KARPAGAM UNIVERSITY
Karpagam Academy Of Higher Education
(Deemed University Established Under Section 3 of UGC 1956)
Coimbatore-21
Faculty of Arts , Science and Humanities
Department of Mathematics
UG Program (CBCS)- (2016 – 2019) Batch

Program: B.Sc Mathematics

SEMESTER – III							
16MMU301	Numerical Methods	04	40	60	100	3	4
16MMU302	Ring Theory and Linear Algebra I	08	40	60	100	3	6
16MMU303	Multivariate Calculus	08	40	60	100	3	6
16MMU304A	Logic and Sets	06	40	60	100	3	4
16MMU304B	Programming with C and C++						
16MMU311	Numerical Methods (Practical)	04	40	60	100	3	2
	Semester total	**30**	**200**	**300**	**500**	**-**	**22**

16MMU301	NUMERICAL METHODS	L T P C 4 0 0 4

Scope: This course provides a deep knowledge to the learners to understand the basic concepts of Numerical Methods which utilize computers to solve Engineering Problems that are not easily solved or even impossible to solve by analytical means.

Objectives: To enable the students to study numerical techniques as powerful tool in scientific computing.

UNIT I

High speed computation: Algorithms, Convergence, Errors: Relative, Absolute, Round off, Truncation. Transcendental and Polynomial equations: Bisection method - -
False Position method - Secant method - Rate of convergence of these methods.

UNIT II

System of linear algebraic equations: Gaussian Elimination - Gauss Jordan methods - Gauss Jacobi method - Gauss Seidel method and their convergence analysis LU decomposition - Power method.

UNIT III

Error bounds - Finite difference operators.
Gregory forward and backward difference interpolation Central difference Lagrange and inverse Lagrange interpolation formula.

UNIT IV

Numerical Differentiation and Integration:
differentiation-

UNIT V

- -

Runge-Kutta methods of orders two and four.

SUGGESTED READINGS

TEXT BOOK

1. Jain. M.K., Iyengar. S.R.K.,and Jain R.K., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

REFERNCES

1. Bradie B., (2007).A Friendly Introduction to Numerical Analysis, Pearson Education, India,
2. Gerald C.F., and Wheatley P.O., (2006). Applied Numerical Analysis, Sixth Edition, Dorling Kindersley (India) Pvt. Ltd., New Delhi.

3. Uri M. Ascher and Chen Greif., (2013). A First Course in Numerical Methods, Seventh Edition., PHI Learning Private Limited.

4. John H., Mathews and Kurtis D. Fink., (2012). Numerical Methods using Matlab, Fourth Edition., PHI Learning Private Limited.
5. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

Which may also be written as

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1}) f(x_k)}{(f(x_k) - f(x_{k-1}))}.$$

Example 5: Using Regula falsi method compute the real root of the equation $xe^x = 2$. Correct to four decimal places.

Solution: Here $f(x) = xe^x - 2$ and $f(0) = -2, f(0.5) = -1.175$

$f(0.8) = -0.2196, f(0.9) = 0.2136$ and $f(1.0) = 0.715$

therefore root lies between 0.8 and 0.9

we take $x_0 = 0.8$ and $x_1 = 0.9$

therefore $x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$

$$x_2 = 0.8 + \frac{0.2196(0.9 - 0.8)}{0.2136 + 6.2196} = 0.851$$

$$\therefore \quad f(x_2) = -0.00697$$

thus $x_3 = 0.851 + \frac{0.00697}{0.2136 + 0.00697}(0.9 - 0.851)$

$$x_3 = 0.851 + 0.0015484 = 0.85256$$

Now $f(x_3) = -0.0001977$

Thus $x_4 = 0.85256 + \frac{0.0001977}{0.2136 + 0.0001977}(0.9 - 0.85256)$

$x_4 = 0.85256 + 0.000043868 = 0.8526$

Again, $f(x_4) = -0.0000239$

$x_5 = 0.8526 + \frac{0.0000239}{0.2136 + 0.0000239}(0.9 - 0.8526)$

$x_5 = 0.8526 + 0.0000053$

$= 0.8526$

$\therefore$ Approximate root is 0.8526

Newton-Raphson Method (or Method of Tangents)

Newton's method gives a better approximation of a root as compared to the approximations obtained by bisection method or Regula falsi method. This method consists of replacing the part of the curve between point $\left[x_0, f(x_0)\right]$ and the x-axis by means of the tangent to the curve at the point and is described graphically in the adjoining Figure 4. The intercept

OT on the x-axis, of the tangent to the curve at the point P is taken as the first approximation.

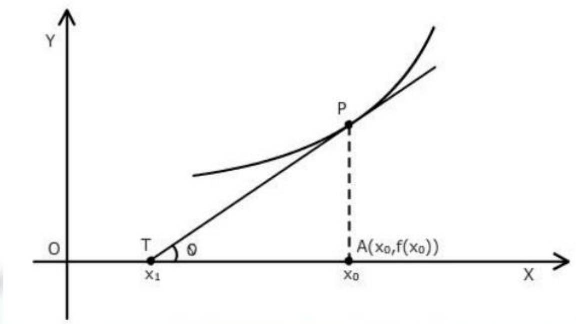

Figure 4

From the given figure 4 we have,

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1}, \quad \text{but} \quad \tan\theta = \frac{dy}{dx} = f'(x_0)$$

This gives, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Repeating the process replacing x_0 by x_1, we get the second approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{and so on.}$$

In general, after $(n+1)$ iterations, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Obviously this method fails if the slope of the tangent to the curve becomes zero.

As an alternative approach

Draw a tangent to the curve at B_0 which meets x-axis at x_1. Then draw a tangent at B_1 which meets x-axis at x_2. Continuing this process, the root ξ is obtained as shown in Fig. 5.

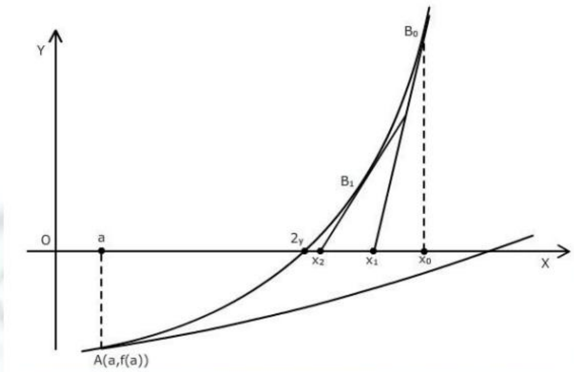

Figure 5

Suppose $\xi = x + h$ where h is a small quantity. Then applying Taylor's formula, we have

$$0 = f(x+h) \approx f(x) + h f'(x)$$

or $$h = \frac{-f(x)}{f'(x)}$$

Thus $\xi = x + h = x - \dfrac{f(x)}{f'(x)}$.

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Example 6: Find the root of the equation

$\cos x - x e^x = 0$ using

(i) Regula falsi Method.

(ii) Newton Raphson Method.

Solution: $f(x)=\cos x - xe^x$

Here $f(0)=1$ and $f(1)=\cos 1 - e = -2.17798$

(i) We take $x_0=0$ and $x_1=1$. By Regula falsi method

$$x_2 = x_1 - \frac{(x_1-x_0)}{(f(x_1)-f(x_0))} f(x_1)$$

$$=1-\frac{(1-0)}{(-2.17798-1)}(-2.17798)$$

= 0.314665

$f(x_2)=0.51987$

$\therefore$ $$x_3 = x_2 - \frac{(x_2-x_1)}{(f(x_2)-f(x_1))} f(x_2)$$

$$=0.314665-\frac{(0.314665-1)}{0.51987}(0.51987)$$

= 0.446728

Continuing the process, we get $x_4=0.491015$, $x_5=0.5099461$, $x_6=0.5152$.

(ii) Let the initial value of the root be $x_0=1$

By Newton-Raphson Method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

or $$x_{n+1}=x_n-\frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - e^{x_n} - x_n e^{x_n}}$$

$$= x_n + \frac{\cos x_n - x_n e^{x_n}}{\sin x_n + (1 + x_n) e^{x_n}}$$

$\therefore \quad x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 + (1 + x_0) e^{x_0}}$

$$= 1 + \frac{\cos 1 - e}{\sin 1 + 2e} = 0.65308$$

$f(x_1) = -0.4606$

$x_2 = x_1 + \frac{\cos x_1 - x_1 e^{x_1}}{\sin x_1 + (1 + x_1) e^{x_1}} = 0.531343$

Continuing the process, we get

$x_3 = 0.51791, \; x_4 = 0.51776$

Example 7: Using Newton-Raphson Method compute $\sqrt{5}$.

Solution: $\sqrt{5}$ will be calculated as the root of the equation $x^2 - 5 = 0$.

So that $f(x) = x^2 - 5$ and $f'(x) = 2x$.

The starting value of the root is obviously 2 hence we take $x_0 = 2$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n}$$

$$= x_n - \frac{1}{2} x_n + \frac{5}{2x_n}$$

$$= \frac{x_n}{2} + \frac{5}{2x_n} = \frac{1}{2}\left(x_n + \frac{5}{x_n} \right)$$

$\therefore$ for $n = 0, \; x_1 = \frac{1}{2}\left(2 + \frac{5}{2} \right) = 2.25$

$$n = 1, x_2 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.236111$$

$$n = 2,\ x_3 = \frac{1}{2}\left(2.236111 + \frac{5}{2.236111}\right) = 2.2360679$$

$$n = 3,\ x_4 = \frac{1}{2}\left(2.2360679 + \frac{5}{2.2360679}\right) = 2.23606797$$

Thus, the value of the root is 2.23606797 correct to nine significant digits.

Rate of Convergence

In numerical analysis, the speed at which a convergent sequence approaches its limit is called the **rate of convergence**. We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

Order of a Convergence

Order of a root

- A root of order m = 1 is called a simple root.
- A root of order m > 1 is called a multiple root.
- A root of order m = 2 is sometimes called as double root and so on.

Rate of Convergence of Secant Method

We assume that ξ is a simple root of $f(x)=0$.

Let us define the error ϵ_k as

$$\epsilon_k = x_k - \xi$$

The secant method reads,

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})f(x_k)}{f(x_k) - f(x_{k-1})} \qquad (3)$$

To figure out the convergence order, we have to find a relation between ϵ_{k+1} and ϵ_k.

Using Taylor's theorem, we have

$$f(x_k) = f(\xi + (x_k - \xi)) = f(\xi + \epsilon_k)$$

$$= f(\xi) + f'(\xi)\epsilon_k + \frac{1}{2}f''(\xi)\epsilon_k^2 + 0(\epsilon_k^3).$$

Similarly we can write $f(x_{k-1})$ as

$$f(x_{k-1}) = f(\xi + (x_{k-1} - \xi)) = f(\xi + \epsilon_{k-1})$$

$$= f(\xi) + f'(\xi)\varepsilon_{k-1} + \frac{1}{2} f''(\xi)\varepsilon_{k-1}^2 + 0(\varepsilon_{k-1}^3).$$

Furthermore, we have

$$x_k - x_{k-1} = (x_k - \xi) - (x_{k-1} - \xi) = \varepsilon_k - \varepsilon_{k-1}.$$

Subtracting ξ from both sides of equation (3) and keeping in mind that by definition we have $f(\xi) = 0$, gives then

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\left(f'(\xi)\varepsilon_k + \frac{1}{2} f''(\xi)\varepsilon_k^2\right)(\varepsilon_k - \varepsilon_{k-1})}{f'(\xi)(\varepsilon_k - \varepsilon_{k-1}) + \frac{1}{2} f''(\xi)\left(\varepsilon_k^2 - \varepsilon_{k-1}^2\right)},$$

Which can be rewritten using,

$$\left(\varepsilon_k^2 - \varepsilon_{k-1}^2\right) = (\varepsilon_k - \varepsilon_{k-1})(\varepsilon_k + \varepsilon_{k-1}) \quad \text{as}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f'(\xi)\varepsilon_k + \frac{1}{2} f''(\xi)\varepsilon_k^2}{f'(\xi) + \frac{1}{2} f''(\xi)(\varepsilon_k + \varepsilon_{k-1})},$$

or $$\varepsilon_{k+1} = \frac{\frac{1}{2} f'(\xi)' \varepsilon_k \ \varepsilon_{k-1}}{f'(\xi) + \frac{1}{2} f''(\xi)(\varepsilon_k + \varepsilon_{k-1})}$$

or $$\varepsilon_{k+1} = \frac{f''(\xi)}{2f'(\xi)} \varepsilon_k \ \varepsilon_{k-1} + 0(\varepsilon_k^{\,3}).$$

The relation $\varepsilon_{k+1} = \frac{f''(\xi)}{2f'(\xi)} \varepsilon_k \ \varepsilon_{k-1} + 0(\varepsilon_k^{\,3})$ is of the form $\varepsilon_{k+1} = c\, \varepsilon_k \ \varepsilon_{k-1}$ (4)

where $c = \frac{f''(\xi)}{2f'(\xi)}$, and higher powers of ε_k are neglected.

The relation (4) is called as *Error equation*. Keeping in view the definition of the rate of convergence, we seek a relation of the form

$$\epsilon_{k+1} = A\,\epsilon_k^{p} \qquad (5)$$

Where A and p are to be determined.

From (5), we have

$$\epsilon_k = A\,\epsilon_{k-1}^{p} \quad \text{or} \quad \epsilon_{k-1} = A^{-1/p}\,\epsilon_k^{1/p}$$

Substituting the values of ϵ_{k+1} and ϵ_{k-1} in eq. (4), we get

$$\epsilon_k^{p} = c\,A^{-(1+1/p)}\epsilon_k^{(1+1/p)} \qquad (6)$$

Comparing the powers of ϵ_k on both sides, we get

$$p = 1 + \frac{1}{p}$$

Which gives $p = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$

Neglecting the minus sign, we find that the rate of convergence for the secant method is p = 1.618. From (6), we also obtain $A = c^{p/(p+1)}$.

The Newton-Raphson formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad (7)$$

Let ξ be a root of $f(x) = 0$ also,

Let us define the error at the kth step to be

$\epsilon_k = x_k - \xi$. We assume f'' is continuous near ξ and use a Taylor approximation about x_k, we have

$$0 = f(\xi) = f(x_k - \epsilon_k) = f(x_k) - \epsilon_k f'(x_k) + \epsilon_k^2 f''(x_k)/2 + 0(\epsilon_k^3)$$

If $f'(x_k) \neq 0$, we may write

$$\frac{-f(x_k)}{f'(x_k)} = -\epsilon_k + \frac{\epsilon_k^2 f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) \qquad (8)$$

Then $\epsilon_{k+1} = x_{k+1} - \xi = \left(x_k - \frac{f(x_k)}{f'(x_k)} \right) - \xi$ using equation (7)

$$\epsilon_{k+1} = x_k - \epsilon_k + \epsilon_k^2 \frac{f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) + \epsilon_k - x_k \qquad \text{using equation (8)}$$

$$\epsilon_{k+1} = \epsilon_k^2 \frac{f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) \qquad (9)$$

We can write (9) as

$\epsilon_{k+1} = C\,\epsilon_k^2$, where $C = \frac{f''(\xi)}{2f'(\xi)}$ as $\begin{matrix} k \to \infty \\ x_k \to \xi \end{matrix}$

and neglecting higher powers of ϵ_k.

Thus the Newton-Raphson method has second order convergence or quadratic convergence.

Possible Questions

PART-A (2 Mark)
UNIT I

1. Define high speed computation Algorithms.
2. Define relative error with example.
3. Write the rate of convergence of the Regula falsi method.
4. Define absolute error with example
5. Define round off error with example.

PART-B (6 Mark)

1. Find the value of 131 using Newton Raphson's Method.
2. Assuming that a root of $x^3-9x+1 = 0$ lies in the interval (2,4) ,find that root by bisection method.
3. Find the real positive root of 3x-cos x –1 =0 by Newton's method correct to 3 decimal places
4. Solve the following for the positive root by False position method $4x=ex$.
5.Find the positive root of the equation $x^3 – 4x – 9 = 0$ by bisection method.
6. Solve the following by Secant method $2x-log_{10}x=7$.
7.Find all the roots of the equation $x^3-6x^2+11x-6=0$ by Newton's method.
8. Solve the equation $x^3 – 4x + 1 = 0$ by Regula Falsi method.
9.Find the positive root of the equation $x^3 – 4x – 9 = 0$ by bisection method.
10.Find all the roots of the equation $x^3- 4x^2+5x-2=0$ by method of false position.

KARPAGAM UNIVERSITY
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DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY: Ms.M.LATHA(KU0969)
SUBJECT: NUMERICAL METHODS
SUBJECT CODE: 16MMU301
CLASS: II B.Sc MATHEMATICS

POSSIBLE QUESTIONS

UNIT I

2 MARKS

1. Define high speed computation Algorithms.
2. Define relative error with example.
3. Write the rate of convergence of the Regula falsi method.
4. Define absolute error with example
5. Define round off error with example.

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POSSIBLE QUESTIONS
UNIT I

6 MARKS

1. Find the value of $\frac{1}{31}$ using Newton Raphson's Method.
2. Assuming that a root of x^3-9x+1 = 0 lies in the interval (2,4) ,find that root by bisection method.
3. Find the real positive root of 3x-cos x –1 =0 by Newton's method correct to 3 decimal places
4. Solve the following for the positive root by False position method $4x = e^x$.
5. Find the positive root of the equation x^3 – 4x – 9 = 0 by bisection method.
6. Solve the following by Secant method $2x - log_{10} x = 7$.
7. Find all the roots of the equation x^3-6x^2+11x-6=0 by Newton's method.

8. Solve the equation x^3 – 4x + 1 = 0 by Regula Falsi method.
9. Find the positive root of the equation x^3 – 4x – 9 = 0 by bisection method.
10. Find all the roots of the equation x^3- 4x^2+5x-2=0 by method of false position.

UNIT-I

1. -------- Method is based on the repeated application of the intermediate value theorem.
    a. Gauss Seidal   b.Bisection   c.Regula Falsi   d.**Newton Raphson**
2. The formula for Newton Raphson method is ------------------.
    a. xn+1 = f (xn) /f ’(xn)   b.xn+1 =xn+ f (xn) /f ’(xn)   c.**xn+1 =xn- f (xn) /f ’(xn)**   d.xn+1 =xn- f '(xn) /f (xn)
3. The order of convergence of Newton Raphson method is ----------
    a. 4   b.2   c.1   d.**0**
4. Graeffe's root squaring method is useful to find -----------------
    a. complex roots   b.single roots   c.unequal roots   d.**polynomial roots**
5. The approximate value of the root of f(x) given by the bisection method is ----
    a. x0 = a + b   b.x0 = f(a) + f(b)   c.**x0 = (a + b)/ 2**   d.X0 = (f(a) + f(b))/2
6. "In Newton Raphson method, the error at any stage is proportional to the -------of the error in the previous stage. "
    a. cube   b.square   c.square root   d.**equa**l
7. The convergence of bisection method is -------.
    a. linear   b.quadratic   c.slow   d.**fast**
8. The order of convergence of Regula falsi method may be assumed to ----------.
    a. 1   b.1.618   c.0   d.**0.5**
9. ------------------ Method is also called method of tangents.
    a. Gauss Seidal   b.Secant   c.Bisection   d.**Newton Raphson**
10. "If f (x) contains some functions like exponential, trigonometric, logarithmic etc.,
11. then f (x) is called -------------- equation."
    a. Algebraic   b.transcendental   c. numerical   d. **polynomial**
12. A polynomial in x of degree n is called an algebraic equation of degree n if -----
    a. f (x) = 0   b.f (x) = 1   c.f (x) <1   d.**f (x) >1**
13. The method of false position is also known as ------------- method.
    a. Gauss Seidal   b.Secant   c.Bisection   d.**Regula falsi**
14. The Newton Rapson method fails if ---------------.
    a. f ' (x) = 0   b.f (x) = 0`   c.f (x) =1   d.**f(x)≠0**
15. The bisection method is simple but ------------------.

a. slowly divergent b.fast convergent c.slowly convergent d.**divergent**

16. ________Method is also called as Bolzano method or interval having method.
    a. Bisection b.false position c.Newton raphson d.**Horner's**
17. The another name of Bisection method is ___________
    a. Bozano b.Regula falsi c.Newtons d.**Giraffes**
18. The convergence of Bisection is Very__________
    a. slow b.fast c.moderate d.**normal**
19. In Regula-Falsi method, to reduce the number of iterations we start with ________ interval
    a. Small b.large c.equal d.n**one**
20. The rate of convergence in Newton-Raphson method is of order ________
    a. 1 b.2 c.3 d.**4**
21. Newton's method is useful when the graph of the function crosses the x-axis is nearly ______.
    a. vertical b.horizontal c.close to zero d.**none**
22. If the initial approximation to the root is not given we can find any two values of x say a and bsuch that f (a) and f(b) are of _________signs.
    a.opposite b.same c.positive d.**negative**
23. The Newton – Raphson method is also known as method of __________
    a. secant b.tangent c.iteration d.**interpolation**
24. If the derivative of f(x) = 0, then _________ method should be used.
    a. Newton – Raphson b.Regula-Falsi c.iteration d.**interpolation**
25. The rate of convergence of Newton – Raphson method is _________
    a. quadratic b.cubic **c.4** d.5
26. If f (a) and f (b) are of opposite signs the actual root lies between _________
    a. (a, b) b.(0, a) c.(0, b) d.**(0, 0)**
27. The convergence of root in Regula-Falsi method is slower than ________
    a. Gauss – Elimination b.Gauss – Jordan c.Newton – Raphson d.**Power method**
28. Regula-Falsi method is known as method of _________
    a. secant b.tangent c.chords d.**elimination**
29. ______________method converges faster than Regula-Falsi method.
    a. Newton – Raphson b.Power method c.elimination d.**interpolation**

30. If f(x) is continuous in the interval (a, b) and if f (a) and f (b) are of opposite signs the equation f(x) = 0 has at least one __________ lying between a and b.

    a. equation b.function c.root d.**polynomial**

31. x^2 + 3x – 3 = 0 is a polynomial of order

    a. 2 b.3 c.1 d.**0**

32. Errors which are already present in the statement of the problem are called____ errors.

    a. Inherent b.Rounding c.Truncation d.**Absolute**

33. Rounding errors arise during__________

    a. Solving b.Algorithm c.Truncation d.**Computation**

34. The other name for truncation error is __________ error.

    a. Absolute b.Rounding c.Inherent d.**Algorithm**

35. Rounding errors arise from the process of __________ the numbers.

    a. Truncating b.Rounding off c.Approximating d.**Solving**

36. Absolute error is denoted by_________

    a. Ea b.Er c.Ep d.**Ex**

37. Truncation errors are caused by using __________ results.

    a. Exact b.True c.Approximate d.**Real**

38. Truncation errors are caused on replacing an infinite process by _________ one.

    a. Approximate b.True c.Finite d.**Exact**

39. If a word length is 4 digits, then rounding off of 15.758 is

    a. 15.75 b.15.76 c.15.758 d.**16**

40. The actual root of the equation lies between a and b when f (a) and f (b) are of ____ signs.

    a. Opposite b. same c.negative d.**positive**

Example 6. Construct the divided difference table for the data

x	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$	1.625	5.875	31.0	131.0	282.125	521.0

Hence, find the interpolating polynomial and an approximation to the value of f (7). We have the following divided difference table

x	*f (x)*	*first order d.d.*	*second order d.d.*	*third order d.d.*	*fourth order d.d.*
0.5	1.625				
		4.25			
1.5	5.875		5.0		
		16.75		1.0	
3.0	31.000		9.5		0
		50.00		1.0	
5.0	131.000		14.5		0
		100.75		1.0	
6.5	282.125		19.5		
		159.25			
8.0	521.000				

We write the divided difference interpolating polynomial as

$$f(x)=f[x_0]+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$$

$$+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

$$=1.625+(x-0.5)(4.25)+5(x-0.5)(x-1.5)$$

$$+(x-0.5)(x-1.5)(x-3.0)$$

$$=(1.625-2.125+3.75-2.25)+x(4.25-10.0+6.75)$$

$$+x^2(5-5)+x^3$$

$$=x^3+x+1.$$

Hence, $f(7.0)=351.$

Finite Difference Operator:

Let the points $x_1, x_2, x_3, ..., x_n$ be equally spaced

$\therefore \quad x_i = x_0 + ih, \quad i = 0,1,2,...,n, \;\; h = x_1 - x_0 = x_2 - x_1 = x_n - x_{n-1}$

We define the following operators

(1) Shift Operator $Ef(x_i) = f(x_i + h)$

(2) Forward difference operator $\Delta f(x_i) = f(x_i + h) - f(x_i)$

(3) Backward difference operator $\nabla f(x_i) = f(x_i) - f(x_i - h)$

(4) Central difference operator $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

(5) The average operator $\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2} \right) + f\left(x_i - \frac{h}{2} \right) \right]$

Relation between the operators (take h = 1).

(i) $\Delta f(x_i) = \nabla f(x_i + 1) = \delta f\left(x_i + \frac{1}{2} \right) = f(x_i + 1) - f(x_i)$

(ii) (a) $\Delta f(x_i) = f(x_i + 1) - f(x_i)$

$= Ef(x_i) - f(x_i)$

$= (E - 1) f(x_i)$

$\Rightarrow$ $\boxed{\Delta = E - 1}$

(b) $\nabla f(x_i) = f(x_i) - f(x_i - 1)$

$= f(x_i) - E^{-1} f(x_i - 1)$

$= (1 - E^{-1}) f(x_i)$

$\Rightarrow$ $\boxed{\nabla = 1 - E^{-1}}$

$\Rightarrow$ $\boxed{E^{-1} = 1 - \nabla}$

(iii) We can also have

$\Delta^n = (E - 1)^n =$

$\nabla^n = (1 - E^{-1})^n =$

Table showing relationship between the operators

	E	Δ	∇	δ
E	E	Δ + 1	$(1-\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$
Δ	E – 1	Δ	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ
μ	$\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}})$	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{\frac{1}{2}}$	$\left(1-\frac{1}{2}\nabla\right)(1-\Delta)^{-\frac{1}{2}}$	$\sqrt{1+\frac{1}{4}\delta^2}$

Example 7. Show that

(i) $\delta=\nabla(1-\nabla)^{-\frac{1}{2}}$ (ii) $\mu=\left(1+\frac{\delta^2}{4}\right)^{1/2}$

(iii) $E=1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$ (iv) $\nabla=-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$

Proof:

(1) R.H.S. = $\nabla(1-\nabla)^{-\frac{1}{2}}$

$$=(1-E^{-1})[1-(1-E^{-1})]^{-\frac{1}{2}} \qquad (\because \nabla=1-E^{-1})$$

$$=(1-E^{-1})(E^{-1})^{-\frac{1}{2}}=(1-E^{-1})(E^{\frac{1}{2}})$$

$$=E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

δ = L.H.S.

(ii) R.H.S. = $\left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2}}$

$$=\left[1+\frac{1}{4}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2\right]^{\frac{1}{2}} \qquad \because \delta=E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

$$=\left[1+\frac{1}{4}(E-E^{-1}-2)\right]^{\frac{1}{2}} \qquad \delta^2=(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2$$

$$= E-E^{-1}-2$$

$$=\left[\frac{4+E+E^{-1}-2}{4}\right]^{1/2}$$

$$=\frac{1}{2}\left[E+E^{-1}+2\right]^{1/2}$$

$$=\frac{1}{2}\left[(E^{\frac{1}{2}}+E^{-\frac{1}{2}})^2\right]^{1/2}$$

$$=\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}})=\mu=L.H.S.$$

(iii) R.H.S. = $1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$

$$=1+\frac{1}{2}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2+(E^{\frac{1}{2}}-E^{-\frac{1}{2}})\times\frac{(E^{\frac{1}{2}}+E^{-\frac{1}{2}})}{2}$$

$\left[\because \mu=\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) = \sqrt{1+\frac{\delta^2}{4}}\ \right]$

$$=1+\frac{1}{2}(E-E^{-1}-2)+\frac{1}{2}(E-E^{-1})$$

$$=\frac{1}{2}[2+E+E^{-1}-2+E-E^{-1}]$$

$$=\frac{1}{2}\times 2E=E=L.H.S.$$

(iv) R.H.S.= $-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$

$$=-\frac{1}{2}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2+(E^{\frac{1}{2}}-E^{-\frac{1}{2}})\frac{(E^{\frac{1}{2}}+E^{-\frac{1}{2}})}{2}$$ (using relation (ii))

$$=-\frac{1}{2}(E+E^{-1}-2)+\frac{1}{2}(E-E^{-1})$$

$$=\frac{1}{2}[-E-E^{-1}-2+E-E^{-1}]$$

$$=\frac{1}{2}[2-2E^{-1}]= [1-E^{-1}]=\nabla=L.H.S.$$

- Similarly we can prove other relations given in the table of operators.

Gregory-Newton Forward Difference Interpolation:

Relation between divided difference and forward difference operator is

$$f[x_0, x_1, ..., x_n] = \frac{1}{n!h^n}\Delta^n f_0 \quad (1)$$

Divided difference interpolating polynomial is written as

$$P_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + ... + (x - x_0(x - x_1)....(x - x_{n-1}) \; f[x_0, x_1, ..., x_n] \quad (2)$$

Using (1) in (2), Interpolating polynomial can be written as

$$P(x) = P_n(x) = f_0 + (x - x_0)\frac{\Delta f_0}{h} + \frac{(x - x_0(x - x_1)\Delta^2 f_0}{2!h^2} + ...$$

$$+ \frac{(x - x_0)(x - x_1) + ...(x - x_{n-1})}{n!h^n}\Delta^n f_0 \quad (3)$$

Polynomial P(x) expressed by equation (3) is known as Gregory-Newton forward differences interpolating polynomial.

Now if we put $u = \frac{(x - x_0)}{h} \Rightarrow hu = (x - x_0)$ and since $x_0, x_1, ..., x_n$ are equally spaced Points that is $x_i = x_0 + ih$

$$\Rightarrow \quad (x - x_i) = (x - (x_0 + ih)) = [(x - x_0) - ih] = (uh - ih) = (u - i)h.$$

$\therefore$ Equation (3) and the truncation error can be written as

$$P(x_0 + hu) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + ... + \frac{u(u-1)....(u-n+1)}{n!}\Delta^n f_0 \quad (4)$$

$$= \sum_{i=0}^{n} \binom{u}{i} \Delta^i f_0 \quad \text{where} \quad \binom{u}{i} = {}^uC_i$$

and truncation error

$$E_n(f; x) = \frac{u(u-1)....(u-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi) \quad (5)$$

Gregory-Newton Backward Difference Interpolation:

We observe that Newton-interpolation with divided differences in terms of backward differences should be in terms of the differences at the end point x_n.

$$f(x) = f\left(x_n + \frac{(x-x_n)}{h} \times h\right) = f(x_n + hu)$$

as we take $\frac{(x-x_n)}{h} = u \Rightarrow x - x_n = hu$

$$\Rightarrow \quad f(x) = f(x_n + uh)$$

$$= E^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n) \qquad [\because E = (1-\nabla)^{-1}]$$

$$= 1 + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \ldots + \frac{u(u+1)-(u+n-1)}{n!}\nabla^n f(x_n) + \ldots \quad (8)$$

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order difference we get the interpolating polynomial as

$$P(x) = P(x_n + hu) = f(x)$$

$$= f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + ... + \frac{u(u+1)....(u+n-1)}{n!} \times \nabla^n f_n.$$

$$= \sum_{i=0}^{n} (-1)^i \binom{-u}{i} \nabla^i f_n \qquad (9)$$

where $\binom{-u}{i} = {}^{-u}C_i$.

Polynomial expressed by relation (9) is known as Gregory-Newton backward difference interpolating polynomial and the truncation error is

$$E_n(f;x) = \frac{u(u+1)....(u+n)}{(n+1)!} h^{n+1} f^{n+1}(\xi)$$

Example9. For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at x = 0.25 and x = 0.35

x	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Solution: The difference table is obtained as

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

The forward difference polynomial is given by

$$P(x) = 1.4 + (x-0.1)\frac{0.16}{0.1} + \frac{(x-0.1)(x-0.2)}{2} \times \frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

The backward difference polynomial is obtained as

$$P(x) = 2.28 + (x-0.5)\frac{0.28}{0.1} + \frac{(x-0.5)(x-0.4)}{2}\frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

Both the polynomials are same.

$\therefore \quad f(0.25) = 1.655, \quad f(0.35) = 1.875.$

We can obtain the interpolated values directly also. So for x = 0.25 we choose x_0 = 0.2 and write

$$u = \frac{x - x_0}{h} = \frac{0.25 - 0.2}{0.1} = 0.5$$

$$\Rightarrow \quad f(0.25) = f(0.2) + (0.5)\Delta f(0.2) + \frac{1}{2}(0.5)(-0.5)\Delta^2 f(0.2)$$

= 1.56 +(0.5(0.20) – (0.125)(0.04) = 1.655

For x = 0.35 we choose x_n = 0.4 and in backward differences as

$$u = \frac{x - x_n}{h} = \frac{0.35 - 0.4}{0.1} = -0.5$$

and $f(0.35) = f(0.4) + (-0.5)\nabla f(0.4) + \frac{1}{2}(0.5)(0.5)\nabla f(0.2)$

= 2.00 – (0.5)(0.24) – (0.125) (0.04)

= 1.875.

Hence the solution.

Possible Questions

PART-A (2 Mark)
UNIT III

1. Prove that $E\Delta=\Delta=\nabla E$.
2. Write any two properties of divided differences.
3. Define Inverse Lagrange's interpolation
4. Prove that $\mu=(1+\delta 24)12$
5. Prove that $\Delta\nabla=\Delta-\nabla=\delta 2$.

PART-B(6 Mark)

1. From the following table, find the value of tan 45 15

x :	45	46	47	48	49	50
tan x :	1.0000	1.0355	1.0723	1.1106	1.1503	1.1917

2. Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

3. From the following table find f(x) and hence f(6) using Newton interpolation formula.

x :	1	2	7	8
f(x) :	1	5	5	4

4. Find the values of y at X=21 and X=28 from the following data.

X:	20	23	26	29
Y:	0.3420	0.3907	0.4384	0.4848

5. Using Newton's divided difference formula. Find the values of f(2),f(8) and f(15) given the following table

X:	4	5	7	10	11	13
f(x):	48	100	294	900	1210	2028

6. Using Lagrange's interpolation formula find the value corresponding to x = 10 from the following table

x :	5	6	9	11
y :	12	13	14	16

7. Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

8. Find the age corresponding to the annuity value 13.6 given the table

Age(x) :	30	35	40	45	50
Annuity Value(y):	15.9	14.9	14.1	13.3	12.5

KARPAGAM UNIVERSITY
COIMBATORE-21
DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY: Ms.M.LATHA(KU0969)
SUBJECT: NUMERICAL METHODS
SUBJECT CODE: 16MMU301
CLASS: II B.Sc MATHEMATICS

POSSIBLE QUESTIONS

UNIT II

2 MARKS

1. Write the formula for method of triangularization.
2..Define iterative method.
3. Define power method.
4. Write the difference between the direct method and iterative method.
5. Define Gauss elimination method.

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POSSIBLE QUESTIONS
UNIT II

6MARKS

1. Solve the following system by Gauss elimination method.

$3x + y - z = 3$
$2x - 8y + z = -5$
$x - 2y + 9z = 8$

2. Solve the following system by Gauss Jacobi method.

$8x + y + z = 8$
$2x + 4y + z = 4$
$x + 3y + 3z = 5$

3. Solve the following system by Gauss Jordan method.

$x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2z = 13$

4. Solve the following system of equations by Gauss-Jacobi method

$10x - 5y - 2z = 3$
$4x - 10y + 3z = -3$
$x + 6y + 10z = -3$

5.Solve the following system by triangularisation method.

$5x - 2y + z = 4$
$7x + y - 5z = 8$
$3x + 7y + 4z = 10$

6. Solve the following system of equations by Gauss-Seidal method.

$28x + 4y - z = 32$
$x + 3y + 10z = 24$
$2x + 17y + 4z = 35$

7.Solve the following system by Gauss Jordan method.

$x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2z = 13$

8. Find the numerically largest Eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and the corresponding Eigen vector.

9.Solve the following system by triangularization method.

$$x+y+5z=16$$
$$2x+3y+z=4$$
$$4x+y-z=4$$

10. Solve the following system of equations by Gauss-Jacobi method

$$4x+2y+z=14$$
$$x+5y-z=10$$
$$x+y+8z=20$$

UNIT II

1. Iterative method is a ------------ method
   a. Direct method **b.InDirect method** c.both 1st & 2nd d.either 1st &2nd
2. -------------- is also a self-correction method.
   a. **Iteration method** b.Direct method c.Interpolation d.none
3. "The condition for convergence of Gauss Seidal method is that the ------should be diagonally dominant"
   a. Constant matrix b.unknown matrix **c.Coefficient matrix** d.Unit matrix
4. In ----------- method, the coefficient matrix is transformed into diagonal matrix
   a. Gauss elimination **b.Gauss jordan** c.Gauss jacobi d.Gauss seidal
5. -------------- Method takes less time to solve a system of equations comparatively than ' iterative method'
   a. **Direct method** b. Indirect method c.Regula falsi d.Bisection
6. The iterative process continues till ------------- is secured.
   a. **convergency** b.divergency c.oscillation d.none
7. "In Gauss elimination method, the solution is getting by means of -----------from which the unknowns are found by back substitution."
   a. " Elementary operations" b." Elementary column operations"

   c." Elementary diagonal operations" **d." Elementary row operations"**

8. "The --------------- is reduced to an upper triangular matrix or a diagonal matrix in direct methods."
   a. Coefficient matrix **b.Constant matrix** c.unknown matrix d.Augment matrix
9. The augment matrix is the combination of ------------------.
   a. **"Coefficient matrix and constant matrix"** c."Unknown matrix and constant matrix"
   b. "Coefficient matrix and Unknown matrix" d." Coefficient matrix, constant matrix and Unknown matrix"
10. The given system of equations can be taken as in the form of -----------
    a. A = B b.BX= A **c.AX= B** d. AB = X
11. Which is the condition to apply Gauss Seidal method to solve a system of equations?
    a. 1st row is dominant b.1st column is dominant **c.diagonally dominant** d.last row dominant
12. Crout's method and triangularisation method are ------ method.
    a. **Direct** b.Indirect c.Iterative d.Interpolation
13. The solution of simultaneous linear algebraic equations are found by using-----

14.

a. Direct method **b.Indirect method** c.both 1st & 2nd d.Bisection

15. The matrix is ____ if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical value of other element in that row.

a. orthogonal b.symmetric **c.diagonally dominant** d.singular

16. If the Eigen values of A are -6, 2, 4 then ______ is dominant.

a. **2** b.-6 c.4 d.-2

17. The Gauss – Jordan method is the modification of ________ method.

a. **Gauss –Elimination** b.Gauss – Jacobi c.Gauss – Seidal d.interpolation

18. x^2 + 5x + 4 = 0 is a ________ equation.

a. **algebraic** b.transcendental c.wave d.heat

19. a + b logx + c sinx + d = 0 is a _________ equation.

a. algebraic **b.transcendental** c.wave d.heat

20. In Gauss – Jordan method, the augmented matrix is reduced into _____matrix

a. upper triangular b.lower triangular **c.diagonal** d.scalar

21. The 1st equation in Gauss – Jordan method, is called __________ equation.

a. **pivotal** b.dominant c.reduced d.normal

22. The element a11 in Gauss – Jordan method is called _________ element.

a. Eigen value b.Eigen vector **c.pivot** d.root

23. The system of simultaneous linear equation in n unknowns AX = B if A is diagonally dominant then the system is said to be __________system

a.dominant **b.diagonal** c.scalar d.singular

24. The convergence of Gauss – Seidal method is roughly _____ that of Gauss – Jacobi method

a. **twice** b.thrice c.once d.4 times

25. Jacobi's method is used only when the matrix is ____

a. **symmetric** b.skew-symmetric c.singular d.non-singular

26. Gauss Seidal method always --------- for a special type of systems.

a. **Converges** b.diverges c.oscillates d.equal

27. Condition for convergence of Gauss Seidal method is ---------------.

a. **"Coefficient matrix is diagonally dominant "** c. pivot element is Zero

b. "Coefficient matrix is not diagonally dominant" d.pivot element is non Zero

28. Modified form of Gauss Jacobi method is -------------------- method.
    a. Gauss Jordan **b. Gauss Siedal** c. Gauss Jacobbi d.Gauss Elimination
29. "In Gauss elimination method by means of elementary row operations, from which the unknowns are found by ----------------- method"
    Forward substitution b**.Backward substitution** c.random d.Gauss Elimination
30. In iterative methods, the solution to a system of linear equations will exist if the absolute value of the largest coefficient is ------------- the sum of the absolute values of all remaining coefficients in each equation.
    a. less than **b. greater than or equal to** c. equal to d.not equal
31. In ------------- iterative method, the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration.
    a. **Gauss Siedal** b. Gauss Jacobi c.Gauss Jordan d.Gauss Elimination
32. The direct method fails if any one of the pivot elements become ----.
    a. **Zero** b.one c.two d.negative
33. In Gauss elimination method the given matrix is transformed into --------------.
    a. Unit matrix b.diagonal matrix **c.Upper triangular matrix** d.lower triangular matrix
34. "If the coefficient matrix is not diagonally dominant, then by ----------- that diagonally dominant coefficient matrix is formed."
    a.Interchanging rows b. Interchanging Columns c. adding zeros **d.Interchangingrow and Columns**
35. Gauss Jordan method is a -----------------.
    a. **Direct method** b. InDirect method c. iterative method d.convergent
36. Gauss Jacobi method is a -----------------.
    a. Direct method b**. InDirect method** c. iterative method d.convergent
37. The modification of Gauss – Jordan method is called ---------------------.
    a. Gauss Jordan b**.Gauss Siedal** c. Gauss Jacobbi d.gauss elemination
38. Gauss Seidal method always converges for ----------- of systems
    a. **Only the special type** b.all types c.quadratic types d.first type
39. In solving the system of linear equations, the system can be written as ---
    a. BX = B b.AX = A **c.AX = B** d.AB = X
40. In solving the system of linear equations, the augment matrix is --------

    a.(A, A) b.(B, B) c.(A, X) **d.(A, B)**

41. "In the direct methods of solving a system of linear equations, at first the given system is written as ------------- form."
    a. **An augment matrix** b.a triangular matrix c.constant matrix d.Coefficient matrix
42. " All the row operations in the direct methods can be carried out on the basis of --"
    a. all elements b**. pivot element** c.negative element d.positiveelement
43. The direct method fails if -----------.
    a. 1st row elements 0 b.1st column elements 0 **c.Either 1st or 2nd** d.2 nd row is dominant
44. "The elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal is called ---------"
    a. Gauss elimination b**. Gauss jordan** c.Gauss jacobi **d**.Gauss siedal
45. In Gauss Jordan method, we get the solution ---------------------
    a."without using back substitution method " c**."By using back substitution method "**
    b,."by using forward substitution method" d."Without using forward substitution method"
46. "If the coefficient matrix is diagonally dominant, then ---------- method converges quickly."
    a. Gauss elimination b.Gauss jordan c. Direct **d.Gauss siedal**
47. Which is the condition to apply Jocobi's method to solve a system of equations
    a. 1st row is dominant b.1st column is dominan t **c.diagonally dominant** d.2 nd row is dominant
48. Iterative method is a ------------ method
    a. Direct method **b.InDirect method** c.Interpolation d.extrapolation
49. "As soon as a new value for a variable is found by iteration it is used immediately in the equations is called --------------."
    a. **Iteration method** b.Direct method c.Interpolation d.extrapolation
50. -------------- is also a self-correction method.
    a. **Iteration method** b.Direct method c.Interpolation d.extrapolation
51. "The condition for convergence of Gauss Seidal method is that the ------should be diagonally dominant"
    a. Constant matrix b.unknown matrix **c.Coefficient matrix** d.extrapolation
52. In ----------- method, the coefficient matrix is transformed into diagonal matrix
    a. Gauss elimination b.**Gauss jordan** c.Gauss jacobi d.Gauss seidal
53. We get the approximate solution from the ---------------.
    a. Direct method **b.InDirect method** c.fast method d.Bisection
54. The iterative process continues till ------------- is secured.
    a. **convergency** b.divergency c.oscillation d.point

55. "In Gauss elimination method, the solution is getting by means of -----------from which the unknowns are found by back substitution."
    a. " Elementary operations" c." Elementary column operations"
    b. " Elementary diagonal operations" **d." Elementary row operations**"
56. "The method of iteration is applicable only if all equation must contain onecoefficient of different unknowns as ---------- than other coefficients."
    a.Smaller **b.larger** C.equal d.non zero
57. "The --------------- is reduced to an upper triangular matrix or a diagonal matrix indirect methods."
    a. Coefficient matrix b.Constant matrix c.unknown matrix **d.Augment matrix**
58. The augment matrix is the combination of ------------------.
    a. **"Coefficient matrix and constant matrix"** c."Unknown matrix and constant matrix"
    b. "Coefficient matrix and Unknown matrix" d." Coefficient matrix, constant matrix and Unknown matrix"
59. The given system of equations can be taken as in the form of -----------
    a. A = B b. BX= A **c.AX= B** d. AB = X
60. "The sufficient condition of iterative methods will be satisfied if the large coefficients are along the ---------------- of the coefficient matrix."
    a.Rows b.Columns **c.Leading Diagonal** d.elements
61. Which is the condition to apply Gauss Seidal method to solve a system of equations?
    a. 1st row is dominant b.1st column is dominant **c.diagonally dominant** d.Leading Diagonal
62. In the absence of any better estimates, the -------------of the function are taken as x = 0, y = 0, z = 0.
    a.**Initialapproximations** b. roots c. points d.final value
63. The solution of simultaneous linear algebraic equations are found by using-
    a. Direct method **b.InDirect method** c.fast method d.Bisection

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DEPARTMENT OF MATHEMATICS

Subject: Numerical Methods	**Semester: III**	**L T P C**
Subject Code: 16MMU301	**Class: II-B.Sc Mathematics**	**4 0 0 4**

UNIT III

Interpolation: Lagrange and Newton's methods. Error bounds - Finite difference operators. Gregory forward and backward difference interpolation – Newton's divided difference – Central difference – Lagrange and inverse Lagrange interpolation formula.

TEXT BOOK

T1. Jain. M.K., Iyengar. S.R.K.,and Jain R.K., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .
R3. Uri M. Ascher and Chen Greif., (2013). A First Course in Numerical Methods, Seventh Edition., PHI Learning Private Limited.
R4. John H., Mathews and Kurtis D. Fink., (2012). Numerical Methods using Matlab, Fourth Edition., PHI Learning Private Limited.

UNIT III

Interpolation: Interpolation means insertion or filling up intermediate terms of series. Interpolation is the method of estimating the value of a function (dependent variable) for any intermediate value of the independent variable when some values of the function corresponding to the values of the variable are given.

that is, given the set of functional values $(x_0, y_0),(x_1, y_1),(x_2, y_2),\ldots,(x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ may not be known, it is required (desired) to find a simpler function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points, such a process is called as interpolation and if $\phi(x)$ happens to be a polynomial than the process is polynomial interpolation. $\phi(x)$ approximates (evaluates) for *f* (*x*)

3. Methods of Interpolation:

Following are the methods of Interpolation

(a) Graphic Method

(b) Method of Curve fitting

(c) Use of finite difference formulae.

Interpolation

Interpolation or interpolating polynomial are having two main uses.

(i) The first use is in reconstructing the function $f(x)$ when it is not given explicitly and only the values of $f(x)$ and for its certain order derivatives at a set of points, called nodes, tabular points or arguments are known.

(ii) The second use is to replace the function $f(x)$ by an interpolating polynomial $\phi(x)$ so that many common operations such as determination of roots, differentiation, integration etc. may be carried out easily using $\phi(x)$.

Lagrange and Newton Interpolation:

Let us assume that $f(x)$ is a function defined and continuous on $[a,b]$ and we have $n + 1$ points.

$a \le x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n \le b$, at these $(n+1)$ points values of $f(x)$ are known.

We want to find the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \qquad \text{(i)}$$

which satisfies the conditions

$$P(x_i) = f(x_i) \qquad i = 0,1,2,\ldots n \qquad \text{(ii)}$$

Putting $(n+1)$ point $x_0, x, \ldots, x_n$ in eqn. (i) & using (ii) we get

$$a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n = P(x_0) = f(x_0)$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n = P(x_1) = f(x_1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n = P(x_n) = f(x_n)$$

This system of equation has a unique solution or polynomial $P(x)$ exists if the Vandermonde's determinant

$$V(x_0, x_1, \ldots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} \ne 0$$

Uniqueness : The polynomial obtained above is unique.

Suppose that there is another polynomial $P^*(x)$ which also satisfies

$$P^*(x_i) = f(x_i) \qquad i = 0,1,2,\ldots,n$$

Consider the polynomial

$$Q(x) = P(x) - P^*(x)$$

Since $P(x)$ & $P^*(x)$ are polynomials of degree *n*.

$\therefore$ $Q(x)$ is also a polynomial of degree $\leq n$.

Also at $x_0, x, \ldots, x_n$

$$Q(x_i) = P(x_i) - P^*(x_i)$$

$$= f(x_i) - f(x_i) \quad i = 0, 1, \ldots, n$$

$$= 0$$

$\Rightarrow$ $Q(x)$ is a polynomial of degree $\leq n$ which has *n* + 1 distinct roots $x_0, x, \ldots, x_n$.

$\Rightarrow$ $Q(x) = 0$ [$\because$ a poly. of degree $\leq n$ cannot have $(n+1)$ roots].

Expanding the determinant equation $\begin{vmatrix} P(x) & x & 1 \\ f(x) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$ we get

$$P(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$\Rightarrow \quad P(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{x - x_0}{(x_1 - x_0)} f(x_1)$$

$$P(x) = \ell_0(x) f(x_0) + \ell_1(x) f(x_1) \qquad \text{(iv)}$$

where

$$\ell_0(x) = \frac{x - x_1}{(x_0 - x_1)}, \quad \ell_1(x) = \frac{x - x_0}{(x_1 - x_0)}$$

$\ell_0(x)$ & $\ell_1(x)$ are called the Lagrange fundamental polynomial satisfying

$$\ell_0(x) + \ell_1(x) = 1$$

$$\ell_0(x_0) = 1, \quad \ell_0(x_1) = 0$$

$$\ell_1(x_0) = 0, \quad \ell_1(x_1) = 1$$

In general $\ell_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Polynomial represented by equation (iv) is called as Lagrange's Interpolating polynomial.

Consider the determinant Equation.

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 \qquad \text{(i)}$$

Expanding along first how we get

$$P(x)(x_0 - x_1) - x(f(x_0) - f(x_1)) + 1(x_1 f(x_0) - x_0 f(x_1)) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) - xf(x_0) + xf(x_1) + x_1 f(x_0) - x_0 f(x_1) - x_0 f(x_0) + x_0 f(x_0) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) + x(f(x_1) - f(x_0)) - x_0(f(x_1) - f(x_0))$$

$$+(x_1 - x_0) f(x_0) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) = (x_0 - x_1) f(x_0) - (x - x_0)(f(x_1) - f(x))$$

$$\Rightarrow \quad P(x) = \frac{(x_0 - x_1) f(x_0)}{(x_0 - x_1)} - \frac{(x - x_0)(f(x_1) - f(x_0))}{-(x_1 - x_0)}$$

$$\Rightarrow \quad P(x) = f(x_0) + (x - x_0)\frac{(f(x_1) - f(x_0))}{(x_1 - x_0)}$$

$$\Rightarrow \quad P(x) = f(x_0) + (x - x_0) f[x_0, x_1] \qquad \text{(ii)}$$

The ratio $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$ is called as first dividend difference of $f(x)$ relative to x_0 and x_1 Polynomial $P(x)$ represented by (ii) is Newton's Dividend Difference Interpolating Polynomial.

Example 1: Given $f(2)=4,\ f(2.5)=5.5$, find the linear interpolating polynomial using

(i) Lagrange Interpolation

(ii) Newton's Dividend difference interpolation

Hence find an approximate value of $f(2.2)$

Solution: We have

$$x_0 = 2, \qquad f(x_0)=4$$

$$x_1 = 2.5, \qquad f(x_1)=5.5$$

(i) The Lagrange fundamental polynomials are given by

$$\ell_0(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-2.5}{-0.5}=-2(x-2.5)$$

$$\ell_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-2}{0.5}=2(x-2)$$

$$P_1(x)=\ell_0(x)f(x_0)+\ell_1(x)f(x_1)$$

$$=-2(x-2.5)(4)+2(x-2)(5.5)$$

$$=(-2x+5)(4)+(2x-4)(5.5)$$

$$=-8x+20+11x-22$$

$$=3x-2.$$

(ii) Newton's dividend difference interpolation

We have

$$f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{5.5-4}{0.5}=3$$

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$
$$= 4 + (x-2)(3) = 4 + 3x - 6 = 3x - 2$$
$$f(2.2) \approx P_1(2.2) = 3 \times (2.2) - 2$$
$$= 6.6 - 2 = 4.6.$$

Example4. Given that $f(0)=1, f(1)=3, f(3)=55$, find the unique polynomial of degree 2 or less, which fits the given data using.

(i) Lagrange Method

(ii) Newton divided difference method

Also find the bound on the error.

Ans (i) We have $x_0 = 0, x_1 = 1, x_2 = 3, f_0 = 1, f_1 = 3$ and $f_2 = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}(x^2-4x+3)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-3)}{(1)(-2)} = \frac{1}{2}(3x+x^2)$$

$$l_2(x) = \frac{(x-x_0)(x-x_2)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{3(2)} = \frac{1}{6}(x^2-x).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$$

$$= \frac{1}{3}(x^2-4x+3) + \frac{3}{2}(3x-x^2) + \frac{55}{6}(x^2-x)$$

$$= 8x^2 - 6x + 1.$$

(ii) The divided differences are given by

$$f[0,1] = \frac{3-1}{1-0} = 2, f[1,3] = \frac{55-3}{3-1} = 26,$$

$$f[0,1,3] = \frac{26-2}{3-0} = 8.$$

The Newton divided difference interpolating polynomial becomes

$$P_2(x) = f[0] + (x-0) f[0,1] + (x-0)(x-1) f[0,1,3]$$

$$= 1 + 2x + 8x(x-1) = 8x^2 - 6x + 1.$$

Example 5. The following values of the function $f(x) = \sin x + \cos x$, are given

x	10°	20°	30°
$f(x)$	1.1585	1.2817	1.3660

Construct the quadratic interpolating polynomial that fits the data. Hence, find $f(\pi/12)$. Compare with the exact value.

Since the value of f at $\pi/12$ radians is required, we convert the data into radian measure. We have

$$x_0 = 10° = \frac{\pi}{18} = 0.1745,\ x_1 = 20° = \frac{\pi}{9} = 0.3491,$$

$$x_2 = 30° = \frac{\pi}{6} = 0.5236.$$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0.3491)(x - 0.5236)}{(-0.1746)(-0.3491)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0.1745)(x - 0.5236)}{(-0.1746)(-0.1745)}$$

$$= -32.8616\left(x^2 - 06981\,x + 0.0914\right)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.1745)(x-0.3491)}{(0.3491)(0.1745)}$$

$$= 16.4155\left(x^2 - 0.5236\,x + 0.0609\right).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = 16.4061\left(x^2 - 0.8727\,x + 0.1828\right)(1.1585)$$

$$-32.8616\left(x^2 - 0.6981\,x + 0.0914\right)(1.2817)$$

$$+16.4155\left(x^2 - 0.5236\,x + 0.0609\right)(1.3660)$$

$$= -0.6887\,x^2 + 1.0751\,x + 0.9903$$

Hence, $f(\pi/12) = f(0.2618) = 1.2246$,

The exact value is $f(0.2618) = \sin(0.2618) + \cos(0.2618) = 1.2247$.

KARPAGAM UNIVERSITY
COIMBATORE-21
DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY: Ms.M.LATHA(KU0969)
SUBJECT: NUMERICAL METHODS
SUBJECT CODE: 16MMU301
CLASS: II B.Sc MATHEMATICS

POSSIBLE QUESTIONS

UNIT II*I*

2 MARKS

1. Prove that $E\Delta = \Delta = \nabla E$.
2.Write any two properties of divided differences.
3. Define Inverse Lagrange's interpolation
4. Prove that $\mu = (1 + \frac{\delta^2}{4})^{\frac{1}{2}}$
5. Prove that $\Delta\nabla = \Delta - \nabla = \delta^2$.

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POSSIBLE QUESTIONS
UNIT III*I*

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6 MARKS

1. From the following table, find the value of tan 45°15′

x°	45	46	47	48	49	50
tan x°	1.0000	1.0355	1.0723	1.1106	1.1503	1.1917

2. Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

3. From the following table find f(x) and hence f(6) using Newton interpolation formula.

x	1	2	7	8
f(x)	1	5	5	4

4. Find the values of y at X=21 and X=28 from the following data.

X:	20	23	26	29
Y:	0.3420	0.3907	0.4384	0.4848

5.Using Newton's divided difference formula. Find the values of f(2),f(8) and f(15) given the following table

X:	4	5	7	10	11	13
f(x):	48	100	294	900	1210	2028

6. Using Lagrange's interpolation formula find the value corresponding to x = 10 from the following table

x	5	6	9	11
y	12	13	14	16

7.Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

8.Find the age corresponding to the annuity value 13.6 given the table

Age(x)	30	35	40	45	50
Annuity Value(y):	15.9	14.9	14.1	13.3	12.5

Unit iii

1. The x values of Interpolating polynomial of newton -Gregory has__________
   a.even space b.**equal space** c.odd space d.unequal
2. The value of E is ___________
   a. delta -1 b.1-delta c.**delta+1** d.delta+2
3. We use the central difference formula such as ____________
   a.lagrange's b.Newton c.Euler d.**bessel's**
4. --------------- Formula can be used for unequal intervals.

   a.Newton's forward b.Newton's backward c. **Lagrange** d.stirling

5. The difference value ∇y1 –∇y0 in a Newton forward differenc table is denoted by
   a. $\boldsymbol{\nabla^2}$ **y0** b. ∇^2 y1 c. ∇y1 d. ∇y0
6. By putting n = 3 in Newton cote's formula we get ----------- rule.

   a.Simpson's 1/3 rule b.**Simpson's 3/8 rule** c.Trapezoidal rule d.Simpson's rule

7. The process of computing the value of a function outside the range is called -----
   a.interpolation b.**extrapolation** c.triangularisation d.integration
8. The process of computing the value of a function inside the range is called ------

   a.interpolation b.extrapolation c.triangularisation d.integration

9. The difference value y2 – y1 in a Newton's forward difference table is denoted by
   a.∇ y0 b. **∇y1** c.∇y2 d.∇^2 y0
10. "__________ Formula can be used for interpolating the value of f(x) near the end of the tabular values."
    a. Newton's forward b. **Newton's backward** c. Lagrange d.stirling
11. The technique of estimating the value of a function for any intermediate value is

    a.interpolation b.extrapolation c.forward method d.backward method

12. The (n+1) th and higher differences of a polynomial of the nth degree are --------

a. a.**zero** b.one c.two d.three

13. Numerical evaluation of a definite integral is called --------

a.integration b.differentiation c.interpolation d.triangularisation

14. "The values of the independent variable are not given at equidistance intervals, weuse --------------- formula."

a. Newton’s forward b. Newton’s backward **c.Lagrange** d.stirling

15. " To find the unknown values of y for some x which lies at the --------- of the table, we use Newton’s Backward formula."

a.beginning b.**end** c.center d.outside

16. " To find the unknown values of y for some x which lies at the --------- of the table, we use Newton’s Forward formula."

a.beginning b.end c.center d.outside

17. " To find the unknown value of x for some y, which lies at the unequal intervals we use ------------------- formula."

a.Newton’s forward b. Newton’s backward c. **Lagrange** d.inverse interpolation

18. "If the values of the variable y are given, then the method of finding the unknown variable x is called ----------------"

a. Newton’s forward b. Newton’s backward c.interpolation d.inverse interpolation

19. In Newton’s backward difference formula, the value of n is calculated by -------.

a.n = (x–xn) / h b. n = (xn–x) / h c.n = (x–x0) / h d.n = (x0–x) / h

20. In Newton’s forward difference formula, the value x can be written as -----------.

a. x0–nh b.xn–nh c.xn + nh d.**x0 + nh**

21. In Newton’s backward difference formula, the value x can be written as ---------

a. x0–nh b. xn–nh c.**xn + nh** d.x0 + nh

22. ------------- Interpolation formula can be used for equal and unequal intervals.

a. Newton’s forward b. Newton’s backward c. **Lagrange** d.none

23. The fourth differences of a polynomial of degree four are ----------.

a.zero b.one c.two d.three

24. If the values x0 = 0, y0 = 0 and h = 1 are given for Newton's forward method, then the value of x is ------------.

A.0 b.1 c.**n** d.X

25. The second difference D2y0 is equal to

a.y2 + 2y1 –y0 b.y2 - 2y1 –y0 c. **y2 - 2y1 + y0** d. y2 + 2y1 + y0

26. The second difference D3y0 is equal to

a.**y3 – 3y2 + 3y1 – y0** b.y3 + 3y2 + 3y1– y0 c. y3 + 3y2 + 3y1+ y0 d. y3 + 3y2 + 3y1+ y3

27. The differences of constant functions are -----

a. Not equal to zero b.**zero** c.one d.two

28. Dy2 = -----

a.y2 – y3 b.y1 – y2 c.y0 – y2 d.**y3– y2**

29. yn = y0 + n Dy0 + n (n-1) / 2! D2y0 + n (n-1) (n-2)/3! D3y0 + ……is known as

a." **Newton's formula for equal intervals** " b.Bessel's formula

c."Newton's formula for unequal intervals " d."Newton's formula for Equal and unequal intervals "

30. In Newton's forward interpolation formula, the first two terms will give the -----

a.extrapolation b. **linear interpolation** c.parabolic interpolation d.interpolation

31. In Newton's forward interpolation formula, the three terms will give the ------

a.extrapolation b. linear interpolation **c.parabolic interpolation** d.interpolation

32. The difference D3f(x) is called -------differences f(x).

a.first b.fourth c.second d.**third**

33. n th difference of a polynomial of n th degree are constant and all higher order difference are

a.constant b.variable **c.zero** d,negative

34. In divided difference the value of any difference is ----- of the order of their argument

a.Independent b.dependent c.Inverse d.direct

35. Central difference equivalent to shift operator is

a. E ½ + E - ½ **b. E ½ - E - ½** c. E ½ . E - ½ d.E

36. The differences Dy are called -------differences f(x).

a.**first** b.fourth c.second d.third

37. The value (delta +1)is ____________

a.E b.h c.h2 d.h4

38. Relation between ,Δ ∇ and E

a. Δ + E = E + ∇ = ∇ + E b. Δ / E = E /∇ = Δ / E c.Δ **E = E ∇ =** Δ d.Δ + E = E +∇ =- E

39. $\Delta^2 y2$ = -----

a.Δy2 – Δy3 b. Δy1 – Δy2 c. y3 – y2 d.**Δy3– Δy2**

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DEPARTMENT OF MATHEMATICS

Subject: Numerical Methods	**Semester: III**	**L T P C**
Subject Code: 16MMU301	**Class: II-B.Sc Mathematics**	**4 0 0 4**

UNIT IV

Numerical Differentiation and Integration: Gregory's Newton's forward and backward differentiation- Trapezoidal rule, Simpson's rule, Simpsons 3/8th rule, Boole's Rule. Midpoint rule, Composite Trapezoidal rule, Composite Simpson's rule.

TEXT BOOK

T1. Jain. M.K., Iyengar. S.R.K.,and Jain R.K., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

R5: Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT IV

Numerical Differentiation:

The problem of differentiation is solved by first approximating the function by an interpolation formula and then differentiating this formula as many times as desired.

Numerical Differentiation Methods Based on Interpolation:

We know that the Lagrange's Interpolation formula is

$$f(x)=\ell_0(x)f(x_0)+\ell_1(x)f(x_1)+\ldots+\ell_n(x)f(x_n) \qquad (1)$$

where

$$\ell_0(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)},$$

$$\ell_1(x)=\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)},$$

$$\ell_n(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}$$

Now, $f'(x)$ can be obtained by differentiating f(x) w.r.t. x

Thus, $f'(x)=\ell_0'(x)f(x_0)+\ell_1'(x)f(x_1)+\ldots+\ell_n'(x)f(x_n)$.

Numerical Differentiation using Newton's Forward Difference Interpolation Formula:

We know that the Newton's forward difference interpolation formula is

$$f(x)=f_0+u\Delta f_0+\frac{u(u-1)}{2!}\Delta^2 f_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 f_0+\ldots \qquad (1)$$

where $u=\dfrac{x-x_0}{h}$ (2)

If we take the approximation of f(x) of order (n) or $O(h^n)$, then

$$f(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-(n-1))}{n!}\Delta^n f_0 \quad (3)$$

and error in this approximation is

$$E(x) = \frac{u(u-1)(u-2)\ldots(u-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi) \quad (4)$$

On differentiating equation (1) w.r.t. u, we have

$$\frac{df(x)}{du} = \Delta f_0 + \frac{2u-1}{2}\Delta^2 f_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 f_0 + \ldots$$

On differentiating equation (2) w.r.t. x, we have

$$\frac{du}{dx} = \frac{1}{h}$$

$$\Rightarrow \quad \frac{df(x)}{dx} = \frac{df(x)}{du}\frac{du}{dx} = \frac{1}{h}\left[\Delta f_0 + \frac{2u-1}{2}\Delta^2 f_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 f_0 + \ldots\right]$$

$$\Rightarrow \quad f'(x) = \frac{df(x)}{dx} = \frac{1}{h}\left[\Delta f_0 + \frac{2u-1}{2}\Delta^2 f_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 f_0 + \ldots\right] \quad (5)$$

and the error in the approximation of the first derivative of order $O(h^n)$ is

$$|E'(x_0)| = |E'(u=0)| \le \frac{h^n}{(n+1)}M_{(n+1)}$$

where $M_{(n+1)} = \max_{x_0 \le x \le x_2} \left|f^{(n+1)}(x)\right|$.

On again differentiating equation (3) w.r.t. x we have

$$f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)\frac{du}{dx}$$

$$\Rightarrow \quad f''(x) = \frac{1}{h^2}\left[\Delta^2 f_0 + (u-1)\Delta^3 f_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 f_0 + \ldots\right]$$

Numerical Differentiation using Newton's Backward Difference Interpolation Formula:

We know that the Newton's backward difference interpolation formula is

$$f(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n + \ldots \quad (1)$$

where $u = \frac{x - x_n}{h}$ (2)

If we take the approximation of f(x) of order (n) or $O(h^n)$, then

$$f(x) = f(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n + \ldots + \frac{u(u+1)(u+2)\ldots(u+(n-1))}{n!}\nabla^n f_0 \quad (3)$$

and error in this approximation is

$$E(x) = \frac{u(u+1)(u+2)\ldots(u+n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi) \quad (4)$$

On differentiating equation (1) w.r.t. u, we have

$$\frac{df(x)}{du} = \nabla f_n + \frac{2u+1}{2}\nabla^2 f_n + \frac{(3u^2+6u+2)}{6}\nabla^3 f_n + \ldots$$

On differentiating equation (2) w.r.t. x, we have

$$\frac{du}{dx} = \frac{1}{h}$$

$$\Rightarrow \quad \frac{df(x)}{dx} = \frac{df(x)}{du}\frac{du}{dx} = \frac{1}{h}\left[\nabla f_n + \frac{2u+1}{2}\nabla^2 f_n + \frac{(3u^2+6u+2)}{6}\nabla^3 f_n + \ldots\right]$$

$$\Rightarrow \quad f'(x) = \frac{df(x)}{dx} = \frac{1}{h}\left[\nabla f_n + \frac{2u+1}{2}\nabla^2 f_n + \frac{(3u^2+6u+2)}{6}\nabla^3 f_n + \ldots\right] \quad (5)$$

and the error in the approximation of the first derivative of order $O(h^n)$ is

$$|E'(x_0)| = |E'(u=0)| \le \frac{h^n}{(n+1)}M_{(n+1)}$$

where $M_{(n+1)} = \max_{x_0 \le x \le x_2} |f^{(n+1)}(x)|$.

On again differentiating equation (3) w.r.t. x we have

$$f''(x) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)\frac{du}{dx}$$

$$\Rightarrow \quad f''(x) = \frac{1}{h^2}\left[\nabla^2 f_n + (u+1)\nabla^3 f_n + \left(\frac{6u^2+18u+11}{12}\right)\nabla^4 f_n + \ldots\right]$$

Example 3: Find $\frac{dy}{dx}$ at x = 0.1 from the following table:

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.9604

Solution: The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		-0.0075		
0.2	0.9900		-0.0049	
		-0.0124		0.0001
0.3	0.9776		-0.0048	
		-0.0172		
0.4	0.9604			

We know that Newton's forward difference interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad (1)$$

where $u = \frac{x - x_0}{h}$

On differentiating w.r.t. x we have

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2)}{6}\Delta^3 y_0\right]$$

here x_0 =0.1, h = 0.1 and x=0.1, thus

$$u = \frac{x - x_0}{h} = \frac{0.1 - 0.1}{0.1} = 0$$

Thus, we have

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right] = \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = -0.050167.$$

Example 4: Find the first and second derivative of the function tabulated below at the point x = 1.1 from the following table:

x	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	0	0.1280	0.5440	1.2960	2.4320	4.0000

Solution: The difference table is:

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1.0	0					
		0.1280				
1.2	0.1280		0.2880			
		0.4160		0.0480		
1.4	0.544		0.3360		0	
		0.7520		0.0480		0
1.6	1.2960		0.3840		0	
		1.1360		0.480		
1.8	2.4320		0.4320			
		1.5680				
2.0	4.0000					

We know that Newton's forward difference interpolation formula is

$$f(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 f_0 + \ . \ . \ . \qquad (1)$$

where $u = \frac{x - x_0}{h}$

on differentiating w.r.t. x we have

$$\frac{df}{dx} = \frac{1}{h}\left[\Delta f_0 + \frac{2u-1}{2}\Delta^2 f_0 + \frac{3u^2 - 6u + 2)}{6}\Delta^3 f_0 + \ . \ . \ .\right]$$

and $$\frac{d^2 f}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f_0 + (u-1)\Delta^3 f_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 f_0 + \ . \ . \ .\right]$$

here x_0 =1.0, h = 0.2 and x=1.1, thus

$$u = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.2} = 0.5$$

Thus, we have

$$\left(\frac{df}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.1280 + \frac{2(0.5)-1}{2}(0.2880) + \frac{3(0.5)^2 - 6(0.5) + 2}{6}(0.0480) + 0\right]$$

$$\Rightarrow \quad \left(\frac{df}{dx}\right)_{x=1.1} = 0.630 .$$

and $$\left(\frac{d^2 f}{dx^2}\right)_{x=1.1} = \frac{1}{(0.2)^2}\left[0.2880 + (0.5-1)(0.0480) + 0\right] = 6.60 .$$

Example 5: Find the first derivative of the function tabulated below at the point x = 5:

x	0	1	2	3	4	5	6
f(x)	0	2.5	8.5	15.5	24.5	36.5	50

Solution: We know that the Newton's backward difference interpolation formula is

$$f(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 f_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f_n +$$

$$\frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f_n + \frac{u(u+1)(u+2)(u+3)(u+4)(u+5)}{6!}\nabla^6 f_n + \ldots$$

where $u = \frac{x - x_n}{h}$

The backward difference table is:

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$
0	0						
		2.5					
1	2.5		3.5				
		6		-2.5			
2	8.5		1		3.5		
		7		1		-3.5	
3	15.5		2		0		1
		9		1		-2.5	
4	24.5		3		-2.5		
		12		-1.5			
5	36.5		1.5				
		13.5					
6	50						

on differentiating w.r.t. x we have

$$f'(x) = \frac{1}{h}\Bigg[\nabla f_n + \frac{2u+1}{2}\nabla^2 f_n + \frac{3u^2+6u+2}{6}\nabla^3 f_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 f_n$$
$$+ \frac{5u^4+40u^3+105u^2+100u+24}{120}\nabla^5 f_n$$
$$+ \frac{6u^5+75u^4+340u^3+675u^2+548u+120}{720}\nabla^6 f_n + \ldots\Bigg]$$

here $x_n = 6$, $h = 1$ and $x=5$, thus

$$u = \frac{x - x_n}{h} = \frac{5-6}{1} = -1$$

Thus, we have

$$\left(f'(x)\right)_{x=5} = \frac{1}{1}\Bigg[13.5 + \frac{2(-1)+1}{2}(1.5) + \frac{3(-1)^2+6(-1)+2}{6}(-1.5) +$$

$$\frac{4(-1)^3+18(-1)^2+22(-1)+6}{24}(-2.5) +$$

$$\frac{5(-1)^4+40(-1)^3+105(-1)^2+100(-1)+24}{120}(-2.5)+$$
$$\left.\frac{6(-1)^5+75(-1)^4+340(-1)^3+675(-1)^2+548(-1)+120}{720}(1)\right]$$

$\Rightarrow \quad \left(f'(x)\right)_{x=5}=13.0917$.

Example 6: From the following table of values of x and f(x), find the first and second derivatives of the function at the point x = 2.2:

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
f(x)	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Solution: The backward difference table is:

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$	$\nabla^6 f$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

We know that the Newton's backward difference interpolation formula is

$$f(x)=f_n+u\nabla f_n+\frac{u(u+1)}{2!}\nabla^2 f_n+\frac{u(u+1)(u+2)}{3!}\nabla^3 f_n+\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f_n+$$
$$\frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 f_n+\frac{u(u+1)(u+2)(u+3)(u+4)(u+5)}{6!}\nabla^6 f_n+\ldots$$

where $u=\frac{x-x_n}{h}$

on differentiating w.r.t. x we have

$$f'(x)=\frac{1}{h}\left[\nabla f_n+\frac{2u+1}{2}\nabla^2 f_n+\frac{3u^2+6u+2}{6}\nabla^3 f_n+\frac{4u^3+18u^2+22u+6}{24}\nabla^4 f_n\right.$$

$$+\frac{5u^4+40u^3+105u^2+100u+24}{120}\nabla^5 f_n$$
$$\left.+\frac{6u^5+75u^4+340u^3+675u^2+548u+120}{720}\nabla^6 f_n+\ldots\right]$$

on again differentiating w.r.t. x we have

$$f''(x)=\frac{1}{h}\left[\nabla^2 f_n+(u+1)\nabla^3 f_n+\frac{12u^2+36u+22}{24}\nabla^4 f_n\right.$$
$$+\frac{20u^3+120u^2+210u+100}{120}\nabla^5 f_n$$
$$\left.+\frac{30u^4+300u^3+1020u^2+1350u+548}{720}\nabla^6 f_n+\ldots\right]$$

here x_n =2.2, h = 0.2 and x=2.2, thus

$$u=\frac{x-x_n}{h}=\frac{2.2-2.2}{0.2}=0$$

Thus, we have

$$(f'(x))_{x=2.2}=\frac{1}{0.2}\left[1.6359+\frac{1}{2}(0.2964)+\frac{1}{3}(0.0535)+\frac{1}{4}(0.0094)+\frac{1}{5}(0.0014)\right.$$
$$\left.+\frac{1}{6}(0.0001)\right]$$

$$\Rightarrow \quad (f'(x))_{x=2.2}=9.0228.$$

and $$(f''(x))_{x=2.2}=\frac{1}{0.2}\left[0.2964+0.0535+\frac{22}{24}(0.0094)+\frac{100}{120}(0.0014)+\frac{548}{720}(0.0001)\right]$$

$$\Rightarrow \quad (f''(x))_{x=2.2}=8.992.$$

In Newton Cotes Quadrature formula x_i are taken as equally spaced points from within the intervals $[a, b]$ and the weights w_i are computed by fitting a function to the $f(x_i)$ data and integrating the resulting function exactly.

The basic procedure for developing Newton – Cotes quadrature rules is to first fix the abscissas $x_0, x_1, x_2, \ldots, x_n \in [a,b]$.

Next interpolate the integrand, *f*, at the abscissas by the polynomial $P_n(x)$. Finally we integrate the interpolating polynomial and set

$I(f)$	$\approx$	$I_n(f)$	$\equiv$	$I(P_n)$
Real value of integral of original integrand		Newton Cotes formula		Real value of integral of interpolating polynomial.

Because we want the final quadrature rule to show a clear dependence on the data values $f(x_i)$, the Lagrange's form of interpolating polynomial will be used

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f(x_i)$$

$$= \sum_{i=0}^{n} \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} f(x_i)$$

$\therefore$ Newton – Cotes Quadrature formulae will take the form

$$I_n(f) = \int_a^b \sum_{i=0}^{n} L_{n,i}(x) f(x_i) dx$$

$$= \sum_{i=0}^{n} \left(\int_a^b L_{n,i}(x) dx \right) f(x_i)$$

$$= \sum_{i=0}^{n} w_i f(x_i) \quad \text{where} \quad w_i = \int_a^b L_{n,i}(x) dx$$

We have two forms of Newton – Cotes formulas which differs in their choice of the abscissas within the interval [*a*, *b*]

(i) Closed Newton – Cotes formulas which include the end points of the integration interval $x = a$ and $x = b$

Here for a given '*n*' we take $\Delta x = (b-a)/n$

and $x_i = a + i\Delta x \quad i = 0,1,2,\ldots,n$

(ii) Open Newton – Cotes formulas which do not include the end points of the integration interval

Here we take $\Delta x = (b-a)/(n+2)$

and then $x_i = a + (i+1)\Delta x \quad i = 0,1,2,\ldots,n$.

Trapezoidal Rule:

In the closed Newton – Cotes formulae

We take $n = 1$

Then $\Delta x = b - a$ and $x_0 = a \qquad x_1 = b$

$\therefore$ Lagrange's Polynomial associated with these points are

$$L_{1,0}(x) = \frac{b-x}{b-a}, \qquad L_{1,1}(x) = \frac{x-a}{b-a}$$

$\Rightarrow$ Quadrature weights are

$$w_0 = \int_a^b \frac{b-x}{(b-a)}dx \quad \text{and} \quad w_1 = \int_a^b \frac{x-a}{(b-a)}dx$$

Put $x = a + t\Delta x \quad \Rightarrow \quad dx = \Delta x\,dt$

when $x = a$, $\therefore \qquad a = a + t\Delta x \quad \Rightarrow \quad t = 0 \quad (\because \Delta x = b - a)$

$x = b \quad \therefore \quad b = a + t\Delta x \quad \Rightarrow \quad (b-a) = t(b-a) \quad \Rightarrow \quad t = 1$

$$\therefore \quad w_0 = \Delta x\int_0^1 t\,dt = \frac{\Delta x}{2} = \frac{(b-a)}{2} = w_1$$

$\therefore$ Closed Newton – Cotes Quadrature formula for $n = 1$ is

$$I(f) \approx I_{1,\text{closed}}(f) = \frac{\Delta x}{2}\left[f(a)+f(b)\right] = \frac{(b-a)}{2}\left[f(a)+f(b)\right]$$

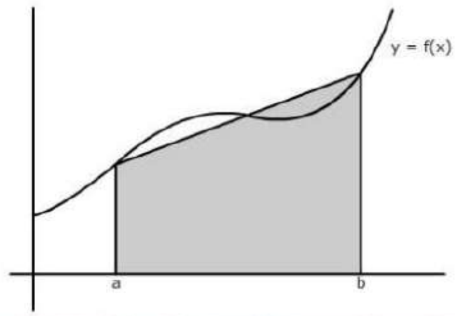

Figure 1: The Trapezoidal Rule

Geometrically, this quadrature rule approximate the value of the definite integral as the area of a trapezoid, so this rule is known as the trapezoidal rule.

Simpson's Rule:

When $n = 2$, the quadrature formulae produces a well known formulae that is, Simpson's Rule.

Here

$$\Delta x = \frac{(b-a)}{2}, \qquad x_0 = a, \qquad x_1 = a + \Delta x = (a+b)/2$$

$$x_2 = a + 2\Delta x = b$$

Weights are calculated as

$$w_0 = \int_a^b L_{2,0}(x)dx = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx$$

Put $x = a + t\Delta x \Rightarrow dx = \Delta x dt$ where $\Delta x = \frac{(b-a)}{2}$

when $x = a \Rightarrow a = a + t\Delta x \Rightarrow t = 0$

when $x = b, \Rightarrow b = a + t\left(\frac{b-a}{2}\right) \Rightarrow (b-a) = t\left(\frac{b-a}{2}\right) \Rightarrow \frac{2(b-a)}{(b-a)} = t$

$\Rightarrow\ t = 2$

$$\therefore \qquad w_0 = \int_0^2 \frac{(a + t\Delta x - a - \Delta x)(a + t\Delta x - a - 2\Delta x)}{(a - a - \Delta x)(a - a - 2\Delta x)} \Delta x\, dt$$

$$= \int_0^2 \frac{(t-1)(t-2)(\Delta x)^3}{2(\Delta x)^2} \qquad = \frac{\Delta x}{2}\int_0^2 \left[t^2 - 3t + 2\right] dt$$

$$= \frac{\Delta x}{2}\left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^2$$

$$= \frac{\Delta x}{2}\left[\frac{8}{3} - 6 + 4\right] = \frac{\Delta x}{2} \times \frac{2}{3} = \frac{\Delta x}{3}$$

Similarly

$$w_1 = \int_a^b L_{2,1}(x)\,dx = -\Delta x \int_0^2 t(t-2)\,dt = \frac{4}{3}\Delta x$$

$$w_2 = \int_a^b L_{2,2}(x)\,dx = \frac{\Delta x}{2}\int_0^2 t(t-1)\,dt = \frac{\Delta x}{3}$$

$$\therefore \qquad I(f) \approx I_{2,\text{closed}}(f) = \frac{\Delta x}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

which is known as Simpson's Rule.

3.1.3. For *n* = 3 we have Simpson's three – eighth rule:

$$I(f) = \frac{(b-a)}{8}\left[f(a) + 3f(a + \Delta x) + 3f(a + 2\Delta x) + f(b)\right]$$

where $\Delta x = \dfrac{(b-a)}{3}$

3.1.4. For *n* = 4 we have Boole's Rule:

$$I(f)=\frac{(b-a)}{90}\left[7f(a)+32f(a+\Delta x)+12f(a+2\Delta x)+32f(a+3\Delta x)+7f(b)\right]$$

where $\Delta x=(b-a)/4$

Mid – Point Rule:

The simplest open Newton – Cotes formulae corresponds to $n = 0$

$$\Rightarrow \quad \Delta x=\frac{(b-a)}{(n+2)}=\frac{(b-a)}{0+2}=(b-a)/2$$

and the only abscissa is $x_0=(a+b)/2$

The quadrature weight is

$$w_0=\int_a^b L_{0,0}(x)dx=\int_a^b dx=b-a$$

$\therefore$ Open Newton – Cotes Quadrature formulae is

$$I(f)\approx I_{0,\text{open}}(f)=(b-a)\times f\left(\frac{a+b}{2}\right) \qquad (1)$$

The formulae given by (1) is known as Mid – Point Rule

3.2.2. For $n = 1$, $\Delta x=\frac{(b-a)}{(n+2)} = \frac{(b-a)}{3}$

and the abscissa are $x_0=a+\Delta x$

$$x_1 = a + 2\Delta x$$

Quadrature weights are

$$w_0 = \int_a^b L_{1.0}(x) = \int_a^b \frac{x - x_1}{x_0 - x_1} dx = \int_a^b \frac{x - (a + 2\Delta x)}{(a + \Delta x - a - 2\Delta x)}$$

$$= \int_a^b \frac{a + 2\Delta x - x}{\Delta x} \quad \text{Put } x = a + t\Delta x, \quad dx = \Delta x dt$$

$$= \Delta x \int_0^3 (2 - t) dt = \Delta x \left[2t - \frac{t^2}{2} \right]_0^3$$

$$= \Delta x \left[6 - \frac{9}{2} \right] = \frac{3\Delta x}{2}$$

when $x = a,$ $t = 0$

$$x = b = t = \frac{b-a}{\Delta x} = \frac{(b-a)}{\left(\frac{b-a}{3}\right)} = 3$$

$$\therefore \quad w_0 = \frac{3\Delta x}{2}$$

Similarly

$$w_1 = \int_a^b L_{1,1}(x) dx = \int_a^b \frac{(x - x_0)}{x_1 - x_0} dx = \int_a^b \frac{x - (a + \Delta x)}{(a + 2\Delta x - a - \Delta x)} dx$$

$$= \int_a^b \frac{x - (a + \Delta x)}{\Delta x} dx = \frac{3\Delta x}{2}$$

Here also we put $x = a + t\Delta x$

we have

$$\Delta x = \frac{(b-a)}{3}$$

$\therefore$ For $n = 1$ Open Newton Cotes formulae is

$$I(f) \approx I_{0,\text{open}}(f) = \frac{(b-a)}{2}\left[f(a+\Delta x) + f(a+2\Delta x)\right]$$

Composite Trapezoidal Rule:

We know that

$$I(f) = I_{1,\text{closed}}(f) + \text{error}$$

$$= \frac{(b-a)}{2}\left[f(a) + f(b)\right] - \frac{(b-a)^3}{12} f''(\xi) \quad (1)$$

If the integration interval [a, b] is split into n subintervals by defining $h = (b-a)/n$ and $x_j = a + jh;\ 0 \le j \le n$, and then the trapezoidal rule is applied on each subinterval $\left[x_{j-1}, x_j\right]$.

We get

$$I(f) = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx$$

$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})}{2}\left[f(x_{j-1}) + f(x_j)\right] - \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^3}{12} f''(\xi_j)$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right] - \frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j)$$

Composite trapezoidal rule Error

$(\because h = x_j - x_{j-1})$ for each j and $x_{j-1} < \xi_j < x_j$ (2)

Composite Simpson's Rule:

Since the basic Simpson's rule divides the interval [a, b] into two pieces

$\therefore$ For Simpson's composite rule, we divide the interval [a, b] into even no. of subinterval

$\therefore$ Let $n = 2m,$ define $h = \frac{b-a}{n} = \frac{b-a}{2m}$

$x_i = a + ih \quad (0 \le i \le 2m)$

and apply Simpson's rule m times once over each subinterval $\left[x_{2j-2}\ x_{2j}\right],\ j = 1,2,...,m$

$$\therefore \quad I(f) = \sum_{j=1}^{m} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$$

$$= \sum_{j=1}^{m} \frac{(x_{2j} - x_{2j-2})}{6}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \sum_{j=1}^{m} \frac{(x_{2j} - x_{2j-2})^5}{2880} f^4(\xi_j)$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2m})\right] - \frac{h^3}{90}\sum_{j=1}^{m} f^{(4)}(\xi_j)$$

$(\because x_{2j} - x_{2j-2} = 2h)$

Provided f has four continuous derivatives.

Similar to composite Trapezoidal Rule

We can find a number $\xi \in [a,b]$ such that $f^4(\xi) = \frac{1}{m}\sum_{j=1}^{m} f^{(4)}(\xi_j)$

$\therefore$ Error Term is $\frac{-h^5 m}{90} f^{(4)}(\xi) = \frac{-(b-a)h^{(4)}(\xi)}{180}$

where $hm = (b-a)/2$

Hence the composite Simpson's rule is

$$I(f) = \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2j}) + f(x_{2m})\right] - \frac{(b-a)h^4}{180} f^{(4)}(\xi)$$

Note: The composite Simpson's Rule has rate of convergence $O(h^4)$.

Evaluate $I = \int_0^1 \frac{1}{1+x}dx$ correct to three decimal places.

Solution: We solve this example by both the Trapezoidal and Simpson's rules with $\Delta x = 0.5, 0.25, 0.125$

$f(x)=\frac{1}{1+x}$

(i) $\Delta x=0.5$ the values of x and $f(x)$ are

x	0	0.5	1.0
$f(x)$	1.0000	0.6667	0.5

(a) Trapezoidal rule give

$$I=\frac{1}{4}\left[1.0000+2(0.6667)+0.5\right]=0.7084$$

(b) Simpson's rule gives

$$I=\frac{1}{6}\left[1.0000+4(0.6667)+0.5\right]=0.6945$$

(ii) $\Delta=0.25$ the tabulated values of x and $f(x)$ are

x	0	0.25	0.50	0.75	1.0
$f(x)$	1.0000	0.8000	0.6667	0.5714	0.5

(a) Trapezoidal rule gives

$$I=\frac{1}{8}\left[1.0+2(0.8000+0.6667+0.5714)+0.5\right]=0.6970$$

(b) Simpson's rule gives

$$I=\frac{1}{2}\left[1.0+4(0.8000+0.5714)+2(0.6667)+0.5\right]=0.6932$$

(iii) Finally we take $\Delta x=0.125$

The tabulated values of x and $f(x)$ are

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1

$f(x)$	1	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{16}\left[1.0 + 2(0.8889 + 0.8000 + 0.7273 + 0.6667 + 0.6154\right.$$

$$\left. + 0.5714 + 0.5333) + 0.5\right]$$

$$= 0.6941$$

(b) Simpson's rule gives

$$I = \frac{1}{24}\left[1.0 + 4\left(0.8889 + 0.7273 + 0.6154 + 0.5333\right)\right.$$

$$\left. + 2\left(0.8000 + 0.6667 + 0.5714\right) + 0.5\right]$$

$$= 0.6932$$

The exact value of Integral $I = \log_e(2) = 0.693147$

$\therefore$ Approximate value can be taken = 0.693

This example demonstrates that in general Simpson's rule yields more accurate results than the Trapezoidal rule.

Possible Questions

PART-A (2 Mark)
UNIT IV

1. Write the formula for Newton forward difference formula for derivatives.
2.Write the formula for Newton backward difference formula for derivatives.
3. Write the Simpson's 3/8th rule formula.
4. Write Boole's rule formula.
5. Write the Simpson's 3/8th rule formula

PART-B (6 Mark)

1.The population of a certain town is given below, Find the rate of growth of population in 1931, 1941, 1961 and 1971.
Year : 1931 1941 1951 1961 1971
Population : 40.62 60.80 79.95 103.56 132.65
in thousands
2.Evaluate using Trapezoidal rule with h = 0.2. Hence obtain the approximate 1 2 0 1 dxx □ □
value of □ .
3.Find the gradient of the road at the middle point of the elevation above a
datum line of seven point of road which are given below:
X : 0 300 600 900 1200 1500 1800
Y : 135 149 157 183 201 205 193
4. By dividing the range into the ten equal parts .Evaluate by Trapezoidal rule and Simpson's rule. 0
sin xdx □ □
5. From the following table of half-yearly premium for policies maturing at
different ages. Estimate the premium for policies maturing at age 46 & 63.
Age x : 45 50 55 60 65
Premium y : 114.84 96.16 83.32 74.48 68.48
6. Find the value of y at x = 1.05 from the table given below.
x : 1.0 1.1 1.2 1.3 1.4 1.5
y: 0.841 0.891 0.932 0.964 0.985 1.015
7. Find the first and second derivative of the function tabulated below atx = 0.6
X : 0.4 0.5 0.6 0.7 0.8
Y : 1.5836 1.7974 2.0442 2.3275 2.6511
8. Evaluate by (i) Trapezoidal rule (ii) Simpson's rule. Also check up the result by actual integration.
9. Given the following data, find and the maximum value of y. '(6) y
X : 0 2 3 4 7 9
Y : 4 26 58 112 466 922
10. Evaluate I= $\int dx\ /\ (1+x)$ 60 using both of the Simpson's rule

KARPAGAM UNIVERSITY
COIMBATORE-21
DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY: Ms.M.LATHA(KU0969)
SUBJECT: NUMERICAL METHODS
SUBJECT CODE: 16MMU301
CLASS: II B.Sc MATHEMATICS

POSSIBLE QUESTIONS
UNIT IV

2 MARKS

1. Write the formula for Newton forward difference formula for derivatives.
2.Write the formula for Newton backward difference formula for derivatives.
3 2. th rule formula.
.
5 2. th rule formula.

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POSSIBLE QUESTIONS
UNIT IV

6 MARKS

1.The population of a certain town is given below, Find the rate of growth of population in 1931, 1941, 1961 and 1971.

Year	: 1931	1941	1951	1961	1971
Population in thousands	: 40.62	60.80	79.95	103.56	132.65

2.Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h = 0.2. Hence obtain the approximate value of π .

3.Find the gradient of the road at the middle point of the elevation above a datum line of seven point of road which are given below:

X	:	0	300	600	900	1200	1500	1800
Y	:	135	149	157	183	201	205	193

4. By dividing the range into the ten equal parts .Evaluate $\int_0^{\pi} \sin x dx$ by Trapezoidal rule and Simpson's rule.

5. From the following table of half-yearly premium for policies maturing at different ages. Estimate the premium for policies maturing at age 46 & 63.

Age x :	45	50	55	60	65
Premium y :	114.84	96.16	83.32	74.48	68.48

6. Find the value of y at x = 1.05 from the table given below.

x :	1.0	1.1	1.2	1.3	1.4	1.5
y:	0.841	0.891	0.932	0.964	0.985	1.015

7. Find the first and second derivative of the function tabulated below atx = 0.6

X	: 0.4	0.5	0.6	0.7	0.8
Y	: 1.5836	1.7974	2.0442	2.3275	2.6511

8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule. Also check up the result by actual integration.

9. Given the following data, find $y'(6)$ and the maximum value of y.

X	:	0	2	3	4	7	9
Y	:	4	26	58	112	466	922

10. Evaluate I= $\int_0^6 dx\ /\ (1+\ x)$ using both of the Simpson's rule.

UNIT-IV

1. " If the given integral is approximated by the sum of ‘n’ trapezoids, then the rule is called as ----------------."
   a.Newton's method b.**Trapezoidal rule** c.simpson's rule d. power
2. The order of error in Trapezoidal rule is -------------.
   a.h b.h^3 c.$\mathbf{h^2}$ d.h^4
3. The general quadratic formula for equidistant ordinates is _________________
   a.raphson **b.Newton-cote's** c.interpolation d.divide difference
4. h/2[(sum of the first and last ordinates)+2(sum of the remaining ordinates)] is______________
   a.simphson's 3/8 b.simphson's 1/3 **c.trapezoidal** d.taylor series
5. Use trapezoidal rule for y(x) ________________

   a.linear b.second degree c.third degree d.degree n

6. " Simpson’s rule is exact for a ----------------- even though it was derived for a Quadratic."
   a.cubic b.less than cubic **c.linear** d.quadratic
7. What is the order of the error in Simpson’s formula?
   A.Four B.three c.two d.one
8. Simpson's 1/3 is findind y(x) upto _____________
   a.linear **b.second degree** c.degree n d.third degree
9. In simpson's 1/3, the number of intervels must be ____________
   a.any integer b.odd **c.even** d.prime
10. In simpson's 1/3, the number of ordinates must be ________________
    a.any integer b**.odd** c.even d.prime
11. Simpson’s one-third rule on numerical integration is called a --------- formula.
    a.closed b.open c.semi closed d.semi opened
12. In simphson's 3/8 rule, we calculate the polynomial of degree _________
    a.degree n b.linear c.second degree d.**third degree**
13. The number of interval is multiple of three the use__________
    a.simpson's 1/3 . b.trapezoidal c.**simpson's 3/8** d.taylor series
14. The number of interval is multiple of six ______________

a. a.simpson's 1/3 b.simphson's 3/8 c.**weddle** d.trapezoidal

15. The error in Simpson's 1/3 is -------------.
    a.h b.h^3 c.h^2 d.**h^4**
16. Modulus of E is ________
    a.<M(b-a)h4/180 b.0 c.>M(b-a)h4/180 d.M(b-a)h4/180
17. The order of error is h^2 for_____________
    a.lagrange's b.**trapezoidal** c.weddle d.simpson's 1/3
18. h^4 is the error of ________
    a.simphson's 3/8 **b.simphson's 1/3** c.trapezoidal d.taylor series
19. The value of integral ex is evaluated from 0 to 0.4 by the following formula. Which method will give the least error ?
    a.Trapezoidal rule with h = 0.2 b. Trapezoidal rule with h = 0.1 c.**Simpson's 1/3 rule with h = 0.1**. d.weddle

20. Using Simpson's rule the area in square meters included between the chain line, irregular boundary and the first and the last offset will be _______
    a. 7.33.28 sq-m b.744.18 sq-m c.880.48 sq-m. **d.820.38 sq-m**
21. By putting n = 1 in Newton cote's formula we get ----------- rule.

    a.Simpson's 1/3 rule b.Simpson's 3/8 rule c.**Trapezoidal rule** d.Simpson's rule

22. " I = (3h / 8) { (y0 + yn) + 3 (y1 + y2 + y4 + y5 + …..)+2(y3 + y6 + y9 + …..) } is known as ------------------."

    a.Simpson's 1/3 rule b.**Simpson's 3/8 rule** c.Trapezoidal rule d.Simpson's rule

23. "I = (h / 3) { (y0 + yn) + 2 (y2+ y4 + y6 + y8 + …..)+ 4(y1 + y3 + y5 + …..) } is known as ------------------."
    a.Simpson's 1/3 rule b.Simpson's 3/8 rule c.Trapezoidal rule d.Simpson's rule
24. The differentiation of logx is ________
    a.**1/x** b.e(x) c.sinx d.cosx
25. ∫ f(x) dx of (a, b) is _____________
    a.F(a) b.F(a+b) c.F(b-a) **d.F(b)-F(a)**
26. h/3[(sum of first and last ordinates)+2(sum of even ordinates)+4(sum of odd ordinates)] is the formula for ___________
    a.trapezoidal b.**simphson's 1/3** **c.**simphson's 3/8 d.taylor series

27. In simpson 1/3 rule, the integral value is h/3[y0+4(y1)+y2] _______________
    a. a.for n=1 b.**for n=2** c.for n=3 d.for n=4
28. Differentiation of sinx is _________________
    a.cosx **b.**tanx c.sinx d.logx
29. Integration of cosx ______________
    a.cosx b.tanx **c.sinx** d.logx
30. If y(x) is linear then use ____________________
    a.simphson's 3/8 b.simphson's 1/3 c.**trapezoidal** d.taylor series
31. The differentiation of secx is _____________
    a.secx tanx b.cotx c.cosecx d.tanx
32. The notation h is ___________ differece of ordinates
    a.sum of ordinates b.number of ordinates c.product of ordinates **d.difference of ordinates**
33. While evaluating the definite integral by Trapezoidal rule, the accuracy can be increased by taking----------
    a.Large number of sub-intervals b.even number of sub-intervals c.multipleof6 d.has multiple of 3
34. Numerical integration when applied to a function of a single variable, it is known as--------
    a.maxima b.minima c.**quadrature** d.quadrant

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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DEPARTMENT OF MATHEMATICS

Subject: Numerical Methods	**Semester: III**	**L T P C**
Subject Code: 16MMU301	**Class: II-B.Sc Mathematics**	**4 0 0 4**

UNIT V

Ordinary Differential Equations: Taylor's series - Euler's method – modified Euler's method - Runge-Kutta methods of orders two and four.

TEXT BOOK

T1. Jain. M.K., Iyengar. S.R.K.,and Jain R.K., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

R2. Gerald C.F. and Wheatley P.O., (2006). Applied Numerical Analysis, Sixth Edition, Dorling Kindersley (India) Pvt. Ltd., New Delhi.

R4. John H., Mathews and Kurtis D. Fink., (2012). Numerical Methods using Matlab, Fourth Edition., PHI Learning Private Limited.

Initial value problem:

A general solution of a differential equation of n^{th} order has n arbitrary constants. It will be of the form $f(x,y,c_1,c_2,\ldots c_n) = 0$. if n conditions are given we can obtain the values of the constants $c_1,c_2,\ldots c_n$. If all the n conditions are specified at the initial point only, then the problem is called an initial value problem

Boundary value problem:

A general solution of a differential equation of n^{th} order has n arbitrary constants. It will be of the form $f(x_1, y_1, c_1, c_2, \ldots c_n) = 0$. If n conditions are given we can obtain the values of the constants c_1, $c_2 \ldots c_n$ If n conditions are specified at more than one point, then the problem is called a boundary value problem

Particular solution:

A most general from of an ordinary differential equation is given by $Q\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0$ we know that the general solution of a differential equation of n^{th} order has n arbitrary constants. If we give particular values to the constants, the solution is said to be a particular solution.

Formula for Taylor series:

$$Y_{n+1} = y_n + \frac{h}{1!} y_n^{I} + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \ldots\ldots$$

Formula for Euler's method or Euler's algorithm:

$Y_{n+1} = y_n + h\, f(x_n, y_n)$, n = 0,1,2,.....

Formula for improved Euler's method?:

$Y_{n+1} = y_n + ½ h [f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$

Formula for modified Eulers method:

$Y_{n+1} = y_n + h[f(x_n + h/2, y_n + h/2 f(x_n, y_n))$

Formula for fourth order Runge-kutta method:

$K_1 = h f(x,y)$

$K_2 = h f(x + h/2, y + k_1/2)$

$K_3 = h f(x + h/2, y + k_2/2)$

$K_4 = h f(x + h, y + k_3)$

$$\Delta y = \frac{1}{6}(K_1 + 2k_2 + 2K_3 + k_4)$$

$$y(x+h) = y(x) + \Delta y$$

Runge-kutta method for simultaneous first order differential equations:

To solve numerically the simultaneous equations

$\frac{dy}{dx} = f_1(x, y, z)$, *and* $\frac{dz}{dx} = f_2(x, y, z)$ *given* the initial conditions $y(x_0) = y_0$,

$z(r_0) = Z_o$

we starting from (x_0, Y_0, z_0) the increments Δy and ΔZ in y and z respectively are given by formulae

$K_1=hf, (x_0,y_0,z_0)$ $l_1 = hf_2 (x_o, y_o, z_o)$

$$K_2 = hf_1(x_0+\frac{h}{2},y_0+\frac{k_1}{2}+Z_0+\frac{l_1}{2}) \quad l_2 = hf_2(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2})$$

$$K_3 = hf_1(x_0+\frac{h}{2},y_0+\frac{k_2}{2}+Z_0+\frac{l_2}{2}) \quad l_3 = hf_2(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2})$$ where $h = \Delta x$

$$K_4 = hf_1(x_0+h,y_0+k_3,Z_0+l_3) \quad l_4 = hf_2(x_0+h, y_0+k_3, z_0+l_3)$$

$$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) \quad \Delta z = \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$y_1=y_0+\Delta y$ and $z_1=z_0+\Delta z$

having got (x_1,y_1,z_1) we get (x_2,y_2,z2) by repeating the above algorithm once again starting from (x_1,y_1,z_1)

Runge-kutta method for second order differential equation (or R-K-method of order from to solve y^{II} = f(x,y, y^1), given $y(x_0)= y_0$ and $y^1(x_0) =y_0^1$?

To solve $y^{II} =f(x,y,y^1)$, given $y(x_0) =y_0$ $y^1(x_0) = y^1{}_0$

Now, set $y^1=Z$ and $y'' = z^1$

Hence, differential equation reduce to $\frac{dy}{dx} = y^1 = z$ and

$\frac{dz}{dx} = z^1 = y''$ = f (x, y, y”) = f (x, y, z)

$\therefore \frac{dy}{dx} = z$ and $\frac{dz}{dx}$ = f (x, y, z) are simultaneous equation Where f_1 (x, y, z) = z, f_2 (x, y, z) = f (x, y, z) given

dx dy

Also y (0) and z (0) are given

Starting from these equations, we can use the R – K method for simultaneous equation and solve the problem.

Milne's predictor formula:

$$Y_{n+1,\,P} = y_{n-3} + \frac{4h}{3}\,(2y_{n-2}^{1} - y^{1}_{n-1} + 2y^{1}_{n})$$

Milne's corrector formula:

$$Y_{n+1,\,C} = y_{n-1} + \frac{h}{3}\,(y^{1}_{n-1} + 4y^{1}_{n} + y^{1}_{n+1})$$

Adam – Bashforth predictor formula:

$$Y_{n+1,\,P} = y_{n} + \frac{h}{24}\,[55y^{1}_{n} - 59y^{1}_{n} + 37y^{1}_{n-2} - 9\,y^{1}_{n-3}]$$

Adam – Bashforth corrector formula:

$$Y_{n+1,\,C} = y_{n} + \frac{h}{24}\,[9y^{1}_{n+1} + 19y^{1}_{n} - 5y^{1}_{n-1} + y^{1}_{n-2}]$$

Relation between Runge – kutta method of second order and modified Euler's method:

In second order Runge – kutta method,

$$\Delta_{y0} = k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$\Delta y_0 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{1}{2}\,h\,f\,(x_0, y_0)\right)$$

$\therefore y_1 = y_0 + \Delta y_0 + y_0 + hf\,(_x$is is exactly the modified Euler method

So, the Runge – kutta method of second order is nothing but the modified Euler method.

Numerical Examples:

01. Using Taylor series method, find correct to four decimal places, the values of y (0.1), given $\frac{dy}{dx} = x^2$ $+y^2$ and y (0) = 1

Solution:

We have $y^1 = x^2 + y^2$

$Y^{ii} = 2x + 2yy'$

$Y^{iii} = 2 + 2yy'' + 2'^2$

$Y^{iv} = 2yy^{iii} + 2y^iy^{ii} + 4y^iy^{ii}$

$= 2yy^{iii} + 6y^iy^{ii}$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = 0.1$, $y_1 = y\ (0.1) = ?$

$Y_0^1 = x_0^2 + y_0^2 = 0 + 1 = 1$

$Y_0^{ii} = 2x_0 + 2y_0y_0^1 = 2$

$Y_0^{iii} = 2 + 2(1)\ (2) + 2\ (1)^2 = 8$

$Y_0^{iv} = 2 \times 1 \times 8 + 6\ (1)\ (2) = 28$

By Taylor series method

$Y_1 = y_0 + \frac{h}{1!} y_0^1 + \frac{h^2}{2!} y_0^2 + \frac{h^3}{3!} y_0^3 + \ldots\ldots$

Y (0.1) = y_1 = 1+

$\frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(28) + \ldots.$

= 1 + 0.1 +0.01 +0.0013333 + 0.000116666

= 1.11144999

= 1.11145

02. Using Taylor series method, find y (1.1) correct to four decimal places given $\frac{dy}{dx}$=x$y^{1/3}$ and y (1) = 1

Solution:

Take x_0 =1, y_0 = 1, h =0.1

$Y^1 = xy^{1/3}$

$Y^{ii} = \frac{1}{3} xy^{-2/3} y^1 + y^{2/3}$

$= \frac{1}{3} x^2 y^{-1/3} + y^{1/3}$

$y^{iii} = \frac{x^2}{3}\left(\frac{-1}{3}\right) y^{-\frac{4}{3}} y^1 + \frac{2x}{3} y^{-\frac{1}{3}} + \frac{1}{3} y^{-\frac{2}{3}} y^1$

$y_0^1 = 1\,(1)^{1/3} = 1$

By Taylor series Y_1 = y (1.1) = 1+0.1 + $\frac{(0.2)^2}{2}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{6}\left(\frac{8}{9}\right) + \ldots\ldots$

= 1+0.1 + 0.00666 + 0.000148 + …….

= 1.10681

03. Using Taylor series method, find y (0.1) given $\frac{dy}{dx} = x^2 - y$, y (0) = 1 (correct to 4 decimal places)

Solution:

$X_0 = 0$, $y_0 = 1$, h =0.1, x_1 =0.1

$Y^1 = x^2 - y$

$Y^{ii} = 2x - y^1$

$Y^{iii} = 2 - y^{ii}$

$Y^{iv} = - y^{iii}$

$Y_0^1 = x_0^2 - y_0 = 0 - 1 = -1$

$Y_0^{11} = 2x_0 - y_0^1 = 0 - (-1) = 1$

$Y_0^{iii} = 2 - 1 = 1$

$Y_0^{iv} = - 1$

$\therefore$ y (o.1) = 1+0.1 (-1)+

$$\frac{0.01}{2}(1) + \frac{(0.001)}{6}(1) + \frac{(0.0001)}{24}(-1, +.....)$$

=0.905125

04. Given $y^1 = - y$ and y (0) = 1, determine the value of y at x = (0.01) (0.01)

(0.04) by Euler method Solution:

$Y^1 = - y$, $x_0 =0$, $y_0 =1$, $x_1 = 0.01$, $x_2 = 0.02$, $x_3 = 0.03$, $x_4 = 0.04$

We have to find y_1, y_2, y_3, y_4 takes h = 0.01

By Euler algorithm, $y_{n+1} = y_n + hy_n^1 = y_n + hf(x_n, y_n)$

$Y_1 = y_0 + h\, f(x_0, y_0) = 1 + (0.01)(-1) = 0.99$

$Y_2 = y_1 + hy_1^1 = 0.99 + (0.01)(-y_1)$

$= 0.99 + (0.01)(-0.99)$

$=0.9801$

$y_3 = y_2 + hf(x_2, y_2) = 0.9801 + (0.01)(-0.9801)$

$= 0.9703$

$y_4 = y_3 + h\, f(x_3, y_3) = 0.9703 + (0.01)(-0.9703) = 0.9606$

05. Compute y at x = 0.25 by modified Euler method

given $y^1 = 2xy$, y (0)= 1 Solution:

Here $f(x, y) = 2xy$, $x_0 = 0$, $y_0 = 1$

Take $h = 0.25$, $x_1 = 0.25$

By modified Euler method

$$Y_1 = y_0 + h\,[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

$f(x_0, y_0) = f(0, 1) = 2(0)(1) = 0$

$\therefore y_1 = 1 + 0.25\, [6\, (0.125, 1)]$

$= 1+0.25\, [2 \times 0.125, 1]$

$=1 = 0.25\, [2 \times 0.125 \times 1]$

$= 1.0625$

06. Solve $\frac{dy}{dx}$ =-2x – y, y (0) = -1 by Taylor series method to find y (0.1) compare it with exact solution?

Solution:

Here $x_0 = 0$, $y_0 = -1$, $h = 0.1$

$Y^1 = -2x - y$

$Y^{ii} = -2 - y^1$

$Y^{iii} = -y^{ii}$

$Y^{iv} = -y^{iii}$

$Y_0^1 = -2x_1 - y_0 = 1$

$Y_0^{11} = -2 - 1 = -3$

$Y_0^{iii} = 3$

$Y_0^{iv} = -3$

$$\therefore y_1 = 1 + \frac{0.1}{1!} \times 1 + \frac{(0.1)^2}{2!} \times (-3) + \frac{(0.1)^3}{3!} \times 3 + \frac{(0.1)^4}{4!} \times (-3) + \ldots.$$

= 1+ 0.1 -0.015 +0.0005 – 0.0000125

= -0.91451

07. Solve $\frac{dy}{dx} = x(1+x^3y)$, y (0) = 3 by Euler's method for y (0.1)

Solution:

$X_0 = 0$, $y_0 = 3$, $h = 0.1$, $x_1 = 0.1$

By Euler's algorithm is $y_1 = y_0 + hf(x_0, y_0)$

= 3 +0.1 f (0, 3) = 3 + 0.1(0)

= 3

Solution:

$X_0 = 0$, $y_0 = 1$, $x_1 = 0.1$

By Euler algorithm, $y_1 = 1 + 0.1\,[2 \times 0 + 3 \times 1] = 1.3$

$Y_2 = y_1 + hf(x_1, y_1)$

$= 1.3 + 0.1f\,[0.1, 1.3]$

$= 1.3 + 0.1\,[2 \times 0.1 + 3 \times 1.3] = 1.71$

08. Obtain the values of y at x = 0.1 using Runge – kutta method of fourth order for the differential equation $y^1 = -y$, given y (0) = 1

Solution:

Here $f(x, y) = -y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$

$K_1 = hf(x_0, y_0) = 0.1\,f(0, 1) = -0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f(0.05, 0.95) = -0.095$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 \frac{K_2}{2}\right) = (0.1)\,f(0.05, 0.9525) = -0.09525$

$K_4 = hf(x_0 + h, y_0 + K_3) = (0.1)\,f(0.1, 0.90475) = -0.090475$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_1 = y_0 + \Delta y = 0.9048375$

10. Compute y (0.3) given $\frac{dy}{dx} + y + xy^2 = 0$, y (0) = 1 by taking h = 0.1 using R.K method of fourth order?

Solution:

$Y^1 = -(xy^2 + y) = f(x, y)$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$ $x_1 = 0.1$

$K_1 = h\,f(x_0, y_0) = 0.1\,[-(x_0 y_0^2 + y_0)] = -0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = -0.1\,[(0.05)(0.95)^2 + 0.95] = -0.0995$

$K_3 = hf\left(x_0 ? \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\,f(o.1, 0.9005) = -0.0982$

$\therefore y_1 1$
$= 1 + 6 [-0.1 + 2 (-0.0995) + 2 (-0.0995) – 0.0982]$

$= 0.9006$

11. What are the values of k_1 and l_1 to solve $y^{11} + xy^1 + y = 0$; $y(0) = 1$, $y^1(0) = 0$ by Runge kutta method of fourth order

$y^{11} = -xy^1 – y$, $x_0 = 0$, $y_0 = 1$

Setting $y^1 = z$, the equation becomes $y^{11} = z^1 = -xz – y$

$\therefore \frac{dy}{dx} = z = 6, (x, y, z)$, $\frac{dz}{dx} = -xz – y = f_2 (x, y, z)$

given $y_0 = 1$, $z_0 = y_0^1 = 0$

By algorithm, $k_1 = hf_1 (x_0, y_0, z_0) = 0.1 f_1 (0, 1, 0) = 0$

$L_1 = hf_2 (x_0, y_0, z_0) = 0.1 f_2 (0, 1, 0) = -1 (0.1) = -0.1$

12. What are the values of k_1 and l_1 solve $y^{11} + 2xy^1 - 4y = 0$, $y(0) = 0.2$, $y^1(0) = 0.5$.

Solution:

Let $\frac{dy}{dx} = z$ then $\frac{d^2 y}{dx^2} = \frac{dz}{dx}$

the given differential equation becomes

$\frac{dz}{dx} = -2xz + 4y$ now $\frac{dy}{dx} = z$ and $\frac{dz}{dx} = 2xz + 4y$

$x_0 = 0, y_0 = 0.2$ $h = 0.2$ $f_1(x,y,z) = z$, $f_2(x_1, x_2, x_3) = -2x2 + 4y$ $K_1 = hf_1(x_0, y_0, z_0) = 0.1 \times 0.5 = 0.05$,

$l_1 = ht_2(x_0, y_0, z_0) = 0.1[-2 \times 0 \times 0.5 + 4 \times .5] = 0.8$

13. What are the values of k_1 and l_1 to solve $y^{11} – x^2 y^1 – 2xy = 1$ $y(0) = 1$, $y^1(0) = 0$

Solution:

dy

Let $\frac{}{dx}$=z

$\therefore$ The given differential equation becomes $\frac{d^2y}{dx^2}$ = x^2y^1 +2xy+1

$\frac{dz}{dx}$ $=x^2z$ +2xy +1, x_0 =0, y_0 = 1, z_0 =0, f_1 (x, y, z) = z

f_2 (x, y, z) = x^2z +2xy +1, h = 0.1

x_1 =$hf_1(x_0, y_0, z_0)$= 0.1 f (0, 1, 0) = 0.1 $\times$ 0 = 0

l_1 =hf_2 (x_0, y_0, z_0) = 0.1

14. What are the values of k_1, k_2, l_1 and l_2 from the system of equations, $\frac{dy}{dx}$ = x +z, $\frac{dz}{dx}$ = x – y given y (0) =2, z (0) = 1 using Runge – Kutta method of fourth order.

Solution:

f_1 (x, y, z) = x + z; f_2 (x, y, z) = x –y

X_0 = 0, y_0 =2, z_0 =1, h = 0.1

Now

K_1 = hf_1 (x_0, y_0, z_0)

= (0.2) f_1 (0, 2, 1)

= (0.1) (0+1)

= 0.1

$K_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= 0.1\ f_1\ (0.05, 2.05, 0.8)$

$= 0.085$

$l_1 = (0.1)\ f_2\ (0, 2, 1)$

$= (0.1)\ (0 - 2^2)$

$= -0.4$

$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1)\ f_2\ (0.05, 2.05, 0.8)$

$= -0.41525$

15. Solve by Euler's method $\frac{dy}{dx} = x^2 + y$, y (0) = 1 of x = 0.02, 0.04

Solution:

Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x^2 + y$, $h = 0.2$

By Euler's algorithm, $y_1 = y_0 + h\ f(x_0, y_0)$

i.e. $y_1 = 1 + 0.02\ (x_0^2 + y_0) = 1.02$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.02 + 0.02\ [(0.02)^2 + 1.02]$

$= 1.04041$

16. Solve $\frac{dy}{dx}$ =x + y, given y (1) = 0 and get y (1.1) by Taylor series method?

Solution:

Here $x_0 = 1$, $y_0 = 0$, h = 0.1

$Y^1 = x + y$

$Y^{ii} = 1+y^1$

$Y^{iii} = y^{ii}$

$Y^{iv} = y^{iii}$

$Y_0^1 = x_0 + y_0 = 1 + 0 = 1$

$Y_0^{11} = 1 + y_0^1 = 2$

$Y_0^{iii} = 2$

$Y_0^{iv} =$

By Taylor series, we have

$$Y_1 = y_0 + \frac{h}{1!} y_0^1 + \frac{h^2}{2!} y_0^{11} + \frac{h^3}{3!} y_0^{111} + \ldots.$$

$Y_1 = y(1.1) = 0 +$

$$\frac{0.1}{1}(1) + \frac{(0.1)^2}{2} 2 + \frac{(0.1)^3}{6} \times 2 + \frac{(0.1)^4}{24} \times 2 + \ldots\ldots$$

= 0.11033847

17. Using Taylor method, compute y (0.2) correct to 4 decimal places given $\frac{dy}{dx}$ = 1 – 2xy and y (0) = 0

Solution:

Here $x_0 = 0$, $y_0 = 0$, h =0.2

$Y^1 = 1 – 2xy$

$Y^{11} = -2 (xy^1 + y)$

$Y^{iii} = -2 [xy^{11} + 2y^1]$

$Y^{iv} = -2 [xy^{iii} + 3y^{11}]$

$Y^v = -2 (xy^{iv} + 4y^{iii}]$

$Y_0^1 = 1 – 2.0.0 = 1$

$Y_0^{11} = 0$

$Y_0^{111} = -4$

$Y_0^{iv} = 0$

$Y_0^v = 32$

By Taylor series,

Y_1 =y (0.2) = 0 +

$$\frac{0.2}{}(1)+\frac{(0.2)^2}{}(0)+\frac{(0.2)^3}{}(-4)+0+\frac{(0.2)^5}{}(32)+ \ldots.$$

1 2 6 120

= 0.1948

18. Solve dy/dx = x+y, given y(1) =0, and get y(1.1), y(1.2) by Taylor series method. Compare your result with the analysis.

Solution:

Here x_0= 1, y_o =0x =0.1

Y^1=x + y y_0^I= x0+y0=1+0 = 1

y^{II} = 1+y^1 y_0^{II}=1xy_0^1 = 2

y^{III} = y^{II} y_0^{III} = y_0^{II} =2

y^{IV} = y^{III} y_0^{iv} = 2 etc

By Taylor series, are have

$$y_1 = y_0 + \frac{h}{1!} y_0^{1} + \frac{h^2}{Z!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \ldots\ldots$$

$$\therefore y_1 = y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24}(2)$$

$$+ \frac{0.1^5}{120}.2 + \ldots. - - - (2)$$

= 0.1+0.01+0.00033+0.00000833+0.000000166+….

Y(1.1) = 0.11033847

Now, take x_0 = 0.1103847

Now, take x_0 = 1.1 h = 0.1,

$$y_2 = y_1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots ---(3)$$

we calculate $y_1^I, y_1^{II}, y_1^{III}, \ldots,$

$x_1 = 1.1,\ y_1 = 0.11033847$

$y_1^I = x_1 + y_1 = 1.1 + 0.11033847 = 1.21033847$

$y_1^{II} = 1 + y_1^I = 2.21033847$

$y_1^{III} = y_1^{II} = y_1^{IV} = y_1^{V} = \ldots = 2.21033847$

using in (3),

$y_2 = y(1.2) = 0.11033847 + 0.1 / 1\ (1.21033847)$

$$+\frac{(0.1)^2}{2}(2.21033847) + \frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^h}{2h}(2.21033847) + \ldots$$

= 0.11033847+2.21033847(0.005+0.0016666+....)

= 0.2461077

The exact solution $\frac{dy}{dx} = x + y$ is y = -x-1+2e^{x-1}

$Y(1.1) = -1\ 1\ -1 + 2e^{0.1}$

= 0.11034

$y(1.2) = -1.2 - 1 + ze^{0.2} = 0.2428$

y (1.1) = 0.11033847

y (1.2) = 0.2461077

Exact values: y(1.1) = 0.110341876

Y(1.2) = 0.24280552

19.Using Taylor method compute y (0.2) and y(0.4) correct to 4 decimal places given

$$\frac{dy}{dx} = 1 - 2xy \text{ and } y(0) = 0$$

Soln

We know $y^1 = 1-2xy$ Here $x_0=0$ $y_0=0$, h=0.2

$y^{II} = -2(xy^1+y)$ $y_0^1 = 1 - zx_0y_0 = 1$

$y^{III} = -2(xy^{II}+2y^I)$ $y_0^{II} = 0$

$y^{IV} = -2(xy^{III} + Jy^{II})$ $y_o^{III} = -4$

$y^V = -2(xy^{iv}+4y^{III})$ $y^{iv}_o = 0$

$y_0^{II} = J\ 32$

by Taylor series

$$y_I = y_0 + \frac{h}{I\,!} y_0^1 + \frac{h^2}{2!} y_o^{II} + \frac{h^3}{3!} y_0^{III} + \ldots ------ (!)$$

(0.2) $(0.2)^2$ $(0.2)^3$ $(0.2)^4$ $(0.2)^5$

$y_1 = y(0.2) = 0 + \frac{}{1} 1 + \frac{}{2}(0) + \frac{}{6} + \frac{0}{24} + \frac{}{120}(32) + ...$

$y^I = Z - x$ | $Z^1 = x + y$

$y^{II} = Z^1 - 1$ $Z^{II} = 1 + y^1$

$y^{III} = Z^{II}$ etc $Z^{III} = y^{II}$ etc

By Taylor series, for y_1 and z we have

$Y_1 = y\ (0.1) = y_0 + hy_0^1 + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + ... --- (1)$

And $Z_1 = Z\ (0.1) = Z_0 + hZ_0^1 + \frac{h^2}{} Z_0^{II} ? h^3 Z_0^{III} + ...(2)$

2 6

$Y_0 = 1$ | $z_0 = 1$

$Y_0^1 = Z_0 - x_0 = 1 - 0 = 1$ | $z_0^1 = x_0 + y_0 + 0 = 1 = 1$

$Y_0^{II} = Z_0^1 - 1 = 1 - 1 = 0$ | $z_0^{II} = 1 + y_o^1 = 1 + 1 = 2$

$Y_0^{III} = z_0^{II} = 2$ | $z_0^{III} = y_0^{II} = 0$

Substituting in (1) and (2), we get $z_0^{IV} = y_0^{III} = 2$

$$Y_1 = y(0.1) = 1 + (0.1) + \frac{(0.01)}{2}(0) + \frac{(0.001)}{6}2 + ...$$

= 1 + 0.1 + 0.000333+…. =1.1007 (correct to 4 decimals)

$z_1 = z(0.1) = 1 + (0.1) +$

$$\frac{(0.01)}{2}2 + \frac{(0.001)}{6}(0) + \frac{0.0001}{24} \times 2 + ...$$

=1+0.1+0.01 +0.0000083+….

=1.1100 (correct to 4 decimal places)

$\therefore$ y (0.1) = 1.1003 and z (0.1) = 1.1100

20. Solve $\frac{dy}{dx} = z - x$, $\frac{dz}{dx} = y + x$ with y (0) = 1, z (0) = 1, by taking h = 0.1, to get y (0.1) and z (0.1).

Here y and z are dependent variables and x is independent.

Solution:

$Y^1 = z - x$ and $z^1 = x + y$

Take $x_0 = 0$, $y_0 = 1$ take $x_0 = 0$, $z_0 = 1$ and h = 0.1

$Y_1 = y(0.1) = ?$ $Z_1 = z(0.1) = ?$

Using in (6)

$Y_1 = y(0.1) = 0 + \frac{0.1}{2}[1+0.9] = \frac{0.19}{2} = 0.095$

$Y_2 = y_1 + \frac{1}{2} h\, [f(x_1, y_1) + f(x_2, y_1 + hf(x_1, y_1)] \rightarrow (7)$

$F(x_1, y_1) = 1 - y_1 = 1 - 0.095 = 0.905$

$F(x_2, y_1 + h f(x_1, y_1) = f(0.2, 0.095 + (0.1)(0.905)) = 0.8145$

Using in (7) we get y_2 = y (0.2) = 0.095 + 0.12

[0.905 +0.8145]

Y (0.2) = 0.18098

$Y_3 = y_2 + \frac{1}{2}$ h [f (x_2, y_2) +6 (x_3, x_2 +h f (x_2, y_2))]→ (8)

Using in (8)

Y_3 =y (0.3) = 0.18098 + $\frac{0.1}{2}$(0.81902 + 1 – 0.26288)

Y (0.3) = 0.258787

The values are tabulated

X	Modified Euler	Improved Euler	Exact solution
0.1	0.095	0.095	0.09516
0.2	0.18098	0.18098	0.18127
0.3	0.258787	0.258787	0.25918

Modified Euler and improved Euler methods give the same values come A to sin decimal places.

21. Evaluate the values of y (0.1) and y (0.2) given $y^{II} - x (y^1)^2 + y^2 = 0$; y (0) =1, y^1 (0) =0 by using Taylor series method?

Solution:

$Y^{II} - x (y^1)^2 + y^2 = 0$

Put $y^1 = z \rightarrow (1)$

Hence the eqn reduces to $z^1 - xz^2 + y^2 = 0$

$\therefore\ z^1 = xz^2 - y^2 \rightarrow (2)$

By initial condition, $y_0 = y(0) = 1$, $z_0 = y_0^1 = 0 \rightarrow (3)$

$Y_1 = 0.2 - 0.00533333 + 0.\ 000085333$

= 0.194752003

Now again starting with x = 0.2 as the starting value so, use again eqn (1)

Now $y_0 = 0.2$, $y_0 = 0.194752003$, h = 0.2

$Y_0^1 = 1 - 2x_0y_0 = 1 - 2(0.2)\ (0.194752003) = 0.9220992$

$Y_0^{II} = -2\ (x_0 y_0^1 + y_0) = -2\ [(0.2)\ (0.9220992) + 0.194752003]$

$= -0.758343686$

$y_0^{III} = -2\ [x_0 y_0^{II} + 2y_0^1]$

$= -2\ [(0.2)\ (-0.758343686) + 2\ (0.9220992)]$

$= -3.38505933$

$y_0^{IV} = -2\ [(0.2)\ (-3.38505933) + 3\ (-0.758343686)]$

$= 5.90408585$

Using eqn (1), again

$Y_2 = y\ (0.4) = 0.194752003 + (0.2)\ (0.9220992)$

$$\frac{(0.2)^2}{2}(-0.758343686) + \frac{(0.2)^3}{6}(-3.38505933) + \frac{(0.2)^4}{24}(5.90408585) = 0.359883723$$

22. Using improved Euler method find y at x =0.1 and y at x =0.2 give $\frac{dy}{dx} = y - \frac{2x}{y}$**,**

y (0) = 1

Solution:

By improved Euler method,

$Y_{n+1} = y_n + —$ 1

2

$h\,[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))] \rightarrow (1)$

$\therefore y_1 = y_0 + \frac{1}{2} h\,[f(x_0, y_0 + h f(x_0, y_0))] \rightarrow (2)$

$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - 0 = 1$

$f(x_1, y_0 + h f(x_0, y_0)) = f(0.1, 1.1) = 1.1 - \frac{2 \times (0.1)}{1.1} = 0.91818$

$y(0.1) = y_1 = 1 + \frac{0.1}{2} [1 + 0.91818] = 1.095909$

$y_2 = y(0.2) = y_1 + \frac{1}{2} h\,[f(x_1, y_1) + f(x_2, x_1 + hf(x_1, y_1))] \rightarrow (3)$

$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1.095909 - \frac{2 \times 0.1}{1.095909}$

$= 0.913412$

$f(x_2, y_1 + h f(x_1, y_1) = f(0.2, 1.095909 + (0.1)(0.9134121)$

$= f(0.2, 1.18732) = 1.18732 - \frac{2 \times 0.2}{1.18732} = 0.8594268$

Using in (3), $y_2 = 1.095909 + \frac{0.1}{2} [0.913412 + 0.850427]$

= 1. 1841009

X	0	0.1	0.2
Y	1	1.095907	1.1841009

23. Apply the fourth order Runge – kutta method, to find y (0.2) given that y^1 = x + y,

y (0) = 1

Solution:

Since h is not mentioned in the question we take h = 0.1

$Y^1 = x + y$; y (0) = 1

$\therefore$ f (x, y) = x+y, $x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $x_2 = 0.2$

By fourth order Runge – kutta method, for the first interative

$K_1 = h\, f(x_0, y_0) = (0.1)(x_0 + y_0) = (10.1)(0+1) = 0.1$

$K_2 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

= (0.1) f (0.05, 1.05) = (1.0) (0.05 +1.05) = 0.11

$k_3 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = (0.1)\, f(0.05, 1.055)$

= (0.1) (0.05 + 1.055) = 0.1105

$k_4 = hf(x_0 + h, y_0 + k_3)$

= (0.1) f (0.1, 11 05) = (0.1) (0.1 + 1.1105)

= 0.12105 $\therefore$ Δy = 1

6

$(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.1 + 0.22 + 0.2210 + 0.12105) = 0.110341667$

$y(0.1) = y_1 = y_0 + \Delta y = 1.110341667 \approx 1.110342$

Now starting from (x_1, y_1) we get (x_2, y_2) again

Apply Runge kutta algorithm replacing (x_0, y_0) by (x_1, y_1)

$K_1 = h f(x_1, y_1) = (0.1)(x_1+y_1) = (0.1)(0.1 + .110342) = 0.1210342$

$K_2 = h f(x_1+\frac{h}{2}, y_1 + \frac{1}{2} k_1) = (0.1) f(0.15, 1.170859)$

$= (0.1)(0.15 + 1.170859) = 0.1320859$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2} k_2) = (0.1) f(0.15, 1.1763848$

$= (0.1)(0.15 +1.1763848) = 0.13262848$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 1.24298048)$

$= 0.144298048$

$Y(0.2) = y(0.1) + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.110342 + \frac{1}{6}(0.794781008$

Y (0.2) = 1.2428055.Correct to four decimals places, y (0.2) = 1.2428

24. Using the Runge – kutta method, tabulate the solution of the system $\frac{dy}{dx}$

z =1 when x = 0 at intervals of h = 0.1 from x = 0.0 to x =0.2.

Solution:

Given $f(x, y, z) = x + z$, $g(x, y, z) = x - y$, $x_0 = 0$, $y_0 = 0$, $z_0 = 1$ and $h = 0.1$

$K_1 = hf(x_0, y_0, z_0)$ $= h(x_0 + z_0)$ $= (0.1)(0 + 1) = 0.1$	$L_1 = hg(x_0, y_0, z_0)$ $= h(x_0 - y_0)$ $= (0.1)(0 - 0) = 0$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$ $= h\left[\left(x_0 + \frac{h}{2}\right) + \left(z_0 + \frac{l_1}{2}\right)\right]$ $= (0.1)\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0}{2}\right)\right]$ $= 0.105$	$L_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$ $= h\left[\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{k_2}{2}\right)\right]$ $= (0.1)\left[\left(0 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$
$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$ $= h\left[\left(x_0 + \frac{h}{2}\right) + \left(z_0 + \frac{l_2}{2}\right)\right]$ $= (0.1)\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0}{2}\right)\right]$ $= 0.105$	$L_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$ $= 4\left[\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{kl2}{2}\right)\right]$ $= (0.1)\left[\left(0 + \frac{0.1}{2}\right) - \left(0 + \frac{0.105}{2}\right)\right]$ $= -0.00026$
$K_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$ $= h[x_0 + h) + (z_0 + l_3)$ $= (0.1)[(0+0.1) + (1 - 0.00026)$ $= 0.1099$	$L_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$ $= h[x_0 + h) - (y_0 + k_3)$ $= (0.1)[(0 + 0.1) - (0 + 0.105)]$ $= -0.0005$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$	$\Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$
$= \frac{1}{6}[0.1+2(0.105)+2(0.105)+0.1099)]$	$= \frac{1}{6}[0+0+2(-0.00026)-0.0005]$
$=0.1050$	$= 0.00017$

$Y_1 = y_0 + \Delta y$	$Z_1 = z_0 + \Delta z$
$= 0 + 0.1050$	$= 1 - 0.00017$
$y(0.1) = 0.1050$	$z(0.1) = 0.9998$

To compute y (0.2) and z (0.2)

Here $x_1 = 0.1$, $y_1 = 0.1050$, $z_1 = 0.9998$

$K_1 = hf(x_1, y_1, z_1)$	$L_1 = hg(x_1, y_1, z_1)$
$= h(x_1 + z_1)$	$= h(x_1 - y_1)$
$= (0.1)(0.1 + 0.9998)$	$= (0.1)(0.1 - 0.1050)$
$= 0.1099$	$= -0.0005$
$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$	$L_2 = hg\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$
$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(z_1 + \frac{l_1}{2}\right)\right]$	$= h\left[\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{k_1}{2}\right)\right]$
$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) + \left(0.9998 + \frac{0.0005}{2}\right)\right]$	$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right)\left(0.105 + \frac{0.1099}{2}\right)\right]$
$= 0.1149$	$= -0.0099$

$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$ $= h\left[\left(x_1 + \frac{h}{2}\right) + \left(z_1 + \frac{l_2}{2}\right)\right]$ $= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) + \left(0.9998 + \frac{0.00099}{2}\right)\right]$	$L_3 = hg\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$ $= h\left[\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{k_2}{2}\right)\right]$ $=(0.1)\left[\left(0.1 + \frac{0.1}{2}\right) - \left(0.1050 + \frac{0.1149}{2}\right)\right]$

— —

= 0.1149	= -0.00125
$K_4 = hf(x_1 + h, y_1 + k_3, z_1 + l_3)$ $= h[(x_1 + h) + z_1 + l_3)]$ = (0.1) [(0.1+0.1) + (0.9998 – 0.00125)] = 0.1198	$L_4 = hg[x_1 + h, y_1 + k_3, z_1 + l_3]$ $= h[(x_1 + h) - (y_1 + k_3)]$ = (0.1) [(0.1 +0.1) – (0.1050 + 0.1149)] — = -0.00199
$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$ $= \frac{1}{6}$ [0.1099 + 2 (0.1149) + 2 (0.1149) + 0.1198)] $= \frac{1}{6}$ [0.1099+0.2298 + 0.2298 + 0.1198] = 0.1149	$\Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$ $= \frac{1}{6}$ [-0.0005 +2(-0.00049)+2 (-0.00125) – 0.001199] $= \frac{1}{6}$ [-0.0005 -0.00198 -0.00199] $= \frac{1}{6}$ [-0.0005 – 0.00198 – 0.00199 = -0.00116

$Y_2 = y_1 + \Delta y$ = 0.1050 +0.1149 = 0.2199 y (0.2) = 0.2199	$Z_2 = z_1 + \Delta z$ = 0.9998 – 0.00116 = 0.9986 z (0.1) = 0.9986

	X=0	X = 0.1	X = 0.2
Y	0	0.1050	0.2199

X	1	0.9998	0.9986

25. Solve $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ using Runge – kutta method for x = 0.2 correct to 4 decimal places.

Initial condition are x = 0, y =1, y^1 = 0

Solution:

Given:

$\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2$ +y^2 = 0 $\rightarrow$ (1)

Put $\frac{dy}{dx}$=z $\rightarrow$ (2)

$\therefore \frac{d^2y}{dx} = \frac{d^2}{dx} \rightarrow$ (3)

Substituting (2) and (3) in (1), we get

$\frac{dz}{} = xz^2 - y$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + h_2)$

$= h(z_0 + l_3) = (0.2)(0 - 0.1958)$

$= -0.0392$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= h[(x_0 + h)(z_0 + l_3)^2 - (y_0 + k_3)^2]$

$= (0.2)[(0.2)(0 - 0.1958)^2 - (1 - 0.01998)^2]$

$= -0.1906$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0 + 2(-0.02) + 2(-0.01998) - 0.0392]$

$= -0.0199$

$\therefore y(0.2) = y_1 = y_0 + \Delta y$

$= 1 - 0.0199$

$= 0.9801$

$\therefore y(0.2) = 0.9801$

26. The differential equation $\frac{dy}{dx} = y - x^2$ is satisfied by y (0) = 1, y (0.2) = 1.12186, y (0.4) = 1.46820, y (0.6) = 1.7379 compute the value of y (0.8) by Milne's predictor corrector formula?

Solution:

Given

$\frac{dy}{dx} = y^1 = y - x^2$ and h = 0.2

$X_0 = 0$ $y_0 = 1$

$X_1 = 0.2$ $y_1 = 1.12186$

$X_2 = 0.4$ $y_2 = 1.46820$

$X_3 = 0.6$ $y_3 = 1.7379$

$X_4 = 0.8$ $y_4 = ?$

By Milne's predictor formula, we have

$Y_{n+1,\,P} = y_{n-3} + \frac{4h}{3}\,[zy_{n-2}{}^1 - y^1{}_{n-1} + 2y^1{}_n] \rightarrow (1)$

To get y_n, put n = 3 in (1) we get

$Y_{n,\,P} = y_0 + \frac{4h}{3}[2y^1{}_1 - y^1{}_2 + 2y^1{}_3] \rightarrow (2)$

Now $y^1{}_1 = (y - x)_1{}^2 = y_1 - x_1{}^2$

$= 1.12186 - (0.2)^2 = 1.08186 \rightarrow (3)$

$y_2{}^1 = (y - x^2)_2 = y_2 - x_1{}^2$

$= 1.\ 46820 - (0.4)^2 = 1.3082 \rightarrow (4)$

$y_3{}^1 = (y - x^2)_3 = y_3 - x_3{}^2$

$= 1.7379 - (0.6)^2 = 1.3779 \rightarrow (5)$

Substituting (3), (4) and (5) and (2), we get

$Y_{h,\,g} = 1 + \frac{4(0.2)}{3}\,[2(1.08186) - 1.3082 + 2\,(1.3779)]$

= 1+0.266 [2.1637 – 1.3082 + 2.7558]

= 1.9630187

$\therefore$ y (0.8) = 1.9630187 (by predictor formula)

By Milne's corrector formula we have

$Y_{n+1, C} = y_{n-1} + \frac{h}{3}(y^1_{n-1} + 4y^1_n + y^1_{n+1})$

To get y_h, put n = 3, we get

$Y_{h, C} = y_2 + \frac{h}{3}(y_2^1 + hy_3^1 + y_n^1) \rightarrow$ (6)

Now $y_n^1 = (y - x^2)_h = y_h - x_h^2$

= 1.96277 – $(0.8)^2$

= 1.3230187 $\rightarrow$ (7)

Substituting (4), (5), (7) in (6) we get

$Y_{4, C} = 1.46820 + \frac{0.2}{3}[1.3082 + 4\,(1.3779) + 1.3230187]$

= 2.0110546

i.e. y (0.8) = 2.0110546

27. Using Taylor's series method, solve $\frac{dy}{dx} = xy + y^2$, y (0) = 1 at x = 0.1, 0.2 and 0.3 continue the solve at

x = 0.4 by Milne's predictor corrector method?

Solution:

Given $y^1 = xy + y^2$, and $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Now $y^1 = xy + y^2$

$Y^{11} = xy^1 + y + 2yy^1$

$Y^{III} = xy^{II} + 2y^1 + 2yy^{II} + 2y^{12}$

To find y (0.1)

By Taylor series we have

$y(0.1) = y_1 + hy_0^1 + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_o^{III} + \ldots\ldots$ (1)

$y_0^{II} = (xy + y^2)_0 = (x_0y_0 + y_0^2) = 1$ ……. (2)

$y_0^{II} = (xy^1 + y + 2yy^1)$

$y_0^{II} = (x_0y_0^1 + y_0 + 2y_0y_0^1) = 3$ ……. (3)

$y_0^{III} = (xy_0^{II} + 2y^1 + 2yy^{II} + 2y^{12})_0 = 10$ …… (4)

Substituting (2). (3) and (4) in (1) we get

$Y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} \times 3 + \frac{(0.1)^3}{6} \times 10$

= 1 + 0.1 + 0.016 + 0.001666

y (0.1) = 1.11666

To find y (0.2)

By Taylor series we have

$\frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \ldots.. (5)$

$Y_2 = y_1 + hy_1^1 +$

Now $y_1^1 = (xy + y^2) = x_1y_1 + y_1^2$

$= (0.1)\,(1.11666) + (1.11666)^2$

$= 0.111666 + 1.2469$

$= 1.3585$ …… (6)

$y_1^{II} = (xy^1 + y + 2yy^1)$

$= x_1y_1^1 + y_1 + 2y_1\, y_1^1$

$= (0.1)\,(1.3585) + 1.11666 + 2\,(1.11666)\,(1.3585)$

$= 0.13585 + 1.11666 + 3.0339$

$= 4.2865$ …… (6)

$y_1^{III} = (xy^{II} + 2y^I + 2yy^{II} + 2y^{12})$

$= (x_1\, y_1^{II} + 2y_1^I + 2y_I y^{\,II} + 2y_1^{12})$

$= (0.1)\,(4.2865) + 2\,(1.3585) + 2\,(1.1167)\,(4.2865) + 2\,(1.3585)^2$

$= 0.4287 + 2.717 + 9.5735 + 3.6916$

$= 16.\,4102$ …… (8)

Substituting (6), (7) and (8) in (5) we get

$$Y\,(0.2) = 1.1167 + (0.1)\,(1.3585) + \frac{(0.1)^2}{2}(4.2865) + \frac{(0.1)^3}{6}\,(16.\,4102)$$

$Y\,(0.2) = 1.1167 + 0.\,13585 + 0.\,0214 + 0.002735$

$Y\,(0.2) = 1.27668$

To find y (0.3)

By Taylor series we have

Now $y_2^1 = (xy + y^2)_2 = (x_2y_2 + y_2^2)$

$= (0.2)(1.2767) + (1.2767)^2$

$= 0.2553 + 1.6299$

$= 1.8852$ …… (10)

$y_2^{II} = (xy^1 + y + 2yy^1)^2$

$= x_2y_2^1 + y_2 + 2y_2y_2^1$

$= (0.2)(1.8852) + 1.2767 + 2(1.2767)(1.8852)$

$= 0.33770 + 1.2767 + 4.8136$

$= 6.4674$ …… (11)

$y_2^{III} = (xy^{III} + 2y^1 + 2yy^{II}\ 2y^{12})_2$

$= (x_2y_2^{II} + 2y_2^1 + 2y_2y_2^{II} + 2y_2^{12})$

$= (0.2)(6.4674) + 2(1.8852) + 2(1.2767)(6.4774) + 2(1.8852)^2$

$= 1.2974 + 3.7704 + 16.5138 + 7.1079$

$= 28.6855$

Substituting (10), (11) and (12) in (9), we get

$$Y(0.3) = 1.2767 + (0.1)(1.8852) + \frac{0.1^2}{2}(6.4674) + \frac{(0.1)^3}{6}(28.6855)$$

= 1.2767 + 0.18852 + 0.0323 + 0.004780

= 1.5023

$\therefore$ y (0.3) = 1.5023

We have the following values

X_0= 0 y_0 =1

X_1 = 0.1 y_1 = 1.11666

X_2 = 0.2 y_2 = 1.27668

X_3 = 0.3 y_3 = 1.50233

To find y (0.4) by Milne's predictor formula

$Y_{n+1, P} = y_{n+3} + \frac{4h}{3}[2y^1{}_{n-2} – y^1{}_{n-2} + 2y^1{}_n]$ ..... (1)

$Y_3^1 = (xy + y_2)_3$

$= (x_3y_3 + y_3^2)$

$= [(0.3)(1.5023) + (1.5023)^2]$

= 0.45069 + 2.2569

= 2. 7076

Putting n =3, we get

$Y_{4, P} = y_0 + \frac{4h}{3}[2y_1^1 – y_2^1 + 2y_3^1]$

$= 1 + \frac{4(0.1)}{3}$ [2 (1.3585) – 1.8852 + 2 (2.7076)]

= 1 + 0.1333 [2.717 – 1.0852 + 5.4152]

$y_{4,P} = 1.8329$

To find y (.04) by Milne's corrector formula

By Milne's corrector formula we have

$y_{n+1, C} = y_{n-1} + [y^1_{n-1} + 4y^1 + y^1_{n+1}]$ …… (143

Now $y_4^1 = (x^2 + y^2)_4 = (x_4y_4 + y_4^2)$

$= [(0.4)(1.8327) + (1.8327)^2]$

$= 0.7330 + 3.3588$

$= 4.0918$

Putting n = 3 in (14) we get

$Y_{4, C} = y_2 + \frac{h}{2}[y_2^1 + 4y_3^1 + y_4^1]$

$Y_{4, C} = 1.27668 + \frac{(0.1)}{3}[1.8852 + 4(2.7076) + 4.0918]$

$= 1.27668 + 0.0333\,[1.8852 + 10.8304 + 4.0918]$

$= 1.8369$

28. Solve and get y (2) given $\frac{dy}{dx} = \frac{1}{2}(x + y)$, y (0) = 2

y (0.5) = 2.636, y (1) = 3.595; y (1.5) = 4.968 by Adam's

method?

Solution:

By Milne's method, we have $y_0^1 = \frac{1}{2}(0 + 2) = 1$

$Y_1^1 = 1.5680$, $y_2^1 = 2.2975$, $y_3^1 = 3.2340$

By Adam's predictor formula

$Y_{n+1, P} = y_n +$

$h \quad \frac{}{24}[55y_n^{\,1} - 59y^1_{n-1} + 37y^1_{n-2} - 9y^1_{n-3}]$

$\therefore y_{4,\,P} = y_3 + \frac{h}{24}[55y^1_n - 59y_2^{\,1} + 37y^1_1 - 9y^1_0]$ ….. (1)

=4.968 +

$\frac{0.5}{24}[55 (3.2340) – 59 (2.2975) + 37 (1.5680) – 9 (1)]$

= 68708

$y_4^1 = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2+6.8708) = 4.4354$

By corrector, $y_{4, C} = y_3 + \frac{h}{24}[9y_n^1 + 19y_3^1 – 5y_2^1 + y_1^1]$.. (2)

$= 4.968 + \frac{0.5}{24}[9 (4.4354) + 19 (3.234) – (2.2975) + 1.5680]$

= 6.8731

29. Find y (0.1), y (0.2), y (0.3) from $\frac{dy}{dx} = xy + y^2$, y (0) = 1 by using Runge – kutta method and hence obtain y (0.4) using Adam's method?

Solution:

$f(x, y) = xy + y^2$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$,

xy = 0.4, $x_4 = 0.4$, $y_0 = 1$

$k_1 = hf(x_0, y_0) = (0.1) f(0, 1) = (0.1) 1 = 0.1$

$k_2 = hf(0.05, y_0 + \frac{k_1}{2}) = (0.1) f(0.05, 1.05)$

$= (0.1) [(0.05)(1.05) + (1.05)^2] = 0.1155$

$k_3 = hf(0.05, y_0 + \frac{k_2}{2}) = (0.1) f(0.05, 1.0578)$

= (0.1) [(0.5) (1.0578) + $(1.0578)^2$]

= 0.1172

k_4 = hf (x_0 + h, y_0 + k_3)

= (0.1) f (0.1, 1.1172)

= (0.1) [(0.10 (1.1172) + $(1.1172)^2$] = 0.13598

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

= 1.1169

y (0.1) = 1.1169

Again, start from y_1

k_1 = hf (x_1, y_1) = (0.1) f (0.1, 1.1169)

= 0.1359

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$ = (0.1) f (0.15, 1.1849)

= 0.1582

$k_3 = hf\left(0.15, y_1 + \frac{k_3}{2}\right)$ = (0.1) f (0.15, 1.196)

= 0.16098

k_4 = (0.1) f (0.2, 1.2779) = 0.1889

$y_2 = 1.1169 + \frac{1}{6}[0.1359 + 2(0.1582 + 0.16098) + 0.1889]$

y (0.2) = 1.2774

Start from (x_2, y_2) to get y_3

$K_1 = hf(x_2, y_2) = (0.1)\, f(0.2, 1.2774) = 0.1887$

K_2 = hf (x_2

— $+\frac{h}{2}$, y_2

$+\frac{k_1}{2}$) = (0.1) f (0.25, 1.3718) = 0.2225

$K_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$

= (0.1) f (0.25, 1.3887) = 0.2274

$k_4 = hf\left(x_3, y_2 + \frac{k_3}{2}\right)$ = (0.1) f (0.3, 1.5048)

=0.2716

$y_3 = 1.2774 + \frac{1}{6}[0.1887 + 2(0.2225) + 2(0.2274) + 0.2716] = 1.5041$

Now we use Adam's predictor formula

$Y_{4,P} = y_3 + \frac{h}{24}[55y_3^{1} - 59y_2^{1} + 37y_1^{1} - 9y_0^{1}]$ …. (2)

$Y_0^1 = x_0y_0 + y_0^2 = 1$

$Y_1^1 = x_1y_1 + y_1^2 = 1.3592$

$Y_2^1 = x_2y_2 + y_2^2 = 1.8872$

$Y_3^1 = x_3y_3 + y_3^2 = 2.7135$

Using (2)

$Y_{4,P} = 1.5041 + \frac{0.1}{2}[55(2.7135) - 59(1.8872) + 37(1.3592) - 9(1)]$

= 1.8341

$y^1_{4,P} = x_4y_4 + y^2_4 = (0.4)(1.8341) + (1.8341)^2 = 4.0976$

$y_{4,P} = y_3 + \frac{h}{2}[9y_4^1 + 19y_3^1 - 5y_2^1 + y_1^1]$

= 1.8389

y (0.4) = 1.8389

30. Solve $y^1 = \frac{y^2 - x^2}{y^2 + x^2}$; y (0) = 1 by Runge – kutta method of fourth order to find y (0.2)

Solution:

$Y^1 = f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, h = 0.2, $x_1 = 0.2$

$f(x_0, y_0) = f(0, 1) = \frac{1-0}{1+0} = 1$

$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = (0.2)\, f(0.1, 1.1)$

$= 0.2 \left[\frac{1.21 - 0.01}{1.21 + 0.01}\right] = 0.9167213$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = (0.2)\, f(0.1, 1.0983606)$

= 0.1967

$k_4 = hf(x_0 + 4, y_0 + k_3) = 0.2\, f(0.2, 1.1967)$

= 0.1891

$\therefore \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.2 + 2 (0.19672) + 2 (1.1967) + 0.1891]$

= 0.19598

$y (0.2) = y_1 = y_0 + \Delta y = 1.19598$

Possible Questions

PART-A (2 Mark)

UNIT V

1. Write the difference between Euler and modified Euler Method.

2.Define Euler method with formula.

3. Write the formula for Milne's predictor – corrector method.

4. Write the formula for Adam's Bash forth predictor – corrector method.

5. Define modified Euler method formula.

PART-B (6 Mark)

1. Solve dy/dx = x + y, given y(1)=0 and get y(1.1),y(1.2) by Taylor's series Method.Compare your result with the explicit method

2. Find y (1.5) taking h=0.5 given $y'=y-1, y(0)=1.1$ by using Euler's Method.

3.Using Adam's method for y (0.4) given $\frac{dy}{dx}=\frac{1}{2}xy$, y(0)=1,y(0.1)=1.01,y(0.2)=1.022, y(0.3)=1.023.

4. Apply fourth order Runge-Kutta method to find y(0.2) given that $y'=x+y, y(0)=1$.

5.Using Taylor method compute y(0.2) and y(0.4) correct to four decimal places given by dy/dx = 1-2xy and y(0)=0

6. Compute y at x=0.25 by modified Euler method. Given y'=2xy,y(0)=1

7. Solve the equation dy/dx=1-y, given y(0)=0 using modified Euler method and tabulate the values at x=0.1, 0.2, 0.3 compare your results with the exact solutions

8. Determine the value of y (0.4) using Milne's methods given $y'=xy+y^2$ use Taylor series to get the values of y(0.1),y(0.2) and y(0.3).

9. Find y (1.1) given y'=2x-y, y (1) =3 by using Taylor series method

10. Obtain the values of y at x=0.1, 0.2 using R-K method of
(i) Second order
(ii) Fourth order
For the differential equation y'= -y given y(0)=1.

KARPAGAM UNIVERSITY
COIMBATORE-21
DEPARTMENT OF MATHEMATICS

NAME OF THE FACULTY: Ms.M.LATHA(KU0969)
SUBJECT: NUMERICAL METHODS
SUBJECT CODE: 16MMU301
CLASS: II B.Sc MATHEMATICS

POSSIBLE QUESTIONS
UNIT V

2 MARKS

1. Write the difference between Euler and modified Euler Method.
2.Define Euler method with formula.
3. Write the formula for Milne's predictor – corrector method.
4. Write the formula for Adam's Bash forth predictor – corrector method.
5. Define modified Euler method formula.

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POSSIBLE QUESTIONS
UNIT V

6 MARKS

1. Solve dy/dx = x +
your result with the explicit method
2. Find y (1.5) taking h=0.5 given $y' = y - 1,\ y(0) = 1.1$
$\frac{dy}{dx} = \frac{1}{2}xy$, y(0)=1,y(0.1)=1.01,y(0.2)=1.022,
y(0.3)=1.023.
4. Apply fourth order Runge-Kutta method to find y(0.2) given that $y' = x + y, y(0) = 1$.
5. Using Taylor method compute y(0.2) and y(0.4) correct to four decimal places given by dy/dx = 1-2xy and y(0)=0
6.
7. Solve the equation dy/dx=1-y, given y(0)=0 using modified Euler method and tabulate the values at x=0.1, 0.2, 0.3 compare your results with the exact solutions
8. $y' = xy + y^2$
use Taylor series to get the values of y(0.1),y(0.2) and y(0.3).
9. -y, y (1) =3 by using Taylor series method
10. Obtain the values of y at x=0.1, 0.2 using R-K method of
    (i) Second order
    (ii) Fourth order

-y given y(0)=1.

UNIT-V

1. The numerical backward differentiation of y w.r.t. x once is ------------.
    a. f ' (x) = (1/h)* (Dy0 + (2r–1)/2 * D2y0 + (3r2–6r+2)/6 * D3y0 + ……….)
    b. b.y = yn + n Ñyn + {n(n+1) / 2!} Ñ2yn + {n(n+1)(n+2) / 3!} Ñ3yn + ……….)
    c. f ' (x) = (1/h)* (Dyn + (2r+1)/2 * D2yn + (3r2+6r+2)/6 * D3yn + ……….)

2. The second derivative of the Newton's forward differentiation is -------------.
    a.y " = (1/h2)* {D2y0 – D3y0 + (11/12) D4y0 ……….} b.y " = (1/h2)* {D2y0 + D3y0 + (11/12) D4y0 ……….}
    c.y " = (1/h)* {D2y0 + D3y0 + (11/12) D4y0 ……….}
3. The second derivative of the Newton's backward differentiation is -------------.
    a.y " = (1/h2)* {D2y0 + D3y0 + (11/12) D4y0 ……….} b.y " = (1/h2)* {D2y0 – D3y0 + (11/12) D4y0 ……….}
    d.y " = (1/h)* {D2y0 – D3y0 + (11/12) D4y0 ……….}

4. The order of error in Trapezoidal rule is -------------.
    a.h b.h^3 c. d.h^4
5. The order of error in Simpson's rule is -------------.
    a.h b.h^3 c.h^2 d.
6. Numerical evaluation of a definite integral is called --------.
    b.Differentiation c.Interpolation d.Triangularization
7. Simpson's ⅜ rule can be applied only if the number of sub interval is in ------.
    a.Equal b.even c. d.unequal
8. By putting n = 2 in Newton cote's formula we get ----------- rule.
    b.Simpson's ⅜ c.Trapezoidal d.Romberg
9. The Newton Cote's formula is also known as --------------- formula.
    a.Simpson's 1/3 b.Simpson's 3/8 c.Trapezoidal d.
10. By putting n = 3 in Newton cote's formula we get ----------- rule.
    a.Simpson's 1/3 c.Trapezoidal d.Romberg
11. By putting n = 1 in Newton cote's formula we get ----------- rule.
    a.Simpson's 1/3 b.Simpson's ⅜ d.Romberg

12. The systematic improvement of Richardon's method is called--------- method
   a.Simpson's 1/3   b.Simpson's ⅜   c.Trapezoidal
13. Simpson's 1/3 rule can be applied only when the number of interval is ------.
   a.Equal   b.   c.multiple of three   d.unequal
14. " Simpson's rule is exact for a ----------------- even though it was derived for a Quadratic."
   a.cubic   b.less than   c.cubic   d.
15. The accuracy of the result using the Trapezoidal rule can be improved by --------
   a." Increasing the interval h"   b." Decreasing the interval h
   d."altering the given function"
16. A particular case of Runge Kutta method of second order is -------------------.
   a.Milne's method   b.Picard's method   c.   d.Runge's method
17. Runge Kutta of first order is nothing but the --------------.
   a.modified Euler method   b.   c.Taylor series   d.none of these
18. In Runge Kutta second and fourth order methods, the values of k1 and k2 are ----
   b.differ   c.always positive   d.always negative
19. The formula of Dy in fourth order Runge Kutta method is given by ---------------.
   a.1/6 * (k1 + 2k2 + 3k3 + 4k4)   b.1/6 * (k1 + k2 + k3 + k4)   c.   d.1/6 * (k1 + 2k2 + 2k3 + k4)
20. ________values are calculated in Runge Kutta fourth order method.
   b.k1, k2 and Dy   c.k1, k2, k3 and Dy   d.none of these
21. The use of Runge kutta method gives ------------- to the solutions of the differential equation than Taylor's series method.
   b.quick convergence   c.oscillation   d.divergence
22. In Runge – kutta method the value x is taken as ----------.
   b.x0= x + h   c.h = x0 + x   d.h = x0 - x
23. In Runge – kutta method the value y is taken as ----------.
   a.y = y0 + h   b.y0= x0 + h   c.y = y0 – Dy
24. In fourth order Runge Kutta method the value of k3 is calculated by ------------.
   a.h f(x - h/2 , y - k2/2)   b.   c.h f(x , y)   d.h f(x – h/2 , y – k1/2)
25. In fourth order Runge Kutta method the value of k4 is calculated by ------------.
   a.h f(x + h/2 , y + k1/2)   b.h f(x + h/2 , y + k2/2)   c   d.h f(x – h , y – k3)
26. ___________is nothing but the modified Euler method.

b.Runge kutta method of third order

c.Runge kutta method of fourth order d.Taylor series method

27. In all the three methods of Rungekutta methods, the values ---------- are same.

a.k4 & k3 b.k3 & k2 c. d.k1, k2, k3 & k4

28. The formula of Dy in third order Runge Kutta method is given by ---------------.

a.1/6 * (k1 + 2k2 + 3k3 + 4k4) b.1/6 * (k1 + k2 + k3 + k4) c. d.1/6 * (k1 + 2k2 + 2k3 + k4)

29. The formula of Dy in second order Runge Kutta method is given by --------------.

a.k1 b. c.k3 d.k4

30. In second order Runge Kutta method the value of k1 is calculated by ------------.

a.h f(x + h/2 , y + k1/2) b.h f(x + h/2 , y + k2/2) d.h f(x – h/2 , y – k1/2)

31. The Runge – Kutta methods are designed to give -------------- and they posses the advantage of requiring only the function values at some selected points on the sub intervals

a.greater accuracy b.lesser accuracy c.average accuracy d.equal

32. If dy/dx is a function x alone, then fourth order Runge – Kutta method reduces to --------------.

a.Trapezoidal rule b.Taylor series c.Euler method d.

33. In Runge Kutta methods, the derivatives of ----------- are not require and we require only the given function values at different points.

b.lower order c.middle order d.zero

34. The use of -------------- method gives quick convergence to the solutions of the differential equation than Taylor's series method.

a.Taylor series b.Euler c. d.Simpson method

35. If dy/dx is a function x alone, then ---------- Runge – Kutta method reduces to Simpson method

b.third order c.second order d.first order

36. If dy/dx is a function of ----------- then fourth order Runge – Kutta method reduces to Simpson method.

a. b.y alone c.both x and y d.none

37. In second order Runge Kutta method the value of k2 is calculated by ------------.

b.h f(x – h/2 , y – k1/2) c. h f(x , y) d.h f(0,0)

Reg.No-----------------

(16MMU301)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Karpagam University

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Third Semester

II Internal Test - Aug'2017

Numerical Methods

Date: .08.17( ) Time: 2 Hours

Class: II B.Sc Mathematics Maximum Marks: 50

PART-A(20X1=20 Marks)

Answer all the Questions:

1. Forward difference operator is denoted by the symbol --------
   a) Δ b) ∇ c) ∑ d) ∏
2. Relation between E and ∇ is ∇ = ----------
   a) E – 1 b) 1 – E^{-1} c) 1 +E^{-1} d) 1 * E^{-1}
3. The n^{th} differences (forward) of a polynomial of the n^{th} degree are -----
   a) constant b) variable c) zero d) one
4. The process of computing the value of a function outside the range is called -----
   a) interpolation b) extrapolation
   c) both d) inverse interpolation
5. ------------- Interpolation formula can be used for equal and unequal intervals.
   a) Newton's forward b) Newton's forward
   c) Lagrange d) Romberg
6. The divided difference operator is -----------
   a) non-linear b) normal c) linear d) translation
7. In difference, f(x+h) – f(x) = --------
   a) Δf(x) b) ∇f(x) c) Δ^2 f(x) d) h(x)
8. The operators are distributive over ------------
   a) subtraction b) multiplication c) division d) addition
9. ------------- Formula can be used for interpolating the value of f(x) near the end of the tabular values.
   a) Newton's forward b) Newton's backward
   c) Lagrange d) divided
10. The -----differences are symmetrical in all their arguments.
    a) forward b) backward c) divided d) central
11. The interval of differencing h , is denoted by---
    a) x_2-x_0 b) x_1-x_0 c) x_3-x_0 d) x_0-x_1
12. The central difference operator is denoted by ---------
    a) D b) δ c) ∇ d) Δ
13. The polynomial x(x-h)(x-2h)(x-3h)……(x-(n-1)h) is defined as ----------
    a) difference of polynomial b) factorial polynomial
    c) forward difference d) backward difference
14. To find the unknown values of y for some x which lies at the --- of the table, we use Newton's Forward formula.
    a) beginning b) end c) centre d) outside
15. In Newton's backward interpolation formula, the value of v is calculated by -------.
    a) v= $(x–x_n)$ / h b) v = $(x_n–x)$ / h
    c) v = $(x–x_0)$ / h d) v = $(x_0–x)$ / h
16. The difference value y_2 – y_1 in a Newton's forward difference table is denoted by ----------.
    a) Δ y_0 b) Δ y_1 c) Δy_2 d) Δ y_0

17. The value of any divided differences is ----------- of the order of the arguments.

a) independent b) dependent c) zero d) one

18. The second difference $\Delta^2 y_0$ is equal to………………

a) $y_2 - 2y_1 - y_0$ b) $y_2 + 2y_1 + y_0$

c) $y_2 - 2y_1 + y_0$ d) $y_2 - 2y_1 + y_0$

19. The x values of interpolating polynomial of Newton - Gregory has________space

a) odd b)even c)equal d)unequal

20. The value of Δy_2 = ………….

a) $y_2 - y_3$ b) $y_2 + y_3$ c) $y_3 - y_2$ d) $y_3 + y_2$

PART-B (3X2= 6 Marks)

ANSWER ALL THE QUESTIONS

21. Define divided differences.
22. Write the formula for Lagrange's interpolation formula for Unequal intervals
23. Prove that $E\Delta = \Delta = \nabla E$.

PART-C (3x8=24 Marks)

ANSWER ALL THE QUESTIONS

24. a) From the following table, find the value of tan 45°15′

x°	: 45	46	47	48	49	50
tan x°	: 1.0000	1.0355	1.0723	1.1106	1.1503	1.1917

(OR)

b) Using inverse interpolation formula, find the value of x when y=13.5.

x:	93.0	96.2	100.0	104.2	108.7
y:	11.38	12.80	14.70	17.07	19.91

25. a) From the following table find f(x) and hence f(6) using Newton interpolation formula.

x :	1	2	7	8
f(x) :	1	5	5	4

(OR)

b) Find the values of y at X=21 and X=28 from the following data.

X:	20	23	26	29
Y:	0.3420	0.3907	0.4384	0.4848

26. a) Find the first two derivatives of $x^{\frac{1}{3}}$ at x=50 and x=56 given the table below:

X:	50	51	52	53	54	55	56
Y:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

(OR)

b) The population of a certain town is given below, Find the rate of growth of population in 1931, 1941, 1961 and 1971.

Year :	1931	1941	1951	1961	1971
Population : (in thousands)	40.62	60.80	79.95	103.56	132.65

KARPAGAM U

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DEPARTMENT OF M

Third Seme

III Internal

NUMERICAL M

Date : .09.17(AN)

Class : II-B.Sc Mathematics

PART - A (20 x

Answer all the questions

1. The use of Runge kutta metho solutions of the differential equa method.
   a) Slow convergence b)
   c) Oscillation d)
2. In this Euler method the actua sequence of short………………
   a)Straight-line b)parabola
3. In Modified Euler's Method av
   a) One step method b)
   c) step by step method d)
4. In R - K method derivatives of ----------
   a) Diagram b) graph c) c
5. In Simpson's 1/3, the number o ……………_
   a) zero b) odd c)eve

15. In Newton cote formula if f(x) is interpolate at equally spaced nodes by a polynomial of degree two then it represents………..

a) Trapezoidal rule b) Simpson s rule
c) midpoint rule d) boole'srule

16. The Euler Method and Modified Euler's Method are -------- convergent

a)slowly b)equally c)fastly d)greater

17. Euler's algorithm formula is ---------------------

a) $Y_{n-1} = y_{n+1} + h f(x_n, y_n)$ n= 0,1,2,3…
b) $Y_{n+1} = y_n - h f(x_n, y_n)$ n= 0,1,2,3…
c) $Y_n = y_0 + f(x_0, y_n)$ n= 0,1,2,3…
d) $Y_{n+1} = y_n + h f(x_n, y_n)$ n= 0,1,2,3…

18. By putting n = 1 in Newton cote's formula we get ----------- rule.

a) Simpson's 1/3 rule b) Simpson's ⅜ rule
c) Trapezoidal rule d) Newton 'method

19. Simpson's ⅜ rule can be applied only if the number of sub interval is in ------.

a) Equal b) even c) multiple of three d) unequal.

20. Runge Kutta of first order is nothing but the --------------.

a) Modified Euler method b) Euler method
c) Taylor series d) Runge's method

PART-B (3X2= 6 Marks)

ANSWER ALL THE QUESTIONS

21. Write the Simpson's 3/8th rule formula.
22. Define modified Euler method formula.
23. Write the formula for Milne's predictor – corrector method.

PART-C (3x8=24 Marks)

ANSWER ALL THE QUESTIONS

24.a) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule. Also check up the result by actual integration.

(OR)

b) Solve dy/dx = x + y, given y(1)=0 and get y(1.1),y(1.2) by Taylor's series Method. Compare your result with the explicit method

25. a) Compute y at x=0.25 by modified Euler method. Given y'=2xy,y(0)=1.

(OR)

b) Find y (1.5) taking h=0.5 given $y' = y - 1,\ y(0) = 1.1$ by using Euler's Method.

26.a) Using Adam's method for y(0.2) ,y(0.3) and y (0.4) given $\frac{dy}{dx} = \frac{1}{2}xy$, y(0)=1,y(0.1)=1.01.

(OR)

b) Apply fourth order Runge-Kutta method to find y(0.2) given that $y' = x + y, y(0) = 1$.