KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: CALCULUS **Subject Code: 17MMU101**

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4	0	0	4

PO: On successful completion of this course the learners gain knowledge about the higher order derivatives and its applications in business, economics and life sciences.

PLO: To enable the students to learn and gain knowledge about concavity, inflection points and its geometrical applications.

UNIT I

Hyperbolic functions, higher order derivatives, Leibniz rule and its applications to problems of type $e^{ax+b}\sin x$, $e^{ax+b}\cos x$, $(ax+b)^n\sin x$, $(ax+b)^n\cos x$.

UNIT II

Reduction formulae, derivations and illustrations of reduction formulae of the type $\int \sin nx\, dx$, $\int \cos nx\, dx$, $\int \tan nx\, dx$,$\int \sec nx\, dx$, $\int \log x^n\, dx$, $\int \sin^n x \sin^m x\, dx$. Curve tracing in Cartesian coordinates, tracing in polar coordinates of standard curves, L'Hospital's rule, applications in business, economics and life sciences.

UNIT III

Volumes by slicing, disks and washers methods, volumes by cylindrical shells, parametric equations, parameterizing a curve, arc length, arc length of parametric curves, area of surface of revolution.

UNIT IV

Concavity and Inflection points, asymptotes. Techniques of sketching conics, reflection properties of conics, rotation of axes and second degree equations, classification into conics using the discriminant, polar equations of conics.

UNIT V

Introduction to vector functions, operations with vector-valued functions, limits and continuity of vector functions, differentiation and integration of vector functions, tangent and normal components of acceleration, modeling ballistics and planetary motion, Kepler's second law.

TEXT BOOK

1. Strauss M.J., Bradley G.L.,and Smith K. J., (2007). Calculus, Third Edition ,Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi.

REFERENCES

1. Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.
2. Anton H., Bivens I., and Davis S.,(2002). Calculus, Seventh Edition, John Wiley and Sons (Asia) P. Ltd., Singapore.
3. Courant R., and John F., (2000). Introduction to Calculus and Analysis (Volumes I & II), Springer- Verlag, New York.

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DEPARTMENT OF MATHEMATICS
Lecture Plan

Subject Name: CALCULUS **Subject Code: 17MMU101**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-I	
1.	1	Introduction to Hyperbolic function	T1:Ch 7;Pg:350-353
2.	1	Inverse hyperbolic function	T1:Ch 7;Pg:353-356
3.	1	Higher order derivatives	R4:Ch 4;Pg:156-158
4.	1	Continuation of Higher order Derivatives	R4:Ch 4;Pg:158-159
5.	1	Leibiniz rule and its applications	R4:Ch 4;Pg:169-173
6.	1	Continuation of Leibiniz rule and its applications	R4:Ch 4;Pg:174-177
7.	1	Problems on type e^{ax+b} sinx, e^{ax+b}cos x,$(ax+b)^n$sinx , $(ax+b)^n$cosx	R4:Ch 4;Pg:178-180
8.	1	Finding concavity	R1:Ch 3;Pg:228-230
9.	1	Finding Inflection point	T1:Ch 4;Pg:124-129
10.	1	Curve Sketching with Asymptotes	R4:Ch 12;Pg:389-402
11.	1	Continuation of Curve Sketching with Asymptotes	R4:Ch 12;Pg:403-409
12	1	Recapitulation and Discussion of possible questions	
Total	**12Hours**		

Text Book:
T1 : M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

Reference Book:
R1 : G.B.Thomas and R.L.Finney., (2005). Calculus , 9th edition, Pearson Edition , Delhi.

R4: Shanti Narayan, P. K. Mittal, Differential Calculus, (2016), Third Edition Vikas publishing House pvt.Ltd.

UNIT-II			
1.	1	Curve tracing in Cartesian Coordinates	R2:Ch 11;Pg:767-770
2.	1	Tracing in polar coordinate for standard curves	R3:Ch 1;Pg:101-103
3.	1	Theorm on L'Hospital's Rule	T1:Ch 4;Pg:148-150
4.	1	Problems based on L'Hospital's Rule	T1:Ch 4;Pg:151-153
5.	1	Continuation of problems on L'Hospital's Rule	T1:Ch 4;Pg:153-155
6.	1	Application in business, economics and life sciences.	T1:Ch 6;Pg:287-290
7.	1	Continuation of Application in business, economics and life sciences.	T1:Ch 6;Pg:291-294
8.	1	Reduction formula – derivation and illustration	R2:Ch 7;Pg:497-498
9	1	Problems based on reduction formula	R2:Ch 7;Pg:500-503
10	1	Continuation of problems on reduction formula	R2:Ch 7;Pg:503-505
11	1	Continuation of problems on reduction formula	R2:Ch 7;Pg:506-507
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

Text Book:

T1 : M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

Reference Book:

R2: H.Anton., I. Bivens ., and S.Davis., (2002). Calculus , 7th edition , John Wiley and sons (Asia) Pvt Ltd, Singapore.

R3: R.Courant and F.John., (2000). Introduction to Calculus and Analysis (Volume I & II), Springer verlag, NewYork.

UNIT-III			
1	1	Volume by slicing	R2:Ch 6;Pg:421-424
2	1	Volume by Disks methods	R1:Ch 5;Pg:397-399
3	1	Volume by washers methods	R1:Ch 5;Pg:400-403
4	1	Volumes by cylindrical shells	R2:Ch 6;Pg:432-434
5	1	Continuation of Volumes by cylindrical shells	R2:Ch 6;Pg:434-436
6	1	Area of a surface of revolution	R2:Ch 6;Pg:444-447
7	1	Parametric Equations	R2:Ch 10;Pg:692-695
8	1	Tangent Lines to Parametric Curves	R2:Ch 10;Pg:695-696
9	1	Continuation of Tangent Lines to Parametric Curves	R2:Ch 10;Pg:696-697
10	1	Arc Length of Parametric Curves	R2:Ch 10;Pg:697-698
11	1	Continuation of Arc Length of Parametric Curves	R2:Ch 10;Pg:699-700
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

Reference Book:

R1 : G.B.Thomas and R.L.Finney., (2005). Calculus , 9th edition, Pearson Edition , Delhi.

R2: H.Anton., I. Bivens ., and S.Davis., (2002). Calculus , 7th edition , John Wiley and sons (Asia) Pvt Ltd, Singapore.

UNIT-IV			
1	1	Introduction to conic section	R2:Ch 10;Pg:730-732
2	1	Techniques of sketching conics	R1:Ch 9;Pg:727-730
3	1	Equations of conics in standard position	R2:Ch 10;Pg:732-735
4	1	Continuation of Equations of conics in standard position	R2:Ch 10;Pg:735-738
5	1	Continuation of Equations of conics in	R2:Ch 10;Pg:738-740

		standard position	
6	1	Translated conics	R2:Ch 10;Pg:740-742
7	1	Reflection properties of the conic sections	R2:Ch 10;Pg:742-744
8	1	Rotation of axes with examples	R2:Ch 10;Pg:750-752
9	1	Classification of conics using discriminant	R1:Ch 9;Pg:748-750
10	1	Polar equation in conics	R2:Ch 10;Pg:755-757
11	1	Continuation of Polar equation in conics	R2:Ch 10;Pg:757-759
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

Reference Book:

R1 : G.B.Thomas and R.L.Finney., (2005). Calculus , 9th edition, Pearson Edition , Delhi.

R2: H.Anton., I. Bivens ., and S.Davis., (2002). Calculus , 7th edition , John Wiley and sons (Asia) Pvt Ltd, Singapore.

UNIT-V

1	1	The Triple product	R1:Ch 10;Pg:824-835
2	1	Introduction to Vector functions	T1:Ch 10;Pg:494-496
3	1	Operation with Vector-valued functions	T1:Ch 10;Pg:496-497
4	1	Limits and continuity of vector functions	T1:Ch 10;Pg:498-500
5	1	Differentiation and integration of vector functions	T1:Ch 10;Pg:502-511
6	1	Tangent and normal components of acceleration	T1:Ch 10;Pg:522-525
7	1	Modeling ballistics and planetary motion	T1:Ch 10;Pg:512-516
8	1	Kepler's second law	T1:Ch 10;Pg:516-519
9	1	Recapitulation and Discussion of possible questions	
10	1	Discussion on Previous ESE Question Papers	
11	1	Discussion on Previous ESE Question Papers	
12	1	Discussion on Previous ESE Question Papers	
Total	**12 Hours**		

Text Book:

T1 : M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

Reference Book:

R1 : G.B.Thomas and R.L.Finney., (2005). Calculus , 9th edition, Pearson Edition , Delhi.

Total no. of Hours for the Course: 60 hours

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DEPARTMENT OF MATHEMATICS

Subject: CALCULUS	**Semester :I**	**L T P C**
Subject Code: 17MMU101	**Class : I- B.Sc Mathematics**	**4 0 0 4**

UNIT I

Hyperbolic functions, higher order derivatives, Leibniz rule and its applications to problems of type $e^{ax+b}\sin x$, $e^{ax+b}\cos x$, $(ax+b)^n\sin x$, $(ax+b)^n\cos x$.

TEXT BOOK

T1 :M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

REFERENCES

R4: Shanti Narayan, P.K. Mittal, Differential Calculus, (2016), Third Edition, Vikas Publishing House Pvt. Ltd.

CALCULUS

UNIT I

DIFFERENTIAL CALCULUS:

Introduction:

The mathematical study of change like motion, growth or decay is calculus. The Rate of change of given function is derivative or differential.

The concept of derivative is essential in day to day life. Also applicable in Engineering, Science, Economics, Medicine etc.

Successive Differentiation:

Let $y = f(x)$ --(1) be a real valued function.

The first order derivative of y denoted by $\frac{dy}{dx}$ or y' or y_1 or Δ^1

The Second order derivative of y denoted by $\frac{d^2y}{dx^2}$ or y'' or y_2 or Δ^2

Similarly differentiating the function (1) n-times, successively,

the n^{th} order derivative of y *exists* denoted by $\frac{d^ny}{dx^n}$ or y^n *or* y_n *or* Δ^n

The process of finding 2nd and higher order derivatives is known as Successive Differentiation.

n^{th} derivative of some standard functions:

1. $y = e^{ax}$

Sol : $y_1 = a\, e^{ax}$

$y_2 = a^2 e^{ax}$

Differentiating Successively

$y_n = a^n e^{ax}$

ie. $D^n[e^{ax}] = a^n e^{ax}$

For, $a = 1$ $D^n[e^x] = e^x$

2. $y = log\,(ax+b)$

Solution: $y_1 = \frac{a}{ax+b}$

$$y_2 = \frac{(-1)a.a}{(ax+b)^2} = \frac{(-1)^1 a^2}{(ax+b)^2}$$

$$y_3 = \frac{(-1)(-2)a^2.a}{(ax+b)^3} = \frac{(-1)^2(1)(2)a^3}{(ax+b)^3} = \frac{(-1)^{3-1}(3-1)!a^3}{(ax+b)^3}$$

$$D^n[\log(ax+b)] = y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Similarly $D^n[\log x] = y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

3. $y = (ax+b)^m$

Solution: $y_1 = m\,(ax+b)^{m-1}\,a$

$$y_2 = m(m-1)\,(ax+b)^{m-2}\,a^2$$

$$y_3 = m(m-1)(m-2)\,(ax+b)^{m-3}\,a^3$$

Similarly

$$y_n = m(m-1)(m-2).......(m-n+1)\,(ax+b)^{m-n}\,a^n$$

$$\ldots(*)$$

Case (i) :- If $m = n$ in (*)

$$D^n[(ax+b)^n] = n(n-1)(n-2)\ldots\ldots 3\,2\,1\,.\,a^n$$

$$= n!\,a^n$$

$$D^n[x^n] = n!$$

Case (ii) :- If $m > n$ in (*)

$$D^n[(ax+b)^m] = \frac{m(m-1)\ldots\ldots(m-n+1)(m-n)(m-n-1)\ldots.321}{(m-n)(m-n-1)\ldots\ldots3.2.1}(ax+b)^{m-n}a^n$$

$$D^n[(ax+b)^m] = \frac{m!}{(m-n)!}(ax+b)^{m-n}a^n$$

$$D^n[x^m] = \frac{m!}{(m-n)!}x^{m-n}a^n$$

Case iii :- If $m < n$ in (*)

$$D^n[(ax+b)^m] = 0$$

Case iv :- If $m = -1$ in (*)

$$D^n\left[\frac{1}{ax+b}\right] = (-1)(-2)\ldots\ldots.(-n)(ax+b)^{-1-n}a^n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = \frac{(-1)^n p(p+1)\ldots\ldots\ldots(p+n-1)a^n}{(ax+b)^{p+n}}$$

$$D^n\left[\frac{1}{(ax+b)^p}\right] = (-1)^n\frac{(p+n-1)!}{(p-1)!}\frac{a^n}{(ax+b)^{p+n}}$$

If $a=1$, $$D^n\left[\frac{1}{x^p}\right] = (-1)^n\frac{(p+n-1)!}{(p-1)!}\frac{1}{x^{p+n}}$$

4. $y = Cos\,(ax+b)$

$$y_1 = -\,Sin\,(ax+b).\,a = a\,Cos\,(ax+b+\pi/2)$$
$$y_2 = -\,Sin\,(ax+b+\pi/2).a^2 = a^2\,Cos\,(ax+b+2\pi/2)$$

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$$y_n = D^n\,[Cos\,(ax+b)] = a^n\,Cos\,(ax+b+n\pi/2)$$

If $a=1,\ b=0$

$$D^n\,[Cos\,x] = Cos\,(x+n\pi/2)$$

5. $y = Sin\,(ax+b)$

$$y_n = D^n\,[Sin\,(ax+b)] = a^n\,Sin\,(ax+b+n\pi/2)$$

If $a=1,\ b=0$

$$D^n\,[Sin\,x] = Sin\,(x+n\pi/2)$$

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6. $y = e^{ax}Sin\,(bx+c)$

$$y_1 = a\,e^{ax}\,Sin\,(bx+c) + b\,e^{ax}Cos\,(bx+c)$$
$$= e^{ax}\,[a\,Sin\,(bx+c) + b\,Cos\,(bx+c)]$$

Put $a = r\cos\theta \quad b = r\,Sin\,\theta$ then $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\frac{b}{a}$

$$y_1 = e^{ax}\,[r\cos\theta\,Sin\,(bx+c) + r\,Sin\,\theta\,Cos\,(bx+c)]$$
$$y_1 = r\,e^{ax}\,[Sin\,(bx+c+\theta)]$$
$$y_2 = r\,[\,e^{ax}\,a\,Sin\,(bx+c+\theta) + e^{ax}\,b\,Cos\,(bx+c+\theta)]$$

Put $a = r\cos\theta \quad b = r\,Sin\,\theta$ then $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\frac{b}{a}$

$$y_2 = r\,e^{ax}\,[\,r\cos\theta\,Sin\,(bx+c+\theta) + r\,Sin\,\theta\,Cos\,(bx+c+\theta)]$$
$$y_2 = r^2\,e^{ax}[Sin\,(bx+c+2\theta)]$$

Similarly,

$y_n = r^n e^{ax}[Sin(bx + c + n\theta)]$

$y_n = D^n[e^{ax} Sin(bx + c)] = (a^2+b^2)^{n/2} e^{ax}[Sin(bx + c + n\tan^{-1}\frac{b}{a})]$

For $a=b=1, c=0$

$D^n[e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$

7. $y = e^{ax}[Cos(bx + c)]$

$y_n = D^n[e^{ax} Cos(bx + c)] = (a^2+b^2)^{n/2} e^{ax}[Cos(bx + c + n\tan^{-1}\frac{b}{a})]$

For $a=b=1, c=0$

$D^n[e^x Sin\, x] = (2)^{n/2} e^x [Sin(x + n\pi/4)]$

8. $y = a^{mx}$

$y_1 = a^{mx}(\log a^m) = a^{mx}(m\log a)$

$y_2 = a^{mx}(m\log a)^2$

Differentiating Successively

$$y_n = a^{mx}(m\log a)^n$$

For $m=1$, $D^n[a^x] = a^x(\log a)^n$

Leibnitz's Theorem :

It provides a useful formula for computing the n^{th} derivative of a product of two functions.

Statement : If u and v are any two functions of x *with* u_n and v_n as their *n^{th} derivative. Then th derivative of uv is*

$$(uv)_n = u_0 v_n + {}^nC_1\, u_1 v_{n-1} + {}^nC_2 u_2 v_{n-2} + \ldots + {}^nC_{n-1} u_{n-1} v_1 + u_n v_0$$

Note : We can interchange u & v $(uv)_n = (vu)_n$,

$${}^nC_1 = n,\quad {}^nC_2 = n(n-1)/2!,\ {}^nC_3 = n(n-1)(n-2)/3! \ldots$$

1. Find the n^{th} derivations of $e^{ax}\cos(bx + c)$

Solution: $y_1 = e^{ax} - b\sin(bx + c) + a\, e^{ax}\cos(bx + c)$, by product rule.

.i.e, $y_1 = e^{ax}\left[a\cos(bx + c) - b\sin(bx + c)\right]$

Let us put $a = r\cos\theta$, and $b = r\sin\theta$.

$\therefore\ a^2 + b^2 = r^2$ and $\tan\theta = b/a$

.ie., $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Now, $y_1 = e^{ax}\left[r\cos\theta\cos(bx + c) - r\sin\theta\sin(bx + c)\right]$

Ie., $y_1 = r\, e^{ax}\cos(\theta + bx + c)$

where we have used the formula $\cos A\cos B - \sin A\sin B = \cos(A + B)$

Differentiating again and simplifying as before,

$y_2 = r^2 e^{ax}\cos(2\theta + bx + c)$.

Similarly $y_3 = r^3 e^{ax}\cos(3\theta + bx + c)$.

....

Thus $y_n = r^n e^{ax}\cos(n\theta + bx + c)$

Where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Thus $D^n[e^{ax}\cos(bx + c)]$

$$= \left[(\sqrt{a^2 + b^2})^n e^{ax}\cos\left[n\tan^{-1}(b/a) + bx + c\right]\right]$$

2. Find the nth derivative of log $\sqrt{4x^2+8x+3}$

Solution : Let y = log $\sqrt{4x^2+8x+3}$ = log $(4x^2 + 8x + 3)^{1/2}$

ie., $y=\frac{1}{2}$ log $(4x^2 + 8x + 3)$ $\because$ log x^n = n log x

$y=\frac{1}{2}$ log { (2x + 3) (2x+1)}, by factorization.

$\because$ $y=\frac{1}{2}$ {log (2x + 3) + log (2x + 1)}

Now $y_n=\frac{1}{2}\left\{\frac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n}+\frac{(-1)^{n-1}(n-1)!2^n}{(2x+1)^n}\right\}$

Ie., $y_n = 2^{n-1}(-1)^{n-1}(n-1)!\left\{\frac{1}{(2x+3)^n}+\frac{1}{(2x+1)^n}\right\}$

3. Find the nth derivative of log$_{10}$ {(1-2x)^3 (8x+1)^5}

Solution : Let y = log$_{10}$ {1-2x)^3 (8x+1)^5}

It is important to note that we have to convert the logarithm to the base e by the property:

$$\log_{10}x=\frac{\log_e x}{\log_e 10}$$

Thus $y=\frac{1}{\log_e 10}\log_e\left\{(1-2x)^3(8x+1)^5\right\}$

Ie., $y=\frac{1}{\log_e 10}\{3\log(1-2x)+5\log(8x+1)\}$

$\therefore y_n=\frac{1}{\log_e 10}\left\{3.\frac{(-1)^{n-1}(n-1)!(-2)^n}{(1-2x)^n}+5\frac{(-1)^{n-1}(n-1)!8^n}{(8x+1)^n}\right\}$

Ie., $y_n=\frac{(-1)^{n-1}(n-1)!2^n}{\log_e 10}\left\{\frac{3(-1)^n}{(1-2x)^n}+\frac{5(4)^n}{(8x+1)^n}\right\}$

4. Find the n^{th} derivative of $e^{2x} \cos^2 x \sin x$

Solution : >> let $y = e^{2x} \cos^2 x \sin x = e^{2x} \left[\frac{1+\cos 2x}{2}\right] \sin x$

ie., $y = \frac{e^{2x}}{2}(\sin x + \sin x \cos 2x)$

$= \frac{e^{2x}}{2}\left\{\sin x + \frac{1}{2}[\sin 3x + \sin(-x)]\right\}$

$= \frac{e^{2x}}{4}(2\sin x + \sin 3x - \sin x) \because \sin(-x) = -\sin x$

$\therefore y = \frac{e^{2x}}{4}(\sin x + \sin 3x)$

Now $y_n = \frac{1}{4}\{D^n(e^{2x}\sin x) + D^n(e^{2x}\sin 3x)\}$

Thus $y_n = \frac{1}{4}\left\{(\sqrt{5})^n e^{2x} \sin[n \tan^{-1}(1/2) + x] + (\sqrt{13})^n e^{2x} \sin[n \tan^{-1}(3/2) + 3x]\right\}$

$\therefore y_n = \frac{e^{2x}}{4}\left\{(\sqrt{5})^n \sin[n \tan^{-1}(1/2) + x] + (\sqrt{13})^n \sin[n \tan^{-1}(3/2) + 3x]\right\}$

5. Find the n^{th} derivative of $e^{2x} \cos^3 x$

Solution : Let $y = e^{2x} \cos^3 x = e^{2x} . \frac{1}{4}(3\cos x + \cos 3x)$

Ie., $y = \frac{1}{4}(3e^{2x}\cos x + e^{2x}\cos 3x)$

$\therefore y_n = \frac{1}{4}\{3D^n(e^{2x}\cos x) + D^n(e^{2x}\cos 3x)\}$

$y_n = \frac{1}{4}\left\{3(\sqrt{5})^n e^{2x} \cos[n \tan^{-1}(1/2) + x] + (\sqrt{13})^n e^{2x} \cos[n \tan^{-1}(3/2) + 3x]\right\}$

Thus $y_n = \frac{e^{2x}}{4}\left\{3(\sqrt{5})^n \cos[n \tan^{-1}(1/2) + x] + (\sqrt{13})^n \cos[n \tan^{-1}(3/2) + 3x]\right\}$

6. **Find the n^th^ derivative of** $\frac{x^2}{(2x+1)(2x+3)}$

Solution : $y = \frac{x^2}{(2x+1)(2x+3)}$ is an improper fraction because; the degree of the numerator being 2 is equal to the degree of the denominator. Hence we must divide and rewrite the fraction.

$y = \frac{x^2}{4x^2+8x+3} = \frac{1}{4}.\frac{4x^2}{4x^2+8x+3}$ for convenience.

$$4x^2+8x+3 \Big) \; 4x^2 \; \Big(1$$
$$\frac{4x^2+8x+3}{-8x-3}$$

$\therefore \quad y = \frac{1}{4}\left[1 + \frac{-8x-3}{4x^2+8x+3}\right]$

Ie., $y = \frac{1}{4} - \frac{1}{4}\left[\frac{8x+3}{4x^2+8x+3}\right]$

The algebraic fraction involved is a proper fraction.

Now $\quad y_n = 0 - \frac{1}{4}D^n\left[\frac{8x+3}{4x^2+8x+3}\right].$

Let $\frac{8x+3}{(2x+1)(2x+3)} = \frac{A}{2x+1} + \frac{B}{2x+3}$

Multiplying by (2x + 1) (2x + 3) we have, 8x + 3 = A (2x + 3) + B (2x + 1) ....(1)

By setting 2x + 1 = 0, 2x + 3 = 0 we get x = -1/2, x = -3/2.

Put x = -1/2 in (1): -1 -1 + A (2) $\Rightarrow$ A = -1/2

Put x = -3/2 in (1): -9 = B (-2) $\Rightarrow$ B = 9/2

$\therefore \quad y_n = -\frac{1}{4}\left\{-\frac{1}{2}D^n\left[\frac{1}{2x+1}\right] + \frac{9}{2}D^n\left[\frac{1}{2x+3}\right]\right\}$

$$= -\frac{1}{8}\left\{(-1)\cdot\frac{(-1)^n n!2^n}{(2x+1)^{n+1}} + 9\cdot\frac{(-1)^n n!2^n}{(2x+3)^{n+1}}\right\}$$

ie., $y_n = \frac{(-1)^{n+1} n!2^n}{8}\left\{\frac{1}{(2x+1)^{n+1}} + \frac{9}{(2x+3)^{n+1}}\right\}$

7. Find the nth derivative of $\frac{x^4}{(x+1)(x+2)}$

Solution : $y = \frac{x^4}{(x+1)(x+2)}$ is an improper fraction.

(deg of nr. = 4 > deg. of dr. = 2)

On dividing x^4 by $x^2 + 3x + 2$, We get

$y = (x^2 - 3x + 7) + \left[\frac{-15x-14}{x^2+3x+2}\right]$

$\therefore \quad y_n = D^n(x^2-3x+7) - D^n\left[\frac{15x-14}{x^2+3x+2}\right]$

But $D = (x^2 - 3x + 7) = 2x - 3$, $D^2(x^2 - 3x + 7) = 2$

$D^3(x^2 - 3x + 7) = 0$......... $D^n(x^2 - 3x + 7) = 0$ if $n > 2$

Hence $y_n = -D^n\left[\frac{15x+14}{(x+1)(x+2)}\right]$

Now, let $D^n \frac{15x+14}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

=> $15x + 14 = A(x+2) + B(x+1)$

Put $x = -1$; $-1 = A(1)$ or $A = -1$

Put $x = -2$; $-16 = B(-1)$ or $B = 16$

$$Y_n = \left\{-D^n\left[\frac{1}{x+1}\right] + 16D^n\left[\frac{1}{x+2}\right]\right\}$$

$$= \frac{(-1)^n n!\, 1^n}{(x+1)^{n+1}} - 16\frac{(-1)^n n!\, 1^n}{(x+2)^{n+1}}$$

$$y_n = (-1)^n n! \left\{\frac{1}{(x+1)^{n+1}} - \frac{16}{(x+2)^{n+1}}\right\} n > 2$$

8. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{n}\right\}$$

Solution : Let y = $\frac{\log x}{x} = \log x.\frac{1}{x}$ and let u = log x, v = $\frac{1}{x}$

We have Leibnitz theorem,

$(uv)_n = uv_n + n_{C_1} u_1 v_{n-1} + n_{C_2} u_2 v_{n-2} + ... + u_n v$ ...... (1)

Now, u = log x $\therefore u_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

$v = \frac{1}{x}$ $\therefore v_n = \frac{(-1)^n n!}{x^{n+1}}$

Using these in (1) by taking appropriate values for n we get,

$$D_n = \left(\frac{\log x}{x}\right) = \log x..\frac{(-1)^n n!}{x^{n+1}} + n\frac{1}{x}.\ \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+\frac{n(n-1)}{1.\ 2}\left(-\frac{1}{x^2}\right)\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$+ ...... + \frac{(-1)^{n-1}(n-1)!}{x^n}.\ \frac{1}{x}$$

Ie.. = log x $\frac{(-1)^n n!}{x^{n+1}} + \frac{(-1)^{n-1} n!}{x^{n+1}}$

$$-\frac{(-1)^{n-2} n!}{2x^{n+1}} + ... + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$-\frac{(-1)^{n-2} n!}{x^{n+1}}\left[\log x(-1)^{-1} - \frac{(-1)^{-2}}{2} + ... + \frac{(-1)^{-1}(n-1)!}{n^1}\right]$$

Note : $(-1)^{-1} = \frac{1}{-1} = -1; (-1)^{-2} = \frac{1}{(-1)^2} = 1$

Also $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$

$$\therefore \frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} \dots - \frac{1}{n}\right]$$

9. If y_n= D^n (x^n logx)

Prove that $y_n = n\, y_{n-1} + (n-1)!$ and hence deduce that

$y_n = n\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$

Solution : $y_n = D^n(x^n \log x) = D^{n-1}\{D(x^n \log x)\}$

$= D^{n-1}\left\{x^n . \frac{1}{x} + nx^{n-1}\log x\right\}$

$= D^{n-1}(x^{n-1}) + nD^{n-1}(x^{n-1} \log x\}$

$\therefore$ $y_n = (n-1)! + ny_{n-1}$. This proves the first part.

Now Putting the values for n = 1, 2, 3...we get

$y_1 = 0! + 1\ \ y_0 = 1 + \log x = 1!(\log x + 1)$

$y_2 = l! + 2y_1 = l+2(l + \log x)$

ie., $y_2 = 21\text{og } x + 3 = 2(\log x + 3/2) = 2!\left(\log x + 1 + \frac{1}{2}\right)$

$y_3 = 2! + 3y_2 = 2 + 3(2\log x + 3)$

ie., $y_3 = 61\text{og } x + ll = 6(\log x + ll/6) = 3!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3}\right)$

……………………………………………………………………………

$$y_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

10. If y = a cos (log x) + b sin (log x), show that

$x^2y_2 + xy_1 + y = 0$. Then apply Leibnitz theorem to differentiate this result n times.

or

If y = a cos (log x) + b sin (log x), show that

Solution : y = a cos (log x) + b sin (log x)

Differentiate w.r.t x

$\therefore$ y_1 = -a sin (log x) $\frac{1}{x}$ + b cos (log x). $\frac{1}{x}$

(we avoid quotient rule to find y_2) .

=> xy_1 = - a sin (log x) + b cos (log x)

Differentiating again w.r.t x we have,

xy_2 + 1 y_1 = - a cos (log x) + b sin (log x) $\frac{1}{x}$

or $x^2y_2 + xy_1$ = - [a cos (log x) + b sin (log x)] = -y

$\therefore$ $x^2y_2+xy_1+y = 0$

Now we have to differentiate this result n times.

ie., $D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$

We have to employ Leibnitz theoreom for the first two terms.

Hence we have,

$$\left\{x^2 . D^n(y_2) + n.\, 2x . D^{n-1}(y_2) + \frac{n(n-1)}{1\,.\,2} .\, 2\,. D^{n-2}(y^{2)})\right\}$$

$$\{x. D^n(y_1) + n.\ 1\ .\ D^{n-1}(y_1)\} + y_n = 0$$

ie., $\{x^2y_{n+2} + 2n\,x\,y_{n+1} + n(n-1)y_n\} + \{xy_{n+1}+ny_n\}+y_n = 0$

ie., $x^2y_{n+2} + 2n\,x\,y_{n+1} + n^2y_n - ny_n + xy_{n+1}+ny_n+y_n = 0$

ie., $x^2y_{n+2} + (2n+l)xy_{n+l} + (n^2+l)y_n = 0$

11. If $\cos^{-1}$ (y/b) = log $(x/n)^n$, then show that

$x^2y_{n+2} + (2n+l)\,xy_{n+l} + 2n^2y_n = 0$

Solution : By data, $\cos^{-1}$ (y/b) = n log (x/n) $\therefore$ $\log(a^m)$ = m log a

=> $\frac{y}{b}$ = cos [n log (x/n)]

or y = b . cos [n log (x/n)]

Differentiating w.r.t x we get,

$y_1 = -b \sin[n \log(x/n)] \cdot n \cdot \frac{1}{(x/n)} \cdot \frac{1}{n}$

or $xy_1 = -n\, b \sin[n \log(x/n)]$

Differentiating w.r.t x again we get,

$xy_2 + 1.\ y_1 = -n\ .\ b \cos[n \log(x/n)]\ n\ \frac{1}{(x/n)} \cdot \frac{1}{n}$

or $x(xy_2+y_1) = n^2 b \cos[n \log(x/n)] = -n^2 y$, by using (1).

or $x^2y_2 + xy_l + n^2y = 0$

Differentiating each term n times we have,

$$D(x^2y_2) + D^n(xy_1) + n^2D^n(y) = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{x^2 y_{n+2} + n.\ 2x.\ y_{n+1} + \frac{n(n-1)}{1.\,2}.\ 2\ .\ y_n\right\} + \{xy_{n+1} + n.\ 1\ .y_n\} + n^2 y_n = 0$$

ie $x^2y_{n+2} + 2\,x\,y_{n+1} + n^2y_n + xy_{n+1} + ny_n + n^2y_n = 0$

or $x^2 y_{n+2} + (2n + l)\,xy_{n+1} + 2n^2y_n = 0$

12. If $y = \sin(\log(x^2 + 2x + 1))$,

or [Feb-03]

If $\sin^{-1} y = 2\log(x + 1)$, show that

$(x+l)^2y_{n+2} + (2n+1)(x+1)y_{n+l} + (n^2 + 4)y_n = 0$

Solution : By data $y = \sin\log(x^2 + 2x + 1)$

$\therefore\ y_1 = \cos\log(x^2 + 2x + 1)\ \frac{1}{(x+1)^2} 2x + 2$

ie., $y1 = \cos\log(x^2 + 2x + 1)\ \frac{1}{x^2 + 2x + 1}\ 2(x + 1)$

ie., $y_1 = \frac{2\cos\log(x^2 + 2x + 1)}{(x+1)}$

or $(x + 1)\,y1 = 2\cos\log(x^2 + 2x + 1)$

Differentiating w.r.t x again we get

$(x+1)y_2 + 1\ \ y_1 = -2 \sin \log (x^2 + 2x + 1)\ \dfrac{1}{(x+1)^2}.\ 2(x+1)$

or $(x + 1)^2 y_2 + (x+1)\ y_1 = -4y$

or $(x+l\ )^2 y_2 + (x+l)\ y_1 + 4y = 0$,

Differentiating each term n times we have,

$$D^n [(x + 1)^2 y_2] + D^n [(x+ 1)y_1] + D^n [y] = 0$$

Applying Leibnitz theorem to the product terms we have,

$$\left\{(x+1)^2 y_{n+2} + n.\ 2(x+1).\ \ y_{n+1} + \frac{n(n-1)}{1.\ 2}.2\ .y_n\right\}$$

$$+ \{(x+l)\ y_{n+1} + n.\ \ 1\ \ .y_n\} + 4y_n = 0$$

ie., $(x+l)^2 y_{n+2} + 2n\ (x+1)y_{n+1}$

$+ n^2 y_n - ny_n + (x+l)y_{n+l} + ny_n + 4y_n = 0$

ie., $(x+l)^2 y_{n+2} + (2n + l)\ (x + l)\ y_{n+1} + (n^2 + 4)y_n = 0$

13. If = log $\left(x+\sqrt{1+x^2}\right)$ prove that

$(1 + x^2)\ y_{n+2} + (2n + 1)\ xy_{n+1} + n^2 y_n = 0$

>> By data, $y = \log \left(x+\sqrt{1+x^2}\right)$

$\therefore y_1 = \dfrac{1}{(x+\sqrt{1+x^2})}\left\{1+\dfrac{1}{2\sqrt{1+x^2}}.\ 2x\right\}$

Ie., $y_1 \dfrac{1}{(x+\sqrt{1+x^2})}\ \dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{1+x^2}}$

or $\sqrt{1+x^2}\,y_1 = 1$

Differentiating w.r.t.x again we get

$$\sqrt{1+x^2 y_2} + \frac{1}{2\sqrt{1+x^2)}}.2x.y_1 = 0$$

or $(1+x^2)y_2 + xy_1 = 0$

Now $D^n [(l+x^2)y_2] + D^n[xy_1] = 0$

Applying Leibnitz theorem to each term we get,

$$\left\{(1+x^2)y_{n+2}+n.\,2x\,.y_{n+1}+\frac{n(n-1)}{1.2}.2\,.y_n\right\}$$

$+ [x . y_{n+1} + n .1 y_n] = 0$

Ie., $(1 + x^2) y_{n+2} + 2 n x y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n = 0$

or $(1+x^2)y_{n+2} + (2n + 1)x y_{n+1} + n^2 y_n = 0$

14. If x = sin t and y = cos mt, prove that

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$ [Feb-04]

Solution : By data x = sin t and y = cos mt

x = sin t => $t = \sin^{-1} x$ and y = cos mt becomes

$y = \cos [m \sin^{-1}x)$

Differentiating w.r.t.x we get

$y_1 = - \sin (m \sin^{-1}x) \dfrac{m}{\sqrt{1-x^2}}$

or $\sqrt{1+x^2}\,y_1 = - m \sin (m \sin^{-1}x)$

Differentiating again w.r.f .x we get,

$$\sqrt{1-x^2 y_2}+\frac{1}{2\sqrt{1-x^2}}(-2x)y_1=-m\cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

or $(1 -x^2)y_2-xy_1 = -m^2y$

or $(1 -x^2)y_2 -xy_1 +m^2y = 0$

Thus $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$

15. If x = tan (log y), find the value of

$(1+x^2)y_{n+1} + (2nx-1) y_n+n(n-1)y_{n-1}$ [July-04]

Solution : By data x = tan(log y) => $\tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$ Since the desired relation involves y_{n+1}, y_n and y_{n-1} we can find y_1 and differentiate n times the result associated with y_1 and y.

Consider $y = e^{\tan^{-1}x}$. $\therefore$ $y. = e^{\tan^{-1}x} \cdot \dfrac{1}{1+x^2}$

or $(1 +x^2)y_1 = y$

Differentiating n times we have

$D^n[(l+x^2)y_1]=D^n[y]$

Anplying Leibnitz theorem onto L.H.S, we have,

$\{(l+x^2)D^n(y_1) + n .2x .D^{n-1} (y_1)$

$+\frac{n(n-1)}{1.2}.2 .D^{n-2}(y_1)\} = y_n$

Ie., $(1+x^2)y_{n+1}+2n x y_n + n (n-1) y_{n-1}-y_n=0$

Or $(l+x^2)y_{n+1} + (2nx-l)y_n + n(n-l)y_{n-1} = 0$

POSSIBLE QUESTIONS

Class : **I – B.Sc. Mathematics**
Subject Name : **Calculus**
Subject Code : **17MMU101**

UNIT – I

2 Mark Questions:

1. Prove that $\mathrm{sech}^2x = 1-\tanh^2x$.
2. Prove that $\mathrm{cosech}^2x = 1-\coth^2x$.
3. Prove that $\cosh^2x - \sinh^2x = 1$..
4. Find that $\int \sinh(6x)\,dx$
5. Prove that $sinh2x = 2sinhxcoshx$.

6Mark Questions :

1. Show that i) $\tanh(x+y) = \frac{tanhx+tanhy}{1+tanhxtanhy}$.
   ii) $\cosh^2 x - \sinh^2 x = 1$.
2. Find the n^{th} derivative for $e^{2x}\sin 3x \sin 4x$.
3. Prove that i) $\sinh(x+y) = sinhxcoshy + coshxsinhy$
   ii) $\cosh(x+y) = coshxcoshy + sinhxsinhy$
4. State and prove Leibniz Rule for n^{th} derivative.
5. Find $\frac{dy}{dx}$ $for\ i)\ y = \cosh^{-1}(secx)\ ii)\ y = \tanh^{-1}(2x)^2$.
6. If $= e^{\operatorname{asin}^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 - a^2y = 0$ and hence show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.
7. Evaluate i)Find the derivatives of $\tanh^{-1} x$.
   ii) If $x = a\cos^3\theta$, $y = a\sin^3\theta$ find $\frac{d^2y}{dx^2}$.
8. If $y = \sin(msin^{-1}x)$, prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and hence show that $(1+x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$
9. If $y = \sin(sinx)\ prove\ that\ \frac{d^2y}{dx^2} + tanx\frac{dy}{dx} + y\cos^2 x = 0$.
10. Find the n^{th} derivative of $cosx\ cos2x \cos 3x$.

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: CALCULUS **Subject Code: 17MMU101**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
The even parts of e^x is called the hyperbolic ______	tangent	cosine	sine	secant	cosine
The odd parts of e^x is called the hyperbolic ______	cosine	tangent	sine	secant	sine
sinh(2x) = ____________	2sinhxcoshx	sinhx + coshx	coshxcoshx	sinhxcoshx	2sinhxcoshx
$\cosh^2x + \sinh^2x$ = ______	tanhx	cosh2x	1	sinh2x	cosh2x
differentiation of sinhx = ______	(- coshx)	sinh2x	coshx	(-sinhx)	coshx
$\cosh^2x - \sinh^2x$ = ______	1	0	cosh2x	sinh2x	1
The slope of a graph ____ on an interval where the graph is concave up	behind	increases	zero	decreases	increases
If the curve $y = x^4$ has no _______ at x = 0	hyperbolic	inflection point	concavity	saddle point	inflection point
The slope of a graph ____ on an interval where the graph is concave down	increases	zero	decreases	behind	decreases
The graph of the function f is concave up on any open interval I where	f"(x) >0	f"(x) < 1	f"(x) <0	f"(x) = 0	f"(x) >0
The graph of the function f is concave down on any open interval I where	f"(x) >0	f"(x) < 1	f"(x) <0	f"(x) = 0	f"(x) <0
A point P(c , f (c)) on a curve is called _______	hyperbolic	inflection point	concavity	saddle point	inflection point
sinh(-x) = _____	(- coshx)	sinh2x	coshx	(- sinhx)	(- sinhx)
coshx coshy + sinhx sinhy = ________	cosh(x +y)	sin(x - y)	cosh(x - y)	sinh (x + y)	cosh(x +y)
Differentiation of y = In (sinhx)	sinhx	cothx	tanhx	coshx	cothx
∫tan hx dx = ________	In(sinhx)	In(sechx)	In(coshx)	cothx	In(coshx)
If x = 0 then sinhx = ___	(-1)	1	0	2	0
If x = 0 then cosh x = ___	(-1)	1	0	2	1
Range of tanhx is	(-1, -1)	(1 , 1)	(0 , 1)	(-1, 1)	(-1, 1)
Range of sechx is	(-1, -1)	[1 , 1)	(0 , 1]	[-1, 1]	(0 , 1]
differentiation of coshx	(- coshx)	sinhx	cosh2x	(-sinhx)	sinhx
sinhx coshy + coshx sinhy = ____	cosh(x +y)	sin(x - y)	cosh(x - y)	sinh (x + y)	sinh (x + y)
$2\cosh^2x - 1$ = ______	tanhx	cosh2x	1	sinh2x	cosh2x
Find the second derivative of e^{2x}	e^{2x}	$2e^{2x}$	$4e^{2x}$	$(-e^{2x})$	$4e^{2x}$
The ____ parts of e^x is called the hyperbolic cosine	positive	even	odd	negative	even
The ____ parts of e^x is called the hyperbolic sine	odd	positive	even	negative	odd
differentiation of (sinh4x)	cosh4x	2cosh2x	2sinh4x	4cosh4x	4cosh4x
If $y = \sinh^{-1}x$ then if and only if ______	y = sinhx	x = coshy	x = sinhy	y = coshx	x = sinhy
2sinhxcoshx = _____	tanhx	cosh2x	1	sinh2x	sinh2x
∫coshx dx = _______	sinhx	cothx	tanhx	sechx	sinhx
The graph of the function f is _____ on any open interval I where f"(x) > 0	local maximum	concave up	local minimum	concave down	concave up
The graph of the function f is _____ on any open interval I where f"(x) < 0	concave up	local minimum	concave down	local maximum	concave down
Differentiation of sech x	cosech x coth x	(- sech x tanh x)	sech x tanh x	cosec hx	(- sech x tanh x)
Differentiation of cosech x	(-sinhx)	cosech x coth x	sech x tanh x	(-cosech x coth x)	(-cosech x coth x)
$\int \text{sech}^2x$ dx = _______	sinhx	cothx	tanhx	sechx	tanhx
∫sec x dx = _______	secx + tanx	log [secx + tanx]	secx + cos x	log [secx + cosec x]	log [secx + tanx]
∫cotx dx = _______	log cosx	log tanx	log secx	logsinx	logsinx
∫sechx tanhx dx = _______	(-cosh x)	(-sech x)	(-sinh x)	(-tanh x)	(-sech x)
∫cosech x coth x dx = _______	(-cosech x)	(-tanh x)	(-sech x)	(-sinh x)	(-cosech x)
For f (x) = sin(x) , find f " (x)	sinhx	cothx	tanhx	sechx	sinhx

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DEPARTMENT OF MATHEMATICS

Subject: CALCULUS	**Semester :I**	**L T P C**
Subject Code: 17MMU101	**Class : I- B.Sc Mathematics**	**4 0 0 4**

UNIT II

Reduction formulae, derivations and illustrations of reduction formulae of the type ∫ sin nx dx, ∫ cos nx dx, ∫ tan nx dx,∫ sec nx dx, ∫ log x^n dx, ∫$\sin^n$ x $\sin^m$x dx. Curve tracing in Cartesian coordinates, tracing in polar coordinates of standard curves, L'Hospital's rule, applications in business, economics and life sciences.

Text Book

T1 :M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

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R2: H.Anton., I. Bivens ., and S.Davis., (2002). Calculus , 7th edition , John Wiley and sons (Asia) Pvt Ltd, Singapore.

R3: R.Courant and F.John., (2000). Introduction to Calculus and Analysis (Volume I & II), Springer verlag, NewYork.

UNIT II

Curvature: The rate of bending of a curve in any interval is called the curvature of the curve in that interval.

Curvature of a circle: The curvature of a circle at any point on it equals the reciprocal of its radius.

Radius of curvature: The radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature

Cartesian form of radius of curvature $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left(\dfrac{d^2y}{dx^2}\right)}$

Parametric equation of radius of curvature $\rho = \dfrac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$

Polar form of radius of curvature $\rho = \dfrac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$

Implicit form of radius of curvature $\rho = \dfrac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}$

Centre of curvature: The circle which touches the curve at P and whose radius is equal to the radius of curvature and its centre is known as centre of curvature.

Equation of circle of curvature: $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

Centre of curvaturee: $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$ $\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$

Evolute: The locus of the centre of curvature is called an evolute

Involute: If a curve C_1 is the evolute of C_2, then C_2 is said to be an involute of a curve C_1.

Parametric equation of some standard curves

Curve	Parametric form
$Y^2 = 4ax$ (parabola)	$X = at^2$, $y = 2at$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse)	$X = a\cos\theta$, $y = b\sin\theta$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (hyperbola)	$X = a\sec\theta$, $y = b\tan\theta$
$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$	$X = a\cos^3\theta$, $y = a\sin^3\theta$
$Xy = c^2$ (rectangular hyperbola)	$X = ct$, $y = \frac{c}{t}$

Envelope: A curve which touches each member of a family of curves is called envelope of that family curves.

Envelope of a family of curves: The locus of the ultimate points of intersection of consecutive members of a family of curve is called the envelope of the family of curves.

Properties of envelope and evolute

Property:1: The normal at any point of a curve is a tangent to its evolute touching at the corresponding centre of curvature.

Property:2 The difference between the radii of curvature at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

Property:3: There is one evolute ,but an infinite number of involutes

Property:4 The envelope of a family of curves touches at each of its point. The corresponding member of that family

Evolute as the envelope of normals: The normals to a curve form a family of straight lines.we know that the envelope of the family of these normals is the locus of the ultimate points of intersection of consecutive normals. But the centre of curvature of a curve is also the point of consecutive normals. Hence the envelope of the normals and the locus of the centres of curvature are the same that is ,the evolute of a curve is the envelope of the normals of the curve.

Part - A

1. **Find the radius of curvature of $y=e^x$ at x=0**

   Solution: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

   $y=e^x$

   $y_1=e^x$ at $x=0$ $y_1=1$

   $y_2=e^x$ at $x=0$ $y_2=1$

   $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}$

2. **Find the radius of curvature of at $x=\frac{\pi}{2}$ on the curve $y=4\sin x - \sin 2x$**

   Solution: $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

   $y_1=4\cos x - 2\cos 2x$ at $x=\frac{\pi}{2}$ $y_1=2$

   $y_2=-4\sin x + \sin 2x$ at $x=\frac{\pi}{2}$ $y_2=-4$

   $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \rho = \frac{(1+4)^{3/2}}{-4} = \frac{5\sqrt{5}}{4}$

3. **Given the coordinates of the centre of curvature of the curve is given as $x = 2a+3at^2$**

$y = 2at^3$ Determine the evolute of the curve

Solution: $\bar{x} = 2a + 3at^2$ $t^2 = (\bar{x} - 2a/3a)$ ------------ 1

$\bar{y} = -2at^3$ $t^3 = \bar{y}/-2a$ ------------ 2

$(\bar{x} - 2a/3a)^3 = (\bar{y}/-2a)^2$

$4(\bar{x}-2a)^3 = 27a\bar{y}^2$

The locus of the centre of curvature (evolute) is $4(x-2a)^3 = 27ay^2$

4. Write the envelope of $Am^2 + Bm + C = 0$, where m is the parameter and A, B and C are functions of x and y. **(NOV 08)**

Solution: Given $Am^2 + Bm + C = 0$..........................(1)

Differentiate (1) partially w.r.t. 'm'

$2Am + B = 0$ $m = -B/2A$............(2)

Substitute (2) in (1) we get

$A(-B/2A)^2 + B(-B/2A) + C = 0$

$AB^2/4A^2 - B^2/2A + C = 0$

$AB^2 - 2AB^2 + 4A^2C = 0$

$-AB^2 + 4A^2C = 0$

Therefore $B^2 - 4AC = 0$ which is the required envelope

5. Find the radius of curvature at any point of the curve $y = x^2$. (NOV 07)

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $y = x^2$ $y_1 = \frac{dy}{dx} = 2x$ and $Y_2 = \frac{d^2y}{dx^2} = 2$

$\rho = \frac{(1+(2x)^2)^{3/2}}{2} = \frac{(1+4x^2)^{3/2}}{2}$

6. Find the envelope of the family of $x \sin \alpha + y \cos \alpha = p$, α being the parameter. **(NOV-07)**

Solution: **Given** $x \sin \alpha + y \cos \alpha = p$................ (1)

Differentiate (1) partially w.r.t. 'α'

$X \cos \alpha - y \sin \alpha = 0$........................(2)

Eliminate α between (1) and (2)

$X \cos \alpha = y \sin \alpha \Rightarrow \frac{\sin \alpha}{\cos \alpha} = \frac{x}{y}$ $\Rightarrow$ **Tan** $\alpha = \frac{x}{y}$

$\sin\alpha = \frac{x}{\sqrt{x^2+y^2}}$ $\cos\alpha = \frac{y}{\sqrt{x^2+y^2}}$

Substitute in (1)

$x.\frac{x}{\sqrt{x^2+y^2}} + y.\frac{y}{\sqrt{x^2+y^2}} = p$

$\sqrt{x^2+y^2} = p$

Squaring on both sides, $x^2+y^2=p^2$ which is the required envelope

7. What is the curvature of $x^2+y^2-4x-6y+10=0$ at any point on it . (JAN-06)

Solution: Given $x^2+y^2-4x-6y+10=0$

The given equation is of the form $x^2+y^2+2gx+2fy+c=0$

Here $2g=-4$ $g=-2$

$2f=-6$ $f=-3$

Centre C(2,3), radius $r=\sqrt{g^2+f^2-c}=\sqrt{4+9-10}=\sqrt{3}$

Curvature of the circle $=\frac{1}{r}$

Therefore Curvature of $x^2+y^2-4x-6y+10=0$ is $\frac{1}{\sqrt{3}}$

8. Find the envelope of the family of straight lines $y=mx\pm\sqrt{m^2-1}$, where m is the parameter (JAN-06)

Solution: Given $y=mx\pm\sqrt{m^2-1}$

$(y-mx)^2=m^2-1$

$Y^2+m^2x^2-2mxy-m^2+1=0$

$m^2(x^2-1)-2mxy+y^2+1=0$ which is quadratic in 'm'

Here, $A=x^2-1$ $B=-2xy$ $C=y^2+1$

The condition is $B^2-4AC=0$

$4x^2y^2-4(x^2-1)(y^2+1)=0$

$4x^2y^2-4x^2y^2-4x^2+4y^2+4=0$

$X^2-y^2=4$ which is the required envelope

9. Find the curvature of the curve $2x^2+2y^2+5x-2y+1=0$ (MAY-05,NOV-07)

Solution: Given $2x^2+2y^2+5x-2y+1=0$

$\div 2$

$x^2+y^2+5/2x-y+1/2=0$

Here $2g=5/2$ $g=5/4$

2f=-1 f=-1/2 centre C (-5/4,1/2) radius $r=\sqrt{g^2+f^2-c}=\sqrt{\frac{25}{16}+\frac{1}{4}-\frac{1}{2}}=\sqrt{\frac{21}{16}}=\frac{\sqrt{21}}{4}$

Therefore Curvature of the circle $2x^2+2y^2+5x-2y+1=0$ is $\frac{1}{r}=\frac{4}{\sqrt{21}}$

10. State any two properties of evolute . **(MAY-05)**

Solution: (i) The normal at any point of a curve is a tangent to its evolute touching at the corresponding centre of curvature. (ii) The difference between the radii of curvature at two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

11. Define the curvature of a plane curve and what the curvature of a straight line. **(JAN 05)**

Solution: The rate at which the plane curve has turned at a point (rate of bending of a curve is called the curvature of a curve. The curvature of a straight line is zero.

12. Define evolute and involute . **(JAN 05)**

Solution: The locus of centre of curvature of a curve ($B_1,B_2,B_3,\ldots$) is called evolute of the given curve.

If a curve C_2 is the evolute of a curve C_1 ,then C_1 is said to be an involute of a curve C_2

13. Find the radius of curvature of the curve $x^2+y^2-6x+4y+6=0$ **(NOV-08)**

Solution: Given $X^2+y^2-6x+4y+6=0$

The given equation is of the form $x^2+y^2+2gx+2fy+c=0$

Here 2g = -6 g=-3

2f =4 f=2

Centre C(3,-2), radius $r=\sqrt{g^2+f^2-c}=\sqrt{4+9-6}=\sqrt{7}$

Radius of Curvature of the circle = radius of the circle=$\sqrt{7}$

14. Find the envelope of the family of circles $(x-\alpha)^2+y^2=4\alpha$,where α is the parameter.(MAY-07)

Solution: Given $(x-\alpha)^2+y^2=4\alpha$

$X^2-2\alpha x+\alpha^2-4\alpha+y^2=0$

$\alpha^2-2\alpha(x+2)+x^2+y^2=0$ which is quadratic in α

The condition is $B^2-4AC=0$

Here A=1 B=-2 (x+2) C= x^2+y^2

$4(x+2)^2-4(x^2+y^2)=0$

$x^2+4x+4-x^2-y^2=0$

$y^2=4x-4$ which is the required envelope.

15. Define evolute . **(MAY-07)**

Solution: The locus of centre of curvature $(\bar{x},\bar{y})$ is called an evolute .

16. Find the envelope of the family of straight lines $y=mx+\frac{a}{m}$ for different values of 'm'.

Solution: Given $y=mx+\frac{a}{m}$ **(NOV 07, May 2009)**

$m^2x-my+a=0$ which is quadratic in 'm'

The condition is $B^2-4AC=0$

Here A=x B=-y C=a

$Y^2-4ax=0$

There fore $y^2=4ax$ which is the required envelope.

17. Find the envelope of the line $\frac{x}{t}+yt=2c$, where 't' is the parameter. **(NOV-02,05)**

Solution: Given $\frac{x}{t}+yt=2c$

$Yt^2-2ct+x=0$ which is quadratic in 't'

The condition is $B^2-4AC=0$

Here A=y B=-2c C=x

$C^2-xy=0$

Therefore $xy=c^2$ which is the required envelope.

18. Find the radius of curvature of the curve y=c cosh(x/c)at the point where it crosses the y-axis.

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ **(NOV 05,May 09)**

Given y=c cosh(x/c) and the curve crosses the y-axis. (i.e.)x=0 implies y=c.

Therefore the point of intersection is (0,c)

$\frac{dy}{dx}$**=c sin h(x/c)(1/c)=sin h (x/c)**

$\frac{dy}{dx}$**(0,c)=sinh 0= 0**

$\frac{d^2y}{dx^2}$**=cos h(x/c)(1/c)**

$\frac{d^2y}{dx^2}$**(0,c)= cos h(0) (1/c) = 1/c**

$\rho = \frac{(1+0)^{\frac{3}{2}}}{\frac{1}{c}} = c$

19. **Find the radius of curvature of the curve $xy=c^2$ at (c,c).** **(NOV 02)**

Solution: Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $xy=c^2$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = -\frac{y}{x}$ implies $\frac{dy}{dx}(c,c) = -1$

$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx} - y.1}{x^2}\right]$

$\frac{d^2y}{dx^2}(c,c) = -\left[\frac{c(-1)-c}{c^2}\right] = \frac{2c}{c^2} = \frac{2}{c}$

$\rho = \frac{(1+(-1)^2)^{3/2}}{2/c} = \frac{c2\sqrt{2}}{2}$

$\rho = c\sqrt{2}$

20. **Find the envelope of the family of straight lines $y = mx \pm \sqrt{a^2m^2+b^2}$, where m is the parameter**

Solution: Given $y = mx \pm \sqrt{a^2m^2+b^2}$ **(Jan 09)**

$(y-mx)^2 = (a^2m^2+b^2)$

$Y^2 + m^2x^2 - 2mxy - a^2m^2 - b^2 = 0$

$m^2(x^2-a^2) - 2mxy + y^2 - b^2 = 0$ which is quadratic in 'm'

Here, $A = x^2 - a^2$ $B = -2xy$ $C = y^2 - b^2$

The condition is $B^2 - 4AC = 0$

$4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is the required envelope

21. **Write down the formula for radius of curvature in terms of parametric coordinate system. (May-09)**

Solution: Radius of curvature $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y'' - y'x''}$

22. **Define the circle of curvature at a point $P(x_1, y_1)$ on the curve $y = f(x)$.** **(Jan-09)**

Solution: The circle of curvature is the circle whose centre is the centre of curvature and radius is the radius of curvature. Therefore the equation of circle of curvature is

$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

PART B

1. **Find the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$.**

Solution: Given $x=a\cos^3\theta$……………………(1) **(NOV-07,MAY-08,MAY-09)**

$Y=a\sin^3\theta$…………………..(2)

Differentiate (1) and (2) w.r.t θ

$\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)=-3a\sin\theta\cos^2\theta$

$$\frac{dy}{d\theta}=3a\sin^2\theta(\cos\theta)=3a\cos\theta\sin^2\theta$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\cos\theta\sin^2\theta}{-3a\sin\theta\cos^2\theta}=-\tan\theta$$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{d\theta}(-\tan\theta).\frac{d\theta}{dx}$

$=-\sec^2\theta.\frac{1}{-3a\sin\theta\cos^2\theta}$

$\frac{d^2y}{dx^2}=\frac{1}{3a\sin\theta\cos^4\theta}$

Radius of curvature $\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{(1+\tan^2\theta)^{3/2}}{\frac{1}{3a\sin\theta\cos^4\theta}}=3a\sin\theta\cos^4\theta\,(\sec^2\theta)^{3/2}$

$=3a\sin\theta\cos^4\theta\sec^3\theta=3a\sin\theta\cos\theta$

$$\rho=3a\sin\theta\cos\theta$$

2. **Find the radius of curvature of the curve $y^2=x^2\frac{(a+x)}{(a-x)}$ at the point (-a, 0).** **(NOV-08)**

Solution: Radius of curvature $\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

Given $y^2=x^2\frac{(a+x)}{(a-x)}=\frac{ax^2+x^3}{a-x}$

Differentiate w.r.t. 'x'

$2y\frac{dy}{dx}=\frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{(a-x)^2}$

$$\frac{dy}{dx}=\frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{2y(a-x)^2}$$

$$\frac{dy}{dx}(-a,0)=\frac{2a(-2a^2+3a^2)+(a^3-a^3)}{0}=\infty$$

$$\therefore \rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$$

$$\frac{dx}{dy} = \frac{y(a-x)^2}{(a^2x+ax^2-x^3)} = \frac{0}{(a^2x+ax^2-x^3)} = 0$$

$$\frac{d^2x}{dy^2} = \frac{(a^2x+ax^2-x^3)\left[y.2(a-x)\left(-\frac{dx}{dy}\right)+(a-x)^2.1\right] - y(a-x)^2\left[a^2\frac{dx}{dy}+2ax\frac{dx}{dy}-3x^2\frac{dx}{dy}\right]}{(a^2x+ax^2-x^3)^2}$$

$$\frac{d^2x}{dy^2}(-a,o) = \frac{(-a^3+a^3+a^3)(4a^2)}{(-a^3+a^3+a^3)^2} = \frac{4a^5}{a^6} = \frac{4}{a}$$

$$\therefore \rho = \frac{\{1+0\}^{3/2}}{\frac{4}{a}} = \frac{a}{4}$$

3. Find the radius of curvature at the point (a,0)on the curve $xy^2 = a^3 - x^3$.(MAY 07)

Solution: Radius of curvature $\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Given $xy^2 = a^3 - x^3$

Differentiate w.r.t.'x'

$2xy\frac{dy}{dx}+y^2.1=-3x^2$ $\quad \frac{dy}{dx} = \frac{-3x^2-y^2}{2xy}$ ……………………(1)

$$\frac{dy}{dx}(a,0) = \frac{-3a^2-0}{2a.0} = \infty$$

Therefore $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$

$\frac{dx}{dy} = \frac{2xy}{-3x^2-y^2}$ ……………(1)

$\frac{dx}{dy}(a,0) = \frac{2a.0}{-3a^2-0} = 0$

Differentiate (2) w.r.t.'y'.

$\frac{d^2x}{dy^2} = \frac{2\left[(-3x^2-y^2)\left(x.1+y\frac{dx}{dy}\right)-xy\left(-6x\frac{dx}{dy}-2y\right)\right]}{(-3x^2-y^2)^2}$

$\frac{d^2x}{dy^2}(a,0) = \frac{2[(-3a^2-0)(a+0)-0]}{(-3a^2-0)^2} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$

Therefore radius of curvature $\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = \frac{\{1+0\}^{3/2}}{-2/3a} = -\frac{3}{2}a$

$\rho = \frac{3}{2}a$ (since the radius of curvature is non-negative)

4. Find the curvature of the parabola $y^2=4x$ at the vertex. **(NOV 07)**

Solution: Radius of curvature $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

Given: $y^2=4x$

Differentiate w.r.t.'x'

$2y\frac{dy}{dx} = 4$

$\frac{dy}{dx} = 2/y$

$$\frac{dy}{dx}(0,0) = \frac{2}{0} = \infty$$

Therefore $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$

$\frac{dx}{dy} = \frac{y}{2}$ ....(1)

$$\frac{dx}{dy}(0,0) = 0$$

Differentiate (1) w.r.t.'y'.

$$\frac{d^2x}{dy^2} = \frac{1}{2}$$

Therefore $\rho = \frac{[1+0]^{3/2}}{1/2} = 2$

Curvature $K = 1/\rho = 1/2$

5. Find the radius of curvature of the curve $27ay^2=4x^3$ at the point where the tangent of the curve makes an angle 45^0 with the X- axis.

Solution: Let (x_1,y_1) be the point on the curve at which the tangent makes an angle 45^0 with the X- axis.

$\frac{dy}{dx}(x_1,y_1) = \text{Tan } 45^0 = 1$ ---------------------------------- (1)

Given $27ay^2 = 4x^3$

Differentiate w.r.t.'x'

$54ay\frac{dy}{dx} = 12x^2$ $\frac{dy}{dx} = \frac{2x^2}{9ay}$

$\frac{dy}{dx}(x_1,y_1) = \frac{dy}{dx} = \frac{2x_1^2}{9ay_1}$-----------------------------------(2)

$\frac{dy}{dx}(x_1,y_1) = \text{Tan } 45^0 = 1 = \frac{2x_1^2}{9ay_1}$

Gives $y_1 = \frac{2x_1^2}{9a}$----------------------------(3)

As (x_1,y_1) lies on the curve $27ay_1^2 = 4x_1^3$ -----------------------------------(4)

Using $y_1 = \frac{2x_1^2}{9a}$ gives $x_1 = 3a$

And using (3) gives $y_1 = 2a$

Y_1 at $(3a,2a) = 1$

$Y_2 = \frac{2}{9a}\left[\frac{y\,2x - x^2 y_1}{y^2}\right]$

$Y_2 = \frac{2}{9a}\left[\frac{2.3a.2a - 9a^2.1}{4a^2}\right] = 1/6a$

Therefore radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1/6a}$

$$\rho = 12a\sqrt{2}$$

6. **Find the evolute of the rectangular hyperbola $xy=c^2$.** **(JAN-06,NOV-08)**

Solution: The equation of the given curve is $xy=c^2$……….(1)

The parametric form of (1) is

$X=ct;\ y=\frac{c}{t}$

$\frac{dx}{dt}=c;\ \frac{dy}{dt}=c\left(\frac{-1}{t^2}\right)=\frac{-c}{t^2}$

$Y_1=\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c} = \frac{-1}{t^2}$

$Y_2=\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-1}{t^2}\right)$

$=\frac{d}{dt}\left(\frac{-1}{t^2}\right).\frac{dt}{dx}$

$$\frac{2}{t^3}.\frac{1}{c} = \frac{2}{ct^3}$$

The co-ordinates of the center of curvature is $(\bar{x},\bar{y})$

Where $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$$= ct - \frac{\left(-1/t^2\right)}{\left(2/ct^3\right)}\left(1+\frac{1}{t^4}\right) = ct + \frac{c(t^4+1)}{2t^3} = \frac{c(3t^4+1)}{2t^3}$$

$\bar{x} = \frac{c}{2}\left(3t + \frac{1}{t^3}\right)$……………………….(2)

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$$

$$= \frac{c}{t} + \frac{1}{\left(\frac{2}{ct^3}\right)}\left(1 + \frac{1}{t^4}\right) = \frac{c}{t} + \frac{c(t^4+1)}{2t} = \frac{c(t^4+3)}{2t}$$

$\bar{y} = \frac{c}{2}\left(t^3 + \frac{3}{t}\right)$……………………………………(3)

Eliminating' between (2) and (3).

(2)+(3)gives

$$\bar{x} + \bar{y} = \frac{c}{2}\left(3t + \frac{1}{t^3}\right) + \frac{c}{2}\left(t^3 + \frac{3}{t}\right) = \frac{c}{2}\left(t^3 + \frac{3}{t} + 3t + \frac{1}{t^3}\right)$$

$\bar{x} + \bar{y} = \frac{c}{2}\left(t + \frac{1}{t}\right)^3$……………………………(4)

(2)-(3)gives

$$\bar{x} - \bar{y} = \frac{c}{2}\left(3t + \frac{1}{t^3}\right) - \frac{c}{2}\left(t^3 + \frac{3}{t}\right) = -\frac{c}{2}\left(t^3 + \frac{3}{t} - 3t - \frac{1}{t^3}\right)$$

$x - y = -\frac{c}{2}\left(t - \frac{1}{t}\right)^3$……………………(5)

$(4)^{2/3}$-$(5)^{2/3}$gives

$$(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[\left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2\right]$$

$$= \frac{(c)^{2/3}}{(2)^{2/3}}(4)$$

Therefore $(\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = (4c)^{2/3}$

The locus of centre of curvature$(x, y)$is

$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}$which is the required evolute of the rectangular hyperbola $xy=c^2$.

7. Find the radius of curvature for the curve r=a(1+cosθ)at $\theta = \frac{\pi}{2}$ *and prove that* $\frac{\rho^2}{r}$is a constant.

Solution: Given r=a(1+cos θ) **(NOV-07,08)**

r'= -a sin θ and r'' = cosθ

The radius of curvature in polar form is $\rho = \frac{(r^2+r'^2)^{3/2}}{(r^2+2r'^2-rr'')}$

$$= \frac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{3/2}}{[a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta]}$$

$$= \frac{a^2(2+2\cos\theta)^{3/2}}{a^2(1+2\cos^2\theta+2\sin^2\theta+3\cos\theta)} = \frac{a^{2}2^{3/2}(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)} = \frac{2\sqrt{2}a(1+\cos\theta)^{1/2}}{3}$$

ρ at $\theta = \frac{\pi}{2}$ is $\rho = \frac{2\sqrt{2}a}{3}$

Also, $\rho^2 = \frac{8a^2}{9}(1+\cos\theta) = \frac{8a^2}{9}r$

Therefore, $\frac{\rho^2}{r} = \frac{8}{9}a = constant.$

8. Considering the evolute as the envelope of normals, find the evolute of the parabola $x^2=4ay$.

Solution: Given $x^2=4ay$ **(NOV-08)**

The parametric equations are $x=2at$, $y=at^2$

$\frac{dx}{dt}=2a$ and $\frac{dy}{dt}=2at$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2at}{2a}=t$$

$m=\frac{dy}{dx}=t$

We know that the equation of normal to the curve is $y-y_1=-\frac{1}{m}(x-x_1)$

$y-at^2=\frac{-1}{t}(x-2at)$ $yt-at^3=-x+2at$

$x+yt=at^3+2at$ ……………….(1)

Differentiate (1) partially w.r.t. 't' we get

$Y=3at^2+2a$ $t^2=\frac{y-2a}{3a}$ $t=\left(\frac{y-2a}{3a}\right)^{1/2}$

Substitute the value of 't' in (1)

$y\left(\frac{y-2a}{3a}\right)^{1/2}+x=a\left(\frac{y-2a}{3a}\right)^{3/2}+2a\left(\frac{y-2a}{3a}\right)^{1/2}$

$x=\left(\frac{y-2a}{3a}\right)^{1/2}\left[a\left(\frac{y-2a}{3a}\right)+2a-y\right]=\left(\frac{y-2a}{3a}\right)^{1/2}\left[\left(\frac{y-2a}{3}\right)+2a-y\right]=\left(\frac{y-2a}{3a}\right)^{1/2}\left[\frac{4a-2y}{3}\right]$

$$3\sqrt{3}\sqrt{a}x=-2(y-2a)^{3/2}$$

Squaring on both sides, we get

$27ax^2=4(y-2a)^3$ which is the required evolute.

9. Obtain the evolute of the parabola $y^2=4ax$. **(NOV-07)**

Solution: Given $y^2=4ax$……………………….(1)

The parametric equations are $x=at^2$, $y=2at$

$\frac{dx}{dt}=2at$ $\frac{dy}{dt}=2a$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2a}{2at}=\frac{1}{t}=y_1$

$Y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{t}\right)$

$\frac{d}{dt}\left(\frac{1}{t}\right).\frac{dt}{dx}=\frac{-1}{t^2}.\frac{1}{2at}=\frac{-1}{2at^3}$

The co-ordinates of the center of curvature Is $(\bar{x},\bar{y})$

Where $\bar{x}=x-\frac{y_1}{y_2}(1+y_1^2)=at^2-\frac{(1/t)}{(-1/2at^3)}\left(1+\frac{1}{t^2}\right)=at^2+\frac{(t^2+1)2at^2}{t^2}=at^2+2at^2+2a$

$\bar{x}=3at^2+2a$……………………….(2)

$$\bar{y}=y+\frac{1}{y_2}(1+y_1^2)$$

$$\bar{y}=2at+\frac{1}{\left(-1/2at^3\right)}\left(1+\frac{1}{t^2}\right)=2at-\frac{(t^2+1)2at^3}{t^2}=2at-2at^3-2at$$

$\bar{y}=-2at^3$……………………………….(3)

Eliminating 't' between (2)and(3),

(2) gives $t^2=\frac{\bar{x}-2a}{3a}$

(3) gives $t^3=-\frac{\bar{y}}{2a}$

$$\left(-\frac{\bar{y}}{2a}\right)^2=\left(\frac{\bar{x}-2a}{3a}\right)^3$$

$$27a\bar{y}^2=4(\bar{x}-2a)^3$$

The locus of centre of curvature$(\bar{x},\bar{y})$is

$27ay^2=4(x-2a)^3$ which is the required evolute.

10. Find the equation of the envelope of $\frac{x}{a}+\frac{y}{b}=1$ *where* $a^2+b^2=c^2$. (NOV-02,07)

Solution: Given that $\frac{x}{a}+\frac{y}{b}=1$…………………(1)

And $a^2+b^2=c^2$……………..(2)

Differentiate (1)and(2) w.r.t 'b'

$\frac{-x}{a^2}\frac{da}{db}-\frac{y}{b^2}=0$…………………(3)

$2a\frac{da}{db}+2b=0$…………………………….(4)

(3)gives $\frac{da}{db}=-\frac{a^2y}{b^2x}$……………..(5)

(4)gives$\frac{da}{db}=\frac{-b}{a}$……………………..(6)

From (5)and (6) $\frac{b}{a}=\frac{a^2y}{b^2x}$

$$\frac{x}{a^2}=\frac{y}{b^2}=\frac{x/a}{a^2}=\frac{y/b}{b^2}=\frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\frac{1}{c^2}$$

$$\frac{x}{a^3} = \frac{1}{c^2} \, and \, \frac{y}{b^3} = \frac{1}{c^2}$$

$$a = (xc^2)^{1/3} \; and \; b = (yc^2)^{1/3}$$

Substitute in (2) we get, $(xc^2)^{2/3} + (yc^2)^{2/3} = c^2$

Therefore $x^{2/3} + y^{2/3} = c^{2/3}$ which is the required envelope.

11. Find the equation of circle of curvature of the parabola $y^2=12x$ at the point (3,6).

Solution: The equation of circle of curvature is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$ **(NOV 07,08,JAN 09)**

Where, $\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$

$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

Given $y^2=12x$

Differentiate w.r.t 'x' we get

$2y\frac{dy}{dx}=12$ implies $\frac{dy}{dx} = \frac{6}{y}$

$Y_1=\frac{dy}{dx}(3,6)=1$ $\quad \frac{d^2y}{dx^2} = \frac{-6}{y^2}\frac{dy}{dx}$

$Y_2=\frac{d^2y}{dx^2}(3,6)=-1/6$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + 1)^{3/2}}{-1/6} = -12\sqrt{2}$$

$\rho = 12\sqrt{2}$ (ρ can not be negative)

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$$

$=3 - \frac{1}{-1/6}(1 + 1) = 15$

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2) = 6 + \frac{1}{-1/6}(1 + 1) = -6$$

Therefore, the equation of circle of curvature is $(x - 15)^2 + (y + 6)^2 = 288$

12. Find the radius of curvature at 't' on $x=e^t\cos t, y=e^t\sin t$. **(JAN-06)**

Solution: Radius of curvature $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

Given $x = e^t\cos t$, $y = e^t\sin t$

$X' = \frac{dx}{dt} = e^t cost - e^t sint = e^t(cost - sint)$

$Y' = \frac{dy}{dt} = e^t cost + e^t sint = e^t(cost + sint)$

$X'' = \frac{d^2x}{dt^2} = e^t(-sint - cost) + e^t(cost - sint) = -2e^t sint$

$Y'' = \frac{d^2y}{dt^2} = e^t(-sint + cost) + e^t(cost + sint) = 2e^t cost$

$\Rightarrow \therefore$ The radius of curvature is $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

$$\rho = \frac{([e^t(cost - sint)]^2 + [e^t(cost + sint)]^2)^{3/2}}{e^t(cost - sint).2e^t cost - e^t(cost + sint).(-2e^t sint)}$$

$= \frac{(e^{2t}[cos^2t + sin^2t - 2sintcost + cos^2t + sin^2t + 2sintcost])^{3/2}}{2e^{2t}[cos^2t - sintcost + sintcost + sin^2t]} = \frac{(2e^{2t})^{3/2}}{2e^{2t}} = \sqrt{2}e^t$

13. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(MAY 05,07)**

Solution: The given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The parametric equations are $x = acos\theta$, $y = bsin\theta$

$$\frac{dx}{d\theta} = -asin\theta, \frac{dy}{d\theta} = bcos\theta$$

$Y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b}{a}cot\theta$

$Y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{b}{a}cot\theta\right)\frac{d\theta}{dx} = \frac{b}{a}cosec^2\theta\left(\frac{-1}{asin\theta}\right) = -\frac{b}{a^2}\frac{cosec^2\theta}{sin\theta}$

$Y_2 = -\frac{b}{a^2}cosec^3\theta$

The Co-ordinate of centre of curvature is $(\bar{x}, \bar{y})$

Where $\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$

$= acos\theta - \frac{\left(-\frac{b}{a}cot\theta\right)\left(1+\frac{b^2}{a^2}cot^2\theta\right)}{-\frac{b}{a^2}cosec^3\theta} = acos\theta - \frac{bcos\theta}{asin\theta}\left(1 + \frac{b^2cos^2\theta}{a^2sin^2\theta}\right)\frac{a^2}{bcosec^3\theta}$

$= acos\theta - \frac{1}{a}cos\theta(a^2sin^2\theta + b^2cos^2\theta) = acos\theta - \frac{1}{a}cos\theta(a^2(1-cos^2\theta) + b^2cos^2\theta)$

$$= acos\theta - \frac{1}{a}cos\theta(a^2 - a^2cos^2\theta + b^2cos^2\theta) = acos\theta - acos\theta + acos^3\theta - \frac{b^2}{a}cos^3\theta$$

$= cos^3\theta\left(a - \frac{b^2}{a}\right) = \left(\frac{a^2-b^2}{a}\right)cos^3\theta$

$\therefore \bar{x} = \left(\frac{a^2-b^2}{a}\right)cos^3\theta$......................(1)

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$$

$$= bsin\theta + \frac{\left(1+\frac{b^2}{a^2}cot^2\theta\right)}{\frac{b}{a^2}cosec^2\theta} = bsin\theta \quad \left(1 + \frac{b^2cos^2\theta}{a^2sin^2\theta}\right)\frac{a^2}{bcosec^2\theta}$$

$$= bsin\theta - \frac{sin\theta}{b}[a^2sin^2\theta + b^2(1 - sin^2\theta)] = bsin\theta - \frac{a^2}{b}sin^3\theta - bsin\theta + bsin^3\theta$$

$$= sin^3\theta\left[b - \frac{a^2}{b}\right] = sin^3\theta\left[\frac{b^2-a^2}{b}\right]$$

$$\bar{y} = -sin^3\theta\left[\frac{a^2-b^2}{b}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Eliminating 'θ' between (1) and (2),we get

$$cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3} \quad and\ sin\theta = \left(\frac{b\bar{y}}{a^2-b^2}\right)^{1/3}$$

we know that, $sin^2\theta + cos^2\theta = 1$

$$sin^2\theta + cos^2\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(\frac{b\bar{y}}{a^2-b^2}\right)^{2/3} = \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} = 1$$

$$\therefore\ (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

The locus of $(\bar{x}, \bar{y})$ is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$ which is the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

14. Find the envelope of $\frac{x}{l} + \frac{y}{m} = 1$ where l and m are connected by $\frac{l}{a} + \frac{m}{b} = 1$ and a,b are constants. (MAY – 05, NOV-05)

Solution: Given that $\frac{x}{l} + \frac{y}{m} = 1$……………….(1) $\quad \frac{l}{a} + \frac{m}{b} = 1$……………….(2)

Differentiating (1) w.r.t 'm', we get

$$x\left(\frac{-1}{l^2}\right)\frac{dl}{dm} + y\left(\frac{-1}{m^2}\right) = 0 \qquad \frac{dl}{dm} = \frac{-yl^2}{xm^2}\ldots\ldots(3)$$

Differentiating (2) w.r.t 'm'

$$\frac{l}{a}\frac{dl}{dm} + \frac{1}{b} = 0 \qquad \frac{dl}{dm} = \frac{-a}{b}\ldots\ldots\ldots(4)$$

From (3) and (4)

$$\frac{-yl^2}{xm^2} = \frac{-a}{b} \Rightarrow \frac{by}{m^2} = \frac{ax}{l^2}$$

$$\frac{\frac{y}{m}}{\frac{m}{b}} = \frac{\frac{x}{l}}{\frac{l}{a}} = \frac{\frac{y}{m}+\frac{x}{l}}{\frac{m}{b}+\frac{l}{a}} = 1$$

$\frac{by}{m^2} = 1, \frac{ax}{l^2} = 1 \Rightarrow m = \sqrt{by}, l = \sqrt{ax}$ Substitute in equation (2),

$\frac{\sqrt{ax}}{a} + \frac{\sqrt{by}}{b} = 1 \quad \Rightarrow \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ which is the required envelope.

15. **Find the points on the parabola $y^2 = 4x$ at which the radius of curvature is $4\sqrt{2}$.** **(MAY – 05)**

Solution: Given $y^2 = 4x$...............(1)

Let, P (a,b) be the point on the curve $y^2 = 4x$ at where $\rho = 4\sqrt{2}$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Differentiate (1) w.r.t. 'x'

$Y_1 = 2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y} \therefore \frac{dy}{dx}(a,b) = \frac{2}{b}$

$Y_2 = \frac{d^2y}{dx^2} = -\frac{2}{y^2}\frac{dy}{dx} \therefore \frac{d^2y}{dx^2}(a,b) = -\frac{4}{b^3}$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(4+b^2)^{3/2}}{4} = 4\sqrt{2}$$

But, $b^2 = 4a \Rightarrow \frac{(4+4a)^{3/2}}{4} = 4\sqrt{2}$ hence $8(1+a)^{3/2} = 16\sqrt{2} \Rightarrow (1+a)^3 = 2^3$

a+1=2 a=1, $b^2 = 4 \Rightarrow b = \pm 2$ $\therefore$ The points are (1,2),(1,-2).

16. Considering the evolute of a curve as the envelope of its normals find the evolute of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(NOV-02,05,MAY-05)

Solution: The given curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The parametric equations are x=acosθ ,y=bsinθ

$\frac{dx}{d\theta}=-asin\theta$,$\frac{dy}{d\theta}=bcos\theta$

m=$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=-\frac{b}{a}cot\theta$

We know that the equation of the normal is $y-y_1=\frac{-1}{m}(x-x_1)$

$$y-bsin\theta=\frac{-1}{-\frac{b}{a}cot\theta}(x-acos\theta)$$

$$y-bsin\theta=\frac{asin\theta}{bcos\theta}(x-acos\theta)$$

$$ybcos\theta-b^2sin\theta cos\theta=axsin\theta-a^2sin\theta cos\theta$$

$\div$ by $sin\theta cos\theta$,we get

$$\frac{yb}{sin\theta}-b^2=\frac{ax}{cos\theta}-a^2$$

$$\frac{ax}{cos\theta}-\frac{yb}{sin\theta}=a^2-b^2 \ldots\ldots\ldots\ldots(1)$$

Differentiate (1) partially w.r.t.' θ', we get

$$\frac{-ax}{cos^2\theta}(-sin\theta)+\frac{by}{sin^2\theta}cos\theta=0$$

$$\frac{sin^3\theta}{cos^3\theta}=\frac{-by}{ax}=tan^3\theta=\frac{-by}{ax}\Rightarrow tan\theta=\frac{-(by)^{1/3}}{(ax)^{1/3}}$$

$$\therefore\ sin\theta=\frac{-(by)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}},cos\theta=\frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3}+(by)^{2/3}}}$$

Substitute in equation (1),we get

$$\frac{ax}{(ax)^{1/3}}\sqrt{(ax)^{2/3}+(by)^{2/3}}-\frac{by}{-(by)^{1/3}}\sqrt{(ax)^{2/3}+(by)^{2/3}}=a^2-b^2$$

$$\sqrt{(ax)^{2/3}+(by)^{2/3}}(ax)^{2/3}+(by)^{2/3}=a^2-b^2$$

$$\left[(ax)^{2/3}+(by)^{2/3}\right]^{3/2}=a^2-b^2$$

$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ which is the required evolute of the ellipse.

17. Find the circle of curvature at (3,4) on xy=12. **(JAN-05)**

Solution: The equation of circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

Where, $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given xy=12

Differentiate w.r.t 'x' we get

$x\frac{dy}{dx} + y = 0$ implies $\frac{dy}{dx} = \frac{-y}{x}$

$Y_1 = \frac{dy}{dx}(3,4) = -4/3$ $\quad \frac{d^2y}{dx^2} = -\frac{\left(x\frac{dy}{dx} - y.1\right)}{x^2}$

$Y_2 = \frac{d^2y}{dx^2}(3,6) = -\frac{\left(3\frac{-4}{3} - 4.1\right)}{3^2} = \frac{8}{9}$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\left(\frac{-4}{3}\right)^2\right)^{3/2}}{8/9} = \frac{(5)^3}{27}.\frac{9}{8} = \frac{125}{24}$

$\rho = \frac{125}{24}$

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$= 3 - \frac{-4/3}{8/9}\left(1+\frac{16}{9}\right) = 3 + \frac{25}{6} = \frac{41}{6}$

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = 4 + \frac{1}{8/9}\left(1+\frac{16}{9}\right) = 4 + \frac{25}{8} = \frac{57}{8}$

Therefore, the equation of circle of curvature is $\left(x - \frac{41}{6}\right)^2 + \left(y - \frac{57}{8}\right)^2 = \left(\frac{125}{24}\right)^2$.

18. Find the curvature for $r = ae^{\theta\cot\alpha}$. **(JAN-05)**

Solution: Given $r = ae^{\theta\cot\alpha}$

$r' = \frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cot\alpha$

$r'' = ae^{\theta\cot\alpha}\cot^2\alpha$

The radius of curvature in polar form is $\rho = \frac{(r^2+r'^2)^{3/2}}{(r^2+2r'^2-rr'')}$

$= \frac{\left(a^2e^{2\theta\cot\alpha} + a^2e^{2\theta\cot\alpha}\cot^2\alpha\right)^{3/2}}{(a^2e^{2\theta\cot\alpha} + 2a^2e^{2\theta\cot\alpha}\cot^2\alpha - a^2e^{2\theta\cot\alpha}\cot^2\alpha)}$

$\frac{a^3e^{3\theta\cot\alpha}(\mathrm{cosec}^2\alpha)^{3/2}}{a^2e^{2\theta\cot\alpha}(1+\cot^2\alpha)} = ae^{\theta\cot\alpha}\mathrm{cosec}\,\alpha \quad (\text{since } (1+\cot^2\alpha) = \mathrm{cosec}^2\alpha)$

$\therefore \rho = r\,\mathrm{cosec}\,\alpha$

Curvature $K = \frac{1}{\rho} = \frac{1}{r\,\mathrm{cosec}\,\alpha}$

19. Find the evolute of the four cusped hyper cycloid $x^{2/3} + y^{2/3} = a^{2/3}$.(JAN-05, NOV-07)

Solution: The equation of the given curve is $x^{2/3} + y^{2/3} = a^{2/3}$……………….(1)

The parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = -3a\sin^2\theta\cos\theta$

$Y_1 = \frac{dy}{d\theta} / \frac{dx}{d\theta} = -\tan\theta$

$Y_2 = -\sec^2\theta / \frac{dx}{d\theta} = ((\sec^4\theta\,\mathrm{cosec}\theta))/3a$

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = a\cos^3\theta - \frac{-\tan\theta}{\frac{(\sec^4\theta\,\mathrm{cosec}\theta)}{3a}}(1+\tan^2\theta)$

$= a\cos^3\theta + 3a\sin^2\theta\cos\theta$--(2)

$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) = a\sin^3\theta + \frac{1}{\frac{(\sec^4\theta\,\mathrm{cosec}\theta)}{3a}}(1+\tan^2\theta)$

$a\sin^3\theta + 3a\cos^2\theta\sin\theta$--(3)

Eliminate θ from 2 & 3

$\bar{x} + \bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta$

$= a(\cos\theta + \sin\theta)^3$------------------------------------(4)

$\bar{x} - \bar{y} = a\cos^3\theta + 3a\sin^2\theta\cos\theta - a\sin^3\theta + 3a\cos^2\theta\sin\theta$

$= a(\cos\theta - \sin\theta)^3$------------------------------------(5)

$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = (a^{2/3})\left(\left((\cos\theta + \sin\theta)^3\right)^{2/3}\right) + \left(\left(\cos\theta - \sin\theta\right)^3\right)^{2/3}\right)$

$-(a^{2/3})(2)$

The locus of centre of curvature is $(x+y)^{2/3} + (x-y)^{2/3} = (a^{2/3})(2)$

20. Find the radius of curvature at the origin of the cycloid x = a (θ + sin θ) and y = a(1 cos θ).

(MAY'07, Nov '08)

Given: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

$x' = a(1 + \cos\theta)$ $y' = a(\sin\theta)$

$x'' = -a\sin\theta$ $y'' = a\cos\theta$

$\Rightarrow\therefore$ The radius of curvature is $\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

$$= \frac{((a(1+\cos\theta))^2+(a\sin\theta)^2)^{3/2}}{a(1+\cos\theta)(a\cos\theta)-a(\sin\theta)(-a\sin\theta)} = 4a\cos\frac{\theta}{2}$$

At $\theta = 0$ $\rho = 4a$

21. Find the envelope of the straight lines represented by the equation x cos α + y sin α = a sec α , α is the parameter **(Nov' 07)**

Solution: Given $x\cos\alpha + y\sin\alpha = a\sec\alpha$

Divided by $\cos\alpha$

$x + y\tan\alpha = a\sec^2\alpha$

$x + y\tan\alpha = a(1 + \tan^2\alpha)$

$a\tan^2\alpha - y\tan\alpha + a - x = 0$

which is quadratic in $\tan\alpha$

$A = a$, $B = -y$ $C = a-x$

The envelope is given by $B^2 - 4AC = 0$

$y^2 = 4a(a-x)$ which is the required envelope

22. Prove that the evolute of the curve x = a (cos θ + log tan $\frac{\theta}{2}$), y=a sin θ is the catenary y = a cosh $\frac{x}{a}$

(Nov '05)

Solution : $x = a(\cos\theta + \log\tan\frac{\theta}{2})$

$x' = a(-\sin\theta + \frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}\frac{1}{2}) = a\cot\theta\cos\theta$

$y = a\sin\theta$

$y' = a\cos\theta$

$y_1 = \frac{y'}{x'} = \tan\theta$

$y_2 = \frac{\sec^2\theta}{x'} = \frac{1}{a}(\sec^4\theta\sin\theta)$

$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$

$= a\,(\cos\theta + \log\tan\frac{\theta}{2}) - \frac{\tan\theta}{\frac{(\sec^4\theta\,\sin\theta)}{a}}(1 + \tan^2\theta)$

$\bar{x} = a\log\tan\frac{\theta}{2}$ ....(1)

$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$

$= a\sin\theta + \frac{1}{\frac{(\sec^4\theta\,\sin\theta)}{a}}(1 - \tan^2\theta)$

$\bar{y} = \frac{a}{\sin\theta}$ ....(2)

Eliminate 'θ' from (1) and (2)

$\tan\frac{\theta}{2} = e^{\bar{x}/a}$ ....(3)

$\frac{\bar{y}}{a} = \frac{(e^{\bar{x}/a} + e^{-\bar{x}/a})}{2} = \cosh\frac{\bar{x}}{a}$

$y = a\cosh\frac{x}{a}$ which is the required evolute

23. Obtain the equation of the evolute of the curve $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$.(May 09)

Solution: Given $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta) = a\,\theta\cos\theta$

$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta) = a\,\theta\sin\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\,\theta\sin\theta}{a\,\theta\cos\theta} = \tan\theta$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta)\,\frac{d\theta}{dx} \quad = \sec^2\theta.\frac{1}{a\,\theta\cos\theta} = \frac{1}{a\,\theta\cos^3\theta}$

The co-ordinates of centre of curvature is $(\bar{x}, \bar{y})$

$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$

$= a(\cos\theta + \theta\sin\theta) - \frac{\sin\theta}{\cos\theta}\,a\,\theta\cos^3\theta.\sec^2\theta$

$\bar{x} = a\cos\theta$ ....(1)

$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$

$= a(\sin\theta - \theta\cos\theta) + a\,\theta\cos^3\theta.\sec^2\theta$

$\bar{y} = a \sin\theta$..................................(2)

Eliminating θ from equations (1) & (2) we get $\bar{x}^2 + \bar{y}^2 = a^2$

The locus of centre of curvature is $x^2 + y^2 = a^2$ which is the required evolute.

24. Find the envelope of the straight line $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are parameters that are connected by the relation a+b = c. **(May 09)**

Solution: Given $\frac{x}{a} + \frac{y}{b} = 1$.....................(1)

And a+b = c..............................(2)

Differentiate Eqns (1) and (2) partially w.r.t 'b'

$-\frac{x}{a^2} \cdot \frac{da}{db} - \frac{y}{b^2} = 0$

$\frac{da}{db} = -\frac{a^2 y}{b^2 x}$..................................(3)

$\frac{da}{db} + 1 = 0$

$\frac{da}{db} = -1$..(4)

Equate (3) & (4) we get

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{x/a}{a} = \frac{y/b}{b} = \frac{x/a + y/b}{a+b} = \frac{1}{c}$$

$\frac{x}{a^2} = \frac{1}{c}$ and $\frac{y}{b^2} = \frac{1}{c}$

$a = \sqrt{xc}$ $\quad$ $b = \sqrt{yc}$

Substitute in eqn (2)

$\sqrt{xc} + \sqrt{yc} = c$

$\sqrt{x} + \sqrt{y} = \sqrt{c}$, which is the required evolute.

25. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ subject to the condition a+b =1.(Jan-09)

Solution: Given $\frac{x}{a} + \frac{y}{b} = 1$.....................(1)

And a+b = 1..............................(2)

Differentiate Eqns (1) and (2) partially w.r.t 'b'

$-\frac{x}{a^2} \cdot \frac{da}{db} - \frac{y}{b^2} = 0$

$\frac{da}{db} = -\frac{a^2 y}{b^2 x}$..............................(3)

$\frac{da}{db} + 1 = 0$

$\frac{da}{db} = -1$..................................(4)

Equate (3) & (4) we get

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{x/a}{a} = \frac{y/b}{b} = \frac{x/a + y/b}{a+b} = 1$$

$\frac{x}{a^2} = 1$ and $\frac{y}{b^2} = 1$

a=$\sqrt{x}$ b=$\sqrt{y}$

Substitute in eqn (2)

$\sqrt{x} + \sqrt{y} = 1$

$\sqrt{x} + \sqrt{y} = 1$, which is the required evolute.

26. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ considering it as the envelope of its normals.

Solution: The parametric equations are x= $a \sec\theta$ and y= b $\tan\theta$. **(Jan-09)**

$\frac{dx}{d\theta} = a \sec\theta \tan\theta$ $\frac{dy}{dx} = b\sec^2\theta$

Slope of the curve m=$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}cosec\,\theta$

Equation of normal to the given curve is $y - y_1 = -\frac{1}{m}(x - x_1)$

y-b $\tan\theta = -\frac{1}{\frac{b}{a}cosec\theta}(x - a\sec\theta)$

y-b$\frac{\sin\theta}{\cos\theta} = -\frac{a\sin\theta}{b}\left(x - \frac{a}{\cos\theta}\right)$

by $\cos\theta - b^2\sin\theta = -ax\sin\theta\cos\theta + a^2\sin\theta$

axcos θ-$by\cot\theta = (a^2 + b^2)$..................................(1)

Differentiate equ (1) w.r.t. θ

-ax $\sin\theta - by\,cosec^2\theta = 0$

$$ax\sin\theta = -\frac{by}{\sin^2\theta}$$

$$\sin^3\theta = -\frac{by}{ax}$$

$$\sin\theta = \left(-\frac{by}{ax}\right)^{\frac{1}{3}}$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-\left(-\frac{by}{ax}\right)^{\frac{2}{3}}} = \sqrt{\frac{(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}}{(ax)^{\frac{2}{3}}}}$$

$$\cot\theta = -\sqrt{\frac{(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}}{(by)^{\frac{2}{3}}}}$$

Substitute in eqn (1) we get,

$$ax\sqrt{\frac{(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}}{(ax)^{\frac{2}{3}}}} - by\sqrt{\frac{(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}}{(by)^{\frac{2}{3}}}} = (a^2+b^2)$$

$$\sqrt{(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}}\left[(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}\right] (a^2+b^2)$$

$$\left[(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}\right]^{3/2} = (a^2+b^2)$$

$(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}} = (a^2+b^2)^{2/3}$, which is the required evolute of the given curve.

27. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$. (Jan 09)

Solution: Given $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiate w.r.t.'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\frac{dy}{dx}_{\left(\frac{a}{4},\frac{a}{4}\right)} = -1$$

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\frac{1}{2\sqrt{x}}}{x}\right]$$

$$\frac{d^2y}{dx^2}_{\left(\frac{a}{4},\frac{a}{4}\right)} = -\frac{1}{2}\left[\frac{-1-1}{a/4}\right] = \frac{4}{a}$$

$$\text{Radius of curvature } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+(-1)^2\right]^{3/2}}{\frac{4}{a}} = \frac{2\sqrt{2}}{4}a = \frac{a}{\sqrt{2}}$$

POSSIBLE QUESTIONS

Class : **I – B.Sc. Mathematics**
Subject Name : **Calculus**
Subject Code : **17MMU101**

UNIT – II

2 Mark Questions:

1. Find the Cartesian coordinate of the point P whose polar coordinates are $(r,\theta) = \left(6\,, \frac{2\pi}{3}\right)$.
2. Find the polar coordinate of the point P whose Cartesian coordinate are $\left(-2, -2\sqrt{3}\right)$
3. State a L'Hospital's Rule.
4. Evaluate $\lim_{x\to 2} \frac{x^7-128}{x^3-8}$.
5. Evaluate $\int \tan^5 x\, dx$

6Mark Questions :

1. Find i) $\lim_{x\to 0} \frac{(1-cosx)sin4x}{x^3 cosx}$ ii) $\lim_{x\to 0} \frac{x-sinx}{x^3}$Derive the reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$.
2. Show that $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$
3. If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$ then prove that $I_n + I_{n-2} = \frac{1}{n-1}$ and hence evaluate I_5
4. Derive the reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$.
5. Evaluate $\int x^4 (logx)^3 dx$.
6. Evaluate i) $\lim_{x\to 0} \frac{x-sinx}{x^3}$ ii) $\lim_{x\to +\infty} \frac{2x^2-3x+1}{3x^2+5x-2}$
7. Derive the reduction formula for $\int \sin^m x \cos^n x\, dx$.
8. Evaluate i) $\int \sec^4 x\, dx$ ii) $\int cosec^7 x\, dx$
9. Derive the reduction formula for $\int_0^{\frac{\pi}{2}} x^n\, sinx\, dx$

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Pollachi Main Road, Eachanari (Po),
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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: CALCULUS **Subject Code: 17MMU101**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
A polar coordinate system in a plane consists of a fixed point O is called the _____	polar	pole	initial ray	parameter	pole
In a polar cooridinates r denotes a ____	distance	area	angle	radius	distance
An Rectangular coordinates means	pole	cartesian coordinate	polar plane	polar coordinate	cartesian coordinate
In a polar cooridinates θ denotes a ____	distance	area	angle	radius	angle
The polar coordinates is denoted by	S(r , θ)	P(r , θ)	R(r , θ)	Q(r , θ)	P(r , θ)
The polar angle is denoted by	θ	O	r	P	θ
If the polar equation is r cosθ = 2 then the cartesian equation is	x = -1	x = 2	x = -2	x = 0	x = 2
The slope of the polar curve = f(θ) is given by ____	2(dy"/dx")	dy'/dx'	dy/dx	dx/dy	dy/dx
A ray emanating from the pole is called the ______	polar curve	polar axis	polar plane	polar coordinate	polar axis
The radial coordinate is denoted by	θ	O	r	P	r
what is another name foe cartesian coordinate ?	square coordinate	rectangular coordinate	polar plane	polar coordinate	rectangular coordinate
$\lim_{x\to 0}$ (sinx / x) =	0	(-1)	1	2	1
$\lim_{x\to 0}$ ((3x - sinx) / x) =	0	(-1)	1	2	2
∫secx tanx dx = ____	tanx	sinx	secx	cos x	secx
$\lim_{x\to 0+}$ (x cot x) =	2	1	0	(-1)	1
$\lim_{x\to 0+} (1 + x)^{1/x}$ =	e	2	1	0	e
In a competitive economy, the total amount that consumers actually spend on a commodity is usually _______ the total amount they would have been willing to spend	equal to	less than	greater than	more than or equal to	less than
∫cos x dx	sinx	(-cos x)	(-sinx)	tanx	(-sinx)
∫u dv = uv - ____	∫ du	∫v du	∫u du	∫dv	∫v du
d(uv) =	uv - vu	uv + vu	udv - v du	udv + v du	udv - v du
A__________system in a plane consists of a fixed point O is called the pole.	polar curve	cartesian coordinate	polar plane	polar coordinate	polar coordinate
A ray emanating from the_____ is called the polar axis	polar	pole	initial ray	parameter	pole
$\int sec^2x$ dx = ______	tanx	sinx	(-cos x)	(-sinx)	tanx
∫(1/x) dx = ___	x	log x	2x	1 - x	log x
∫cotx dx = _______	log cosx	log tanx	log secx	logsinx	logsinx
∫sec x dx = _______	secx + tanx	log [secx + tanx]	secx + cos x	log [secx + cosec x]	log [secx + tanx]
∫log x dx = _______	x log x	log x + x	x log x - x	x log x + x	x log x - x
$\lim_{x\to 0}$ (tanx / x) =	0	(-1)	1	2	1

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DEPARTMENT OF MATHEMATICS

Subject: CALCULUS	**Semester :I**	**L T P C**
Subject Code: 17MMU101	**Class : I- B.Sc Mathematics**	**4 0 0 4**

UNIT III

Volumes by slicing, disks and washers methods, volumes by cylindrical shells, parametric equations, parameterizing a curve, arc length, arc length of parametric curves, area of surface of revolution.

REFERENCES

1. Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.
2. Anton H., Bivens I., and Davis S.,(2002). Calculus, Seventh Edition, John Wiley and Sons (Asia) P. Ltd., Singapore.

UNIT III

INTEGRATION

The process of integration reverses the process of differentiation. In differentiation, if $f(x) = 2x^2$ then $f'(x) = 4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t$ is t^2.

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d(the variable) after the function to be integrated.

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x',

and $\int 2t\,dt$ means 'the integral of $2t$ with respect to t'

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\,dx = 2x^2$. However, the differential coefficient of $2x^2 + 7$ is also $4x$. Hence $\int 4x dx$ is also equal to $2x^2 + 7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\int 4x\,dx = 2x^2 + c$ and $\int 2t\,dt = t^2 + c$

'c' is called the **arbitrary constant of integration.**

The general solution of integrals of the form $\int ax^n dx$, where a and n are constants is given by:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

This rule is true when n is fractional, zero, or a positive or negative integer, with the exception of $n = -1$.
Using this rule gives:

(i) $\int 3x^4 dx = \frac{3x^{4+1}}{4+1} + c = \frac{3}{5}x^5 + c$

(ii) $\int \frac{2}{x^2} dx = \int 2x^{-2} dx = \frac{2x^{-2+1}}{-2+1} + c$

$= \frac{2x^{-1}}{-1} + c = \frac{-2}{x} + c$, and

(iii) $\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}\sqrt{x^3} + c$

Standard integrals

(i) $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)

(ii) $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

(iii) $\int \sin ax \, dx = -\frac{1}{a} \cos ax + c$

(iv) $\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + c$

(v) $\int \operatorname{cosec}^2 ax \, dx = -\frac{1}{a} \cot ax + c$

(vi) $\int \operatorname{cosec} ax \cot ax \, dx = -\frac{1}{a} \operatorname{cosec} ax + c$

(vii) $\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + c$

(viii) $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

(ix) $\int \frac{1}{x} dx = \ln x + c$

Determine: (a) $\int 5x^2 dx$ (b) $\int 2t^3 dt$

The standard integral, $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$

(a) When $a = 5$ and $n = 2$ then

$$\int 5x^2 dx = \frac{5x^{2+1}}{2+1} + c = \frac{5x^3}{3} + c$$

(b) When $a = 2$ and $n = 3$ then

$$\int 2t^3 dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \frac{1}{2}t^4 + c$$

Determine $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$

$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$ may be written as

$\int 4\,dx + \int \frac{3}{7}x\,dx - \int 6x^2\,dx$

i.e. each term is integrated separately. (This splitting up of terms only applies, however, for addition and subtraction).

Hence $\int \left(4 + \frac{3}{7}x - 6x^2\right)dx$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$

$$= \mathbf{4x + \frac{3}{14}x^2 - 2x^3 + c}$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Determine (a) $\int \frac{2x^3 - 3x}{4x}dx$ (b) $\int (1-t)^2 dt$

(a) Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x}dx = \int \frac{2x^3}{4x} - \frac{3x}{4x}dx$$

$$= \int \frac{x^2}{2} - \frac{3}{4}dx = \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$

$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \mathbf{\frac{1}{6}x^3 - \frac{3}{4}x + c}$$

(b) Rearranging $\int (1-t)^2 dt$ gives:

$$\int (1 - 2t + t^2)dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

$$= \mathbf{t - t^2 + \frac{1}{3}t^3 + c}$$

Determine the following integrals:

1. (a) $\int 4\,dx$ (b) $\int 7x\,dx$

$$\left[\text{(a) } 4x + c \quad \text{(b) } \frac{7x^2}{2} + c\right]$$

2. (a) $\int \frac{2}{5}x^2 dx$ (b) $\int \frac{5}{6}x^3 dx$

$$\left[\text{(a) } \frac{2}{15}x^3 + c \quad \text{(b) } \frac{5}{24}x^4 + c\right]$$

3. (a) $\int \left(\frac{3x^2 - 5x}{x}\right) dx$ (b) $\int (2+\theta)^2 d\theta$

$$\left[\begin{array}{l}\text{(a) } \frac{3x^2}{2} - 5x + c \\ \text{(b) } 4\theta + 2\theta^2 + \frac{\theta^3}{3} + c\end{array}\right]$$

4. (a) $\int \frac{4}{3x^2} dx$ (b) $\int \frac{3}{4x^4} dx$

$$\left[\text{(a) } \frac{-4}{3x} + c \quad \text{(b) } \frac{-1}{4x^3} + c\right]$$

5. (a) $2\int \sqrt{x^3} dx$ (b) $\int \frac{1}{4}\sqrt[4]{x^5} dx$

$$\left[\text{(a) } \frac{4}{5}\sqrt{x^5} + c \quad \text{(b) } \frac{1}{9}\sqrt[4]{x^9} + c\right]$$

6. (a) $\int \frac{-5}{\sqrt{t^3}} dt$ (b) $\int \frac{3}{7\sqrt[5]{x^4}} dx$

$$\left[\text{(a) } \frac{10}{\sqrt{t}} + c \quad \text{(b) } \frac{15}{7}\sqrt[5]{x} + c\right]$$

7. (a) $\int 3\cos 2x\,dx$ (b) $\int 7\sin 3\theta\,d\theta$

$$\left[\begin{array}{l}\text{(a) } \frac{3}{2}\sin 2x + c \\ \text{(b) } -\frac{7}{3}\cos 3\theta + c\end{array}\right]$$

8. (a) $\int \frac{3}{4}\sec^2 3x\,dx$ (b) $\int 2\,\text{cosec}^2 4\theta\,d\theta$

$$\left[\text{(a) } \frac{1}{4}\tan 3x + c \quad \text{(b) } -\frac{1}{2}\cot 4\theta + c\right]$$

Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit.

The operation of applying the limits is defined as: $[x]_a^b = (b) - (a)$

The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2\,dx$

Applying the limits gives:

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3} + c\right]_1^3 = \left(\frac{3^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right)$$
$$= (9 + c) - \left(\frac{1}{3} + c\right) = 8\frac{2}{3}$$

Evaluate (a) $\int_1^2 3x\,dx$

(a) $$\int_1^2 3x\,dx = \left[\frac{3x^2}{2}\right]_1^2 = \left\{\frac{3}{2}(2)^2\right\} - \left\{\frac{3}{2}(1)^2\right\}$$
$$= 6 - 1\frac{1}{2} = 4\frac{1}{2}$$

(b) $\int_{-2}^3 (4 - x^2)dx$

(b) $$\int_{-2}^3 (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^3$$
$$= \left\{4(3) - \frac{(3)^3}{3}\right\} - \left\{4(-2) - \frac{(-2)^3}{3}\right\}$$
$$= \{12 - 9\} - \left\{-8 - \frac{-8}{3}\right\}$$
$$= \{3\} - \left\{-5\frac{1}{3}\right\} = 8\frac{1}{3}$$

Evaluate $\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}}\right) d\theta$,

$$\int_1^4 \left(\frac{\theta+2}{\sqrt{\theta}}\right) d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}} + \frac{2}{\theta^{\frac{1}{2}}}\right) d\theta$$

$$= \int_1^4 \left(\theta^{\frac{1}{2}} + 2\theta^{-\frac{1}{2}}\right) d\theta$$

$$= \left[\frac{\theta^{(\frac{1}{2})+1}}{\frac{1}{2}+1} + \frac{2\theta^{(-\frac{1}{2})+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta}\right]_1^4$$

$$= \left\{\frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4}\right\} - \left\{\frac{2}{3}\sqrt{(1)^3} + 4\sqrt{1}\right\}$$

$$= \left\{\frac{16}{3} + 8\right\} - \left\{\frac{2}{3} + 4\right\}$$

$$= 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Evaluate: $\int_0^{\pi/2} 3 \sin 2x \, dx$

$$\int_0^{\frac{\pi}{2}} 3 \sin 2x \, dx$$

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\} - \left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$= \left\{-\frac{3}{2}\cos \pi\right\} - \left\{-\frac{3}{2}\cos 0\right\}$$

$$= \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

Find: $\int \frac{4}{(5x-3)} dx$

Let $u = (5x - 3)$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

$$\text{Hence} \quad \int \frac{4}{(5x-3)}dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}du$$
$$= \frac{4}{5}\ln u + c$$
$$= \mathbf{\frac{4}{5}\ln(5x - 3) + c}$$

Evaluate: $\int_0^{\pi/6} 24\sin^5\theta\cos\theta d\theta$

Let $u = \sin\theta$ then $\frac{du}{d\theta} = \cos\theta$ and $d\theta = \frac{du}{\cos\theta}$

$$\text{Hence} \quad \int 24\sin^5\theta\cos\theta\, d\theta$$
$$= \int 24u^5\cos\theta\frac{du}{\cos\theta}$$
$$= 24\int u^5 du, \text{ by cancelling}$$
$$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$$
$$= 4\sin^6\theta + c$$

$$\text{Thus} \quad \int_0^{\pi/6} 24\sin^5\theta\cos\theta\, d\theta$$
$$= [4\sin^6\theta]_0^{\pi/6} = 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$$
$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \mathbf{\frac{1}{16}} \text{ or } \mathbf{0.0625}$$

Find: $\int \frac{x}{2+3x^2}dx$

Let $u = 2 + 3x^2$ then $\frac{du}{dx} = 6x$ and $dx = \frac{du}{6x}$

Hence $\int \frac{x}{2+3x^2} dx$

$= \int \frac{x}{u} \frac{du}{6x} = \frac{1}{6} \int \frac{1}{u} du$, by cancelling,

$= \frac{1}{6} \ln u + x$

$= \frac{1}{6} \ln(2 + 3x^2) + c$

Determine: $\int \frac{2x}{\sqrt{4x^2 - 1}} dx$

Let $u = 4x^2 - 1$ then $\frac{du}{dx} = 8x$ and $dx = \frac{du}{8x}$

Hence $\int \frac{2x}{\sqrt{4x^2-1}} dx$

$= \int \frac{2x}{\sqrt{u}} \frac{du}{8x} = \frac{1}{4} \int \frac{1}{\sqrt{u}} du$, by cancelling

$= \frac{1}{4} \int u^{-1/2} du$

$= \frac{1}{4} \left[\frac{u^{(-1/2)+1}}{-\frac{1}{2}+1} \right] + c = \frac{1}{4} \left[\frac{u^{1/2}}{\frac{1}{2}} \right] + c$

$= \frac{1}{2}\sqrt{u} + c = \frac{1}{2}\sqrt{4x^2 - 1} + c$

Determine: $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

Let $x = a \sin\theta$, then $\frac{dx}{d\theta} = a \cos\theta$ and $dx = a\cos\theta\, d\theta$.

Hence $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

$= \int \frac{1}{\sqrt{a^2 - a^2 \sin^2\theta}} a\cos\theta\, d\theta$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1 - \sin^2\theta)}}$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}}$, since $\sin^2\theta + \cos^2\theta = 1$

$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + c$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Hence $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c$

Evaluate $\int_0^3 \frac{1}{\sqrt{9 - x^2}} dx$

From Problem 13, $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3 \quad \text{since } a = 3$$

$$= (\sin^{-1}1 - \sin^{-1}0) = \frac{\pi}{2} \text{ or } \mathbf{1.5708}$$

Find: $\int \sqrt{a^2 - x^2}\,dx$

Let $x = a\sin\theta$ then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$

Hence $\int \sqrt{a^2 - x^2}dx$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{a^2\cos^2\theta}\,(a\cos\theta\,d\theta)$$

$$= \int (a\cos\theta)(a\cos\theta\,d\theta)$$

$$= a^2\int \cos^2\theta\,d\theta = a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

(since $\cos 2\theta = 2\cos^2\theta - 1$)

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c$$

since from Chapter 27, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1 - \left(\frac{x}{a}\right)^2}$$

$$= \sqrt{\frac{a^2 - x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

Thus $\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}\right] + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Evaluate: $\int_0^4 \sqrt{16 - x^2}\,dx$

From Problem 15, $\int_0^4 \sqrt{16-x^2}\,dx$

$$= \left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right]_0^4$$

$$= \left[8\sin^{-1}1 + 2\sqrt{0}\right] - \left[8\sin^{-1}0 + 0\right]$$

$$= 8\sin^{-1}1 = 8\left(\frac{\pi}{2}\right)$$

$$= \mathbf{4\pi} \text{ or } \mathbf{12.57}$$

Determine: $\int \frac{1}{(a^2+x^2)}\,dx$

Let $x = a\tan\theta$ then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\,d\theta$

Hence $\int \frac{1}{(a^2+x^2)}\,dx$

$$= \int \frac{1}{(a^2 + a^2\tan^2\theta)}(a\sec^2\theta\,d\theta)$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta} \text{ since } 1+\tan^2\theta = \sec^2\theta$$

$$= \int \frac{1}{a}d\theta = \frac{1}{a}(\theta) + c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\frac{x}{a}$

Hence $\int \frac{1}{(a^2+x^2)}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Evaluate: $\int_0^2 \frac{1}{(4+x^2)}\,dx$

From Problem 17, $\int_0^2 \frac{1}{(4+x^2)}\,dx$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a = 2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8} \text{ or } \mathbf{0.3927}$$

Integration by parts

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where u and v are both functions of x.

Rearranging gives: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

ie
$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

or
$$\int u\,dv = uv - \int v\,du$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x dx$, $\int t \sin t\, dt$, $\int e^\theta \cos\theta\, d\theta$ and $\int x \ln x\, dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to dv'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: 'If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

Determine: $\int x \cos x \, dx$

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\frac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\,dx$, from which $v = \int \cos x\,dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int \boxed{u}\ \boxed{dv} = \boxed{u}\ \boxed{v} - \int \boxed{v}\ \boxed{du}$$
$$\int \boxed{x}\ \boxed{\cos x\,dx} = \boxed{(x)}\ \boxed{(\sin x)} - \int \boxed{(\sin x)}\ \boxed{(dx)}$$

i.e. $\int x \cos x\,dx = x \sin x - (-\cos x) + c$

$$= \mathbf{x \sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side,

i.e. $\frac{d}{dx}(x \sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$

using the product rule

$= x \cos x$, which is the function being integrated

Find: $\int 3te^{2t}\,dt$

Let $u = 3t$, from which, $\frac{du}{dt} = 3$, i.e. $du = 3\,dt$ and

let $dv = e^{2t}\,dt$, from which, $v = \int e^{2t}\,dt = \frac{1}{2}e^{2t}$

Substituting into $\int u\,dv = u\,v - \int v\,du$ gives:

$$\int 3te^{2t}dt = (3t)\left(\frac{1}{2}e^{2t}\right) - \int\left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\int e^{2t}\,dt$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + c$$

Hence $\int 3te^{2t}\,dt = \frac{3}{2}e^{2t}\left(t - \frac{1}{2}\right) + c,$

which may be checked by differentiating.

Determine: $\int x^2 \sin x\,dx$

Let $u = x^2$, from which, $\frac{du}{dx} = 2x$, i.e. $du = 2x\,dx$, and

let $dv = \sin x\,dx$, from which, $v = \int \sin x\,dx = -\cos x$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2 \sin x\,dx = (x^2)(-\cos x) - \int(-\cos x)(2x\,dx)$$

$$= -x^2\cos x + 2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

Find: $\int x \ln x\,dx$

The logarithmic function is chosen as the 'u part' Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$ i.e. $du = \frac{dx}{x}$

Letting $dv = x\,dx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x \ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $$\int x \ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

or $$\frac{x^2}{4}(2\ln x - 1) + c$$

Find: $\int e^{ax} \cos bx\,dx$

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u = e^{ax}$, from which $\frac{du}{dx} = ae^{ax}$, i.e. $du = ae^{ax}\,dx$

and let $dv = \cos bx\,dx$, from which,

$$v = \int \cos bx\,dx = \frac{1}{b}\sin bx$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int e^{ax}\cos bx\,dx$$

$$= (e^{ax})\left(\frac{1}{b}\sin bx\right) - \int \left(\frac{1}{b}\sin bx\right)(ae^{ax}dx)$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[\int e^{ax}\sin bx\,dx\right] \qquad (1)$$

$\int e^{ax} \sin bx\, dx$ is now determined separately using integration by parts again:

Let $u = e^{ax}$ then $du = ae^{ax}\, dx$, and let $dv = \sin bx\, dx$, from which

$$v = \int \sin bx\, dx = -\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\int e^{ax} \sin bx\, dx = (e^{ax})\left(-\frac{1}{b}\cos bx\right) - \int \left(-\frac{1}{b}\cos bx\right)(ae^{ax}\, dx)$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\, dx$$

Substituting this result into equation (1) gives:

$$\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\, dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\int e^{ax}\cos bx\, dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

i.e. $$\left(\frac{b^2 + a^2}{b^2}\right)\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{b^2}(b\sin bx + a\cos bx)$$

Hence $\int e^{ax} \cos bx \, dx$

$$= \left(\frac{b^2}{b^2 + a^2}\right)\left(\frac{e^{ax}}{b^2}\right)(b \sin bx + a \cos bx)$$

$$= \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c$$

Using a similar method to above, that is, integrating by parts twice, the following result may be proved:

$$\int e^{ax} \sin bx \, dx$$

$$= \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c \qquad (2)$$

Evaluate $\int_0^{\frac{\pi}{4}} e^t \sin 2t \, dt$, correct to 4 decimal places

Comparing $\int e^t \sin 2t \, dt$ with $\int e^{ax} \sin bx \, dx$ shows that $x = t$, $a = 1$ and $b = 2$.
Hence, substituting into equation (2) gives:

$$\int_0^{\frac{\pi}{4}} e^t \sin 2t \, dt$$

$$= \left[\frac{e^t}{1^2 + 2^2}(1 \sin 2t - 2 \cos 2t)\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}\left(\sin 2\left(\frac{\pi}{4}\right) - 2 \cos 2\left(\frac{\pi}{4}\right)\right)\right] - \left[\frac{e^0}{5}(\sin 0 - 2 \cos 0)\right]$$

$$= \left[\frac{e^{\frac{\pi}{4}}}{5}(1 - 0)\right] - \left[\frac{1}{5}(0 - 2)\right] = \frac{e^{\frac{\pi}{4}}}{5} + \frac{2}{5}$$

$= \mathbf{0.8387}$, correct to 4 decimal places

POSSIBLE QUESTIONS

Class : **I – B.Sc. Mathematics**
Subject Name : **Calculus**
Subject Code : **17MMU101**

UNIT – III

2 Mark Questions:

1. Write down the surface area formula for the Revolution about the X-axis.
2. Define surface area of revolution.
3. Define a length of a parameterized curve .
4. Write down a formula for finding Volumes using Disks and Washers methods.
5. Convert the polar equation to Cartesian equation for $r = \frac{25}{2sin\theta - 3cos\theta}$

6Mark Questions :

1. Find the Volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines y = 1 , x = 4 about the line y = 1.
2. Find the area of the surface generated by revolving the curve $y = x^3$, $0 \leq x \leq \frac{1}{2}$ about the x-axis.
3. Find the Volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line x = 3 about the line x = 3.
4. Find arc length for the circumference of a circle of radius a form the parametric equations $x = a\cos t$, $y = asint$ $(0 \leq t \leq 2\pi)$.
5. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x – axis to generate a solid. Find the volume of the solid.
6. In a disastrous first flight, an experimental paper airplane follows the trajectory of the particle in $x = t - 3\ sint$, $y = 4 - 3\ cost$, $t \geq 0$. but crashes into a wall at a time t = 10.
   i) At what times was the airplane flying horizontally?
   ii) At what time was it flying vertically?
7. Use Cylindrical shells to find the volume of the solid generate when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x – axis is revolved about the y – axis.
8. Find $\frac{d^2y}{dx^2}$ for the parametric equation $x = t - t^2\ and\ y = t - t^3$.
9. Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$ about the x-axis .
10. Find the length of the asteroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$.

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: CALCULUS **Subject Code: 17MMU101**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
The volume of the cylinder is _____	base - height	base x height	2(base + height)	(base x height) / 2	base x height
A function with a continuous first derivative is said to be smooth and its graph is called ______	smooth curve	length	smooth plane	smooth derivative	smooth curve
If a right cylinder is generated by translating a region of area A through a distance h, then h is called	circumference	base	height	length	height
A function with a continuous first derivative is said to be	length	smooth derivative	smooth	smooth curve	smooth
A piece of cone is called a	frustum	surface	area	radil	frustum
(base circumference x slant height) / 2 =	volume of cone	lateral surface area	volume of solid	area of revolution	lateral surface area
Volume of a right circular cylinder is	πr^2	$2\pi r^2 h$	$2\pi r$	$\pi r^2 h$	$\pi r^2 h$
_______ is a solid that generayte when a plane region is translated along a line or axis that is perpendicular to the region	sphere	right cylinder	cone	pyramid	right cylinder
A right cylinder is a solid that generayte when a plane region is translated along a line or axis that is ______ to the region	perpendicular	bounded	parallel	linear	perpendicular
The volume of a solid can be obtained by integrating the ________ from one end of the solid to the other .	length	height	cross sectional area	surface area	cross sectional area
volume of a sphere is	$4/3\ \pi r^3$	$1/2\ \pi r^2 h$	$\pi r^2 h$	$2\pi r$	$4/3\ \pi r^3$
_______ is a solid enclosed by two concentric right circular cylinders	right cylinder	surface area	cylindrical shell	cone	cylindrical shell
volume of a cylindrical shell = ________	2π	πr^2	$2\pi r^2 h$	$2\pi r$	2π
A_________ is a surface that is generated by revolving a plane curve about an axis thatb lies in the same plane as the curve.	lateral surface area	surface of revolution	area of revolution	cross sectional area	surface of revolution
The direction in which the graph of a pair of parametric equation is traced as the parameter increaseas is called the	parametric curve	cross sectional area	orientation	surface of revolution	orientation
A curve with an orientation imposed on it by a set of parametric equation is called	orientation	surface of revolution	parametric curve	area of revolution	parametric curve
An tangent lines to the parametric curve is	2(dy"/dx")	dy'/dx'	dy/dx	dx/dy	dy/dx
The curve represented by the parametric equations $x = t^2$ and $y = t^3$ is called	ellipse	semicubical parabola	hyperbola	parabola	semicubical parabola
If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$ is called a equation of ______	hyperbola	parabola	cycloid	solid	cycloid
The parametric equation of _______ is x = acost and y = asint	ellipse	circle	hyperbola	parabola	circle
The parametric equation of _______ is x = acost and y = bsint	hyperbola	parabola	ellipse	circle	ellipse
If x = sect and y = tan t the find dy/dx	1/tan t	sect / tant	tant / sect	1/sect	sect / tant
The parametric equation of _______ is x = asect and y = btant	hyperbola	parabola	ellipse	circle	hyperbola
If f (x) = x + sin x , then f ′(x) =	sin x − x cos x	1+ cos x	cos x	1− cos x	1+ cos x
An horizontal tangent for parametric equations	dy/dt =0	dy/dx =0	dx/dt =0	dx/dy=0	dy/dt =0
An vertical tangent for parametric equations	dy/dt =0	dy/dx =0	dx/dt =0	dx/dy=0	dx/dt =0
A cylindrical shell is a solid enclosed by two concentric __________	right cylinder	cone	right circular cylinders	parametric curve	right circular cylinders
The parametric equation of a circle is	x = a cost , y = a sint	x = a tant , y = a sect	x = a cost , y = a sect	x = a cost , y = b sint	x = a cost , y = a sint
The parametric equation of a ellipse is	x = a tant , y = a sect	x = a cost , y = a sint	x = a tant , y = a sect	x = a cost , y = b sint	x = a cost , y = b sint
The parametric equation of a hyperbola is	x = a cost , y = a sint	x = a cost , y = b sint	x = a sect , y = b tant	x = a tant , y = a sect	x = a sect , y = b tant
A cylinderical shell is a solid enclosed by_____concentric right circular cylinders	four	two	three	one	two
A solid of ______ is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region	cone	revolution	perpendicular	parallel	revolution
The line of a solid of revolution is a _____ of revolution	ray	distance	axis	plane	axis
A function f is smooth on [a, b] if f' is______ on [a, b].	discontinuous	parallel	perpendicular	continuous	continuous

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DEPARTMENT OF MATHEMATICS

Subject: CALCULUS	**Semester :I**	**L T P C**
Subject Code: 17MMU101	**Class : I- B.Sc Mathematics**	**4 0 0 4**

UNIT IV

Concavity and Inflection points, asymptotes. Techniques of sketching conics, reflection properties of conics, rotation of axes and second degree equations, classification into conics using the discriminant, polar equations of conics.

TEXT BOOK

1. Strauss M.J., Bradley G.L.,and Smith K. J., (2007). Calculus, Third Edition ,Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi.

REFERENCES

1. Thomas G.B., and Finney R.L., (2005). Calculus, Ninth Edition, Pearson Education, Delhi.
2. Anton H., Bivens I., and Davis S.,(2002). Calculus, Seventh Edition, John Wiley and Sons (Asia) P. Ltd., Singapore.

MULTIPLE INTEGRAL:

INTRODUCTION: When a function f(x) is integrated with respect to x between the limits a and b, we get the definite integral $\int_a^b f(x)dx$

If the integrand is a function f(x,y) and if it is integrated with respect to x and y repeatedly between the limits x_0 and x_1 (for x) between the limits y_0 and y_1(for y)

We get a *double integral* that is denoted by the symbol $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy$

Extending the concept of double integral one step further, we get the *triple integral* $\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\,dx\,dy\,dz$

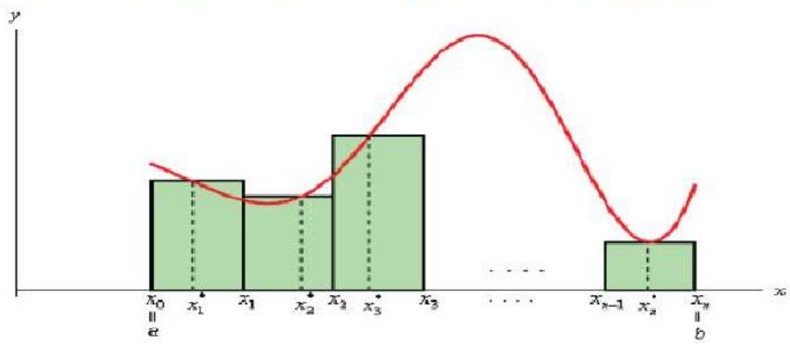
EVALUATION OF DOUBLE INTEGRALS

Before starting on double integrals let's do a quick review of the definition of a definite integrals for functions of single variables. First, when working with the

Integral, $$\int_a^b f(x)\,dx$$

We think of x's as coming from the interval $a \le x \le b$. For these integrals we can say that we are integrating over the interval $a \le x \le b$. Note that this does assume that $a < b$, however, if we have $b < a$ then we can just use the interval $b \le x \le a$.

Now, when we derived the definition of the definite integral we first thought of this as an area problem. We first asked what the area under the curve was and to do this we broke up the interval $a \le x \le b$ into n subintervals of width Δx and choose a point, x_i^*, from each interval as shown below,



Each of the rectangles has height of $f(x_i^*)$ and we could then use the area of each of these rectangles to approximate the area as follows.

$$A \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_i^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

To get the exact area we then took the limit as n goes to infinity and this was also the definition of the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy$, we first integrate f(x,y) with respect to x partially, treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be a function of y. Then we integrate this function of y with respect to y between the limits y_0 and y_1 as usual.

1. **Evaluate** $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}} dydx$

Solution: Let I = $\int_0^2 \left(xe^{\frac{y}{x}}\right)_0^{x^2} dx = \int_0^2 x(e^x - 1)dx$

$$= \left[(x)(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^2$$

$$= 2e^2 - 4 - e^2 + 2 + 1$$

$$= (2)(e^2 - 2) - [(e^2 - 2) - (1)]$$

$$= e^2 - 1$$

2. **Evaluate:** $\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$

Solution:Let I = $\int_0^a y \int_0^{\sqrt{a^2-x^2}} dx = \int_0^a \sqrt{a^2 - x^2}dx$

$$= \left[\frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$$

$$\frac{= a^2}{2} sin^{-1}(1)$$

$$= \frac{\pi a^2}{4}$$

1. **Evaluate:** $\iint xy(x + y)\, dxdy$ **over the area between y =** x^2 **and y = x.**

Solution:The limits are: x varies from 0 to 1 and y varies from x^2 to x.

$$I = \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\, dydx$$

$$= \int_0^1 \left(x^2 \left(\frac{y^2}{2}\right)_{x^2}^{x} + x \left(\frac{y^3}{3}\right)_{x^2}^{x} \right) dx$$

$$= \int_0^1 x \left\{ \left(\frac{x^3}{2} + \frac{x^3}{3}\right) - \left(\frac{x^5}{2} + \frac{x^6}{3}\right) \right\} dx$$

$$= \left(\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right)_0^1$$

$$= \frac{3}{56}$$

2. Evaluate: $\iint x^2y^2dxdy$ over the region in the first quadrant of the circle $x^2+y^2=1$.

Solution: In the given region, y varies from 0 to $\sqrt{1-x^2}$ and x varies from 0 to 1.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2y^2\, dydx$$

$$= \int_0^1 x^2 \left(\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}} dx = \frac{1}{3}\int_0^1 x^2(1-x^2)^{3/2}\, dx$$

Put x = sinθ. Then dx = cosθdθ. θ varies from 0 to π/2.

$$\therefore I = \frac{1}{3}\int_0^{\pi/2} sin^2\theta cos^4\theta\, d\theta = \frac{1}{3}\int_0^{\pi/2} (1-cos^2\theta)\, cos^4\theta d\theta = \frac{1}{3}\left[\int_0^{\pi/2} cos^4\theta d\theta - \int_0^{\pi/2} cos^6\theta d\theta\right]$$

$$= \frac{1}{3}\left[\frac{3}{4}\frac{1}{2}\frac{\pi}{2} - \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right] = \frac{\pi}{96}$$

3. Evaluate: $\int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2) dydx\right]$

Solution : Let $I = \int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2) dydx\right] = \int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2) dydx\right] = \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_x^{\sqrt{x}} dx$

$$= \int_0^1 \left[\left(\frac{x^3}{2} + \frac{x.xy^{3/2}}{3}\right) - \left(\frac{x^4}{2} + \frac{x^4}{3}\right)\right] dx = \left[\frac{x^4}{8} + \frac{x^{7/2}}{(7/2)(3)} - \frac{5}{6}\left(\frac{x5}{5}\right)\right]_0^1$$

$$= \left(\frac{1}{8} + \frac{2}{21} + \frac{1}{6}\right) - (0) = \left(\frac{21+16-28}{168}\right) = \frac{9}{168} = \frac{3}{56}$$

DOUBLE INTEGRATION IN POLAR COORDINATES:

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)dr\, d\theta$, we first integrate w.r.to r between the limits r_1 and r_2. Keeping θ is fixed and the resulting expression is integrated w.r.to θ from θ_1 to θ_2.

In this integral r_1 and r_2 $are\ functions\ of\ \theta$ and $\theta_1, \theta_2 are\ constants.$

1. **Evaluate:** $\int_0^{\pi/2} \int_0^{\infty} \frac{r dr d\theta}{(r^2+a^2)^2}$

Solution: $Let\ I = \int_0^{\pi/2} \left[\int_0^{\infty} \frac{rdr}{(r^2+a^2)^2}\right] d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{-1}{(r^2+a^2)^2}\right)_0^{\infty} d\theta$$

$$= \frac{1}{2a^2}(\theta)_0^{\pi/2}$$

$$= \frac{\pi}{4a^2}$$

2. **Evaluate:** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{2cos\theta} r^2 dr d\theta$

Solution: Let I = $\int_{-\pi/2}^{\pi/2} \left(\frac{r^3}{3}\right)_0^{2cos\theta} d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2} (8cos^3\theta - 0)\, d\theta$

$$= \frac{8}{3} \times 2\int_0^{\pi/2} cos^3\theta\, d\theta$$

$$= \frac{16}{3}\frac{2}{3} = \frac{32}{9}$$

1. **Evaluate:** $\iint r^2 sin\theta dr d\theta$ **over the cardioids r = a (1+cosθ).**

Solution: The limits of r: 0 to a (1+cosθ) and The limits of θ: 0 to π.

$I = \int_0^{\pi} \int_0^{a(1+cos\theta)} r^2 sin\theta\, dr d\theta = \int_0^{\pi} \left(\frac{r^3}{3}\right)_0^{a(1+cos\theta)} sin\theta d\theta$

$$= \frac{a^3}{3}\int_0^{\pi} sin\theta(1+cos\theta)^3 d\theta$$

Put 1+cosθ = t then –sinθ dθ = dt

When $\theta = 0$, $t = 2$

When $\theta = \pi$, $t = 0$.

$$\therefore I = \frac{a^3}{3}\int_0^2 t^3\,dt = \frac{a^3}{3}\left(\frac{a^4}{4}\right)_0^2 = \frac{4a^3}{3}$$

2. **Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dxdy$ using polar cordinates**

Solution: $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$ are the polar coordinates for the above integral

$$I = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\ \frac{1}{2}\,d(r^2)d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{2}\left[-e^{-r^2}\right]_0^\infty d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[e^{-r^2}\right]_0^\infty d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}}[0-1]d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4}$$

CHANGE THE ORDER OF INTEGRATION:

The double integral $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$ will take the form $\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$ when the order of integration is changed. This process of converting a given double integral into its equivalent double integral by changing the order of integration is often called change of order of integration. To effect the change of order of integration, the region of integration is identified first, a rough sketch of the region is drawn and then the new limits are fixed.

1. **Find the limits of integration in the double integral $\iint_R f(x,y)dxdy$, where R is in the first quadrant and bounded by: $x = 0, y = 0, x + y = 1$.**

   Solution: The limits are: y varies from 0 to 1 and x varies from 0 to 1-y.

2. **Change the order of integration $\int_0^a \int_y^a f(x,y)dxdy$**

   Solution: The given region of integration is bounded by y=0, y=a, x=y & x=a.

   After changing the order, we have, I = $\int_0^a \int_0^x f(x,y)dydx$

3. **Change the order of integration for the double integral $\int_0^1 \int_0^x f(x,y)dxdy$**

   Solution: $\int_0^1 \int_y^1 f(x,y)dxdy$

1. **Change the order of integration in I = $\int_0^1 \int_{x^2}^{2-x} xy\,dydx$ and hence evaluate it.**

Solution: Let I = $\int_0^1 \int_{x^2}^{2-x} xy\,dydx$

The given region of integration is bounded by x=0, x=1, $y=x^2$ and x+y=2.

In the given integration x is fixed and y is varying.

So, after changing the order we have to keep y fixed and x should vary.

After changing the order we've two regions R_1 & R_2

I = I_1+I_2

I = $\int_0^1 \int_0^{\sqrt{y}} xy\,dxdy + \int_1^2 \int_1^{2-x} xy\,dxdy$

$$=\frac{1}{2}\left[\int_0^1 (x^2y)_0^{\sqrt{y}}\,dy \int_1^2 (x^2y)_0^{2-y}\,dy\right]$$

$$=\frac{1}{2}\left[\int_0^1 y^2\,dy + \int_1^2 y(2-y)^2\,dy\right]$$

$$=\frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1 + \left(2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right)_1^2\right] = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dydx$ by changing the order of integration.

Solution: The given region is bounded by x=0, x=1, y=x and $x^2+y^2=2$.

$I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dydx$

After changing the order we've,

The region R is splinted into two regions R_1& R_2.

In R_1: limits of x: 0 to y & limits of y: 0 to 1

In R_2: limits of x: 0 to $\sqrt{2-y^2}$ & limits of y: 1 to $\sqrt{2}$

$I = I_1 + I_2$

$I_1 = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dxdy$

$$= \int_0^1 \left(\sqrt{x^2+y^2}\right)_0^y dy = (\sqrt{2}-1)\int_0^1 y\,dy = (\sqrt{2}-1)\left(\frac{1}{2}\right)$$

$I_2 = \int_0^{\sqrt{2}} \int_x^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dxdy$

$$= \int_1^{\sqrt{2}} \left(\sqrt{x^2+y^2}\right)_0^{\sqrt{2-y^2}} dy = \int_1^{\sqrt{2}} (\sqrt{2}-y)dy = (\sqrt{2}-1)\left(\frac{y2}{2}\right)_0^1 + \sqrt{2}(y)_1^{\sqrt{2}} - \left(\frac{y2}{2}\right)_1^{\sqrt{2}}$$

$$= (2-\sqrt{2}) - \frac{1}{2}$$

$I = (\sqrt{2}-1)\left(\frac{1}{2}\right) + (2-\sqrt{2}) - \frac{1}{2}$

$$= 1 - \frac{1}{\sqrt{2}}$$

3. Evaluate by changing the order of integration in $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

Solution: Let $I = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dydx$

The given region of integration is bounded by x=0, x=4, $y = x^2/4$, $y^2 = 4x$

After changing the order we've

Limits of x: $y^2/4$ to $2\sqrt{y}$

Limits of y: 0 to 4

$I = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dxdy = 16/3.$

4. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)dydx$

Solution: Given $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)dydx$

The given region of integration is bounded by x=0, x=1, $y=x^2$ and x+y=2

In the given integration x is fixed and y is varying

So, after changing the order we have to keep y fixed and x should vary.

After changing the order we have two regions R_1 & R_2

$I = I_1 + I_2$

$$I = \int_0^1 \int_0^{\sqrt{y}} f(x,y)dxdy + \int_1^2 \int_1^{2-x} f(x,y)dxdy$$

EVALUATION OF TRIPLE INTEGRALS

To evaluate $\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x,y,z)\, dx\, dy\, dz$, we first integrate $f(x,y,z)$ with respect to x , treating y and s as constants temporarily. The limits x_0 and x_1 may be constants or functions of y and z, so that the resulting function got after the innermost integration may be a function of y and z. Then we perform the middle integration with respect to y, treating z as a constant temporarily. The limits y_0 and y_1 may be constants or functions of z, so that the resulting function got after the middle integration may be a function of z only. Finally we perform the outermost integration with respect to z between the constant limits z_0 and z_1.

1. Evaluate: $\int_0^a \int_0^b \int_0^c (x+y+z)\, dzdydx$

Solution: Let I = $\int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dydx$

$$= \int_0^a \int_0^b \left(cz + cy + \frac{c^2}{2}\right) \; dydx$$

$$= \int_0^a \left(cxy + \frac{cy^2}{2} + \frac{c^2y}{2}\right)_0^b dx$$

$$= \int_0^a \left(bcx + \frac{cb^2}{2} + \frac{c^2b}{2}\right) \; dx$$

$$= \left(\frac{bcx^2}{2} + \frac{cb^2x}{2} + \frac{c^2bx}{2}\right)_0^a$$

$$= \frac{bca^2}{2} + \frac{cb^2a}{2} + \frac{c^2ba}{2}$$

$$= \frac{abc}{2}(a+b+c)$$

2. Evaluate: $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy$

Solution: Let I= $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dxdy$

$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \left(\frac{(a^2-y^2)}{2}\right) sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi a^3}{6}$$

3. Evaluate: $\int_0^1 \int_0^{\sqrt{1^2-x^2}} \int_0^{\sqrt{1^2-x^2-y^2}} xyzdzdydx$

Solution: Let I=$\int_0^1 \int_0^{\sqrt{1^2-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{1^2-x^2-y^2}} dzdydx$

$$= \frac{1}{2}\int_0^1 x\left[\int_0^{\sqrt{1^2-x^2}} y(1-x^2-y^2)dy\right]dx$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1^2-x^2)^2}{2} - \frac{(1^2-x^2)^2}{4}\right]dx = \frac{1}{8}\int_0^1 x\,(1^2-x^2)^2 dx$$

$$= \frac{-1}{16}\int_0^1 (1^2-x^2)^2(-2xdx) = \frac{-1}{16}\left[\frac{(1^2-x^2)^3}{3}\right]_0^1 = \frac{1}{48}$$

4. Evaluate: $\int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^4 sin\varphi drd\varphi d\theta$

Solution: Let I =$\int_0^{2\pi} d\theta \int_0^{\pi} \left(\frac{r^5}{5}\right)_0^a sin\varphi d\varphi$

$$= \frac{a^5}{5}\int_0^{2\pi} d\theta \int_0^{\pi} sin\varphi d\varphi = \frac{a^5}{5}\int_0^{2\pi} (-cos\varphi)_0^{\pi}\, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$$

1. Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z} dzdydx$

Solution : $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z} dzdydx = \int_0^{\log a}\int_0^{x} [e^{x+y+z}]_0^{x+y} dydx = \int_0^{\log a}\int_0^{x} (e^{2(x+y)} - e^{x+y})dydx$

$$= \int_0^{\log a} \left(\frac{e^{2(x+y)}}{2} - e^{x+y} \right)_0^x dx = \int_0^{\log a} \left[\left(\frac{1}{2} e^{4x} - e^{2x} \right) - \left(\frac{e^{2x}}{2} - e^x \right) \right] dx$$

$$= \int_0^{\log a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) dx = \left[\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right]_0^{\log a}$$

$$= \left(\frac{1}{8} e^{4\log a} - \frac{3}{4} e^{2\log a} + e^{\log a} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$= \frac{1}{8} a^4 - \frac{3}{4} a^2 + a - \frac{3}{8}$$

2. Evaluate $\int_0^a \int_0^b \int_0^c \left(x^2 + y^2 + z^2 \right) dx dy dz$

Solution : $I = \int_0^a \int_0^b \left[\frac{x^3}{3} + y^2 x + z^2 x \right]_0^c dy dz \quad = \int_0^a \int_0^b \left[\frac{c^3}{3} + cy^2 + cz^2 \right] dy dz$

$$= \int_0^a \left[\frac{c^3 y}{3} + \frac{cy^3}{3} + cyz^2 \right]_0^b dz \quad = \int_0^a \left[\frac{c^3 b}{3} + \frac{cb^3}{3} + cbz^2 \right] dz$$

$$= \left[\frac{c^3 bz}{3} + \frac{cb^3 z}{3} + \frac{cbz^3}{3} \right]_0^a = \frac{c^3 ba}{3} + \frac{cb^3 a}{3} + \frac{cba^3}{3} = \frac{abc}{3} [c^2 + b^2 + a^2]$$

POSSIBLE QUESTIONS

Class : **I – B.Sc. Mathematics**
Subject Name : **Calculus**
Subject Code : **17MMU101**

UNIT – IV

2 Mark Questions:

1. Find where the graph of $f(x) = x^3 + 3x + 1$ is concave up and it is concave down.
2. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.
3. Find the focus and directrix of the parabola $y^2 = 10x$.
4. Find the inflection point for the function $f(x) = 3x^5 - 5\,x^3 +\ 2$.
5. Define discriminant test.

6Mark Questions :

1. Sketch the graph of the parabolas $i)\ x^2 = 12y \quad ii)\ y^2 + 8x = 0$ and show that focus and directrix of each.
2. Identify and sketch the curve $153x^2 - 193xy + 97y^2 - 30x - 40y - 200 = 0$.
3. Sketch the graph of the ellipse $i)\frac{x^2}{9} + \frac{y^2}{16} = 1\ ii)\ x^2 + 2y^2 = 4$. and showing the foci of each.
4. Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3 , 0) and the line x = 1 as the corresponding directrix.
5. Find the equation of the curve $x^2 - xy + y^2 - 6 = 0$ in $x'y'$ - coordinates. if the coordinate axes are rotated through an angle of $\theta = 45°$
6. Find the constants a, b and c for the ellipse $r = \frac{6}{2+cos\theta}$
7. Describe the graph of the equation $16x^2 + 9y^2 - 64x - 54y + 1 = 0$.
8. Find a polar equation for an ellipse with semimajor axis 39.44 AU and eccentricity size of pluto's orbit around the sun.
9. Identify and sketch the curve xy = 1
10. i) Find the directrix of the parabola $r = \frac{25}{10+10\ cos\theta}$

    ii) Find the equation for the hyperbola with eccentricity 3/2 and directrix x = 2.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: CALCULUS **Subject Code: 17MMU101**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
Which one is the example for conic section ?	parabola	solid	triangle	rectangle	parabola
_______ is the set of all points in the plane that are equiditant from a fixed line and fixed point not on the line .	ellipse	hyperbola	parabola	circle	parabola
_______ is the set of all points in the plane, the sum of whose distance from two fixed point is a given positive constant that is greater than the distance between the fixed point.	ellipse	hyperbola	parabola	circle	ellipse
_______ is the set of all points in the plane, the sum of whose distance from two fixed point is a given positive constant that is less than the distance between the fixed point.	ellipse	hyperbola	parabola	circle	hyperbola
In a ellipse the midpoint of the line segment joining the foci is called ______	vertices	axis	symmetry	center	center
In an ellipse ,the end point of the major axis is called ______	minor axis	vertices	symmetry	center	vertices
The midpoint of the line segment joining the foci is called the _____ of the hyperbola	vertices	axis	symmetry	center	center
The hyperbola intersect the focal axis at two points is called the ______	center	vertices	axis	symmetry	vertices
In a hyperbola the line through the center that is perpendicular to the focal axis is called _____	asymptotes	vertices	conjugate axis	standard position	conjugate axis
______ is the set of points in a plane whose distance from a given fixed point in the plane is constant.	ellipse	hyperbola	parabola	circle	circle
The fixed point is the _____ of the circle.	center	vertices	radius	standard position	center
Find the focus of the parabola $y^2 = 10x$	(0 , 5/2)	(-5/2 , 0)	(5/2 , 0)	(0 , -5/2)	(5/2 , 0)
The eccentricity of a parabola is	e < 1	e >1	e = 0	e =1	e =1
The eccentricity of a hyperbola is	e < 1	e >1	e = 0	e =1	e >1
Find the radius of the circle r = 4 cos θ	2	8	4	16	2
The eccentricity of a ellipse is	e < 1	e >1	e = 0	e =1	e < 1
If $B^2 - 4AC = 0$ then it is called	ellipse	hyperbola	parabola	circle	parabola
If $B^2 - 4AC < 0$ then it is called	ellipse	hyperbola	parabola	circle	ellipse
If $B^2 - 4AC > 0$ then it is called	ellipse	hyperbola	parabola	circle	hyperbola
An ellipse is the set of all points in the plane, the sum of whose distance from two fixed point is a given positive constant that is _______ the distance between the fixed point.	greater than	equal to	less than	not equal to	greater than
An hyperbola is the set of all points in the plane, the sum of whose distance from two fixed point is a given positive constant that is _______ the distance between the fixed point.	greater than	equal to	less than	not equal to	less than
If e =1 then it is the eccentricity of a _______	ellipse	hyperbola	parabola	circle	parabola
If e > 1 then it is the eccentricity of a _______	ellipse	hyperbola	parabola	circle	hyperbola
If e < 1 then it is the eccentricity of a _______	ellipse	hyperbola	parabola	circle	ellipse
The constant distance is the _____ of the circle	center	vertices	radius	standard position	radius
Find the radius of the circle r = 6 cos θ	2	3	6	36	3
In a ellipse the midpoint of the line segment joining the_____is called center	minor axis	foci	major axis	vertices	foci
Which one is the not a example for conic section ?	rectangle	ellipse	hyperbola	parabola	rectangle
Intersection of two straight lines is --------------	Surface	Curve	Plane	Point	Plane
Plane is a --------------- surface	1 -D	2 -D	3 - D	Dimensionless	2 -D
The angle between the asymptotes of a rectangular hyperbola is	30	45	60	90	90
The intersection of a cone with a plane gives	Point	Line	Conic Section	Two points	Point
A point where the graph pf a function has a tangent line and where the concavity changes is callled _______	hyperbolic	inflection point	concavity	saddle point	inflection point
A point where the graph pf a function has a _____ and where the concavity changes is callled a point of inflection	circle	tangent line	staright line	curve	tangent line
The slope of a graph increases on an interval where the graph is ______	local maximum	concave up	local minimum	concave down	concave up

The slope of a graph decreases on an interval where the graph is ________	concave up	local minimum	concave down	local maximum	concave down
If the curve $y = x^4$ has no inflection point at _____	x = 2	x =0	x = 1	x = -1	x =0
A point P(c , f (c)) on a ______ is called an inflection point	straight line	curve	cone	circle	curve
Find the vertical asymptote for f(x) = log (2 − x).	x = 2	x =0	x = 1	x = -1	x = 2
Find the horizontal asymptote of $y = e^x + 5$.	y =1	y = -5	y =5	y =0	y =5
______can also be thought of as a tangent to the curve infinity.	point of inflection	asymptotes	concave down	local maximum	asymptotes

KARPAGAM ACADEMY OF HIGHER EDUCATION
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DEPARTMENT OF MATHEMATICS

Subject: CALCULUS	**Semester :I**	**L T P C**
Subject Code: 17MMU101	**Class : I- B.Sc Mathematics**	**4 0 0 4**

UNITV

Introduction to vector functions, operations with vector-valued functions, limits and continuity of vector functions, differentiation and integration of vector functions, tangent and normal components of acceleration, modeling ballistics and planetary motion, Kepler's second law.

Text Book

T1 : M.J.Strauss., G.L.Bradley and K.J.Smith.,(2007). Calculus, third edition , dorling Kindersley(India) Pvt Ltd. (Pearson Edition), Delhi.

Cal ulus with ve tor fun tions

A vector function $\mathrm{r}(t) = (f(t), g(t), h(t))$ is a function of one variable—that is, there is only one "input" value. What makes vector functions more complicated than the function $y = f(x)$ that we studied in the first part of this book is of course that the "output" values are now three-dimensional vectors instead of simply numbers. It is natural to wonder if there is a corresponding notion of derivative for vector functions. In the simpler case of a function $y = s(t)$, in which t represents time and $s(t)$ is position on a line, we have seen that the derivative $s'(t)$ represents velocity; we might hope that in a similar way the derivative of a vector function would tell us something about the velocity of an object moving in three dimensions.

One way to approach the question of the derivative for vector functions is to write down an expression that is analogous to the derivative we already understand, and see if we can make sense of it. This gives us

$$\begin{aligned}
\mathrm{r}'(t) &= \lim_{\Delta t\to 0} \frac{\mathrm{r}(t+\Delta t) - \mathrm{r}(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{(f(t+\Delta t) - f(t), g(t+\Delta t) - g(t), h(t+\Delta t) - h(t))}{\Delta t} \\
&= \lim_{\Delta t\to 0} \left(\frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right) \\
&= (f'(t), g'(t), h'(t)),
\end{aligned}$$

if we say that what we mean by the limit of a vector is the vector of the individual coordinate limits. So starting with a familiar expression for what appears to be a derivative, we find that we can make good computational sense out of it—but what does it actually mean?

We know how to interpret $\mathrm{r}(t+\Delta t)$ and $\mathrm{r}(t)$—they are vectors that point to locations in space; if t is time, we can think of these points as positions of a moving object at times that are Δt apart. We also know what $\Delta\mathrm{r} = \mathrm{r}(t+\Delta t) - \mathrm{r}(t)$ means—it is a vector that points from the head of $\mathrm{r}(t)$ to the head of $\mathrm{r}(t+\Delta t)$, assuming both have their tails at the origin. So when Δt is small, $\Delta\mathrm{r}$ is a tiny vector pointing from one point on the path of the object to a nearby point. As Δt gets close to 0, this vector points in a direction that is closer and closer to the direction in which the object is moving; geometrically, it approaches a vector tangent to the path of the object at a particular point.

Unfortunately, the vector $\Delta \mathbf{r}$ approaches 0 in length; the vector $(0, 0, 0)$ is not very informative. By dividing by Δt, when it is small, we effectively keep magnifying the length of $\Delta \mathbf{r}$ so that in the limit it doesn't disappear. Thus the limiting vector $(f'(t), g'(t), h'(t))$ will (usually) be a good, non-zero vector that is tangent to the curve.

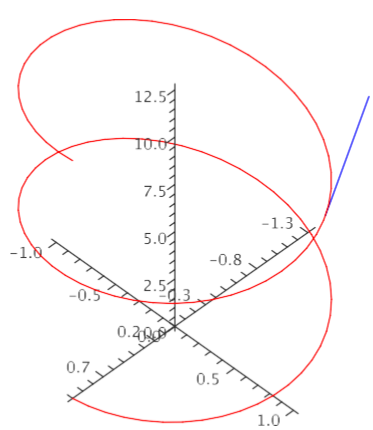
What about the length of this vector? It's nice that we've kept it away from zero, but what does it measure, if anything? Consider the length of one of the vectors that approaches the tangent vector:

$$\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right| = \frac{|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)|}{|\Delta t|}$$

The numerator is the length of the vector that points from one position of the object to a "nearby" position; this length is approximately the distance traveled by the object between times t and $t + \Delta t$. Dividing this distance by the length of time it takes to travel that distance gives the average speed. As Δt approaches zero, this average speed approaches the actual, instantaneous speed of the object at time t.

So by performing an "obvious" calculation to get something that looks like the derivative of $\mathbf{r}(t)$, we get precisely what we would want from such a derivative: the vector $\mathbf{r}'(t)$ points in the direction of travel of the object and its length tells us the speed of travel. In the case that t is time, then, we call $\mathbf{v}(t) = \mathbf{r}'(t)$ the velocity vector. Even if t is not time, $\mathbf{r}'(t)$ is useful—it is a vector tangent to the curve.

EXAMPLE 13.2.1 We have seen that $\mathbf{r} = (\cos t, \sin t, t)$ is a helix. We compute $\mathbf{r}' = (-\sin t, \cos t, 1)$, and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. So thinking of this as a description of a moving object, its speed is always $\sqrt{2}$; see figure 13.2.2. □



EXAMPLE 13.2.2 The velocity vector for $(\cos t, \sin t, \cos t)$ is $(-\sin t, \cos t, -\sin t)$. As before, the first two coordinates mean that from above this curve looks like a circle. The z coordinate is now also periodic, so that as the object moves around the curve its height oscillates up and down. In fact it turns out that the curve is a tilted ellipse, as shown in figure 13.2.3. □

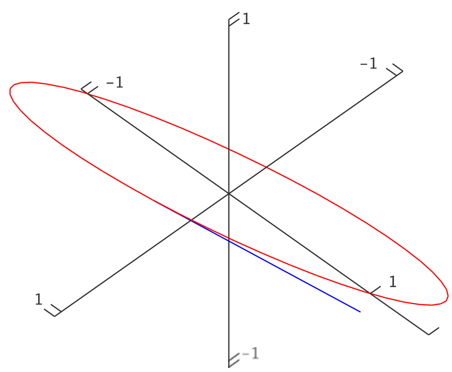


Figure 13.2.3 The ellipse $\mathbf{r} = (\cos t, \sin t, \cos t)$. (AP)

EXAMPLE 13.2.3 The velocity vector for $(\cos t, \sin t, \cos 2t)$ is $(-\sin t, \cos t, -2\sin 2t)$. The z coordinate is now oscillating twice as fast as in the previous example, so the graph is not surprising; see figure 13.2.4. □

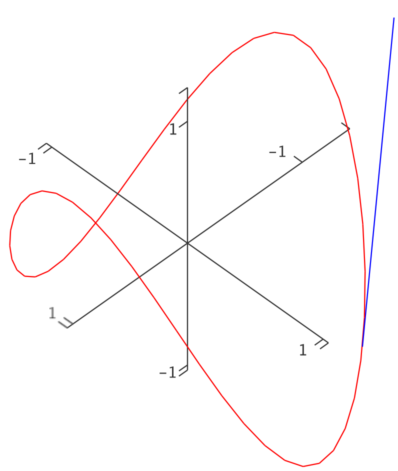


Figure 13.2.4 $(\cos t, \sin t, \cos 2t)$. (AP)

EXAMPLE 13.2.4 Find the angle between the curves $(t, 1-t, 3+t^2)$ and $(3-t, t-2, t^2)$ where they meet.

The angle between two curves at a point is the angle between their tangent vectors—any tangent vectors will do, so we can use the derivatives. We need to find the point of intersection, evaluate the two derivatives there, and finally find the angle between them.

To find the point of intersection, we need to solve the equations

$$t = 3 - u$$
$$1 - t = u - 2$$
$$3 + t^2 = u^2$$

Solving either of the first two equations for u and substituting in the third gives $3 + t^2 = (3 - t)^2$, which means $t = 1$. This together with $u = 2$ satisfies all three equations. Thus the two curves meet at $(1, 0, 4)$, the first when $t = 1$ and the second when $t = 2$.

The derivatives are $(1, -1, 2t)$ and $(-1, 1, 2t)$; at the intersection point these are $(1, -1, 2)$ and $(-1, 1, 4)$. The cosine of the angle between them is then

$$\cos\theta = \frac{-1 - 1 + 8}{\sqrt{6}\sqrt{18}} = \frac{1}{\sqrt{3}},$$

so $\theta = \arccos(1/\sqrt{3}) \approx 0.96$.

□

The derivatives of vector functions obey some familiar looking rules, which we will occasionally need.

THEOREM 13.2.5 Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable functions, $f(t)$ is a differentiable function, and a is a real number.

a. $\frac{d}{dt} a\mathbf{r}(t) = a\mathbf{r}'(t)$

b. $\frac{d}{dt}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

c. $\frac{d}{dt} f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

d. $\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$

e. $\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$

f. $\frac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

■

Note that because the cross product is not commutative you must remember to do the three cross products in formula (e) in the correct order.

When the derivative of a function $f(t)$ is zero, we know that the function has a horizontal tangent line, and may have a local maximum or minimum point. If $\mathbf{r}'(t) = \mathbf{0}$, the geometric interpretation is quite different, though the interpretation in terms of motion is similar. Certainly we know that the object has speed zero at such a point, and it may thus be abruptly changing direction. In three dimensions there are many ways to change direction; geometrically this often means the curve has a cusp or a point, as in the path of a ball that bounces off the floor or a wall.

EXAMPLE 13.2.6 Suppose that $\mathbf{r}(t) = \langle 1 + t^3, t^2, 1\rangle$, so $\mathbf{r}'(t) = \langle 3t^2, 2t, 0\rangle$. This is $\mathbf{0}$ at $t = 0$, and there is indeed a cusp at the point $(1, 0, 1)$, as shown in figure 13.2.5. □

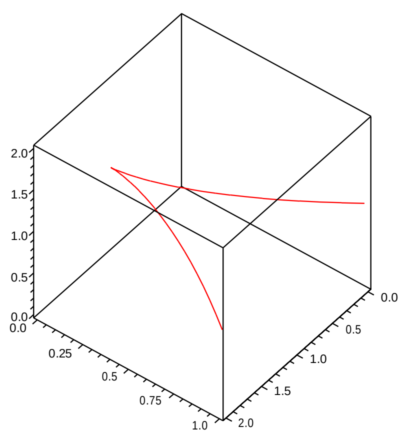


Figure 13.2.5 $\langle 1 + t^3, t^2, 1\rangle$ has a cusp at $(1, 0, 1)$. (AP)

Sometimes we will be interested in the direction of $\mathbf{r}'$ but not its length. In some cases, we can still work with $\mathbf{r}'$, as when we find the angle between two curves. On other occasions it will be useful to work with a unit vector in the same direction as $\mathbf{r}'$; of course, we can compute such a vector by dividing $\mathbf{r}'$ by its own length. This standard unit tangent vector is usually denoted by $\mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}.$$

In a sense, when we computed the angle between two tangent vectors we have already made use of the unit tangent, since

$$\cos\theta = \frac{\mathbf{r}'\cdot\mathbf{s}'}{|\mathbf{r}'||\mathbf{s}'|} = \frac{\mathbf{r}'}{|\mathbf{r}'|}\cdot\frac{\mathbf{s}'}{|\mathbf{s}'|}$$

Now that we know how to make sense of $\mathbf{r}'$, we immediately know what an antiderivative must be, namely

$$\int \mathbf{r}(t)\, dt = \left(\int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt \right),$$

if $\mathbf{r} = (f(t), g(t), h(t))$. What about definite integrals? Suppose that $\mathbf{v}(t)$ gives the velocity of an object at time t. Then $\mathbf{v}(t)\Delta t$ is a vector that approximates the displacement of the object over the time Δt: $\mathbf{v}(t)\Delta t$ points in the direction of travel, and $|\mathbf{v}(t)\Delta t| = |\mathbf{v}(t)||\Delta t|$ is the speed of the object times Δt, which is approximately the distance traveled. Thus, if we sum many such tiny vectors:

$$\sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t$$

we get an approximation to the displacement vector over the time interval $[t_0, t_n]$. If we take the limit we get the exact value of the displacement vector:

$$\lim \sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t = \int_{t_0}^{t_n} \mathbf{v}(t)\, dt = \mathbf{r}(t_n) - \mathbf{r}(t_0).$$

Denote $\mathbf{r}(t_0)$ by $\mathbf{r}_0$. Then given the velocity vector we can compute the vector function $\mathbf{r}$ giving the location of the object:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(u)\, du.$$

EXAMPLE 13.2.7 An object moves with velocity vector $(\cos t, \sin t, \cos t)$, starting at $(1, 1, 1)$ at time 0. Find the function $\mathbf{r}$ giving its location.

$$\begin{aligned}\mathbf{r}(t) &= (1, 1, 1) + \int_0^t (\cos u, \sin u, \cos u)\, du \\ &= (1, 1, 1) + (\sin u, -\cos u, \sin u)\Big|_0^t \\ &= (1, 1, 1) + (\sin t, -\cos t, \sin t) - (0, -1, 0) \\ &= (1 + \sin t, 2 - \cos t, 1 + \sin t)\end{aligned}$$

See figure13.2.6. □

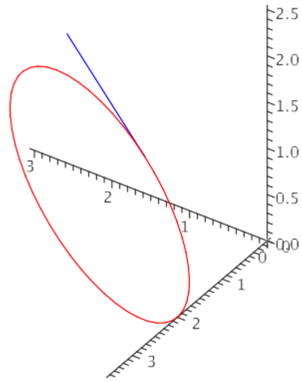


Figure 13.2.6 Path of the object with its initial velocity vector. (AP)

Exercises 13.2.

1. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle t^2, 1, t\rangle$. ⇒
2. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos t, \sin 2t, t^2\rangle$. ⇒
3. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos(e^t), \sin(e^t), \sin t\rangle$. ⇒
4. Find a vector function for the line tangent to the helix $\langle \cos t, \sin t, t\rangle$ when $t = \pi/4$. ⇒
5. Find a vector function for the line tangent to $\langle \cos t, \sin t, \cos 4t\rangle$ when $t = \pi/3$. ⇒
6. Find the cosine of the angle between the curves $\langle 0, t^2, t\rangle$ and $\langle \cos(\pi t/2), \sin(\pi t/2), t\rangle$ where they intersect. ⇒
7. Find the cosine of the angle between the curves $\langle \cos t, -\sin(t)/4, \sin t\rangle$ and $\langle \cos t, \sin t, \sin(2t)\rangle$ where they intersect. ⇒
8. Suppose that $|\mathbf{r}(t)| = k$, for some constant k. This means that $\mathbf{r}$ describes some path on the sphere of radius k with center at the origin. Show that $\mathbf{r}$ is perpendicular to $\mathbf{r}'$ at every point. Hint: Use Theorem 13.2.5, part (d).
9. A bug is crawling along the spoke of a wheel that lies along a radius of the wheel. The bug is crawling at 1 unit per second and the wheel is rotating at 1 radian per second. Suppose the wheel lies in the y-z plane with center at the origin, and at time $t = 0$ the spoke lies along the positive y axis and the bug is at the origin. Find a vector function $\mathbf{r}(t)$ for the position of the bug at time t, the velocity vector $\mathbf{r}'(t)$, the unit tangent $\mathbf{T}(t)$, and the speed of the bug $|\mathbf{r}'(t)|$. ⇒
10. An object moves with velocity vector $\langle \cos t, \sin t, t\rangle$, starting at $\langle 0, 0, 0\rangle$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒
11. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$, $t \geq 0$. When is the speed of the particle a minimum? ⇒
12. A particle moves so that its position is given by $\langle \cos t, \sin t, \cos(6t)\rangle$. Find the maximum and minimum speeds of the particle. ⇒
13. An object moves with velocity vector $\langle t, t^2, \cos t\rangle$, starting at $\langle 0, 0, 0\rangle$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒

14. What is the physical interpretation of the dot product of two vector valued functions? What is the physical interpretation of the cross product of two vector valued functions?
15. Show, using the rules of cross products and differentiation, that

$$\frac{d}{dt}(\mathrm{r}(t)\times \mathrm{r}'(t)) = \mathrm{r}(t)\times \mathrm{r}''(t).$$

16. Determine the point at which $\mathrm{f}(t) = (t, t^2, t^3)$ and $\mathrm{g}(t) = (\cos(t), \cos(2t), t+1)$ intersect, and find the angle between the curves at that point. (Hint: You'll need to set this one up like a line intersection problem, writing one in s and one in t.) If these two functions were the trajectories of two airplanes on the same scale of time, would the planes collide at their point of intersection? Explain. ⇒
17. Find the equation of the plane perpendicular to the curve $\mathrm{r}(t) = (2\sin(3t), t, 2\cos(3t))$ at the point $(0, \pi, -2)$. ⇒
18. Find the equation of the plane perpendicular to $(\cos t, \sin t, \cos(6t))$ when $t = \pi/4$. ⇒
19. At what point on the curve $\mathrm{r}(t) = (t^3, 3t, t^4)$ is the plane perpendicular to the curve also parallel to the plane $6x + 6y - 8z = 1$? ⇒
20. Find the equation of the line tangent to $(\cos t, \sin t, \cos(6t))$ when $t = \pi/4$. ⇒

13.3 Ar length and urvature

Sometimes it is useful to compute the length of a curve in space; for example, if the curve represents the path of a moving object, the length of the curve between two points may be the distance traveled by the object between two times.

Recall that if the curve is given by the vector function r then the vector $\Delta\mathrm{r} = \mathrm{r}(t + \Delta t) - \mathrm{r}(t)$ points from one position on the curve to another, as depicted in figure 13.2.1. If the points are close together, the length of $\Delta\mathrm{r}$ is close to the length of the curve between the two points. If we add up the lengths of many such tiny vectors, placed head to tail along a segment of the curve, we get an approximation to the length of the curve over that segment. In the limit, as usual, this sum turns into an integral that computes precisely the length of the curve. First, note that

$$|\Delta\mathrm{r}| = \frac{|\Delta\mathrm{r}|}{\Delta t}\Delta t \approx |\mathrm{r}'(t)|\,\Delta t,$$

when Δt is small. Then the length of the curve between $\mathrm{r}(a)$ and $\mathrm{r}(b)$ is

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}|\Delta\mathrm{r}| = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{|\Delta\mathrm{r}|}{\Delta t}\Delta t = \lim_{n\to\infty}\sum_{i=0}^{n-1}|\mathrm{r}'(t)|\,\Delta t = \int_a^b |\mathrm{r}'(t)|\,dt.$$

(Well, sometimes. This works if between a and b the segment of curve is traced out exactly once.)

EXAMPLE 13.3.1 Let's find the length of one turn of the helix $\mathbf{r} = \langle\cos t, \sin t, t\rangle$ (see figure 13.1.1). We compute $\mathbf{r}' = \langle -\sin t, \cos t, 1\rangle$ and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, so the length is

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

□

EXAMPLE 13.3.2 Suppose $y = \ln x$; what is the length of this curve between $x = 1$ and $x = \sqrt{3}$?

Although this problem does not appear to involve vectors or three dimensions, we can interpret it in those terms: let $\mathbf{r}(t) = \langle t, \ln t, 0\rangle$. This vector function traces out precisely $y = \ln x$ in the x-y plane. Then $\mathbf{r}'(t) = \langle 1, 1/t, 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + 1/t^2}$ and the desired length is

$$\int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{t^2}}\, dt = 2 - \sqrt{2} + \ln(\sqrt{2} + 1) - \frac{1}{2}\ln 3.$$

(This integral is a bit tricky, but requires only methods we have learned.) □

Notice that there is nothing special about $y = \ln x$, except that the resulting integral can be computed. In general, given any $y = f(x)$, we can think of this as the vector function $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + (f')^2}$. The length of the curve $y = f(x)$ between a and b is thus

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Unfortunately, such integrals are often impossible to do exactly and must be approximated. One useful application of arc length is the arc length parameterization. A vector function $\mathbf{r}(t)$ gives the position of a point in terms of the parameter t, which is often time, but need not be. Suppose s is the distance along the curve from some fixed starting point; if we use s for the variable, we get $\mathbf{r}(s)$, the position in space in terms of distance along the curve.

We might still imagine that the curve represents the position of a moving object; now we get the position of the object as a function of how far the object has traveled.

EXAMPLE 13.3.3 Suppose $\mathbf{r}(t) = \langle\cos t, \sin t, 0\rangle$. We know that this curve is a circle of radius 1. While t might represent time, it can also in this case represent the usual angle between the positive x-axis and $\mathbf{r}(t)$. The distance along the circle from $(1, 0, 0)$ to $(\cos t, \sin t, 0)$ is also t—this is the definition of radian measure. Thus, in this case $s = t$ and $\mathbf{r}(s) = \langle\cos s, \sin s, 0\rangle$. □

EXAMPLE 13.3.4 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. We know that this curve is a helix. The distance along the helix from $(1, 0, 0)$ to $(\cos t, \sin t, t)$ is

$$s = \int_0^t |\mathbf{r}'(u)|\, du = \int_0^t \sqrt{\cos^2 u + \sin^2 u + 1}\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t.$$

Thus, the value of t that gets us distance s along the helix is $t = s/\sqrt{2}$, and so the same curve is given by $\hat{\mathbf{r}}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. □

In general, if we have a vector function $\mathbf{r}(t)$, to convert it to a vector function in terms of arc length we compute

$$s = \int_a^t |\mathbf{r}'(u)|\, du = f(t),$$

solve $s = f(t)$ for t, getting $t = g(s)$, and substitute this back into $\mathbf{r}(t)$ to get $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$.

Suppose that t is time. By the Fundamental Theorem of Calculus, if we start with arc length

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

and take the derivative, we get

$$s'(t) = |\mathbf{r}'(t)|.$$

Here $s'(t)$ is the rate at which the arc length is changing, and we have seen that $|\mathbf{r}'(t)|$ is the speed of a moving object; these are of course the same.

Suppose that $\mathbf{r}(s)$ is given in terms of arc length; what is $|\mathbf{r}'(s)|$? It is the rate at which arc length is changing *relative to arc length*; it must be 1! In the case of the helix, for example, the arc length parameterization is $\langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$, the derivative is $\langle -\sin(s/\sqrt{2})/\sqrt{2}, \cos(s/\sqrt{2})/\sqrt{2}, 1/\sqrt{2}\rangle$, and the length of this is

$$\sqrt{\frac{\sin^2(s/\sqrt{2})}{2} + \frac{\cos^2(s/\sqrt{2})}{2} + \frac{1}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

So in general, $\mathbf{r}'$ is a unit tangent vector.

Given a curve $\mathbf{r}(t)$, we would like to be able to measure, at various points, how sharply curved it is. Clearly this is related to how "fast" a tangent vector is changing direction, so a first guess might be that we can measure curvature with $|\mathbf{r}''(t)|$. A little thought shows that this is flawed; if we think of t as time, for example, we could be tracing out the curve more or less quickly as time passes. The second derivative $|\mathbf{r}''(t)|$ incorporates this notion of time, so it depends not simply on the geometric properties of the curve but on how quickly we move along the curve.

EXAMPLE 13.3.5 Consider $\mathbf{r}(t) = \langle\cos t, \sin t, 0\rangle$ and $\mathbf{s}(t) = \langle\cos 2t, \sin 2t, 0\rangle$. Both of these vector functions represent the unit circle in the x-y plane, but if t is interpreted as time, the second describes an object moving twice as fast as the first. Computing the second derivatives, we find $|\mathbf{r}''(t)| = 1$, $|\mathbf{s}''(t)| = 4$. □

To remove the dependence on time, we use the arc length parameterization. If a curve is given by $\mathbf{r}(s)$, then the first derivative $\mathbf{r}'(s)$ is a unit vector, that is, $\mathbf{r}'(s) = \mathbf{T}(s)$. We now compute the second derivative $\mathbf{r}''(s) = \mathbf{T}'(s)$ and use $|\mathbf{T}'(s)|$ as the "official" measure of curvature, usually denoted κ.

EXAMPLE 13.3.6 We have seen that the arc length parameterization of a particular helix is $\mathbf{r}(s) = \langle\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. Computing the second derivative gives $\mathbf{r}''(s) = \langle-\cos(s/\sqrt{2})/2, -\sin(s/\sqrt{2})/2, 0\rangle$ with length $1/2$. □

What if we are given a curve as a vector function $\mathbf{r}(t)$, where t is not arc length? We have seen that arc length can be difficult to compute; fortunately, we do not need to convert to the arc length parameterization to compute curvature. Instead, let us imagine that we have done this, so we have found $t = g(s)$ and then formed $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$. The first derivative $\hat{\mathbf{r}}'(s)$ is a unit tangent vector, so it is the same as the unit tangent vector $\mathbf{T}(t) = \mathbf{T}(g(s))$. Taking the derivative of this we get

$$\frac{d}{ds}\mathbf{T}(g(s)) = \mathbf{T}'(g(s))g'(s) = \mathbf{T}'(t)\frac{dt}{ds}.$$

The curvature is the length of this vector:

$$\kappa = |\mathbf{T}'(t)|\left|\frac{dt}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|ds/dt|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

(Recall that we have seen that $ds/dt = |\mathbf{r}'(t)|$.) Thus we can compute the curvature by computing only derivatives with respect to t; we do not need to do the conversion to arc length.

EXAMPLE 13.3.7 Returning to the helix, suppose we start with the parameterization $\mathbf{r}(t) = \langle\cos t, \sin t, t\rangle$. Then $\mathbf{r}'(t) = \langle-\sin t, \cos t, 1\rangle$, $|\mathbf{r}'(t)| = \sqrt{2}$, and $\mathbf{T}(t) = \langle-\sin t, \cos t, 1\rangle/\sqrt{2}$. Then $\mathbf{T}'(t) = \langle-\cos t, -\sin t, 0\rangle/\sqrt{2}$ and $|\mathbf{T}'(t)| = 1/\sqrt{2}$. Finally, $\kappa = 1/\sqrt{2}/\sqrt{2} = 1/2$, as before. □

EXAMPLE 13.3.8 Consider this circle of radius a: $\mathbf{r}(t) = \langle a\cos t, a\sin t, 1\rangle$. Then $\mathbf{r}'(t) = \langle-a\sin t, a\cos t, 0\rangle$, $|\mathbf{r}'(t)| = a$, and $\mathbf{T}(t) = \langle-a\sin t, a\cos t, 0\rangle/a$. Now $\mathbf{T}'(t) = \langle-a\cos t, -a\sin t, 0\rangle/a$ and $|\mathbf{T}'(t)| = 1$. Finally, $\kappa = 1/a$: the curvature of a circle is everywhere the inverse of the radius. It is sometimes useful to think of curvature as describing what circle a curve most resembles at a point. The curvature of the helix in the previous example is 1/2; this means that a small piece of the helix looks very much like a circle of radius 2, as shown in figure 13.3.1. □

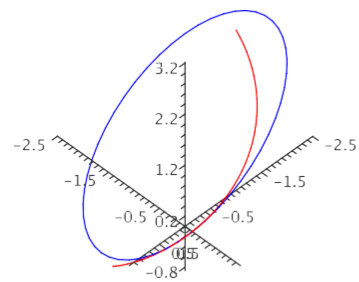


Figure 13.3.1 A circle with the same curvature as the helix. (AP)

EXAMPLE 13.3.9 Consider $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t\rangle$, as shown in figure 13.2.4. $\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\sin(2t)\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + 4\sin^2(2t)}$, so

$$\mathbf{T}(t) = \left\langle \frac{-\sin t}{\sqrt{1 + 4\sin^2(2t)}}, \frac{\cos t}{\sqrt{1 + 4\sin^2(2t)}}, \frac{-2\sin 2t}{\sqrt{1 + 4\sin^2(2t)}} \right\rangle.$$

Computing the derivative of this and then the length of the resulting vector is possible but unpleasant. □

Fortunately, there is an alternate formula for the curvature that is often simpler than the one we have:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

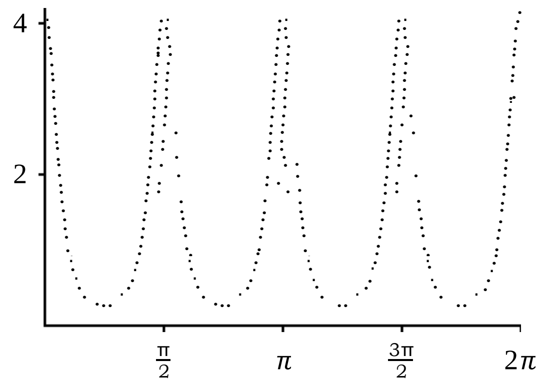
EXAMPLE 13.3.10 Returning to the previous example, we compute the second derivative $\mathbf{r}''(t) = \langle -\cos t, -\sin t, -4\cos(2t)\rangle$. Then the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ is

$$\langle -4\cos t\cos 2t - 2\sin t\sin 2t,\ 2\cos t\sin 2t - 4\sin t\cos 2t,\ 1\rangle.$$

Computing the length of this vector and dividing by $|\mathbf{r}'(t)|^3$ is still a bit tedious. With the aid of a computer we get

$$\kappa = \frac{\sqrt{48\cos^4 t - 48\cos^2 t + 17}}{(-16\cos^4 t + 16\cos^2 t + 1)^{3/2}}.$$

Graphing this we get



Compare this to figure 13.2.4—you may want to load the Java applet there so that you can see it from different angles. The highest curvature occurs where the curve has its highest and lowest points, and indeed in the picture these appear to be the most sharply curved portions of the curve, while the curve is almost a straight line midway between those points. □

Let's see why this alternate formula is correct. Starting with the definition of $\mathbf{T}$, $\mathbf{r}' = |\mathbf{r}'|\mathbf{T}$ so by the product rule $\mathbf{r}'' = |\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T}'$. Then by Theorem 12.4.1 the cross product is

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= |\mathbf{r}'|\mathbf{T} \times |\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T} \times |\mathbf{r}'|\mathbf{T}' \\ &= |\mathbf{r}'||\mathbf{r}'|'(\mathbf{T} \times \mathbf{T}) + |\mathbf{r}'|^2(\mathbf{T} \times \mathbf{T}') \\ &= |\mathbf{r}'|^2(\mathbf{T} \times \mathbf{T}')\end{aligned}$$

because $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, since $\mathbf{T}$ is parallel to itself. Then

$$\begin{aligned}|\mathbf{r}' \times \mathbf{r}''| &= |\mathbf{r}'|^2|\mathbf{T} \times \mathbf{T}'| \\ &= |\mathbf{r}'|^2|\mathbf{T}||\mathbf{T}'|\sin\theta \\ &= |\mathbf{r}'|^2|\mathbf{T}'|\end{aligned}$$

using exercise 8 in section 13.2 to see that $\theta = \pi/2$. Dividing both sides by $|\mathbf{r}'|^3$ then gives the desired formula.

We used the fact here that $\mathbf{T}'$ is perpendicular to $\mathbf{T}$; the vector $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$ is thus a unit vector perpendicular to $\mathbf{T}$, called the unit normal to the curve. Occasionally of use is the unit binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.

Exercises 13.3.

1. Find the length of $\langle 3\cos t, 2t, 3\sin t\rangle$, $t \in [0, 2\pi]$. ⇒
2. Find the length of $\langle t^2, 2, t^3\rangle$, $t \in [0, 1]$. ⇒
3. Find the length of $\langle t^2, \sin t, \cos t\rangle$, $t \in [0, 1]$. ⇒
4. Find the length of the curve $y = x^{3/2}$, $x \in [1, 9]$. ⇒
5. Set up an integral to compute the length of $\langle \cos t, \sin t, e^t\rangle$, $t \in [0, 5]$. (It is tedious but not too difficult to compute this integral.) ⇒
6. Find the curvature of $\langle t, t^2, t\rangle$. ⇒
7. Find the curvature of $\langle t, t^2, t^2\rangle$. ⇒
8. Find the curvature of $\langle t, t^2, t^3\rangle$. ⇒
9. Find the curvature of $y = x^4$ at $(1, 1)$. ⇒

13.4 Motion along a urve

We have already seen that if t is time and an object's location is given by $\mathbf{r}(t)$, then the derivative $\mathbf{r}'(t)$ is the velocity vector $\mathbf{v}(t)$. Just as $\mathbf{v}(t)$ is a vector describing how $\mathbf{r}(t)$ changes, so is $\mathbf{v}'(t)$ a vector describing how $\mathbf{v}(t)$ changes, namely, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ is the acceleration vector.

EXAMPLE 13.4.1 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, 1\rangle$. Then $\mathbf{v}(t) = \langle -\sin t, \cos t, 0\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t, 0\rangle$. This describes the motion of an object traveling on a circle of radius 1, with constant z coordinate 1. The velocity vector is of course tangent to the curve; note that $\mathbf{a}\cdot\mathbf{v} = 0$, so $\mathbf{v}$ and $\mathbf{a}$ are perpendicular. In fact, it is not hard to see that $\mathbf{a}$ points from the location of the object to the center of the circular path at $(0, 0, 1)$. □

Recall that the unit tangent vector is given by $\mathbf{T}(t) = \mathbf{v}(t)/|\mathbf{v}(t)|$, so $\mathbf{v} = |\mathbf{v}|\mathbf{T}$. If we take the derivative of both sides of this equation we get

$$\mathbf{a} = |\mathbf{v}|'\mathbf{T} + |\mathbf{v}|\mathbf{T}'. \tag{13.4.1}$$

Also recall the definition of the curvature, $\kappa = |\mathbf{T}'|/|\mathbf{v}|$, or $|\mathbf{T}'| = \kappa|\mathbf{v}|$. Finally, recall that we defined the unit normal vector as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa|\mathbf{v}|\mathbf{N}$. Substituting into equation 13.4.1 we get

$$\mathbf{a} = |\mathbf{v}|'\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}. \tag{13.4.2}$$

The quantity $|\mathbf{v}(t)|$ is the speed of the object, often written as $v(t)$; $|\mathbf{v}(t)|'$ is the rate at which the speed is changing, or the scalar acceleration of the object, $a(t)$. Rewriting equation 13.4.2 with these gives us

$$\mathbf{a} = a\mathbf{T} + \kappa v^2\mathbf{N} = a_T\,\mathbf{T} + a_N\,\mathbf{N};$$

a_T is the tangential component of acceleration and a_N is the normal component of acceleration. We have already seen that a_T measures how the speed is changing; if you are riding in a vehicle with large a_T you will feel a force pulling you into your seat. The other component, a_N , measures how sharply your direction is changing *with respect to time*. So it naturally is related to how sharply the path is curved, measured by κ, and also to how fast you are going. Because a_N includes v^2, note that the effect of speed is magnified; doubling your speed around a curve quadruples the value of a_N . You feel the effect of this as a force pushing you toward the outside of the curve, the "centrifugal force."

In practice, if want a_N we would use the formula for κ:

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}|\mathbf{r}'|^2 = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}.$$

To compute a_T we can project $\mathbf{a}$ onto $\mathbf{v}$:

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}.$$

EXAMPLE 13.4.2 Suppose $\mathbf{r} = \langle t, t^2, t^3\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N.

Taking derivatives we get $\mathbf{v} = \langle 1, 2t, 3t^2\rangle$ and $\mathbf{a} = \langle 0, 2, 6t\rangle$. Then

$$a_T = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}} \quad \text{and} \quad a_N = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{1 + 4t^2 + 9t^4}}.$$

□

Exercises 13.4.

1. Let $\mathbf{r} = \langle \cos t, \sin t, t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
2. Let $\mathbf{r} = \langle \cos t, \sin t, t^2\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
3. Let $\mathbf{r} = \langle \cos t, \sin t, e^t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
4. Let $\mathbf{r} = \langle e^t, \sin t, e^t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
5. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 0\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
6. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 0\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
7. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 1\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
8. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 1\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒

9. Describe a situation in which the normal component of acceleration is 0 and the tangential component of acceleration is non-zero. Is it possible for the tangential component of acceleration to be 0 while the normal component of acceleration is non-zero? Explain. Finally, is it possible for an object to move (not be stationary) so that both the tangential and normal components of acceleration are 0? Explain.

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE
DEPARTMENT OF MATHEMATICS
POSSIBLE QUESTIONS

Name of the Faculty : Pavithra. K
Class : I – B.Sc. Mathematics
Subject Name : Calculus
Subject Code : 17MMU101

UNIT – V

2 Mark Questions:

1. Define vector valued functions.
2. Find $\lim_{t\to 2} F(t)$, where $F(t) = (t^2 - 3)i + e^t j + (sin\pi t)k.$
3. For what values of t is $G(t) = |t|i + (cost)j + (t-5)k$ differentiable.
4. Find $\frac{d}{dt}[2F(t) + t^3 G(t)]$ if $F(t) = i + e^t j + t^2 k$ and $G(t) = 3t^2 i + e^{-t} j - 2tk.$
5. Find $\int_0^\pi [ti + 3j - (sint)k]dt.$
6. Write down the polar formulas for velocity and acceleration.
7. Write down the tangential and normal components of acceleration.

6Mark Questions :

1. Find the Volume of the Parallelepipied determined by the vectors $u = i - 2j + 3k$, $v = -4i + 7j - 11k$, $w = 5i + 9j - k$
2. If the position vector of a moving body is $R(t) = 2ti - t^2 j$ for $t \geq 0$. Express R and the velocity vector V(t) in terms of u_r and u_θ.
3. Let $F(t) = t^2 i + tj - (sint)k$ and $G(t) = ti + \frac{1}{t}j + 5k.$ find
   i) (F+G)(t) ii) (F X G)(t) iii) (F.G)(t)
4. State and prove Kepler's second law of motion.
5. Show that $\lim_{t\to 1}[F(t)XG(t)] = \left(\lim_{t\to 1} F(t)\right) X \left(\lim_{t\to 1} G(t)\right)$ for the vector functions $F(t) = ti + (1-t)j + t^2 k$ and $G(t) = e^t i - (3 + e^t)k$
6. Find the tangential and normal components of the acceleration of an object the moves with position vector $R(t) = \langle t^3, t^2, t\rangle.$
7. Find the second and third derivative of the vector function
   $i) F(t) = e^t i + (sint)j + (t^3 + 5t)k.$
   $ii) F(t) = e^{2t} i + (1 - t^2)j + (cos2t)k.$
8. A boy standing at the edge of a cliff throws a ball upwards at a 30° angle with an initial speed of 64 ft/s. suppose that when the ball leaves the boy's hand, it is 48 ft above the ground as the base of the cliff.
   i) what are the time of flight of the ball and its range?
   ii) what are the velocity of the ball and its speed at impact?
9. Let $F(t) = i + tj + t^2 k$ anf $G(t) = ti + e^t j + 3k$.Verify that $(FXG)'(t) = (F'XG)(t) + (FXG')(t)$
10. If the velocity of a particle moving in space is $V(t) = e^t i + t^2 j + (cos2t)k$. Find the particle's position as a function of t if the position at time t=0 is $R(0) = 2i + j - k$.

$ti + (1-t)j + t^2k \; and \; G(t) = e^t i - (3+e^t)k$

6. Find the tangential and normal components of the acceleration of an object the moves with position vector $R(t) = \langle t^3, t^2, t\rangle$.
7. Find the second and third derivative of the vector function
   $i)\ F(t) = e^t i + (sint)j + (t^3 + 5t)k.$
   $ii)\ F(t) = e^{2t} i + (1-t^2)j + (cos2t)k.$
8. A boy standing at the edge of a cliff throws a ball upwards at a $30°$ angle with an initial speed of 64 ft/s. suppose that when the ball leaves the boy's hand, it is 48 ft above the ground as the base of the cliff.
   i) what are the time of flight of the ball and its range?
   ii) what are the velocity of the ball and its speed at impact?
9. Let $F(t) = i + tj + t^2k \; anf \; G(t) = ti + e^t j + 3k$.Verify that $(FXG)'(t) = (F'XG)(t) + (FXG')(t)$
10. If the velocity of a particle moving in space is $V(t) = e^t i + t^2 j + (cos2t)k$. Find the particle's position as a function of t if the position at time t=0 is $R(0) = 2i + j$ –k.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: CALCULUS **Subject Code: 17MMU101**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
If u , v and w are vectors in R then u x (v + w) =	(u x v) + (u x w)	u.v + u. w	uv +(u +w)	u + w	(u x v) + (u x w)
(F + G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) + G(t)
(F - G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) - G(t)
(F x G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) / G(t)	F(t) x G(t)
(*f* F)(t)	*f*(t)F	*f*(t)F(t)	*f*(t)	F(t)	*f*(t)F(t)
(F . G) (t) =	F(t) - G(t)	F(t) + G(t)	F(t) x G(t)	F(t) . G(t)	F(t) . G(t)
The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the length of the _______ of its orbit.	minor axis	semi major axis	major axis	semi minor axis	semi major axis
lim [F(t) + G(t)] =	lim F(t) - G(t)	lim F(t) - lim G(t)	[lim F(t)] [lim G(t)	lim F(t) + lim G(	lim F(t) + lim G(t)
lim [F(t) - G(t)] =	lim F(t) - G(t)	lim F(t) - lim G(t)	[lim F(t)] [lim G(t)	lim F(t) + lim G(	lim F(t) - lim G(t)
lim [F(t) . G(t)] =	lim F(t) - lim G(t)	[lim F(t)] x [lim G(t)]	lim F(t) + lim G(t)	[lim F(t)] [lim G	[lim F(t)] [lim G(t)]
lim [F(t) x G(t)] =	lim F(t) - lim G(t)	[lim F(t)] x [lim G(t)]	lim F(t) + lim G(t)	[lim F(t)] [lim G	[lim F(t)] x [lim G(t)]
A vector function F(t) is said to be _______ at t if t is in the domain	bounded	continuous	differentiable	derivative	continuous
lim F(t) + lim G(t) =	lim [F(t) + G(t)]	lim [F(t) x G(t)]	lim [F(t) . G(t)]	lim [F(t) - G(t)]	lim [F(t) + G(t)]
The planets moves about the sun in elliptical orbit , with the sun at _	one	two	three	four	one
mass x acceleration =	speed	velocity	force	momentum	force
The derivative of velocity is equal to	speed	acceleration	force	momentum	acceleration
The square of the time of ___complete revolution of a planet about its orbit is proportional to the cube of the length of the semi major axis of its orbit.	four	one	two	three	one
The magnitude of velocity is a	momentum	force	speed	acceleration	speed
The derivative of position is a	momentum	velocity	speed	acceleration	velocity
[lim F(t)] x [lim G(t)]	lim [F(t) + G(t)]	lim [F(t) x G(t)]	lim [F(t) . G(t)]	lim [F(t) - G(t)]	lim [F(t) x G(t)]
If a , b and c are vectors in R then (c.a)b - (b.a)c =	a x (b x c)	(a x c) x b	(b x c) x a	c x (a x b)	a x (b x c)
If v is a vector the v x 0 =	1	0	v	(-v)	0
If v and w are vectors and s and t are scalars then st(v x w) =	st(v) . st(w)	st(v) x st(w)	s(v) x t(w)	s(v) . t(w)	s(v) x t(w)
(F . G)' (t) =	(F' . G)(t) - (F . G')(t)	F'(t) . G'(t)	F' G' (t)	(F' . G)(t) + (F . G')(t)	(F' . G)(t) + (F . G')(t)
(F x G)' (t) =	(F' x G)(t) + (F x G')(t)	F'(t) x G'(t)	F'x G' (t)	(F' . G)(t) + (F . G')(t)	(F' x G)(t) + (F x G')(t)
Range of the projectile is	v^2 sinα	2v / g	v^2 sinα/g	2v sinα/g	v^2 sinα/g
Time of flight of a projectile is	v^2 sinα/g	2v sinα/g	v^2 sinα	2v / g	2v sinα/g
The square of the time of one complete revolution of a planet about its orbit is ______ to the cube of the length of the semi major axis of its orbit.	greater than	proportional	less than	equal to	proportional
The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the ______of the semi major axis of its orbit.	radius	length	distance	center	length
[lim F(t)] [lim G(t)] =	lim [F(t) + G(t)]	lim [F(t) x G(t)]	lim [F(t) . G(t)]	lim [F(t) - G(t)]	lim [F(t) . G(t)]
_____of the projectile is v^2 sinα/g	speed	Range	Distance	Time of flight	Range
_____of the projectile is 2vsinα/g	speed	Range	Distance	Time of flight	Time of flight
F(t) . G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F . G) (t)
F(t) x G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F x G) (t)
F(t) - G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F - G) (t)
F(t) + G(t) =	(F + G) (t)	(F - G) (t)	(F x G) (t)	(F . G) (t)	(F + G) (t)

Reg No----------------------

[17MMU101]

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
First SEMESTER
I INTERNAL TEST-Jul'17
Calculus

Date : .07.2017 **Time: 2 Hours**
Class : I B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
Answer all the questions

1. $\cosh^2 x - \sinh^2 x$ = ______

a) tanhx b) cosh2x c) 1 d) sinh2x

2. The odd parts of e^x is called the hyperbolic ______

a) cosine b) tangent c) sine d) secant

3. 1. coshx coshy + sinhx sinhy = ________

a) cosh(x +y) b) sin(x - y) c) cosh(x - y) d) sinh (x + y)

4. Differentiation of y = In (sinhx)

a) sinhx b) cothx c) tanhx d) coshx

5. ∫tan hx dx = ________

a) In(sinhx) b) In(sechx) c) In(coshx) d) cothx

6. If x = 0 then sinhx = ___

a) (-1) b) 1 c) 0 d) 2

7. If x = 0 then cosh x = ___

a) (-1) b) 1 c) 0 d) 2

8. Range of tanhx is

a) (-1, -1) b) (1 , 1) c) (0 , 1) d) (-1, 1)

9. d(uv) =

a) uv – vu b) uv + vu c) udv - v du d) udv + v du

10. The even parts of e^x is called the hyperbolic ______

a) cosine b) tangent c) sine d) secant

11. sinh(-x) = _____

a)(- coshx) b) sinh2x c) coshx d) (- sinhx)

12. $2\cosh^2 x - 1$ = ______

a) tanhx b) cosh2x c) 1 d) sinh2x

13. Find the second derivative of e^{2x}

a) e^{2x} b) $2e^{2x}$ c) $4e^{2x}$ d) $(-e^{2x})$

14. ∫coshx dx = _______

a) sinhx b) cothx c) tanhx d) sechx

15. $\int \operatorname{sech}^2 x\, dx$ = _______

a) sinhx b) cothx c) tanhx d) sechx

16. In a polar cooridinates r denotes a ____

a) distance b) area c) angle d) radius

17. $\int \log x \, dx$ = ________

a) x log x b) log x + x c) x log x – x d) x log x + x

18. $\int \sec x \tan x \, dx$ = _____

a) tanx b) sinx c) secx d) cos x

19. The polar coordinates is denoted by

a) S(r , θ) b) P(r , θ) c) R(r , θ) d) Q(r , θ)

20. A polar coordinate system in a plane consists of a fixed point O is called the ______

a) polar b) pole c) initial ray d) parameter

Part-B(3x2= 6 Marks)

Answer all the questions

21. Prove that $sinh2x = 2sinhxcoshx$.

22. Evaluate $\int \tan^5 x \, dx$

23. Find the inflection point for the function $f(x) = 3x^5 - 5\,x^3 + 2$.

PART – C(3x8 =24 Marks)

Answer all the questions

24.a) Prove that i) $\sinh(x+y) = sinhxcoshy + coshxsinhy$

ii) $\cosh(x+y) = coshxcoshy + sinhxsinhy$

(OR)

b) State and prove Leibniz Rule for n^{th} derivative.

25. a) Find $\frac{dy}{dx}$ $for\ i)\ y = \cosh^{-1}(secx)$ $ii)\ y = \tanh(x^2+1)$.

(OR)

b) Find the n^{th} derivative of $cosx\, cos2x \cos 3x$.

26. a) Evaluate i) $\int \sec^4 x \, dx$ ii) $\int cosec^7 x \, dx$

(OR)

b) Derive the reduction formula for $\int_0^{\frac{\pi}{2}} x^n \, sinx \, dx$

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[17MMU101]

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
First SEMESTER
II INTERNAL TEST-Jul'17
Calculus

Date : .08.2017 **Time: 2 Hours**
Class : I B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
Answer all the questions

1. A polar coordinate system in a plane consists of a fixed point O is called the _____.
   a) polar b) pole c) initial ray d) parameter
2. In a polar cooridinates r denotes a ____
   a) distance b) area c) angle d) radius
3. An Rectangular coordinates means
   a) pole b) cartesian coordinate
   c) polar plane d) polar coordinate
4. $\lim_{x\to 0}(\sin x / x) =$
   a) 0 b) (-1) c) 1 d) 2
5. The volume of the cylinder is _____
   a) base – height b) base x height
   c) 2(base + height) d) (base x height) / 2
6. A function with a continuous first derivative is said to be smooth and its graph is called
   a) smooth curve b) length
   c) smooth plane d) smooth derivative
7. In a polar coordinates θ denotes a ____
   a) distance b) area c) angle d) radius
8. If the polar equation is r cosθ = 2 then the Cartesian equation is
   a) x = -1 b) x = 2 c) x = -2 d) x = 0
9. The slope of the polar curve = f(θ) is given by ____
   a) 2(dy"/dx") b) dy'/dx' c) dy/dx d) dx/dy
10. A ray emanating from the pole is called the ______
    a) polar curve b) polar axis
    c) polar plane d) polar coordinate
11. The radial coordinate is denoted by
    a) θ b) O c) r d) P
12. What is another name foe cartesian coordinate ?
    a) square coordinate b) rectangular coordinate
    c) polar plane d) polar coordinate
13. $\int \sec x \tan x \, dx =$ ____
    a) tanx b) sinx c) secx d) cos x
14. $\lim_{x\to 0+}(x \cot x) =$
    a) 2 b) 1 c) 0 d) (-1)
15. A right cylinder is a solid that generayte when a plane region is translated along a line or axis that is ______ to the region

a) perpendicular b) bounded

c) parallel d) linear

16. $\lim_{x\to 0+} (1 + x)^{1/x} =$

a) e b) 2 c) 1 d) 0

17. volume of a cylindrical shell = _________

a) 2π b) πr^2 c) $2\pi r^2 h$ d) $2\pi r$

18. A__________ is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve.

a) lateral surface area b) surface of revolution

c) area of revolution d) cross sectional area

19. An tangent lines to the parametric curve is

a) 2(dy''/dx'') b) dy'/dx' c) dy/dx d) dx/dy

20. The curve represented by the parametric equations $x = t^2$ and $y = t^3$ is called

a) ellipse b) semicubical parabola

c) hyperbola d) parabola

PART-B (3 x 2 =6 Marks)

Answer All the Questions:

21. Convert the polar equation to Cartesian equation for $r = \frac{25}{2sin\theta - 3cos\theta}$

22. Define surface area of revolution.

23. Find the Cartesian coordinate of the point P whose polar coordinates are $(r, \theta) = \left(6, \frac{2\pi}{3}\right)$.

PART-C (3 x 8 =24 Marks)

Answer All the Questions:

24. a) Evaluate i) $\lim_{x\to 0} \frac{1-cosx}{secx}$ ii) $\lim_{x\to +\infty} \frac{2x^2-3x+1}{3x^2+5x-2}$

(OR)

b) Derive the reduction formula for $\int \sin^m x \cos^n x \, dx$.

25. a) Find $\frac{d^2y}{dx^2}$ for the parametric equation $x = t - t^2$ *and* $y = t - t^3$.

(OR)

b) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ then prove that $I_n + I_{n-2} = \frac{1}{n-1}$ and hence evaluate I_5.

26. a) Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ about the x-axis .

(OR)

b) Find the length of the asteroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 2\pi$.

Reg No----------------------

[17MMU101]

KARPAGAM UNIVERSITY
KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
First Semester
III INTERNAL TEST-Sep'17
Calculus

Date : .09.2017 **Time: 2 Hours**
Class : I B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)

Answer all the questions

1. Which one is the example for conic section ?

a) parabola b) solid c) triangle d) rectangle

2. _______ is the set of all points in the plane that are equidistant from a fixed line and fixed point not on the line .

a) ellipse b) hyperbola c) parabola d) circle

3. In an ellipse ,the end point of the major axis is called ______

a) minor axis b) vertices c) symmetry d) center

4. The midpoint of the line segment joining the foci is called the _____ of the hyperbola.

a) vertices b) axis c) symmetry d) center

5. If u , v and w are vectors in R then u x (v + w) = _______________

a) (u x v) + (u x w) b) u.v + u. w

c) uv +(u +w) d) u + w

6. (F + G) (t) = _____________

a) F(t) - G(t) b) F(t) + G(t) c) F(t) x G(t) d) F(t) / G(t)

7 The derivative of velocity is equal to ______________

a) speed b) acceleration c) force d) momentum

8. A vector function F(t) is said to be _______ at t if t is in the domain of F.

a) bounded b) continuous c) differentiable d) derivative

9. _______ is the set of all points in the plane, the sum of whose distance from two fixed point is a given positive constant that is less than the distance between the fixed point.

a) ellipse b) hyperbola c) parabola d) circle

10.In an ellipse the midpoint of the line segment joining the foci is called ______

a) vertices b) axis c) symmetry d) center

11. The fixed point is the _____ of the circle.

a) center b) vertices c) radius d) standard position

12. Find the focus of the parabola $y^2 = 10$ x.

a) (0 , 5/2) b) (-5/2 , 0) c) (5/2 , 0) d) (0 , -5/2)

13. The planets moves about the sun in elliptical orbit , with the sun at ____ focus.

a) one b) two c) three d) four

14. mass x acceleration = __________

a) speed b) velocity c) force d) momentum

15. (F - G) (t) = _____________

a) F(t) - G(t)   b) F(t) + G(t)   c) F(t) x G(t)   d) F(t) / G(t)

16. lim [F(t) x G(t)] =___________

a) lim F(t) - lim G(t)   b) [lim F(t)] x [lim G(t)]

c) lim F(t) + lim G(t)   d) [lim F(t)] [lim G(t)]

17. The eccentricity of a parabola is ____________

a) e < 1   b) e >1   c) e = 0   d) e =1

18.Find the radius of the circle r = 4 cos θ.

a) 2   b) 8   c) 4   d) 16

19. If B^2 - 4AC =0 then it is called _________________

a) ellipse   b) hyperbola   c) parabola   d) circle

20. The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the length of the _______ of its orbit.

a) minor axis   b) semi major axis

c) major axis   d) semi minor axis

PART-B (3 x 2 =6 Marks)

Answer all the Questions:

21. Find $\lim_{t\to 2} F(t)$, where $F(t) = (t^2 - 3)i + e^t j + (sin\pi t)k$.

22.Define discriminant test.

23. Find the focus and directrix of the parabola $y^2 = 10x$.

PART-C (3 x 8 =24 Marks)

Answer all the Questions:

24. a) Sketch the graph of the ellipse $i)\frac{x^2}{9} + \frac{y^2}{16} = 1$ $ii)$ $x^2 + 2y^2 = 4$. and show the foci of each.

(OR)

b) Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3 , 0) and the line x = 1 as the corresponding directrix.

25. a) Find the equation of the curve $x^2 - xy + y^2 - 6 = 0$ in $x'y'$ - coordinates. if the coordinate axes are rotated through an angle of $\theta = 45°$.

(OR)

b) Find the tangential and normal components of the acceleration of an object the moves with position vector $R(t) = \langle t^3, t^2, t\rangle$.

26. a) Let $F(t) = t^2 i + tj - (sint)k$ and $G(t) = ti + \frac{1}{t} j + 5k$.

find i) (F+G)(t)   ii) (F X G)(t)   iii) (F.G)(t)

(OR)

b) State and prove the Kepler's second law of motion.