2-3D'}{4}\right)+\frac{1}{16}(D'^4+9D'^2-6D'^3)+\ldots\right]z^2$$

$$=-8\left[1+\left(\frac{D'^2-3D'}{4}\right)+\frac{1}{16}(9D'^2)\right]z^2$$ **Omitting Hr. Derivatives**

$$= -8\left[z^2+\frac{1}{4}(2)-\frac{3}{4}(2z)+\frac{1}{16}[9(2)]\right]$$

$$=-8\left[z^2-\frac{3}{2}z+\frac{13}{8}\right]$$

The general solution of (2) is $y(z) = \text{C.F+P.I}= Ae^{-3z} + Be^{4z} + -8\left[z^2-\frac{3}{2}z+\frac{13}{8}\right]$

$\therefore\ y(x) = Ax^{-3} + Bx^4 - 8\left[(\log x)^2-\frac{3}{2}\log x+\frac{13}{8}\right]$ **is** the required general solution of (1).

7. Solve $(x^2D^2 - xD+1)y = \left(\frac{\log x}{x}\right)^2$ ***. (Nov./Dec 2005)***

Solution:

The given ODE is $(x^2D^2 - xD+1)y = \left(\frac{\log x}{x}\right)^2$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1)-D'+1)y = \left(ze^{-z}\right)^2 = e^{-2z}z^2$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2-2D'+1)y = e^{-2z}z^2$ **----------------(2)**

The A.E of (2) is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1,1$

$\therefore$ C.F= $(Az + B)e^z$

Now P.I= $\dfrac{1}{f(D')} e^{-2z} z^2 = \dfrac{1}{(D'-1)^2} e^{-2z} = \dfrac{1}{0} e^z$ (Ordinary rule fails)

$= z \dfrac{1}{2(D'-1)} e^z$ (Ordinary rule fails)

$= z^2 \dfrac{1}{2} e^z$

The general solution of (2) is $y(z) =$ C.F+P.I= $(Az + B)e^z + z^2 \dfrac{1}{2} e^z$

$\therefore\ y(x) = (A\log x + B)e^{\log x} + (\log x)^2 \dfrac{e^{\log x}}{2}$

$y(x) = (A\log x + B)x + (\log x)^2 \dfrac{x}{2}$ is the required general Solution of (1)

8. i)Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$ **. (AU June 2010)**

Solution:

The given ODE is $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) - 2D' - 4)y = e^{2z} + 2z$,where

$\Rightarrow (D'^2 - 3D' - 4)y = 32z^2$ **---------------- (2)**

$D = \dfrac{d}{dx}; D' = \dfrac{d}{dz}$

The A.E of (2) is $m^2 - 3m - 4 = 0 \Rightarrow (m-4)(m+1) = 0 \Rightarrow m = -1,\ 4$

$\therefore$ C.F= $Ae^{-z} + Be^{4z}$

Now P.I= $\dfrac{1}{f(D')}(e^{2z} + 2z) = \dfrac{1}{D'^2 - 3D' - 4} e^{2z} + \dfrac{1}{D'^2 - 3D' - 4} 2z$

$= P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D'^2 - 3D' - 4} e^{2z}$$

$$= \frac{1}{2^2 - 3(2) - 4} e^{2z}$$

$$= -\frac{e^{2z}}{6}.$$

$$P.I_2 = \frac{1}{D'^2 - 3D' - 4} 2z$$

$$= \frac{2}{-4} \frac{1}{\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]} z$$

$$= \frac{-1}{2}\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} z$$

$$= \frac{-1}{2}\left[1 + \left(\frac{D'^2 - 3D'}{4}\right) + \left(\frac{D'^2 - 3D'}{4}\right)^2 + \ldots\right] z$$

$$= \frac{-1}{2}\left[1 - \frac{3D'}{4}\right] z$$ **{Omitting Hr. Derivatives}**

$$= \frac{-1}{2}\left[z - \frac{3}{4}\right]$$

$$P.I = P.I_1 + P.I_2$$

$$= -\frac{e^{2z}}{6} - \frac{1}{2}\left[z - \frac{3}{4}\right]$$

The general solution of (2) is $y(z) = \text{C.F+P.I} = Ae^{-z} + Be^{4z} - \frac{e^{2z}}{6} - \frac{1}{2}\left[z - \frac{3}{4}\right]$

$\therefore\ y(x) = Ax^{-1} + Bx^4 - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$ **i**s the required general solution of (1).

ii) Solve $x^2 y'' + 3xy' + 5y = x\cos(\log x) + 3$. ***(Nov./Dec. 2006, May / June 2009)***

Solution:

The given ODE is $(x^2D^2 + 3xD + 5)y = x\cos(\log x) + 3$ ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ xD = D';\ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) + 3D' + 5)y = e^z\cos z + 3$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 + 2D' + 5)y = e^z\cos z + 3$ **----------------(2)**

The A.E of (2) is $m^2 + 2m + 5 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm 2i$

$\therefore$ C.F= $e^{-z}(A\cos 2z + B\sin 2z)$

Now P.I= $\frac{1}{f(D')}(e^z\cos z + 3) = \frac{1}{f(D')}e^z\cos z + 3\frac{1}{f(D')}e^{0z}$

$= P.I_1 + P.I_2$

Now $P.I_1 = \frac{1}{D'^2 + 2D' + 5}e^z\cos z$

$= e^z\frac{1}{(D'+1)^2 + 2(D'+1) + 5}\cos z$

$= e^z\frac{1}{D'^2 + 4D' + 8}\cos z$ Replace D'^2 $by - a^2$

$= e^z\frac{1}{-1 + 4D' + 8}\cos z$

$= e^z\frac{1}{-1 + 4D' + 8}\cos z$

$$= e^z \frac{1}{4D'+7} \cos z = e^z \frac{4D'-7}{16D'^2 - 49} \cos z$$

$$= e^z \frac{(-4\sin z - 7\cos z)}{-65}$$

$$= e^z \frac{(4\sin z + 7\cos z)}{65}$$

The general solution of (2) is

$$y(z) = \text{C.F+P.I} = e^{-z}(A\cos 2z + B\sin 2z) + e^z \frac{(4\sin z + 7\cos z)}{65}$$

$$\therefore\ y(x) = \frac{1}{x}[A\cos(\log x^2) + B\sin(\log x^2)] + \frac{x(4\sin(\log x) + 7\cos(\log x))}{65}$$

is the required general solution of (1).

9. **Solve** $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$. **.(AU Dec 2010)**

Solution:

The given ODE is $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$. ---(1)

To solve (1) use $x = e^z \Rightarrow z = \log x,\ \ xD = D';\ \ x^2D^2 = D'(D'-1)$

$\therefore$ (1) becomes $(D'(D'-1) - 3D'+4)y = e^{2z}\cos z$,where $D = \frac{d}{dx}; D' = \frac{d}{dz}$

$\Rightarrow (D'^2 - 4D'+4)y = e^{2z}\cos z$ **-----------------(2)**

The A.E of (2) is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$\Rightarrow (m-2)(m-2) = 0 \Rightarrow m = 2, 2$

$\therefore$ C.F= $e^{2z}(Az + B)$

Now P.I $= \frac{1}{f(D')}(e^{2z}\cos z)$

$$= \left[\frac{1}{D'^2 - 4D'+4}\right] e^{2z}\cos z$$

$$= e^{2z}\left[\frac{1}{(D'+2)^2 - 4(D'+2)+4}\right]\cos z$$

$$= e^{2z}\frac{1}{D'^2}\cos z$$

$$= -e^{2z}\cos z$$

The general solution of (2) is $y(z) = \text{C.F+P.I} = e^{2z}(Az+B) - e^{2z}\cos z$

$\therefore\ y(x) = x^2[A\log x + B] - x^2\cos(\log x)$ **i**s the required general solution of (1).

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: MATHEMATICS-II **Subject Code: 17PHU602**

Class : III - B.Sc. Physics **Semester : VI**

Unit II

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression ………………...is called a linear combination of f1,f2… … …fm.	c1 f1+c2 f2+…………+cm fm	c1 f1*c2 f2*…………*cm fm	c1 f1/c2 f2/…………/cm fm	c1 f1-c2 f2-…………-cm fm	c1 f1+c2 f2+…………+cm fm
If f1,f2… … ….fm are m given functions and c1,c2… … … …cm are m constants then the expression c1 f1+c2 f2+ … … … ….+cm fm is called a …………….. of f1,f2… … …fm.	non linear combination	homogeneous equation	non homogeneous equation	linear combination	linear combination
Any ………….. combination of solutions of the homogeneous linear differential equation is also a solution of homogeneous equation.	linear	nonlinear	zero	separable	linear
Any lienar combination of solutions of the ……………… linear differential equation is also a solution of homogeneous equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
Any lienar combination of solutions of the homogeneous linear differential equation is also a …………………..of homogeneous equation.	value	separable	solution	exact	solution

The n functions f1,f2… … ….fn are called …………….. on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	linearly dependent	linearly independent	finite	infinite	linearly dependent
The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not ………………..,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x.	all zero	one zero	two zero	n zero	all zero
The n functions f1,f2… … ….fn are called linearly dependent on a ≤ x ≤ b ifthere existsa constants c1,c2… … … …cn not all zero,such that c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=…….. for all x.	1	2	3	0	0
The functions f1,f2… … ….fn are called ………………….. on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=0.	linearly dependent	linearly independent	finite	infinite	linearly independent
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)=0 for all x implies that c1=c2=… … … …=cn=…………..	0	1	2	3	0
The functions f1,f2… … ….fn are called linearly independent on a ≤ x ≤ b if the relation c1 f1(x)+c2 f2(x)+ … … … ….+cn fn (x)…………………….. for all x implies that c1=c2=… … … …=cn=0	equal to 0	< 0	> 0	not equal to 0	equal to 0
The nth order ………………….linear differential equations always possess n solutions that are linealy independent.	homogeneous	non homogeneous	singular	non singular	homogeneous

The nth order homogeneous linear …………………. equations always possess n solutions that are linealy independent.	differential	integral	bernoulli	euler	differential
The nth order homogeneous linear differential equations always possess …………………….solutions that are linealy independent.	zero	finite	inifinite	n	n
The nth order homogeneous linear differential equations always possess n solutions that are …………………	linearly dependent	linearly independent	finite	infinite	linearly independent
Let f1, f2,.. .. ..fn be n……………..functions each of which has an (n-1)st derivative on real interval $a \leq x \leq b$	real	complex	finite	infinite	real
Let f1, f2,.. .. ..fn be n real functions each of which has an ------------derivative on real interval $a \leq x \leq b$	n	n-1	n+1	n+2	n-1
Let f1, f2,.. .. ..fn be n real functions each of which has an (n-1)st derivative on ------------ interval $a \leq x \leq b$	real	complex	finite	infinite	real
The ……………...solution of homogeneous equation is called the complementary function of equation.	explicit	implicit	general	particular	general
The general solution of -------------- equation is called the complementary function of equation.	homogeneous	non homogeneous	singular	non singular	homogeneous
The general solution of homogeneous equation is called the ------------------ function of equation.	real	complex	complementary	particular	complementary
Any ………………….solution of linear differential equation involving no arbitrary constants is called particular integralof this equation.	explicit	implicit	general	particular	particular
Any particular solution of linear differential equation involving ------------- arbitrary constants is called particular integralof this equation.	finite	infinite	no	one	no

Any particular solution of linear differential equation involving no arbitrary constants is called ……………. integralof this equation.	general	particular	finite	infinite	particular
The soluation--------------- is called the general solutionsof linear differential equations.	yc-yp	yc+yp	yc*yp	yc/yp	yc+yp
The soluation yc+yp is called the --------------- solutionsof linear differential equations.	explicit	implicit	general	particular	general
In general solution yc+yp where yc is ……………..function	real	complex	complementary	particular	complementary
In general solution yc+yp where yp is ……………..function	explicit	implicit	general	particular	particular

UNIT-III

Partial Differential Equations: Formation of Partial Differential Equation by eliminating arbitrary constants and arbitrary functions – Solutions of Partial Differential Equations by direct integration – Solution of standard types of first order partial differential equations.

INTRODUCTION:

If z=f(x,y), then z is the dependent variable and x and y are independent variables. The partial derivatives of z w.r.to x and y are $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ etc.we shall employ the following notations: $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$

A partial differential equation in z is one which contains the variable z and its partial derivatives

3.1 FORMATION OF P.D.E BY ELIMINATING ARBITRARY CONSTANTS

3.1.1 Form a partial differential equation by eliminating the arbitrary constants a & b from $z = a(x + y) + b$

Solution:

Given $z = a(x + y) + b$ …(1)

Differentiate (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a$$

$$p = a \dots (2)$$

Differentiate (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = a$$

$$q = a \dots (3)$$

From equation (2) & (3) we get

$$p = q$$

3.1.2 Form a partial differential equation by eliminating the arbitrary constants a & b from

$$\boldsymbol{z = ax + by}$$

Solution:

Given $z = ax + by$ …(1)

Differentiate (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = a$$

$$p = a$$

Differentiate (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = b$$

$$q = b$$

Substituting in equation (1) we get

$$z = px + qy$$

3.1.3 Find the PDE of all planes having equal intercepts on the x and y axis.

Solution:

Intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given a = b [since equal intercepts on the x and y- axis]

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

Here a and c are the two arbitrary constants.

Differentiate (1) partially with respect to x, we get, $\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$

$$\frac{1}{a} + \frac{1}{c}p = 0$$

$$\frac{1}{a} = -\frac{1}{c}p \ldots (2)$$

Differentiate (1) partially with respect to y, we get

$$0 + \frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0$$

$$\frac{1}{a} + \frac{1}{c}q = 0$$

$$\frac{1}{a} = -\frac{1}{c}q \ldots (3)$$

From equation (2) & (3) we get

$$-\frac{1}{c}p = -\frac{1}{c}q$$

$$p = q$$

3.1.4 Form partial differential equation by eliminating the arbitrary constants a and b from the equation $(x-a)^2 + (y-b)^2 + z^2 = 1$

Solution:

Given $(x-a)^2 + (y-b)^2 + z^2 = 1$ …(1)

Differentiate (1) partially with respect to x, we get

$$2(x-a)+0+2z\frac{\partial z}{\partial x}=0$$

$$(x-a)+zp=0 \qquad \dots(2)$$

Differentiate (1) partially with respect to y, we get

$$0+2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$(y-b)+zq=0 \qquad \dots(3)$$

Substituting (2) & (3) in equation (1) we get

$$(-zp)^2+(-zq)^2+z^2=1$$

$$z^2(p^2+q^2+1)=1$$

3.1.5 Form partial differential equation by eliminating the arbitrary constants a and b from the equation $(x-a)^2+(y-b)^2=z^2cot^2\alpha$

Solution:

Given $(x-a)^2+(y-b)^2=z^2cot^2\alpha$ …(1)

Differentiate (1) partially with respect to x,

we get, $2(x-a)+0=2z\frac{\partial z}{\partial x}cot^2\alpha$

$$(x-a)=zpcot^2\alpha \dots(2)$$

Differentiate (1) partially with respect to y, we get

$$0+2(y-b)=2z\frac{\partial z}{\partial y}cot^2\alpha$$

$$(y-b)=zqcot^2\alpha \dots(3)$$

Substituting (2) & (3) in equation (1) we get

$$(zpcot^2\alpha)^2 + (zqcot^2\alpha)^2 = z^2cot^2\alpha$$

$$z^2cot^4\alpha(p^2 + q^2) = z^2cot^2\alpha$$

$$p^2 + q^2 = \frac{1}{cot^2\alpha}$$

$$p^2 + q^2 = tan^2\alpha$$

3.1.6 Eliminate the arbitrary constants a & b from $z = (x^2 + a)(y^2 + b)$

Solution:

Given $z = (x^2 + a)(y^2 + b)$ …(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b)$$

$$p = 2x(y^2 + b)$$

$$\frac{p}{2x} = y^2 + b \ldots (2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a)$$

$$q = 2y(x^2 + a)$$

$$\frac{q}{2y} = x^2 + a \ldots (3)$$

Substituting (2) & (3) in equation (1) we get

$$z = \left(\frac{q}{2y}\right)\left(\frac{p}{2x}\right)$$

$$4xyz = pq$$

3.1.7 Form partial differential equation by eliminating the arbitrary constants a and b from the equation $z = ax^n + by^n$

Solution:

Given$z = ax^n + by^n$ …(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = anx^{n-1}$$

$$\frac{px}{n} = ax^n \dots (2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = bny^{n-1}$$

$$\frac{qy}{n} = by^n \dots (3)$$

Substituting (2) & (3) in equation (1) we get

$$z = \frac{px}{n} + \frac{qy}{n}$$

$$zn = px + qy$$

3.1.8 Form a partial differential equation by eliminating a and b from the expression $(x-a)^2 + (y-b)^2 + z^2 = c^2$

Solution:

Given $(x-a)^2 + (y-b)^2 + z^2 = c^2 \dots (1)$

Here a and b are two arbitrary constants

Differentiate (1) with respect to x, we get

$$2(x-a)+0+2z\frac{\partial z}{\partial x}=0$$

$$(x-a)+zp=0$$

$$(x-a)=-zp \ldots..(2)$$

Differentiate (1) with respect to y, we get

$$0+2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$(y-b)+zq=0$$

$$(y-b)=-zq \ldots..(3)$$

Eliminating a and b from (1), (2) and (3) we get

$$(-zp)^2+(-zq)^2+z^2=c^2$$

$$z^2p^2+z^2q^2+z^2=c^2$$

$$z^2(p^2+q^2+1)=c^2$$

3.2 FORMATION OF P.D.E BY ELIMINATING ARBITRARY FUNCTIONS

3.2.1 Form the partial differential equation by eliminating the arbitrary function $z=f\left(\frac{x}{y}\right)$

Solution:

Given $z=f\left(\frac{x}{y}\right)$ …(1)

Differentiate (1) partially with respect to x, we get

$$p=\frac{\partial z}{\partial x}=f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \ldots.(2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \ldots .(3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)}{f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)}$$

$$\frac{p}{q} = -\frac{y}{x}$$

$$px = -qy$$

$$px + qy = 0$$

3.2.2 Form the partial differential equation by eliminating the arbitrary function $z = xy + f(x^2 + y^2)$

Solution:

Given $z = xy + f(x^2 + y^2)$ …(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = y + f'(x^2 + y^2)(2x)$$

$$p - y = f'(x^2 + y^2)(2x) \ldots (2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = x + f'(x^2 + y^2)(2y)$$

$$q - x = f'(x^2 + y^2)(2y) \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p - y}{q - x} = \frac{f'(x^2 + y^2)(2x)}{f'(x^2 + y^2)(2y)}$$

$$\frac{p - y}{q - x} = \frac{x}{y}$$

$$(p - y)y = (q - x)x$$

$$py - qx = y^2 - x^2$$

3.2.3 Form the PDE by eliminating the arbitrary function from $z = f(x^2 + y^2)$

Solution:

Given $z = f(x^2 + y^2)$ …(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 + y^2)(2x)$$

$$p = f'(x^2 + y^2)(2x) \ldots (2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = f'(x^2 + y^2)(2y)$$

$$q = f'(x^2 + y^2)(2y) \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2 + y^2)(2x)}{f'(x^2 + y^2)(2y)}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = qx$$

3.2.4 Form the PDE from $z = f(2x - 6y)$

Solution:

Given $z = f(2x - 6y)$ …(1)

Differentiate (1) partially with respect to x, we get,

$$p = \frac{\partial z}{\partial x} = f'(2x - 6y)(2)$$

$$p = 2f'(2x - 6y) \ldots (2)$$

Differentiate (1) partially with respect to y, we get,

$$q = \frac{\partial z}{\partial y} = f'(2x - 6y)(-6)$$

$$q = -6f'(2x - 6y) \qquad \ldots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{2f'(2x - 6y)}{-6f'(2x - 6y)}$$

$$\frac{p}{q} = \frac{-1}{3}$$

$$3p = -q$$

$$3p + q = 0$$

3.2.5 Form the PDE from $z = x + y + f(xy)$

Solution: Given $z = x + y + f(xy)$ …(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = 1 + f'(xy)(y)$$

$$p - 1 = yf'(xy) \ldots (2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = 1 + f'(xy)(x)$$

$$q-1 = xf'(xy) \dots (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p-1}{q-1} = \frac{yf'(xy)}{xf'(xy)}$$

$$\frac{p-1}{q-1} = \frac{y}{x}$$

$$x(p-1) = y(q-1)$$

$$xp - x = yq - y$$

$$xp - yq = x - y$$

3.2.6 Form the PDE by eliminating the functions from $z = f(x+t) + g(x-t)$

Solution:

Given $z = f(x+t) + g(x-t)$ …(1)

Differentiate (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = f'(x+t) + g'(x-t) \dots (2)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+t) + g''(x-t) \dots (3)$$

$$\frac{\partial z}{\partial t} = f'(x+t) - g'(x-t) \dots (4)$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+t) + g''(x-t) \dots (5)$$

From equation (3) & (4) we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$

3.2.7 Form the partial differential equation by eliminating the arbitrary function

from $\boldsymbol{\varphi\left(z^2 - xy, \frac{x}{z}\right) = 0}$

Solution:

Given $\varphi\left(z^2 - xy, \frac{x}{z}\right) = 0$

Let $u = z^2 - xy, v = \frac{x}{z}$

$$\frac{\partial u}{\partial x} = 2z\frac{\partial z}{\partial x} - y = 2zp - y$$

$$\frac{\partial u}{\partial y} = 2z\frac{\partial z}{\partial y} - x = 2zq - x$$

$$\frac{\partial v}{\partial x} = \frac{z(1) - x\frac{\partial z}{\partial x}}{z^2} = \frac{z - px}{z^2}$$

$$\frac{\partial v}{\partial y} = -\frac{x}{z^2}\frac{\partial z}{\partial x} = \frac{-xq}{z^2}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

$$(2zp - y)\left(\frac{-xq}{z^2}\right) - (2zq - x)\left(\frac{z - px}{z^2}\right) = 0$$

$$-\frac{2xpq}{z} + \frac{xyq}{z^2} - (2zq - x)\left(\frac{z - px}{z^2}\right) = 0$$

$$x^2p - (xy - 2z^2)q = xz$$

3.2.8 Form the P.D.E by eliminating f and φ from $z = xf\left(\frac{y}{x}\right) + y\varphi(x)$ Solution:

Given $z = xf\left(\frac{y}{x}\right) + y\varphi(x)$ …(1)

$$p = \frac{\partial z}{\partial x} = xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\varphi'(x) \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \varphi(x) = f'\left(\frac{y}{x}\right) + \varphi(x) \ldots (3)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \varphi'(x) \ldots (4)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \ldots (5)$$

$$(2)x + (3)y implies$$

$$px + qy = -yf'\left(\frac{y}{x}\right) + xf\left(\frac{y}{x}\right) + xy\varphi'(x) + yf'\left(\frac{y}{x}\right) + y\varphi(x)$$

$$= xy\varphi'(x) + xf\left(\frac{y}{x}\right) + y\varphi(x)$$

$$px + qy = xy\varphi'(x) + z \ldots (6)$$

Use (5) in (4), we get

$$s = -\frac{y}{x}t + \varphi'(x)$$

$$\frac{xs + yt}{x} = \varphi'(x)$$

Use in (6) we get

$$px + qy = xy\left[\frac{xs + yt}{x}\right] + z$$

$$px + qy = xys + y^2 t + z$$

$$z = px + qy - xys - y^2t$$

3.2.9 Form the partial differential equation by eliminating the arbitrary function f and g in $\boldsymbol{z = x^2 f(y) + y^2 g(x)}$

Solution:

Given $z = x^2 f(y) + y^2 g(x) \ldots .(1)$

$$p = \frac{\partial z}{\partial x} = 2xf(y) + y^2 g'(x) \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2yg(x) \qquad \ldots (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2\, f(y) + \; y^2 g''(x) \ldots (4)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = 2xf'(y) + 2yg'(x) \ldots (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2\, g(x) \qquad \ldots (6)$$

$$(2)x + (3)y implies$$

$$px + qy = 2x^2 f(y) + xy^2 g'(x) + yx^2 f'(y) + 2y^2 g(x)$$

$$px + qy = 2[x^2 f(y) + y^2 g(x)] + xy[yg'(x) + xf'(y)]$$

$$px + qy = 2z + xy\left(\frac{s}{2}\right)$$

$$2px + 2qy = 4z + xys$$

$$4z = 2px + 2qy - xys$$

3.2.10 Form a partial differential equation by eliminating arbitrary functions from $z = xf(2x+y) + g(2x+y)$

Solution: Given $z = xf(2x+y) + g(2x+y)$

$$p = \frac{\partial z}{\partial x} = x[f'(2x+y)2] + f(2x+y)(1) + g'(2x+y)2$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2x[f''(2x+y)2] + f'(2x+y)2$$

$$+f'(2x+y)2 + 2g''(2x+y)2]$$

$$= 4xf''(2x+y) + 4f'(2x+y) + 4g''(2x+y)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 4[xf''(2x+y) + g''(2x+y)] + 4f'(2x+y) \dots (1)$$

$$q = \frac{\partial z}{\partial y} = xf'(2x+y) + g'(2x+y)$$

$$t = \frac{\partial^2 z}{\partial y^2} = xf''(2x+y) + g''(2x+y) \dots.(2)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = 2xf''(2x+y) + f'(2x+y) + 2g''(2x+y) \dots (3)$$

Equation (1) implies

$$\frac{\partial^2 z}{\partial x^2} = 4\frac{\partial^2 z}{\partial y^2} + 4f'(2x+y) \dots.(4)$$

Equation (3) implies

$$\frac{\partial^2 z}{\partial x \partial y} = 2\frac{\partial^2 z}{\partial y^2} + f'(2x+y) \dots.(5)$$

(4) – 2(5) implies

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} = 4\frac{\partial^2 z}{\partial y^2} + 4f'(2x+y) - 8\frac{\partial^2 z}{\partial y^2} - 4f'(2x+y)$$

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} = -4\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

3.4.1 Complete integral:

A solution containing as many arbitrary constants as there are independent variables is called complete integral.

3.4.2 Singular integral:

The equation of the envelop of the surface represented by the complete integral of given PDE is called its singular integral.

Thus if f(x,y,z,a,b)=0 is the complete integral of given PDE then the singular integral is obtained by eliminating a,b from f(x,y,z,a,b)=0

$$\frac{\partial f}{\partial a} = 0$$
$$\frac{\partial f}{\partial b} = 0$$

3.4.3 General solution:

If f(x,y,z,a,b)=0 is the complete integral of PDE g(x,y,z,p,q)=0 then put b=$\phi(a)$ and eliminate 'a' from f(x,y,z,a, $\phi(a)$)=0 and $\frac{\partial f}{\partial a} = 0$.

SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

1. Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

Solution:

Given$\frac{\partial^2 z}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$$

Integrating with respect to x we get

$$\frac{\partial z}{\partial y} = f(y)$$

Integrating with respect to y we get

$$z = x\, f(y) + g(x)$$

$$z = +x\, f(y) + g(x)$$

Where $f(y)$ and $g(y)$ are arbitrary.

2. Solve $\frac{\partial^2 z}{\partial x^2} = \sin y$

Solution:

Given

$$\frac{\partial^2 z}{\partial x^2} = \sin y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \sin y$$

Integrating with respect to x we get

$$\frac{\partial z}{\partial x} = x \sin y + f(y)$$

Integrating with respect to x we get

$$z = \sin y \frac{x^2}{2} + x\, f(y) + g(y)$$

$$z = \frac{x^2}{2} \sin y \ + x\, f(y) + g(y)$$

Where $f(y)$ and $g(y)$ are arbitrary.

TYPE I

3.4.1 PROBLEM BASED ON FIRST ORDER P.D.E F[p,q]=0

1. Find the complete solution of the partial differential equation $\sqrt{\boldsymbol{p}} + \sqrt{\boldsymbol{q}} = \mathbf{1}$

Solution:

Given $\sqrt{p} + \sqrt{q} = 1$ …(1)

This equation is of the form f(p,q)=0

Hence the trial solution is $z = ax + by + c$

To get the complete integral we have to eliminate any one of the arbitrary constants.

Since in a complete integral

Number of arbitrary constant = number of independent variable

$$z = ax + by + c$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Substituting in equation (1) we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

Hence the complete solution is

$$z = ax + (1 - \sqrt{a})^2\, y + c$$

2. Find the complete integral of p – q = 0

Solution:

Given p – q = 0 …(1)

This equation is of the form f(p,q)=0

Hence the trial solution is $z = ax + by + c$

To get the complete integral we have to eliminate any one of the arbitrary constants.

Since in a complete integral

Number of arbitrary constant = number of independent variable

$$z = ax + by + c$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Substituting in equation (1) we get

$$a - b = 0$$

$$a = b$$

Hence the complete solution is

$$z = ax + a\,y + c$$

3. Find the complete solution of the partial differential equation $p^2 + q^2 - 4pq = 0$

Solution:

Given $p^2 + q^2 = 4pq$

This equation is of the form f(p,q)=0

Hence the trial solution is $z = ax + by + c$

To get the complete integral we have to eliminate any one of the arbitrary constants.

Since in a complete integral

Number of arbitrary constant = number of independent variable

$$z = ax + by + c$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Substituting in equation (1) we get

$$a^2 + b^2 - 4ab = 0$$

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2}$$

$$= \frac{4a \pm \sqrt{12a^2}}{2}$$

$$= \frac{4a \pm 2\sqrt{3}a}{2}$$

$$= 2a \pm \sqrt{3}a$$

$$b = a(2 \pm \sqrt{3})$$

Hence the complete solution is

$$z = ax + a(2 \pm \sqrt{3})\, y + c$$

TYPE II

3.4.2 PROBLEM BASED ON F(x,p,q)=0

1. Find the complete integral of $p = 2qx$

Solution:

Given $p = 2qx$

This equation is of the form $f(x, p, q) = 0$

Let $q = a$

Then $p = 2ax$

We know that

$$dz = p\, dx + q\, dy$$

$$dz = 2ax\,dx + a\,dy$$

Integrating on both sides

$$\int dz = \int 2ax\,dx + \int a\,dy$$

$$z = 2a\left(\frac{x^2}{2}\right) + ay + c$$

$$z = ax^2 + ay + c$$

This is the required complete integral.

2. Solve $\boldsymbol{p(1-q^2) = q(1-z)}$

Solution:

Given $p(1-q^2) = q(1-z)$ ...(1)

The equation is of the form $f(z,p,q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \qquad q = a\frac{dz}{du}$$

Substituting in equation (1) we get

$$\frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2\right] = a\frac{dz}{du}[1-z]$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a(1-z)$$

$$1 - a(1 - z) = a^2 \left(\frac{dz}{du}\right)^2$$

$$1 - a + az = a^2 \left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{a^2}[1 - a + az]$$

$$\frac{dz}{du} = \frac{1}{a}\sqrt{1 - a + az}$$

$$\int \frac{a}{\sqrt{1 - a + az}} dz = \int du$$

$$2\sqrt{1 - a + az} = u + c$$

$$4(1 - a + az) = (u + c)^2$$

$$4(1 - a + az) = (x + ay + c)^2$$

This is the complete integral of the given equation.

3. Solve $p(1 + q) = qz$

Solution: Given $p(1 + q) = qz$

The equation is of the form$f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \qquad q = a\frac{dz}{du}$$

Substituting in equation (1) we get

$$\frac{dz}{du}\left(1+a\frac{dz}{du}\right)=a\frac{dz}{du}z$$

$$1+a\frac{dz}{du}=az$$

$$a\frac{dz}{du}=az-1$$

$$\frac{dz}{du}=\frac{az-1}{a}$$

$$\frac{du}{dz}=\frac{a}{az-1}$$

$$du=\frac{a}{az-1}dz$$

Integration on both sides $\int du=\int\frac{a}{az-1}dz$

$$u=\log(az-1)+\log c$$

$$x+ay=\log c(az-1)$$

This is the complete integral of the given equation.

4. Solve $z^2=1+p^2+q^2$

Solution:

Given $z^2=1+p^2+q^2$ ---------- (1)

The equation is of the form $f(z,p,q)=0$

Let $u=x+ay$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \qquad q = a\frac{dz}{du}$$

Substituting in equation (1) we get

$$z^2 = 1 + \left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2$$

$$z^2 - 1 = \left[\frac{dz}{du}\right]^2 [1 + a^2]$$

$$\left[\frac{dz}{du}\right]^2 = \frac{z^2 - 1}{1 + a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z^2 - 1}{1 + a^2}}$$

$$\frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integration on both sides

$$\int \frac{dz}{\sqrt{z^2 - 1}} = \int \frac{du}{\sqrt{1 + a^2}}$$

$$cosh^{-1}z = \frac{1}{\sqrt{1 + a^2}}u + b$$

$$cosh^{-1}z = \frac{1}{\sqrt{1 + a^2}}(x + ay) + b$$

This is the complete integral

TYPE III

3.4.3 PROBLEM BASED ON f(x,p,)=g(y,q)

1. Find the complete integral of pq = xy

Solution: Given pq = xy

$$\frac{p}{x} = \frac{y}{q}$$

This equation is of the form f(x, p) = g(y, q)

$$\frac{p}{x} = \frac{y}{q} = a$$

$$\therefore p = ax \text{ and } q = \frac{y}{a}$$

We know that

$$dz = pdx + qdy$$

$$dz = ax\,dx + \frac{y}{a}dy$$

Integrating on both sides

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + c$$

$$2az = a^2x^2 + y^2 + b$$

This is the required complete integral.

2. Find the complete solution of the PDE $p^2 + q^2 = x + y$

Solution:

Given $p^2 + q^2 = x + y$

$$p^2 - x = -q^2 + y$$

This equation is of the form f(x, p) = g(y, q)

$$p^2 - x = -q^2 + y = a$$

$$p^2 - x = a \qquad -q^2 + y = a$$

$$p^2 = x + a \qquad q^2 = y - a$$

$p = \sqrt{x+a}$ $\qquad q = \sqrt{y-a}$

We know that

$dz = pdx + qdy$

$$dz = (x+a)^{\frac{1}{2}}\,dx + (y-a)^{\frac{1}{2}}dy$$

Integrating on both sides

$$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + c$$

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$$

This is the required complete integral.

3. Find the solution of $px - qy = x$

Solution:

Given $px - qy = x$

$$px - x = qy$$

$$x(p-1) = qy$$

$$p - 1 = \frac{a}{x} \qquad qy = a$$

$$p = \frac{a}{x} + 1 \qquad q = \frac{a}{y}$$

We know that

$$dz = pdx + qdy$$

$$dz = \left(\frac{a}{x} + 1\right) dx + \left(\frac{a}{y}\right) dy$$

Integrating on both sides

$$z = a\log x + x + a\log y + b$$

$$z = a\log xy + x + b$$

This is the required complete integral.

4. Solve $\sqrt{p} + \sqrt{q} = x + y$

Solution: Given $\sqrt{p} + \sqrt{q} = x + y$

$$\sqrt{p} - x = y - \sqrt{q}$$

Let $\sqrt{p} - x = y - \sqrt{q} = a$

$$\sqrt{p} - x = a \quad and \quad y - \sqrt{q} = a$$

$$\sqrt{p} = x + a \quad and \quad -\sqrt{q} = a - y$$

$$p = (x+a)^2 \quad and \quad q = (y-a)^2$$

We know that $dz = p\,dx + q\,dy$

$$dz = (x+a)^2\,dx + (y-a)^2 dy$$

Integrating on both sides

$$\int dz = \int (x+a)^2\,dx + \int (y-a)^2\,dy$$

$$z = \left(\frac{(x+a)^3}{3}\right) + \frac{(y-a)^3}{3} + c$$

This is the required complete integral.

TYPE IV

3.4.4 CLAIRAUT'S FORM z=px + qy + f (p,q)

1. Find the complete solution of the partial differential equation $z = px + qy + p^2 + q^2$

Solution:

Given $z = px + qy + p^2 + q^2$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + a^2 + b^2$

Where a and b are arbitrary constants

2. Find the complete solution of the partial differential equation $z = px + qy + (pq)^{3/2}$

(or)

Find the complete solution of the partial differential equation $\frac{z}{pq} = \frac{x}{p} + \frac{y}{q} + \sqrt{pq}$

Solution:

Given $\frac{z}{pq} = \frac{x}{p} + \frac{y}{q} + \sqrt{pq}$

$$\frac{z}{pq} = \frac{px + qy + pq\sqrt{pq}}{pq}$$

$$z = px + qy + (pq)^{3/2}$$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + (ab)^{3/2}$

Where a and b are arbitrary constants

3. Find the singular solution of the partial differential equation $z = px + qy + p^2 - q^2$

Solution:

Given $z = px + qy + p^2 - q^2$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + a^2 - b^2$ …(1)

Where a and b are arbitrary constants

Differentiating (1) p.w.r.to a we get

$0 = x + 2a$

$a = -\frac{x}{2}$

Differentiating (1) p.w.r.to b we get

$0 = y - 2b$

$b = \frac{y}{2}$

Substituting a and b value in equation (1) we get

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$$

$$z = \frac{-x^2 + y^2}{4}$$

$$4z = y^2 - x^2$$

This is the required singular solution.

4. Find the singular solution of the partial differential equation $z = px + qy + 3pq$

Solution:

Given $z = px + qy + 3pq$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + 3ab$ …(1)

Where a and b are arbitrary constants

Differentiating (1) p.w.r.to a we get

$0 = x + 3b$

$b = -\frac{x}{3}$

Differentiating (1) p.w.r.to b we get

$0 = y + a$

$a = -\frac{y}{3}$

Substituting a and b value in equation (1) we get

$$z = \left(-\frac{y}{3}\right)x + \left(-\frac{x}{3}\right)y + \left(-\frac{y}{3}\right)(-\frac{x}{3})$$

$$z = -\frac{yx}{3} - \frac{xy}{3} + \frac{xy}{9}$$

$9z = -5xy$

$9z + 5xy = 0$

This is the required singular solution.

5. Find the singular solution of the partial differential equation $z = px + qy + pq$

Solution:

Given $z = px + qy + pq$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + ab$ …(1)

Where a and b are arbitrary constants

Differentiating (1) p.w.r.to a we get

$0 = x + b$

$b = -x$

Differentiating (1) p.w.r.to b we get

$0 = y + a$

$a = -y$

Substituting a and b value in equation (1) we get

$$z = (-y)x + (-x)y + (-y)(-x)$$

$z = -yx - xy + xy$

$z = -xy$

$z + xy = 0$

This is the required singular solution.

6. Solve $z = px + qy + pq$

Solution:

Given $z = px + qy + pq$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + ab$

Where a and b are arbitrary constants

Singular solution is found as follows

$z = ax + by + ab$ …(1)

Differentiating with respect to 'a', we get

$0 = x + b$

$$b = -x$$

Differentiating (1) with respect to 'b', we get

$$0 = y + a$$

$$a = -y$$

Substituting in equation (2) we get

$$z = -yx - xy + xy$$

$$z = -xy$$

$$z + xy = 0$$

This is the singular integral

To get the general integral

Put $b = \varphi(a)$ in equation (1), we get

$$z = ax + \varphi(a)y + a\varphi(a) \quad \dots (2)$$

Differentiating w.r.to a we get

$$0 = x + \varphi'(a)y + a\varphi'(a) + \varphi(a) \quad \dots (3)$$

Eliminate a between (2) & (3) we get the general solution.

7. Solve $\boldsymbol{z = px + qy + (pq)^{\frac{3}{2}} (or) \frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}}$

Solution:

Given $z = px + qy + (pq)^{\frac{3}{2}}$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + (ab)^{\frac{3}{2}}$

Where a and b are arbitrary constants

Singular solution is found as follows

$z = ax + by + (ab)^{\frac{3}{2}}$ …(1)

Differentiating with respect to ‘a’, we get

$$0 = x + \frac{3}{2}a^{\frac{1}{2}}b^{\frac{3}{2}}$$

$$x = -\frac{3}{2}a^{\frac{1}{2}}b^{\frac{3}{2}}$$

$$x^3 = -\frac{27}{8}a^{\frac{3}{2}}b^{\frac{9}{2}}$$

Differentiating with respect to ‘b’, we get

$$0 = y + \frac{3}{2}a^{\frac{3}{2}}b^{\frac{1}{2}}$$

$$y = -\frac{3}{2}a^{\frac{3}{2}}b^{\frac{1}{2}}$$

$$\frac{x^3}{y} = \frac{-\frac{27}{8}a^{\frac{3}{2}}b^{\frac{9}{2}}}{-\frac{3}{2}a^{\frac{3}{2}}b^{\frac{1}{2}}}$$

$$\frac{x^3}{y} = \frac{9}{4}b^4$$

$$b^4 = \frac{4}{9}\frac{x^3}{y}$$

$$b = \sqrt[4]{\frac{4}{9}\frac{x^3}{y}}$$

$$\frac{x}{y^3} = \frac{-\frac{3}{2}a^{\frac{1}{2}}b^{\frac{3}{2}}}{\frac{27}{8}a^{\frac{9}{2}}b^{\frac{3}{2}}}$$

$$a^4 = \frac{4}{9}\frac{y^3}{x}$$

$$a = \sqrt[4]{\frac{4}{9}\frac{y^3}{x}}$$

Substituting a and b value in equation (1) we get

$$z = \left(\frac{4}{9}\frac{y^3}{x}\right)^{\frac{1}{4}} x + \left(\frac{4}{9}\frac{x^3}{y}\right)^{\frac{1}{4}} + \left[\left(\frac{4}{9}\frac{y^3}{x}\frac{4}{9}\frac{x^3}{y}\right)^{1/4}\right]^{3/2}$$

$$z = \left(\frac{4}{9}\right)^{\frac{1}{4}} y^{\frac{3}{4}}x^{\frac{3}{4}} + \left(\frac{4}{9}\right)^{\frac{1}{4}} x^{\frac{3}{4}}y^{\frac{3}{4}} + \left(\frac{4}{9}xy\right)^3$$

$$z = 2\left(\frac{4}{9}\right)^{\frac{1}{4}} (xy)^{\frac{3}{4}} + \left(\frac{4}{9}xy\right)^3$$

This is the required singular integral.

8. Solve $z = px + qy + p^2q^2$

Solution:

Given $z = px + qy + p^2q^2$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + a^2b^2$

Where a and b are arbitrary constants

Singular solution is found as follows

$$z = ax + by + a^2b^2 \qquad \ldots(1)$$

Differentiating with respect to 'a', we get

$$0 = x + 2ab^2$$

$$x = -2ab^2 \qquad \ldots(2)$$

$$\frac{x}{b} = -2ab$$

Differentiating with respect to 'b', we get

$$0 = y + 2a^2b$$

$$y = -2a^2b$$

$$\frac{y}{a} = -2ab \qquad \ldots(3)$$

$$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k}(say)$$

$$a = ky; \quad b = kx$$

Put in equation (2) we get

$$x = -2k^3yx^2$$

$$k^3 = -\frac{1}{2xy}$$

Put a & b in equation (1) we get

$$z = kxy + kxy + k^4x^2y^2$$

$$z = 2kxy + x^2y^2\left(-\frac{1}{2xy}\right)$$

$$z = 2kxy - \frac{k}{2}xy = \frac{3}{2}kxy$$

$$z^3 = \frac{27}{8}k^3x^3y^3 = \frac{27}{8}\left(-\frac{1}{2xy}\right)x^3y^3$$

$$z^3 = -\frac{27}{16}x^2y^2$$

$$16z^3 + 27x^2y^2 = 0$$

This is the singular solution.

To get the general integral

Put $b = \varphi(a)$ in equation (1), we get

$$z = ax + \varphi(a)y + a^2(\varphi(a))^2 \qquad \dots(4)$$

Differentiating w.r.to a we get

$$0 = x + \varphi'(a)y + a^2 2\varphi(a)\varphi'(a) + [\varphi(a)]^2 2a \quad \dots(5)$$

Eliminate a between (4) & (5) we get the general solution.

9. Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution:

Given $z = px + qy + \sqrt{1 + p^2 + q^2}$

This is of form $z = px + qy + f(p, q)$

Hence the complete integral is $z = ax + by + \sqrt{1 + a^2 + b^2}$

Where a and b are arbitrary constants

Singular solution is found as follows

$z = ax + by + \sqrt{1 + a^2 + b^2}$ …(1)

Differentiating with respect to 'a', we get

$$0 = x + 0 + \frac{1}{2}\frac{0 + 2a + 0}{\sqrt{1 + a^2 + b^2}}$$

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$x = -\frac{a}{\sqrt{1 + a^2 + b^2}} \quad \ldots (2)$$

Differentiating (1) with respect to 'b', we get

$$0 = 0 + y + \frac{1}{2}\frac{0 + 2b + 0}{\sqrt{1 + a^2 + b^2}}$$

$$0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}}$$

$$y = -\frac{b}{\sqrt{1 + a^2 + b^2}} \quad \ldots (3)$$

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - x^2 - y^2 = \frac{1 + a^2 + b^2 - a^2 - b^2}{1 + a^2 + b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

$$\sqrt{1 - x^2 - y^2} = \frac{1}{\sqrt{1 + a^2 + b^2}}$$

$$\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

Substituting in equation (2) we get

$$x = -a\sqrt{1 - x^2 - y^2}$$

$$a = -\frac{x}{\sqrt{1 - x^2 - y^2}}$$

Substituting in equation (3) we get

$$y = -b\sqrt{1 - x^2 - y^2}$$

$$b = -\frac{y}{\sqrt{1 - x^2 - y^2}}$$

Substituting in equation (1) we get

$$z = -\frac{x^2}{\sqrt{1 - x^2 - y^2}} - \frac{y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}$$

$$z = \sqrt{1 - x^2 - y^2}$$

$$z^2 = 1 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 1$$

This is the required singular solution

To get the general integral

Put $b = \varphi(a)$ in equation (1), we get

$$z = ax + \varphi(a)y + \sqrt{1 + a^2 + [\varphi(a)]^2} \qquad \ldots(4)$$

3.4.5 EQUATIONS REDUCIBLE TO STANDARD FORM

1. Find the complete integral of $x^2p^2 + y^2q^2 = z^2$

Solution:

Given $x^2p^2 + y^2q^2 = z^2$

$$(xp)^2 + (yq)^2 = z^2 \qquad \ldots.(1)$$

This equation is of the form $f(z, x^m p, y^n q) = 0$

Here $m = 1, n = 1$

Put $X = \log x$

$$\frac{\partial X}{\partial x} = \frac{1}{x}$$

$$P = \frac{\partial z}{\partial X}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$p = P\frac{1}{x}$$

$$xp = P$$

Put $Y = \log y$

$$\frac{\partial Y}{\partial y} = \frac{1}{y}$$

$$Q = \frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$q = Q\frac{1}{y}$$

$yq = Q$

Substituting in equation (1) we get

$P^2 + Q^2 = z^2 \qquad \ldots(2)$

This equation is of the form $f(z, P, Q) = 0$

Let $u = X + aY$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = a$$

$$P = \frac{dz}{du}, \qquad Q = a\frac{dz}{du}$$

Substituting in equation (1) we get

$$\left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2 = z^2$$

$$\left[\frac{dz}{du}\right]^2 [1 + a^2] = z^2$$

$$\left[\frac{dz}{du}\right]^2 = \frac{z^2}{1 + a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{dz}{z} = \frac{du}{\sqrt{1+a^2}}$$

Integration on both sides

$$\int \frac{dz}{z} = \int \frac{du}{\sqrt{1+a^2}}$$

$$\log z = \frac{1}{\sqrt{1+a^2}} u + b$$

$$\log z = \frac{1}{\sqrt{1+a^2}} (X + aY) + b$$

$$\log z = \frac{1}{\sqrt{1+a^2}} (\log x + a \log y) + b$$

This is the complete integral

Part – A: Questions

1. Family of sphere having their centre on the line x = y = z
2. $z = ax + by$
3. centre lie on xy plane with radius "r" (or) $(x - a)^2 + (y - b)^2 + z^2 = r$
4. $z = (x+a)^2+(y+b)^2$.
5. $z = (x^2 + a^2)(y^2 + b^2)$
6. $z = x^2 f(y) + y^2 g(x)$
7. $z = f(y) + \phi(x+y+z)$
8. Eliminate the arbitrary function f from $z = f(xy/z)$
9. Solve $\frac{\partial^2 z}{\partial x^2} = \sin y$
10. Form a PDE by eliminating the arbitrary constants a and b from the equation $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$.
11. Form a partial differential equation by eliminating arbitrary constants a and b from $z = (x+a)^2 + (y+b)^2$
12. $p + q = pq$
13. $z = px + qy + p^2 - q^2$
14. $z = px + qy + \sqrt{pq}$
15. $p + q = x + y$
16. $z = 1 + p^2 + q^2$
17. $(1-x)p + (2-y)q = 3-z$.
18. Solve :$pq=y$
19. Define General and Complete intégrals of a partial differential equaitons.
20. Define singular integral of a partial differential equations.

Part – B : Questions

1. Form a P.D.E by eliminating arbitrary functions from z =xf(2x+y) + g(2x+y).
2. Solve : $p^2y(1+x^2) = qx^2$
3. Find the singular integral of $z = px+qy+p^2-q^2$
4. Solve : $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$
5. Solve : $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y} + 4sin(x+y)$
6. $x (y – z)p + y (z – x) q = z (x – y)$
7. $(3z - 4y)p +(4x - 2z)q =2y - 3x$
8. $(y – z) p - (2x + y) q = 2x + z$
9. $px (y^2 + z)+ qy (x^2 + z) = z (x^2 - y^2)$
10. $y^2 p – xy q = x (z -2y)$
11. $(x^2 – y^2 –z^2) p + 2xy q = 2zx$
12. $(D^3 – 3 D D'^2 + 2 D'^3) Z = 0$
13. $(D^2 - DD' + D' - 1)z = 0$
14. $(D^2-2DD'+D'^2)z = 0$
15. $4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$
16. $(D^2 - D D'- 20 D'^2) Z = e^{5x + y} + \sin (4x – y)$
17. $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$
18. $(D^3 – 7 D D'^2 - 6 D'^3) Z = e^{2x + y} +\sin (x +2y)$
19. $(D^2 + 4DD' - 5D'^2)z = 3e^{2x-y} + \sin(x - 2y)$
20. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$
21. (2 D2 – 5 D D'+ 2 D'2) Z = 5 sin (2x + y)
22. $(D^2 – 2 DD') Z = x^3 y + e^{2x}$
23. $(D^2 + 2DD' + D'^2)z = x^2 y + e^{x-y}$
24. $(D^2 - 5DD' + 6D'^2)z = y \sin x$
25. Solve $(D^2 + 3DD' - 4D'^2)z = x + \sin y$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: MATHEMATICS-II **Subject Code: 17PHU602**

Class : III - B.Sc. Physics **Semester : VI**

Unit III

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
In a PDE, there will be one dependent variable and ____ independent variables	only one	two or more	no	infinite number of	two or more
The ______ of a PDE is that of the highest order derivative occurring in it	degree	power	order	ratio	order
The degree of the a PDE is ______of the higest order derivative	power	ratio	degree	order	power
Afirst order PDE is obtained if______	Number of arbitrary constants is equal Number of independent variables	Number of arbitrary constants is lessthan Number of independent variables	Number of arbitrary constants is greater than Number of independent variables	Number of arbitrary constants is not equal to Number of independent variables	Number of arbitrary constants= Number of independent variables
In the form of PDE, f(x,y,z,a,b)=0. What is the order?	1	2	3	4	1
What is form of the z=ax+by+ab by eliminating the arbitrary constants?	z=qx+py+pq	z=px+qy+pq	z=px+qy+p	z=py+qy+q	z=px+qy+pq
General solution of PDE F(x,y,z,p,q)=0 is any arbitray function F of specific functions u,v is____satisfying given PDE	F(u,v)=0	F(x,y,z)=0	F(x,y)=0	F(p,q)=0	F(u,v)=0

The PDE of the first order can be written as--------------	F(x,y,s,t)	F(x,y,z,p,q)=0	F(x,y,z,1,3,2)=0	F(x,y)=0	F(x,y,z,p,q)=0
The complete solution of clairaut's equation is _____	z=bx+ay+f(a,b)	z=ax+by+f(a,b)	z=ax+by	z=f(a,b)	z=ax+by+f(a,b)
The Clairaut's equation can be written in the form --------	z=px+qy+f(p,q)	z=(p-1)x+qy+f(x,y)	z=Pp+Qq	Pq+Qp=r	z=px+qy+f(p,q)
From the PDE by eliminating the arbitrary function from z=f(x^2 -y^2) is	xp+yq=0	p=-(x/y)	q=yp/x	yp+xq=0	yp+xq=0
Which of the following is the type f(z,p,q)=0 ?	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2x f(y+2x)	p(1+q)=qz
The equation (D^2 z+2xy(Dz)^2+D'=5 is of order ____and degree____	2 and 2	2 and 1	1 and 1	0 and 1	2 and 1
The complementry function of (D^2 -4DD'+4D'^2)z=x+y is	f(y+2x)+xg(y+2x)	f(y+x)+xg(y+2x)	f(y+x)+xg(y+x)	f(y+4x)+xg(y+4x)	f(y+2x)+xg(y+2x)
The solution of xp+yq=z is _____	f(x^2,y^2)=0	f(xy,yz)	f(x,y)=0	f(x/y ,y/z)=0	f(x/y ,y/z)=0
The solution of p+q=z is ____	f(xy,ylogz)=0	f(x+y, y+logz)=0	f(x-y, y-logz)=0	f(x-y,y+logz)=0	f(x-y, y-logz)=0
A solution which contains the maximum possible number of arbitrary functions is called------------- integral.	singular	complete	general	particular	general
The lagrange's linear equation can be written in the form -------------	Pq+Qp=r	Pq+Qp=R	Pp+Qq=R	F(x,y)=0	Pp+Qq=R
The complete solution of the PDE 2p+3q =1 is ----------------	z=ax+[(1-2a)/3]y+c	z=ax+y+c	z=ax+(1-2x)/y+c	z=ax+b	z=ax+[(1-2a)/3]y+c
The complete solution of the PDE pq=1 is ------------------	z=ax+(1/a)y+b	z=ax+y+b	z=ax+ay/b+c	z=ax+b	z=ax+(1/a)y+b
The solution got by giving particular values to the arbitrary constants in a complete integral is called a --------	general	singular	particular	complete	particular
The general solution of Lagrange's equation is denoted as-------	f(u,v)=0	zx	f (x,y)	F(x,y,s,t)=0	f(u,v)=0
The subsidiary equations are px+qy=z is --------------	dx/y=dy/z=dz/x	dx/x=dy/y=dz/z	xdx=ydy=zdz	dz/z=dx/y=dy/x	dx/x=dy/y=dz/z
The general solution of equation p+q=1 is ------------	f(xyz,0)	f(x-y,y-z)	f(x-y,y+z)	F(x,y,s,t)=0	f(x-y,y-z)
The separable equation of the first order PDE can be written in the form of ------------	f(x,y)=g(x,y)	f(a,b)=g(x,y)	f(x,p)=g(y,q)	f(x)=g(a)	f(x,p)=g(y,q)

Complementary function is the solution of ---------------------	f(a,b)	f(1,0)=0	f(D,D')z=0	f(a,b)=F(x,y)	f(D,D')z=0
C.F+P.I is called ------------------- solution	singular	complete	general	particular	general
Particular integral is the solution of -------------	f(a,b)=F(x,y)	f(1,0)=0	[1/f(D,D')]F(x,y)	f(a,b)=F(u,v)	[1/f(D,D')]F(x,y)
Which is independent varible in the equation z= 10x+5y	x&y	z	x,y,z	x alone	x&y
Which is dependent varible in the equation z=2x+3y	x	z	y	x&y	z
Which of the following is the type f(z,p,q)=0	p(1+q)=qx	p(1+q)=qz	p(1+q)=qy	p=2xf'(x^2)-(y^2))	p(1+q)=qz
Which is complete integral of z=px+qy+(p^2)(q^2)	z=ax+by+(a^2)(b^2)	z=a+b+ab	z=ax+by+ab	z=a+f(a)x	z=ax+by+(a^2)(b^2)
The complete integral of PDE of the form F(p,q)=0 is	z=ax+f(a)y+c	z=ax+f(a)+b	z=a+f(a)x	z=ax+f(a)	z=ax+f(a)y+c
The relation between the independent and the dependent variables which satisfies the PDE is called-------	solution	complet solution	general solution	singular solution	solution
A solution which contains the maximum possible number of arbitrary constant is called---------	general	complete	solution	singular	complete
The equations which do not contain x & y explicitly can be written in the form---------------	f(z,p,q)=0	f(p,q)=0	(p,q)=0	f(x,p,q)=0	f(z,p,q)=0
The subsidiary equations of the lagranges equation 2y(z-3)p + (2x-z)q = y(2x-3)	dx/2y(z-3) = dy/(2x-z) =dz/y(2x-3)	dx/(2x-z) =dy/2y(z-3) =dz/y(2x-3)	dx/2y=dz/(z-3)	dx/2y=dz/(z-3)=dy/2x	dx/2y(z-3) =dy/(2x-z) =dz/y(2x-3)
A PDE ., the partial derivatives occuring in which are of the first degree is said to be --------------	linear	non-linear	order	degree	linear
A PDE., the partial derivatives occuring in which are of the 2 or more than 2 degree is said to be------	linear	non-linear	order	degree	non-linear
If z=(x^2+a)(y^2+b) then differentiating z partially with respect to x is -----	2x	3x(y^2+b)	2x(y^2+b)	3x+y	2x(y^2+b)
If z=ax+by+ab then differentiating z partially with respect to y is -----	a	a+b	0	b	b

The complete solution of the PDE p=2qx is ----------------	z=ax+ay+c	ax+b	z = ax^2+ay+c	z= ax+(b+c)	z = ax^2+ay+c
The general solution of px-qy=xz is	f(u,v)=0	f(xy,x-logz)=0	f(x-y,y-z)=0	f(x-y,y+z)=0	f(xy,x-logz)=0
If z= f(x^2+y^2) then differentiating z partially with respect to x is -----	p=2xf ’ (x^2+y^2)	p=2xf(x^2+y^2)	p=2xf’(x^2- y^2)	p(1+q)=qy	p=2xf ’ (x^2+y^2)
If z= f(x^2+y^2 +z^2) thendifferentiating z partially with respect to y is -----	q=2xf(x^2+y^2)	q=(2y+2zz') f'(x^2+y^2 +z^2)	q=2y	q=0	q=(2y+2zz') f'(x^2+y^2 +z^2)
The solution of differentiating z partially with respect to x twice gives -----	ax	ax+by+c	ax+b	ax=p	ax+b
The general solution of PDE is of the form	C.F+P.I	C.F-P.I	C.F*P.I	C.F/P.I	C.F+P.I
The Equation is of the form Z=px+qy+f(p,q) is called__________	clairaut	charpit	crout	separable	clairaut
f(x,p)=g(y,q) is called _______equation	clairaut	charpit	crout	separable	separable
Reducible equation is defined as te product of _______factors.	linear	nonlinear	polynomial	recursive	linear
The order of PDE to be the order of the derivative of ______ order occurring in it.	lowest	highest	first	second	highest
The solution of the PDE consists ______ main parts	2	3	4	5	2

UNIT-IV

Laplace Transforms: Definition – Laplace Transforms of standard functions – Linearity property – First Shifting Theorem – Transform of $tf(t), \frac{f(t)}{t}, f^{'}(t), f^{''}(t)$.Inverse Laplace Transforms – Applications to solutions of First Order and Second Order Differential Equations with constant coefficients.

LAPLACE TRANSFORM

Definition

Let a function *f*(t) be continuous and defined for all positive values of 't'. The Laplace transform of *f*(t) associates a function by the equations

$$\varphi(s) = \int_0^{-st} f(t).\,dt$$

Here $\varphi(s)$ is said to be the Laplace Transform of $f(t)$ and it is written as L[*f*(t)].

Thus $\varphi(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt,$ t>0

PROPERTIES OF LAPLACE TRANSFORMS

Property 1: Change of scale property

If $L[f(t)] = \Phi(s)$, then $L[f(at)] = \frac{1}{a}.F(\frac{s}{a})$

Property 2: First shifting Property

If $L[f(t)] = F(s)$ then (i) $L[e^{-at} f(t)] = F(s+a)$

(ii) $L[e^{at} f(t)] = F(s-a)$

Property 3: (i) Laplace Transform of Derivative

$$L[f'(t)] = sL[f(t)] - f(0).$$

(ii) Laplace Transform of derivative of order n.

$$L[f''(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots\dots\dots - f^{(n-1)}(0)$$

Property 4: Laplace Transform of integrals.

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Property 5: Laplace Transform of $t - f(t)$

If $L[f(t)] = F(s)$. then $L[t.f(t)] = \frac{-d}{ds}.F(s)$

Note: In general $L[t^n.f(t)] = (-1)^n \frac{d^n}{ds^n}.F(s)$.

Property 6: Laplace Transform of $\left[\frac{f(t)}{t}\right]$

If $L[f(t)] = F$ and if $\lim_{t\to 0}\frac{f(t)}{t}$ exists then.

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s).ds$$

Find the Laplace transform of $\frac{e^{at}-\cos bt}{t}$ and $\sin t\mu_{\pi}(t)$ where $\mu_{\pi}(t)$ is the unit step function.

Solution:

$$L\left(\frac{e^{at}-\cos bt}{t}\right)=\int_s^{\infty}\left(\frac{1}{s-a}-\frac{s}{s^2+b^2}\right)ds$$

$$=\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)_s^{\infty}$$

$$=\log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right)$$

$$L^{-1}\left(\sin t\mu_{\pi}(t)\right)=L^{-1}\left(\sin(\pi-t)\right)$$

$$=e^{-\pi s}\frac{1}{s^2+1}$$

Verify the initial value theorem for the function $1+e^{-2t}$.

$$\underset{t\to 0}{lt}\, f(t) = \underset{s\to\infty}{lt}\, sF(s)$$

$$\underset{t\to 0}{lt}\, f(t) = \underset{t\to 0}{lt}(1+e^{-2t}) = 1+1 = 2$$

$$\underset{s\to\infty}{lt}\, sF(s) = \underset{s\to\infty}{lt}\, s\,L[f(t)]$$

$$= \underset{s\to\infty}{lt}\, s\,L[1+e^{-2t}]$$

$$= \underset{s\to\infty}{lt}\, s\left[L(1)+L(e^{-2t})\right]$$

$$= \underset{s\to\infty}{lt}\, s\left[\frac{1}{s}+\frac{1}{s+2}\right]$$

$$= \underset{s\to\infty}{lt}\left[s\frac{1}{s}+s\frac{1}{s+2}\right]$$

$$= \underset{s\to\infty}{lt}\left[1+\frac{s}{s\left(1+2/s\right)}\right]$$

Initial value theorem:

$$=1+1$$

$$=2$$

Hence initial value theorem is verified.

Verify the final value theorem for the function $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]$

Solution:

$$f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right] = \int_0^t te^{-2t}\,dt = \left(t\frac{e^{-2t}}{-2}-\frac{e^{-2t}}{4}\right)_0^t = \left(t\frac{e^{-2t}}{-2}-\frac{e^{-2t}}{4}+\frac{1}{4}\right)$$

FVT: $\underset{t\to\infty}{Lim}\, f(t) = \underset{s\to 0}{Lim}\; sF(s)$

LHS: $\underset{t\to\infty}{Lim}\left(t\frac{e^{-2t}}{-2}-\frac{e^{-2t}}{4}+\frac{1}{4}\right) = \frac{1}{4}$

$$sF(s) = \left[\frac{s}{s(s+2)^2}\right] = \frac{1}{(s+2)^2}$$

$$\underset{s\to 0}{Lim}\ sF(s) = \frac{1}{4} = RHS$$

Hence Proved

CONVOLUTION THEOREM

5.3.1 Using convolution theorem, find the inverse Laplace transform of

a. $\dfrac{1}{s^2(s^2+25)}$

Solution:

$$L^{-1}\left[\frac{1}{s^2}\frac{1}{(s^2+25)}\right] = L^{-1}\left[\frac{1}{s^2}.\frac{1}{(s^2+25)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{(s^2+25)}\right]$$

$$= t * \frac{\sin 5t}{5}$$

$$= \frac{1}{5}\int_0^t (t-u)\sin 5u\ du$$

$$= \frac{1}{5}\left[(t-u)\left(\frac{-\cos 5u}{5}\right) - (-1)\left(\frac{-\sin 5u}{25}\right)\right]_0^t$$

$$= \frac{1}{5}\left(\frac{-\sin 5t}{25} + \frac{t}{5}\right)$$

b. $\frac{1}{s^3(s+5)}$

Solution:

$$L^{-1}\left[\frac{1}{s^3(s+5)}\right] = L^{-1}\left[\frac{1}{s^3}\frac{1}{(s+5)}\right]$$
$$= L^{-1}\left[\frac{1}{s^3}\right] * L^{-1}\left[\frac{1}{(s+5)}\right]$$
$$= \frac{t^2}{2} * e^{-5t}$$
$$= \frac{1}{2}\int_0^t u^2\, e^{-5(t-u)}\, du$$
$$= \frac{1}{2}e^{-5t}\int_0^t u^2 e^{5u}\, du$$
$$= \frac{1}{2}e^{-5t}\left[u^2\frac{e^{5u}}{5} - (2u)\frac{e^{5u}}{25} + (2)\frac{e^{5u}}{125}\right]_0^t$$
$$= \frac{e^{-5t}}{250}\left[25t^2e^{5t} - 10te^{5t} + 2e^{5t} - 2\right]$$

c. $\frac{1}{(s^2+a^2)^2}$

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}.\frac{1}{(s^2+a^2)}\right]$$
$$L^{-1}\left[F(s)G(s)\right] = \int_0^t f(u).g(t-u)\,du$$
$$F(s) = \frac{1}{s^2+a^2} \Rightarrow f(t) = \frac{\sin at}{a}$$
$$G(s) = \frac{1}{s^2+a^2} \Rightarrow g(t) = \frac{\sin at}{a}$$
$$\therefore\ L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{a^2}\int_0^t \sin au\,.\sin a(t-u)du$$
$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - \cos at]du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[\left(\frac{\sin(2au-at)}{2a}\right)_0^t - \cos at\,(u)_0^t\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a}+\frac{\sin at}{2a} - t\cos at\right]$$

$$= \frac{\sin at - at\cos at}{2a^3}$$

d. $\frac{s}{(s^2+a^2)^2}$

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}.\frac{1}{(s^2+a^2)}\right]$$

$$L^{-1}\left[F(s)G(s)\right] = \int_0^t f(u).g(t-u)\,du$$

$$F(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = \cos at$$

$$G(s) = \frac{1}{s^2+a^2} \Rightarrow g(t) = \frac{\sin at}{a}$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}\int_0^t \cos au\,.\sin\frac{a(t-u)}{a}\,du$$

$$= \frac{1}{2a}\int_0^t [\sin(at) - \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\sin at\,.(u)_0^t + \left[\frac{\cos(2au-at)}{2a}\right]_0^t\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}[\cos at - \cos(-at)]\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}[\cos at - \cos at]\right]$$

$$= \frac{t\sin at}{2a}$$

e. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}.\frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[F(s).G(s)\right]$$

$$F(s)=\frac{s}{s^2+a^2} \Rightarrow f(t)=L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$$

$$G(s)=\frac{s}{s^2+b^2} \Rightarrow g(t)=L^{-1}\left[\frac{s}{s^2+b^2}\right]=\cos bt$$

$$\therefore\ L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \cos au.\cos b(t-u)\,du$$

$$=\frac{1}{2}\int_0^t [\cos((a-b)u+bt)+\cos((a+b)u-bt)]\,du$$

$$=\frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{(a-b)}+\frac{\sin[(a+b)u-bt]}{(a+b)}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right]$$

$$=\frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$

$$=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

f. $\frac{1}{s^2(s+5)}$. ***(May/June 2005)***

solution:

$$L^{-1}\left[\frac{1}{s^2(s+5)}\right] = \int_0^t\int_0^t e^{-5t}dt\ \ dt = \int_0^t \left(\frac{e^{-5t}}{-5}\right)_0^t dt$$

$$= \int_0^t \left(\frac{-e^{-5t}+1}{5}\right)dt$$

$$= \frac{1}{5}\left(\frac{-e^{-5t}}{-5}+t\right)_0^t$$

$$= \frac{1}{25}\left(e^{-5t}+5t-1\right)$$

g. $\frac{s+2}{(s^2+4s+13)^2}$. ***(May/June 2006)***

solution:

$$\frac{s+2}{(s^2+4s+13)^2} = \frac{s+2}{((s+2)^2+9)^2}$$

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{((s+2)^2+9)^2}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= e^{-2t}\left(\frac{t\sin 3t}{6}\right)$$

h. $\frac{1}{s(s^2-a^2)}$

solution

$$L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \int_0^t \frac{\sinh at}{a}dt$$

$$= \frac{1}{a}\left(\frac{\cosh at}{a}\right)_0^t$$

$$= \frac{1}{a^2}\left(\cosh at - 1\right)$$

INVERSE LAPLACE TRANSFORM

Definition: If F(s) is the Laplace transform of a function $f(t) i.e., L[f(t)] = F(s)$, then *f*(t) is called the inverse Laplace transform of the function F(s) and is written as

$f(t) = L^{-1}[F(s)] L^{-1}$ is called the inverse Laplace transform operator.

Important Results in Laplace Transform

Result 1. Linearly property:

If a and b are any constant while F(s) and G(s) are the Laplace transform of *f*(t) and *g*(t) respectively.

Then $L^{-1}[a.F(s) + b.G(s)] = a.L^{-1}[F(s)] + b.L^{-1}[G(s)]$

Result 2. First shifting property:

(i)$L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$

(ii)$L^{-1}[F(s-a)] = e^{at} L[F(s)]$

WORKED EXAMPLE

Ex.1. Find $L^{-1}\left[\frac{1}{s-2}\right]$

Sol. $L^{-1}\left[\frac{1}{s-2}\right] = e^{2t} L^{-1}\left[\frac{1}{s}\right] = e^{2t}(1) = e^{2t}$

Ex.2. Find $L^{-1}\left[\frac{1}{s^2+25}\right]$

Sol. $L^{-1}\left[\frac{1}{s^2+25}\right] = \frac{1}{5} L^{-1}\left[\frac{5}{s^2+25}\right] = \frac{1}{5}\sin 5t$

Ex.3. Find $L^{-1}\left[\frac{s}{s^2+9}\right]$

Sol. $L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t$

Ex.4. Find $L^{-1}\left[\frac{1}{s^2+9}\right]$

Sol. $L^{-1}\left[\frac{1}{s^2+9}\right] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}L^{-1}\left[\frac{3}{s^2+3^2}\right] = \frac{1}{3}\sinh 3t$

SOLUTION OF LINEAR ODE OF SECOND ORDER WITH CONSTANT COEFFICIENTS USING LAPLACE TRANSFORMATION TECHNIQUES.

Using Laplace transform solve $\frac{dy}{dt} - 3y = e^{2t}$ **subject to** $y(0) = 1$**.**

Solution:

$$\frac{dy}{dt} - 3y = e^{2t}$$

Taking L.T. on both sides,

$$L\left(\frac{dy}{dt}\right) - 3L(y) = L(e^{2t})$$

$$s\bar{y} - y(0) - 3\bar{y} = \frac{1}{s-2} \Rightarrow s\bar{y} - 1 - 3\bar{y} = \frac{1}{s-2}$$

$$\bar{y}(s-3) = \frac{1}{s-2} + 1$$

$$\bar{y} = \frac{s-1}{(s-2)(s-3)}$$

$$y = L^{-1}\left(\frac{s-1}{(s-2)(s-3)}\right)$$

Now consider $\frac{s-1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3} \Rightarrow A = -1,\ B = 2$

$$y = L^{-1}\left(\frac{s-1}{(s-2)(s-3)}\right) = L^{-1}\left(\frac{-1}{s-2}\right) + L^{-1}\left(\frac{2}{s-3}\right)$$

$$y = -e^{2t} + 2e^{3t}$$

Solve the following initial value problem using

Laplace transforms $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=2e^{-3t}, y=0, y'(0)=-2$.

Solution:

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=2e^{-3t} \Rightarrow s^2\bar{y}-sy(0)-y'(0)+6(s\bar{y}-y(0))+9\bar{y}=\frac{2}{s+3}$$

$$\bar{y}(s^2+6s+9)+2=\frac{2}{s+3}$$

$$\bar{y}=\frac{\frac{2}{s+3}-2}{(s+3)^2}=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{(s+3)^3}$$

$$-2s-4=A(s+3)^2+B(S+3)+C$$

$$when\ s=-3 \Rightarrow C=2$$

$$Comparing\ the\ coefficients of\ s^2,\ A=0$$

$$Comparing\ the\ coefficient\ of\ s,\ 6A+B=-2 \Rightarrow B=-2$$

$$Comparing\ the\ cons\tan t\ coefficient,\ 9A+3B+C=-4$$

$$\Rightarrow C=-4+6=2$$

$$\bar{y}=\frac{-2}{(s+3)^2}+\frac{2}{(s+3)^3}$$

$$y=L^{-1}\left(\frac{-2}{(s+3)^2}\right)+L^{-1}\left(\frac{2}{(s+3)^3}\right)$$

$$\Rightarrow y=2e^{-3t}\left(\frac{t^2}{2}-t\right)$$

Solve using Laplace transforms $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=te^{-t}, y(0)=0, y'(0)=-1$.

Solution:

$$y''+4y'+4y=te^{-t}$$

$$s^2\bar{y}-y(0)-y'(0)+4[s\bar{y}-y(0)]+4\bar{y}=\frac{1}{(s+1)^2}$$

$$\bar{y}(s^2+4s+4)=\frac{1}{(s+1)^2}$$

$$\bar{y}=\frac{1}{(s+1)^2(s+2)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+2)}+\frac{D}{(s+2)^2}$$

$$when\ \ s=-1 \Rightarrow B=1$$
$$when\ \ s=-2 \Rightarrow D=1$$
$$Comparing\ \ s^2\ \ coefficients,\ \ 5A+B+4C+D=1 \Rightarrow 5A+4C=-1$$
$$Comparing\ \ s^3\ coefficients,\ A+C=0 \Rightarrow C=1 \Rightarrow A=-1$$
$$y=L^{-1}\left(\frac{1}{\left(s+1\right)^2(s+2)^2}\right)=L^{-1}\left(\frac{-1}{s+1}\right)+L^{-1}\left(\frac{1}{(s+1)^2}\right)+L^{-1}\left(\frac{1}{(s+2)}\right)+L^{-1}\left(\frac{1}{(s+2)^2}\right)$$
$$y=e^{-t}\left(-1+t\right)+e^{-2t}\left(1+t\right)$$

Using Laplace transform solve $y''+2y'-3y=3, y(0)=4, y'(0)=-7$.

Solution:

$$y''+2y'-3y=3,$$
$$s^2\overline{y}-sy(0)-y'(0)+2\left(s\overline{y}-y(0)\right)-3\overline{y}=\frac{3}{s}$$
$$\overline{y}\left(s^2+2s-3\right)-4s+7-8=\frac{3}{s}$$
$$\overline{y}\left(s^2+2s-3\right)=\frac{3}{s}+1+4s$$

$$\bar{y} = \frac{\frac{3}{s}+1+4s}{(s+3)(s-1)} = \frac{3+s+4s^2}{s(s+3)(s-1)} = \frac{A}{s}+\frac{B}{s+3}+\frac{C}{(s-1)}$$

$$when\ s=0 \Rightarrow -3A=3 \Rightarrow A=-1$$
$$when\ s=1 \Rightarrow 4C=8 \Rightarrow C=2$$
$$when\ s=-3 \Rightarrow 12B=36 \Rightarrow B=3$$

$$\bar{y} = \frac{3+s+4s^2}{s(s+3)(s-1)} = \frac{-1}{s}+\frac{3}{s+3}+\frac{2}{(s-1)}$$

$$y = L^{-1}\left(\frac{-1}{s}\right)+L^{-1}\left(\frac{3}{s+3}\right)+L^{-1}\left(\frac{2}{(s-1)}\right)$$

$$y = -1+3e^{-3t}+2e^{t}$$

Using Laplace transform solve $y''+3y'+2y=e^{-t}, y(0)=1, y'(0)=0$**.**

Solution:

$$y''+3y'+2y=e^{-t}$$

$$s^2\bar{y}-sy(0)-y'(0)+3\left(s\bar{y}-y(0)\right)+2\bar{y}=\frac{1}{s+1}$$

$$\bar{y}\left(s^2+3s+2\right)-s-3=\frac{1}{s+1}$$

$$\bar{y} = \frac{\frac{1}{s+1}+s+3}{(s+1)(s+2)} = \frac{s^2+4s+4}{(s+1)^2(s+2)} = \frac{s+2}{(s+1)^2} = \frac{A}{s+1}+\frac{B}{(s+1)^2}$$

$$when\ s=-1 \Rightarrow B=1$$

$$Comparing\ \ cons\tan t\ coefficients,\ A+B=2 \Rightarrow A=1$$

$$\bar{y} = \frac{s+2}{(s+1)^2} = \frac{1}{s+1}+\frac{1}{(s+1)^2}$$

$$y = L^{-1}\left(\frac{1}{s+1}\right)+L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}(1+t)$$

Using Laplace transform solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ **with** $y = 2, \frac{dy}{dx} = -1$ **at** $x = 0$.

Solution:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \Rightarrow s^2\bar{y} - sy(0) - y'(0) - 2\big(s\bar{y} - y(0)\big) + \bar{y} = \frac{1}{s-1}$$

$$\bar{y}(s^2 - 2s + 1) - 2s + 1 + 4 = \frac{1}{s-1}$$

$$\bar{y} = \frac{\frac{1}{s-1} - 5 + 2s}{(s-1)^2} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

But $2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C$

when $s = 1 \Rightarrow C = 6$

Comparing the coefficients of s^2, $A = 2$

Comparing the coefficients of s, $-2A + B = -7 \Rightarrow B = -7 + 4 \Rightarrow B = -3$

$$\bar{y} = \frac{2}{s-1} + \frac{-3}{(s-1)^2} + \frac{6}{(s-1)^3}$$

$$y = L^{-1}\left(\frac{2}{s-1} + \frac{-3}{(s-1)^2} + \frac{6}{(s-1)^3}\right) = 2e^t - 3te^t + 3t^2e^t = e^t(2 - 3t + 3t^2)$$

Solve by using Laplace transform $\ddot{y} - 3\dot{y} + 2y = 4$ **given that** $y(0) = 0, y'(0) = 0$.

Solution:

$$\ddot{y} - 3\dot{y} + 2y = 4 \Rightarrow s^2\bar{y} - sy(0) - y'(0) - 3\left(s\bar{y} - y(0)\right) + 2\bar{y} = \frac{4}{s}$$

$$\bar{y}\left(s^2 - 3s + 2\right) = \frac{4}{s}$$

$$\bar{y} = \frac{4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \Rightarrow 1 = A(s-1)(s-2) + B(s)(s-2) + C(s)(s-1)$$

$$When \;\; s = 0 \Rightarrow A = \frac{1}{2}$$

$$When \;\; s = 1 \Rightarrow B = -1$$

$$when \;\; s = 2 \Rightarrow C = \frac{1}{2}$$

$$\bar{y} = \frac{4}{s(s-1)(s-2)} = \frac{1/2}{s} + \frac{-1}{s-1} + \frac{1/2}{s-2}$$

$$y = L^{-1}\left(\frac{1/2}{s} + \frac{-1}{s-1} + \frac{1/2}{s-2}\right) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t}$$

Solve the differential equation using Laplace transform $y''+4y'+4y = e^{-t}$ **given that** $y(0) = 0, y'(0) = 0$.

Solution:

$$y''+4y'+4y = e^{-t}$$

$$s^2\bar{y} - y(0) - y'(0) + 4\left[s\bar{y} - y(0)\right] + 4\bar{y} = \frac{1}{s+1}$$

$$\bar{y}\left(s^2 + 4s + 4\right) = \frac{1}{s+1}$$

$$\bar{y} = \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$$

$when \ \ s = -1 \Rightarrow A = 1$

$when \ \ s = -2 \Rightarrow C = -1$

$Comparing \ \ s^2 \ \ coefficients, \ \ A + B = 0 \Rightarrow B = -1$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right) = L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{-1}{(s+2)}\right) + L^{-1}\left(\frac{-1}{(s+2)^2}\right)$$

$$y = e^{-t} - e^{-2t} - te^{-2t}$$

Using Laplace transform technique, solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t, y = 0, y'(0) = 0$ **when** $t = 0$.

Solution:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$$

Taking Laplace equation on both sides,

$$s^2\overline{y} - sy(0) - y'(0) + 2(s\overline{y} - y(0)) + 5\overline{y} = \frac{1}{(s+1)^2 + 1}$$

$$\overline{y}\left(s^2 + 2s + 5\right) = \frac{1}{(s^2 + 2s + 2)}$$

$$\overline{y} = \frac{1}{(s^2 + 2s + 2)\left(s^2 + 2s + 5\right)}$$

$$\Rightarrow y = L^{-1}\left(\frac{1}{\left((s+1)^2 + 1\right)\left((s+1)^2 + 4\right)}\right)$$

$$y = e^{-t}L^{-1}\left(\frac{1}{\left(s^2+1\right)\left(s^2+4\right)}\right)$$

$$\frac{1}{\left(s^2+1\right)\left(s^2+4\right)} = \sin t * \frac{\sin 2t}{2}$$

$$= \frac{1}{2}\int_0^t \sin 2u \ \sin(t-u)du$$

$$\sin 2u\ \sin(t-u) = \frac{1}{2}\left[\cos(3u-t) - \cos(u+t)\right]$$

$$\sin t * \frac{\sin 2t}{2} = \frac{1}{2}\int_0^t \frac{1}{2}\left[\cos(3u-t) - \cos(u+t)\right]du$$
$$= \frac{1}{4}\left(\frac{\sin(3u-t)}{3} - \sin(u+t)\right)$$
$$= \frac{1}{4}\left[\left(\frac{\sin(2t)}{3} - \sin 2t\right) - \left(\frac{-\sin t}{3} - \sin t\right)\right]$$
$$= \frac{1}{4}\left[\left(\frac{-2\sin(2t)}{3}\right) - \left(\frac{-4\sin t}{3}\right)\right]$$

$$y = e^{-t}L^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right) = e^{-t}\frac{1}{12}\left[4\sin t - 2\sin 2t\right] = e^{-t}\frac{1}{6}\left[2\sin t - \sin 2t\right]$$

Using convolution, solve the initial value problem $y''+9y = \sin 3t, y(0)=0, y'(0)=0$ **.**

Solution:

$$y''+9y=\sin 3t$$
$$L\left(y''\right)+9L(y)=L(\sin 3t)$$
$$\Rightarrow s^2\overline{y}-sy(0)-y'(0)+9\overline{y}=\frac{3}{s^2+9}$$
$$\left(s^2+9\right)\overline{y}=\frac{3}{s^2+9}$$
$$\Rightarrow \overline{y}=\frac{3}{\left(s^2+9\right)^2}$$
$$y=L^{-1}\left(\frac{3}{\left(s^2+9\right)^2}\right)$$
$$=L^{-1}\left(\frac{1}{2s}.\frac{6s}{\left(s^2+9\right)^2}\right)$$
$$=\frac{1}{2}\int_0^t t\sin 3t\ dt$$
$$\frac{1}{2}\int_0^t t\sin 3t\ dt=\frac{1}{2}\left[t\left(\frac{-\cos 3t}{3}\right)-1\left(\frac{-\sin 3t}{9}\right)\right]_0^t$$
$$=\frac{1}{2}\left[\left(\frac{-t\cos 3t}{3}\right)+\left(\frac{\sin 3t}{9}\right)\right]$$
$$=\frac{1}{2}\left[\frac{-3t\cos 3t+\sin 3t}{9}\right]$$

Using Laplace transform find the solution of $y'+3y+2\int_0^t ydt=t, y(0)=1$

Solution:

$$y'+3y+2\int_0^t ydt=t$$

Taking L. T. on both sides,

$$L(y') + 3L(y) + 2L\left(\int_0^t y\,dt\right) = L(t)$$

$$s\,\bar{y} - y(0) + 3\bar{y} + 2\frac{1}{s}\,\bar{y} = \frac{1}{s^2}$$

$$\bar{y}\left(s + 3 + \frac{2}{s}\right) = \frac{1}{s^2} + 1$$

$$\bar{y}\left(\frac{s^2 + 3s + 2}{s}\right) = \frac{1 + s^2}{s^2}$$

$$\bar{y} = \frac{1 + s^2}{\left(s^2 + 3s + 2\right)s} = \frac{1 + s^2}{s(s+1)(s+2)}$$

$$\frac{1 + s^2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Using partial fractions, A= $\frac{1}{2}$, B= -2, C = $\frac{5}{2}$

$$\bar{y} = \frac{1 + s^2}{\left(s^2 + 3s + 2\right)s}$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1 + s^2}{s(s+1)(s+2)}\right)$$

$$L^{-1}\left(\frac{1 + s^2}{s(s+1)(s+2)}\right) = L^{-1}\left(\frac{1/2}{s}\right) + L^{-1}\left(\frac{-2}{s+1}\right) + L^{-1}\left(\frac{5/2}{s+2}\right)$$

$$y(t) = \frac{1}{2}\ (1) - 2\ e^{-t} + \frac{5}{2}e^{-2t}$$

Find the inverse Laplace transform of $\frac{1}{(s+1)(s^2+4)}$ ***(May / June 2009)***

Solution:

$$L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = L^{-1}\left(\frac{1}{s^2+4}\cdot\frac{1}{s+1}\right)$$

$$= L^{-1}\left[F(s)\,.\,G(s)\right]$$

$$= f(t) * g(t) = \sin 2t * e^{-t}$$

$$= \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \sin 2u\; e^{-(t-u)}du$$

$$= e^{-t}\int_0^t \sin 2u\; e^{u}du$$

$$= e^{-t}\,\frac{e^{u}}{1+4}\left(\sin 2u - 2\cos 2u\right)\Big|_0^t$$

$$= e^{-t}\left[\frac{e^{t}}{5}\left(\sin 2t - 2\cos 2t\right) + \frac{2}{5}\right]$$

$$= \frac{1}{5}\left(\sin 2t - 2\cos 2t\right) + \frac{2e^{-t}}{5}$$

Solve the equation $y''+9y = \cos 2t, y(0) = 1, y\left(\frac{\pi}{2}\right) = -1$**, using Laplace transform(*May/ June 2009)***

Solution:

Given $y''+9y = \cos 2t$

$$L\left(y''\right)+9L(y)=L(\cos 2t)$$

$$s^2L(y(t))-sy(0)-y'(0)+9L(y(t))=\frac{s}{s^2+4}$$

$$s^2Y(s)-s(1)-k+9Y(s)=\frac{s}{s^2+4}, \qquad Assume \quad L(y(t))=Y(s)\ \&\ y'(0)=k$$

$$Y(s)\left(s^2+9\right)=\frac{s}{s^2+4}+s+k$$

$$Y(s)=\frac{\frac{s}{s^2+4}+s+k}{\left(s^2+9\right)}=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{\left(s^2+9\right)}$$

$$y(t)=L^{-1}\left[\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{\left(s^2+9\right)}\right]=L^{-1}\left[\frac{1}{5}\left(\frac{s}{\left(s^2+4\right)}-\frac{s}{\left(s^2+9\right)}\right)+\frac{s+k}{\left(s^2+9\right)}\right]$$

$$y(t)=L^{-1}\left(\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}\right)+L^{-1}\left(\frac{s}{\left(s^2+9\right)}\right)+L^{-1}\left(\frac{k}{\left(s^2+9\right)}\right)$$

$$y=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{k}{3}\sin 3t$$

$$Given \quad y\left(\frac{\pi}{2}\right)=-1 \Rightarrow k=\frac{12}{5}$$

$$y=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t$$

Using Convolution theorem find $L^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right]$ ***(Nov/ Dec. 2009)***

Solution:

$$L^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right]=L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s+1}\right)$$

$$= L^{-1}\left[F(s)\,.\,G(s)\right]$$

$$= f(t) * g(t) = \sin t * e^{-t}$$

$$= \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \sin u \; e^{-(t-u)} du$$

$$= e^{-t}\int_0^t \sin u \; e^{u} du$$

$$= e^{-t}\,\frac{e^{u}}{1+1}\left(\sin u - \cos u\right)\Big|_0^t$$

$$= e^{-t}\left[\frac{e^{t}}{2}\left(\sin t - \cos t\right) + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left(\sin t - \cos t\right) + \frac{e^{-t}}{2}$$

Solve the differential equation $\frac{d^2y}{dt^2} + y = \sin 2t, y(0) = 0, y'(0) = 0$ by using Laplace transform method*(Nov/Dec. 2009)*

Solution:

Given $\frac{d^2y}{dt^2} + y = \sin 2t$

Applying Laplace on both sides,

$$L(y'') + L(y) = L(\sin 2t)$$

$$s^2L(y(t)) - sy(0) - y'(0) + L(y(t)) = \frac{2}{s^2+4}$$

Let $L(y(t)) = Y(s) \Rightarrow s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$

$$\Rightarrow s^2Y(s)+Y(s)=\frac{2}{s^2+4}$$

$$Y(s)\left(s^2+1\right)=\frac{2}{s^2+4}$$

$$Y(s)=\frac{2}{\left(s^2+4\right)\left(s^2+1\right)}$$

$$y(t) \quad = L^{-1}\left[\frac{2}{\left(s^2+4\right)\left(s^2+1\right)}\right]$$

$$= L^{-1}\left[\frac{2}{\left(s^2+4\right)}\cdot\frac{1}{\left(s^2+1\right)}\right]$$

$$= L^{-1}\left[F(s)\ .\ G(s)\right]$$

$$= f(t) * g(t) = \sin 2t * \sin t$$

$$= \int_0^t f(u)g(t-u)du$$

$$= \int_0^t \sin 2u \ \ \sin(t-u)\ du$$

$$= \int_0^t \frac{1}{2}\left(\cos(2u-t+u)-\cos(u+t)\right)du$$

$$= \frac{1}{2}\int_0^t \left(\cos(3u-t)-\cos(u+t)\right)du$$

$$= \frac{1}{2}\left(\frac{\sin(3u-t)}{3}-\sin(u+t)\right)_0^t$$

$$= \frac{1}{2}\left(\frac{\sin(3t-t)}{3}-\sin(t+t)-\frac{\sin(-t)}{3}+\sin(t)\right)$$

$$= \frac{1}{2}\left(\frac{\sin(2t)}{3}-\sin(2t)+\frac{\sin t}{3}+\sin(t)\right)$$

$$= \frac{1}{3}\left(-\sin 2t+2\sin t\right)$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: MATHEMATICS-II **Subject Code: 17PHU602**

Class : III - B.Sc. Physics **Semester : VI**

Unit IV

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The operator L that transforms f(t) into F(s) is called the -------- operator.	Fourier	Hankel	Laplace operator	Z	Laplace operator
The Laplace transform is said to exist if the integral is --------- for some value of s; otherwise it does not exist.	discontinuous	divergent	closed	convergent	convergent
If f(t) is --------- on every finite interval in $(0,\infty)$ and is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	unifromly continuous	piecewise continuous	convergent	divergent	piecewise continuous
If f(t) is piecewise continuous on every -------- and is of exponentialorder 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	closed interval [0,1]	Half open interval [0,1)	infinite interval in $(0,\infty)$	finite interval in $(0,\infty)$	finite interval in $(0,\infty)$
If f(t) is piecewise continuous on every finite interval in $(0,\infty)$ and is of ------- 'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a.	exponential order	quadratic order	cubic order	n th order	exponential order

If f(t) is piecewise continuous on every finite interval in (0,∞) and is ofexponentialorder'a' for t>0, then the Laplace transform of f(t) exists for all s>a, ie F(s) exists for every s>a. This condition is	necessary	non sufficient	Sufficient	both necessary and sufficient	Sufficient
L[1] =	n! / s^(n+1)	1/s , s > 0	1/(t+1)	1/ (s-a)	1/s , s > 0
L[t^n] =	2/(s-1)	n!	1/ s^(n+1)	n! / s^(n+1)	n! / s^(n+1)
L[e^(at)] =	1/ (s-a)	1/s , s > 0	n! / s^(n+1)	a/(s-a)	1/ (s-a)
L[e^(-at)] =	F(s-a)	s^2 F(s)-s f(0)- f '(0)	1/ (s+a)	n! / s^(n+1)	1/ (s+a)
L[sinat]=	a/(s^2 +a^2)	1/(s^2 +a^2)	(s^2 +a^2)	a/(s^3+a^3)	a/(s^2 +a^2)
L[cosat]=	n! / s^(n+1)	s^(n+1)	t^(n+1)	s/(s^2 +a^2)	s/(s^2 +a^2)
L[coshat]=	s/(s^2 -a^2)	1/(s^3 -a^3)	s/(s^2 +a^2)	1/a F(s/a)	s/(s^2 -a^2)
L[af(t) + bg(t)]=	aF(s)+bG(s)	aF(s)-bG(s)	bF(s)-aG(s)	bF(s) * aG(s)	aF(s)+bG(s)
L[af(t) + bg(t)]= aF(s)+bG(s) is called ------property	quasi linear	non-linear	Linearity	homogenous	Linearity
Lineraity property is	L[af(t) + bg(t)]= aF(s) * bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)	1/a F(s/a)	L[af(t) + bg(t)]= aF(s)-bG(s)	L[af(t) + bg(t)]= aF(s)+bG(s)
If L[f(t)]=F(s) then L[e^at f(t)]=	aF(s)+bG(s)	F(s+a)	1-s	F(s-a)	F(s-a)
First Shifting property is if L[f(t)] = F(s) then ------	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)	s^2 F(s)-s f(0)- f '(0)	s^(n+1)	L[e^at f(t)]=F(s-a)
If L[f(t)]=F(s) then L[e^at f(t)]=F(s-a) is called ----property	linear	convolution	First shifting property	non homogenous	First shifting property
If L[f(t)]= F(s) then L[f(at)]=1/a F(s/a) is called ________ property.	Change of scale	convolution	First shifting property	non homogenous	Change of scale
If L[f(t)]= F(s) then L[f(at)]=	F(s/a)	1/a F(s/a)	F(s-a)	a F(s/a)	1/a F(s/a)
______is called the change of scale property	L[f(at)]= t-1	L[f(at)]= 1/(s^3 -a^3)	L[f(at)]= 1/a F(s/a)	L[e^at f(t)]=F(s-a)	L[f(at)]= 1/a F(s/a)
Change of scale property is -----	L[f(at)]= 1/a F(s/a)	L[f(at)]= F(s/a)	L[f(at)]= F(a/s)	L[f(at)]= a F(s/a)	L[f(at)]= 1/a F(s/a)
If L[f(t)]= F(s) then L[f ' (t)] =	F(s)-f(0)	s F(s)-+(0)	s F(s)-f(0)	F(s)+f(0)	s F(s)-f(0)
If L[f(t)]= F(s) then L[f " (t)] =	s^2 F(s)-s f(0)	s^2 F(s)-s f(0)- f '(0)	s^2 F(s)-s f(0)+ f '(0)	s^2 F(s)+s f(0)+ f '(0)	s^2 F(s)-s f(0)- f '(0)
L[5 (t^3)] =	1	1/s , s > 0	3/ (s^4)	30/ (s^4)	30/ (s^4)
L[6 t] =	6	6/(s^2)	6/s	6-s	6/(s^2)
L[2 e ^ (-6 t)] =	2/(s+6)	2	2/(s-6)	2/s	2/(s+6)
L[7] =	7/s	1/s , s > 0	(-7/s)	7	7/s

L[10 sin2t]=	20/(s^2-4)	2/(s^2+4)	2/(s^2-4)	20/(s^2+4)	20/(s^2+4)
L[7 cosh3t]=	7s/(s^2-9)	7/(s^2-9)	s/(s^2-9)	7s/(s^2+9)	7s/(s^2-9)
The inverse laplace transform of 1/s is =	0	-1	s+a	1	1
The inverse laplace transform of 1/(s-a) is =	e^(-at)	1/e^(at)	e^(at)	1/e^(-at)	e^(at)
The inverse laplace transform of 1/(s+a) is =	e^(-at)	1/e^(at)	1/e^(-at)	e^(at)	e^(-at)
If L[f(t)]=F(s) then f(t) is called -------- laplace transform of F(s)	Linear	non-linear	inverse	quasi linear	inverse
If L is linear then -------- is Linear.	L+1	L^(-1)	1/L	(-1/L)	L^(-1)
If L is linear then L inverse is --------	non-linear	Linear	divergent	quasi linear	Linear
The convolution of f*g of f(t) and g(t) is defined as	(f*g)(t)=∫ (from 0 to t) f(u) g(t+u) du	(f*g)(t)=∫ (from 0 to t) f(u) du	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)=∫ (from 0 to t) g(t-u) du	(f*g)(t)=∫ (from 0 to t) f(u) g(t-u) du
_______is called the convolution theorem.	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du	(f*g)(t)=1-t	(f*g)(t)=e^(-at)	(f*g)(t)=L^(-1)(1)	(f*g)(t)=∫ from 0 to t f(u) g(t-u) du
A function f(t) is said to be -----with period T>0 if f(t+T)=f(t) for all t	even	projection	odd	peroidic	periodic
L[k] =	k/s	k/s , s > 0	(-1/s)	k	k/s
L[sinhat]=	a/(s^2 -a^2)	1/(s^3 -a^3)	a/(s^2 +a^2)	1/a F(s/a)	a/(s^2 -a^2)
L[e^(8t)] =	1/ (s-8)	1/s , s > 0	n! / s^(n+1)	8/(s-8)	1/ (s-8)

UNIT-V

Interpolation with unequal intervals problems-Lagrange's interpolation problems-
Newton's divided difference interpolation problems-Newton's forward and backward difference problems

Introduction

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x : x_1 x_2 x_3............x_n

y : y_1 y_2 y_3...........y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 to x_n

Let y = $f(x)$ be a function taking the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the *values* $x_0, x_1, x2, \ldots\ldots x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases , we replace f(x) by simple fan arbitrary function and let Φ(x) denotes an arbitrary function which satisfies the set of values given in the table above . The function Φ(x) is called interpolating function or smoothing function or interpolation formula.

Newton's forward interpolation formula(or) Gregory-Newton forward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots, x_n.$

Let suppose that the values of x i.e., $x_0, x_1, x_2 \ldots\ldots, x_n$ are equidistant .

$x_1 = x_0 + h$; $x_2 = x_1 + h$; *and so on* $x_n = x_{n-1} + h$;

Therefore $xi = x0 + i\,h$, *where* $i = 1,2, \ldots, n$

Let $P_n(x)$ be a polynomial of the n^{th} degree in which x is such that

$$y_I = f(x_i) = P_n(x_i), \quad I = 0,1,2, \ldots. n$$

Let us assume Pn(x) in the form given below

$$P_n(x) = a_0 + a_1 (x - x_0)^{(1)} + a_2 (x - x_0)^{(2)} + \ldots\ldots. + a_r (x - x_0)^{(r)} + \ldots\ldots. + \ldots\ldots + a_n (x - x_0)^{(n)} \quad \ldots\ldots(1)$$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \ldots\ldots a_n$ can be found as follows :

$P_n(x_0) = y_0 = a_0$ (setting x = x0, in (1))

Similarly $y_1 = a_0 + a_1 (x_1 - x_0)$

$$y_2 = a_0 + a_1 (x_2 - x_0) + a_2 (x_2 - x_0)$$

From these, we get the values of $a_0, a_1, a_2, \ldots\ldots a_n$

i.e.,

Therefore, $a_0 = y_0$

$\Delta y_0 = y_1 - y0 = a_1 (x_1 - x_0)$

$$= a_1 h$$

$$\Rightarrow a_1 = \Delta y_0 / h$$

lly $\Rightarrow a_2 = (\Delta y_1 - \Delta y_0) / 2h^2 = \Delta^2 y_0 / 2!\, h^2$

lly $\Rightarrow a_3 = \Delta^3 y_0 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = = y_0 + (x - x_0)^{(1)}\Delta y_0 /h + (x - x_0)^{(2)}\Delta^2 y_0/(2!\, h^2) + \ldots\ldots + (x - x_0)^{(r)}\Delta^r y_0/(r!\, h^r) + \ldots\ldots + (x - x_0)^{(n)}\Delta^r y_0/(n!\, h^n)$$

By substituting $\frac{x - x_0}{h} = u$ *, the above equation becomes*

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad \ldots$$

By substituting $u = u^{(1)}$,

$u(u-1) = u^{(2)}$,

$u(u-1)(u-2) = u^{(3)}$, *… in the above equation, we get*

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

The above equation is known as **Gregory-Newton forward formula or Newton's forward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.

2. This formula apply forward differences of y_0, hence this is used to interpolate the values of y nearer to beginning value of the table (i.e., x lies between x0 to ^{x}1 or x1 to x_2)

Example.

Find the values of y at x = 21 from the following data.

x: 20 23 26

x: 0.3420 0.3907

0.4384

29

0.4848

Solution.

Step 1. Since x = 21 is nearer to beginning of the table. Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

Step 3. Write down the formula and put the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots. + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u^{(1)} = (x - x_0) / h = (21 - 20) / 3 = 0.3333$

$u(2) = u(u-1) = (0.3333)(0.6666)$

$P_n\,(x=21) = y(21) = 0.3420 + (0.3333)(0.0487) + (0.3333)(-0.6666)(-0.001)$

$+ (0.3333)(-0.6666)(-1.6666)(-0.0003)$

$= \mathbf{0.3583}$

Example: . From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step1. Since$x = 46$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$..Hence we apply Newton's forward formula.

Step 2. Construct the difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6.00			
65	68.48				

Step 3. Write down the formula and put the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{u^{(r)}}{r!}\Delta^r y0 + \ldots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u = (x - x_0) / h$ = (46 – 45) / 5 = 01/5 = 0.2

$$P_n(x=46) = y(46) = 114.84 + [0.2\ (-18.68)] + [0.2\ (-0.8)\ (5.84)/\ 3]$$
$$+ [0.2\ (-0.8)\ (-1.8)(-1.84)/6\ ]$$
$$+ [0.2\ (-0.8)\ (-1.8)(-2.8)(0.68)]$$

= 114.84 – 3.7360 – 0.4672 – 0.08832 – 0.228

= **110.5257**

Example . From the following table , find the value of tan $45^0\ 15'$

x^0:	45	46	47	48	49	50
tan x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since $x = 45^o\ 15'$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 1$. Hence we apply Newton's forward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$

45^0	1.0000					
		0.03553				
460	1.03553		0.00131			
		0.03684		0.00009		
47^0	1.07237		0.00140		0.00003	
		0.03824		0.00012		-0.00005
48^0	1.11061		0.00152		-0.00002	
		0.03976		0.00010		
49^0	1.15037		0.00162			
		0.04138				
50^0	1.19175					

Step 3. Write down the formula and substitute the various values :

$$P_n(x) = P_n y(x_0 + uh) = y_0 + u^{(1)}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \ldots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \ldots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

Where $u = (45^o\,15' - 45^0) / 1^0$

$= 15' / 1^0$

$= 0.25$ ……………(since $1^0 = 60'$)

$$y\,(x=45^o15') = P_5\,(45^o15') = 1.00 + (0.25)(0.03553) + (0.25)(-0.75)(0.00131)/2$$
$$+(0.25)(-0.75)(-1.75)(0.00009)/6$$
$$+(0.25)(-0.75)(-1.75)(-2.75)(0.0003)/24$$
$$+(0.25)(-0.75)(-1.75)(-2.75)(-3.75)(-0.00005)/120$$
$$= 1.000 + 0.0088825 - 0.0001228 + 0.0000049$$

$=\boldsymbol{1.00876}$

Newton's backward interpolationformula(or) Gregory-Newton backward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots\ldots\ldots, y_n$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots\ldots, x_n.$

Let suppose that the values of x i.e., $x_0, x_1, x_2 \ldots\ldots\ldots, x_n$ are equidistant .

$x_1 = x_0 + h$; $x_2 = x_1 + h$; and so on $x_n = x_{n-1} + h$;

Therefore $xi = x0 + i\,h$, where $i = 1,2, \ldots, n$

Let $P_n(x)$ be a polynomial of the n^{th} degree in which $x$is such that

$y_I = f(x_i) = P_n(x_i)$, $I = 0,1,2, \ldots. n$

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)(x - x_{n-1})^{)} + \ldots\ldots\ldots + a_n(x - x_n)(x - x_{n-1}) \ldots(x - x_1) \ldots\ldots(1)$$

Let us assume Pn(x) in the form given below

$$P_n(x) = a_0 + a_1(x - x_n)^{(1)} + a_2(x - x_n)^{(2)} + \ldots\ldots\ldots + a_r(x - x_n)^{(r)} + \ldots\ldots + \ldots\ldots + a_n(x - x_n)^{(n)} \ldots\ldots(1.1)$$

This polynomial contains the n + 1 constants $a_0, a_1, a_2, \ldots\ldots\ldots a_n$ can be found as follows :

$P_n(x_n) = \quad y_n = a_0$ (setting x = xn, in (1))

Similarly$y_{n-1} = a_0 + a_1(x_{n-1} - x_n)$

$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)$

From these, we get the values of $a_0, a_1, a_2, \ldots\ldots a_n$

Therefore, $a_0 = y_n$

$\breve{}yn = y_n\text{-}yn\text{-}1 = a_1(x_{n-1} - x_n)$

$= a_1 h$

$\Rightarrow a_1 = y_n / h$

lly $\Rightarrow a_2 = (\breve{}y_1 - \breve{}y_n) / 2h^2 = {}^2 y_n / 2!\, h^2$

lly$\Rightarrow a_3 = {}^3 / 3!\, h^3$

Putting these values in (1), we get

$$P_n(x) = = y_n + (x - x_n)^{(n)} \nabla y_n / h + (x - x_n)^{(2)} \nabla^2 y_n / (2!\, h^2) + (x - x_n)^{(r)} \nabla^r y_n / (r!\, h^r) + \ldots + (x - x_n)^{(n)} \nabla^r y_n / (n!\, h^n)$$

By substituting $\dfrac{x - x_n}{h} = v$ *, the above equation becomes*

$$y(x_n + vh) = y_n + v \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \ldots\ldots$$

By substituting $v = v^{(1)}$,

$v(v+1) = v^{(2)}$,

$v(v+1)(v+2) = v^{(3)}$, *… in the above equation, we get*

$$P_n(x) = P_n y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)}}{2!} \nabla^2 y_n + \frac{v^{(3)}}{3!} \nabla^3 y_n + \ldots + \frac{v^{(r)}}{r!} \nabla^r y_n + \ldots + \frac{v^{(n)}}{n!} \Delta^n y_n$$

The above equation is known as **Gregory-Newton backward formula or Newton's backward interpolation formula.**

Note : 1. This formula is applicable only when the interval of difference is uniform.

2. This formula apply backward differences of y_n, hence this is used to interpolate the values of y nearer to the end of a set tabular values. (i.e., x lies between xn to xn- 1 and xn-1 to x_{n-2})

Example: Find the values of y at x = 28 from the following data.

x:	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Solution.

Step 1. Since x = 28 is nearer to beginning of the table. Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$
20	0.3420	(0.3420-0.3907)		
		0.0487	(0.0477-0.0487)	
23	0.3907		-0.001	
		0.0477		
26	0.4384			-0.0003
			-0.0013	
29	0.4848	0.0464		

Step 3. Write down the formula and put the various values :

$$P_3(x) = P_3 y(x_n + vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}\nabla^2 y_n}{2!} + \frac{v^{(3)}\nabla^3 y_n}{3!}$$

Where $v^{(1)} = (x - x_n) / h = (28 - 29) / 3 = -0.3333$

$v^{(2)} = v(v+1) = (-0.333)(0.6666)$

$v^{(3)} = v(v+1)(v+2) = (-0.333)(0.6666)(1.6666)$

$P_n (x=28) = y(28) = 0.4848 + (-0.3333)(0.0464) + (-0.3333)(0.6666)(-0.0013)/2$

$+(-0.3333)\ (0.6666)(1.6666)\ \ (-0.0003)/6$

$$= 0.4848 - 0.015465 + 0.0001444 + 0.0000185$$

$$= \mathbf{0.4695}$$

Example: From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 63.

Age	x:	45	50	55	60	65
Premium	y:	114.84	96.16	83.32	74.48	68.48

Solution.

Step 1. Since$x = 63$ is nearer to beginning of the table and the values of x is equidistant i.e., $h = 5$..Hence we apply Newton's backward formula.

Step 2. Construct the difference table

x	y	∇y_0	$\nabla^2 y0$	$\nabla^3 y_0$	$\nabla^4 y_0$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.12		4.00		-
				1.16	
		-8.84			
60	74.48		2.84		
		-6.00			
65	68.48				
					0.68

Step 3. Write down the formula and put the various values :

$$P_3\,(x) = P_3 y(x_n + vh) = y_n + v^{(1)}\nabla y_n + \frac{v^{(2)}}{2!}\nabla^2 y_n + \frac{v^{(3)}}{3!}\nabla^3 yn + \frac{v^{(4)}}{4!}\nabla^4 y_n$$

Where $v^{(1)} = (x - x_n) / h = (63 - 65) / 5 = -2/5 = -0.4$

$$v(2) = v(v+1) = (-0.4)(1.6)$$

$$v(3) = v(v+1)\,(v+2) = (-0.4)(1.6)\,(2.6)$$

$$v(4) = v(v+1)\ (v+2))\ (v+3) = (-0.4)(1.6)\ (2.6)(3.6)$$

$$P_4\ (x=63)=y(63) = 68.48 + [(-0.4)\ (-6.0)] +[(-0.4)\ (1.6)\ (2.84)/ 2]$$
$$+ [(-0.4)\ (1.6)\ (2.6)(-1.16)/6\ ]$$
$$+ [(-0.4)\ (1.6)\ (2.6)(3.6)\ (0.68)/24\ ]$$

$$= 68.48 +2.40 - 0.3408 +0.07424 – 0.028288$$

= **70.5852**

Example: From the following table , find the value of tan 49^0 15'

x^0:	45	46	47	48	49	50
tan x^0:	1.0	1.03553	1.07237	1.11061	1.15037	1.19175

Solution.

Step 1. Since$x = 49^o\ 45'$is nearer to beginning of the table and the values of x is equidistant i.e., $h =1$.Hence we apply Newton's backward formula.

Step 2. Construct the difference table to find various Δ's

x	y	Δy_0	$\Delta^2 y0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$

45^0	1.0000					
		0.03553				
46	1.03553					
			0.00131			
47^0	1.07237	0.03684		0.00009		
			0.00140		0.00003	
48^0	1.11061	0.03824		0.00012		-0.00005
			0.00152		-0.00002	
49^0	1.15037	0.03976		0.00010		
			0.00162			
50^0	1.19175	0.04138				

Step 3. Write down the formula and substitute the various values :

$$P_5(x) = P_5 y(x_n + vh) = y_n + v^{(1)} \nabla y_n + \frac{v^{(2)}}{2!} \nabla^2 y_n + \frac{v^{(3)}}{3!} \nabla^3 yn + \frac{v^{(4)}}{4!} \nabla^4 yn + \frac{v^{(5)}}{5!} \nabla^5 y_n$$

Where $v = (49^o\, 45' - 50^0) / 1^0$

$= -\,15' / 1^0$

$= -\,0.25$ ……………(since $1^0 = 60'$)

$v(2) = v(v+1) \qquad = (-0.25)\,)(\,0.75)$

$\qquad\qquad\qquad\qquad = (-0.25)\,(0.75)(1.75)$

$v(3) = v(v+1)\,(v+2)$

$$v(4) = v(v+1)(v+2))(v+3) = (-0.25)(0.75)(1.75)(2.75)$$

$$y(x=49^o 15') = P_5(49^o 15') = 1.19175 + (-0.25)(0.04138) + (-0.25)(0.75)(0.00162)/2$$
$$+(-0.25)(0.75)(1.75)(0.0001)/6$$

$$+(-0.25)(0.75)(1.75)(2.75)(-0.0002)/24$$
$$+(-0.25)(0.75)(1.75)(2.75)(3.75)(-0.00005)/120$$
$$= 1.19175 - 0.010345 - 0.000151875 + 0.000005 + \ldots$$
$$= \mathbf{1.18126}$$

Lagrange's Interpolation Formula

Interpolation means the process of computing intermediate values of a function a given set of tabular values of a function. Suppose the following table represents a set of values of x and y.

x:	x_0	x_1	x_2	$x3$	………..	x_n
y:	y_0	y_1	y_2	$y3$	………..	y_n

We may require the value of $y = y_i$ for the given $x = x_i$, where x lies between x_0 to x_n

Let y = $f(x)$ be a function taking the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the *values* $x_0, x_1, x2, \ldots\ldots x_n$. Now we are trying to find $y = y_i$ for the given $x = x_i$ under assumption that the function f(x) is not known. In such cases, x_i 's are not equally spaced we use *Lagrange's interpolation formula.*

Newton's Divided Difference Formula:

The divided difference $f[x_0, x_1, x_2, \ldots, x_n]$, sometimes also denoted $[x_0, x_1, x_2, \ldots, x_n]$, on $n+1$ points

X_0, x_1, ..., x_n of a function $f(x)$ is defined by $f[x_0] \equiv f(x_0)$ and

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_0, \dots, x_{n-1}] - f[x_1, \dots, x_n]}{x_0 - x_n}$$

for $n \geq 1$. The first few differences are

$$f[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1}$$

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$$

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_0, \dots, x_{n-1}] - f[x_1, \dots, x_n]}{x_0 - x_n}.$$

Defining

$\pi_n(x) \equiv (x - x_0)(x - x_1)\cdots(x - x_n)$ and taking the derivative

$\pi'_n(x_k) = (x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)$ gives the identity

$$f[x_0, x_1, \dots, x_n] = \sum_{k=0}^{n} \frac{f_k}{\pi'_n(x_k)}.$$

Lagrange's interpolationformula(for unequal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \dots\dots\dots, y_n$ corresponding to the values $x_0, x_1, x_2 \dots\dots\dots, x_n.$

Let suppose that the values of x *i.e.,* $x_0, x_1, x_2 \dots\dots\dots, x_n$. are not equidistant .

$$y_I = f(x_i) \quad I = 0,1,2, \dots. N$$

Now, there are (n+1) paired values$(x_i,y_i,)$, *I = 0,1,2, … n and hence f(x) can be represented by a polynomial function of degree n in x.*

Let us consider f(x) as follows

$$f(x) = a_0(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)$$
$$+ a_1(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)$$
$$+ a_2(x-x_0)(x-x_3)(x-x_4)\ldots(x-x_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+a_n(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_{n-1}) \quad \ldots\ldots\ldots(1)$$

Substituting $x = x_0$, $y = y_0$, *in the above equation*

$$y_0 = a_0(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)$$

which implies $a_0 = y_0/(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)$

Similarly $a_1 = y_1/(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)$

$$a_2 = y_2/(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n = y_n\,(x_n-x_0)(x_n-x_2)(x_n-x_3)\ldots(x_n-x_{n-1})$$

Putting these values in (1), we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)}\,y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}\,y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_2)(x_1-x_3)\ldots(x_1-x_n)}\,y_2$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ \ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_3)\ \ldots(x_n-x_{n-1})} y_n$$

The above equation is called ***Lagrange's interpolation formula*** for unequal intervals.

Note : 1. This formula is will be more useful when the interval of difference is not uniform.

POSSIBLE QUESTIONS

1. Prove that $E\Delta = \Delta = \nabla E$.
2. Write Gregory Newton backward interpolation formulae.
3. Define Inverse Lagrange's interpolation
4. Prove that $\mu = (1+\frac{\delta^2}{4})^{\frac{1}{2}}$
5. Prove that $\Delta\nabla = \Delta - \nabla = \delta^2$.
6. From the following table, find the value of tan 45°15′

x° :	45	46	47	48	49	50
tan x° :	1.0000	1.0355	1.072	1.1106	1.1503	1.1917

7. Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

8. From the following table find f(x) and hence f(6) using Newton interpolation formula.

x :	1	2	7	8
f(x) :	1	5	5	4

9. Find the values of y at X=21 and X=28 from the following data.

X:	20	23	26	29
Y:	0.3420	0.3907	0.4384	0.4848

10. Using Newton's divided difference formula. Find the values of f(2),f(8) and f(15) given the following table

x:	4	5	7	10	11	13
f(x):	48	100	294	900	1210	2028

11. Using Lagrange's interpolation formula find the value corresponding to x = 10 from the following table

x :	5	6	9	11
y :	12	13	14	16

12. From the following table of half-yearly premium for policies maturing at different ages. Estimate the premium for policies maturing at age 46 & 63.

Age x :	45	50	55	60	65
Premium y :	114.84	96.16	83.32	74.48	68.48

13. Find the value of y at x = 1.05 from the table given below.

x :	1.0	1.1	1.2	1.3	1.4	1.5
y :	0.841	0.891	0.932	0.964	0.985	1.015

14. Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

15. Find the age corresponding to the annuity value 13.6 given the table

Age(x) :	30	35	40	45	50
Annuity Value(y):	15.9	14.9	14.1	13.3	12.5

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: MATHEMATICS-II **Subject Code: 17PHU602**

Class : III - B.Sc. Physics **Semester : VI**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The process of computing the value of the function inside the given range is called ________	Interpolation	extrapolation	reduction	expansion	Interpolation
If the point lies inside the domain [x_0, x_n], then the estimation of f(y) is called _________	Interpolation	extrapolation	reduction	expansion	Interpolation
The process of computing the value of the function outside the given range is called ________	Interpolation	extrapolation	reduction	expansion	extrapolation
If the point lies outside the domain [x_0, x_n], then the estimation of f(y) is called _________	Interpolation	extrapolation	reduction	expansion	extrapolation
In the forward difference table y_0 is called ___________ element.	leading	ending	middle	positive	leading
In the forward difference table forward symbol ((y_0)), forward symbol(^2(y_0)) , ... are called ___________ difference.	leading	ending	middle	positive	leading
The difference of first forward difference is called ________.	divided difference	2nd forward differ	3rd forward differe	4th forward difference	2nd forward difference
Gregory Newton forward interpolation formula is also called as Gregory Newton forward_____________ formula.	Elimination	iteration	difference	distance	difference

Gregory Newton backward interpolation formula is also called as Gregory Newton backward_____________ formula	Elimination	iteration	difference	distance	difference
Gregory Newton backward interpolation formula is also called as Gregory Newton backward _____________ formula .	Elimination	iteration	difference	distance	difference
The divided differences are _________ in their arguments.	constant	symmetrical	varies	singular	symmetrical
In Gregory Newton forward interpolation formula 1st two terms of this series give the result for the _________ interpolation.	Ordinary linear	ordinary differenti	parabolic	central	Ordinary linear
Gregory Newton forward interpolation formula 1st three terms of this series give the result for the _________ interpolation.	Ordinary linear	ordinary differenti	parabolic	central	parabolic
Gregory Newton forward interpolation formula is mainly used for interpolating the values of y near the _____________ of the set of tabular values.	beginning	end	centre	side	beginning
Gregory Newton backward interpolation formula is mainly used for interpolating the values of y near the _______of the set of tabular values.	beginning	end	centre	side	end
From the definition of divided difference (u-u_0)/(x-x_0) we have __________ =	(y,y_0)	(x,y)	(x_0, y_0)	(x,x_0)	(x_0, y_0)
If f(x) =0, then the equation is called ___________	Homogenous	non-homogenous	first order	second order	Homogenous
The order of y_(x+3) - 5 y_(x+2) + 7y_(x+1)+y_x= 10x is	2	0	1	3	3
A function which satisfies the difference equation is a ________of the difference equation.	Solution	general solution	complementary so	particular solution	Solution
The degree of the difference equation is _________	The highest powe	The difference be	The difference bet	The highest value of	The highest powers of
The degree of the difference equation is _________	2	0	1	3	1
The order of y (x+3) -y (x+2) = 5x^2 is	3	2	1	0	1
The difference between the highest and lowest subscripts of y are called ______ of the difference equation	degree	order	power	value	order

E-1=	backward differen	forward symbol	μ	δ	forward symbol
Which of the following is the central difference operator?	backward differen	forward symbol	μ	δ	δ
1+(forward symbol)=	backward differen	E	μ	δ	E
μ is called the ______ operator	Central	average	backward	displacement	average
The other name of shifting operator is ______ operator	Central	average	backward	displacement	displacement
The difference of constant functions are____________	0	1	2	3	0
The nth order divided difference of x_n will be a polynomial of degree ________.	0	1	2	3	2
The operator forward symbol is ___________	homogenous	heterogeneous	linear	a variable	linear

y’s