KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
SYLLABUS

		Semester – I
		L T P C
17MMU102	**ALGEBRA**	6 1 0 6

Course Objective :

On successful completion of course the learners will be enriched with the concept of De Moivre's theorem, Rings, fields, linear transformation, which are very useful for their research.

Course Outcome :

To enable the students to learn and gain knowledge about functions, relations, systems of linear equations and linear transformations.

UNIT I

Polar representation of complex numbers, nth roots of unity, De Moivre's theorem for rational Indices and its applications. Sets –Finite and infinite sets-Equality sets-Subsets-Comparability -Proper subsets-Axiomatic development of set theory-Set operations.

UNIT II

Equivalence relations, Functions, Composition of functions, Invertible functions, One to one Correspondence and cardinality of a set, Well-ordering property of positive integers.

UNIT III

Division algorithm, Divisibility and Euclidean algorithm, Congruence relation between integers, Principles of Mathematical Induction, Statement of Fundamental Theorem of Arithmetic.

UNIT IV

Systems of linear equations, row reduction and echelon forms, vector equations, the matrix equation Ax=b, solution sets of linear systems, applications of linear systems, linear independence.

UNIT V

Introduction to linear transformations, matrix of a linear transformation, inverse of a matrix, characterizations of invertible matrices. Subspaces of Rn, dimension of subspaces of Rn and rank of a matrix, Eigen values, Eigen Vectors and Characteristic Equation of a matrix.

SUGGESTED READINGS

TEXT BOOKS

1. Titu Andreescu., and Dorin Andrica,(2006). Complex Numbers from A to Z, Birkhauser. Library of Congress Cataloging-in-Publication Data Andreescu, Titu, (**For Unit –I**).

2. Edgar G. Goodaire and Michael M. Parmenter, ,(2005). Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.(**For Unit –II)**

3.David C. Lay., (2007). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint. **(For Unit III, IV and V)**

REFERENCE

1. Kenneth Hoffman., Ray Kunze., (2003).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

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LECTURE PLAN

Subject: Algebra **Subject Code: 17MMU102**

Class : I – B. Sc. Mathematics **Semester : I**

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
	UNIT-I		
1.	1	Introduction to Polar representation of complex numbers	T1:Ch: 2; Pg.No:33-34
2.	1	Continuation on Polar representation of complex numbers	T1:Ch: 2; Pg.No:35-36
3.	1	Continuation on Polar representation of complex numbers	T1:Ch: 2; Pg.No:36-37
4.	1	n^{th} roots of unity	T1:Ch: 2; Pg.No:38-39
5.	1	Continuation of n^{th} roots of unity	T1:Ch: 2; Pg.No:40-41
6.	1	Continuation of n^{th} roots of unity	T1:Ch: 2; Pg.No:42-43
7.	1	Tutorial- I	
8.	1	Continuation of Problems on n^{th} roots of unity	T1:Ch: 2; Pg.No:44-45
9.	1	De Moivre's Theorem for rational indices	T1:Ch: 2; Pg.No:46-47
10.	1	Continuation on De Moivre's Theorem for rational indices	T1:Ch: 2; Pg.No:48-49
11.	1	Continuation on De Moivre's Theorem for rational indices	T1:Ch: 2; Pg.No:50-52
12.	1	De Moivre's Theorem and its applications	T1:Ch: 2; Pg.No:53-54
13.	1	Continuation on De Moivre's Theorem and its applications	T1:Ch: 2; Pg.No:54-55
14.	1	Tutorial- II	
15.	1	Continuation on De Moivre's Theorem and its applications	T1:Ch: 2; Pg.No:33-36
16.	1	Continuation on De Moivre's Theorem and its applications	T1:Ch: 2; Pg.No:37-39
17.	1	Sets and its types	T2: Ch: 2; Pg.No:37-39
18.	1	Subsets and Proper subsets	T2: Ch: 2; Pg.No:40-41
19.	1	Set Operations with examples	T2: Ch: 2; Pg.No:43-46
20.	1	Continuation on Set Operations with examples	T2: Ch: 2; Pg.No:47-49
21.	1	Tutorial- III	
22.	1	Recapitulation and discussion of possible questions on unit I	
Total no. of lecture hours planned : 22 hrs			

Prepared By : R. Praveen Kumar, Department of Mathematics, KAHE

T1:Titu Andreescuand Dorin Andrica,2006.ComplexNumbers from A to Z,Birkhauser,Library of congress cataloging-in –publication dataAndreescu, Titu,1956.

T2:EdgerG.Goodaire and Michael M.Parameter,2005. Discrete Mathematics with graph theory,3rd edition,Pearson Educaion(Singapore) P.Ltd.,Indian Reprint.

UNIT-II			
1.	1	Basic concepts of Equivalence relations	R1:Ch 11:Pg.NO:391-393
2.	1	Continuation of Equivalence relations	T2: Ch: 2; Pg. No :56-57
3.	1	Continuation of Equivalence relations	T2: Ch: 2; Pg. No :57-58
4.	1	Functions: definitions and properties	T2: Ch: 2; Pg. No :59-60
5.	1	Continuation on functions	T2: Ch: 2; Pg. No :61-62
6.	1	Tutorial- I	
7.	1	Composition of functions	T2: Ch: 3; Pg. No :71-73
8.	1	Continuation on Composition of functions	T2: Ch: 3; Pg. No :74-75
9.	1	Invertible functions	T2: Ch: 3; Pg. No :76-77
10.	1	Continuation on Invertible functions	T2: Ch: 3; Pg. No :78-79
11.	1	Continuation on Invertible functions	T2: Ch: 3; Pg. No :80-81
12.	1	One to one correspondence	T2: Ch: 2; Pg. No :59-60
13.	1	Tutorial- II	
14.	1	Problems on one to one correspondence	T2: Ch: 2; Pg. No :61-62
15.	1	Cardinality of a set	T2: Ch: 3; Pg. No :66-67
16.	1	Continuation on Cardinality of a set	T2: Ch: 3; Pg. No :68-69
17.	1	Continuation on Cardinality of a set	T2: Ch: 3; Pg. No :70-71
18.	1	Well-ordering property of positive integers	T2: Ch:3; Pg. No :72-73
19.	1	Continuation on Well-ordering property of positive integers	T2: Ch:3; Pg. No :74-75
20.	1	Tutorial- III	
21.	1	Recapitulation and discussion of possible questions	
Total no. of lecture hours planned : 21 hrs			

T2:EdgerG.Goodaire and Michael M.Parameter,2005. Discrete Mathematics with graph theory,3rd edition,Pearson Educaion(Singapore) P.Ltd.,Indian Reprint.

R1. Kenneth Hoffman., Ray Kunze., 2003. Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

UNIT-III			
1.	1	Introduction on Division algorithm	T2: Ch: 4; Pg. No :97-104
2.	1	Divisibility algorithm	T2: Ch: 4; Pg. No:105-106
3.	1	Euclidean algorithm	T2: Ch: 4; Pg. No:107-108
4.	1	Continuation of Euclidean algorithm	T2: Ch: 4; Pg. No:109-110
5.	1	Problems on Divisibility and Euclidean algorithm	T2: Ch: 4; Pg. No:115-116
6.	1	Tutorial –I	
7.	1	Continuation on Divisibility and Euclidean algorithm	T2: Ch: 4; Pg. No:116-117
8.	1	Continuation on Divisibility and Euclidean algorithm	T2: Ch: 4; Pg. No:118-119

9.	1	Continuation on Divisibility and Euclidean algorithm	T2: Ch: 4; Pg. No:119-120
10.	1	Congruence relation between integers	T2: Ch: 4; Pg. No :121-122
11.	1	Continuation on Congruence relation between integers	T2: Ch: 4; Pg. No :123-124
12.	1	Tutorial –II	
13.	1	Continuation on Congruence relation between integers	T2: Ch: 4; Pg. No :124-126
14.	1	Continuation on Congruence relation between integers	T2: Ch:4, Pg. No:127-130
15.	1	Continuation on Congruence relation between integers	T2: Ch:4, Pg. No:131-138
16.	1	Principles of Mathematical Induction	T2: Ch: 4; Pg. No :139-145
17.	1	Continuation on Principles of Mathematical Induction	T2: Ch: 5; Pg. No :149-151
18.	1	Statement of Fundamental Theorem of Arithmetic	T2: Ch: 5; Pg. No :152-154
19.	1	Tutorial- III	
20.	1	Recapitulation and discussion of possible questions	
Total no. of lecture hours planned : 20 hrs			
T2:EdgerG.Goodaire and Michael M.Parameter,2005. Discrete Mathematics with graph theory,3rd edition,Pearson Educaion(Singapore) P.Ltd.,Indian Reprint.			
UNIT-IV			
1.	1	Introduction and basic concepts of systems of linear equations	T3:Ch:1; Pg,No:1-4
2.	1	Problems on systems of linear equations	T3:Ch:1; Pg,No:5-9
3.	1	Continuation of Problems on systems of linear equations	T3:Ch:1; Pg,No:10-12
4.	1	Row reduction	T3:Ch:1; Pg,No:13-15
5.	1	Continuation on Row reduction	T3:Ch:1; Pg,No:16-19
6.	1	Tutorial –I	
7.	1	Continuation on Row reduction	T3:Ch:1; Pg,No:20-23
8.	1	Echelon forms	T3:Ch:1; Pg,No:24-27
9.	1	Continuation on Echelon forms	T3:Ch:1; Pg,No:28-30
10.	1	Continuation on Echelon forms	T3:Ch:1; Pg,No:29-34
11.	1	Vector equations	T3:Ch:1; Pg,No:35-38
12.	1	Tutorial –II	
13.	1	The matrix equation Ax=b	R1:Ch:1; Pg.No:6-8
14.	1	Problems on Ax =b form	R1:Ch:1; Pg.No:9-10
15.	1	Solution sets of linear systems	T3:Ch:1; Pg,No:39-43
16.	1	Applications of linear systems	T3:Ch:1; Pg,No:44-46
17.	1	Linear independance	T3:Ch:1; Pg,No:50-52
18.	1	Continuation on Linear independance	T3:Ch:1; Pg,No:53-55
19.	1	Tutorial- III	
20.	1	Recapitulation and discussion of possible questions	
Total no. of lecture hours planned : 20 hrs			

T3: David C.Lay,2007.Linear Algebra and itsapplications 3rd edition, Pearson Educaion(Asia) P.Ltd.,Indian Reprint. **R1**. Kenneth Hoffman., Ray Kunze., 2003. Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.			
UNIT-V			
1.	1	Introduction to linear transformations	R1:Ch:3.1; Pg.No:67-69
2.	1	Matrix of a linear transformations	R1:Ch:3.1; Pg.No:70-72
3.	1	Continuation on Matrix of a linear transformations	R1:Ch:3.1; Pg.No:73-75
4.	1	Inverse of a matrix	R1:Ch:3.1; Pg.No:76-78
5.	1	Problems on Inverse of a matrix	R1:Ch:3.1; Pg.No:79-80
6.	1	Tutorial –I	
7.	1	Characterizations of invertible matrices	R1:Ch:3.1; Pg.No:81-84
8.	1	Continuation on Characterizations of invertible matrices	R1:Ch:3.1; Pg.No:85-87
9.	1	Subspaces of R^n	R1:Ch:2.2; Pg.No:34-36
10.	1	Dimensions of subspaces of R^n	R1:Ch:2.2; Pg.No:37-38
11.	1	Tutorial –II	
12.		Continuation on Dimensions of subspaces of R^n	R1:Ch:2.2; Pg.No:39-40
13.		Rank of matrix	T3:Ch:5; Pg.No:264-266
14.		Eigen values ,Eigen vectors	T3:Ch:5; Pg.No:267-268
15.		Continuation on Eigen values ,Eigen vectors	T3:Ch:5; Pg.No:269-270
16.		Characteristic Equation of a matrix	T3:Ch:5; Pg.No:271-272
17.		Problems on Finding Characteristic Equation of a matrix	T3:Ch:5; Pg.No:273-275
18.	1	Tutorial- III	
19.	1	Recapitulation and discussion of important questions	
20.	1	Discuss on Previous ESE question papers	
21.	1	Discuss on Previous ESE question papers	
22.	1	Discuss on Previous ESE question papers	
Total no. of lecture hours planned : 22 hrs			
T3: David C.Lay,2007.Linear Algebra and itsapplications 3rd edition, Pearson Educaion(Asia) P.Ltd.,Indian Reprint. **R1**. Kenneth Hoffman., Ray Kunze., 2003. Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.			

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T3: David C.Lay,2007.Linear Algebra and itsapplications 3rd edition, Pearson Educaion(Asia) P.Ltd.,Indian Reprint.

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Department of Mathematics

Subject : Algebra	**Subject Code : 17MMU102**	**L T P C**
Class : I – B.Sc. Mathematics	**Semester : I**	**6 1 0 6**

UNIT I

Polar representation of complex numbers, nth roots of unity, De Moivre's theorem for rational Indices and its applications. Sets –Finite and infinite sets-Equality sets-Subsets-Comparability -Proper subsets-Axiomatic development of set theory-Set operations.

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UNIT – I
Complex Number in Polar Form
1. Complex Numbers

In algebra we discovered that many equations are not satisfied by any real numbers. Examples are:

$$x^2 = -2 \qquad \text{or} \qquad x^2 - 2x + 40 = 0$$

We must introduce the concept of complex numbers.

Definition: A **complex number** is an ordered pair $z = (x, y)$ of real numbers x and y. We call x the **real part** of z and y the **imaginary part**, and we write

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

Example 1: $\operatorname{Re}(4,6) = 4$ and $\operatorname{Im}(4,6) = 6$

Two complex numbers are equal where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$:

$$z_1 = z_2 \quad \text{if and only if} \quad x_1 = x_2 \text{ and } y_1 = y_2$$

Addition and Subtraction of Complex Numbers: We define for two complex numbers, the **sum** and **difference** of $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$:

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \quad \text{and} \quad z_1 - z_2 = (x_1 - x_2, y_1 - y_2).$$

Multiplication of two complex numbers is defined as follows:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Example 2: Let $z_1 = (3,4)$ and $z_2 = (5,-6)$ then

$$z_1 + z_2 = (3+5, 4+(-6)) = (8,-2)$$

and

$$z_1 - z_2 = (3-5, 4-(-6)) = (-2,10)$$

and

$$z_1 z_2 = (3\cdot 5 - 4\cdot(-6), 3\cdot(-6) + 4\cdot 5) = (39,2).$$

We need to represent complex numbers in a manner that will make addition and multiplication easier to do.

Complex numbers represented as $z = x + iy$

A complex number whose imaginary part is 0 is of the form $(x, 0)$ and we have

$$(x_1,0)+(x_2,0)=(x_1+x_2,0) \text{ and } (x_1,0)-(x_2,0)=(x_1-x_2,0)$$

and

$$(x_1,0)\cdot(x_2,0)=(x_1x_2,0)$$

which looks like real addition, subtraction and multiplication. So we identify $(x,0)$ with the real number x and therefore we can consider the real numbers as a subset of the complex numbers.

We let the letter $i=(0,1)$ and we call ***i*** a purely imaginary number.

Now consider $i^2 = i\cdot i=(0,1)\cdot(0,1)=(-1,0)$ and so we can consider the complex number $i^2=-1=$ the real number -1. We also get $yi = y\cdot(0,1)=(0,y)$

And so we have: $(x,y)=(x,0)+(0,y)=x+iy$

Now we can write addition and multiplication as follows:

$$z_1+z_2=(x_1+x_2,y_1+y_2)=x_1+x_2+i(y_1+y_2)$$

and $z_1z_2=(x_1x_2-y_1y_2,x_1y_2+x_2y_1)= x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$.

Example 3: Let $z_1=(2,3)=2+3i$ and $z_2=(5,-4)=5-4i$, then

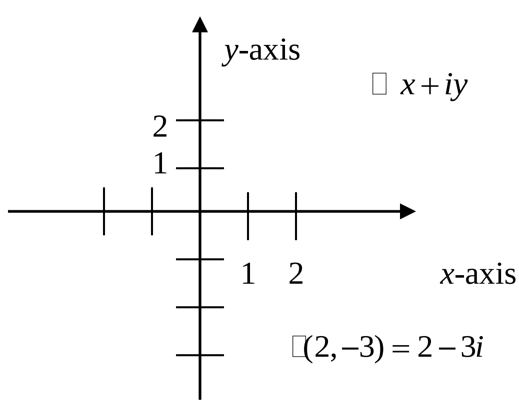
$$z_1+z_2=(2+3i)+(5-4i)=7-i$$

and

$$z_1\cdot z_2=(2+3i)\cdot(5-4i)=10+15i-8i-12i^2=22+7i$$

The Complex Plane

The geometric representation of complex numbers is to represent the complex number (x,y) as the point (x,y).



So the real number $(x,0)$ is the point on the horizontal *x*-axis, the purely imaginary number $yi=(0,y)$ is on the vertical y-axis. For the complex number (x,y), x is the real part and y is the imaginary part.

Example 4. Locate 2-3i on the graph above.

How do we **divide** complex numbers? Let's introduce the conjugate of a complex number then go to division.

Given the complex number $z = x + iy$, define the conjugate $\bar{z} = \overline{x+iy} = x - iy$

We can divide by using the following:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Example 5. $\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{6+12i^2+8i+9i}{9-16i^2} = -\frac{6}{25} + i\frac{17}{25}$

Complex Numbers in Polar Form

It is possible to express complex numbers in polar form. If the point $z = (x, y) = x + iy$ is represented by polar coordinates $r,\ \theta$, then we can write $x = r\cos\theta$, $y = r\sin\theta$ and $z = r\cos\theta + ir\sin\theta = re^{i\theta}$. r is the modulus or absolute value of z, $|z| = r = \sqrt{x^2 + y^2}$, and θ is z_2 the argument of *z*,

$\theta = \arctan\left(\frac{y}{x}\right)$. The values of *r* and θ determine *z* uniquely, but the converse is not true. The modulus *r* is determined uniquely by *z*, but θ is only determined up to a multiple of 2π. There are infinitely many values of θ which satisfy the equations $x = r\cos\theta,\ y = r\sin\theta$, but any two of them differ by some multiple of 2π. Each of these angles θ is called an **argument** of *z*, but, by convention, one of them is called the **principal argument**.

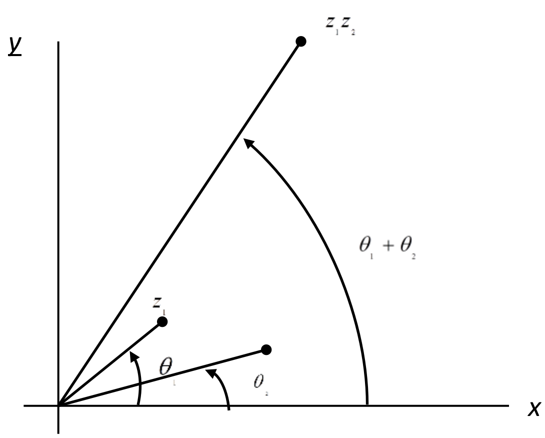
Definition If *z* is a non-zero complex number, then the unique real number θ, which satisfies

$$x = |z|\cos\theta,\ y = |z|\sin\theta,\quad -\pi < \theta \le \pi$$

is called the **principal argument** of *z*, denoted by $\theta = \arg(z)$.

Note: The distance from the origin to the point (x, y) is $|z|$, the modulus of *z*; the argument of *z* is the angle $\theta = \arctan\frac{y}{x}$. Geometrically, θ is the directed angle measured from the positive *x*-axis to the line segment from the origin to the point (x, y). When $z = 0$, the angle θ is undefined.

The polar form of a complex number allows one to multiply and divide complex number more easily than in the Cartesian form. For instance, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$, $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$. These formulae follow directly from DeMoivre's formula.



Example 6. For $z = 1 + i$, we get $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4}$. The principal value of θ is $\frac{\pi}{4}$, but $\frac{9\pi}{4}$ would work also.

Multiplication and Division in Polar Form

Let $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then we have

$z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

Example 7: $z_1 = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and $z_2 = \sqrt{3} - i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Then $z_1 z_2 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

Since $\frac{\pi}{4} + \frac{\pi}{6} = \frac{10\pi}{24} = \frac{5\pi}{12}$

And

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

We can use $z^2 = z \cdot z = r \cdot r(\cos(\theta+\theta) + i\sin(\theta+\theta) = r^2(\cos 2\theta + i\sin 2\theta)$
And so:

DeMoivre's Theorem:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

where n is an positive integer.
We want to prove that, for all positive integers n,
$(\mathrm{i}\sin x + \cos x)^n = \mathrm{i}\sin nx + \cos nx$

Step 1: case *n* = 1

Trivially, $(\mathrm{i}\sin x + \cos x)^1 = \mathrm{i}\sin x + \cos x$. So the result holds for $n = 1$.

Step 2: arbitrary *n*

We assume the induction hypothesis, that is, we assume
$(\mathrm{i}\sin x + \cos x)^{n-1} = \mathrm{i}\sin(n-1)x + \cos(n-1)x$
Now we have

$$\begin{aligned}(\mathrm{i}\sin x + \cos x)^n &= (\mathrm{i}\sin x + \cos x)(\mathrm{i}\sin x + \cos x)^{n-1} \\ &= (\mathrm{i}\sin x + \cos x)(\mathrm{i}\sin(n-1)x + \cos(n-1)x) \\ &= \cos x\cos(n-1)x - \sin x\sin(n-1)x \\ &\qquad + \mathrm{i}\left[\sin x\cos(n-1)x + \cos x\sin(n-1)x\right] \\ &= \cos nx + \mathrm{i}\sin nx\end{aligned}$$

using the compound angle identities.
This proves the induction step, so by the principle of mathematical induction,
$(i\sin x + \cos x)^n = i\sin nx + \cos nx$
for all positive integers, *n*.

Let $r = 1$ to get: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Example 1: Compute $(1+i)^6$

$$\begin{aligned}(1+i)^6 &= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^6 \\ &= \sqrt{2}^6\left(\cos 6\cdot\frac{\pi}{4} + i\sin 6\cdot\frac{\pi}{4}\right) \\ &= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) \\ &= -8i\end{aligned}$$

n^th Roots of Complex Numbers:

Consider $z = r(\cos\theta + i\sin\theta) = w^n = R^n(\cos n\phi + i\sin n\phi)$ (Equation 1)

where $w = R(\cos\phi + i\sin\phi)$. Then $R = \sqrt[n]{r}$, and so $\theta = n\phi$ or $\phi = \frac{\theta}{n}$.

However $n\phi = \theta + 2\pi$ also satisfies Equation 1 and so $\phi = \frac{\theta}{n} + \frac{2\pi}{n}$. And $n\phi = \theta + 4\pi$ implies $\phi = \frac{\theta}{n} + \frac{4\pi}{n}$. However $n\phi = \theta + 6\pi$ implies $\phi = \frac{\theta}{n} + \frac{6\pi}{n}$.

And continuing $n\phi = \theta + k\pi$ implies $\phi = \frac{\theta}{n} + \frac{k\pi}{n}$. for k any integer up to n.

We get $\sqrt[n]{z} = \sqrt[n]{x}\left(\cos\left(\frac{\theta + k2\pi)}{n}\right) + i\sin\left(\frac{\theta + k2\pi)}{n}\right)\right)$, k=0, 1, 2, 3, …, (n-1).

Example 2. Find the sixth root of -1.

There will be six roots:

$$z_1 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$z_2 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{6}\right)\right) = i$$

$$z_3 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6} + \frac{4\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{6}\right)\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$z_4 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6} + \frac{6\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{6\pi}{6}\right)\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$z_5 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6} + \frac{8\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{8\pi}{6}\right)\right) = -i$$

$$z_6 = \sqrt[6]{1}\left(\cos\left(\frac{\pi}{6} + \frac{10\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{10\pi}{6}\right)\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

ELEMENTARY SET THEORY

I. BASIC CONCEPTS

Example 3: Find the square roots of i.

Since $i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$, we let

$w_1 = \sqrt{1}\left(\cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) + i\sin\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ is one square root of i. The second square root of i is :

1 **Definition 1**: A ***set*** is a collection of objects together with some rule to determine whether a given object belongs to this collection. Any object of this collection is called an ***element*** of the set.

1. Each element of the set is listed within a set of brackets: { }.
2. Within the brackets, the first few elements are listed, with dots following to show that the set continues with the selection of the elements following the same rule as the first few.
3. Within the brackets, the set is described by writing out the exact rule by which elements are chosen. The name given each element is separated from the selection rule with a vertical line.

Examples:

(a) Denote by *A* the set of natural numbers with are greater than 25. The set could be written in the following ways:

{26,27,28….} (using the second notation listed above)

$\{x \mid x$ is a natural number and $x > 25\}$ (using the third notation above)

The above description is read as "the set of all *x* such that *x* is a natural number and $x > 25$".

Note that 32 is an element of A. We write $\mathbf{32 \in A}$, where "∈" denotes "*is an element of*." Also, $\mathbf{6 \notin A}$, where "∉" denotes "*is not an element of*."

(b) Let *B* be the set of numbers {3,5,15,19,31,32}. Again the elements of the set are natural numbers. However, the rule is given by actually listing each element of the set (as in the first notation above). We see that $15 \in B$, but $23 \notin B$.

(c) Let *C* be the set of all natural numbers which are less than 1. In this set, we observe that there are no elements. Hence, *C* is said to be an ***empty set***. A set with no elements is denoted by ∅.

Definition: A set *A* is said to be a ***subset*** of a set *B* if *every* element of *A* is an element of *B*.

Notation: To indicate that set *A* is a subset of set *B*, we use the expression **A ⊂ B,** where "⊂" denotes "*is a subset of*". **A ⊄ B** means that *A* is *not* a subset of *B*.

Examples:

(a) Let *B* be the set of natural numbers. Let *A* be the set of even natural numbers. Clearly, *A* is a subset of *B*. However, *B* is not a subset of *A*, for $3 \in B$, but $3 \notin A$.

(b) An empty set ∅ is a subset of *any* set *B*. If this were not so, there would be some element $x \in \emptyset$ such that $x \notin B$. However, this would contradict with the definition of an empty set as a set with no elements.

Theorem: Properties Of Sets

Let A, B, and C be sets.

1. For any set A, $A \subset A$ (Reflexive Property)

2. If $A \subset B$ and $B \subset C$, then $A \subset C$ (Transitive Property)

Definition: Two sets, A and B, are said to be **equal** if and only if A is a subset of B and B is a subset of A. To indicate that two sets, A and B, are equal, we use the symbol **A = B**. This means that sets A and B contain *exactly the same elements*. **A ≠ B** means that A and B are not equal sets.

Example:

Let A be the set of even natural numbers and B be the set of natural numbers which are multiples of 2. Clearly, $A \subset B$ and $B \subset A$. Therefore, since A and B contain exactly the same elements, $A = B$.

Remarks:

(a) Two equal sets always contain the same elements. However, the rules for the sets may be written differently, as in the above example.

(b) Since any two empty sets are equal, we will refer to any empty set as ***the*** empty set.

(c) A is said to be a ***proper subset*** of B is and only if:
(i) $A \subset B$
(ii) $A \neq B$, and
(iii) $A \neq \emptyset$.

Theorem: Properties of Set Equality

(a) For any set A, $A = A$. (Reflexive Property)

(b) If $A = B$, then $B = A$. (Symmetric Property)

(c) If $A = B$ and $B = C$, then $A = C$. (Transitive Property)

Definition: Let A and B be subsets of a set X. The ***intersection*** of A and B is the set of all elements in X common to both A and B.

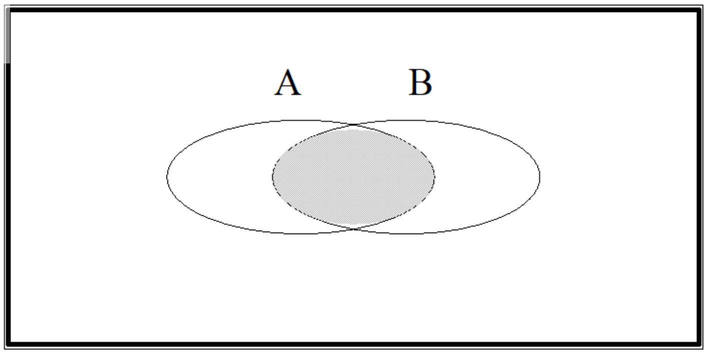
Notation: "$A \cap B$" denotes "*A* intersection *B*" or the intersection of sets *A* and *B*.

Thus, $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$, or $A \cap B = \{x \mid x \in A \wedge x \in B\}$.

Examples:

a. Given that the box below represents *X*, the shaded area represents $A \cap B$:

X:



b. Let $A = \{2,4,5\}$ and $B = \{1,4,6,8\}$ Then, $A \cap B = \{4\}$.

Note: A set that has only one element, such as {4}, is sometimes called a singleton set.

c. Let $A = \{2,4,5\}$ and $B = \{1,3\}$. Then $A \cap B = \emptyset$.

Remarks:

a. If, as in the above example 1.11c, *A* and *B* are two sets such that $A \cap B$ is the empty set, we say that *A* and *B* are ***disjoint.***

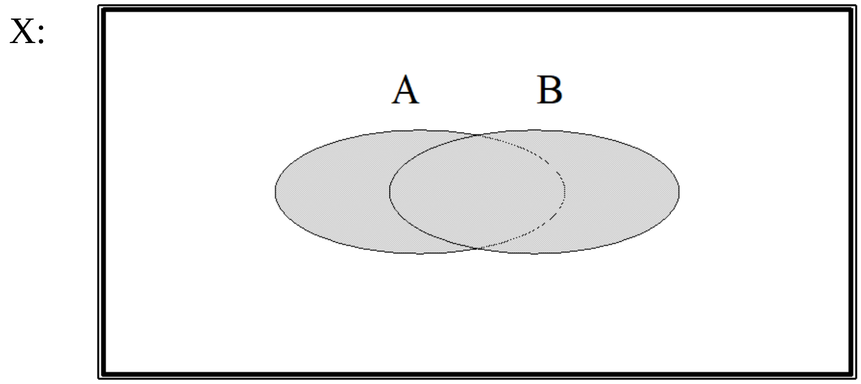
b. Given sets *A* and *B*. $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Definition: Let *A* and *B* be subsets of a set *X*. The ***union*** of *A* and *B* is the set of all elements belonging to *A* or *B*.

Notation: "$A \cup B$" denotes "*A* union *B*" or the union of sets *A* and *B*.
Thus, $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$. Or $A \cup B = \{x \mid x \in A \vee x \in B\}$.

Examples:

a. Given that the box below represents X, the shaded area represents $A \cup B$:



b. Let A = {2,4,5} and B = {1,4,6,8}.
Then, A ∪ B = {1,2,4,5,6,8}

Remark:

Given sets A and B. $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Definition: Let A and B be subsets of a set X. The set $B - A$, called the ***difference*** of B and A, is the set of all elements in B which are not in A.

Thus, $B - A = \{x \in X \mid x \in B \text{ and } x \notin A\}$.

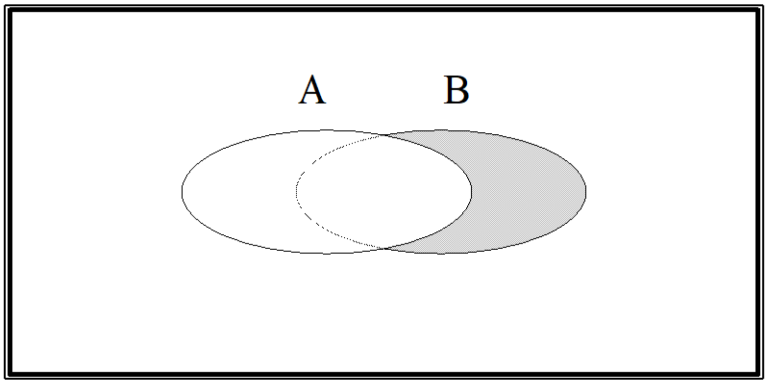
Examples:

a. Let $B = \{2,3,6,10,13,15\}$ and $A = \{2,10,15,21,22\}$.
Then $B - A = \{3,6,13\}$.

b. Let X be the set of natural numbers and A be the set of odd natural numbers. Then, $X - A$ = the set of even natural numbers; or $X - A = \{x \mid x \text{ is a natural number and } x \text{ is even}\}$.

c. Given that the box below represents X, the shaded area represents $B - A$.

X:



Definition: If $A \subset X$, then $X - A$ is sometimes called the ***complement*** of A with respect to X.

Notation: The following symbols are used to denote the complement of A with respect to X:

$$\complement_x A,\ \complement A,\ \sim A,\ \tilde{A}, \text{ and } A'$$

Thus, $\complement_x A = \{x \in X \mid x \notin A\}$.

Theorem: Let A and B be subsets of a set X.

Then, $A - B = A \cap \complement B$.

SUB- SET

Let set A be a set containing all students of your school and B be a set containing all students of class XII of the school. In this example each element of set B is also an element of set A. Such a set B is said to be subset of the set A. It is written as B Í A

Consider D ={1, 2, 3, 4,........}

E = {..... -3 -2, -1, 0, 1, 2, 3, ...}

Clearly each element of set D is an element of set E also \ D Í E

If A and B are any two sets such that each element of the set A is an element of the set B also, then A is said to be a subset of B.

Remarks

(i) Each set is a subset of itself i.e. $A \subseteq A$.

(ii) Null set has no element so the condition of becoming a subset is automatically satisfied. Therefore null set is a subset of every set.

(iii) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iv) If $A \subseteq B$ and $A \neq B$ then A is said to be a proper subset of B and B is said to be a super set of A. i.e. $A \subset B$ or $B \supset A$.

Example If A = {x : x is a prime number less than 5} and

B = {y : y is an even prime number} then is B a proper subset of A ?

Solution : It is given that

A = {2, 3 }, B = {2}.

Clearly $B \subseteq A$ and $B \neq A$

We write $B \subset A$

and say that B is a proper subset of A.

Example If A = {1, 2, 3, 4}, B = {2, 3, 4, 5}.

is $A \subseteq B$ or $B \subseteq A$?

Solution : Here $1 \in A$ but $1 \notin B \Rightarrow A \not\subseteq B$.

Also $5 \in B$ but $5 \notin A \Rightarrow B \not\subseteq A$.

Hence neither A is a subset of B nor B is a subset of A.

POWER SET

Let A = {a, b}

Subset of A are ϕ , {a}, {b} and {a, b}.

If we consider these subsets as elements of a new set B (say) then

B = {ϕ,{a},{b},{a,b}}

B is said to be the power set of A.

Notation : Power set of a set A is denoted by P(A).

Power set of a set A is the set of all subsets of the given set.

Example Write the power set of each of the following sets :

(i) A = {x : xÎ R and x^2 + 7 = 0 }.

(ii) B = {y : yÎ N and1 £ y £ 3}.

Solution :

(i) Clearly A = f (Null set)

\ f is the only subset of given set \ P (A)={f }

(ii) The set B can be written as {1, 2, 3}
\P (B) = { f , {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} } .

UNIVERSAL SET

Consider the following sets.

A = {x : x is a student of your school}

B = {y : y is a male student of your school}

C = {z : z is a female student of your school}

D = {a : a is a student of class XII in your school}

Clearly the set B, C, D are all subsets of A.

CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

A={1, 2}, B= {3, 4, 5}.

Set of all ordered pairs of elements of A and B is {(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)}

This set is denoted by A × B and is called the cartesian product of sets A and B.

i.e. A×B ={(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)}

Cartesian product of B sets and A is denoted by B×A.

In the present example, it is given by

B×A = {(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)}

Clearly A×B ¹ B×A.

In the set builder form :

A×B = {(a,b) : a Î A and b Î B }

B×A = {(b,a) : b Î B and a Î A }

Note : If A = f or B = f or A , B = f

then A ´ B = B ´A = f .

Example

(1) Let A={a,b,c}, B={d,e}, C={a,d}.

Find (i) A×B(ii) B×A (iii) A×(B È C) (iv) (A Ç C) ´ B

(v) (A Ç B) ´C (vi) A ´ (B - C) .

Solution : (i) A×B ={(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)}.

(ii) B×A = {(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)}.

(iii) A = {a, b, c}, B È C ={a,d,e}.
× (B È C) ={(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e).

(iv) A ÇC = {a}, B={d, e}.

\(A ÇC)×B={(a, d), (a, e)}
(v) A ÇB = f , c={a,d}, \ A Ç B ´ c = f

(vi) A = {a,b,c}, B - C = {e}.\A ´ (B - C) = {(a,e),(b,e),(c,e)}

PART – B (5 x 2 =10)

Possible Questions (2 marks)

1. Find the polar representations for the complex number z=3-2i.
2. Find the polar representations for the complex number $z=6+6i\sqrt{3}$.
3. Find the polar representations for the complex number z=-4i.
4. Find the polar representations for the complex number $z=-\frac{1}{4}+i\frac{\sqrt{3}}{4}$.
5. Find the polar representations for the complex number z=cos a- i sin a .
6. State the De Moivre's theorem
7. Find the square roots of the complex numbers z=1+i.
8. Find the square roots of the complex numbers z=i.
9. Compute $(1+i)^{1000}$
10. Find the cube roots of the complex numbers z= -i.
11. Find the cube roots of the complex numbers z= 27.
12. Compute $(-1+i)^4$
13. Define finite and infinite sets
14. Define Complement of a set
15. Prove that if A and B are finite sets, then $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

PART – C (5 x 6 =30)

Possible Questions (6 marks)

1) Find the Polar representation of the complex number z=1+cos a +i sin a, a ϵ (0,2Π).

2) Compute $z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$

3) i) Find polar representations for the complex number z=$-\frac{1}{2} - i$ √

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Subject: Algebra **Subject Code: 17MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit I

Complex number in Polar form

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

S.No	Question	Choice 1	Choice 2	Choice 3	Choice 4		Answer
1	A complex number $z=x+iy$ write the polar representation in the form……………	$z=r(\cos\Theta+i\sin\Theta)$	$z=r(\cos\Theta)$	$z=(\cos\Theta+i\sin\Theta)$	$z=r(\cos\Theta-i\sin\Theta)$		$z=r(\cos\Theta+i\sin\Theta)$
2	The polar representation $z=r(\cos\Theta+i\sin\Theta)$ where $r\epsilon$……………	$[0,\infty]$	$[0,1)$	$[1,\infty)$	$[0,\infty)$		$[0,\infty)$
3	The polar representation $z=r(\cos\Theta+i\sin\Theta)$ where $\Theta\epsilon$……………	$[0,\Pi]$	$(0,2\Pi]$	$[0,2\Pi)$	$[0,2\Pi]$		$[0,2\Pi)$
4	The polar argument Θ of the geometric imageof z is called the……………………of z.	angle	argument	theta	coordinate		argument
5	The polar argument r of the geometric imageof z is called the……………………of z.	root	real	modulus	imaginary		modulus
6	For …………………the modulus and argument of z are uniquely determined	$z=0$	$z>0$	$z<0$	$z\neq0$		$z\neq0$
7	For $z\neq0$ the modulus and argument of z are ……………...determined	unique	double	triple	zero		unique

8	Two complex numbers z1 and z2≠0 are equal if and only if ……………..	r1=r2	r1<r2	r1>r2	r1/r2		r1=r2
9	Two complex numbers z1 and z2≠0 are …………….. if and only if r1=r2	one	equal	not equal	multiple		equal
10	Two complex numbers z1 and z2≠0 are equal if and only if r1=r2 and t1-t2=………..,for an integer k.	kΠ	-2	k/Π	2kΠ		2kΠ
11	The set Arg z is called the ……………………argument of the complexnumber z.	finite	infinite	extended	singular		extended
12	Any complexnumber z can be represented as z=r(cosΘ+isinΘ) ,where r………………	≥0	≤0	>0	<0		≥0
13	Any complexnumber z can be represented as z=r(cosΘ+isinΘ) ,where r ≥0 and Θϵ………………	Z	R	W	N		R
14	The modulus of the numbers z= -1+i√3 is………………..	2	-2	1	-1		2
15	The modulus of the numbers z= 1-i√3 is………………..	#REF!	1	2	-2		2
16	The modulus of the numbers z= 2+2i is………………..	√2	3√2	4√2	2√2		2√2
17	The modulus of the numbers z= -1-i is………………..	√2	3√2	4√2	2√2		√2
18	The argument of the numbers z= -1+i√3 is………………..	Π/3	2Π/3	5Π/3	4Π/3		5Π/3
19	The argument of the numbers z= 1-i√3 is………………..	Π/3	2Π/3	Π	4Π/3		2Π/3

20	The argument of the numbers z= 2+2i is………………..	Π/4	7Π/4	5Π/4	3Π/4		Π/4
21	The argument of the numbers z= -1-i is………………..	Π/4	7Π/4	5Π/4	3Π/4		5Π/4
22	The modulus of the numbers z= 2i is………………..	0	1	2	3		2
23	The modulus of the numbers z= -1 is………………..	1	2	3	4		1
24	The modulus of the numbers z= 2 is………………..	1	2	3	4		2
25	The modulus of the numbers z= -3i is………………..	0	3	6	9		3
26	The argument of the numbers z= 2i is………………..	Π/2	7Π/2	5Π/2	3Π/2		Π/2
27	The argument of the numbers z= -1 is………………..	Π/4	Π/2	Π/3	Π		Π
28	The argument of the numbers z= 2 is………………..	0	Π	Π/2	Π/4		0
29	The argument of the numbers z= -3i is………………..	Π/2	7Π/2	5Π/2	3Π/2		3Π/2
30	cos0 +isin 0=……………	1	-1	2	-2		1
31	cos Π/2 + i sin Π/2 =……………	1	-1	i	negative i		i
32	cos Π + i sin Π =……………	1	-1	i	negative i		-1

33	cos 3Π/2 + i sin 3Π/2 =…………..	1	-1	i	negative i		negative i
34	The complex number z=(1+ cos z+sin a) if a=Π then z=……………..	0	1	2	3		0
35	The complex number z=(1+ cos z+sin a) if ……………. then z=0	a<Π	a>Π	a=Π	a≠Π		a=Π
36	In De Moivre's theorem the power ofcomplex number z^n=…………..	r^n/ (cos nΘ+isin nΘ)	(cos nΘ+i sin nΘ)	r^n (cos nΘ-isin nΘ)	r^n (cosΘ+i sin Θ)		r^n (cos nΘ+i sin nΘ)
37	$\|z^n\|$=………..	$\|z\|^n$	$\|-z\|^n$	$\|1/z\|^n$	$\|z\|$		$\|z\|^n$
38	If r=1 then $(\cos n\Theta+i\sin n\Theta)^n$=……………………..	(cos nΘ+i sin nΘ)	(cos nΘ-i sin nΘ)	(cos Θ/n+i sin Θ/n)	(cos Θ+i sin Θ)		(cos nΘ+i sin nΘ)
39	If …………….. then $(\cos n\Theta+i\sin n\Theta)^n$=(cos nΘ+i sin nΘ)	r=0	r=1	r=-1	r=2		r=1
40	The value of $(1+i)^{1000}$=…………..	2^500	1^1000	2^1000	1^500		2^500
41	In the field of real numbers Z^n-z0=…………..	0	1	2	3		0
42	In the field of real numbers Z^n-z0=0 is used for defining the …………roots of number z0.	1st	2nd	n th	(n+1) th		n th
43	In the field of real numbers Z^n-z0=0 is used for defining the n th ………….. of number z0.	numbers	real	equations	roots		roots
44	Any solution Z of the equation Z^n-z0=0 an …………root of the complex number z0.	1st	2nd	n th	(n+1) th		n th
45	Any solution Z of the equation Z^n-z0=0 an n th root of the ……….. number z0.	real	complex	imaginary	rational		complex

46	Any solution Z of the equation Z^n-z0=0 an n th root of the complex number ………….....	z0	z1	z2	z3		z0
47	the root of the equation Z^n-1=0 are called the n th root of ………....	unity	finite	infinite	equation		unity
48	If A = {1,2,3,4,…} then the set A is	finite	composite	infinite	equality		infinite
49	If a finite set S has 'n' elements then the power set of S has ____ elements	n	2^n	n-1	n+1		2^n
50	If A = {1,2,3,4,5}and B = {3,7,9} then A\B =	{1,2,4,5)	{1,2,3,4,5,7,9}	{7,9}	{3}		{1,2,4,5)
51	If A={a,b,c,d} and B ={f,b,d,g} then A∩B =	{a,b,c}	{a,b,c,d,f}	{b,d}	{f,g,d}		{b,d}
52	n(A∪B)=	n(A)+n(B)	n(A)+n(B)-n(A∩B)	n(A)-n(B)	n(A)-n(B)+n(A∩B)		n(A)+n(B)-n(A∩B)

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Department of Mathematics

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Class : I – B.Sc. Mathematics	**Semester : I**	**6**	**1**	**0**	**6**

UNIT II

Equivalence relations, Functions, Composition of functions, Invertible functions, One to one Correspondence and cardinality of a set, Well-ordering property of positive integers.

SUGGESTED READINGS

TEXT BOOKS

1. Titu Andreescu., and Dorin Andrica,(2006). Complex Numbers from A to Z, Birkhauser. Library of Congress Cataloging-in-Publication Data Andreescu, Titu.

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UNIT – II

Relations and Functions

RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim}

B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly R Í A ´ B, i.e.R = {(a,b):a Î A,b Î B and aRb}

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) R = f , R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) R = { (a,a) " a Î A } , is called the identity relation.

Domain and Range of a Relation

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

Consider previous example given above.

Domain = {Mohan, Sohan, David, Karim}

Range = {Rita, Marry, Fatima}

Example 1 Given that A = {2, 4, 5, 6, 7}, B = {2, 3}.

R is a relation from A to B defined by
R = {(a, b) : a Î A, b Î B and a is divisible by b}

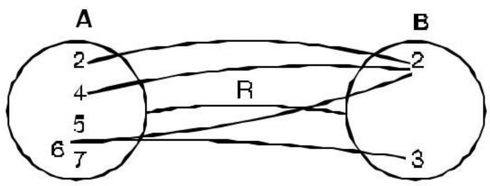
find (i) R in the roster form
(ii) Domain of R
(iii) Range of R
(iv) Repersent R diagramatically.

Solution : (i) R = {(2, 2), (4, 2), (6, 2), (6, 3)}

(ii) Domain of R = {2, 4, 6}

(iii) Range of R = {2, 3}

(iv)



Example 2 If R is a relation 'is greater than' from A to B, where A= {1, 2, 3, 4, 5} and B = {1,2,6}. Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {3, 4, 5}

(iii) Range of R = {1, 2}

2.1 Overview

This chapter deals with linking pair of elements from two sets and then introduce relations between the two elements in the pair. Practically in every day of our lives, we pair the members of two sets of numbers. For example, each hour of the day is paired with the local temperature reading by T.V. Station's weatherman, a teacher often pairs each set of score with the number of students receiving that score to see more clearly how well the class has understood the lesson. Finally, we shall learn about special relations called functions.

2.1.1 *Cartesian products of sets*

Definition : Given two non-empty sets A and B, the set of all ordered pairs (x, y), where $x \in$ A and $y \in$ B is called Cartesian product of A and B; symbolically, we write

A × B = {(*x*, *y*) | $x \in$ A and $y \in$ B}

If A = {1, 2, 3} and B = {4, 5}, then

A × B = {(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)}

And B × A = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal, i.e. $(x, y) = (u, v)$ if and only if $x = u$, $y = v$.

(ii) If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = p \times q$.

(i) A × A × A = {(*a*, *b*, *c*) : *a*, *b*, *c* ∈ A}. Here (*a*, *b*, *c*) is called an ordered triplet.

2.1.2 *Relations* A Relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product set A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B.

The set of all first elements in a relation R, is called the domain of the relation R, and the set of all second elements called images, is called the range of R.

For example, the set R = {(1, 2), (– 2, 3), ($\frac{1}{2}$, 3)} is a relation; the domain of

R = {1, – 2, $\frac{1}{2}$ } and the range of R = {2, 3}.

(i) A relation may be represented either by the Roster form or by the set builder form, or by an arrow diagram which is a visual representation of a relation.

(ii) If $n(A) = p$, $n(B) = q$; then the $n(A \times B) = pq$ and the total number of possible relations from the set A to set B = 2^{pq}.

2.1.3 *Functions* A relation *f* from a set A to a set B is said to be **function** if every element of set A has one and only one image in set B.

In other words, a function *f* is a relation such that no two pairs in the relation has the same first element.

The notation $f : X \rightarrow Y$ means that *f* is a function from X to Y. X is called the **domain** of *f* and Y is called the **co-domain** of *f*. Given an element $x \in X$, there is a unique element

y in Y that is related to *x*. The unique element *y* to which *f* relates *x* is denoted by *f* (*x*) and is called *f* of *x*, or the **value of *f* at *x***, or the *image of x under f*.

The set of all values of *f* (*x*) taken together is called the **range of *f*** or image of X under *f*. Symbolically.

range of $f = \{ y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}$

Definition : A function which has either **R** or one of its subsets as its range, is called a real valued function. Further, if its domain is also either **R** or a subset of **R**, it is called a real function.

2.1.4 *Some specific types of functions*

(i) **Identity function:**

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = x$ for each $x \in \mathbf{R}$ is called the **identity function.**

Domain of $f = \mathbf{R}$

Range of $f = \mathbf{R}$

(ii) **Constant function:** The function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = C$, $x \in \mathbf{R}$, where C is a constant $\in \mathbf{R}$, is a **constant function**.

Domain of $f = \mathbf{R}$

Range of $f = \{C\}$

(iii) **Polynomial function:** A real valued function $f : \mathbf{R} \to \mathbf{R}$ defined by $y = f(x) = a_0 + a_1x + ... + a_nx^n$, where $n \in \mathbf{N}$, and $a_0, a_1, a_2...a_n \in \mathbf{R}$, for each $x \in \mathbf{R}$, is called Polynomial functions.

(iv) **Rational function:** These are the real functions of the type $\frac{f(x)}{g(x)}$, where f(x) and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$. For example $f : \mathbf{R} - \{-2\} \to \mathbf{R}$ defined by $f(x) = \frac{x+1}{x+2}$, $x \in \mathbf{R} - \{-2\}$ is a rational function.

(v) **The Modulus function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$x \in \mathbf{R}$ is called the modulus function.

Domain of $f = \mathbf{R}$

Range of $f = \mathbf{R}^+ \cup \{0\}$

(vi) **Signum function:** The real function

$f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x)\quad \frac{|x|}{x},\ x\ \ 0 \qquad 1,\ \text{if}\ \ x\ \ 0$$

$$0,\ \text{if}\ \ x\ \ 0$$

$$0,\ x\ \ 0 \qquad 1,\ \text{if}\ \ x\ \ 0$$

is called the **signum function**. Domain of f = **R**, Range of f = {1, 0, – 1}

(vii) **Greatest integer function:** The real function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = [x]$, $x \in \mathbf{R}$ assumes the value of the greatest integer less than or equal to x, is called the **greatest integer function**.

Thus $f(x) = [x] = -1$ for $-1 \le x < 0$ $f(x) = [x] = 0$ for $0 \le x < 1$

$[x] = 1$ for $1 \le x < 2$

$[x] = 2$ for $2 \le x < 3$ and so on

2.1.5 ***Algebra of real functions***

(i) ***Addition of two real functions***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \in \mathbf{R}$.

Then we define $(f + g) : X \to \mathbf{R}$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

(ii) ***Subtraction of a real function from another***

Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$.

Then, we define $(f - g) : X \to \mathbf{R}$ by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) ***Multiplication by a Scalar***

Let $f : X \to \mathbf{R}$ be a real function and α be any scalar belonging to **R**. Then the product αf is function from X to **R** defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(iv) ***Multiplication of two real functions***

Let $f : X \to \mathbf{R}$ and $g : x \to \mathbf{R}$ be any two real functions, where $X \subseteq \mathbf{R}$. Then product of these two functions i.e. $fg : X \to \mathbf{R}$ is defined by $(fg)(x) = f(x)\, g(x)$ $x \in X$.

(v) ***Quotient of two real function***

Let f and g be two real functions defined from X → **R**. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined from X → **R** as

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in$ X.

- **Note** Domain of sum function $f + g$, difference function $f - g$ and product function fg.

$= \{x : x \in D_f \cap D_g\}$

where D_f = Domain of function f

D_g = Domain of function g

$F = \{x : x \in D_f \cap D_g \text{ and } g(x) \neq 0\}$

2.2 Solved Examples

Short Answer Type

Example 1 Let A = {1, 2, 3, 4} and B = {5, 7, 9}. Determine

(i) A × B (ii) B × A

(iii) Is A × B = B × A ? (iv) Is n (A × B) = n (B × A) ?

(i) A × B = {(1, 5), (1, 7), (1, 9), (2, 5), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9)}

(ii) B × A = {(5, 1), (5, 2), (5, 3), (5, 4), (7, 1), (7, 2), (7, 3), (7, 4), (9, 1), (9, 2), (9, 3), (9, 4)}

(iii) No, A × B ≠ B × A. Since A × B and B × A do not have exactly the same ordered pairs.

(iv) n (A × B) = n (A) × n (B) = 4 × 3 = 12
n (B × A) = n (B) × n (A) = 4 × 3 = 12

Hence n (A × B) = n (B × A)

Example 2 Find x and y if:

(i) $(4x + 3, y) = (3x + 5, -2)$ (ii) $(x - y, x + y) = (6, 10)$

Solution

(i) Since $(4x + 3, y) = (3x + 5, -2)$, so

$4x + 3 = 3x + 5$

or $x = 2$

and $y = -2$

(ii) $x - y = 6$

$x + y = 10$

$\therefore 2x = 16$

r $x = 8$

$8 - y = 6$

$\therefore y = 2$

Example 3 If A = {2, 4, 6, 9} and B = {4, 6, 18, 27, 54}, $a \in$ A, $b \in$ B, find the set of ordered pairs such that 'a' is factor of 'b' and $a < b$.

Solution Since A = {2, 4, 6, 9}

B = {4, 6, 18, 27, 54},

we have to find a set of ordered pairs (a, b) such that a is factor of b and $a < b$.

Since 2 is a factor of 4 and 2 < 4.

So (2, 4) is one such ordered pair.

Similarly, (2, 6), (2, 18), (2, 54) are other such ordered pairs. Thus the required set of ordered pairs is

{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54,), (9, 18), (9, 27), (9, 54)}.

FUNCTION

A **Function** assigns to each element of a set, exactly one element of a related set. Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few. The third and final chapter of this part highlights the important aspects of functions.

Function - Definition

A function or mapping (Defined as $f:X\to Y f:X\to Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each $x\in X x\in X$, there exists a unique $y\in Y y\in Y$ such that $(x,y)\in R (x,y)\in R$. 'x' is called pre-image and 'y' is called image of function f.

A function can be one to one or many to one but not one to many.

Injective / One-to-one function

A function $f:A\to B f:A\to B$ is injective or one-to-one function if for every $b\in B b\in B$, there exists at most one $a\in A a\in A$ such that $f(s)=t f(s)=t$.

This means a function **f** is injective if $a1\neq a2 a1\neq a2$ implies $f(a1)\neq f(a2) f(a1)\neq f(a2)$.

Example

- $f:N\to N, f(x)=5x f:N\to N, f(x)=5x$ is injective.
- $f:N\to N, f(x)=x2 f:N\to N, f(x)=x2$ is injective.
- $f:R\to R, f(x)=x2 f:R\to R, f(x)=x2$ is not injective as $(-x)2=x2 (-x)2=x2$

Surjective / Onto function

A function $f:A\to B f:A\to B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in B b\in B$, there exists some $a\in A a\in A$ such that $f(a)=b f(a)=b$. This means that for any y in B, there exists some x in A such that $y=f(x) y=f(x)$.

Example

- $f:N\to N, f(x)=x+2 f:N\to N, f(x)=x+2$ is surjective.

- $f:R\to R, f(x)=x^2$ is not surjective since we cannot find a real number whose square is negative.

Bijective / One-to-one Correspondent

A function $f:A\to B$ is bijective or one-to-one correspondent if and only if **f** is both injective and surjective.

Problem

Prove that a function $f:R\to R$ defined by $f(x)=2x–3$ is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x_1)=f(x_2)$, then $2x_1–3=2x_2–3$ and it implies that $x_1=x_2$.

Hence, f is **injective**.

Here, $2x–3=y$

So, $x=(y+5)/3$ which belongs to R and $f(x)=y$.

Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

Inverse of a Function

The **inverse** of a one-to-one corresponding function $f:A\to B$, is the function $g:B\to A$, holding the following property −

$f(x)=y\Leftrightarrow g(y)=x$

The function f is called **invertible**, if its inverse function g exists.

Example

- A Function $f:Z\to Z, f(x)=x+5$, is invertible since it has the inverse function $g:Z\to Z, g(x)=x-5$.
- A Function $f:Z\to Z, f(x)=x^2$ is not invertiable since this is not one-to-one as $(-x)^2=x^2$.

Composition of Functions

Two functions $f:A\to B$ and $g:B\to C$ can be composed to give a composition gof. This is a function from A to C defined by $(gof)(x)=g(f(x))$

Example

Let f(x)=x+2f(x)=x+2 and g(x)=2x+1g(x)=2x+1,

find (fog)(x)(fog)(x) and (gof)(x)(gof)(x).

Solution

(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3(fog)(x)=f(g(x))=f(2x+1)=2x+1+2=2x+3

(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5(gof)(x)=g(f(x))=g(x+2)=2(x+2)+1=2x+5

Hence, (fog)(x)≠(gof)(x)(fog)(x)≠(gof)(x)

Some Facts about Composition

- If f and g are one-to-one then the function (gof)(gof) is also one-to-one.
- If f and g are onto then the function (gof)(gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

Theorem A total function has an inverse if and only if it is bijective.

Proof

Suppose $f : A \rightarrow B$ has an inverse $f-1$. Then we show that f is bijective.

We first show that f is one to one. Suppose $f(x_1) = f(x_2)$ then

$$f^{-1}(f(x_1)) = f^{-1}(f(x_2)),$$

$$\Rightarrow f^{-1}of(x_1) = f^{-1}of(x_2),$$

$$\Rightarrow 1_A(x_1) = 1_A(x_2),$$

$$\Rightarrow x_1 = x_2.$$

Next we first show that f is onto. Let $b \in B$ and let $a = f-1\,(b)$ then

$f(a) = f(f^{-1}(b)) = b$ and so f is surjective.

The second part of the proof is concerned with showing that if $f : A \rightarrow B$ is bijective then it has an inverse $f-1$. Clearly, since f is bijective we have that for each $a \in A$ there exists a unique b B such that $f(a) \in b$.

Define g: $B \quad A$ by letting $g(b)$ be the unique a in A such that $f(a)\ b$. Then we have that:

$$gof(a) = g(b) = a \text{ and } f\,og(b) = f(a) = b.$$

Therefore, g is the inverse of f.

Some Discrete Examples

EXAMPLE 2 Suppose $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ and

$$f = \{(1, x), (2, y), (3, z), (4, y)\}.$$

Then f is a function $A \to B$ with domain A and target B. Since rng $f = \{x, y, z\} = B$, f is onto. Since $f(2) = f(4)$ $(= y)$ but $2 \neq 4$, f is not one-to-one. [In fact, there can exist no one-to-one function $A \to B$. Why not? See Exercise 25(a).] ▲

EXAMPLE 3 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$ and

$$f = \{(1, w), (2, y), (3, x)\}.$$

Then $f : A \to B$ is a function with domain A and range $\{w, y, x\}$. Since rng $f \neq B$, f is not onto. [No function $A \to B$ can be onto. Why not? See Exercise 25(b).] This function is one-to-one because $f(1)$, $f(2)$, and $f(3)$ are all different: If $f(a_1) = f(a_2)$, then $a_1 = a_2$. ▲

EXAMPLE 4 Suppose $A = \{1, 2, 3\}$, $B = \{x, y, z\}$,

$$f = \{(1, z), (2, y), (3, y)\} \quad \text{and} \quad g = \{(1, z), (2, y), (3, x)\}.$$

Then f and g are functions from A to B. The domain of f is A and dom $g = A$ too. The range of f is $\{z, y\}$, which is a proper subset of B, so f is not onto. On the other hand, g is onto because rng $g = \{z, y, x\} = B$. This function is also one-to-one because $g(1)$, $g(2)$, and $g(3)$ are all different: If $g(a_1) = g(a_2)$, then $a_1 = a_2$. Notice that f is not one-to-one: $f(2) = f(3)$ $(= y)$, yet $2 \neq 3$. ▲

EXAMPLE 5 Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x - 3$. Then dom $f = \mathbb{Z}$. To find rng f, note that

$$\begin{aligned} b \in \text{rng } f &\leftrightarrow b = 2a - 3 && \text{for some integer } a \\ &\leftrightarrow b = 2(a - 2) + 1 && \text{for some integer } a \end{aligned}$$

and this occurs if and only if b is odd. Thus, the range of f is the set of odd integers. Since rng $f \neq \mathbb{Z}$, f is not onto. It is one-to-one, however: If $f(x_1) = f(x_2)$, then $2x_1 - 3 = 2x_2 - 3$ and $x_1 = x_2$. ▲

EXAMPLE 6 Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 2x - 3$. This might look like a perfectly good function, as in the last example, but actually there is a difficulty. If we try to calculate $f(1)$, we obtain $f(1) = 2(1) - 3 = -1$ and $-1 \notin \mathbb{N}$. Hence, no function has been defined. ▲

PROBLEM 7. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2 - 5x + 5$. Determine whether or not f is one-to-one and/or onto.

Solution. To determine whether or not f is one-to-one, we consider the possibility that $f(x_1) = f(x_2)$. In this case, $x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$, so $x_1^2 - x_2^2 = 5x_1 - 5x_2$ and $(x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$. This equation indeed has solutions with $x_1 \neq x_2$: Any x_1, x_2 satisfying $x_1 + x_2 = 5$ will do, for instance, $x_1 = 2$, $x_2 = 3$. Since $f(2) = f(3) = -1$, we conclude that f is not one-to-one.

Is f onto? Recalling that the graph of $f(x) = x^2 - 5x + 5$, $x \in \mathbb{R}$, is a parabola with vertex $(\frac{5}{2}, -\frac{5}{4})$, clearly any integer less than -1 is not in the range of f. Alternatively, it is easy to see that 0 is not in the range of f because $x^2 - 5x + 5 = 0$ has no integer solutions (by the quadratic formula). Either argument shows that f is not onto. ∎

PROBLEM 8. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = 3x^3 - x$. Determine whether or not f is one-to-one and/or onto.

Solution. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$. Then $3x_1^3 - x_1 = 3x_2^3 - x_2$, so $3(x_1^3 - x_2^3) = x_1 - x_2$ and

$$3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2.$$

If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{3}$, which is impossible since x_1 and x_2 are integers. Thus, $x_1 = x_2$ and f is one-to-one.

Is f onto? If yes, then the equation $b = f(x) = 3x^3 - x$ has a solution in $\mathbb{Z}$ for every integer b. This seems unlikely and, after a moment's thought, it occurs to us that the integer $b = 1$, for example, cannot be written this way: $1 = 3x^3 - x$ for some integer x implies $x(3x^2 - 1) = 1$. But the only pairs of integers whose product is 1 are the pairs $1, 1$ and $-1, -1$. So here, we would require $x = 3x^2 - 1 = 1$ or $x = 3x^2 - 1 = -1$, neither of which is possible. The integer $b = 1$ is a counterexample to the assertion that f is onto, so f is not onto. ∎

EXAMPLE

Let $g: \mathsf{R} \to \mathsf{R}$ be defined by $g(x) = x^2$. The domain of g is R; the range of g is the set of nonnegative real numbers. Since this is a proper subset of R, g is not onto. Neither is g one-to-one since $g(3) = g(-3)$, but $3 \neq -3$. ▲

Define $h: [0, \infty) \to \mathsf{R}$ by $h(x) = x^2$. This function is identical to the function g of the preceding example except for its domain. By *restricting the domain* of g to the nonnegative reals we have produced a function h which is one-to-one since $h(x_1) = h(x_2)$ implies $x_1^2 = x_2^2$ and hence $x_1 = \pm x_2$. Since $x_1 \geq 0$ and $x_2 \geq 0$, we must have $x_1 = x_2$. ▲

The Identity Function

For any set A, the *identity function on* A is the function $\iota_A: A \to A$ defined by $\iota_A(a) = a$ for all $a \in A$. In terms of ordered pairs,

$$\iota_A = \{(a, a) \mid a \in A\}.$$

When there is no possibility of confusion about A, we will often write ι, rather than ι_A. (The Greek symbol ι is pronounced "yōta", so that "ι_A" is read "yota sub A."

The graph of the identity function on R is the familiar line with equation $y = x$. The identity function on a set A is indeed a function $A \to A$ since, for any $a \in A$, there is precisely one pair of the form $(a, y) \in \iota$, namely, the pair (a, a).

INVERSES AND COMPOSITION

The Inverse of a Function

Suppose that f is a one-to-one onto function from A to B. Given any $b \in B$, there exists $a \in A$ such that $f(a) = b$ (because f is onto) and only one such a (because f is one-to-one). Thus, for each $b \in B$, there is precisely one pair of the form $(a, b) \in f$. It follows that the set $\{(b, a) \mid (a, b) \in f\}$, obtained by reversing the ordered pairs of f, is a function from B to A (since each element of B occurs precisely once as the first coordinate of an ordered pair).

EXAMPLE 13 If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, then

$$f = \{(1, x), (2, y), (3, z), (4, t)\}$$

is a one-to-one onto function from A to B and, reversing its pairs, we obtain a function $B \to A$: $\{(x, 1), (y, 2), (z, 3), (t, 4)\}$. ▲

DEFINITION A function $f : A \to B$ *has an inverse* if and only if the set obtained by reversing the ordered pairs of f is a function $B \to A$. If $f : A \to B$ has an inverse, the function

$$\boxed{f^{-1} = \{(b, a) \mid (a, b) \in f\}}$$

is called the *inverse* of f.

We pronounce f^{-1}, "f inverse," terminology which should not be confused with $\frac{1}{f}$: f^{-1} is simply the name of a certain function, the inverse of f.[2]

If $f : A \to B$ has an inverse $f^{-1} : B \to A$, then f^{-1} also has an inverse because reversing the pairs of f^{-1} gives a function, namely f: thus, $(f^{-1})^{-1} = f$.

EXAMPLE 14 If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, and

$$f = \{(1, x), (2, y), (3, z), (4, t)\}$$

then

$$f^{-1} = \{(x, 1), (y, 2), (z, 3), (t, 4)\}$$

and $(f^{-1})^{-1} = \{(1, x), (2, y), (3, z), (4, t)\} = f$. ▲

PROPOSITION

A function $f : A \to B$ has an inverse $B \to A$ if and only if f is one-to-one and onto.

For any function g, remember that $(x, y) \in g$ if and only if $g(x) = y$; in particular, $(b, a) \in f^{-1}$ if and only if $a = f^{-1}(b)$. Thus,

$$a = f^{-1}(b) \leftrightarrow (b, a) \in f^{-1} \leftrightarrow (a, b) \in f \leftrightarrow f(a) = b.$$

The equivalence of the first and last equations here is very important:

(2) $$a = f^{-1}(b) \text{ if and only if } f(a) = b.$$

For example, if for some function f, $\pi = f^{-1}(-7)$, then we can conclude that $f(\pi) = -7$. If $f(4) = 2$, then $4 = f^{-1}(2)$.

The solution to the equation $2x = 5$ is $x = \frac{5}{2} = 2^{-1} \cdot 5$. Generally, to solve the equation $ax = b$, we ask if $a \neq 0$, and if this is the case, we multiply each side of the equation by a^{-1}, obtaining $x = a^{-1}b = \frac{b}{a}$. Since all real numbers except 0 have a multiplicative inverse, checking that $a \neq 0$ is just checking that a has an inverse.

Look again at statement (2). We solve the equation $f(x) = y$ for x in the same way we solve $ax = b$ for x. We first ask if f has an inverse, and if it does, apply f^{-1} to each side of the equation, obtaining $x = f^{-1}(y)$.

The "application" of f^{-1} to each side of the equation $y = f(x)$ is very much like multiplying each side by f^{-1}. "Multiplying by f^{-1}" may sound foolish, but there is a context (called *group theory*) in which it makes good sense. Our intent here is just to provide a good way to remember the fundamental relationship expressed in (2).

EXAMPLE

If $f : \mathsf{R} \to \mathsf{R}$ is defined by $f(x) = 2x - 3$, then f is one-to-one and onto, so an inverse function exists. According to (2), if $y = f^{-1}(x)$, then $x = f(y) = 2y - 3$. Thus, $y = \frac{1}{2}(x + 3) = f^{-1}(x)$. ▲

Let $A = \{x \in \mathsf{R} \mid x \leq 0\}$, $B = \{x \in \mathsf{R} \mid x \geq 0\}$ and define $f : A \to B$ by $f(x) = x^2$. This is just the squaring function with domain restricted so that it is one-to-one as well as onto. Since f is one-to-one and onto, it has an inverse. To obtain $f^{-1}(x)$, let $y = f^{-1}(x)$, deduce [by the relationship expressed in (2)] that $f(y) = x$ and so $y^2 = x$. Solving for y, we get $y = \pm\sqrt{x}$. Since $x = f(y)$, $y \in A$, so $y \leq 0$. Thus, $y = -\sqrt{x}$; $f^{-1}(x) = -\sqrt{x}$. ▲

PROBLEM 18. Let $A = \{x \mid x \neq \frac{1}{2}\}$ and define $f : A \to \mathrm{R}$ by $f(x) = \dfrac{4x}{2x-1}$.

Is f one-to-one? Find rng f. Explain why $f : A \to$ rng f has an inverse. Find dom f^{-1}, rng f^{-1}, and a formula for $f^{-1}(x)$.

Solution. Suppose $f(a_1) = f(a_2)$. Then $\dfrac{4a_1}{2a_1-1} = \dfrac{4a_2}{2a_2-1}$, so $8a_1a_2 - 4a_1 = 8a_1a_2 - 4a_2$, hence $a_1 = a_2$. Thus f is one-to-one.

Next,

$$
\begin{aligned}
y \in \text{rng } f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\
&\leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{4x}{2x-1} \\
&\leftrightarrow \text{there is an } x \in A \text{ such that } 2xy - y = 4x \\
&\leftrightarrow \text{there is an } x \in A \text{ such that } x(2y-4) = y.
\end{aligned}
$$

If $y = 2$, the equation $x(2y-4) = y$ becomes $0 = 2$ and no x exists. On the other hand, if $y \neq 2$, then $2y - 4 \neq 0$ and so, dividing by $2y-4$, we obtain $x = \dfrac{y}{2y-4}$. (It is easy to see that such x is never $\frac{1}{2}$; that is, $x \in A$.) Thus $y \in$ rng f if and only if $y \neq 2$. So rng $f = B = \{y \in \mathrm{R} \mid y \neq 2\}$.

Since $f : A \to B$ is one-to-one and onto, it has an inverse $f^{-1} : B \to A$. Also, dom f^{-1} = rng $f = B$ and rng f^{-1} = dom $f = A$. To find $f^{-1}(x)$, set $y = f^{-1}(x)$. Then

$$x = f(y) = \frac{4y}{2y-1}$$

and, solving for y, we get $y = \dfrac{x}{2x-4} = f^{-1}(x)$. ∎

Composition of Functions

DEFINITION If $f : A \to B$ and $g : B \to C$ are functions, then the *composition of* g *and* f is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

EXAMPLE 19 If $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{u, v, w\}$, and if $f : A \to B$ and $g : B \to C$ are the functions

$$f = \{(a, x), (b, y), (c, x)\}, \quad g = \{(x, u), (y, w)\},$$

then

$$(g \circ f)(a) = g(f(a)) = g(x) = u,$$
$$(g \circ f)(b) = g(f(b)) = g(y) = w,$$
$$(g \circ f)(c) = g(f(c)) = g(x) = u$$

and so $g \circ f = \{(a, u), (b, w), (c, u)\}$. ▲

EXAMPLE 20 If f and g are the functions $\mathsf{R} \to \mathsf{R}$ defined by

$$f(x) = 2x - 3, \quad g(x) = x^2 + 1,$$

then both $g \circ f$ and $f \circ g$ are defined and we have

$$(g \circ f)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 1$$

and

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) - 3.$$ ▲

EXAMPLE 21 In the definition of $g \circ f$, it is required that $\operatorname{rng} f \subseteq B = \operatorname{dom} g$. If $f : \mathsf{R} \to \mathsf{R}$ and $g : \mathsf{R} \setminus \{1\} \to \mathsf{R}$ are the functions defined by

$$f(x) = 2x - 3 \quad \text{and} \quad g(x) = \frac{x}{x - 1},$$

then $g \circ f$ is not defined because $\operatorname{rng} f = \mathsf{R} \not\subseteq \operatorname{dom} g$. On the other hand, $f \circ g$ is defined and

$$(f \circ g)(x) = 2\left(\frac{x}{x - 1}\right) - 3.$$ ▲

PROPOSITION Composition of functions is an associative operation.

Proof We must prove that $(f \circ g) \circ h = f \circ (g \circ h)$ whenever each of the two functions—$(f \circ g) \circ h$ and $f \circ (g \circ h)$—is defined. Thus, we assume that for certain sets A, B, C, and D, h is a function $A \to B$, g is a function $B \to C$, and f is a function $C \to D$. A direct proof is suggested.

Since the domain of $(f \circ g) \circ h$ is the domain of $f \circ (g \circ h)$ (namely, the set A), we have only to prove that $((f \circ g) \circ h)(a) = (f \circ (g \circ h))(a)$ for any $a \in A$. For this, we have

$$((f \circ g) \circ h)(a) = (f \circ g)(h(a)) = f(g(h(a)))$$

and

$$(f \circ (g \circ h))(a) = f((g \circ h)(a)) = f(g(h(a)))$$

as desired. ∎

If $f: A \to B$ has an inverse $f^{-1}: B \to A$, then, recalling (2),

$$f^{-1}(b) = a \text{ if and only if } b = f(a).$$

So for any $a \in A$,

$$a = f^{-1}(b) = f^{-1}(f(a)) = f^{-1} \circ f(a).$$

In other words, the composition $f^{-1} \circ f = \iota_A$, the identity function on A. Similarly, for any element $b \in B$,

$$b = f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b).$$

Thus, the composition $f \circ f^{-1} = \iota_B$ is the identity function on B. We summarize.

PROPOSITION Functions $f: A \to B$ and $g: B \to A$ are inverses if and only if $g \circ f = \iota_A$ and $f \circ g = \iota_B$; that is, if and only if

$$g(f(a)) = a \text{ and } f(g(b)) = b \quad \text{for all } a \in A \text{ and all } b \in B.$$

PROBLEM 23. Show that the functions $f: \mathbb{R} \to (1, \infty)$ and $g: (1, \infty) \to \mathbb{R}$ defined by

$$f(x) = 3^{2x} + 1, \qquad g(x) = \tfrac{1}{2}\log_3(x - 1)$$

are inverses.

Solution. For any $x \in \mathbb{R}$,

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(3^{2x} + 1) \\ &= \tfrac{1}{2}(\log_3[(3^{2x} + 1) - 1]) \\ &= \tfrac{1}{2}(\log_3 3^{2x}) = \tfrac{1}{2}2x = x\end{aligned}$$

and for any $x \in (1, \infty)$,

$$(f \circ g)(x) = f(g(x)) = f(\tfrac{1}{2}\log_3(x-1))$$
$$= 3^{2(\frac{1}{2}\log_3(x-1))} + 1$$
$$= 3^{\log_3(x-1)} + 1 = (x-1) + 1 = x.$$

ONE-TO-ONE CORRESPONDENCE AND THE CARDINALITY OF A SET

DEFINITIONS A *finite* set is a set which is either empty or in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$ of the first n natural numbers, for some $n \in \mathsf{N}$. A set which is not finite is called *infinite*.

DEFINITION Sets A and B have the *same cardinality* and we write $|A| = |B|$, if and only if there is a *one-to-one correspondence* between them; that is, if and only if there exists a one-to-one onto function from A to B (or from B to A).

EXAMPLES 25

- $a \mapsto x$, $b \mapsto y$ is a one-to-one correspondence between $\{a, b\}$ and $\{x, y\}$; hence, $|\{a, b\}| = |\{x, y\}|$ $(= 2)$.
- The function $f : \mathsf{N} \longrightarrow \mathsf{N} \cup \{0\}$ defined by $f(n) = n - 1$ is a one-to-one correspondence between N and $\mathsf{N} \cup \{0\}$; so $|\mathsf{N}| = |\mathsf{N} \cup \{0\}|$.
- The function $f : \mathsf{Z} \longrightarrow 2\mathsf{Z}$ defined by $f(n) = 2n$ is a one-to-one correspondence between the set Z of integers and the set 2Z of even integers; thus, Z and 2Z have the same cardinality. ▲

PROBLEM 26. Show that the set R^+ of positive real numbers has the same cardinality as the open interval $(0, 1) = \{x \in \mathsf{R} \mid 0 < x < 1\}$.

Solution. Let $f : (0, 1) \to \mathsf{R}^+$ be defined by

$$f(x) = \frac{1}{x} - 1.$$

We claim that f establishes a one-to-one correspondence between $(0, 1)$ and R^+.

To show that f is onto, we have to show that any $y \in \mathsf{R}^+$ is $f(x)$ for some $x \in (0, 1)$. But

$$y = \frac{1}{x} - 1 \text{ implies } x = \frac{1}{1+y}$$

which is in $(0, 1)$ since $y > 0$. Therefore,

$$y \in \mathsf{R}^+ \text{ implies } y = f\left(\frac{1}{1+y}\right)$$

so f is indeed onto. Also, f is one-to-one because

$$f(x_1) = f(x_2) \to \frac{1}{x_1} - 1 = \frac{1}{x_2} - 1$$
$$\to \frac{1}{x_1} = \frac{1}{x_2}$$
$$\to x_1 = x_2.$$

DEFINITIONS A set A is *countably infinite* if and only if $|A| = |\mathrm{N}|$ and *countable* if and only if it is either finite or countably infinite. A set which is not countable is *uncountable*.

PROBLEM 27. Show that $|\mathrm{Z}| = \aleph_0$.

Solution. The set of integers is infinite. To show they are countably infinite, we list them: $0, 1, -1, 2, -2, 3, -3, \ldots$. This list is just $f(1), f(2), f(3), \ldots$ where $f : \mathrm{N} \to \mathrm{Z}$ is defined by

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ -\frac{1}{2}(n-1) & \text{if } n \text{ is odd,} \end{cases}$$

which is certainly both one-to-one and onto. ∎

PROBLEM 28. Show that $|\mathrm{N} \times \mathrm{N}| = |\mathrm{N}|$.

Solution. The elements of $\mathrm{N} \times \mathrm{N}$ can be listed by the scheme illustrated in Fig 3.4. The arrows indicate the order in which the elements of $\mathrm{N} \times \mathrm{N}$ should be listed—(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), ... Wherever the arrows terminate, there is no difficulty in continuing, so each ordered pair acquires a definite position. ∎

WELL ORDERING PRINCIPLE

(Well-Ordering Principle).
Every non-empty subset of natural numbers contains its least element.
Proof:
To prove the weak form of the principle of mathematical induction. The proof is based on contradiction. That is, suppose that we need to prove that “whenever the statement P holds true, the statement Q holds true as well”. A proof by contradiction starts with the assumption that “the statement P holds true and the statement Q does not hold true” and tries to arrive at a contradiction to the validity of the statement P being true

PART – B (5 x 2 =10 Marks)

Possible Questions (2 Mark)

1. Define Equivalence relations.
2. Define functions with examples
3. Define composition functions with examples.
4. Define Ivertible functions
5. Define one-to-one correspondence with example
6. Define cardinality of a set.
7. State the two properities of composition functions
8. Write the various types of Functions.
9. Define domain & co domain of the function.
10. Define range of the function.
11. Define equality of two functions.
12. Define denumerable sets.
13. Define countable set
14. Define Identity Mapping.
15. Define constant mapping

PART – C (5 x 6 =10 Marks)

Possible Questions (6 Mark)

1). If ρ and σ are equivalence relations defined on a set S, Prove that $\rho \cap \sigma$ is an equivalence relation.

2) Show that the following functions are 1-1

i) f : R → R given by f(x)= $5x^2 - 1$

ii) f: Z → Egiven by f(n)=$3x^3 - x$

3)If the function f: R→ R is given by f(x)= cos x and g: R → R is given by g(x)=x^3 find $(g \circ f)(x)$ *and* $(f \circ g)(x)$ and show that they are not equal.

4) Explain about types of relation with examples.

5) Let A={1,2,3} and f,g,h and s be functions from A to A given by

f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) } ;

h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find $f \circ g$, $g \circ f$, $f \circ h \circ g$, $g \circ s$, $s \circ s$, $f \circ s$.

6) Let S={1,2,3,4,5} and T={1,2,3,8,9} and define the functions f : S→ T and g: S → S by f={(1,8), (3,9),(4,3),(2,1),(5,2)} and g={(1,2),(3,1),(2,2),(4,3),(5,2)} ,then find the values of the following $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$.

7) Let f, g and h: R → R be defined by f(x)=x+2 ,g(x)= $\frac{1}{x^2+1}$ and h(x)=3

Compute i)$h \circ g \circ f(x)$ ii)$g \circ h \circ f(x)$ iii) $g \circ f^{-1} \circ f(x)$.

8) If f: X→Y and A, B are two subsets of Y, then prove that

i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

9) For integers a,b define aRb if and only if a – b is divisible by m. Show that R defines an equivalence relation on Z.

10).Let A be the set A={xϵR \ x>0} and define f ,g, h :A→ R by f(x)=$\frac{x}{x+1}$,g(x)=$\frac{1}{x}$,h(x)=x+1 find $g \circ f$, $f \circ g$, $h \circ g \circ f$ and $f \circ g \circ h$.

11) Write about the types of function with example

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Subject: Algebra **Subject Code: 17MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit II

Relations and Functions

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

S.No	Question	Choice 1	Choice 2	Choice 3	Choice 4		Answer
1	If f:A→B hence f is called a …………….	function	form	formula	fuzzy		function
2	If the function f is otherwise called as ……………………	limit	mapping	lopping	inverse		mapping
3	If f:A→B in this set A is called the ……………..of the function f.	domain	co domain	set	element		domain
4	If f:A→B in this set B is called the ……………..of the function f.	domain	co domain	set	element		co domain
5	The value of the function f for a and is denoted by ……………………………	a(f)	f(a)	a	f		f(a)
6	If aϵA then the element in B which is assignedto ais called the ………………of a	B-image	a-image	A-image	f-image		f-image
7	The element a may be referred to as the …………………..of f(a)	f-image	pre-image	domain	codomain		pre-image

8	The ………….. of a function as the image of its domain	domain	range	co domain	image		range
9	The range of a function as the……………. of its domain	range	domain	image	preimage		image
10	The range of a function as the image of its ……………..	co domain	image	domain	range		domain
11	Let f be a mapping of A to B,Each element of A has a …………. and each element in B need not be appear as the image of an element in A.	unique preimage	unique image	unique zero	unique range		unique image
12	Let f be a mapping of A to B,Each element of …... has a unique image and each element in B need not be appear as the image of an element in A.	A	B	f	f(A)		A
13	Let f be a mapping of A to B,Each element of A has a unique image and each element in………. need not be appear as the image of an element in A.	A	B	f	f(A)		B
14	Let f be a mapping of A to B,Each element of A has a unique image and each element in B need not be appear as the …………. of an element in A.	domain	range	co domain	image		image
15	One-to-one mapping is also sometimes known as…………………	injection	bijection	surjection	imjection		injection
16	A mapping f:A→B is said to be ………….. if different elements in A have different f-images in B	zero	one-one	onto	into		one-one
17	A mapping f:A→B is said to be 1-1 if ……………….elements in A have different f-images in B	same	different	not equal	one		different
18	A mapping f:A→B is said to be 1-1 if different elements in A have different ……………. in B	pre images	f-images	B-images	A-images		f-images
19	In one-one mappings an element in B has only……………..preimage in A	zero	one	two	three		one

20	In …………...mappings an element in B has only one preimage in A	one-one	onto	into	one-oneonto		one-one
21	One-one onto mapping is also sometimes known as…………………	injection	bijection	surjection	imjection		bijection
22	A mapping f:A→B is said to be ………….. if different elements in A have same f-images in B	one-one	onto	into	many one		many one
23	In many-one mappings some elements in B has more than………………preimage in A	zero	one	two	three		one
24	In many-one mappings some elements in B has ………….. one preimage in A	equal	more than	less than	only		more than
25	Two sets A and B are said to have the same number of elements iff a one-one mapping of A onto B exists, such sets are said to be ………………..	equivalent	merely equivalent	cardinally equivalent	notequivalent		cardinally equivalent
26	Two sets A and B are said to have the same number of elements iff a ……….. mapping of A onto B exists, such sets are said to be cardinally equivalent	one-one	many one	onto	into		one-one
27	Two sets A and B are said to have the same number of elements iff a one-one mapping of A ………. B exists, such sets are said to be cardinally equivalent	one-one	many one	onto	into		onto
28	Two sets A and B are said to have the …….number of elements iff a one-one mapping of A onto B exists, such sets are said to be cardinally equivalent	same	different	zero	finite		same
29	Cardinally eqivalent can be written as……………	A+B	A-B	A~B	A/B		A~B
30	Cardinally eqivalent sets are to have the ……….. cardinal number.	zero	one	same	finite		same
31	Cardinally eqivalent sets are to have the same ……………. number.	rational	complex	real	cardinal		cardinal

32	If $f:A\to B$ is one-one onto, then $f^{-1}:B\to A$ .the mapping f^{-1} is called the ……………mapping of the mapping of f.	integral	inverse	invert	reverse		inverse
33	Only one-one and onto mapping posses……………..mappings.	integral	inverse	invert	reverse		inverse
34	Only ………….. mapping posses inverse mappings.	one-one and into	one-one	one-one and many one	one-one and onto		one-one and onto
35	If $f:A\to B$ is one-one onto, then $f^{-1}:B\to A$ is also …………………	one-one and into	one-one	one-one and many one	one-one and onto		one-one and onto
36	If $f:A\to B$ is one-one onto, then the inverse mapping of f is ……………...	zero	unique	different	same		unique
37	If $f:X\to Y$ and $g:Y\to Z$then the ………………..of the function f and g demoted by $(g_o f):X\to Z$.	inverse	composite	different	one-one		composite
38	If $f:X\to Y$ and $g:Y\to Z$then the composite of the function f and g demoted by ……………	$(f_o g):X\to Z$.	$(f_o g):X\to Y$.	$(g_o f):y\to Z$.	$(g_o f):X\to Z$.		$(g_o f):X\to Z$.
39	In general $g_o f$ …………… f_o g	equal	notequal	less than	more than		notequal
40	If xRx ,forevery x϶A since every triangle is congruent to it self.Thus R is ……………	reflexive	symmetic	transitive	anti-symmetric		reflexive
41	If xRy and yRz→ x Rz,since if triangle x is congruent to y and triangley is congrugent to z then,trainglex is congruent to z.Then R is	reflexive	symmetic	transitive	anti-symmetric		transitive
42	If xRy →y R z since if triangle x is congruent to y and triangle y is congrugent to x.Then R is ……………	reflexive	symmetic	transitive	anti-symmetric		symmetic
43	If R is reflexive,symmetric and transitive therefore R is an ……………….relation	one-one	onto	equivalence	equal		equivalence

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Department of Mathematics

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Class : I – B.Sc. Mathematics	**Semester : I**	**6 1 0 6**

UNIT III

Division algorithm, Divisibility and Euclidean algorithm, Congruence relation between integers, Principles of Mathematical Induction, Statement of Fundamental Theorem of Arithmetic.

SUGGESTED READINGS

TEXT BOOKS

1. Titu Andreescu., and Dorin Andrica,(2006). Complex Numbers from A to Z, Birkhauser. Library of Congress Cataloging-in-Publication Data Andreescu, Titu.

2. Edgar G. Goodaire and Michael M. Parmenter, ,(2005)**.** Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.

3.David C. Lay., (2007). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint.

REFERENCE

1. Kenneth Hoffman., Ray Kunze., (2003).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

UNIT – III

THE INTEGERS

DIVISIBILITY THEORY IN THE INTEGERS

Well- Ordering Principle

Every non empty set S of nonnegative integers contains a least element. That is, there exists some integer a in S such that $a \le b$ for all b in S.

THE DIVISION ALGORITHM

Division Algorithm, the result is familiar to most of us roughly, it asserts that an integer a can be "divided" by a positive integer b in such a way that the remainder is smaller than b. The exact statement of this fact is Theorem 1.:

Theorem 1. *Given integers a and b, with* $b > 0$, *there exist unique integers q and r satisfying*

$$a = qb + r \qquad 0 \le r < b$$

The integers q and r are called, respectively, the quotient and remainder in the division of a by b.

Proof. Let a and b be integers with $b > 0$ and consider the set

$$S = \{a - xb : x is an integer;\ a - xb \ge 0\}.$$

Claim: The set S is nonempty

It suffices to find a value x which making $a - xb$ nonnegative. Since $b \ge 1$, we have $|a|b \ge |a|$ and so, $a - (-|a|)b = a + |a|b \ge a + |a| \ge 0$. For the choice $x = -|a|$, then $a - xb$ lies in S. Therefore S is nonempty, hence the claim.
Therefore by Well-Ordering Principle, S contains a small integer, say r. By the definition of S there exists an integer q satisfying

$$r = a - qb \qquad 0 \le r.$$

Claim: $r < b$

Suppose $r \ge b$. Then we have

$$a - (q + 1)b = (a - qb) - b = r - b \ge 0.$$

This implies that, $a - (q + 1)b \in S$. But $a - (q + 1)b = r - b < r$, since $b > 0$, leading to a contradiction of the choice of r as the smallest member of S. Hence, $r < b$, hence the claim.

Next we have to show that the uniqueness of q and r. Suppose that a as two representations of the desired form, say,

$$a = qb + r = q'b + r',$$

where $0 \le r < b$ and $0 \le r' < b$. Then $(r' - r) = b(q - q')$. Taking modulus on both sides,

$$|(r' - r)| = |b(q - q')| = |b||(q - q')| = b|(q - q')|.$$

But we have $-b < -r \le 0$ and $0 \le r' < b$, upon adding these inequalities we obtain $-b < r' - r < b$. This implies $b|(q - q')| < b$, which yields $0 \le |q - q'| < 1$. Because $|q - q'|$ is a nonnegative integer, the only possibility is that $|q - q'| = 0$, hence, $q = q'$. This implies $|r' - r| = 0$, that is, $r = r'$. Hence the proof. □

Corollary 1. *If a and b are integers, with $b \neq 0$, then there exists integers q and r such that*

$$a = qb + r \qquad 0 \le r < |b|.$$

Proof. It is enough to consider the case in which b is negative. Then $|b| > 0$, and Theorem 1. produces unique integers q' and r for which

$$a = q'|b| + r \qquad 0 \le r < |b|.$$

Noting that $|b| = -b$, we may take $q = -q'$ to arrive at $a = qb + r$, with $0 \le r < |b|$. □

Application of the Division Algorithm

1. Square of any integer is either of the form $4k$ or $4k + 1$. That is, the square of integer leaves the remainder 0 or 1 upon division by 4.

   Solution: Let a be any integer. If a is even, we can let $a = 2n$, n is an integer, then $a^2 = (2n)^2 = 4n^2 = 4k$. If a is odd, we can let $a = 2n+1$, n is an integer, then $a^2 = (2n+1)^2 = 4n^2 + 4n+1 = 4(n^2 + n)+1 = 4k+1$.

2. The square of any odd integer is of the form $8k + 1$.

   Solution: Let a be an integer and let $b = 4$, then by division algorithm

   a is representable as one of the four forms: $4q$, $4q + 1$, $4q + 2$, $4q + 3$. In this representation, only those integers of the forms $4q + 1$ and $4q + 3$ are odd. If $a = 4q + 1$, then

   $$a^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 8(2q^2 + q) + 1 = 8k + 1.$$

   If $a = 4q + 3$, then

   $$a^2 = (4q+3)^2 = 16q^2+24q+9 = 16q^2+24q+8+1 = 8(2q^2+3q+1)+1 = 8k+1.$$

3. For all integer $a \geq 1$, $\frac{a(a^2+2)}{3}$ is an integer.

Solution: Let $a \geq 1$ be an integer. According to division algorithm, a is of the form $3q$, $3q + 1$ or $3q + 2$. If $a = 3q$, then

$$\frac{3q((3q)^2 + 2)}{3} = 9q^3 + 2q,$$

which is clearly an integer. Similarly we can prove other two cases also.

THE GREATEST COMMON DIVISOR

Definition 1. *An integer b is said to be divisible by an integer* $a \neq 0$, *in symbols* $a|b$, *if there exists some integer c such that* $b = ac$. *We write* $a \nmid b$ *to indicate that b is not divisible by a.*

Thus, for example, −22 is divisible by 11, because −22 = 11(−2). However, 22 is not divisible by 3; for there is no integer c that makes the statement $22 = 3c$ true.

There is other language for expressing the divisibility relation $a|b$. We could say that a is a divisor of b, that a is a factor of b, or that b is a multiple of a. Notice that in Definition 1 there is a restriction on the divisor a: Whenever the notation $a|b$ is employed, it is understood that a is different from zero.
If a is a divisor of b, then b is also divisible by $-a$ (indeed, $b = ac$ implies that $b = (-a)(-c)$), so that the divisors of an integer always occur in pairs.
To find all the divisors of a given integer, it is sufficient to obtain the positive divisors and then adjoin to them the corresponding negative integers. For this reason, we shall usually limit ourselves to a consideration of positive divisors.
It will be helpful to list some immediate consequences of Definition 1.

Theorem 2. *For integers a, b, c, the following hold:*

1. $a|0$, $1|a$, $a|a$.
2. $a|1$ *if and only if* $a = \pm 1$.
3. *If* $a|b$ *and* $c|d$, *then* $ac|bd$.
4. *If* $a|b$ *and* $b|c$, *then* $a|c$.
5. $a|b$ *and* $b|a$ *if and only if* $a = \pm b$.
6. *If* $a|b$ *and* $b \neq 0$, *then* $|a| \leq |b|$.
7. *If* $a|b$ *and* $a|c$, *then* $a|(bx + cy)$ *for arbitrary integers x and y.*

Proof. 1. Since $0 = a.0$, $a|0$. Since $a = 1.a$, $1|a$. Since $a = a.1$, $a|a$.

2. We have $a|1$ if and only if $1 = a.c$ for some c, this is if and only if $a = \pm 1$.

3. Clear from definition.

4. Clear from definition.

5. Clear from definition.

6. If $a|b$, then there exists an integer c such that $b = ac$; also, $b \neq 0$ implies that $c \neq 0$. Upon taking absolute values, we get $|b| = |ac| = |a||c|$. Because $c \neq 0$, it follows that $|c| \geq 1$, whence $|b| = |a||c| \geq |a|$.

7. The relations $a|b$ and $a|c$ ensure that $b = ar$ and $c = as$ for suitable integers r and s. But then whatever the choice of x and y, $bx + cy = arx + asy = a(rx + sy)$. Because $rx + sy$ is an integer, this says that $a|(bx + cy)$, as desired. □

Definition 2. *Let a and b be given integers, with at least one of them different from zero. The greatest common divisor of a and b, denoted by* $gcd(a, b)$, *is the positive integer d satisfying the following:*

(i) $d|a$ *and* $d|b$.

(ii) *If* $c|a$ *and* $c|b$, *then* $c \leq d$.

Example: The positive divisors of −12 are 1, 2, 3, 4, 6, 12, whereas those of 30 are 1, 2, 3, 5, 6, 10, 15, 30; hence, the positive common divisors of −12 and 30 are 1, 2, 3, 6. Because 6 is the largest of these integers, it follows that $gcd(-12, 30) = 6$. In the same way, we can show that $gcd(-5, 5) = 5$, $gcd(8, 17) = 1$, $gcd(-8, -36) = 4$.

Theorem 3. *Given integers a and b, not both of which are zero, there exist integers x and y such that*

$$gcd(a, b) = ax + by.$$

Proof. Consider the set S of all positive linear combinations of a and b :

$$S = \{au + bv : au + bv > 0;\ u, v \text{ integers}\}.$$

Since, if $a \neq 0$ then $|a| = au+b.0 \in S$, where $u = 1$, *if* $a > 0$; $u = -1$, *if* $a < 0$, S is nonempty. Therefore by the Well-Ordering Principle, S must contain a smallest element, say d. Thus, from the very definition of S, there exist integers x and y for which $d = ax + by$. Claim: $d = gcd(a, b)$

By using the Division Algorithm, we can obtain integers q and r such that $a = qd + r$, where $0 \leq r < d$. Then r can be written in the form:

$$\begin{aligned} r &= a - qd \\ &= a - q(ax + by) \\ &= a(1 - qx) + b(-qy). \end{aligned}$$

If r were positive, then this representation would imply that r is a member of S, contradicting the fact that d is the least integer in S (recall that $r < d$). Therefore, $r = 0$, and so $a = qd$, or equivalently $d|a$. By similar reasoning, $d|b$, this implies d is a common divisor of a and b.

Now if c is an arbitrary positive common divisor of the integers a and b, then part (7) of Theorem 2 allows us to conclude that $c|(ax + by)$; that is, $c|d$. By part (6) of the same theorem, $c = |c| \leq |d| = d$, so that d is greater than every positive common divisor of a and b. Hence $d = gcd(a, b)$. Hence the claim. Therefore $gcd(a, b) = ax + by$. □

Corollary 2. *If a and b are given integers, not both zero, then the set*

$$T = ax + by : x, y \text{ are integers}$$

is precisely the set of all multiples of $d = gcd(a, b)$.

Proof. Because $d|a$ and $d|b$, we know that $d|(ax + by)$ for all integers x, y. Thus, every member of T is a multiple of d. Conversely, d may be written as $d = ax_0 + by_0$ for suitable integers x_0 and y_0 , so that any multiple nd of d is of the form

$$nd = n(ax_0 + by_0) = a(nx_0) + b(ny_0).$$

Hence, nd is a linear combination of a and b, and, by definition, lies in T. □

Definition 3. *Two integers a and b, not both of which are zero, are said to be relatively prime whenever* $gcd(a, b) = 1$.

Theorem 4. *Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that* $1 = ax + by$.

Proof. If a and b are relatively prime so that $gcd(a, b) = 1$, then Theorem 3 guarantees the existence of integers x and y satisfying $1 = ax+by$. Conversely, suppose that $1 = ax + by$ for some choice of x and y, and that $d = gcd(a, b)$. Because $d|a$ and $d|b$, Theorem 2 yields $d|(ax+by)$, or $d|1$. This implies $d = \pm 1$. But d is a positive integer, $d = 1$. That is a and b are relatively prime. □

Corollary 3. *If* $gcd(a, b) = d$, *then* $gcd(a/d, b/d) = 1$.

Proof. Since $d|a$ and $d|b$, a/d and b/d are integers. We have, if $gcd(a, b) = d$, then there exists x and y such that $d = ax + by$. Upon dividing each side of this equation by d, we obtain the expression

$$1 = (a/d)x + (b/d)y.$$

Because a/d and b/d are integers, a/d and b/d are relatively prime. Therefore $gcd(a/d, b/d) = 1$. □

Corollary 4. *If* $a|c$ *and* $b|c$, *with* $gcd(a, b) = 1$, *then* $ab|c$.

Proof. Since $a|c$ and $b|c$, we can find integers r and s such that $c = ar = bs$. Given that $gcd(a, b) = 1$, so there exists integers x and y such that $1 = ax+by$.

Multiplying the last equation by c, we get,

$$c = c1 = c(ax + by) = acx + bcy.$$

If the appropriate substitutions are now made on the right-hand side, then

$$c = a(bs)x + b(ar)y = ab(sx + ry).$$

This implies, $ab|c$.
Theorem 5. *(Euclid's lemma.) If* $a|bc$, *with* $gcd(a, b) = 1$, *then* $a|c$.

Proof. Since $gcd(a, b) = 1$, we have $1 = ax + by$ for some integers x and y. Multiplication of this equation by c produces

$$c = 1c = (ax + by)c = acx + bcy.$$

Since $a|bc$ and $a|ac$, we have $a|acx + bcy$. This implies $a|c$. □

Note: If a and b are not relatively prime, then the conclusion of Euclid's

lemma may fail to hold. For example: 6|9.4 but 6 - 9 and 6 - 4.

Theorem 6. *Let* a, b *be integers, not both zero. For a positive integer* d, $d = gcd(a, b)$ *if and only if*

(i) $d|a$ *and* $d|b$.

(ii) *Whenever* $c|a$ *and* $c|b$, *then* $c|d$.

Proof. Suppose that $d = gcd(a, b)$. Certainly, $d|a$ and $d|b$, so that (i) holds. By Theorem 3, d is expressible as $d = ax + by$ for some integers x, y. Thus, if $c|a$ and $c|b$, then $c|(ax + by)$, or rather $c|d$. This implies, condition (ii) holds. Conversely, let d be any positive integer satisfying the stated conditions (i) and (ii). Given any common divisor c of a and b, we have $c|d$ from hypothesis (ii). This implies that $d \geq c$, and consequently d is the greatest common divisor of a and b. □

THE EUCLIDEAN ALGORITHM

Lemma 1. *If $a = qb + r$, then $gcd(a, b) = gcd(b, r)$.*

Proof. If $d = gcd(a, b)$, then the relations $d|a$ and $d|b$ together imply that $d|(a - qb)$, or $d|r$. Thus, d is a common divisor of both b and r. On the other hand, if c is an arbitrary common divisor of b and r, then $c|(qb + r)$, whence $c|a$. This makes c a common divisor of a and b, so that $c \le d$. It now follows from the definition of $gcd(b, r)$ that $d = gcd(b, r)$. □

The Euclidean algorithm

The Euclidean Algorithm may be described as follows: Let a and b be two integers whose greatest common divisor is desired. Because $gcd(|a|, |b|) = gcd(a, b)$,, with out loss of generality we may assume $a \ge b > 0$. The first step is to apply the Division Algorithm to a and b to get

$$a = q_1 b + r_1 \quad 0 \le r_1 < b.$$

If it happens that $r_1 = 0$, then $b|a$ and $gcd(a, b) = b$. When $r_1 \ne 0$, divide b by r_1 to produce integers q_2 and r_2 satisfying

$$b = q_2 r_1 + r_2 \quad 0 \le r_2 < r_1.$$

If $r_2 = 0$, then we stop; otherwise, proceed as before to obtain

$$r_1 = q_3 r_2 + r_3 \quad 0 \le r_3 < r_2.$$

This division process continues until some zero remainder appears, say, at the $(n + l)^{th}$ stage where r_{n-1} is divided by r_n (a zero remainder occurs sooner or later because the decreasing sequence $b > r_1 > r_2 > \cdots \ge 0$ cannot contain more than b integers). The result is the following system of equations:

$$\begin{aligned}
a &= q_1 b + r_1 & 0 &\le r_1 < b \\
b &= q_2 r_1 + r_2 & 0 &\le r_2 < r_1 \\
r_1 &= q_3 r_2 + r_3 & 0 &\le r_3 < r_2 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n & 0 &\le r_n < r_{n-1} \\
r_{n-1} &= q_{n+1} r_n + 0.
\end{aligned}$$

By Lemma 1,

$$gcd(a, b) = gcd(b, r_1) = = gcd(r_{n-1}, r_n) = gcd(r_n, 0) = r_n.$$

Note:Start with the next-to-last equation arising from the Euclidean Algorithm, we can determine x and y such that $gcd(a, b) = ax + by$.

Example: Let us see how the Euclidean Algorithm works in a concrete case by calculating, say, *gcd*(12378, 3054). The appropriate applications of the Division Algorithm produce the equations

$$12378 = 4.3054 + 162$$

$$3054 = 18.162 + 138$$

$$162 = 1.138 + 24$$

$$138 = 5.24 + 18$$

$$24 = 1.18 + 6$$

$$18 = 3.6 + 0$$

This tells us that the last nonzero remainder appearing in these equations, namely, the integer 6, is the greatest common divisor of 12378 and 3054:

$$6 = gcd(12378, 3054).$$

To represent 6 as a linear combination of the integers 12378 and 3054, we start with the next-to-last of the displayed equations and successively eliminate the remainders 18, 24, 138, and 162:

$$\begin{aligned} 6 &= 24 - 18 \\ &= 24 - (138 - 5.24) \\ &= 6.24 - 138 \\ &= 6(162 - 138) - 138 \\ &= 6.162 - 7.138 \\ &= 6.162 - 7(3054 - 18.162) \\ &= 132.162 - 7.3054 \\ &= 132(12378 - 4.3054) - 7.3054 \\ &= 132.12378 + (-535)3054 \end{aligned}$$

Thus, we have

$$6 = gcd(12378, 3054) = 12378x + 3054y,$$

where $x = 132$ and $y = -535$. Note that this is not the only way to express the integer 6 as a linear combination of 12378 and 3054; among other possibilities, we could add and subtract 3054.12378 to get

$$6 = (132 + 3054)12378 + (-535 - 12378)3054 = 3186.12378 + (-12913)3054.$$

Theorem 7. *If* $k > 0$, *then* $gcd(ka, kb) = k\, gcd(a, b)$.

Proof. If each of the equations appearing in the Euclidean Algorithm for a and b, multiplied by k, we obtain

$$ak = q_1(bk) + r_1k \qquad 0 \leq r_1k < bk$$
$$bk = q_2(r_1k) + r_2k \qquad 0 \leq r_2k < r_1k$$
$$\vdots$$
$$r_{n-2}k = q_n(r_{n-1}k) + r_nk \qquad 0 \leq r_nk < r_{n-1}k$$
$$r_{n-1}k = q_{n+1}(r_nk) + 0.$$

But this is clearly the Euclidean Algorithm applied to the integers ak and bk, so that their greatest common divisor is the last nonzero remainder r_nk; that is,

$$gcd(ka, kb) = r_nk = k\, gcd(a, b),$$

Hence the theorem. □

Corollary 5. *For any integer* $k \neq 0$, $gcd(ka, kb) = |k|\, gcd(a, b)$.

Proof. We already have, if $k > 0$, then $gcd(ka, kb) = k\, gcd(a, b)$. Therefore it suffices to consider the case in which $k < 0$. Then $-k = |k| > 0$ and, by Theorem 7,

$$\begin{aligned} gcd(ak, bk) &= gcd(-ak, -bk) \\ &= gcd(a|k|, b|k|) \\ &= |k|\, gcd(a, b). \end{aligned}$$

Hence the result. □

De nition 4. *The least common multiple of two nonzero integers a and b, denoted by* $lcm(a, b)$, *is the positive integer m satisfying the following:*

(i) $a|m$ *and* $b|m$.

(ii) *If* $a|c$ *and* $b|c$, *with* $c > 0$, *then* $m \leq c$.

As an example, the positive common multiples of the integers -12 and 30 are 60, 120, 180, ... hence, $lcm(-12, 30) = 60$.

Theorem 8. *For positive integers a and b*

$$gcd(a, b)\, lcm(a, b) = ab.$$

Proof. Let $d = gcd(a, b)$ and let $m = ab/d$, then $m > 0$.

Claim: $m = lcm(a, b)$

Since d is the common divisor of a and b we have $a = dr$, $b = ds$ for in-tegers r and s. Then $m = as = rb$. This implies, m a (positive) common multiple of a and b.

Now let c be any positive integer that is a common multiple of a and b, then $c = au = bv$ for some integers u and v. As we know, there exist integers x and y satisfying d =ax+ by. In consequence,

$$\frac{c}{m} = \frac{cd}{ab} = \frac{c(ax + by)}{ab} = (\frac{c}{b})x + (\frac{c}{a})y = vx + uy.$$

This equation states that $m|c$, this implies, $m \leq c$. By the definition of least common multiple, we have $m = lcm(a, b)$. Hence the claim. Therefore $gcd(a, b)\ lcm(a, b) = ab$. □

Corollary 6. *For any choice of positive integers a and b,* $lcm(a, b) = ab$ *if and only if* $gcd(a, b) = 1$.

Definition 5. *If a, b, c, are three integers, not all zero,* $gcd(a, b, c)$ *is defined to be the positive integer d having the following properties:*

(i) *d is a divisor of each of a, b, c.*

(ii) *If e divides the integers a, b, c, then* $e \leq d$.

For example $gcd(39, 42, 54) = 3$ and $gcd(49, 210, 350) = 7$.

Example: Consider the linear Diophantine equation

$$172x + 20y = 1000$$

Applying the Euclidean's Algorithm to the evaluation of $gcd(172, 20)$, we find that

$$\begin{aligned} 172 &= 8.20 + 12 \\ 20 &= 1.12 + 8 \\ 12 &= 1.8 + 4 \\ 8 &= 24, \end{aligned}$$

whence $gcd(172, 20) = 4$. Because $4|1000$, a solution to this equation exists. To obtain the integer 4 as a linear combination of 172 and 20, we work backward through the previous calculations, as follows:

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 212 - 20 \\ &= 2(172 - 8.20) - 20 \\ &= 2.172 + (-17)20 \end{aligned}$$

Upon multiplying this relation by 250, we arrive at

$$\begin{aligned} 1000 &= 250.4 \\ &= 250(2.172 + (-17)20) \\ &= 500.172 + (-4250)20, \end{aligned}$$

so that $x = 500$ and $y = -4250$ provide one solution to the Diophantine equation in question. All other solutions are expressed by

$$x = 500 + (20/4)t = 500 + 5t$$

$$y = -4250 - (172/4)t = -4250 - 43t,$$

for some integer t.

If we want to find positive solution,if any happen to exist. For this, t must be chosen to satisfy simultaneously the inequalities

$$5t + 500 > 0 \qquad -43t - 4250 > 0$$

or

$$-98\frac{36}{43} > t > -100.$$

Because t must be an integer, we are forced to conclude that $t = -99$. Thus, our Diophantine equation has a unique positive solution $x = 5$, $y = 7$ corresponding to the value $t = -99$.

THE FUNDAMENTAL THEOREM OF ARITHMETIC

Definition 6. *An integer $p > 1$ is called a prime number, or simply a prime, if its only positive divisors are 1 and p. An integer greater than 1 that is not a prime is termed composite.*

Among the first ten positive integers, 2, 3, 5, 7 are primes and 4, 6, 8, 9, 10 are composite numbers. Note that the integer 2 is the only even prime, and according to our definition the integer 1 plays a special role, being neither prime nor composite.

Theorem 1. *If p is a prime and $p|ab$, then $p|a$ or $p|b$.*

Proof. If $p|a$, then we need go no further, so let us assume that p - a. Because the only positive divisors of p are 1 and p itself, this implies that $gcd(p, a) = 1$. Hence, by Euclid's lemma, we get $p|b$. □

Corollary 8. If p is a prime and $p|a_1a_2 \cdots a_n$, then $p|a_k$ for some k, where $1 \le k \le n$.

Proof. We proceed by induction on n, the number of factors. When $n = 1$, the stated conclusion obviously holds; whereas when $n = 2$, the result is the content of Theorem 10. Suppose, as the induction hypothesis, that $n > 2$ and that whenever p divides a product of less than n factors, it divides at least one of the factors. Now let $p|a_1a_2 \cdots a_n$. From Theorem 10, either $p|a_n$ or $p|a_1a_2 \cdots a_{n-1}$ If $p|a_n$, then we are through. As regards the case where $p|a_1a_2 \cdots a_{n-1}$, the induction hypothesis ensures that $p|a_k$ for some choice of k, with $1 \le k \le n - 1$. In any event, p divides one of the integers $a_1, a_2, \cdots, a_n$.

Theorem 2. *(Fundamental Theorem of Arithmetic.) Every positive integer $n > 1$ can be expressed as a product of primes; this representation is unique, apart from the order in which the factors occur.*

Proof. Either n is a prime, there is nothing to prove. If n is composite, then there exists an integer d satisfying $d|n$ and $1 < d < n$. Among all such integers d, choose p_1 to be the smallest (this is possible by the Well-Ordering Principle). Then P_1 must be a prime number. Otherwise it too would have a divisor q with $1 < q < p_1$; but then $q|p_1$ and $p_1|n$ imply that $q|n$, which contradicts the choice of p_1 as the smallest positive divisor, not equal to 1, of n. We therefore may write $n = p_1n_1$, where p_1 is prime and $1 < n_1 < n$.If n_1

happens to be a prime, then we have our representation. In the contrary case, the argument is repeated to produce a second prime number p_2 such that $n_1 = p_2 n_2$; that is,

$$n = p_1 P_2 n_2 \qquad 1 < n_2 < n_1.$$

If n_2 is a prime,then it is not necessary to go further. Otherwise, write $n_2 = p_3 n_3$, with p_3 a prime:

$$n = p_1 P_2 p_3 n_3 \quad 1 < n_3 < n_2.$$

The decreasing sequence $n > n_1 > n_2 > \cdots > 1$ cannot continue indefinitely, so that after a finite number of steps n_{k-1} is a prime, call it, p_k. This leads to the prime factorization

$$n = p_1 p_2 \cdots p_k.$$

To establish the second part of the proof-the uniqueness of the prime factorization, let us suppose that the integer n can be represented as a product of primes in two ways, say,

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s \quad r \leq s,$$

where the p_i and q_j are all primes, written in increasing magnitude so that

$$p_1 \leq p_2 \leq \cdots \leq p_r \qquad q_1 \leq q_2 \leq \cdots \leq q_s.$$

Because $p_1 | q_1 q_2 \cdots q_s$, Corollary 9 tells us that $p_1 = q_k$ for some k; but then $p_1 \geq q_1$. Similar reasoning gives $q_1 \geq p_1$, whence $p_1 = q_1$. We may cancel this common factor and obtain

$$p_2 p_3 \cdots p_r = q_2 q_3 \cdots q_s.$$

Now repeat the process to get $p_2 = q_2$ and, in turn,

$$p_3 p_4 \cdots p_r = q_3 q_4 \cdots q_s.$$

Continue in this fashion. If the inequality $r < s$ were to hold, we would eventually arrive at

$$1 = q_{r+1} q_{r+2} \cdots q_s,$$

which is absurd, because each $q_j > 1$. Hence, $r = s$ and

$$p_1 = q_1, p_2 = q_2, \cdots , p_r = q_r,$$

making the two factorizations of n identical. The proof is now complete.

THE THEORY OF CONGRUENCES

Definition 1. *Let n be a fixed positive integer. Two integers a and b are said to be congruent modulo n, symbolized by*

$$a \equiv b(mod n)$$

if n divides the difference $a - b$; that is, provided that $a - b = kn$ for some integer k.

Theorem 1. For arbitrary integers a and b, $a \equiv b(mod n)$ if and only if a and b leave the same nonnegative remainder when divided by n.

Proof. Suppose $a \equiv b(mod n)$, so that $a = b + kn$ for some integer k. Upon division by n, b leaves a certain remainder r; that is, $b = qn + r$, where $0 \leq r < n$. Therefore,

$$a = b + kn = (qn + r) + kn = (q + k)n + r$$

which indicates that a has the same remainder as b.
On the other hand, suppose we can write $a = q_1 n + r$ and $b = q_2 n + r$, with the same remainder r $(0 \leq r < n)$. Then

$$a - b = (q_1 n + r) - (q_2 n + r) = (q_1 - q_2)n,$$

whence $n|a - b$. That is, $a \equiv b(mod n)$. □

Theorem 2. *Let $n > 1$ be fixed and a, b, c, d be arbitrary integers. Then the following properties hold:*

1. $a \equiv a(mod n)$.
2. *If* $a \equiv b(mod n)$, *then* $b \equiv a(mod n)$.
3. *If* $a \equiv b(mod n)$ *and* $b \equiv c(mod n)$, *then* $a \equiv c(mod n)$.
4. *If* $a \equiv b(mod n)$ *and* $c \equiv d(mod n)$, *then* $a + c \equiv b + d(mod n)$ *and* $ac \equiv bd(mod n)$.
5. *If* $a \equiv b(mod n)$, *then* $a + c \equiv b + c(mod n)$ *and* $ac \equiv bc(mod n)$.
6. *If* $a \equiv b(mod n)$, *then* $a^k \equiv b^k(mod n)$ *for any positive integer* k.

Problem 1: Show that $41|2^{20} - 1$.
Solution: We have

$$2^5 \equiv -9(mod\ 41).$$

Therefore

$$(2^5)^4 \equiv (-9)^4(mod\ 41).$$

This implies that

$$2^{20} \equiv (-9)^4(mod\ 41).$$

But we have $(-9)^4 = 81.81$ and $81 \equiv -1(mod\ 41)$. Therefore

$$2^{20} \equiv (-1)(-1)(mod\ 41).$$

This implies $41|2^{20} - 1$.
Problem 2: Find the remainder obtained upon dividing the sum 1!

$$+ 2! + 3! + 4! + \cdots + 99! + 100!$$

by 12.
Solution: We have $4! \equiv 24 \equiv 0(mod\ 12)$; thus, for $k \geq 4$,

$$k! \equiv 4!.5.6 \cdots k \equiv 0.5..6 \cdots k \equiv 0(mod\ 12).$$

Therefore

$$1! + 2! + 3! + 4! + \cdots + 100! \equiv 1! + 2! + 3! + 0 + \cdots + 0 \equiv 9(mod\ 12).$$

The remainder 9.

Theorem 3. *If* $ca \equiv cb(mod\ n)$*, then* $a \equiv b(mod\ n/d)$*, where* $d = gcd(c, n)$
Proof. By hypothesis, we can write

$$c(a - b) = ca - cb = kn, \tag{3.1}$$

for some integer k. Knowing that $gcd(c, n) = d$, there exist relatively prime integers r and s satisfying $c = dr$, $n = ds$. When these values are substituted in Eq. 3.1 and the common factor d canceled, the net result is

$$r(a - b) = ks.$$

Hence, $s|r(a - b)$ and $gcd(r, s) = 1$. Euclid's lemma yields $s|(a - b)$, which implies $a \equiv b(mod\ s)$; in other words, $a \equiv b(mod\ n/d)$. □

Corollary 12. *If* $ca \equiv cb(mod\ n)$ *and* $gcd(c, n) = 1$*, then* $a \equiv b(mod\ n)$.

Corollary 13. *If* $ca \equiv cb(mod\ p)$ *and* p - c*, where* p *is a prime number, then* $a \equiv b(mod\ p)$.

Proof. The conditions p - c and p a prime imply that $gcd(c, p) = 1$. Then by Corollary 12, $a \equiv b(mod\ p)$.

PRINCIPLE OF MATHEMATICAL INDUCTION

The principle of mathematical induction

Let P(n) be a given statement involving the natural number n such that

3. The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and
4. If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + ... + (2n – 1) = n2$

Solution Let the given statement P(n) be defined as P(n) : $1 + 3 + 5 +...+ (2n – 1) = n^2$, for $n \in \mathbf{N}$. Note that P(1) is true, since

P(1) : $1 = 1^2$

Assume that P(k) is true for some $k \in \mathbf{N}$, i.e., P(k) : 1 + 3

$+ 5 + ... + (2k – 1) = k\ 2$

Now, to prove that P(k + 1) is true, we have

$1 + 3 + 5 + ... + (2k – 1) + (2k + 1)$

$= k^2 + (2k + 1)$ (Why?)

$= k2 + 2k + 1 = (k + 1)2$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all $n \in \mathbf{N}$.

Example 2 $2^{2n} – 1$ is divisible by 3.

Solution Let the statement P(n) given as

P(n) : $22n – 1$ is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$2^2 – 1 = 4 – 1 = 3.1$ is divisible by 3.

Assume that P(n) is true for some natural number k, i.e., P(k): 2^{2k} –
1 is divisible by 3, i.e., $2^{2k} – 1 = 3q$, where $q \in \mathbf{N}$ Now, to prove that P(k + 1) is true, we have

P(k + 1) : $22(k+1) – 1 = 22k + 2 – 1 = 22k . 22 – 1$

$= 2^{2k} . 4 – 1 = 3.2^{2k} + (2^{2k} – 1)$

$= 3.2^{2k} + 3q$
$= 3\,(2^{2k} + q) = 3m$, where $m \in \mathbf{N}$
Thus P(k + 1) is true, whenever P(k) is true.
Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Example 3 $2n + 1 < 2^n$, for all natual numbers n ε 3.

Solution Let P(n) be the given statement, i.e., P(n) : $(2n + 1) < 2^n$ for all natural numbers, n ε 3. We observe that P(3) is true, since
2.3 + 1 = 7 < 8 = 23

Assume that P(n) is true for some natural number k, i.e., $2k + 1 < 2^k$
To prove P(k + 1) is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have 2(k + 1) + 1 = 2 k + 3

8.2k + 1 + 2 < 2k + 2 < 2k . 2 = 2k + 1.

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, n ε 3.

Long Answer Type

Example 4 Define the sequence a_1, a_2, a_3... as follows :
$a_1 = 2$, $a_n = 5\,a_{n-1}$, for all natural numbers n ε 2.

(iii) Write the first four terms of the sequence.
(iv) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Solution

r We have $a_1 = 2$
$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$ $a_3 = 5a_{3-1} = 5a_2 =$
$5.10 = 50$ $a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

r Let P(n) be the statement, i.e.,
P(n) : a_n = 2.5 n–1 for all natural numbers. We observe that P(1) is true
Assume that P(n) is true for some natural number k, i.e., P(k) : $a_k = 2.5^{k-1}$. Now to prove that P (k + 1) is true, we have

P(k + 1) : $a_{k+1} = 5.a_k = 5\,.\,(2.5^{k-1})$
$= 2.5^k = 2.5^{(k+1)-1}$
Thus P(k + 1) is true whenever P (k) is true.
Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

Example 5 The distributive law from algebra says that for all real numbers c, a_1 and a_2, we have $c\,(a_1 + a_2) = ca_1 + ca_2$.
Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if c, $a_1, a_2, ..., a_n$ are any real numbers, then
$c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... + ca_n$

Solution Let P(n) be the given statement, i.e.,

$P(n) : c(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... ca_n$ for all natural numbers $n \geq 2$, for $c, a_1, a_2, ... a_n \in \mathbf{R}$.

We observe that P(2) is true since

$$c(a_1 + a_2) = ca_1 + ca_2$$ (by distributive law)

Assume that P(*n*) is true for some natural number *k*, where $k > 2$, i.e.,

$$P(k) : c(a_1 + a_2 + ... + a_k) = ca_1 + ca_2 + ... + ca_k$$

Now to prove P(*k* + 1) is true, we have

$P(k+1) : c(a_1 + a_2 + ... + a_k + a_{k+1})$

$= c((a_1 + a_2 + ... + a_k) + a_{k+1})$

$= c(a_1 + a_2 + ... + a_k) + ca_{k+1}$ (by distributive law)

$= ca_1 + ca_2 + ... + ca_k + ca_{k+1}$

Thus P(*k* + 1) is true, whenever P (*k*) is true.

Hence, by the principle of Mathematical Induction, P(*n*) is true for all natural numbers $n \geq 2$.

Example 7 Prove by the Principle of Mathematical Induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers *n*.

Solution Let P(*n*) be the given statement, that is,

$P(n) : 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers *n*. Note that P (1) is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that P(*n*) is true for some natural number *k*, i.e.,

$P(k) : 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k + 1)! - 1$

To prove P (*k* + 1) is true, we have

$P(k + 1) : 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k + 1) \times (k + 1)!$

(i) $(k + 1)! - 1 + (k + 1)! \times (k + 1)$

(ii) $(k + 1 + 1)(k + 1)! - 1$

(iii) $(k + 2)(k + 1)! - 1 = ((k + 2)! - 1$

Thus P (*k* + 1) is true, whenever P (*k*) is true. Therefore, by the Principle of Mathematical Induction, P (*n*) is true for all natural number *n*.

Example 8

Prove, by Mathematical Induction, that

$$(n+1)^2 + (n+2)^2 + (n+3)^2 + ... + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$$

is true for all natural numbers n.

Discussion

Some readers may find it difficult to write the L.H.S. in P(k + 1). Some cannot factorize the L.H.S. and are forced to expand everything.

For P(1),

L.H.S. = $2^2 = 4$, R.H.S. = $\frac{1 \times 3 \times 8}{6} = 4$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$(k+1)^2 + (k+2)^2 + (k+3)^2 + \ldots + (2k)^2 = \frac{k(2k+1)(7k+1)}{6}$$

…. (1)

For P(k + 1) ,

$$(k+2)^2 + (k+3)^2 + \ldots + (2k)^2 + (2k+1)^2 + (2k+2)^2$$ (There is a missing term in front

and two more terms at the back.)

$$= (k+2)^2 + (k+3)^2 + \ldots + (2k)^2 + (2k+1)^2 + 4(k+1)^2$$

$$= (k+1)^2 + (k+2)^2 + (k+3)^2 + \ldots + (2k)^2 + (2k+1)^2 + 3(k+1)^2$$

$$= \frac{k(2k+1)(7k+1)}{6} + (2k+1)^2 + 3(k+1)^2$$, by (1)

$$= \frac{(2k+1)}{6}\left[k(7k+1) + 6(2k+1)\right] + 3(k+1)^2$$ (Combine the first two terms)

$$= \frac{(2k+1)}{6}\left[7k^2 + 13k + 6\right] + 3(k+1)^2$$

$$= \frac{(2k+1)}{6}(7k+6)(k+1) + 3(k+1)^2$$

$$= \frac{(k+1)}{6}\left[(2k+1)(7k+6) + 18(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[14k^2 + 37k + 24\right]$$

$$= \frac{(k+1)}{6}(2k+3)(7k+8) = \frac{(k+1)[2(k+1)+1][7(k+1)+1]}{6}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Example 9

Prove, by Mathematical Induction, that

$$1 \cdot n + 2(n-1) + 3(n-2) + \ldots + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$$

is true for all natural numbers n.

Discussion

The "up and down" of the L.H.S. makes it difficult to find the middle term, but you can avoid this.

Solution

Let P(n) be the proposition:

$$1 \cdot n + 2(n-1) + 3(n-2) + \ldots + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$$

For P(1),

L.H.S. = 1, R.H.S. = $\frac{1}{6}\times 1\times 2\times 3=1$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$1\cdot k+2(k-1)+3(k-2)+...+(k-2)\cdot 2+k\cdot 1=\frac{1}{6}k(k+1)(k+2)$$

…. (1)

For P(k + 1) ,

$$1\cdot(k+1)+2k+3(k-1)+...+(k-1)\cdot 3+k\cdot 2+(k+1)\cdot 1$$
$$=1\cdot(k+1)+2[(k-1)+1]+3[(k-2)+1]+...+(k-1)\cdot[2+1]+k\cdot[1+1]+(k+1)\cdot 1$$
$$=1\cdot k+2(k-1)+3(k-2)+...+(k-2)\cdot 2+k\cdot 1$$
$$+1 \quad +2 \quad +3 \quad +...+ \quad (k-1) \quad +k \quad +(k+1)$$

(The bottom series is arithmetic)

$$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2) \text{ , by (1)}$$
$$=\frac{1}{6}(k+1)(k+2)[k+3]=\frac{1}{6}(k+1)[(k+1)+1]\,[(k+1)+2]$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Example 10

Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) is divisible by 24, for all natural numbers n.

Discussion

Mathematical Induction cannot be applied directly. Here we break the proposition into three parts. Also note that 24 = 4×3×2×1 = 4!

Solution

Let P(n) be the proposition:

1. n(n + 1) is divisible by 2! =2.
2. n(n + 1)(n + 2) is divisible by 3! = 6.
3. n(n + 1)(n + 2)(n + 3) is divisible by 4! = 24.

For P(1),

1. 1×2 = 2 is divisible by 2.
2. 1×2×3 = 6 is divisible by 3.
3. 1×2×3×4 = 24 is divisible by 24. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

1. k(k + 1) is divisible by 2, that is, k(k + 1) = 2a …. (1)
2. k(k + 1)(k + 2) is divisible by 6 , that is, k(k + 1)(k + 2) = 6b …. (2)
3. k(k + 1)(k + 2)(k + 3) is divisible by 24 , that is, k(k + 1)(k + 2)(k + 3) = 24c …. (3)

where a, b, c are natural numbers.

For P(k + 1) ,

1. (k + 1)(k + 2) = k(k + 1) + 2(k + 1) = 2a + 2(k + 1) , by (1)
   = 2 [a + k + 1] …. (4)
   , which is divisible by 2.
2. (k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)
   = 6b + 3 × 2[a + k + 1] , by (2), (4)
   = 6 [b + a + k + 1] …. (5)
   , which is divisible by 6.
3. (k + 1)(k + 2)(k + 3)(k + 4) = k(k + 1)(k + 2)(k + 3) + 4(k + 1)(k + 2)(k + 3)
   = 24c + 4 × 6[b + a + k + 1] , by (3) , (5)
   = 24 [c + b + a + k + 1]
   , which is divisible by 24 .

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

PART – B (5 x 2 =10 Marks)

Possible Questions (2 Mark)

1. Define the divisibility over a field.
2. Define the greatest common divisor of two polynomials over a field.
3. State the Division Algorithm.
4. Define relatively prime polynomials.
5. Define quotient and remainder.
6. State the Euclidean algorithm.
7. Define reducible.
8. Define irreducible.
9. State the principles of mathematical induction.
10. State the Fundamental theorem of Arithmetic.
11. Write the any two basic properties of the Greatest Common divisor.
12. Write the any two basic properties of the Prime factors.
13.Define residue.
14.Write any two properties of congruence relation.

PART – C (5 x 6 =10 Marks)

Possible Questions (6 Mark)

1) Prove that $1^2+2^2+3^2+\ldots +n^2= n(n+1)(2n+1)/6$ by Principle of Mathematical induction.

2) Find a+b (mod n), ab (mod n) and $(a + b)^2$(mod n) if a=4003, b=-127, n=85.

3) Prove that the sum of the first n odd integers is n^2.

4) State and prove the Principles of Mathematical Induction.

5) Find the quotient q and the remainder r as defined in the Division algorithm

i) a=500, b=17 ii)a=-500,b=17 iii)a=-500 ,b=-17

6) Define greatest common divisor& Find the greatest common divisor of a and b and express it in the form ma+nb for suitable integers m and n .

i) a=26 ,b=118.

7) State and prove the Division Algorithm.

8) Solve the following congruence i) $3x \equiv 1 \ (mod\ 5)$ ii) i) $3x \equiv 1 \ (mod\ 6)$

9)State and prove the fundamental theorem of Arithmetic .

10) Prove that ,if $a\equiv x \ (mod\ n) and\ b \equiv y(mod\ n),$ then

i) $a+b\equiv x + y(mod\ n)$ and ii) $ab\equiv xy(mod\ n)$.

11. State and prove Euclidean Algorthim.

12. State and prove Euclidean theorem.

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Subject: Algebra **Subject Code: 17MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit III

The Integer

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

S.No	Question	Choice 1	Choice 2	Choice 3	Choice 4		Answer
1	Let f(x),g(x)≠0 be any two polynomials of the polynmial domain F[x],over the field F. Then there exist uniquely two polynomials q(x) & r(x) in F[x] such that ………………..	f(x)= q(x)g(x)+r(x)	f(x)= q(x)+r(x)	f(x)=q(x)g(x)	f(x)=g(x)+r(x)		f(x)= q(x)g(x)+r(x)
2	Let f(x),g(x)≠0 be any two polynomials of the polynmial domain F[x],over the field F. Then there exist uniquely two polynomials q(x) & r(x) in F[x] such that f(x)=q(x)g(x)+r(x) where r(x)….	equal to zero	not equal to zero	less than zero	more than zero		equal to zero
3	Division algorithm for polynomials over a field deg r(x) ………..deg g(x)	<	>	=	≠		=
4	In the division algorithm, the polynomial q(x) is called the ……….on dividing f(x) by g(x)	quotient	remainder	divisior	diviend		quotient
5	In the division algorithm, the polynomial q(x) is called the quotient on dividing f(x) by g(x) and the polynomail r(x)is called the ……………………	quotient	remainder	divisior	diviend		remainder
6	A polynomial domain F[x] over a field F is a principal………………	commutatice ring	ideal ring	associative ring	division ring		ideal ring

7	A polynomial …………….. F[x] over a field F is a principal idea lring	domain	range	co domain	quotient		domain
8	A polynomial domain F[x] over a ………….. F is a principal ideal ring	ring	domain	range	field		field
9	In a Euclidean algorithm ,Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are ……………………….	zero	one	two	three		zero
10	In a Euclidean algorithm ,Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are zero.Then f(x and g(x) have a ………………………d(x)	common divisor	greatest common divisor	least common divisor	equal divisor		greatest common divisor
11	Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are zero.Then f(x and g(x) have a greatest common divisor d(x),it can be expressed in the form……..	d(x)=m(x)f(x)+n(x)g(x)	d(x)=m(x)f(x)-n(x)g(x)	d(x)=f(x)+n(x)g(x)	d(x)=m(x)f(x)+n(x)		d(x)=m(x)f(x)+n(x)g(x)
12	In a Euclidean algorithm,the expression d(x)=m(x)f(x)+n(x)g(x) for ………………..m(x) and n(x) in F[x].	ring	field	polynomials	domain		polynomials
13	The greatest common divisor should be a ………………polynomial	zero	monic	double	triple		monic
14	If a(x)≠0 and f(x) are elements of F[x] then a(x) is a ……………..of f(x)	quotient	remainder	divisor	dividend		divisor
15	If a(x)≠0 and f(x) are elements of F[x] then a(x) is a divisor of f(x)iff there is a polynomial b(x) be in f[x] then ……………………	f(x)=a(x)+b(x)	f(x)=a(x)-b(x)	f(x)=a(x)b(x)	f(x)=a(x)/b(x)		f(x)=a(x)b(x)
16	The divisor of f(x) symbolically write ………………….	a(x)/f(x)	f(x)/a(x)	b(x)/f(x)	a(x)/b(x)		a(x)/f(x)
17	A ……….is an element of F[x] which has a multiplicative inverse.	zero	unit	two	three		unit

18	A unit is an element of F[x] which has ………………. inverse.	finite	infinite	multiplicative	zero		multiplicative
19	A unit is an element of F[x] which has a multiplicative …………….....	ring	field	range	inverse		inverse
20	All the polynomials of ……………. degree belonging to F[x] are units of F[x].	1st	2nd	zero	nth		zero
21	All the polynomials of zero degree belonging to F[x] are……………...of F[x].	units	field	ring	range		units
22	The……………….. elements of F are the only units of F[x].	zero	non zero	finite	infinite		non zero
23	The non zero elements of F are the ……………..of F[x].	only units	not only units	double units	zero units		only units
24	If f(x) and g(x) are polynomials in F[x], then we call f(x) and g(x) associates if ………………..for some $0 \neq c \in F$.	f(x)=g(x)	f(x)=c/g(x)	f(x)=c+g(x)	f(x)=cg(x)		f(x)=cg(x)
25	If f(x) and g(x) are ……………….. in F[x], then we call f(x) and g(x) associates if f(x) =c g(x) for some $0 \neq c \in F$.	field	ring	polynomials	domain		polynomials
26	If f(x) and g(x) are polynomials in F[x], then we call f(x) and g(x) associates if f(x) =c g(x) for some ……………....	$0 = c \in F$	$0 > c \in F$	$0 < c \in F$	$0 \neq c \in F$		$0 \neq c \in F$
27	Two non zero polynomials f(x) and g(x) in F[x] are associates iff …………………….. And ………………………	f(x)+g(x) & g(x)/f(x)	f(x)g(x) & g(x)f(x)	f(x)/g(x) & g(x)-f(x)	f(x)/g(x) & g(x)/f(x)		f(x)/g(x) & g(x)/f(x)
28	Two non zero polynomials f(x) and g(x) in F[x] are ……………. iff f(x)/g(x) and g(x)/f(x)	commutates	associates	divisible	distributive		associates
29	The divisorsof f(x) are called its……………………divisors.	proper	improper	finite	infinite		improper

30	All other divisors of f(x), if there are any , are called its……………………divisors.	proper	improper	finite	infinite		proper
31	If f(x) be a ……………...of positive degree, then f(x) is said to be irreducible over F.	function	domain	polynomial	range		polynomial
32	If f(x) be a polynomial of ………….. degree, then f(x) is said to be irreducible over F.	zero	positive	negative	infinite		positive
33	If f(x) be a polynomial of positive degree, then f(x) is said to be ………………. over F.	irreducible	reducible	singular	non singular		irreducible
34	An irreducible polynomial is otherwise called as……………..	point	prime	power	degree		prime
35	It has ………….. proper divisors in F[x]; f(x) is irreducible over F.	no	0ne	two	infinite		no
36	It has no proper divisors in F[x]; f(x) is …………………. over F.	irreducible	reducible	singular	non singular		irreducible
37	It has a ……………... divisors in F[x]; f(x) is reducible over F.	finite	infinite	proper	improper		proper
38	It has a proper divisors in F[x]; f(x) is………………….. over F.	irreducible	reducible	singular	non singular		reducible
39	……………………. depends on the field.	irreducibility	reducibility	singularity	non singularity		irreducibility
40	Irreducibility depends on the …………….....	field	domain	range	ring		field
41	Two polynomials are said to be relatively prime if their greatest common divisor is …………....	0	1	2	3		1
42	………………. polynomials are said to be relatively prime if their greatest common divisor is 1.	zero	one	two	three		two

43	Two polynomials are said to be …………………. if their greatest common divisor is 1.	field	prime	relatively prime	uniquely prime		relatively prime
44	Two polynomials are said to be relatively prime if their ……………….divisor is 1.	zero	greatest common	leatest common	infinite		greatest common
45	Let m be any fixed positive integer.Then an integer a is said to be congruent to another integer b modulo m if ……………..	m/(ab)	m/(a-b)	m/(a+b)	m/a		m/(a-b)
46	Let m be any fixed ………….. integer.Then an integer a is said to be congruent to another integer b modulo m if m/(a-b).	positive	negative	zero	infinite		positive
47	Let m be any fixed positive integer.Then an integer a is said to be ………… to another integer b modulo m if m/(a-b).	division	range	congruent	domain		congruent
48	Let m be any fixed positive integer.Then an integer a is said to be congruent to another integer b …………… m if m/(a-b).	multiplication	addition	division	modulo		modulo

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Subject : Algebra	**Subject Code : 17MMU102**	**L T P C**
Class : I – B.Sc. Mathematics	**Semester : I**	**6 1 0 6**

UNIT IV

Systems of linear equations, row reduction and echelon forms, vector equations, the matrix equation Ax=b, solution sets of linear systems, applications of linear systems, linear independence.

SUGGESTED READINGS

TEXT BOOKS

1. Titu Andreescu., and Dorin Andrica,(2006). Complex Numbers from A to Z, Birkhauser. Library of Congress Cataloging-in-Publication Data Andreescu, Titu.

2. Edgar G. Goodaire and Michael M. Parmenter, ,(2005). Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.

3.David C. Lay., (2007). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint.

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1. Kenneth Hoffman., Ray Kunze., (2003).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

UNIT – IV

SYSTEM OF LINEAR EQUATIONS

A linear equation in variables $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \cdots a_n$ and b are constant real or complex numbers. The constant a_i is called the coefficient of x_i and b is called the constant term of the equation.

A system of linear equations (or linear system) is a finite collection of linear equations in same variables. For instance, a linear system of m equations in n variables $x_1, x_2, \cdots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn} &= b_m \end{aligned} \qquad (9.1)$$

A solution of a linear system is a n-tuple $(s_1, s_2, \cdots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \cdots, s_n$ are sub-stituted for $x_1, x_2, \cdots, x_n$, respectively. The set of all solutions of a linear system is called the solution set of the system.

Any system of linear equations has one of the following exclusive conclusions.

(a) No solution.

(b) Unique solution.

(c) Infinitely many solutions.

A linear system is said to be consistent if it has at least one solution and is said to be inconsistent if it has no solution.

The system of equations (9.1) is said to be homogeneous if all b_j are zero; otherwise, it is said to be non-homogeneous.

The system of equations (9.1) can be expressed as the single matrix equation

$$AX = B, \qquad (9.2)$$

vector (column matrix) X that satisfies the matrix equation (9.2) is also the solution of the system.

Definition 21. *The matrix* $[AB]$ *which is obtained by placing the constant column matrix B to the right of the matrix A is called the augmented matrix. Thus the augmented matrix of the system AX = B is*

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Theorem 35. *The system $AX = B$ is consistent if and only if A and $[AB]$ have the same rank.*

System of non-homogeneous Equations

If we are given with a system of m equations in n unknowns, proceed as follows:

1. Write down the corresponding matrix equation $AX = B$.
2. By elementary row transformations obtain row echelon matrix of the augmented matrix $[AB]$.
3. Examine whether the rank of A and the rank of $[AB]$ are the same or not.

Case 1 If rank of $A \neq$ rank of $[AB]$, then the system is inconsistent and has no solution. otherwise, it is said to be non-homogeneous.

The system of equations (9.1) can be expressed as the single matrix equation

$$AX = B, \tag{9.2}$$

Any vector (column matrix) X that satisfies the matrix equation (9.2) is also the solution of the system.

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System of non-homogeneous Equations

If we are given with a system of m equations in n unknowns, proceed as follows:

1. Write down the corresponding matrix equation $AX = B$.

2. By elementary row transformations obtain row echelon matrix of the augmented matrix [*AB*].

3. Examine whether the rank of *A* and the rank of [*AB*] are the same or not.

Case 1 If rank of *A* ≠ rank of [*AB*], then the system is inconsistent and has no solution.

Case 2 If rank of *A* = rank of [*AB*], then the system is consistent.

Case 2a If rank of *A* = rank of [*AB*] = *n* = number of unknowns, then the system has unique solution.

Case 2b If rank of *A* = rank of [*AB*] < *n* = number of unknowns, then the system has infinitely many solutions. We assign arbi-trary values to ($n - r$) unknowns and determine the remaining *r* unknowns uniquely.

Solution of System of Linear Equations

Any given system of linear equations may be written in term of matri*x*, such that

$$AX = B \qquad \text{…(i)}$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A is known as co-efficient matri*x*.

If we multiply both sides of (i) by the reciprocal matri*x* A^{-1}, then we get $A^{-1}AX = A^{-1}B$

$$(A^{-1}A)X = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ where } \Delta \neq 0$$

$$= \frac{1}{\Delta}\begin{bmatrix} A_1d_1 + A_2d_2 + A_3d_3 \\ B_1d_1 + B_2d_2 + B_3d_3 \\ C_1d_1 + C_2d_2 + C_3d_3 \end{bmatrix} \qquad \text{…(ii)}$$

Hence from (ii) equating the values of *x*, y and z we get the desired result.

This method is true only when (i) $\Delta \neq 0$ (ii) number of equations and number of unknowns (e.g. *x*, y, z etc.) are the same.

Example 1. Solve the equations with the help of determinants :
x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6.

Sol. The co-efficient determinant is $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \neq 0$

$$\therefore \quad x = \frac{1}{2}\begin{bmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{bmatrix} \Rightarrow \quad x = \frac{1}{2} \times 4 = 2$$

$$y = \frac{1}{2}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{bmatrix} \Rightarrow \quad y = \frac{1}{2}(2) = 1 \Rightarrow y = 1$$

$$z = \frac{1}{2}\begin{bmatrix} 1 & 1 & +3 \\ 1 & 2 & +4 \\ 1 & 4 & +6 \end{bmatrix} \Rightarrow \quad z = \frac{1}{2}[-4 + 6 + (4-6)] = 0 \Rightarrow \quad z = 0$$

$\therefore$ Solution is $x = 2$, $y = 1$, $z = 0$.

Row reduced Echelon Form:

In addition to the above three conditions, if a matrix satisfies the following conditions:
Each column which contains a leading entry of a row has all other entries zeros, then the matrix is said to be in row reduced echelon matrix.

Row Rank and Column Rank of a Matrix

Row rank of a matrix, say A is the number of non zero rows in the row echelon matrix A and is denoted by $\rho_R(A)$.

Column Rank of a matrix, say A is the number of non zero columns in the column echelon matrix A and is denoted by $\rho_C(A)$.

Note: (i) Every matrix is row equivalent to row echelon matrix.
(ii) Every matrix is column equivalent to a column echelon matrix.
(iii) If a matrix A is in row echelon form, then its transpose is in column echelon form.

Example. 1: Reduce the matrix $A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 & 3 \\ 0 & 2 & 6 & 1 & 3 & 9 \\ 0 & 4 & 12 & -2 & 10 & 7 \end{bmatrix}$ to the row reduced echelon form and hence find its rank.

Solution: Applying $R_2 \to R_2\text{-}R_1, R_3 \to R_3\text{-}2R_1$, and $R_4 \to R_4\text{-}4R_1$ on the matrix A,

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 & 7 \\ 0 & 0 & 0 & 2 & -2 & 3 \end{bmatrix}$$

Applying $R_1 \to R_1\text{+}R_2, R_3 \to R_3\text{-}3R_2$, and $R_4 \to R_4\text{-}2R_2$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Applying $R_1 \to R_1\text{-}3R_3, R_2 \to R_2\text{-}2R_3$, and $R_4 \to R_4\text{+}R_3$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required row reduced echelon form of the matrix A. Since, the number of non zero rows is 3, thus row rank of A is 3.

System of Linear Equations and Matrices

Linear Equation

$$y = m x \qquad .1$$

is an equation, in which variable y is expressed in terms of x and the constant m, is called Linear Equation. In Linear Equation exponents of the variable is always ' one'.

Equation 1 is also called equation of line.

Linear Equation in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b \qquad .2$$

where $x_1, x_2, x_3, ..., x_n$ are variables and
$a_1, a_2, a_{13}, ..., a_n$ and b are constants.

Linear System:

A Linear System of m linear equations and n unknowns is:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \qquad .3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m
\end{aligned}$$

where $x_1, x_2, x_3, ..., x_n$ are variables or unknowns and a's and b's are constants.

Solution:

Solution of the linear system (3) is a sequence of n numbers

$s_1, s_2, s_3, ..., s_n$, which satisfies system (3) when we substitute $x_1 = s_1, x_2 = s_2, x_3 = s_3, ..., x_n = s_n$.

Example.1. Solve the system of equations

$$x - 3y = -3 \qquad \rightarrow 1$$
$$2x + y = 8 \qquad \rightarrow 2$$

Solution:

$-2E_1 + E_2 \Rightarrow$

$$\begin{aligned} -2x + 6y &= 6 \\ 2x + y &= 8 \end{aligned}$$

$+7y = 14 \Rightarrow y = 2$

From equation 1

$x = -3 + 3y$
$x = -3 + 6 = 3$

Solution is $x = 3$ and $y = 2$

Check Substitute the solution in Equations 1 and 2

Equation 1 $\Rightarrow$ $3 - 3(2) = 3 - 6 = -3$
Equation 2 $\Rightarrow$ $2(3) + 2 = 6 + 2 = 8$.

Example.2. Solve the system of equations

$$x - 3y = -7 \qquad \to 1$$
$$2x - 6y = 7 \qquad \to 2$$

Solution:

$2E_1 - E_2 \Rightarrow$

$$\begin{aligned} 2x - 6y &= -7 \\ -2x + 6y &= -14 \end{aligned}$$

$0 + 0 = -21$

This makes no sense as $0 \neq -21$, hence there is no solution.

NOTE: **Inconsistent** , the system of equations is inconsistent, if the system has no solution.

Consistent, the system of equations is consistent if the system has at least one solution.

Example: ***Inconsistent and consistent system of equations***

For the system of linear equations which is represented by straight lines:

$$a_1x - b_1y = c_1 \qquad \to l_1$$
$$a_2x - b_2y = c_2 \qquad \to l_2$$

There are three possibilities:

No solution one solution infinite many solutions [inconsistent] [consistent] [consistent]

Note:1. A system will have unique solution (only one solution)when number of unknowns is equal to number of equations

Note:2. A system is over determined , if there are more equations then unknowns and it will be mostly inconsistent.

Note:3. A system is under determined if there are less equations then unknowns and it may turn inconsistent.

Augmented Matrix

System of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_{31} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_{31} = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_{31} = b_3$$

can be written in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or we may write it in the form AX=b,

where A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, b = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Augmented matrix is $[A:b] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

Example: 4. Write the matrix and augmented form of the system of linear equations

3x – y + 6z = 6
x + y + z = 2
2x + y +4z = 3

Solution: Matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

Augmented form is $[A:b] = \begin{bmatrix} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

Elementary Row operations:

Elementary row operations are steps for solving the linear system of equations:

I. Interchange two rows
II. Multiply a row with non zero real number
III. Add a multiple of one row to another row

SYSTEM WITH NO SOLUTION

Example: 6 . Solve the system of linear equations

$$x - 2y + z - 4u = 1$$
$$x + 3y + 7z + 2u = 2$$
$$x - 12y - 11z - 16u = 5$$

Solution:

Augmented matrix is:

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix}$$

Reducing it to row echelon form (using Gaussian - elimination method)

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \quad R_2 - R_1,\ R_3 - R_1$$

$$\approx \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \quad -R_3 + 2R_2$$

Last equation is

$$0x + 0y + 0z + 0u = -3$$

but $0 \neq -3$

hence there is no solution for the given system of linear equations.

Conditions on Solutions

Example:7. For which values of 'a' will be following system

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

(i) infinitely many solutions?
(ii) No solution?
(iii) Exactly one solution?

Solution:

Augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a + 2 \end{bmatrix}$$

Reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & -14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{bmatrix} \quad R_2\text{-}3R_1,\ R_3\text{-}4R_1$$

$$\approx \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2 - 16 & a - 4 \end{bmatrix} \quad -\tfrac{1}{7} R_2,\ R_3\text{-}R_2$$

writing in the equation form,

$$x + 2y - 3z = 4 \quad \rightarrow 1$$
$$y - 2z = \tfrac{10}{7} \quad \rightarrow 2$$
$$(a^2 - 16)z = a - 4 \quad \rightarrow 3$$

or equation 3 can be written as

$$(a+4)(a-4)z = a-4$$

CASE I .

$$a = 4 \quad \Rightarrow 0z = 0$$

$$x + 2y - 3z = 4$$
$$y - 2z = \tfrac{10}{7}$$

as number of equations are less than number of unknowns, hence the system has infinite many solutions,

let $z = t$

$$y = \tfrac{10}{7} + 2t$$
$$x = 4 + 3t - 4t - \tfrac{20}{7} = -t + \tfrac{8}{7}$$

where ‘t’ is any real number.

CASE II

$a = -4 \quad \Rightarrow \quad 0z = -8$, but $0 \neq -8$, hence, there is no solution.

CASE III

$a \neq 4, a \neq -4,$ let $a = 1$

Equatins .3. $\Rightarrow (1-4)(1+4)z = 1-4$

$$-15z = -3$$
$$z = \tfrac{1}{5}$$

$$y = \tfrac{10}{7} + \tfrac{2}{5} = \tfrac{64}{35}$$
$$x = 4 + \tfrac{3}{5} - 2(\tfrac{64}{35}) = \tfrac{47}{35}$$

the system will have unique solution when $a \neq 4$ and $a \neq -4$
and for $a=1$ the solution is

$$x = \tfrac{47}{35}, y = \tfrac{64}{35} \quad \text{and} \quad z = \tfrac{1}{5}.$$

NOTE: (i) $a=-4$, no solution,
(ii) $a=4$, infinite many solutions and
(iii) $a \neq 4$, $a \neq -4$, exactly one solution .

Example:8. What conditions must a, b, and c satisfy in order for the system of equations

$$\begin{aligned} x + y + 2z &= a \\ x \quad + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

to be consistent.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$ reducing it to reduced row echelon form

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \quad R_2\text{-}R_1,\ R_3\text{-}2R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-a-b \end{bmatrix} \quad R_3\text{-}R_1$$

The system will be consistent if only if $c - a - b = 0$

or $c = a + b$

Thus the required condition for system to be consistent is

$c = a + b.$

Solution of a system $AX=b$

Let $AX = b$ be a given $m \times n$ system. The $m \times (n + 1)$ matrix $[A|b]$ is called the **augmented matrix**for the system $AX = b$. Let $[\tilde{A}|\tilde{b}]$ be the row echelon from [A|b]. The following conclusion is now obvious from the earlier discussions.

Let A**X** = **b** be a m$\times$n system of linear equation and let $[\tilde{A}|\tilde{\boldsymbol{b}}]$ be the row echelon form [A|**b**], and let r be the number of nonzero rows of $[\tilde{A}|\tilde{\boldsymbol{b}}]$. Note that 1 $\leq$ min {m, n}. Then the following hold:For the system A**X** = b

(i) The system is inconsistent, i.e., there is no solution if among the nonzero rows of $[\tilde{A}|\tilde{\boldsymbol{b}}]$there is a row with zero everywhere except at the last place. That is (n+1)th column is not a pivot column for $[\tilde{A}|\tilde{\boldsymbol{b}}]$.

(ii) The system is solvable if $[\tilde{A}|\tilde{\boldsymbol{b}}]$ has r nonzero rows with r $\leq$ n. There is a unique solution if r = n i.e., $[\tilde{A}|\tilde{\boldsymbol{b}}]$ has exactly n- nonzero rows, the number of variables. And, there are infinitely many solutions if $[\tilde{A}|\tilde{\boldsymbol{b}}]$ has r-nonzero rows, with r < n. In fact, one can compute these solutions as follows: for 1 $\leq$ i $\leq$ r, let p_i^{th} column be the pivot column. Then, assign arbitrary values to each of the variable x_j, j $\neq p_i$ and compute the values of the variable x_{p_i}, 1 $\leq$ i $\leq$ r in terms of these (as in example 2.2.2). Thus, the general solution will have n - r variables taking arbitrary values.

Examples:

(i) Consider the system *AX* = ***b*** where

$$A = \begin{bmatrix} 1 & 1 & 2 & -5 \\ 2 & 5 & -1 & -9 \\ 2 & 1 & -1 & 3 \\ 1 & 3 & 2 & 7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ -3 \\ -11 \\ -5 \end{bmatrix}.$$

It is early to verify that the augmented matrix

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & 3 & 2 & 7 & -5 \end{array}\right]$$

is equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then by theorem 2.4.1, the system A***X*** = *b* is consistent and has infinite number of solutions. In fact, if

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

Here, we can give arbitrary value to the variable x_4, and other variable can be computed by :

$$\begin{aligned} x_1 \quad + 2x_4 &= -5 \\ x_2 \quad - 3x_4 &= 2 \\ x_3 - 2x_4 &= 3 \end{aligned},$$

i.e., $$x_3 = 5 - 2x_4$$ $$x_2 = 2 + 3x_4$$ $$x_1 = -5 - 2x_4,$$

where x_4 can be assigned any arbitrary value.

(ii) Consider the system *AX* = ***b***, where

$$A = \begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}.$$

The augmented matrix in this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right].$$

It is easy to see that this is equivalent to

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 1 \\ 2 & -3 & 2 & 8 \\ 5 & -8 & 7 & 5/2 \end{array}\right]$$

Since, the last row is identically zero for the position of *A* and non-zero for the portion of *B*, the system is inconsistent.

(iii) Consider the system *AX* = ***b***, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

The augmented matrix [A|**b**] of the system can be shown to be equivalent to

$$[\tilde{A} \mid \tilde{\boldsymbol{b}}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right],$$

When $\tilde{A}$ is the reduced row echelon form of *A*. Then, *AX* = *b* has unique solution, namely

$$\tilde{\boldsymbol{b}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

LINEAR DEPENDENCE AND INDEPENDENCE OF ROW & COLUMN MATRICES.

Any quantity having n components is called a vector of order n. If $a_1, a_2, ...a_n$ are elements of fields (F, +, .), then these numbers written in a particular order form a vector.

Thus an n-dimensional vector X over a field (F,+, .) is written as X= $(a_1, a_2, ...a_n)$ where $a_i \in F$.

Row matrix of type $1\times n$ is n—dimensional vector written as $X=[a_1, a_2, ...a_n]$

Column matrix of type $n\times 1$ is also n dimensional vector written as

$$X=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ or } [a_1 \quad a_2 \quad .. \quad a_n]$$

As the vectors are considered as either row matrix or column matrix, the operation of addition of vectors will have the same properties as the addition of matrices.

Linear Dependence:

The set of vectors $\{v_1, v_2, ...v_n\}$ are said to be linearly dependent if there exist scalars $a_1, a_2, ...a_n$ not all zero such that $a_1v_1+a_2v_2+.....+a_nv_n = 0$

Linear Independence:

The set of vectors $\{v_1, v_2, ...v_n\}$ are said to be linearly independent if there exist scalars $a_1, a_2, ...a_n$ such that $a_1v_1+a_2v_2+.....+a_nv_n = 0$ gives $a_1=a_2=.....=a_n = 0$.

Example1: Show that the vectors u=(1,3,2), v=(1,-7,-8) and w=(2,1,-1) are linearly independent.

Proof: The vectors are said to be linearly dependent if

au+ bv +cw=0 where a, b, c are not all zero.

means a(1,3,2)+b(1,-7,-8)+c(2,1,-1)=(0,0,0) (1)

(a+b+2c, 3a-7b+c, 2a-8b-c)= (0, 0, 0)

which gives a+b+2c=0 (2)

3a-7b+c=0 (3)

2a-8b-c=0 (4)

Adding (3) and (4), we have

5a-15b=0 $\Rightarrow$ a=3b

$\therefore$ From (3) 3(3b)-7b+c=0 $\Rightarrow$ 9b-7b+c $\Rightarrow$ c=-2b

Putting a=3b and c=-2b in (2), we get

3b+b-4b=0, which is true. Giving different real value to b we get infinite non zero real values of a and c. So a, b, c are not all zero.

Hence given vectors u, v and w are linearly independent.

Theorem 1: If two vectors are linearly dependent then one of them is scalar multiple of other.

Proof: Let u, v be the two linearly dependent set of vectors. Then there exists scalars a, b(not both zero) such that

a. u + b. v = 0 (1)

Case 1. When $a \neq 0$

From (1), au = - bv $\Rightarrow$ $u = -\frac{b}{a}v$

Hence u is scalar multiple of v.

Case II. When $b \neq 0$

From (1), bv = - au $\Rightarrow$ $v=-\frac{a}{b}u$

Hence v is scalar multiple of u. Thus in both cases one of them are scalar multiple of other.

Theorem 2: Every superset of a linearly dependent set is linearly dependent.

Proof: Let $S_n = \{X_1, X_1......X_n\}$ be set of n vectors which are linearly dependent.

Let $S_r = \{X_1, X_1......, X_n, X_{n+1}...., X_r\}$ where r > n be any super set of S_n.

As $\{X_1, X_1......X_n\}$ is linearly dependent set

$\therefore$ There are scalars $a_1, a_2, a_3, ..., a_n$ not all zero such that

$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$

$\Rightarrow \quad a_1X_1 + a_2X_2 + ... + a_nX_n + 0.X_{n+1} + 0.X_{n+2} + ... + 0.X_r = 0$

As $a_1, a_2, a_3, ..., a_n$ are not all zero

$\therefore$ Set $S_r = \{X_1, X_2......, X_n, X_{n+1}...., X_r\}$ is linearly dependent set.

Hence every set of linearly dependent set is linearly dependent.

Theorem 3: Every subset of linearly independent set is linearly independent.

Proof: Let $S_n = \{X_1, X_1......X_n\}$ be set of n vectors which are linearly independent.

Let $S_r = \{X_1, X_1......, X_r\}$ where r < n be any subset of S_n.

As $\{X_1, X_1......X_n\}$ is linearly independent set thus

$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$ gives

$a_1 = a_2 = a_3, ... = a_n = 0$

$a_1X_1 + a_2X_2 + ... + a_rX_r = 0$ where $a_1 = a_2 = a_3, ... = a_r = 0$

$\therefore$ Set $S_r = \{X_1, X_1......, X_r\}$ is linearly independent set.

Hence every subset of linearly independent set is linearly independent.

Theorem 4: If vectors $X_1, X_1......X_n$ are linearly dependent, then at least one of them may be written as linear combination of the rest.

Proof: Since the vectors $X_1, X_1......X_n$, are linearly dependent, therefore there exist scalars

$a_1, a_2, a_3, ..., a_n$ not all zero, such that

$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$ or $a_1X_1 + a_2X_2 + ... + a_iX_i + a_{i+1}X_{i+1} ... + a_nX_n = 0$

Suppose $a_i \neq 0$

$-a_iX_i = a_1X_1 + a_2X_2 + ... a_{i-1}X_{i-1} + a_{i+1}X_{i+1} ... + a_nX_n$

or $X_i = \frac{a_1}{-a_i}X_1 + \frac{a_2}{-a_i}X_2 + ... + \frac{a_{i-1}}{-a_i}X_{i-1} + \frac{a_{i+1}}{-a_i}X_{i+1} ... + \frac{a_n}{-a_i}X_n$

Hence vector X_i is a linear combination of the rest.

Theorem 5: If the set $\{X_1, X_1......X_n\}$ is linearly independent and the set $\{X_1, X_1......X_n, Y\}$ is linearly dependent, then Y is linear combination of the vectors $X_1, X_1......X_n$.

Proof: Consider the relation

$$a_1X_1 + a_2X_2 + ... + a_nX_n + aY = 0 \qquad (1)$$

As set $\{X_1, X_1......X_n, Y\}$ is linearly dependent

$\therefore \quad a_1, a_2, a_3, ..., a_n, a$ are not all zero

We claim that $a \neq 0$. If a=0, then (1) becomes

$$a_1X_1 + a_2X_2 + ... + a_nX_n = 0$$

As set $\{X_1, X_1......X_n\}$ is linearly independent

$\therefore \qquad a_1 = a_2 = a_3, ... = a_n = 0$

Then from (1), the set $\{X_1, X_1......X_n, Y\}$ is linearly independent which a contradiction to the given condition is. Thus a = 0 is not possible. Hence $a \neq 0$

From (1), we have $-aY = a_1X_1 + a_2X_2 + ... + a_nX_n$

or $Y=\frac{a_1}{-a}X_1+\frac{a_2}{-a}X_2+.......+\frac{a_n}{-a}X_n$, which proves the result.

Theorem 6: The kn-vectors $A_1, A_2, ..., A_k$ are linearly dependent iff the rank of the matrix $A=[A_1, A_2, ..., A_k]$ with the given vectors as columns is less than k.

Proof: Let $x_1A_1 + x_2A_2, ...+ x_kA_k = 0$

where $x_1, x_2, ..., x_k$ are scalars

$$\Rightarrow x_1\begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{n1} \end{bmatrix} + x_1\begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{n2} \end{bmatrix} + ... + x_k\begin{bmatrix} a_{1k} \\ a_{2k} \\ : \\ a_{nk} \end{bmatrix} = O$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 + ... + a_{1k}x_k = 0$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2k}x_k = 0$$

$$...$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nk}x_k = 0$$

Which can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1k} \\ a_{21} & a_{22} & ... & a_{2k} \\ : & : & : & : \\ : & : & : & : \\ a_{n1} & a_{n2} & ... & a_{nk} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ : \\ : \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ : \\ 0 \end{bmatrix}$$

$$\Rightarrow AX=O$$

Let the vectors $A_1, A_2, ..., A_k$ be linearly dependent.

So, from the relation (i), scalars $x_1, x_2, ..., x_k$ are not all zero and thus the homogeneous system of equations given by (ii) has non-trivial solution. Hence $\rho(A)<k$ .Converse of this theorem is also true.

Theorem 7: A square matrix A is singular iff its columns (rows) are linearly dependent.

Proof: Let n be the order of the square matrix A and $A_1, A_2, ..., A_n$ be its columns.

$\therefore$ $A=[A_1, A_2, ..., A_n]$

Proceed in same way as above theorem to prove $\rho(A)<n$

Since $\rho(A)<n$, thus $|A| = 0$ and hence A is singular matrix.

Conversely, the column vectors of A are linearly dependent.

Theorem 8: The kn-vectors $A_1, A_2, ..., A_k$ are linearly independent if the rank of the matrix $A=[A_1, A_2, ..., A_k]$ is equal to k.

Proof: Proceed in the same way as above theorem to obtain $AX=O$. Now suppose .

Then $|A| \neq 0$ and homogeneous system of equations given by (ii) has trivial solution only.

$\therefore$ $x_1 = x_2 = ... = x_k = 0$

Thus, the vectors $A_1, A_2, ..., A_k$ are linearly independent.

Theorem 9: The number of linearly independent solution of the equation AX=O is (n-r) where r is the rank of matrix A.

Proof: Given that rank of A is r which means A has r linearly independent columns. Let first r columns are linearly independent.

Now, $A=[C_1, C_2, ...C_r, ..., C_n]$, where $C_1, C_2, ..., C_n$ are column vectors of A.

$$\therefore [C_1, C_2, ..., C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow C_1x_1 + C_2x_2 + ... + C_nx_n = 0 \qquad ...(i)$$

As the set $[C_1, C_2, ..., C_r]$ is linearly independent, thus each vector $C_r, C_{r+1}, ..., C_n$ can be written as linear combination of $C_1, C_2, ..., C_r$.

Now, $C_{r+1} = a_{11}C_1 + a_{12}C_2 + ... + a_{1r}C_r$

$C_{r+2} = a_{21}C_1 + a_{22}C_2 + ... + a_{2r}C_r$

....

$C_n = a_{k1}C_1 + a_{k2}C_2 + ... + a_{kr}C_r$, where k=n-r ...(ii)

From (i) and (ii), we get

$$X_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1r} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, ..., X_{n-r} = \begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

Thus, AX=O has (n-r) solutions.

Check Your Progress

1. Find the vector p if the given vectors are linearly dependent $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ p \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Ans. p=2.

LINEAR SYSTEM OF EQUATIONS

System of Non Homogeneous Linear Equation

If

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \end{array}\right\} \quad \dots(1)$$

be given system of m linear equations then (1) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_m \end{bmatrix}$$

$\Rightarrow$ AX = B and $C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} : b_1 \\ a_{21} & a_{22} & \dots & a_{2n} : b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} : b_m \end{bmatrix}$

then [A : B] or C is called augmented matrix. Sometime we also write A : B for [A : B]

Consistent Equations.

(i) If rank of A = rank of [A : B] and there is unique solution when rank of A = rank of [A : B] = n

(i) rank of A = rank of [A : B] = r < n.

Inconsistent Equations.

If rank of A $\neq$ rank of [A : B] i.e. have no solution.

Example 1. Discuss the consistency of the following system of equation 2x + 3y + 4z = 11, x + 5y + 7z = 15, 3x + 11y + 13z = 25, if consistent, solve.

Sol. The augmented matrix $[A : B] = \begin{bmatrix} 2 & 3 & 4:11 \\ 1 & 5 & 7:15 \\ 3 & 11 & 13:25 \end{bmatrix}$

R_{12} operation is done so $\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 2 & 3 & 4:11 \\ 3 & 11 & 13:25 \end{bmatrix}$

Next operating $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & -7 & -10:-19 \\ 0 & -4 & -8:-20 \end{bmatrix}$$

Again, operating $R_2 \rightarrow -\frac{1}{7} R_2$ and $R_3 \rightarrow -\frac{1}{4} R_3$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & 1 & \frac{10}{7}:\frac{19}{7} \\ 0 & 1 & 2:5 \end{bmatrix}$$

Next operating $R_3 \rightarrow R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7 : 15 \\ 0 & 1 & \frac{10}{7} : \frac{19}{7} \\ 0 & 0 & \frac{4}{7} : \frac{16}{7} \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + 5y + 7z &= 15 \\ y + \frac{10}{7}z &= \frac{19}{7} \\ \frac{4}{7}z &= \frac{16}{7} \end{aligned} \qquad \ldots(M)$$

From which we get rank of A = 3 as well as rank of A : B = 3. Hence the system of equations is consistent and has unique solution $\frac{4}{7}z = \frac{16}{7} \Rightarrow z = 4$

And $y + \frac{10}{7}z = \frac{19}{7} \Rightarrow y + \frac{10}{7} \times 4 = \frac{19}{7} \Rightarrow y = -\frac{21}{7} = -3$

And from (M), we have $x + 5y + 7z = 15 \Rightarrow x = 2$

i.e. we have the solution $x = 2$, $y = -3$ and $z = 4$, which is the required result.

Example 2. Test the following equations for consistency and hence solve these equations 2x – 3y + 7z = 5, 3x + y – 3z = 13 and 2x + 19y–47z = 32.

Sol. The above equations may be written as AX = B.

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Operating $R_2 \to 2R_2 - 3R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

Next, we operate $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

This indicate the rank of A = 2 which is less than 3 (the number of variables) i.e. $\rho(A) = 2 < 3$

So, the given equations are not consistent and so infinite number of solutions can be obtained.

Example 3. Show that if λ ≠ –5, the system of equation 3x – y + 4z = 3, x + 2y – 3z = –2 and 6x + 5y + λz = –3 have a unique solution. If λ = –5, show that the equations are consistent. Determine the solution, in each case.

Sol. The given equations are

$$\left.\begin{aligned} 3x - y + 4z &= 3, \\ x + 2y - 3z &= -2 \\ \text{and} \quad 6x + 5y + \lambda z &= -3 \end{aligned}\right\} \qquad \ldots(1)$$

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$ such that AX = B from (1)

Then augmented matrix A : B = $\begin{bmatrix} 3 & -1 & 4:3 \\ 1 & 2 & -3:-2 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$

Operating R_{12}(i.e. interchanging R_1 and R_2)

$$A : B = \begin{bmatrix} 1 & 2 & -3:-2 \\ 3 & -1 & 4:3 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$$

Now operating $R_2 - 3R_1$ [i.e. R_2, 1(–3)] and R_3, 1(–6) i.e. $R_3 - 6R_1$, we get

$$A : B \sim \begin{bmatrix} 1 & 2 & 3 & :-2 \\ 0 & -7 & 13 & :9 \\ 0 & -7 & \lambda+18 & :9 \end{bmatrix}$$

Next, $R_3 - R_2$[(i.e. R_3, 2(–1)], we get

$$\sim \begin{bmatrix} 1 & 2 & -3:-2 \\ 0 & -7 & 13:9 \\ 0 & 0 & \lambda+5:0 \end{bmatrix} \quad \text{…(2)}$$

If $\lambda = -5$, then rank of A becomes $\rho(A) = 2$ which is less than 3, (the number of unknowns) and hence the equations will be consistent and will have infinite number of solutions

Next, operating, $R_1 + \frac{2}{7}R$, we get

$$\sim \begin{bmatrix} 1 & 0 & 5:\frac{4}{7} \\ 0 & -7 & 13:9 \\ 0 & 0 & \lambda+5:0 \end{bmatrix}$$ from this matrix, if $\lambda \neq -5$

then rank is 3 and the equation will be consistent and we get

$$x + \frac{5}{7}z = \frac{4}{7} \;;\; -7y + 13z = 9 \text{ and } (\lambda + 5)z = 0 \text{ i.e. } z = 0$$

$\Rightarrow \quad -7y = 9 \Rightarrow \quad y = -\frac{9}{7}$ and $x + 0 = \frac{4}{7}$ i.e. $x = \frac{4}{7}$.

i.e. unique solution is $x = \frac{4}{7}$, $y = -\frac{9}{7}$, $z = 0$, which is required result.

If $\lambda = -5$, then from (2), we have $x + 2y - 3z = -2$, $-7y + 13z = 9$ …(3)

If we take z = k than from (3),

$$y = \frac{13k-9}{7} \quad \text{and} \quad z = \frac{3k + 2\left(\frac{13k-9}{7}\right) - 2}{3} - \frac{4-5k}{7}$$

Example 4. Examine whether the following equations are consistent and solve them if they are consistent $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$ and $6y - 18z + 1 = 0$.

Sol. The above equations may be written in the form

$$AX = B \text{ which is } \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} \quad \text{…(1)}$$

Now the augmented matrix may be written as

$$A : B = \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \quad \ldots(2)$$

Operating $R_2 \to R_2 - 3R_1$, we get

$$A : B \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

Now, operating $R_3 \to R_3 - 3R_2$, we get

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

Hence rank of A = $\rho(A) = 2$ and $\rho(A : B) = 3$. So, $\rho(A) = 2 < 3$ (number of variables). This indicated that given equation are in consistent and so it has no unique solution.

Example 5. Solve the following system of equations by matrix method $x + y + z = 8$, $x - y + 2z = 6$ and $3x + 5y - 7z = 14$.

Sol. The above equations written in the form AX = B.

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix}$

So, we may write augmented matrix as

$$A : B = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 2 & : & 6 \\ 3 & 5 & -7 & : & 14 \end{bmatrix} \quad \ldots(1)$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$, we have

$$A : B \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 2 & 10 & : & 10 \end{bmatrix} \quad \ldots(2)$$

Again $R_3 \to R_3 + R_2$, we have

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 0 & -9 & : & -12 \end{bmatrix}$$

this implies that

$$\left.\begin{aligned} x + y + z &= 8 \\ -2y + z &= -2 \\ \text{and} \quad -9z &= -12 \end{aligned}\right\} \quad \ldots(3)$$

$\Rightarrow \quad z = \frac{4}{3}$ and $2y = z + 2 = \frac{4}{3} + 2 = \frac{10}{3} \quad \therefore \quad y = \frac{5}{3}$

Using 1st equation of (3), we get $x + y + z = 8$

$\Rightarrow \quad x + \frac{5}{3} + \frac{4}{3} = 8 \quad \Rightarrow \quad x = 8 - 3 = 5$

From (2) we see that $\rho(A) = 3$ = number of variables so, the system of equations are consistent and solutions are $x = 5$, $y = \frac{5}{3}$, $z = \frac{4}{3}$.

Example 6. Determine for what values of λ and μ the following equations have (i) no solution ii) a unique solution (iii) infinite number of solution : $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$.

Sol. The above equations may be written in the form AX = B.

i.e. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix $[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

Again operating $R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

$\Rightarrow$ we get $x + y + z = 6$, $y + 2z = 4$ and $(\lambda - 3)z = \mu - 10$.

(i) If $R(A) \neq R[A : B]$ i.e. if $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, then rank of A $\neq$ rank of [A : B]. Since $\rho(A) = 2$ and $\rho(A : B) = 3$. The equation have no solution.

(ii) The equations have unique solution if rank of A = rank of [A : B] = 3, i.e. if $\lambda - 3 \neq 0$ and $\mu - 3 \neq 0$.

(iii) If $\rho(A) = \rho(A : B) = 2$ i.e. when $\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e. when $\lambda = 3$ and $\mu = 10$. Then these are infinite number of solution.

System of Homogeneous Linear Equations

If

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = \end{array}\right\} \quad \ldots(1)$$

be given system of m linear equations then (1) may be written as $AX=O$

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}$$

Here A is called the coefficient matrix and the given system of equations $AX=O$ is called linear homogeneous system of equations.

Working rule for determining solution of m homogeneous equations in n variables.

Firstly we find the rank of coefficient matrix A. Then

1. There is only a trivial solution which is $x_1 = x_2 = \ldots\ldots = x_n = 0$ if $\rho(A) = n$.

2. A can be reduced to a matrix which has (n-r) zero rows and r non zero rows and if $\rho(A) < n$ so the system is consistent and has infinite number of solutions.

Thus, the given system of equations has a non- trivial solution iff $|A| = 0$

Example 1: Solve the following system of equations

x - y + z = 0

x + 2y - z = 0

2x + y + 3z = 0

Solution. Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Operating $R_2 \to R_2 + (-R_1)$ and $R_3 \to R_3 + (-2)R_1$,

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + (-R_2)$, $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

Operating $R_2 \to R_2 \times \left(\frac{1}{3}\right)$ and $R_3 \to R_3 \times \left(\frac{1}{3}\right)$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \ \rho(A)=3$ = number of variables and hence the given system of equations has only trivial solution, x = y = z = 0.

Example: Solve the following system of equations:

x - y + 2z - 3w = 0

3x + 2y - 4z + w = 0

4x - 2y + 9w = 0

Solution: Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}$

Operating $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_2 \to R_2\left(\frac{1}{5}\right)$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \end{bmatrix}$$

$\therefore$ $\rho(A)=3$, Here n = 4 (the number of unknowns)

Now $\rho(A)<4$. Thus the system of equations has infinite solutions. The solutions will contain 4 – 3=1 arbitrary constant.

Equation corresponding to the matrix are

$$x - y + 2z - 3w = 0 \qquad (1)$$

$$y - 2z + 2w = 0 \qquad (2)$$

$$-4z + 17w = 0 \qquad (3)$$

From (3), $z = \frac{17}{4}w$

$\therefore$ From (2), $y - \frac{17}{2}w + 2w = 0 \Rightarrow y = \frac{13}{2}w$

$\therefore$ From (1), $x - \frac{13}{2}w + \frac{17}{2}w - 3w = 0 \Rightarrow x = w$

Putting w = k, we get x =k, $y = \frac{13}{2}$k, $z = \frac{17}{4}$k, which is the general solution, where k is an arbitrary parameter.

Check Your Progress

1. Solve the following system of liear equation

$x - y + z = 0$

$x + 2y - z = 0$

$2x + y + 3z = 0$

Ans. x = y = z = 0.

2. Find the values of a and b for which the following system of linear equations

$2x + by - z = 3$

$5x + 7y + z = 7$.

$ax + y + 3z = a$

Ans. a = 1 and b = 3.

APPLICATION OF LINEAR SYSTEM

Three by three systems of linear equations are also used to solve real-life problems. The given problem is expressed as a system of linear equations and then solved to determine the value of the variables. Sometimes, the system will consist of three equations but not every equation will have three variables. Example three is one such problem.

Example 1: Solve the following problem using your knowledge of systems of linear equations.

Jesse, Maria and Charles went to the local craft store to purchase supplies for making decorations for the upcoming dance at the high school. Jesse purchased three sheets of craft paper, four boxes of markers and five glue sticks. His bill, before taxes was $24.40. Maria spent $30.40 when she bought six sheets of craft paper, five boxes of markers and two glue sticks. Charles, purchases totaled $13.40 when he bought three sheets of craft paper, two boxes of markers and one glue stick. Determine the unit cost of each item.

Let **p** represent the number of sheets of craft paper.
Let **m** represent the number of boxes of markers.
Let **g** represent the number of glue sticks.

Express the problem as a system of linear equations:

$$\begin{aligned} 3p + 4m + 5g &= \$24.40 \\ 6p + 5m + 2g &= \$30.40 \\ 3p + 2m + g &= \$13.40 \end{aligned}$$

Solve the system of linear equations to determine the unit cost of each item.

$$\begin{matrix} 3p+4m+5g=24.40 \\ 3p+2m+g=13.40 \end{matrix} \Rightarrow \begin{matrix} 3p+4m+5g=24.40 \\ -5(3p+2m+g=13.40) \end{matrix} \Rightarrow \begin{matrix} 3p+4m+5g=24.40 \\ \underline{-15p-10m-5g=-67.00} \\ -12p-6m=-42.60 \end{matrix}$$

$$\begin{matrix} 6p+5m+2g=30.40 \\ 3p+2m+g=13.40 \end{matrix} \Rightarrow \begin{matrix} 6p+5m+2g=30.40 \\ -2(3p+2m+g=13.40) \end{matrix} \Rightarrow \begin{matrix} 6p+5m+2g=30.40 \\ \underline{-6p-4m-2g=-26.80} \\ m=3.60 \end{matrix}$$

$$\begin{aligned} &-12p-6m=-42.60 \\ &-12p-6(3.60)=-42.60 \\ &-12p-21.60=-42.60 \\ &-12p-21.60+21.60=-42.60+21.60 \\ &-12p=-21 \\ &\frac{-12}{-12}p=\frac{-21}{-12} \\ &p=1.75 \end{aligned}$$

$$\begin{aligned} &3p+2m+g=13.40 \\ &3(1.75)+2(3.60)+g=13.40 \\ &5.25+7.20+g=13.40 \\ &12.45+g=13.40 \\ &12.45-12.45+g=13.40-12.45 \\ &g=.95 \end{aligned}$$

The unit cost of each item is: 1 sheet of craft paper = $1.75
1 box of markers = $3.60
1 glue stick = $0.95

Example 2: Solve the following problem using your knowledge of systems of linear equations.

Rafael, an exchange student from Brazil, made phone calls within Canada, to the United States, and to Brazil. The rates per minute for these calls vary for the different countries. Use the information in the following table to determine the rates.

Month	Time within Canada (min)	Time to the U.S. (min)	Time to Brazil (min)	Charges ($)
September	90	120	180	$252.00
October	70	100	120	$184.00
November	50	110	150	$206.00

Let **c** represent the rate for calls within Canada.
Let **u** represent the rate for calls to the United States.
Let **b** represent the rate for calls to Brazil.

Express the problem as a system of linear equations:

$$90c + 120u + 180b = \$252.00$$
$$70c + 100u + 120b = \$184.00$$
$$50c + 110u + 150b = \$206.00$$

$$\begin{aligned} 90c + 120u + 180b &= 252.00 \\ 70c + 100u + 120b &= 184.00 \end{aligned} \Rightarrow \begin{aligned} &2(90c + 120u + 180b = 252.00) \\ &-3(70c + 100u + 120b = 184.00) \end{aligned}$$

$$\Rightarrow \begin{array}{l} 180c + 240u + 360b = 504.00 \\ \underline{-210c - 300u - 360b = -552.00} \\ \quad -30c - 60u = -48.00 \end{array}$$

$$\begin{aligned} 70c + 100u + 120b &= 184.00 \\ 50c + 110u + 150b &= 206.00 \end{aligned} \Rightarrow \begin{aligned} &-5(70c + 100u + 120b = 184.00) \\ &4(50c + 110u + 150b = 206.00) \end{aligned}$$

$$\Rightarrow \begin{array}{l} -350c - 500u - 600b = -920.00 \\ \underline{200c + 440u + 600b = 824.00} \\ \quad -150c - 60u = -96.00 \end{array}$$

$$\begin{aligned}-30c-60u&=-48.00\\-150c-60u&=-96.00\end{aligned} \Rightarrow \begin{aligned}-1(-30c-60u&=-48.00)\\-150c-60u&=-96.00\end{aligned} \Rightarrow \begin{aligned}30c+60u&=48.00\\-150c-60u&=-96.00\\\hline -120c&=-48.00\end{aligned}$$

$$\frac{-120}{-120}c=\frac{-48.00}{-120}$$

$$c=.40$$

$$-30c-60u=-48.00$$
$$-30(.40)-60u=-48.00$$
$$-12.00-60u=-48.00$$
$$-12.00+12.00-60u=-48.00+12.00$$
$$-60u=-36.00$$
$$\frac{-60}{-60}u=\frac{-36.00}{-60}$$
$$u=.60$$

$$70c+100u+120b=184.00$$
$$70(.40)+100(.60)+120b=184.00$$
$$28.00+60.00+120b=184.00$$
$$88.00+120b=184.00$$
$$88.00-88.00+120b=184.00-88.00$$
$$120b=96.00$$
$$\frac{120}{120}b=\frac{96.00}{120}$$
$$b=.80$$

The cost of minutes within Canada is \$0.40/min. The cost of minutes to the United States is \$0.60/min. The cost of minutes to Brazil is \$0.80/min.

PART – B (5 x 2 =10)

Possible Questions (2 marks)

1. Define the systems of Linear equations
2. Define the row reduction echelon matrix with example.
3. Define the row equivalent matrix.
4. What do you mean by Linear Independence?
5. When we say that the system is homogeneous.
6. In which case the linear equations are equivalent.
7. What do you mean by Linear dependence?
8. When we say that the system is Non-homogeneous.

PART – C (5 x 6 =30)

Possible Questions (6 marks)

1) Determine if b is a linear combination of a_1 and a_2 where $a_1=\begin{bmatrix}1\\-2\\3\end{bmatrix}$, $a_2=\begin{bmatrix}5\\-13\\-3\end{bmatrix}$ and $b=\begin{bmatrix}-3\\8\\1\end{bmatrix}$

2) Determine the system is consistent

$$x_1-6x_2=5$$
$$x_2-4x_3+x_4=0$$
$$-x_1+6x_2+x_3+5x_4=3$$
$$-x_2+5x_3+4x_4=0$$

3) Determine if the system is consistent $\begin{bmatrix}1 & 5 & 2 & -6\\0 & 4 & -7 & 2\\0 & 0 & 5 & 0\end{bmatrix}$

4) Let $A=\begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}$,$u=\begin{bmatrix}4\\-1\end{bmatrix}$ and $v=\begin{bmatrix}-3\\5\end{bmatrix}$ Verify i) A(u+v)=Au+Av ii)A(5u)= 5A(u).

5) Find the general solutions of the system whose augmented matrix is $\begin{bmatrix}3 & -4 & 2 & 0\\-9 & 12 & -6 & 0\\-6 & 8 & -4 & 0\end{bmatrix}$

6) Describe the solution of $AX = B$ where $A = \begin{bmatrix}3 & 5 & 6\\-3 & -2 & 1\\6 & 1 & -8\end{bmatrix}$ and $b = \begin{bmatrix}7\\-1\\-4\end{bmatrix}$

7) If $A=\begin{pmatrix}3 & -1 & 2\\2 & 1 & 1\\1 & -3 & 0\end{pmatrix}$ find all solutions of AX=0 by row reducing A.

8) In $V_3(R)$ the vectors (1,2,1) ,(2,1,0) and (1,-1,2) are linearly independent or not

9) Find a row reduced echelon matrix which is row equivalent to

$A = \begin{bmatrix}1 & -i\\2 & 2\\i & 1+i\end{bmatrix}$ What are the solutions of AX=0?

10) Let $v_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2=\begin{bmatrix}4\\5\\6\end{bmatrix}$ and $v_3=\begin{bmatrix}2\\1\\0\end{bmatrix}$,

i) Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

ii) If possible, find a linear dependence relation among v_1, v_2, and v_3

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Subject: Algebra **Subject Code: 17MMU102**

Class : I - B.Sc. Mathematics **Semester : I**

Unit IV
System of Linear Equation

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

S.No	Question	Choice 1	Choice 2	Choice 3	Choice 4		Answer
1	Any n-tuple of elements of F which satisfies each of the equations in linear equation is called a …………………………of the system.	value	root	solution	function		solution
2	Any………..-tuple of elements of F which satisfies each of the equations in linear equation is called a solution of the system.	1	2	3	n		n
3	Any n-tuple of elements of F which satisfies each of the ………… in linear equation is called a solution of the system.	functions	equations	roots	solutions		equations
4	If y1=y2=……….=ym=0 then the system is ………………………………..	homogeneous	non homogeneous	linear	nonlinear		homogeneous
5	If y1=y2=……….=ym=………….then the system is homogeneous.	0	1	2	3		0
6	The most fundamental technique for finding the solution of a system of linear equations is the technique of …………………………	substitution	elimination	integration by parts	differention		elimination

7	The most fundamental technique for finding the solution of a system of……………….equations is the technique of elimination.	integral	differential	linear	nonlinear		linear
8	The most fundamental technique for finding the ……………. of a system of linear equations is the technique of elimination.	function	root	solution	value		solution
9	………… systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.	one	Two	three	four		Two
10	Two systems of linear equations are ……... if each equation in each system is a linear combination of the equations in the other system.	zero	equivalent	different	division		equivalent
11	Two systems of linear equations are equivalent if each equation in each system is a ………………. combination of the equations in the other system.	linear	non linear	homogeneous	non homogeneous		linear
12	Two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the …………system.	first	same	other	finite		same
13	………………….systems of linear equations have exactly the same solutions.	linear	nonlinear	Equivalent	homogeneous		Equivalent
14	Equivalent systems of ………………...equations have exactly the same solutions.	linear	non linear	homogeneous	non homogeneous		linear
15	Equivalent systems of linear equations have exactly the……………. solutions.	zero	same	different	finite		same
16	An …………...matrix R is called a row reduced echelon matrix if R is row reduced.	mxm	nxn	mxn	nxm		mxn
17	An mxn matrix R is called a ………………….. matrix if R is row reduced.	row reduced echelon	column reduced echelon	echelon	null		row reduced echelon
18	An mxn matrix R is called a row reduced echelon matrix if R is ……………	unit	null	column reduced	row reduced		row reduced

19	In the row reduced echelon form every ………….. R which has all its entries 0 occurs below every row has a non zero entry.	row	column	unit	singular		row
20	In the row reduced echelon form every row R which has all its entries …………. occurs below every row has a non zero entry.	0	1	2	3		0
21	In the row reduced echelon form every row R which has all its entries 0 occurs below every row has a ……………...entry.	zero	non zero	unit	diagonal		non zero
22	In the ……………….. form every row R which has all its entries 0 occurs below every row has a non zero entry.	row reduced echelon	column reduced echelon	echelon	null		row reduced echelon
23	An ……….. matrix R is called row reduced if the first non zero entry in each non zero row of R is equal to 1	mxm	nxn	mxn	nxm		mxn
24	An mxn matrix R is called ……………….. if the first non zero entry in each non zero row of R is equal to 1	row reduced echelon	column reduced echelon	rowreduced	column reduced		rowreduced
25	An mxn matrix R is called row reduced if the first …………. entry in each non zero row of R is equal to 1	zero	non zero	diagonal	unit		non zero
26	An mxn matrix R is called row reduced if the first non zero entry in each non zero row of R is equal to …………..	0	1	2	3		1
27	In row reduced, each ……………... of R which contains the leading non zero entry of some row has all its other entries 0.	row	column	diagonal	first		column
28	In row reduced, each column of R which contains the…………….. non zero entry of some row has all its other entries 0.	first	second	third	leading		leading
29	In row reduced, each column of R which contains the leading …………….. entry of some row has all its other entries 0.	zero	non zero	diagonal	unit		non zero
30	In row reduced, each column of R which contains the leading non zero entry of some ……………. has all its other entries 0.	row	column	diagonal	first		row
31	In row reduced, each column of R which contains the leading non zero entry of some row has all its other entries………………..	0	1	2	3		0

32	Every ……………….. matrix A is row equivalent to a row reduced echelon matrix.	mxm	nxn	mxn	nxm		mxn
33	Every mxn matrix A is ………………….equivalent to a row reduced echelon matrix.	row	column	diagonal	leading		row
34	Every mxn matrix A is row equivalent to a ………………….. matrix.	row reduced echelon	column reduced echelon	echelon	null		row reduced echelon
35	If A is an mxn matrix and ……………..,then the homogeneous system of linear equations AX=0 has a non- trivial solution.	m<n	m>n	m=n	m-n		m<n
36	If A is an mxn matrix and m<n,then the………………...system of linear equations AX=0 has a non- trivial solution.	homogeneous	non homogeneous	linear	nonlinear		homogeneous
37	If A is an mxn matrix and m<n,then the homogeneous system of linear equations AX=………… has a non- trivial solution.	0	1	2	3		0
38	If A is an mxn matrix and m<n,then the homogeneous system of linear equations AX=0 has a …………solution.	trivial	non- trivial	zero	non- zero		non- trivial
39	If A is an ………….. matrix,then A is row equivalent to the nxn identity matrix iff the system of equations AX=0 has only the trivial solution.	mxm	nxn	mxn	nxm		nxn
40	If A is an nxn matrix,then A is ……….....to the nxn identity matrix iff the system of equations AX=0 has only the trivial solution.	row equivalent	column equivalent	diagonal	leading		row equivalent
41	If A is an nxn matrix,then A is row equivalent to the nxn ………………... matrix iff the system of equations AX=0 has only the trivial solution.	zero	identity	row	column		identity
42	If A is an nxn matrix,then A is row equivalent to the nxn identity matrix iff the system of equations …………….. has only the trivial solution.	AX=I	AX=0	AX=R	AX=B		AX=0
43	If A is an nxn matrix,then A is row equivalent to the nxn identity matrix iff the system of equations AX=0 has only the ……………….solution.	trivial	non- trivial	zero	non- zero		trivial

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Subject : Algebra **Subject Code : 17MMU102** **L T P C**

Class : I – B.Sc. Mathematics **Semester : I** **6 1 0 6**

UNIT V

Introduction to linear transformations, matrix of a linear transformation, inverse of a matrix, characterizations of invertible matrices. Subspaces of Rn, dimension of subspaces of Rn and rank of a matrix, Eigen values, Eigen Vectors and Characteristic Equation of a matrix.

SUGGESTED READINGS

TEXT BOOKS

1. Titu Andreescu., and Dorin Andrica,(2006). Complex Numbers from A to Z, Birkhauser. Library of Congress Cataloging-in-Publication Data Andreescu, Titu.

2. Edgar G. Goodaire and Michael M. Parmenter, ,(2005). Discrete Mathematics with Graph Theory, 3rd Edition, Pearson Education (Singapore) P. Ltd., Indian Reprint.

3.David C. Lay., (2007). Linear Algebra and its Applications, Third Edition, Pearson Education Asia, Indian Reprint.

REFERENCE

1. Kenneth Hoffman., Ray Kunze., (2003).Linear Algebra, Second edition, Prentice Hall of India Pvt Ltd, New Delhi.

UNIT – V
LINEAR TRANSFORMATIONS AND MATRICES

Linear Transformation:

Definition of linear transformation:
A linear transformation L of the vector space V into the vector space W is a *function* (denoted by $L:V\rightarrow W$) such that for $u,v\in V, k\in R$,

(a) $L(u+v)=L(u)+L(v)$.

(b) $L(ku)=kL(u)$.

Note:
If $L:V\rightarrow V$ and L is a linear transformation, L is also called a *linear operator* on V.

Note:
$L(u)$ is called the image of u .

Example:

Let

$$u=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}, v=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}, u^*=\begin{bmatrix}u_1^*\\u_2^*\\\vdots\\u_m^*\end{bmatrix}, v^*=\begin{bmatrix}v_1^*\\v_2^*\\\vdots\\v_m^*\end{bmatrix}.$$

A linear transformation L of R^n (V) into R^m (W) is a function such that

(a) $L(u+v)=u^*+v^*=L(u)+L(v)$, where $L(u)=u^*$ and $L(v)=v^*$.

(b) $L(ku)=kL(u)=ku^*$, where $k\in R$.

Several special cases of the above linear transformation are the following:

1. Projection: $L:R^3\rightarrow R^2$ is defined by

$$L\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}.$$

L is a *linear transformation* since

(a) for any $u=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, v=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$,

$$L(u+v)=L\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{bmatrix}\right)=\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}=\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1\\v_2\end{bmatrix}=L\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)+L\left(\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\right)=L(u)+L(v).$$

(b) for $k\in R$,

$$L(ku)=L\left(\begin{bmatrix}ku_1\\ku_2\\ku_3\end{bmatrix}\right)=\begin{bmatrix}ku_1\\ku_2\end{bmatrix}=k\begin{bmatrix}u_1\\u_2\end{bmatrix}=kL\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=kL(u).$$

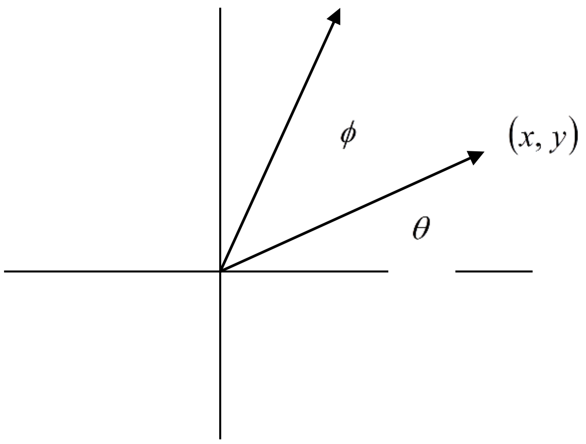
2.
Dilation: $L_1:R^3\to R^3$ is defined by

$$L_1(u)=L_1\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=r\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=ru,\ r>1.$$

Constraction: $L_2:R^3\to R^3$ is defined by

$$L_2(u)=L_2\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=r\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=ru,\ 0<r<1$$

$\Rightarrow$ Both L_1 and L_2 are *linear transformations.*
3.



Let $u=\begin{bmatrix}x\\y\end{bmatrix}$ and $r=\|u\|$. Then, $x=r\cos(\theta), y=r\sin(\theta)$.

$$\Rightarrow x' = r\cos(\theta+\phi) = r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi)$$
$$y' = r\sin(\theta+\varphi) = r\sin(\theta)\cos(\phi) + r\cos(\theta)\sin(\phi).$$

$$\Rightarrow \quad x' = x\cos(\phi) - y\sin(\phi)$$
$$y' = x\sin(\phi) + y\cos(\phi)$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Rotation: $L: R^2 \to R^2$ is defined by

$$L(u) = L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

L is a *linear transformation.*

4. Let A be fixed $m \times n$ matrix. Then,
$L: R^n \to R^m$ defined by

$$L(u) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = Au$$

is a *linear transformation* since
(a) for any $u, v \in R^n$,

$$L(u+v) = A(u+v) = Au + Av = L(u) + L(v).$$

(b) for $k \in R$,

$$L(ku) = A(ku) = k(Au) = kL(u).$$

Example:

Let
$L: P_2 \to P_1, L(a_2x^2 + a_1x + a_0) = (a_2 + a_1)x + a_0$,
where P_n is the set of all the polynomials of degrees $\leq n$. Is L a linear transformation?

[solution:]

L is a *linear transformation* since
(a) for any $u = a_2x^2 + a_1x + a_0, v = b_2x^2 + b_1x + b_0$ in P_2,

$$\begin{aligned}L(u+v) &= L((a_2+b_2)x^2+(a_1+b_1)x+(a_0+b_0)) \\ &= [(a_2+b_2)+(a_1+b_1)]x+(a_0+b_0) \\ &= [(a_2+a_1)x+a_0]+[(b_2+b_1)x+b_0] \\ &= L(a_2x^2+a_1x+a_0)+L(b_2x^2+b_1x+b_0) \\ &= L(u)+L(v)\end{aligned}$$

(b) for $k \in R$,

$$\begin{aligned}L(ku) &= L(k(a_2x^2+a_1x+a_0)) = L(ka_2x^2+ka_1x+ka_0) \\ &= (ka_2+ka_1)x+ka_0 = k[(a_2+a_1)x+a_0] \\ &= kL(a_2x^2+a_1x+a_0) \\ &= kL(u)\end{aligned}$$

Example:

Let $L: P_n \to P_n$, L is the operation of taking the derivative, for example, $L(x^2) = 2x$.

Is L a linear transformation?

[solution:]

L is a *linear transformation* since

(a) for any $u = a_nx^n + a_{n=1}x^{n-1} + \cdots + a_0, v = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0$ in P_n,

$$\begin{aligned}L(u+v) &= L((a_n+b_n)x^n+(a_{n-1}+b_{n-1})x^{n-1}+\cdots+(a_0+b_0)) \\ &= n(a_n+b_n)x^{n-1}+(n-1)(a_{n-1}+b_{n-1})x^{n-2}+\cdots+(a_1+b_1) \\ &= [na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1]+[nb_nx^{n-1}+(n-1)b_{n-1}x^{n-2}+\cdots+b_1] \\ &= L(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0)+L(b_nx^n+b_{n-1}x^{n-1}+\cdots+b_0) \\ &= L(u)+L(v)\end{aligned}$$

(b) for $k \in R$,

$$\begin{aligned}L(ku) &= L(ka_nx^n+ka_{n-1}x^{n-1}+\cdots+ka_0) \\ &= nka_nx^{n-1}+(n-1)ka_{n-1}x^{n-2}+\cdots+ka_1 \\ &= k[na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1] \\ &= kL(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0) \\ &= kL(u)\end{aligned}$$

Example:

$L: R^3 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 u_2 \\ u_3 \end{bmatrix}.$$

Is L a linear transformation?

[solution:]

L is *not* a linear transformation since

(a) for any $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$,

$$L(u+v) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)(u_2+v_2) \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_1u_2+u_1v_2+u_2v_1+v_1v_2 \\ u_3+v_3 \end{bmatrix}$$

$$\neq \begin{bmatrix} u_1u_2+v_1v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} u_1u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1v_2 \\ v_3 \end{bmatrix} = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = L(u) + L(v)$$

Important result:

Let $L: V \to W$ be a linear transformation. Then,

- $L(0_V) = 0_W$, where 0_V is the zero vector in V and 0_W is the zero vector in W.
- $L(u-v) = L(u) - L(v)$.
- For *any vectors* $v_1, v_2, \ldots, v_k$ in V and any scalars $c_1, c_2, \ldots, c_k$, then
$L(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1L(v_1) + c_2L(v_2) + \cdots + c_kL(v_k)$.
- If V is an n-dimensional vector space and $S = \{w_1, w_2, \ldots, w_n\}$ be a basis for V. If u is any vector in V, then $L(u)$ is a linear combination of $L(w_1), L(w_2), \ldots, L(w_n)$.

Example:

Let $L : R^3 \to R^2$ defined by

$$L(x) = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax.$$

Let

$$S = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}, T = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Then, since

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow [x]_S = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x$$

and

$$L(x) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_1 + 3x_3 \end{bmatrix} = (x_1 + x_2 + x_3)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_1 + 2x_2 + 3x_3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then

$$\Rightarrow [L(x)]_T = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 \end{bmatrix} = L(x) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}[x]_S = A[x]_S$$

$$L(x) = [L(x)]_T = A[x]_S = Ax.$$

Example:

Let $L : R^3 \to R^2$ defined by

$$L(x) = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_3 \end{bmatrix}.$$

Let

$$S = \{v_1, v_2, v_3\} = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}, T = \{w_1, w_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}.$$

Find the matrix of L with respect to the bases S and T.
[solution:]

$$A = \left[[L(v_1)]_T \quad [L(v_2)]_T \quad [L(v_3)]_T \right]$$

Thus,

$$L(v_1) = L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1+1\\2\cdot 0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\2\end{bmatrix} = 2w_1 + 0w_2$$

$$\Rightarrow [L(v_1)]_T = \begin{bmatrix}2\\0\end{bmatrix}.$$

$$L(v_2) = L\left(\begin{bmatrix}0\\1\\4\end{bmatrix}\right) = \begin{bmatrix}0+1\\2\cdot 4\end{bmatrix} = \begin{bmatrix}1\\8\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\2\end{bmatrix} = 1w_1 + 4w_2$$

$$\Rightarrow [L(v_2)]_T = \begin{bmatrix}1\\4\end{bmatrix}$$

$$L(v_3) = L\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}1+2\\2\cdot 3\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\2\end{bmatrix} = 3w_1 + 3w_2$$

$$\Rightarrow [L(v_3)]_T = \begin{bmatrix}3\\3\end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix}2 & 1 & 3\\0 & 4 & 3\end{bmatrix}.$$

General Procedure for Computing A:

Let $L : R^n \to R^m$ be a linear transformation. Let

$$S = \{v_1, v_2, \ldots, v_n\} \text{ and } T = \{w_1, w_2, \ldots, w_m\}$$

be bases for R^n and R^m, respectively. Then, the matrix of L with respect to the bases S and T can be obtained via the following steps:

1. Form the $m \times (n+m)$ augmented matrix

$$\left[w_1 \quad w_2 \quad \cdots \quad w_m \mid L(v_1) \quad L(v_2) \quad \cdots \quad L(v_n) \right].$$

2. Transform the augmented matrix into the reduced row echelon matrix,

$$\left[I_{n\times n} \mid A \right].$$

The matrix A is the matrix of L with respect to the bases S and T.

Inverse of Matrix

If A is a non singular matrix, then inverse of matrix A exist and is defined as matrix A^{-1} satisfies $AA^{-1}=A^{-1}A=I$, where I is unit matrix of same order as that of the matrix A. To find the inverse of matrix A write $A=IA$, then perform same suitable elementary row (column) operations on the matrix A and on the right hand side till we reach the result $I=BA$. Then $A^{-1}=B$.

Example 1: Find the inverse of matrix $A=\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 1\\ 5 & 2 & 3\end{bmatrix}$ using the elementary operations.

Solution. We write $A=IA$ i.e., $\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 1\\ 5 & 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$

Operating $R_3 \to R_3+(-5)R_1, R_2 \to R_2 \times \frac{1}{4}$

we get, $\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & \frac{1}{4}\\ 0 & -13 & -7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} & 0\\ -5 & 0 & 1\end{bmatrix}A$

Operating $R_1 \to R_1+(-3)R_2, R_3 \to R_3+13R_2$,

$$\begin{bmatrix}1 & 0 & \frac{5}{4}\\ 0 & 1 & \frac{1}{4}\\ 0 & 0 & -\frac{15}{4}\end{bmatrix}=\begin{bmatrix}1 & -\frac{3}{4} & 0\\ 0 & \frac{1}{4} & 0\\ -5 & \frac{13}{4} & 1\end{bmatrix}A$$

Operating $R_3 \to R_3 \times \left(\frac{-4}{15}\right), R_1 \to R_1+\left(\frac{-5}{4}\right)R_3, R_2 \to R_2+\left(\frac{-1}{4}\right)R_3$,

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{7}{15} & \frac{1}{15}\\ \frac{4}{3} & -\frac{13}{15} & -\frac{4}{15}\end{bmatrix}A=\frac{1}{15}\begin{bmatrix}-10 & 5 & 5\\ -5 & 7 & 1\\ 20 & -13 & -4\end{bmatrix}A$$

$$A^{-1}=\frac{1}{15}\begin{bmatrix}-10 & 5 & 5\\ -5 & 7 & 1\\ 20 & -13 & -4\end{bmatrix}.$$

Problems to Check The Progrress

1. Using elementary operation, find the inverse of the following matrices.

$A=\begin{pmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{pmatrix}$ Ans. $A^{-1}=\frac{1}{14}\begin{pmatrix}3 & -1 & 5\\ 5 & 3 & -1\\ -1 & 5 & 3\end{pmatrix}$.

Characterizations of Invertible Matrices

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A, they are either all true or all false).

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. A is expressible as a product of elementary matrices.

f. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

g. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

h. There is an $n \times n$ matrix C such that $CA = I_n$.

i. There is an $n \times n$ matrix D such that $AD = I_n$.

j. A^T is an invertible matrix.

EXAMPLE: Use the Invertible Matrix Theorem to determine if A is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix} \sim \cdots \sim \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 16 \end{bmatrix}}_{\text{3 pivots positions}}$$

Circle correct conclusion: Matrix A *is* / *is not* invertible.

Theorem

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

If not, find all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is ______ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make -$2b_1 + b_3$ nonzero.

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

Subspaces of $\mathbb{R}^n$ and Their Dimensions

Vector Space $\mathbb{R}^n$

Definition 1.1. The **vector space** $\mathbb{R}^n$ is a set of all n-tuples (called vectors)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $x_1, x_2, \cdots, x_n$ are real numbers , together with two binary operations, vector addition and scalar multiplication defined as follows:

(1) **Vector addition**: To every $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

(2) **Scalar multiplication**: To every number k and vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$$k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}$$

Ex. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{y} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$. Find $2x + y$.

Properties

(1) **Vector addition**: For all vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$

(i) vector addition is associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$;

(ii) vector addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$;

(iii) there exists an element (additive identity or origin)

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

such that $\mathbf{x} + 0 = \mathbf{x}$ for every vector $\mathbf{x}$;

(2) **Scalar multiplication**: To numbers a, b and vector $\mathbf{x}$

(i) scalar multiplication is associative: $a(b\mathbf{x}) = (ab)\mathbf{x}$;

(ii) $1\mathbf{x} = \mathbf{x}$ for every vector $\mathbf{x}$.

(3) **Scalar multiplication distributes over vector addition:** $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

(4) **Scalar multiplication distributes over addition of scalars:** $(a+b)\mathbf{x} = a\mathbf{x} + a\mathbf{y}$.

Subspaces of $\mathbb{R}^n$

Definition 1.2. Subspaces of $\mathbb{R}^n$ A subset W of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it has the following properties:

a. W contains the zero vector in $\mathbb{R}^n$.

b. W is closed under addition: if $\mathbf{w}_1$, $\mathbf{w}_2$ are both in W, then so is $\mathbf{w}_1 + \mathbf{w}_2$.

c. W is closed under scalar multiplication: If $\mathbf{w}$ is in W and k is an arbitrary scalar, then $k\mathbf{w}$ is in W.

2 Null and Column spaces of Matrices

2.1 Homogeneous system

Consider the following homogeneous linear system of m equations and n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Or

$$A\mathbf{x} = 0,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is the coefficient matrix and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Then $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution. Moreover, it $\mathbf{x}$ and $\mathbf{y}$ are two solutions of the system, so are

1. $\mathbf{x}+\mathbf{y}$
2. $k\mathbf{x}$,

where k is any number. Therefore,

$$a\mathbf{x}+b\mathbf{y}$$

are also solutions to the system, where a, b are numbers. $a\mathbf{x}+b\mathbf{y}$ is called a linear combination of $\mathbf{x}$ and $\mathbf{y}$.

The set of all solutions to the linear system $A\mathbf{x}=0$, is called the Null space of matrix A, denoted by $Null(A)$ or $N(A)$. It is a subspace of $\mathbb{R}^n$. It is also called the kernel of A, denoted by $Ker(A)$.

2.2 Inhomogeneous System

For $\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$, the system of linear equation

$$A\mathbf{x}=\mathbf{b}$$

may or may not be compatible. When it is compatible, assume that $\mathbf{x}_0$ is any solution of $A\mathbf{x}=\mathbf{b}$, then any solution $\mathbf{x}$ of $A\mathbf{x}=\mathbf{b}$ can be written $\mathbf{x}=\mathbf{x}_0+\mathbf{y}$, where $\mathbf{y}$ is a solution of the homogeneous system $A\mathbf{y}=0$.

That is, the solution *set* of $A\mathbf{x}=\mathbf{b}$ is $\mathbf{x}_0+N(A)$.

2.3 Span

Definition 2.1. A vector $\mathbf{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there are scalars $c_1, c_n \cdots, c_k$ such that

$$\mathbf{b}=c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_k\mathbf{x}_k.$$

Ex. Note that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ x_2 \end{bmatrix}=x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}+x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Ex. $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}=2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Definition 2.2. The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is a subspace. It is called the span of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, denoted by

$$Span\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}.$$

2.4 Column space of matrices

Let A be am $m \times n$ matrix and $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be the column vectors of matrix A. The span of column vectors:

$$span\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$$

is called the column space of matrix A. It is also called the range of A, denoted by $R(A)$.

Definition 14. *Let A be am $m \times n$ matrix. Then the matrix A or any matrix obtained by deleting some rows or columns of A is called sub-matrix of A.*

Definition 15. *Let A be am $m \times n$ matrix given by*

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

If we retain any t rows and t columns of A and deleting $m-t$ rows and $n-t$ columns, we obtain a $t \times t$ square sub-matrix of A. The determinant of this square sub-matrix of order t is called a minor of A of order t.

Definition 16. *The number r is called the rank of the matrix A if it satisfies the following properties:*

1. *There is at least one non-zero minor of order r.*
2. *Every minor of order $r+1$ is zero.*

Problem 1: Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

Solution: Since $|A| = 0$, $\rho(A) = 1$

Problem 2: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$

Solution: Since there is no minors of order 4 and 3, and hence $\rho(A) < 3$. Now A has a minor $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \neq 0$, and since its order is 2, $\rho(A) = 2$.

Problem 3: Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 6 & 8 & 2 \\ 1 & 2 & 5 & 9 \\ 1 & 2 & 5 & 9 \end{bmatrix}$

Solution: Left as exercise.

Problem 4: Prove that the rank of matrix every element of which is unity is 1.

Solution; Since all elements are 1, square matrix of every order will have determinant 0, except the square matrix [1] of order 1.

Problem 5: Show that no skew-symmetric matrix can be of rank 1.

Solution: Let A be a skew-symmetric matrix. If A is zero matrix, then $\rho(A) = 0 \neq 1$. If A is nonzero matrix, then there exists at least one minor of the form $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2 \neq 0$. Hence rank of A is not 1.

CHARACTERISTICS MATRIX

If A be a square matrix of order n, then we can form the matrix $[A - \lambda I]$, where I is the unit matrix of order n and λ is scalar. The determinant corresponding to this matrix equated to zero is called the characteristic equation i.e. if $A - \lambda I$ be the matrix then

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \dots(1)$$

is the characteristic equation of A.

On expanding the determinant (1), the characteristic equation may be written as

$(-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-1} \lambda + a_n = 0$

which is n^{th} degree equation in λ.

The roots of (1) are called eigen values or characteristic roots or latent roots of the matrix A.

Eigen Vectors

We take the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$

and if $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$ where $x_1, x_2, \dots, x_n$ are vectors

then the linear transformation $Y = AX$ …(2), transforms the column vector X into the column vector Y. Generally, it is required to find such vectors which either transform it is into them selves or to a scalar multiple of them selves. If X be such a vector which is transformed into λX using the transformation (2) then $\lambda X = AX$

$\Rightarrow AX - \lambda X = 0$

i.e. $[A - \lambda I]X = 0$ …(3)

The matrix equation (3) represents n homogeneous linear equations.

$$\begin{aligned} &(a_{11} - \lambda)x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n = 0 \\ &a_{21} x_1 + (a_{22} - \lambda)x_2 + a_{23} + x_3 + \dots + a_{2n} x_n = 0 \\ &a_{31}x_1 + a_{32} x_2 + (a_{33} - \lambda)x_3 + \dots + a_{3n} x_n = 0 \qquad \dots(4) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1} x_1 + a_{n2} x_2 + (a_{n3} - \lambda)x_3 + \dots + a_{nn}{-}\lambda\, x_n = 0 \end{aligned}$$

This equation (4) will have a non-trivial solution only if to co-efficient matrix is singular i.e. if the determinant $|A - \lambda I| = 0$.

This equation is also called characteristic equation of the transformation and is also the same as the characteristic equation (1) of matrix A. This characteristic equation has n roots which are eigen values of A corresponding to each root of (1), the equation (3) has non-zero solution.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}$$

which is known as an eigen vector or latent vector. So, if X is a solution of (3) then KX is also a solution, where K is an arbitrary constant. So, we see that the eigen vector corresponding to an eigen value is not unique.

Example 1. Find the eigen values and eigen vectors of the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Sol. The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

i.e. $(1 - \lambda)(4 - \lambda) - 4 = 0 \quad \Rightarrow \quad \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda(\lambda - 5) = 0$

i.e. $\lambda = 0, 5 \quad \therefore \quad$ eigen values of A are 0 and 5.

So, corresponding to $\lambda = 0$ eigen vectors are given by $\begin{vmatrix} 1-0 & 2 \\ 2 & 4-0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

i.e. $x_1 + 2x_2 = 0$ and $2x_1 + 4x_2 = 0$

i.e. single equation $x_1 + 2x_2 = 0 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1}$ so for $\lambda = 0$ eigen vectors are $(2, -1)$ and for $\lambda = 5$, we have

$\begin{vmatrix} 1-5 & 2 \\ 2 & 4-5 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$ and $2x_1 - x_2 = 0$.

i.e. eigen vectors are $\frac{x_1}{1} = \frac{x_2}{2}$ i.e. (1, 2) are eigen vectors corresponding to λ=5.

Properties of Eigen Values

(I) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal. We will prove this property for a matrix of order 3 and the method can be extended for the matrices of any finite order.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ …(1)

Then characteristic matrix $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(\quad) + \dots = 0$ …(2)

If λ_1, λ_2 and λ_3 be eigen values of A then

$|A - \lambda I| = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \dots + (-1)^3 \lambda_1 \lambda_2 \lambda_3$ …(3)

Equating the co-efficients of λ^2 from (2) and (3), we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ which is the required result.

(II) The product of the eigen values of a matrix A is equal to its determinants. If take $\lambda = 0$ in (3) then, we get $|A - 0| = -\lambda_1\lambda_2\lambda_3$ which is the required result.

(III) If λ is an eigen values of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of inverse matrix A^{-1}. If X be the eigen vector corresponding to the eigen value λ then

$AX = \lambda X$ …(4)

Pre-multiplying (4) by A^{-1}, we get $A^{-1}AX = A^{-1}\lambda X$

i.e. $IX = \lambda A^{-1}X \Rightarrow X = \lambda(A^{-1}X) \Rightarrow A^{-1}X = \frac{1}{\lambda}X$

This is of the same form as that in (1) from which we get that $\frac{1}{\lambda}$ is an eigen value of the inverse matrix A^{-1}.

(IV) If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}. As A is an orthogonal matrix so A^{-1} will be same as the transpose of matrix A i.e. $A' = A^{-1}$. So, $\frac{1}{\lambda}$ is an eigen value of A'. But the matrix A and A' have the same eigen values.

[since we know that $|A - \lambda I| = |A' - \lambda I|$]. Hence $\frac{1}{\lambda}$ is also an eigen value of A.

(V) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of a matrix A then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ where m is a positive ineteger.

If A_i be the eigen value of A and X_i be the corresponding eigen vector, then

$$AX_i = \lambda_i X_i \quad \ldots(1)$$

Consider $A^2 X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i (AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i$ similarly, we proceed and find $A^3 X_i = \lambda_i^3 X_i$ and so on such that in general we get

$$A^m X_i = \lambda_i^m X_i \quad \ldots(2)$$

which has the same form as (1). Hence λ_i^m is an eigen-value of A^m and the corresponding eigen vector is the same as that of X_i.

Example 2. Find the characteristic roots and characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Sol. The characteristic equation of matrix A is $| A - \lambda I | = 0$ i.e.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $(8 - \lambda) [(7 - \lambda) (3 - \lambda) - 16] + 6[(-6) (3 - \lambda) + 8] + 2[24 - 2(7 - \lambda)] = 0$

i.e. $(8 - \lambda) [21 + \lambda^2 - 10\lambda - 16] + 6[-10 + 6\lambda] + 2[24 - 14 + 2\lambda] = 0$

i.e. $-\lambda^3 + 18\lambda^2 - 85\lambda + 40 - 60 + 36\lambda + 20 + 4\lambda = 0$

i.e. $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ i.e. $\lambda = 0, 3, 15$.

$\therefore$ Corresponding to $\lambda = 0$, eigen vectors are given by

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. equations are

$$8x_1 - 6x_2 + 2x_3 = 0 \quad \ldots(1)$$
$$-6x_1 + 7x_2 - 4x_3 = 0 \quad \ldots(2)$$
$$2x_1 - 4x_2 + 3x_3 = 0 \quad \ldots(3)$$

From (2) and (3) we get

$$\frac{x_1}{21-16} = \frac{x_2}{-8+18} = \frac{x_3}{24-14} \quad \text{i.e.} \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

i.e. eigen vector are (1, 2, 2)

Similarly from (1) and (2) we get the same vectors

Now for λ = 3, eigen vectors are obtained from $\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e. $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e. equations are

$5x_1 - 6x_2 + 2x_3 = 0$ …(4)

$-6x_1 + 4x_2 - 4x_3 = 0$ …(5)

and $2x_1 - 4x_2 = 0$ …(6)

From (4) and (5), we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

i.e. $\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

i.e. eigen vectors are (2, 1, –2) and for λ = 15, eigen vectors are given by

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. equation are $-7x_1 - 6x_2 + 2x_3 = 0$ …(7)

$6x_1 + 8x_2 + 4x_3 = 0$ …(8)

and $2x_1 - 4x_2 + 2x_3 = 0$ …(9)

From (7) and (8), we get

$$\frac{x_1}{12+8} = \frac{x_2}{-6-14} = \frac{x_3}{28-18} \quad \text{i.e.} \quad \frac{x_1}{20} = \frac{x_2}{-20} = \frac{x_3}{10}$$

i.e. eigen vectors are (2, –2, 1) corresponding to λ = 15.

Example 3. Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Sol. Let the given matrix be A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

So, the characteristic equation of A is |A – λI| = 0

i.e. $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$ …(1)

$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$\Rightarrow (6-\lambda)[9 - 6\lambda + \lambda^2 - 1] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

$\Rightarrow -\lambda^3 + \lambda^2[6 + 6] - \lambda[36 - 8 + 8] + [48 - 8 - 8] = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 16\lambda - 32 = 0$

$\Rightarrow (\lambda - 2)^2(\lambda - 8) = 0$ i.e. λ = 2, 2 and 8.

which are the characteristic roots of (1).

Now corresponding to the eigen values λ = 2, 2, 8 the given eigen vectors are obtained from [A – λI]X = 0.

i.e. $$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(2)$$

(2) may be written as

$(6 - \lambda)x_1 - 2x_2 + 2x_3 = 0,$ …(A)

$-2x_1 + (3 - \lambda)x_2 - x_3 = 0,$ …(B)

and $2x_1 - x_2 + (3 - \lambda)x_3$ …(C)

we now, consider different cases.

Case I. When $\lambda = 2$, then (A), (B) and (C) may be written as

$4x_1 - 2x_2 + 2x_3 = 0$ …(A_1)

$-2x_1 + x_2 + x_3 = 0$ …(B_1)

$2x_1 - x_2 + x_3 = 0$ …(C_1)

If $x_3 = 0$, then from (A_1) and (B_1), we get

$-2x_1 + x_2 = 0$ i.e. $\frac{x_1}{1} = \frac{x_2}{2}$

and so eigen vector for $\lambda = 2$, for $x_3 = 0$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

and when $x_2 = 0$, then from (A_1) and (B_1) for $\lambda = 2$,

$2x_1 + x_3 = 0 \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_3}{-2}$

$\therefore$ another eigen vector for $\lambda = 2$ is $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Case II. When $\lambda = 8$, equations (A), (B) and (C) become

$-2x_1 - 2x_2 + 2x_3 = 0$ …(A_{11})

$-2x_1 - 5x_2 - x_3 = 0$ …(B_{11})

$2x_1 - x_2 - 5x_3 = 0$ …(C_{11})

eliminating x_3 from (A_{11}) and (B_{11}), we get

$x_1 + 2x_2 = 0$ i.e. $\frac{x_1}{2} = \frac{x_2}{-1}$ …(M)

and by eliminating x_1 from (A_{11}) and (B_{11}), we get

$x_2 + x_3 = 0$ i.e. $\frac{x_2}{-1} = \frac{x_3}{1}$ …(N)

Using (M) and (N), we get $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$

i.e. corresponding to $\lambda = 8$, eigen vector is $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Example 1. Find the eigen values and eigen vectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Sol. The characteristic equation of the given matrix is

$$|A - \lambda I| = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

i.e. $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$

i.e. eigen values are $\lambda = -3, -3, 5$

∴ If x, y and z be the eigen vectors. Corresponding to the eigen values λ

(I) We have $\begin{bmatrix} -1-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 0$ …(1)

Now for $\lambda = 5$ we have

$-7x + 2y - 3z = 0 \qquad 2x - 4y - 6z = 0$

$-x - 2y - 5z = 0$

from (1) and (2) $\dfrac{x}{-12-12} = \dfrac{y}{-6-42} = \dfrac{z}{28-4}$

Hence eigen vector is [1, 2, –1] $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}$

(II) If $\lambda = -3$, then from (1), we get $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ which gives only one independent $x + 2y - 3z = 0$ …(3)

if we take y = 0, we get $x - 3z = 0 \Rightarrow \dfrac{x}{3} = \dfrac{y}{0} = \dfrac{z}{1}$

∴ for $\lambda = -3$, eigen vector is (3, 0, 1) when y = 0.

at when z = 0, (3) gives $x + 2y = 0 \Rightarrow \dfrac{x}{2} = \dfrac{y}{-1} = \dfrac{z}{0}$

i.e. eigen vector in this case is (2, –1, 0)

∴ the eigen vectors obtained are (1, 2, –1), (3, 0, 1) and (2, –1, 0)

which are the required result.

Example 2. Find the sum and the product of eigen values of A = $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

Sol. The characteristic equation of matrix A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 3 & -2 \\ -2 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

i.e. $(2 - \lambda)(1 - \lambda)(2 - \lambda) + 3[1 + 2(2 - \lambda)] + (2)(0 - \overline{1-\lambda}) = 0$

$\Rightarrow \quad (2 - \lambda)(\lambda^2 - 3\lambda + 2 + 3) - 6\lambda + 15 + 2 - 2\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 11\lambda + 10 - 6\lambda + 15 + 2 - 2\lambda = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 19\lambda + 19 = 0$

∴ sum of the eigen value $\lambda_1 + \lambda_2 + \lambda_3 = -(-5) = 5$

and the product of the eigen values is $\lambda_1 \lambda_2 \lambda_3 = -19$.

1. Determine the charecteristics roots and the corresponding characteristics vectors of the matrix $A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Ans. Characteristics roots are 0, 3, 15.

Example 3 Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$***. Show that the characteristic equation is satisfied by A and hence obtain the inverse of the given matrix.***

Sol. The characteristic equation is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$ …(1)

we have to show that A satisfies (1) i.e. $A^3 - 4A^2 - 20A - 35I = 0$ …(2)

Consider

$$A^2 = A.A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now, we consider $A^3 - 4A^2 - 20A - 35I$, which is

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60 & 232-92-140 \\ 140-60-80 & 163-88-40-35 & 208-148-60 \\ 60-40-40 & 76-36-20 & -56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Equation (2) is satisfied and $A^{-1} = \frac{1}{35}[A^2 - 4A - 20I]$

i.e. $$A^{-1} = \frac{1}{35}\left\{\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right\}$$

$$= \frac{1}{35}\begin{bmatrix} 20-4-20 & 23-12 & 23-28 \\ 15-16 & 23-8-20 & 37-12 \\ 10-14 & 9-8 & 14-4-20 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

i.e. $$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$ is the required result.

PART – B (5 x 2 =10)

Possible Questions (2 marks)

1. Define linear transformation with example.

2. Define null space.

3. Define rank of a matrix

4. Define inverse of a matrix with example.

5. Define the subspace .

6. Define symmetric matrix with example.

7. Define self adjoint with example.

8. Define characteristic equation of a matrix.

9. Define the Eigen value and Eigen vector of a matrix.

10. Write any two properties of Eigen values.

PART – C (5 x 6 =30)

Possible Questions (6 marks)

1) Find the characteristic vectors corresponding to each characteristic root if A= $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

2) Find the inverse of the matrix A= $\begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

3) Find the eigen values and eigen vectors of the matrix A= $\begin{pmatrix} 5 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

4) Let A= $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and define T: $\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).Find the images under T of u= $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and v= $\begin{bmatrix} a \\ b \end{bmatrix}$.

5) Defined T: $\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).find a vector x whose image under T is b.

If A= $\begin{pmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{pmatrix}$, b= $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$.

6) Compute the inverse of the matrix A= $\begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$

7) Let A= $\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$, u= $\begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix}$ and v= $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ Define T: $\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).Find T(u) and T(v).

8) Show that a square matrix A is orthogonal iff $A^{-1} = A^T$.

9) Let A= $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and define T: $\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).Find the images under T of u= $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and v= $\begin{bmatrix} a \\ b \end{bmatrix}$.

10) Find the rank of the matrix A= $\begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{pmatrix}$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.

Subject: Algebra | **Subject Code: 17MMU102**

Class : I - B.Sc. Mathematics | **Semester : I**

Unit V
Linear Transformation and Matrices

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

S.No	Question	Choice 1	Choice 2	Choice 3	Choice 4		Answer
1	Let V& W be vector spaces over the field F. A linear transformation from V into W is a function T from V into W such that T(cu+v)=………………. for all u,v in V and all scalars c in F.	T(u)+T(v)	cT(u)+cT(v)	T(u)+cT(v)	cT(u)+T(v)		cT(u)+T(v)
2	Every …………….. transformation is a linear transformation.	matrix	row	column	unit		matrix
3	Every matrix transformation is a………….transformation.	linear	non linear	homogeneous	non homogeneous		linear
4	……………... transformation preserve the operations of vector addition and scalar multiplication.	linear	non linear	matrix	row		linear
5	Linear transformation preserve the ………….. of vector addition and scalar multiplication.	addition	functions	operations	values		operations
6	Linear transformation preserve the operations of ……………….. and scalar multiplication.	vector addition	vector subtraction	vector multiplication	vector division		vector addition
7	Linear transformation preserve the operations of vector addition and ……………..	vector multiplication	scalar multiplication	matrix multiplication	vector division		scalar multiplication

8	If T is a linear transformation , then T(0)=…………	0	1	2	3		0
9	T(cu+dv)=……………..	T(cu)+T(dv)	cT(u)-dT(v)	T(u)+T(v)	cT(u)+dT(v)		cT(u)+dT(v)
10	Let T be a linear transformation then there exists a unique matrix A such that T(x)=………….. for all x in R	0	Ax	x	1		Ax
11	Let T be a linear transformation then there exists a ……………... matrix A such that T(x)=Ax for all x in R	zero	unique	identity	diagonal		unique
12	An nxn matrix B such that BA=I is called a ………………..of A	zero	left inverse	right inverse	identity		left inverse
13	An ……………..matrix B such that BA=I is called a left inverse of A	mxm	nxn	mxn	nxm		nxn
14	An nxn matrix B such that AB=I is called a ………………..of A	zero	left inverse	right inverse	identity		right inverse
15	An ……………..matrix B such that AB=I is called a right inverse of A	mxm	nxn	mxn	nxm		nxn
16	If AB=BA=I then B is called a …………...inverse of A.	two sided	left inverse	right inverse	identity		two sided
17	If AB=BA=………….. then B is called a two sided inverse of A.	0	1	I	-1		I
18	A two sided inverse of Aand Ais said to be ………………..	invertible	inverse	identity	vertible		invertible
19	If A is invertible,so is A^{-1} and $(A^{-1})^{-1}$=…………..	A^{-1}	A	0	I		A

20	If A is ……………….,so is A^{-1} and $(A^{-1})^{-1}=A$	invertible	inverse	identity	vertible		invertible
21	If both A and B are invertible ,so is AB,and $(AB)^{-1}$=……………	B^{-1}	A^{-1}	BA	$B^{-1}A^{-1}$		$B^{-1}A^{-1}$
22	If both A and B are ………….. ,so is AB,and $(AB)^{-1}=B^{-1}A^{-1}$	invertible	inverse	identity	vertible		invertible
23	A ……………….. of invertible matrices is invertible	addition	subtraction	product	division		product
24	A product of invertible ……………. is invertible	matrices	functions	vectors	equations		matrices
25	A product of invertible matrices is ……………..	invertible	unity	identity	vertible		invertible
26	An ……………….matrix is invertible.	null	identity	elementary	singular		elementary
27	An elementary matrix is………………..	invertible	inverse	identity	vertible		invertible
28	A ……………... of V is a subset W of V which is itself a vectorspace over F with the operations of vector addition and scalar multiplication on V.	subspace	space	vector	function		subspace
29	A subspace of V is a subset W of V which is itself a vectorspace over F with the ……………….. of vector addition and scalar multiplication on V.	functions	operations	scalar	vector		operations
30	A subspace of V is a subset W of V which is itself a vectorspace over F with the operations of …………and scalar multiplication on V.	vector addition	vector subtraction	vector multiplication	vector division		vector addition
31	A subspace of V is a subset W of V which is itself a vectorspace over F with the operations of vector addition and …………….. on V.	vector multiplication	scalar multiplication	matrix multiplication	vector division		scalar multiplication

32	The ………………consisting of the zero vector alone is a subspace of V, called zero subspace of V.	subset	set	space	subspace		subset
33	The subset consisting of the……………. vector alone is a subspace of V, called zero subspace of V.	zero	unit	finite	infinite		zero
34	The subset consisting of the zero vector alone is a subspace of V, called ………………….. of V.	zero subspace	zero space	zero subset	zero set		zero subspace
35	An ………….. matrix A over the field F is symmetric if Aij=Aji for each i and j.	mxm	nxn	mxn	nxm		nxn
36	An nxn matrix A over the ……………. F is symmetric if Aij=Aji for each i and j.	field	scalar	vector	matrix		field
37	An nxn matrix A over the field F is ………..if Aij=Aji for each i and j.	symmetric	non symmetric	singular	non singular		symmetric
38	An nxn matrix A over the field F is symmetric if ……………. for each i and j.	Aij<Aji	Aij>Aji	Aij=Aji	Aij≠Aji		Aij=Aji
39	Any set which contains a lineary dependent set is ……………....	linearly dependent	linearly independent	linear	non linear		linearly dependent
40	Any subset of a lineary independent set is ……………....	linearly dependent	linearly independent	linear	non linear		linearly independent
41	Any set which contains the ……….vector is linearly dependent.	0	unit	inverse	complex		0
42	Any set which contains the 0 vector is…………….	linearly dependent	linearly independent	linear	non linear		linearly dependent
43	A set S of vectors is ………………. iff each finite subset of S is linearly independent.	linearly dependent	linearly independent	linear	non linear		linearly independent
44	A set S of vectors is linearly independent iff each …………. subset of S is linearly independent.	one	finite	infinite	null		finite

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Reg. No --------------

(17MMU102)

KARPAGAM ACDEMY OF HIGHER EDUCATION

COIMBATORE - 21

Department of Mathematics

First Semester

First Internal Test – July - 2017

Algebra

Date : 27.07.2017 (AN) **Time: 2 Hours**

Class : I - B.Sc. Mathematics **Maximum: 50 Marks**

PART-A (20 x 1 =20 Marks)

Answer All the Questions

1. A complex number z=x+iy is write in the polar representation as…...

a) z=r(cosΘ+isinΘ b) z=r(cosΘ) c) z=(cosΘ+isinΘ) d) z=r(isinΘ)

2. The polar representation z=r(cosΘ+isinΘ) where r∈…………….

a) [0,∞] b) [0,1) c) [1,∞) d) [0,∞)

3. The modulus of the numbers z= 2i is………………..

a) 0 b) 1 c) 2 d) 3

4. Two complex numbers z1 and z2≠0 are …… if and only if r1=r2

a) one b) equal c) not equal d) multiple

5. Any complex number z can be represented as z = r (cosΘ+isinΘ), where r………………

a) ≥0 b) ≤0 c) >0 d) <0

6. The set Arg z is called the ……argument of the complex number z.

a) finite b) infinite c) extended d) singular

7. For ………the modulus and argument of z are uniquely determined

a) z=0 b) z>0 c) z<0 d) z≠0

8. The polar representation z=r(cosΘ+isinΘ) where Θ∈…………….

a) [0,Π] b) (0,2Π) c) [0,2Π) d) [0,2Π]

9. The modulus of the numbers z= 1-i√3 is………………..

a)-1 b)1 c)2 d)-2

10. cos0 +i sin 0=……………

a)0 b)1 c)2 d)3

11. In the field of real numbers Z^n-z_0=…………..

a)0 b)1 c)2 d)3

12. The argument of the numbers z= -1-i is………………..

a) Π/4 b)3Π/4 c) 5Π/4 d) 7Π/4

13. cos Π + i sin Π =……………

a) 0 b)1 c) -1 d) i

14. The polar argument Θ of the geometric image of z is called ………of z.

a) angle b) argument c) theta d) coordinate

15. In the field of real numbers Z^n-z_0=0 is used for defining the ………..roots of number z_0.

a) $(n-1)^{th}$ b) $(n+1)^{th}$ c) n^{th} d) $(n-2)^{th}$

16. If f:A→B hence f is called a ……………..

a) function b) form c) formula d) fuzzy

17. The ………….. of a function as the image of its domain

a) domain b) range c) co domain d) image

18. If the function f is otherwise called as ……………………

a) limit b) mapping c) lopping d) inverse

19. If f:A→B in this set B is called the …………of the function f.

a) domain b) co domain c) set d) element

20. If R is reflexive,symmetric and transitive therefore R is an ……………..relation

a) one-one b) onto c) equivalence d) equal

PART-B (3 x 2 = 6 Marks)

Answer All the Questions

21. Find the polar representations for the complex number z=3-2i.

22. Define finite and infinite set.

23. Define Equivalence relations.

PART-B (3 x 8 = 24 Marks)

Answer All the Questions

24. a) Find the Polar representation of the complex number

z=1+cos a +i sin a ,aϵ(0,2Π).

(OR)

b) Compute $z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$

25. a) State and Prove De Moivre's theorem.

(OR)

b) Find$|z|$, arg z, Arg z, arg $\bar{z}$, arg (-z) for z=(7-7$\sqrt{3}$ i)(-1-i).

26. a) Find the Fourth roots for the complex number z=-i

(OR)

b) Let S={1,2,3,4,5} and T={1,2,3,8,9} and define the functions f : S→ T and g: S → S by f={(1,8), (3,9),(4,3),(2,1),(5,2)} and g={(1,2),(3,1),(2,2),(4,3),(5,2)} ,then find the values of the following $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$.

Reg. No --------------

(17MMU102)

KARPAGAM ACADEMY OF HIGHER EDUCTAION
COIMBATORE - 21
Department of Mathematics
First Semester
First Internal Test – July - 2017
Algebra

Date : 19.08.2017 (AN) **Time: 2 Hours**
Class : I - B.Sc. Mathematics **Maximum: 50 Marks**

PART-A (20 x 1 =20 Marks)

Answer All the Questions

1. The value of the function f for a and is denoted by ……………
a) a(f) b) f(a) c) a d) f
2. If a϶A then the element in B which is assignedto ais called the ……...of a
a) B-image b) a-image c) A-image d) f-image
3. One-to-one mapping is also sometimes known as…………….
a) injection b) bijection c) surjection d) imjection
4. In one-one mappings an element in B has only……preimage in A
a) zero b) two c) one d) three
5. If f:A→B in this set B is called the ……………..of the function f.
a) domain b) co domain c) set d) element
6. The element a may be referred to as the ………..of f(a)
a) f-image b) pre-image c) domain d) codomain
7. A mapping f:A→B is said to be ….. if different elements in A have different f-images in B
a) zero b) one-one c) onto d) into
8. In many-one mappings some elements in B has ……. one preimage in A
a) equal b) more than c) less than d) only
9. Let f(x),g(x)≠0 be any two polynomials of the polynmial domain F[x],over the field F. Then there exist uniquely two polynomials q(x) & r(x) in F[x] such that ………………..
a) f(x)=q(x)g(x)+r(x) b) f(x)=q(x)+r(x)
c) f(x)=q(x)g(x) d) f(x)=g(x)+r(x)
10. Division algorithm for polynomials over a field deg r(x) …….deg g(x)
a) < b) > c) = d) ≠
11. A polynomial domain F[x] over a field F is a principal………
a) commutative ring b) ideal ring
c) associative ring d) division ring
12. In a Euclidean algorithm ,Let F be a field and f(x) and g(x) be any two polynomials in F[x], not both of which are ………………….
a) zero b) one c) two d) three
13. In the division algorithm, the polynomial q(x) is called the …….on dividing f(x) by g(x)
a) quotient b) remainder c) divisor d) dividend
14. The divisor of f(x) symbolically write ……………….
a) f(x)/a(x) b) b(x)/f(x) c) a(x)/b(x) d) a(x)/f(x)
15. A ……….is an element of F[x] which has a multiplicative inverse.
a) zero b) unit c) two d) three
16. The non zero elements of F are the ……………..of F[x].
a) only units b) not only units c) double units d) zero units
17. If f(x) and g(x) are polynomials in F[x], then we call f(x) and g(x) associates if ……………….for some 0≠c ϵ F.
a) f(x)=g(x) b) f(x)=c/g(x) c) f(x)=c+g(x) d) f(x)=cg(x)
18. Only one-one and onto mapping posses……………..mappings.
a) integral b) inverse c) invert d) reverse
19. The divisorsof f(x) are called its………………………divisors.
a) proper b) improper c) finite d) infinite
20. An irreducible polynomial is otherwise called as……………..
a) point b) prime c) power d) degree

PART-B (3 x 2 = 6 Marks)

Answer All the Questions

21. Write the various types of Functions.
22. State the Euclidean algorithm.
23. Define the greatest common divisor of two polynomials over a field.

PART-B (3 x 8 = 24 Marks)

Answer All the Questions

24. a) Show that the following functions are 1-1

i) f : R → R given by f(x)= $5x^2$ - 1

ii) f: Z → Egiven by f(n)=$3x^3$ - x

(OR)

b) Let A be the set A={xϵR \ x>0} and define f ,g, h :A→ R by

f(x)=$\frac{x}{x+1}$,g(x)=$\frac{1}{x}$,h(x)=x+1.find g ∘ f , f ∘ g, h ∘ g ∘ f and

f ∘ g ∘ h .

27. a) Prove that the sum of the first n odd integers is n^2.

(OR)

b) State and prove the Division Algorithm

28. a) Define greatest common divisor& Find the greatest common divisor of a and b and express it in the form ma+nb for suitable integers m and n .

i) a=26 ,b=118. ii) a=427 , b=616.

(OR)

b) Solve the following congruence

i) $3x \equiv 1 \ (mod\ 5)$

ii) $3x \equiv 1 \ (mod\ 6)$

Reg. No --------------
(17MMU102)

KARPAGAM ACDEMY OF HIGHER EDUCATION

Coimbatore - 21
Department of Mathematics
First Semester
Third Internal Test – July - 2017
Algebra

Date : 07.10.2017 (AN) **Time: 2 Hours**
Class : I - B.Sc. Mathematics **Maximum: 50 Marks**

PART-A (20 x 1 =20 Marks)

Answer All the Questions

1. Any n-tuple of elements of F which satisfies each of the ………… in linear equation is called a solution of the system.
a) functions b) equations c) roots d) solutions
2. If y1=y2=………..=ym=0 then the system is ………………………………..
a) homogeneous b) non homogeneous c) linear d) nonlinear
3. The most fundamental technique for finding the ……………. of a system of linear equations is the technique of elimination.
a) function b) root c) solution d) value
4. …………. systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in the other system.
a) one b) two c) three d) four
5. ………………….systems of linear equations have exactly the same solutions.
a) linear b) nonlinear c) Equivalent d) homogeneous
6. In the ……………….. form every row R which has all its entries 0 occurs below every row has a nonzero entry.
a) row reduced echelon b) column reduced echelon
c) echelon d) null
7. An ……….. matrix R is called row reduced if the first nonzero entry in each non zero row of R is equal to 1
a) mxm b) nxn c) mxn d) nxm
8. An mxn matrix R is called row reduced if the first nonzero entry in each non zero row of R is equal to …………..
a) 0 b) 1 c) 2 d) 3
9. Equivalent systems of linear equations have exactly the…………….. solutions.
a) zero b) same c) different d) finite
10. An nxn matrix B such that AB=I is called a ……………..of A
a) zero b) left inverse c) right inverse d) identity
11. If A is ………………,so is A^{-1} and $(A^{-1})^{-1}=A$
a) invertible b) inverse c) identity d) vertible
12. A product of invertible matrices is ………………
a) invertible b) unity c) identity d) vertible
13. An ………………..matrix is invertible.
a) null b) identity c) elementary d) singular
14. If AB=BA=I then B is called a …………...inverse of A.
a) two sided b) left inverse c) right inverse d) identity
15. If A is invertible, so is A^{-1} and $(A^{-1})^{-1}$=…………..
a) A^{-1} b) A c) 0 d) i
16. If T is a linear transformation , then T(0)=…………
a) 0 b) 1 c) 2 d) 3
17. Every …………….. transformation is a linear transformation.
a) matrix b) row c) column d) unit
18. Linear transformation preserve the …….... of vector addition and scalar multiplication.
a) addition b) functions c) operations d) values
19. If AB=BA=………….. then B is called a two sided inverse of A.
a) 0 b) 1 c) i d) -1
20. A two sided inverse of A and A is said to be ………………..
a) invertible b) inverse c) identity d) vertible

PART-B (3 x 2 = 6 Marks)

Answer All the Questions

21. Define the systems of Linear equations
22. Define null space.
23. When we say that the system is homogeneous?

PART-B (3 x 8 = 24 Marks)

Answer All the Questions

24. a) Determine if b is a linear combination of a_1 and a_2 where

$a_1=\begin{bmatrix}1\\-2\\3\end{bmatrix}$, $a_2=\begin{bmatrix}5\\-13\\-3\end{bmatrix}$ and $b=\begin{bmatrix}-3\\8\\1\end{bmatrix}$

(OR)

b) Determine the system is consistent

$$x_1-6x_2=5$$
$$x_2-4x_3+x_4=0$$
$$-x_1+6x_2+x_3+5x_4=3$$
$$-x_2+5x_3+4x_4=0$$

25. a) Defined $T:\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).find a vector x whose image under T is b. If $A=\begin{pmatrix}1 & 0 & -3\\-3 & 1 & 6\\2 & -2 & -1\end{pmatrix}$, $b=\begin{pmatrix}-2\\3\\-1\end{pmatrix}$.

(OR)

b) Compute the inverse of the matrix $A=\begin{pmatrix}2 & -1 & 1\\-15 & 6 & -5\\5 & -2 & 2\end{pmatrix}$

26. a) Describe the solution of $AX = B$ where $A = \begin{bmatrix}3 & 5 & 6\\-3 & -2 & 1\\6 & 1 & -8\end{bmatrix}$ and $b = \begin{bmatrix}7\\-1\\-4\end{bmatrix}$

(OR)

b) Let $A=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ and define $T:\mathbb{R}^2 \to \mathbb{R}^2$ by T(x)=A(x).Find the images under T of $u=\begin{bmatrix}1\\-3\end{bmatrix}$ and $v=\begin{bmatrix}a\\b\end{bmatrix}$.

Reg. No................................

[16MMU102]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2016 onwards)

B.Sc., DEGREE EXAMINATION, NOVEMBER 2016

First Semester

MATHEMATICS

ALGEBRA

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 2 = 10 Marks) (2 ½ Hours)
Answer ALL the Questions

21. Write the polar form of a complex number.
22. State the well-ordering principle.
23. State the principle of mathematical induction.
24. Define linear independence.
25. Define rank of a matrix A.

PART C (5 x 6 = 30 Marks)
Answer ALL the Questions

26. a) Convert the complex number $z = 2 - 2\sqrt{3}i$ to polar form.

Or

b) If ω is any nth roots of unity, show that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

27. a) Let S be the set of all integers. Define a relation on S as follows: for any $a, b \in S, a \sim b \Leftrightarrow a - b$ is an even number. Then prove that this relation is an equivalence relation.

Or

b) Prove that a function f has an inverse $\Leftrightarrow$ f is one-to-one and onto.

28. a) Prove that the congruence has an equivalence relation.

Or

b) State Division algorithm, Euclidean algorithm and Fundamental theorem of Arithmetic.

29. a) Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & -3 & 5 & -9 & -7 \end{bmatrix}$$

Or

b) Given $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Find $4u, (-3)v$ and $4u + (-3)v$. Also represent it graphically.

30. a) Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a ono-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

Or

b) Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.
