KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT Of MATHEMATICS

Subject: Ring Theory and Linear Algebra I **Subject Code: 16MMU302**

L	T	P	C
6	2	0	6

PO: On successful completion of course the learners gain about the linear transformations, homomorphism, isomorphism and its properties.

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PLO: To enable the students to learn and gain knowledge about rings, subrings, vector spaces, subspaces, algebra of subspaces, isomorphism and its properties.

UNIT I

Definition and examples of rings, properties of rings, subrings, integral domains and fields, characteristic of a ring. Ideal, ideal generated by a subset of a ring, factor rings, operations on ideals, prime and maximal ideals.

UNIT II

Ring homomorphisms, properties of ring homomorphisms, Isomorphism theorems I, II and III, field of quotients.

UNIT III

Vector spaces, subspaces, algebra of subspaces, quotient spaces, linear combination of vectors, linear span, linear independence, basis and dimension, dimension of subspaces.

UNIT IV

Linear transformations, null space, range, rank and nullity of a linear transformation, matrix representation of a linear transformation, algebra of linear transformations.

UNIT V

Isomorphism: Isomorphism theorems, invertibility and isomorphisms, change of coordinate matrix.

SUGGESTED READINGS

TEXT BOOK

1. Fraleigh.J.B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

2. Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

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DEPARTMENT OF MATHEMATICS

Lecture Plan

Subject Name: Ring theory and Linear algebra I **Subject Code: 16MMU302**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-I	
1	1	Definition and examples of rings	R1: Ch 12, 237-239
2	1	Properties of rings	R1: Ch 12, 239
3	1	Properties of rings	R1: Ch 12, 240
4	1	Tutorial	
5	1	Theorems on subrings	R1: Ch 12, 241
6	1	Theorems on subrings	R1: Ch 12, 242
7	1	integral domains	R1: Ch 13, 249-250
8	1	Tutorial.	
9	1	integral domains	R1: Ch 13, 250
10	1	Theorems on fields	R1: Ch 13, 250
11	1	Theorems on fields	R1: Ch 13, 251-252
12	1	Tutorial.	
13	1	Theorems on ideal	R1: Ch 13, 252
14	1	Theorems on ideal	R1: Ch 13, 253
15	1	Theorems on ideal generated by a subset of a ring	R1: Ch 13, 253
16	1	Tutorial.	
17	1	Theorems on ideal generated by a subset of a ring	R1: Ch 13, 254
18	1	Theorems on factor rings	R1: Ch 13, 255-256
19	1	Theorems on factor rings	R1: Ch 13, 256
20	1	Tutorial.	
21	1	operations on ideals	R1: Ch 14, 262-263
22	1	Theorems on prime ideals	R1: Ch 14, 264-266
23	1	Theorems on prime maximal ideals	R1: Ch 14, 267-268
24	1	Recapitulation and Discussion of possible questions	
Total	**24 Hours**		

Text book

T Fraleigh. J. B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

Reference

R1 Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa

Publishing House, New Delhi.			
UNIT-II			
1	1	Definitions and examples on ring homomorphisms	R1: Ch 15, 280-281
2	1	Theorems on ring homomorphisms	R1: Ch 15, 281
3	1	Theorems on ring homomorphisms	R1: Ch 15, 282
4	1	Tutorial.	
5	1	Theorems on ring homomorphisms	R1: Ch 15, 283
6	1	Theorems on ring homomorphisms	R1: Ch 15, 284
7	1	properties of ring homomorphisms	R1: Ch 15, 285
8	1	Tutorial.	
9	1	properties of ring homomorphisms	R1: Ch 15, 286
10	1	properties of ring homomorphisms	R1: Ch 15, 287
11	1	properties of ring homomorphisms	R1: Ch 15, 288-289
12	1	Tutorial.	
13	1	properties of ring homomorphisms	R1: Ch 15, 289
14	1	properties of ring homomorphisms	R1: Ch 15, 290
15	1	Isomorphism theorem I	T1: Ch 7, 301
16	1	Tutorial.	
17	1	Isomorphism theorem I	T1: Ch 7, 302
18	1	Isomorphism theorem II	T1: Ch 7, 303-305
19	1	Isomorphism theorem III	T1: Ch 7, 306-309
20	1	Tutorial.	
21	1	Theorems on field of quotients	T1: Ch 7, 310
22	1	Theorems on field of quotients	T1: Ch 7, 311
23	1	Theorems on field of quotients	T1: Ch 7, 312
24	1	Recapitulation and Discussion of possible questions	
Total	**24Hours**		
Text book T Fraleigh. J. B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore. **Reference** R1 Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.			
UNIT-III			
1	1	Introduction to Vector spaces	R1: Ch 19, 345
2	1	Theorems on subspaces	R1: Ch 19, 346
3	1	Theorems on subspaces	R1: Ch 19, 347
4	1	Tutorial	
5	1	Theorems on subspaces	R1: Ch 19, 348
6	1	Theorems on subspaces	R1: Ch 19, 349
7	1	Theorems on subsapces	R1: Ch 19, 350

8	1	Tutorial.	
9	1	Theorems on subsapces	R1: Ch 19, 351
10	1	properties of subspaces	T1: Ch 6, 283
11	1	properties of subspaces	T1: Ch 6, 284
12	1	Tutorial	
13	1	Theorems on algebra of subspaces	T1: Ch 6, 285
14	1	Theorems on algebra of subspaces	T1: Ch 6, 286
15	1	Theorems on quotient spaces	T1: Ch 6, 287
16	1	Tutorial.	
17	1	Theorems on quotient spaces	T1: Ch 6, 288
18	1	Theorems on linear span	T1: Ch 6, 289
19	1	Theorems on linear span	T1: Ch 6, 290
20	1	Tutorial.	
21	1	Theorems on linear independence	T1: Ch 6, 291-292
22	1	Theorems on basis and dimension	T1: Ch 6, 293-294
23	1	Theorems on dimension of subspaces	T1: Ch 6, 294
24	1	Recapitulation and Discussion of possible questions	
Total	**24 Hours**		

Text book

T Fraleigh. J. B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

Reference

R1 Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

UNIT-IV

1	1	Introduction to Linear transformations	T: Ch 2, 33
2	1	Theorems on linear transformations	R1: Ch 9, 212
3	1	Theorems on linear transformations	R1: Ch 9, 213
4	1	Tutorial.	
5	1	Theorems on null space	R1: Ch 9, 214
6	1	Theorems on null space	R1: Ch 9, 215
7	1	Theorems on null space	R1: Ch 9, 216
8	1	Tutorial.	
9	1	Theorems on null space	R1: Ch 9, 217-218
10	1	properties of null space	R1: Ch 9, 218
11	1	properties of null space	R1: Ch 9, 219
12	1	Tutorial.	
13	1	Theorems on range	R2: Ch 11, 320
14	1	Theorems on range	R2: Ch 11, 321
15	1	Theorems on rank of a linear transformation	R2: Ch 11, 322
16	1	Tutorial.	
17	1	Theorems on rank of a linear transformation	R2: Ch 11, 323

18	1	Theorems on nullity of a linear transformation	R2: Ch 11, 324-325
19	1	Theorems on nullity of a linear transformation	R2: Ch 11, 325
20	1	Tutorial.	
21	1	Theorems on matrix representation of a linear transformation	R2: Ch 11, 326
22	1	Theorems on algebra of linear transformations	R2: Ch 11, 327
23	1	Theorems on algebra of linear transformations	R2: Ch 11, 328-329
24	1	Recapitulation and Discussion of possible questions	
Total	**24 Hours**		

Text book

T Fraleigh. J. B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

Reference

R1 Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

R2 Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

UNIT-V

1	1	Isomorphism theorems	R2: Ch 12, 340
2	1	Isomorphism theorems	R2: Ch 12, 341
3	1	Isomorphism theorems	R2: Ch 12, 342-343
4	1	Tutorial.	
5	1	Isomorphism theorems	R2: Ch 12, 344
6	1	Isomorphism theorems	R2: Ch 12, 344-345
7	1	Isomorphism theorems	R2: Ch 12, 346
8	1	Tutorial.	
9	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 350
10	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 351-352
11	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 353
12	1	Tutorial.	
13	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 353-354
14	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 355
15	1	Theorems on invertibility and isomorphisms	R2: Ch 12, 356-357
16	1	Tutorial.	
17	1	Theorems on change of coordinate matrix	R2: Ch 12, 357
18	1	Theorems on change of coordinate matrix	R2: Ch 12, 357-358
19	1	Theorems on change of coordinate matrix	R2: Ch 12, 358-359
20	1	Tutorial.	
21	1	Theorems on change of coordinate matrix	R2: Ch 12, 360
22	1	Recapitulation and Discussion of possible	

		questions	
23	1	Discussion on Previous ESE Question Papers	
24	1	Discussion on Previous ESE Question Papers	
Total	**24 Hours**		

Text book

T Fraleigh. J. B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

Reference

R1 Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

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Total no. of Hours for the Course: 120 hours

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DEPARTMENT OF MATHEMATICS

		L	T	P	C
Subject: Ring theory & Linear algebra I	**Semester: III**				
Subject Code: 16MMU302	**Class: II-B.Sc. Mathematics**	**6**	**2**	**0**	**6**

UNIT I

Definition and examples of rings, properties of rings, subrings, integral domains and fields, characteristic of a ring. Ideal, ideal generated by a subset of a ring, factor rings, operations on ideals, prime and maximal ideals.

*

TEXT BOOK

1. Fraleigh.J.B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

2. Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

1 Rings

1.1 Definitions and examples

We now move on to something completely different — rings. In a ring, we are allowed to add, subtract, multiply but not divide. Our canonical example of a ring would be , the integers, as studied in IA Numbers and Sets.

In this course, we are only going to consider rings in which multiplication is commutative, since these rings behave like “number systems”, where we can study number theory. However, some of these rings do not behave like . Thus one major goal of this part is to understand the different properties of , whether they are present in arbitrary rings, and how different properties relate to one another.

Definition 1 (Ring) *A* ring *is a quintuple* $(R, +, , 0_R, 1_R)$ *where* $0_R, 1_R \in R$*, and* $+, : R \times R \to R$ *are binary operations such that*

1. $(R, +, 0_R)$ *is an abelian group.*
2. *The operation* $: R \times R \to R$ *satisfies associativity, i.e.*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c,$$

and identity:

$$1_R \cdot r = r \cdot 1_R = r.$$

3. *Multiplication distributes over addition, i.e.*

$$r_1 \cdot (r_2 + r_3) = (r_1 \cdot r_2) + (r_1 \cdot r_3)$$
$$(r_1 + r_2) \cdot r_3 = (r_1 \cdot r_3) + (r_2 \cdot r_3).$$

If R is a ring and $r \in R$, we write $-r$ for the inverse to r in $(R, +, 0_R)$. This satisfies $r + (-r) = 0_R$. We write $r - s$ to mean $r + (-s)$ etc. Some people don’t insist on the existence of the multiplicative identity, but we will for the purposes of this course.

Since we can add and multiply two elements, by induction, we can add and multiply any finite number of elements. However, the notions of infinite sum and product are undefined. It doesn't make sense to ask if an infinite sum converges.

Definition 2 (Commutative ring) *We say a ring R is* commutative *if $a \cdot b = b \cdot a$ for all $a, b \in R$.*

From now onwards, all rings in this course are going to be commutative.

Just as we have groups and subgroups, we also have subrings.

Definition 3 (Subring) *Let $(R, +, , 0_R, 1_R)$ be a ring, and $S \subseteq R$ is a subset. We say S is a* subring *of R if $0_R, 1_R \in S$, and the operations $+,$ make S into a ring in its own right. In this case we write $S \leq R$.*

Example 1 *The familiar number systems are all rings: we have $\leq\leq\leq$, under the usual* $0, 1, +,$.

Example 2 *The set $[i] = \{a + ib : a, b \in\} \leq$ is the* Gaussian integers*, which is a ring.*

We also have the ring $[\sqrt{2}] = \{a + b\sqrt{2} \in: a, b \in\} \leq$.

We will use the square brackets notation quite frequently. It should be clear what it should mean, and we will define it properly later.

In general, elements in a ring do not have inverses. This is not a bad thing. This is what makes rings interesting. For example, the division algorithm would be rather contentless if everything in had an inverse. Fortunately, only has two invertible elements — 1 and -1. We call these *units*

Definition 4 (Unit) *An element $u \in R$ is a* unit *if there is another element $v \in R$ such that $u \cdot v = 1_R$.*

It is important that this depends on R, not just on u. For example, $2 \in$ is not a unit, but $2 \in$ is a unit (since $\frac{1}{2}$ is an inverse).

A special case is when (almost) everything is a unit.

Definition 5 (Field) *A* field *is a non-zero ring where every $u \neq 0_R \in R$ is a unit.*

We will later show that 0_R cannot be a unit unless in a very degenerate case.

Example 3 *is not a field, but , , are all fields.*

Similarly, $[i]$ *is not a field, while* $[\sqrt{2}]$ *is.*

Example 4 *Let R be a ring. Then $0_R + 0_R = 0_R$, since this is true in the group $(R, +, 0_R)$. Then for any $r \in R$, we get*

$$r \cdot (0_R + 0_R) = r \cdot 0_R.$$

We now use the fact that multiplication distributes over addition. So

$$r \cdot 0_R + r \cdot 0_R = r \cdot 0_R.$$

Adding $(-r \cdot 0_R)$ to both sides give

$$r \cdot 0_R = 0_R.$$

This is true for any element $r \in R$. From this, it follows that if $R \neq \{0\}$, then $1_R \neq 0_R$ — if they were equal, then take $r \neq 0_R$. So

$$r = r \cdot 1_R = r \cdot 0_R = 0_R,$$

which is a contradiction.

Note, however, that $\{0\}$ forms a ring (with the only possible operations and identities), the zero ring, albeit a boring one. However, this is often a counterexample to many things.

Definition 6 (Product of rings) *Let R, S be rings. Then the* product $R \times S$ *is a ring via*

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s) \cdot (r', s') = (r \cdot r', s \cdot s').$$

The zero is $(0_R, 0_S)$ and the one is $(1_R, 1_S)$.

We can (but won't) check that these indeed are rings.

Definition 7 (Polynomial) *Let R be a ring. Then a* polynomial *with coefficients in R is an expression*

$$f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n,$$

with $a_i \in R$. The symbols X^i are formal symbols.

We identify f and $f + 0_R \cdot X^{n+1}$ as the same things.

Definition 8 (Degree of polynomial) *The* degree *of a polynomial f is the largest m such that $a_m \neq 0$.*

Definition 9 (Monic polynomial) *Let f have degree m. If $a_m = 1$, then f is called* monic.

Definition 10 (Polynomial ring) *We write $R[X]$ for the set of all polynomials with coefficients in R. The operations are performed in the obvious way, i.e. if $f = a_0 + a_1X + \cdots + A_nX$ and $g = b_0 + b_1X + \cdots + b_kX^k$ are polynomials, then*

$$f + g = \sum_{r=0}^{\max\{n,k\}} (a_i + b_i)X^i,$$

and

$$f \cdot g = \sum_{i=0}^{n+k} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) X^i,$$

We identify R with the constant polynomials, i.e. polynomials $\sum a_iX^i$ with $a_i = 0$ for $i > 0$. In particular, $0_R \in R$ and $1_R \in R$ are the zero and one of $R[x]$.

This is in fact a ring.

Note that a polynomial is just a sequence of numbers, interpreted as the coefficients of some formal symbols. While it does indeed induce a function in the obvious way, we shall not identify the polynomial with the function given by it, since different polynomials can give rise to the same function.

For example, in $/2[X]$, $f = X^2 + X$ is not the zero polynomial, since its coefficients are not zero. However, $f(0) = 0$ and $f(1) = 0$. As a function, this is identically zero. So $f \neq 0$ as a polynomial but $f = 0$ as a function.

Definition 11 (Power series) *We write $R[[x]]$ for the ring of power series on R, i.e.*

$$f = a_0 + a_1X + a_2X^2 + \cdots ,$$

where each $a_i \in R$. This has addition and multiplication the same as for polynomials, but without upper limits.

A power series is very not a function. We don't talk about whether the sum converges or not, because it is not a sum.

Example 5 *Is $1 - X \in R[X]$ a unit? For every $g = a_0 + \cdots + a_nX^n$ (with $a_n \neq 0$), we get*

$$(1 - X)g = \mathit{stuff} + \cdots - a_nX^{n+1},$$

which is not 1*. So g cannot be the inverse of* $(1 - X)$*. So* $(1 - X)$ *is not a unit.*

However, $1 - x \in R[[X]]$ is *a unit, since*

$$(1 - X)(1 + X + X^2 + X^3 + \cdots) = 1.$$

Definition 12 (Laurent polynomials) *The* Laurent polynomials *on R is the set $R[X, X^{-1}]$, i.e. each element is of the form*

$$f = \sum_{i\in} a_iX^i$$

where $a_i \in R$ and only finitely many a_i are non-zero. The operations are the obvious ones.

We can also think of Laurent series, but we have to be careful. We allow infinitely many positive coefficients, but only finitely many negative ones. Or else, in the formula for multiplication, we will have an infinite sum, which is undefined.

Example 6 *Let X be a set, and R be a ring. Then the set of all functions on X, i.e. functions $f : X \to R$ is a ring given by*

$$(f + g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x) \cdot g(x).$$

Here zero is the constant function 0 *and one is the constant function* 1.

Usually, we don't want to consider all functions $X \to R$. Instead, we look at some subrings of this. For example, we can consider the ring of all continuous functions $\to$. This contains, for example, the polynomial functions, which is just $[X]$ *(since in , polynomials* are *functions).*

Possible Questions
8 marks

1. Prove that $(\mathbb{Z}_n, \oplus, \odot)$ is a ring
2. Let R be the ring with identity. Prove that the set of all units in R is a group under multiplication
3. Let F be any filed. Prove that the only ideals of F are F and $\{0\}$
4. Prove that the characteristic of an integral domain is either prime or 0
5. Prove that any finite integral domain is a field
6. Prove that $\mathbb{Z}_n$ is an integral domain iff n is prime
7. Prove that the only isomorphism $f : \mathbb{Q} \to \mathbb{Q}$ is the identity map
8. Prove that $\mathbb{Z}_n$ is an field iff n is prime
9. If U is an ideal of R and $1 \in U$ then prove that $U = R$
10. Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is an integral domain if and only if A is prime.
11. If two operations $*$ and $\bigcirc$ on the set of integers $\mathbb{Z}$ are defined by

$$a * b = a + b + 1$$

and

$$a \bigcirc b = a + b + ab$$

for all $a, b \in \mathbb{Z}$. Show that $(\mathbb{Z}, *, \bigcirc)$ is a commutative ring. What is the zero of the ring? Is it ring with unity?

12. Classify the ring $(S, \oplus_{10}, \odot_{10})$ where $S = \{0, 2, 4, 6, 8\}$. What is the unity of the ring. Is it a ring with or without zero divisors?

13. Show that a commutative ring with the cancellation property (under multiplication) has no zero-divisors.

14. List all zero-divisors in $\mathbb{Z}_{20}$

15. Prove that a finite integral domain is a field.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Ring theory and Linear Algebra I **Subject Code: 16MMU302**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
The unique identity of the additive group (R,+) is denoted by ___	0	1	2	3	0
{0} with __ binary operation	1	2	3	4	2
0 satisfies all the condition of ___	ring	ideal	Integral domain	Zero divisor	ring
(p(s),u,n) is ____	ring	Not a ring	Boolean ring	field	Not a ring
Boolean ring example	(p(s),u,n)	(p(s),Δ,n)	Q	R	(p(s),Δ,n)
A ring R is said to be _____ring if ab=ba	commutative	boolean	Null ring	B and a	Commutative
A ring R is said to be commutative ring if_____	ab≠ba	ab=ba	a=2a	a=b	ab=ba
The familiar rings Z,Q,R are all rings with _____	identity	unit	Zero divisor	ideal	identity
M2(R)IS A Ring with unity	(1 0	(0 0	(1 2	(0 1	(1 0
A ring with identity . then tha identity element is	unique	differnce	2	Either b or c	unique
A ring (R, +,.) all the _____ is a unit	Non zero	zero	evennumber	oddnumber	nonzero
A commutative skew field is called	Integral domain	field	ideal	Zero divisor	field
In a ring (Z12,+,.). the one of the zero divisor is	10	11	5	2	2
Skew field has ______	Zero divisor	No zero divisor	Integral domain	b and c	b and c
Zn is an integral domain iff n is _____	composite	prime	even	odd	prime
Any field F is an ____	Integral domain	Not integral domain	ideal	Either b or c	Integral domain
Any finite integral domain is ____	Not a field	field	Right ideal	ideal	Field
Z is not a ___	Ring	subring	field	Either b or c	Field
The only idempotent element of an integral domain are	2and 0	1 and 2	0 and1	3and5	0 and 1

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DEPARTMENT OF MATHEMATICS

Subject: Ring theory & Linear algebra I	**Semester: III**	**L T P C**
Subject Code: 16MMU302	**Class: II-B.Sc. Mathematics**	**6 2 0 6**

UNIT II

Ring homomorphisms, properties of ring homomorphisms, Isomorphism theorems I, II and III, field of quotients

TEXT BOOK

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REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

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1 Homomorphisms, ideals and quotients

Just like groups, we will come up with analogues of homomorphisms, normal subgroups (which are now known as ideals), and quotients.

Definition 1 (Homomorphism of rings) *Let R, S be rings. A function $\phi : R \to S$ is a* ring homomorphism *if it preserves everything we can think of, i.e.*

1. $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$,
2. $\phi(0_R) = 0_S$,
3. $\phi(r_1 \cdot r_2) = \phi(r_1) \cdot \phi(r_2)$,
4. $\phi(1_R) = 1_S$.

Definition 2 (Isomorphism of rings) *If a homomorphism $\phi : R \to S$ is a bijection, we call it an* isomorphism.

Definition 3 (Kernel) *The* kernel *of a homomorphism $\phi : R \to S$ is*

$$\ker(\phi) = \{r \in R : \phi(r) = 0_S\}.$$

Definition 4 (Image) *The* image *of $\phi : R \to S$ is*

$$(\phi) = \{s \in S : s = \phi(r) \textit{ for some } r \in R\}.$$

A homomorphism $\phi : R \to S$ is injective if and only if $\ker \phi = \{0_R\}$.

A ring homomorphism is in particular a group homomorphism $\phi : (R, +, 0_R) \to (S, +, 0_S)$ of abelian groups. So this follows from the case of groups.

In the group scenario, we had groups, subgroups and *normal* subgroups, which are special subgroups. Here, we have a special kind of subsets of a ring that act like normal subgroups, known as *ideals*.

Definition 5 (Ideal) *A subset $I \subseteq R$ is an* ideal*, written $I \lhd R$, if*

1. *It is an additive subgroup of* $(R, +, 0_R)$*, i.e. it is closed under addition and additive inverses.* *(additive closure)*

2. *If* $a \in I$ *and* $b \in R$*, then* $a \cdot b \in I$*.* *(strong closure)*

We say I *is a proper ideal if* $I \neq R$*.*

Note that the multiplicative closure is stronger than what we require for subrings — for subrings, it has to be closed under multiplication by its own elements; for ideals, it has to be closed under multiplication by everything in the world. This is similar to how normal subgroups not only have to be closed under internal multiplication, but also conjugation by external elements.

If $\phi : R \to S$ is a homomorphism, then $\ker(\phi) \lhd R$.

Since $\phi : (R, +, 0_R) \to (S, +, 0_R)$ is a group homomorphism, the kernel is a subgroup of $(R, +, 0_R)$.

For the second part, let $a \in \ker(\phi)$, $b \in R$. We need to show that their product is in the kernel. We have

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) = 0 \cdot \phi(b) = 0.$$

So $a \cdot b \in \ker(\phi)$.

Example 1 *Suppose* $I \lhd R$ *is an ideal, and* $1_R \in I$*. Then for any* $r \in R$*, the axioms entail* $1_R \cdot r \in I$*. But* $1_R \cdot r = r$*. So if* $1_R \in I$*, then* $I = R$*.*

In other words, every proper ideal does not contain 1*. In particular, every proper ideal is not a subring, since a subring must contain* 1*.*

We are starting to diverge from groups. In groups, a normal subgroup is a subgroup, but here an ideal is not a subring.

Example 2 *We can generalize the above a bit. Suppose* $I \lhd R$ *and* $u \in I$ *is a unit, i.e. there is some* $v \in R$ *such that* $uv = 1_R$*. Then by strong closure,* $1_R = u \cdot v \in I$*. So* $I = R$*.*

Hence proper ideals are not allowed to contain any unit at all, not just 1_R*.*

Example 3 *Consider the ring of integers. Then every ideal of is of the form*

$$n = \{\cdots, -2n, -n, 0, n, 2n, \cdots\} \subseteq .$$

It is easy to see this is indeed an ideal.

To show these are all the ideals, let $I \lhd$. If $I = \{0\}$, then $I = 0$. Otherwise, let $n \in N$ be the smallest positive element of T. We want to show in fact $I = n$. Certainly $n \subseteq I$ by strong closure.

Now let $m \in I$. By the Euclidean algorithm, we can write

$$m = q \cdot n + r$$

with $0 \leq r < n$. Now $n, m \in I$. So by strong closure, $m, qn \in I$. So $r = m - q \cdot n \in I$. As n is the smallest positive element of I, and $r < n$, we must have $r = 0$. So $m = q \cdot n \in n$. So $I \subseteq n$. So $I = n$.

The key to proving this was that we can perform the Euclidean algorithm on . Thus, for any ring R in which we can "do Euclidean algorithm", every ideal is of the form $aR = \{a \cdot r : r \in R\}$ for some $a \in R$. We will make this notion precise in later.

Definition 6 (Generator of ideal) *For an element $a \in R$, we write*

$$(a) = aR = \{a \cdot r : r \in R\} \lhd R.$$

This is the ideal generated by a.

In general, let $a_1, a_2, \cdots, a_k \in R$, we write

$$(a_1, a_2, \cdots, a_k) = \{a_1 r_1 + \cdots + a_k r_k : r_1, \cdots, r_k \in R\}.$$

This is the ideal generated by $a_1, \cdots, a_k$.

We can also have ideals generated by infinitely many objects, but we have to be careful, since we cannot have infinite sums.

Definition 7 (Generator of ideal) *For $A \subseteq R$ a subset, the* ideal generated by *A is*

$$(A) = \left\{ \sum_{a \in A} r_a \cdot a : r_a \in R, \text{ only finitely-many non-zero} \right\}.$$

These ideals are rather nice ideals, since they are easy to describe, and often have some nice properties.

Definition 8 (Principal ideal) *An ideal I is a* principal ideal *if $I = (a)$ for some $a \in R$.*

So what we have just shown for is that all ideals are principal. Not all rings are like this. These are special types of rings, which we will study more in depth later.

Example 4 *Consider the following subset:*

$$\{f \in [X] : \text{ the constant coefficient of } f \text{ is } 0\}.$$

This is an ideal, as we can check manually (alternatively, it is the kernel of the "evaluate at 0" homomorphism). It turns out this is a principal ideal. In fact, it is (X).

We have said ideals are like normal subgroups. The key idea is that we can divide by ideals.

Definition 9 (Quotient ring) *Let $I \lhd R$. The* quotient ring *R/I consists of the (additive) cosets $r + I$ with the zero and one as $0_R + I$ and $1_R + I$, and operations*

$$(r_1 + I) + (r_2 + I) = (r_1 + r_2) + I$$
$$(r_1 + I) \cdot (r_2 + I) = r_1 r_2 + I.$$

The quotient ring is a ring, and the function

$$R \to R/I$$
$$r \mapsto r + I$$

is a ring homomorphism. This is true, because we defined ideals to be those things that can be quotiented by. So we just have to check we made the right definition.

Just as we could have come up with the definition of a normal subgroup by requiring operations on the cosets to be well-defined, we could have come up with the definition of an ideal by requiring the multiplication of cosets is well-defined, and we will end up with the strong closure property.

We know the group $(R/I, +, 0_{R/I})$ is well-defined, since I is a (normal) subgroup of R. So we only have to check multiplication is well-defined.

Suppose $r_1 + I = r_1' + I$ and $r_2 + I = R_2' + I$. Then $r_1' - r_1 = a_1 \in I$ and $r_2' - r_2 = a_2 \in I$. So

$$r_1' r_2' = (r_1 + a_1)(r_2 + a_2) = r_1 r_2 + r_1 a_2 + r_2 a_1 + a_1 a_1.$$

By the strong closure property, the last three objects are in I. So $r_1' r_2' + I = r_1 r_2 + I$.

It is easy to check that $0_R + I$ and $1_R + I$ are indeed the zero and one, and the function given is clearly a homomorphism.

Example 5 *We have the ideals $n \lhd$. So we have the quotient ring $/n$. The elements are of the form $m + n$, so are just*

$$0 + n, 1 + n, 2 + n, \cdots, (n - 1) + n.$$

Addition and multiplication is just what we are used to — it is addition and multiplication modulo n.

Note that it is easier to come up with ideals than normal subgroups — we can just pick up random elements, and then take the ideal generated by them.

Example 6 *Consider $(X) \lhd [X]$. What is $[X]/(X)$? Elements are represented by*

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n + (X).$$

But everything but the first term is in (X). So every such thing is equivalent to $a_0 + (X)$. It is not hard to convince yourself that this representation is unique. So in fact $[X]/(X) \cong$, with the bijection $a_0 + (X) \leftrightarrow a_0$.

If we want to prove things like this, we have to convince ourselves this representation is unique. We can do that by hand here, but in general, we want to be able to do this properly.

[Euclidean algorithm for polynomials] Let be a field and $f, g \in [X]$. Then there is some $r, q \in [X]$ such that

$$f = gq + r,$$

with $\deg r < \deg g$. This is like the usual Euclidean algorithm, except that instead of the absolute value, we use the degree to measure how “big” the polynomial is.

Let $\deg(f) = n$. So

$$f = \sum_{i=0}^{n} a_i X^i,$$

and $a_n \neq 0$. Similarly, if $\deg g = m$, then

$$g = \sum_{i=0}^{m} b_i X^i,$$

with $b_m \neq 0$. If $n < m$, we let $q = 0$ and $r = f$, and done.

Otherwise, suppose $n \geq m$, and proceed by induction on n.

We let

$$f_1 = f - a_n b_m^{-1} X^{n-m} g.$$

This is possible since $b_m \neq 0$, and is a field. Then by construction, the coefficients of X^n cancel out. So $\deg(f_1) < n$.

If $n = m$, then $\deg(f_1) < n = m$. So we can write

$$f = (a_n b_m^{-1} X^{n-m}) g + f_1,$$

and $\deg(f_1) < \deg(f)$. So done. Otherwise, if $n > m$, then as $\deg(f_1) < n$, by induction, we can find r_1, q_1 such that

$$f_1 = gq_1 + r_1,$$

and $\deg(r_1) < \deg g = m$. Then

$$f = a_n b_m^{-1} X^{n-m} g + q_1 g + r_1 = (a_n b_m^{-1} X^{n-m} + q_1)g + r_1.$$

So done. Now that we have a Euclidean algorithm for polynomials. So we should be able to show that every ideal of $[X]$ is generated by one polynomial. We will not prove it specifically here, but later show that in *general*, in every ring where the Euclidean algorithm is possible, all ideals are principal.

We now look at some applications of the Euclidean algorithm.

Example 7 *Consider* $[X]$*, and consider the principal ideal* $(X^2+1) \lhd [X]$*. We let* $R = [X]/(X^2+1)$*.*

Elements of R are polynomials

$$\underbrace{a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n}_{f} + (X^2+1).$$

By the Euclidean algorithm, we have

$$f = q(X^2+1) + r,$$

with $\deg(r) < 2$*, i.e* $r = b_0 + b_1 X$*. Thus* $f + (X^2+1) = r + (X^2+1)$*. So every element of* $[X]/(X^2+1)$ *is representable as* $a + bX$ *for some* $a, b \in R$*.*

Is this representation unique? If $a + bX + (X^2+1) = a' + b'X + (X^2+1)$*, then the difference* $(a - a') + (b - b')X \in (X^2+1)$*. So it is* $(X^2+1)q$ *for some* q*. This is possible only if* $q = 0$*, since for non-zero* q*, we know* $(X^2+1)q$ *has degree at least* 2*. So we must have* $(a - a') + (b - b')X = 0$*. So* $a + bX = a' + b'X$*. So the representation is unique.*

What we've got is that every element in R is of the form $a + bX$*, and* $X^2 + 1 = 0$*, i.e.* $X^2 = -1$*. This sounds like the complex numbers, just that we are calling it X instead of i.*

To show this formally, we define the function

$$\begin{aligned} \phi : [x]/(X^2+1) &\to \\ a + bX + (X^2+1) &\mapsto a + bi. \end{aligned}$$

This is well-defined and a bijection. It is also clearly additive. So to prove this is an isomorphism, we have to show it is multiplicative. We check this manually. We have

$$\begin{aligned}
&\phi((a + bX + (X^2 + 1))(c + dX + (X^2 + 1)))\\
&= \phi(ac + (ad + bc)X + bdX^2 + (X^2 + 1))\\
&= \phi((ac - bd) + (ad + bc)X + (X^2 + 1))\\
&= (ac - bd) + (ad + bc)i\\
&= (a + bi)(c + di)\\
&= \phi(a + bX + (X^2 + 1))\phi(c + dX + (X^2 + 1)).
\end{aligned}$$

So this is indeed an isomorphism.

Possible Questions
8 marks

1. Prove that the field of quotients F of an integral domain D is the smallest field containing D
2. State and prove first theorem isomorphism of rings
3. Let $f : \mathbb{Z} \to \mathbb{Z}_n$ be defined by $f(x) = r$ if $x = qn + r, 0 \leq r < n$. Prove that f is a homomorphism
4. State and prove fundamental theorem of homomorphism of rings
5. State and prove two properties of homomorphism
6. Let $f : R \to R'$ be a homomorphism. Prove that $Ker\ f$ is an ideal of R
7. If D and D' are isomorphic integral domains then prove that their field of quotients are also isomorphic
8. State and prove fundamental theorem of homomorphism
9. State and prove first theorem isomorphism of rings
10. If D and D'

    are isomorphic integral domains then prove that their field of quotients are also isomorphic
11. Let $f : R \to R'$ be a homomorphism and K be the kernal of f. Prove that K is an ideal of R
12. Prove that any integral domain D can be embedded in a field F and every element of F can be expressed as a quotient of two elements of D
13. State and prove isomorphism theorem I

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Ring theory and Linear Algebra I **Subject Code: 16MMU302**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
For any positive integer n, the set nZ={0,±n,±2n,…}is an ideal of ____	Z	-2Z	Zn	-3Z	Z
Φ is a ring homo morphism . a,bϵΦ.	Φ(a)+Φ(-b)	Φ(-a)+Φ(-b)	Φ(a)+Φ(b)	Φ(a)-Φ(b)	Φ(a)+Φ(b)
Let R be a commutative ring of characteristic 2. Then the mapping f:a to a^2 is a ring	Homomorphism	Isomorphism	Endomorphism	monomorphism	Homomorphism
Let Φ be a ring homo morphism from R TO S.then ker Φ={rϵR\Φ(r)=0} is	An integral domain of R	An ideal of R	field	Not field	An ideal of R
If G is a group of order____,where p is prime then G is abilian	P^3	P^5	P^2	P^4	P^2
A ___ with 2 binary operation	Group	Abelian group	Ring	B and c	Ring
___ satisfies all the condition of ring	{2,3}	{0}	N	w	{0}
Let a,b,c belong to a ring R .then a0=0=	1a	2a	3a	0a	oa
Let a belong to a ring R .then (-1)a=?	-1	0	-a	a	-a
The ring of integer is _____	field	Integral domain	Zero divisor	a or c	Integral domain
Skew field has _____	Zero divisor	No zero divisor	Atleast One zero divisor	Many zero divisor	No zero divisor
Zn is field iff n is	composite	prime	Even	odd	prime
Zn is____ iff n is prime	field	Integral domain	ideal	a and b	a and b
{Z} is finite but Z is not	Integral domain	Field	Not field	Not integral domain	Not fieldfd
Any Boolean ring charectreistic is	2	4	6	0	2
Q is a subring of	W	Z	2Z	R	R
The _____ of two subring of a ring is a subring	union	intersection	Symmetric difference	A nad c	intersection
2Z AND 3Z are _______ of Z	subring	Not a subring	Either a or b	Neither a nor b	Not a subring
If F is a_____ its only ideals are {0} and F	FIELD	Not field	Integral domain	Integral domain	field
Principal ideal is a and itsdenoted by	(a)	{a}	[a]	a	(a)
R is a commutative ring with identity 1.then aϵ	(a)	{a}	[a]	a	(a)
___ is a maximal ideal of Z	-2	-4	-6	-8	-2
___is a maximal ideal of Z	-3	-6	-9	-12	(3
A Homomorphism that is both one one and onto is called _______________ homomorphism.	Ring	Polynomial	Group	Cyclic	Ring
For any positive integer n, the mapping K→K mod n is ___________ from ℤ to ℤ□	Ring	Polynomial	Group	Cyclic	Ring
The mapping from ℤ to ℤ□ is called _____________	Ring	Cyclic	Natural	polynomial	Natural
Every ideal of a ring is of a ring homomorphism of ℝ	Kernel	Subring	Homomorphism	Maximal ideal	Kernel
If ℝ is a ring with unity and the characteristic of ℝ is n>0, then ℝ contain a subring ____________ to ℤ□	isomorphic	monomorphic	homomorphic	epimorphic	Isomorphic
Let F be a field, aϵF and f(x)ϵf[x] then f(a) is the remainder in the division of f(x) by ______________	a-x	x-a	a x^2	x^2 -a	x-a
A polynomial degree n over a field has almost n zeros ____________ multiplicity	Counting	Not defined	finite	infinite	Counting
Let F be a field aϵF and f(x)ϵf[x] then a is a Zero of f(x) iff (x,a) is a ___________ of f(x)	Dividend	Factor	multiplier	Quotient	Factor
A principal ideal domain is an integral domain ℝ in which every ideal has the form ___________ for some a in ℝ	<a>=[ra:rϵ ℝ]	<a>=[ar:rϵ ℝ]	<a>=[r/a:rϵ ℝ]	<a>=[r/a:rϵ ℝ]	<a>=[ra:rϵ ℝ]
Let F be a field I a non Zero ideal in f(x) and g(x) an element of f(x) then I=g(x) is a non Zero polynomial of __________ degree in I	minimum	maximum	least	Highest	minimum
Let F be field aϵF and f(x) ϵf(x) then f(a) is the division of f(x) by (x-a)	Quotient	Remainder	multiplier	dividend	remainder
Let D be an integral domain. Then there exits a fields F that contains a subring _____________ to D	isomorphic	monomorphic	Epimorphic	homomorphic	isomorphic
Let ℝ be a ring with unity 1. The mapping ℤ→ℝ given by n→n:1 is a ____________ homomorphism	Ring	cyclic	isomorphic	ideal	ring

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DEPARTMENT OF MATHEMATICS

Subject: Ring theory & Linear algebra I	**Semester: III**	**L T P C**
Subject Code: 16MMU302	**Class: II-B.Sc. Mathematics**	**6 2 0 6**

UNIT III

Vector spaces, subspaces, algebra of subspaces, quotient spaces, linear combination of vectors, linear span, linear independence, basis and dimension, dimension of subspaces.

TEXT BOOK

1. Fraleigh.J.B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

2. Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

[First isomorphism theorem] Let $\phi : R \to S$ be a ring homomorphism. Then $\ker(\phi) \lhd R$, and

$$\frac{R}{\ker(\phi)} \cong (\phi) \leq S.$$

We have already seen $\ker(\phi) \lhd R$. Now define

$$\begin{aligned} \Phi : R/\ker(\phi) &\to (\phi) \\ r + \ker(\phi) &\mapsto \phi(r). \end{aligned}$$

This well-defined, since if $r + \ker(\phi) = r' + \ker(\phi)$, then $r - r' \in \ker(\phi)$. So $\phi(r - r') = 0$. So $\phi(r) = \phi(r')$.

We don't have to check this is bijective and additive, since that comes for free from the (proof of the) isomorphism theorem of groups. So we just have to check it is multiplicative. To show Φ is multiplicative, we have

$$\begin{aligned} \Phi((r + \ker(\phi))(t + \ker(\phi))) &= \Phi(rt + \ker(\phi)) \\ &= \phi(rt) \\ &= \phi(r)\phi(t) \\ &= \Phi(r + \ker(\phi))\Phi(t + \ker(\phi)). \end{aligned}$$

This is more-or-less the same proof as the one for groups, just that we had a few more things to check.

Since there is the *first* isomorphism theorem, we, obviously, have more coming.

[Second isomorphism theorem] Let $R \leq S$ and $J \lhd S$. Then $J \cap R \lhd R$, and

$$\frac{R + J}{J} = \{r + J : r \in R\} \leq \frac{S}{J}$$

is a subring, and

$$\frac{R}{R \cap J} \cong \frac{R + J}{J}.$$

Define the function

$$\begin{aligned}\phi : R &\to S/J\\ r &\mapsto r + J.\end{aligned}$$

Since this is the quotient map, it is a ring homomorphism. The kernel is

$$\ker(\phi) = \{r \in R : r + J = 0, \text{ i.e. } r \in J\} = R \cap J.$$

Then the image is

$$(\phi) = \{r + J : r \in R\} = \frac{R + J}{J}.$$

Then by the first isomorphism theorem, we know $R \cap J \lhd R$, and $\frac{R+J}{J} \leq S$, and

$$\frac{R}{R \cap J} \cong \frac{R + J}{J}.$$

Before we get to the third isomorphism theorem, recall we had the subgroup correspondence for groups. Analogously, for $I \lhd R$,

$$\begin{aligned}\{\text{subrings of } R/I\} &\longleftrightarrow \{\text{subrings of } R \text{ which contain } I\}\\ L \leq \frac{R}{I} &\longrightarrow \{x \in R : x + I \in L\}\\ \frac{S}{I} \leq \frac{R}{I} &\longleftarrow I \lhd S \leq R.\end{aligned}$$

This is exactly the same formula as for groups.

For groups, we had a correspondence for normal subgroups. Here, we have a correspondence between ideals

$$\{\text{ideals of } R/I\} \longleftrightarrow \{\text{ideals of } R \text{ which contain } I\}$$

It is important to note here quotienting in groups and rings have different purposes. In groups, we take quotients so that we have a simpler group to work with. In rings, we often take quotients to get more interesting rings. For example, $[X]$ is quite boring, but $[X]/(X^2 + 1) \cong$ is more interesting. Thus this ideal correspondence allows us to occasionally get interesting ideals from boring ones.

[Third isomorphism theorem] Let $I \lhd R$ and $J \lhd R$, and $I \subseteq J$. Then $J/I \lhd R/I$ and

$$\left(\frac{R}{I}\right)/\left(\frac{J}{I}\right) \cong \frac{R}{J}.$$

We define the map

$$\begin{aligned} \phi : R/I &\to R/J \\ r + I &\mapsto r + J. \end{aligned}$$

This is well-defined and surjective by the groups case. Also it is a ring homomorphism since multiplication in R/I and R/J are “the same”. The kernel is

$$\ker(\phi) = \{r + I : r + J = 0, \text{ i.e. } r \in J\} = \frac{J}{I}.$$

So the result follows from the first isomorphism theorem.

Note that for any ring R, there is a unique ring homomorphism $\to R$, given by

$$\begin{aligned} \iota : &\to R \\ n \geq 0 &\mapsto \underbrace{1_R + 1_R + \cdots + 1_R}_{n \text{ times}} \\ n \leq 0 &\mapsto -(\underbrace{1_R + 1_R + \cdots + 1_R}_{-n \text{ times}}) \end{aligned}$$

Any homomorphism $\to R$ must be given by this formula, since it must send the unit to the unit, and we can show this is indeed a homomorphism by distributivity. So the ring homomorphism is unique. In fancy language, we say is the initial object in (the category of) rings.

We then know $\ker(\iota) \lhd$. Thus $\ker(\iota) = n$ for some n.

Definition 1 (Characteristic of ring) *Let R be a ring, and $\iota :\to R$ be the unique such map. The* characteristic *of R is the unique non-negative n such that* $\ker(\iota) = n$.

Example 1 *The rings , , , all have characteristic* 0*. The ring* $/n$ *has characteristic* n*. In particular, all natural numbers can be characteristics.*

The notion of the characteristic will not be too useful in this course. However, fields of non-zero characteristic often provide interesting examples and counterexamples to some later theory.

0.1 Integral domains, field of factions, maximal and prime ideals

Many rings can be completely nothing like . For example, in , we know that if $a, b \neq 0$, then $ab \neq 0$. However, in, say, $/6$, we get $2, 3 \neq 0$, but $2 \cdot 3 = 0$. Also, has some nice properties such as every ideal is principal, and every integer has an (essentially) unique factorization. We will now classify rings according to which properties they have.

We start with the most fundamental property that the product of two non-zero elements are non-zero. We will almost exclusively work with rings that satisfy this property.

Definition 2 (Integral domain) *A non-zero ring* R *is an* integral domain *if for all* $a, b \in R$*, if* $a \cdot b = 0_R$*, then* $a = 0_R$ *or* $b = 0_R$*.*

An element that violates this property is known as a *zero divisor*.

Definition 3 (Zero divisor) *An element* $x \in R$ *is a* zero divisor *if* $x \neq 0$ *and there is a* $y \neq 0$ *such that* $xy = 0 \in R$*.*

In other words, a ring is an integral domain if it has no zero divisors.

Example 2 *All fields are integral domains, since if* $a \cdot b = 0$*, and* $b \neq 0$*, then* $a = a \cdot (b \cdot b^{-1}) = 0$*. Similarly, if* $a \neq 0$*, then* $b = 0$*.*

Example 3 *A subring of an integral domain is an integral domain, since a zero divisor in the small ring would also be a zero divisor in the big ring.*

Example 4 *Immediately, we know , , , are integral domains, since is a field, and the others are subrings of it. Also,* $[i] \leq$ *is also an integral domain.*

These are the nice rings we like in number theory, since there we can sensibly talk about things like factorization.

It turns out there are no interesting finite integral domains. Let R be a finite ring which is an integral domain. Then R is a field.

Let $a \in R$ be non-zero, and consider the ring homomorphism

$$\begin{aligned} a \cdot - : R &\to R \\ b &\mapsto a \cdot b \end{aligned}$$

We want to show this is injective. For this, it suffices to show the kernel is trivial. If $r \in \ker(a \cdot -)$, then $a \cdot r = 0$. So $r = 0$ since R is an integral domain. So the kernel is trivial.

Since R is finite, $a \cdot -$ must also be surjective. In particular, there is an element $b \in R$ such that $a \cdot b = 1_R$. So a has an inverse. Since a was arbitrary, R is a field.

So far, we know fields are integral domains, and subsets of integral domains are integral domains. We have another good source of integral domain as follows: Let R be an integral domain. Then $R[X]$ is also an integral domain.

We need to show the product of two non-zero elements are non-zero. Let $f, g \in R[X]$ be non-zero, say

$$\begin{aligned} f &= a_0 + a_1X + \cdots + a_nX^n \in R[X] \\ g &= b_0 + b_1X + \cdots + b_mX^m \in R[X], \end{aligned}$$

with $a_n, b_m \neq 0$. Then the coefficient of X^{n+m} in fg is a_nb_m. This is non-zero since R is an integral domain. So fg is non-zero. So $R[X]$ is an integral domain. So, for instance, $[X]$ is an integral domain.

We can also iterate this.

Write $R[X,Y]$ for $(R[X])[Y]$, the polynomial ring of R in two variables. In general, write $R[X_1,\cdots,X_n] = (\cdots((R[X_1])[X_2])\cdots)[X_n]$.

Then if R is an integral domain, so is $R[X_1,\cdots,X_n]$.

We now mimic the familiar construction of from . For any integral domain R, we want to construct a field F that consists of "fractions" of elements in R. Recall that the subring of any field is an integral domain. This says the converse — every integral domain is the subring of some field.

Definition 4 (Field of fractions) *Let R be an integral domain. A* field of fractions *F of R is a field with the following properties*

1. *$R \leq F$*
2. *Every element of F may be written as $a \cdot b^{-1}$ for $a,b \in R$, where b^{-1} means the multiplicative inverse to $b \neq 0$ in F.*

For example, is the field of fractions of .

Every integral domain has a field of fractions.

The construction is exactly how we construct the rationals from the integers — as equivalence classes of pairs of integers. We let

$$S = \{(a,b) \in R \times R : b \neq 0\}.$$

We think of $(a,b) \in S$ as $\frac{a}{b}$. We define the equivalence relation $\sim$ on S by

$$(a,b) \sim (c,d) \Leftrightarrow ad = bc.$$

We need to show this is indeed a equivalence relation. Symmetry and reflexivity are obvious. To show transitivity, suppose

$$(a,b) \sim (c,d), \quad (c,d) \sim (e,f),$$

i.e.

$$ad = bc, \quad cf = de.$$

We multiply the first equation by f and the second by b, to obtain

$$adf = bcf, \quad bcf = bed.$$

Rearranging, we get

$$d(af - be) = 0.$$

Since d is in the denominator, $d \neq 0$. Since R is an integral domain, we must have $af - be = 0$, i.e. $af = be$. So $(a, b) \sim (e, f)$. This is where being an integral domain is important.

Now let

$$F = S/\sim$$

be the set of equivalence classes. We now want to check this is indeed the field of fractions. We first want to show it is a field. We write $\frac{a}{b} = [(a, b)] \in F$, and define the operations by

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd}. \end{aligned}$$

This *is* well-defined, and makes $(F, +, \cdot, \frac{0}{1}, \frac{1}{1})$ into a ring. There are many things to check, but those are straightforward, and we will not waste time doing that here.

Finally, we need to show every non-zero element has an inverse. Let $\frac{a}{b} \neq 0_F$, i.e. $\frac{a}{b} \neq \frac{0}{1}$, or $a \cdot 1 \neq b \cdot 0 \in R$, i.e. $a \neq 0$. Then $\frac{b}{a} \in F$ is defined, and

$$\frac{b}{a} \cdot \frac{a}{b} = \frac{ba}{ba} = 1.$$

So $\frac{a}{b}$ has a multiplicative inverse. So F is a field.

We now need to construct a subring of F that is isomorphic to R. To do so, we need to define an injective isomorphism $\phi : R \to F$. This is given by

$$\begin{aligned} \phi : R &\to F \\ r &\mapsto \frac{r}{1}. \end{aligned}$$

This is a ring homomorphism, as one can check easily. The kernel is the set of all $r \in R$ such that $\frac{r}{1} = 0$, i.e. $r = 0$. So the kernel is trivial, and ϕ is injective. Then by the first isomorphism theorem, $R \cong (\phi) \subseteq F$.

Finally, we need to show everything is a quotient of two things in R. We have

$$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = \frac{a}{1} \cdot \left(\frac{b}{1}\right)^{-1},$$

as required. This gives us a very useful tool. Since this gives us a field from an integral domain, this allows us to use field techniques to study integral domains. Moreover, we can use this to construct new interesting fields from integral domains.

Example 5 *Consider the integral domain* $[X]$. *Its field of fractions is the field of all rational functions* $\frac{p(X)}{q(X)}$, *where* $p, q \in [X]$.

To some people, it is a shame to think of rings as having elements. Instead, we should think of a ring as a god-like object, and the only things we should ever mention are its ideals. We should also not think of the ideals as containing elements, but just some abstract objects, and all we know is how ideals relate to one another, e.g. if one contains the other.

Under this philosophy, we can think of a field as follows: A (non-zero) ring R is a field if and only if its only ideals are $\{0\}$ and R. Note that we don't need elements to define the ideals $\{0\}$ and R. $\{0\}$ can be defined as the ideal that all other ideals contain, and R is the ideal that contains all other ideals. Alternatively, we can reword this as "R is a field if and only if it has only two ideals" to avoid mentioning explicit ideals.

($\Rightarrow$) Let $I \lhd R$ and R be a field. Suppose $x \neq 0 \in I$. Then as x is a unit, $I = R$.

($\Leftarrow$) Suppose $x \neq 0 \in R$. Then (x) is an ideal of I. It is not $\{0\}$ since it contains x. So $(x) = R$. In other words $1_R \in (x)$. But (x) is defined to be $\{x \cdot y : y \in R\}$. So there is some $u \in R$ such that $x \cdot u = 1_R$. So x is a unit. Since x was arbitrary, so R is a field. This is another reason why fields are special. They have the simplest possible ideal structure.

This motivates the following definition:

Definition 5 (Maximal ideal) *An ideal I of a ring R is* maximal *if $I \neq R$ and for any ideal J with $I \leq J \leq R$, either $J = I$ or $J = R$.*

The relation with what we've done above is quite simple. There is an easy way to recognize if an ideal is maximal.

An ideal $I \lhd R$ is maximal if and only if R/I is a field.

R/I is a field if and only if $\{0\}$ and R/I are the only ideals of R/I. By the ideal correspondence, this is equivalent to saying I and R are the only ideals of R which contains I, i.e. I is maximal. So done. This is a nice result. This makes a correspondence between properties of ideals I and properties of the quotient R/I. Here is another one:

Definition 6 (Prime ideal) *An ideal I of a ring R is* prime *if $I \neq R$ and whenever $a, b \in R$ are such that if $a \cdot b \in I$, then $a \in I$ or $b \in I$.*

This is like the opposite of the property of being an ideal — being an ideal means if we have something in the ideal and something outside, the product is always in the ideal. This does the backwards. If the product of two random things is in the ideal, then one of them must be from the ideal.

Example 6 *A non-zero ideal $n \lhd$ is prime if and only if n is a prime.*

To show this, first suppose $n = p$ is a prime, and $a \cdot b \in p$. So $p \mid a \cdot b$. So $p \mid a$ or $p \mid b$, i.e. $a \in p$ or $b \in p$.

For the other direction, suppose $n = pq$ is a composite number ($p, q \neq 1$). Then $n \in n$ but $p \notin n$ and $q \notin n$, since $0 < p, q < n$.

So instead of talking about prime numbers, we can talk about prime ideals instead, because ideals are better than elements.

We prove a result similar to the above: An ideal $I \lhd R$ is prime if and only if R/I is an integral domain.

Let I be prime. Let $a + I, b + I \in R/I$, and suppose $(a + I)(b + I) = 0_{R/I}$. By definition, $(a + I)(b + I) = ab + I$. So we must have $ab \in I$. As I is prime, either $a \in I$ or $b \in I$. So $a + I = 0_{R/I}$ or $b + I = 0_{R/I}$. So R/I is an integral domain.

Conversely, suppose R/I is an integral domain. Let $a, b \in R$ be such that $ab \in I$. Then $(a + I)(b + I) = ab + I = 0_{R/I} \in R/I$. Since R/I is an integral domain, either $a + I = 0_{R/I}$ or $b + I = 0_{R/i}$, i.e. $a \in I$ or $b \in I$. So I is a prime ideal.

Prime ideals and maximal ideals are the main types of ideals we care about. Note that every field is an integral domain. So we immediately have the following result: Every maximal ideal is a prime ideal.

$I \lhd R$ is maximal implies R/I is a field implies R/I is an integral domain implies I is prime. The converse is not true. For example, $\{0\} \subseteq$ is prime but not maximal. Less stupidly, $(X) \in [X, Y]$ is prime but not maximal (since $[X, Y]/(X) \cong [Y]$). We can provide a more explicit proof of this, which is essentially the same.

[Alternative proof] Let I be a maximal ideal, and suppose $a, b \notin I$ but $ab \in I$. Then by maximality, $I + (a) = I + (b) = R = (1)$. So we can find some $p, q \in R$ and $n, m \in I$ such that $n + ap = m + bq = 1$. Then

$$1 = (n + ap)(m + bq) = nm + apm + bqn + abpq \in I,$$

since $n, m, ab \in I$. This is a contradiction.

Let R be an integral domain. Then its characteristic is either 0 or a prime number.

Consider the unique map $\phi :\to R$, and $\ker(\phi) = n$. Then n is the characteristic of R by definition.

By the first isomorphism theorem, $/n = (\phi) \leq R$. So $/n$ is an integral domain. So $n \lhd$ is a prime. So $n = 0$ or a prime number.

0.2 Factorization in integral domains

We now move on to tackle the problem of factorization in rings. For sanity, we suppose throughout the section that R is an integral domain. We start by making loads of definitions.

Definition 7 (Unit) *An element $a \in R$ is a* unit *if there is an $b \in R$ such that $ab = 1_R$. Equivalently, if the ideal $(a) = R$.*

Definition 8 (Division) *For elements $a, b \in R$, we say a* divides *b, written $a \mid b$, if there is a $c \in R$ such that $b = ac$. Equivalently, if $(b) \subseteq (a)$.*

Definition 9 (Associates) *We say $a, b \in R$ are* associates *if $a = bc$ for some unit c. Equivalently, if $(a) = (b)$. Equivalently, if $a \mid b$ and $b \mid a$.*

In integers, this can only happen if a and b differ by a sign, but in more interesting rings, more interesting things can happen.

When considering division in rings, we often consider two associates to be "the same". For example, in , we can factorize 6 as

$$6 = 2 \cdot 3 = (-2) \cdot (-3),$$

but this does not violate unique factorization, since 2 and -2 are associates (and so are 3 and -3), and we consider these two factorizations to be "the same".

Definition 10 (Irreducible) *We say $a \in R$ is* irreducible *if $a \neq 0$, a is not a unit, and if $a = xy$, then x or y is a unit.*

For integers, being irreducible is the same as being a prime number. However, "prime" means something different in general rings.

Definition 11 (Prime) *We say $a \in R$ is* prime *if a is non-zero, not a unit, and whenever $a \mid xy$, either $a \mid x$ or $a \mid y$.*

It is important to note all these properties depend on the ring, not the element itself.

Example 7 *$2 \in$ is a prime, but $2 \in$ is not (since it is a unit).*

Similarly, the polynomial $2X \in [X]$ is irreducible (since 2 is a unit), but $2X \in [X]$ not irreducible.

We have two things called prime, so they had better be related. A principal ideal (r) is a prime ideal in R if and only if $r = 0$ or r is prime.

($\Rightarrow$) Let (r) be a prime ideal. If $r = 0$, then done. Otherwise, as prime ideals are proper, i.e. not the whole ring, r is not a unit. Now suppose $r \mid a \cdot b$. Then $a \cdot b \in (r)$. But (r) is prime. So $a \in (r)$ or $b \in (r)$. So $r \mid a$ or $r \mid b$. So r is prime.

($\Leftarrow$) If $r = 0$, then $(0) = \{0\} \lhd R$, which is prime since R is an integral domain. Otherwise, let $r \neq 0$ be prime. Suppose $a \cdot b \in (r)$. This means $r \mid a \cdot b$. So $r \mid a$ or $r \mid b$. So $a \in (r)$ and $b \in (r)$. So (r) is prime.

Note that in , prime numbers exactly the irreducibles, but prime numbers are also prime (surprise!). In general, it is not true that irreducibles are the same as primes. However, one direction is always true.

Let $r \in R$ be prime. Then it is irreducible.

Let $r \in R$ be prime, and suppose $r = ab$. Since $r \mid r = ab$, and r is prime, we must have $r \mid a$ or $r \mid b$. wlog, $r \mid a$. So $a = rc$ for some $c \in R$. So $r = ab = rcb$. Since we are in an integral domain, we must have $1 = cb$. So b is a unit.

We now do a long interesting example.

Example 8 *Let*

$$R = [\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in\} \leq .$$

By definition, it is a subring of a field. So it is an integral domain. What are the units of the ring? There is a nice trick we can use, when things are lying inside . Consider the function

$$N : R \to_{\geq 0}$$

given by

$$N(a + b\sqrt{-5}) \mapsto a^2 + 5b^2.$$

It is convenient to think of this as $z \mapsto z\bar{z} = |z|^2$. This satisfies $N(z \cdot w) = N(z)N(w)$. This is a desirable thing to have for a ring, since it *immediately implies all units have norm* 1 *— if* $r \cdot s = 1$*, then* $1 = N(1) = N(rs) = N(r)N(s)$*. So* $N(r) = N(s) = 1$*.*

So to find the units, we need to solve $a^2 + 5b^2 = 1$*, for a and b units. The only solutions are* ± 1*. So only* $\pm 1 \in R$ *can be units, and these obviously are units. So these are all the units.*

Next, we claim $2 \in R$ *is irreducible. We again use the norm. Suppose* $2 = ab$*. Then* $4 = N(2) = N(a)N(b)$*. Now note that nothing has norm* 2*.* $a^2 + 5b^2$ *can never be* 2 *for integers* $a, b \in$*. So we must have, wlog,* $N(a) = 4, N(b) = 1$*. So b must be a unit. Similarly, we see that* $3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ *are irreducible (since there is also no element of norm* 3*).*

We have four irreducible elements in this ring. Are they prime? No! Note that

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3.$$

We now claim 2 *does not divide* $1 + \sqrt{-5}$ *or* $1 - \sqrt{-5}$*. So* 2 *is not prime.*

To show this, suppose $2 \mid 1 + \sqrt{-5}$*. Then* $N(2) \mid N(1 + \sqrt{-5})$*. But* $N(2) = 4$ *and* $N(1 + \sqrt{-5}) = 6$*, and* $4 \nmid 6$*. Similarly,* $N(1 - \sqrt{-5}) = 6$ *as well. So* $2 \nmid 1 \pm \sqrt{-5}$*.*

There are several life lessons here. First is that primes and irreducibles are not the same thing in general. We've always thought they were the same because we've been living in the fantasy land of the integers. But we need to grow up.

The second one is that factorization into irreducibles is not necessarily unique, since $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ are two factorizations into irreducibles.

However, there is one situation when unique factorizations holds. This is when we have a Euclidean algorithm available.

Possible Questions
8 marks

1. Prove that the union of two subspaces of a vector space is a subspace iff one is contained in the other

2. Let V be a finite dimensional vector space over a field F. Let A be a subspace of V. Prove that there exists a subspace B of V such that $V = A \oplus B$

3. Let H be a nonempty subset of a vector space V. Then prove that H is a subspace of V if and only if H is closed under addition and scalar multiplication

4. Prove that vectors $v_1, v_2, \cdots, v_k, k \geq 2$ are linearly dependent if and only if one of the vectors is a linear combination of the others

5. State and prove basis theorem.

6. Prove that $\mathbb{R} \times \mathbb{R}$ is a vector space over $\mathbb{R}$

7. Let W be a nonemepty subset of a vector space V. State and prove the necessary and sufficient condition for W to be a subspace of V

8. Let $S = \{(6, 2, 1), (-1, 3, 2)\}$. Determine, if S is linearly independent or dependent?

9. Let $S = \{v_1, \cdots, v_n\}$ be a basis for V. Then prove that every subset of V contains more than n elements is linearly dependent

10. Let $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$. Determine, if S is linearly independent or dependent?

11. Determine, whether $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ is a basis of $\mathbb{R}^3$ or not

12. Determine, whether $\{(1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ is a basis of $\mathbb{R}^3$ or not

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Ring theory and Linear Algebra I **Subject Code: 16MMU302**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
For any positive integer n, the set nZ={0,±n,±2n,…}is an ideal of ____	Z	-2Z	Zn	-3Z	Z
Φ is a ring homo morphism . a,bєΦ.	Φ(a)+Φ(-b)	Φ(-a)+Φ(-b)	Φ(a)+Φ(b)	Φ(a)-Φ(b)	Φ(a)+Φ(b)
Let R be a commutative ring of characteristic 2. Then the mapping f:a to a^2 is a ring	Homomorphism	Isomorphism	Endomorphism	monomorphism	Homomorphism
Let Φ be a ring homo morphism from R TO S.then ker Φ={rєR\Φ(r)=0} is	An integral domain of R	An ideal of R	field	Not field	An ideal of R
If G is a group of order____,where p is prime then G is abilian	P^3	P^5	P^2	P^4	P^2
A ___ with 2 binary operation	Group	Abelian group	Ring	B and c	Ring
___ satisfies all the condition of ring	{2,3}	{0}	N	w	{0}
Let a,b,c belong to a ring R .then a0=0=	1a	2a	3a	0a	oa
Let a belong to a ring R .then (-1)a=?	-1	0	-a	a	-a
The ring of integer is _____	field	Integral domain	Zero divisor	a or c	Integral domain
Skew field has _____	Zero divisor	No zero divisor	Atleast One zero divisor	Many zero divisor	No zero divisor
Zn is field iff n is	composite	prime	Even	odd	prime
Zn is____ iff n is prime	field	Integral domain	ideal	a and b	a and b
{Z} is finite but Z is not	Integral domain	Field	Not field	Not integral domain	Not fielfd
Any Boolean ring charectreistic is	2	4	6	0	2
Q is a subring of	W	Z	2Z	R	R
The _____ of two subring of a ring is a subring	union	intersection	Symmetric difference	A nad c	intersection
2Z AND 3Z are _______ of Z	subring	Not a subring	Either a or b	Neither a nor b	Not a subring
If F is a_____ its only ideals are {0} and F	FIELD	Not field	Integral domain	Integral domain	field
Principal ideal is a and itsdenoted by	(a)	{a}	[a]	a	(a)
R is a commutative ring with identity 1.then aє	(a)	{a}	[a]	a	(a)
___ is a maximal ideal of Z	-2	-4	-6	-8	-2
___is a maximal ideal of Z	-3	-6	-9	-12	(3
A Homomorphism that is both one one and onto is called _______________ homomorphism.	Ring	Polynomial	Group	Cyclic	Ring
For any positive integer n, the mapping K→K mod n is ___________ from ℤ to ℤ□	Ring	Polynomial	Group	Cyclic	Ring
The mapping from ℤ to ℤ□ is called _____________	Ring	Cyclic	Natural	polynomial	Natural
Every ideal of a ring is of a ring homomorphism of ℝ	Kernel	Subring	Homomorphism	Maximal ideal	Kernel
If ℝ is a ring with unity and the characteristic of ℝ is n>0, then ℝ contain a subring ____________ to ℤ□	isomorphic	monomorphic	homomorphic	epimorphic	Isomorphic
Let F be a field, aєF and f(x)єf[x] then f(a) is the remainder in the division of f(x) by ______________	a-x	x-a	a x^2	x^2 -a	x-a
A polynomial degree n over a field has almost n zeros ____________ multiplicity	Counting	Not defined	finite	infinite	Counting
Let F be a field aєF and f(x)єf[x] then a is a Zero of f(x) iff (x,a) is a ___________ of f(x)	Dividend	Factor	multiplier	Quotient	Factor
A principal ideal domain is an integral domain ℝ in which every ideal has the form ___________ for some a in ℝ	<a>=[ra:rє ℝ]	<a>=[ar:rє ℝ]	<a>=[r/a:rє ℝ]	<a>=[r/a:rє ℝ]	<a>=[ra:rє ℝ]
Let F be a field I a non Zero ideal in f(x) and g(x) an element of f(x) then I=g(x) is a non Zero polynomial of __________ degree in I	minimum	maximum	least	Highest	minimum
Let F be field aєF and f(x) єf(x) then f(a) is the division of f(x) by (x-a)	Quotient	Remainder	multiplier	dividend	remainder
Let D be an integral domain. Then there exits a fields F that contains a subring _____________ to D	isomorphic	monomorphic	Epimorphic	homomorphic	isomorphic
Let ℝ be a ring with unity 1. The mapping ℤ→ℝ given by n→n:1 is a ____________ homomorphism	Ring	cyclic	isomorphic	ideal	ring

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DEPARTMENT OF MATHEMATICS

Subject: Ring theory & Linear algebra I	**Semester: III**	**L T P C**
Subject Code: 16MMU302	**Class: II-B.Sc. Mathematics**	**6 2 0 6**

UNIT IV

Linear transformations, null space, range, rank and nullity of a linear transformation, matrix representation of a linear transformation, algebra of linear transformations.

TEXT BOOK

1. Fraleigh.J.B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

2. Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

1 Definition of Vector Space

We shall study structures with two operations, an addition and a scalar multiplication, that are subject to some simple conditions. We will reflect more on the conditions later but on first reading notice how reasonable they are. For instance, surely any operation that can be called an addition (e.g., column vector addition, row vector addition, or real number addition) will satisfy conditions (1) through (5) below.

1.1 Definition and Examples

A vector space (over $\Re$) consists of a set V along with two operations '+' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all scalars $r, s \in \Re$:

the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$

vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

there is a zero vector $\in V$ such that $\vec{v} + = \vec{v}$ for all $\vec{v} \in V$

each $\vec{v} \in V$ has an additive inverse $\vec{w} \in V$ such that $\vec{w} + \vec{v} =$

the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$

addition of scalars distributes over scalar multiplication, $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

scalar multiplication distributes over vector addition, $r\cdot(\vec{v}+\vec{w}) = r\cdot\vec{v}+r\cdot\vec{w}$

ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

The definition involves two kinds of addition and two kinds of multiplication, and so may at first seem confused. For instance, in condition (7) the '+' on the left is addition of two real numbers while the '+' on the right is addition of two vectors in V. These expressions aren't ambiguous because of context; for example, r and s are real numbers so '$r + s$' can only mean real number addition. In the same way, item (9)'s left side 'rs' is ordinary real number multiplication, while its right side '$s \cdot \vec{v}$' is the scalar multiplication defined for this vector space.

The best way to understand the definition is to go through the examples below and for each, check all ten conditions. The first example includes that check, written out at length. Use it as a model for the others. Especially important are the closure conditions, (1) and (6). They specify that the addition and scalar multiplication operations are always sensiblethey are defined for every pair of vectors and every scalar and vector, and the result of the operation is a member of the set.

This subset of $\Re^2$ is a line through the origin.

$$L = xyy = 3x$$

We shall verify that it is a a vector space, under the usual meaning of '+' and '·'.

$$x_1y_1 + x_2y_2 = x_1 + x_2y_1 + y_2 \qquad r \cdot xy = rxry$$

These operations are just the ones of $\Re^2$, reused on its subset L. We say that L inherits these operations from $\Re^3$.

We shall check all ten conditions. The paragraph having to do with addition has five conditions. For condition (1), closure under addition, suppose that we start with two vectors from the line L,

$$\vec{v}_1 = x_1 y_1 \quad \vec{v}_2 = x_2 y_2$$

so that they satisfy the restrictions that $y_1 = 3x_1$ and $y_2 = 3x_2$. Their sum

$$\vec{v}_1 + \vec{v}_2 = x_1 + x_2 y_1 + y_2$$

is also a member of the line L because the fact that its second component is three times its first $y_1 + y_2 = 3(x_1 + x_2)$ follows from the restrictions on $\vec{v}_1$ and $\vec{v}_2$. For (2), that addition of vectors commutes, just compare

$$\vec{v}_1 + \vec{v}_2 = x_1 + x_2 y_1 + y_2 \quad \vec{v}_2 + \vec{v}_1 = x_2 + x_1 y_2 + y_1$$

and note that they are equal since their entries are real numbers and real numbers commute. (That the vectors satisfy the restriction of lying in the line is not relevant for this condition; they commute just because all vectors in the plane commute.) Condition (3), associativity of vector addition, is similar.

$$(x_1$$

$$y_1 + x_2$$

$$y_2) + x_3$$

$$y_3 = (x_1 + x_2) + x_3$$

$$(y_1 + y_2) + y_3$$

The checks for the five conditions having to do with scalar multiplication are similar. For (6), closure under scalar multiplication, suppose that $r \in$ and $\vec{v} \in L$

$$\vec{v} = xy$$

so that it satisfies the restriction $y = 3x$. Then

$$r \cdot \vec{v} = r \cdot xy = rxry$$

is also a member of L: the fact that its second component is three times its first $ry = 3(rx)$ follows from the restriction on $\vec{v}$. Next, this checks (7).

$$(r + s) \cdot xy = (r + s)x(r + s)y = rx + sxry + sy = r \cdot xy + s \cdot xy$$

For (8) we have this.

$$r \cdot (x_1y_1 + x_2y_2) = r(x_1 + x_2)r(y_1 + y_2) = rx_1 + rx_2ry_1 + ry_2 = r \cdot x_1y_1 + r \cdot x_2y_2$$

The ninth

$$(rs) \cdot xy = (rs)x(rs)y = r(sx)r(sy) = r \cdot (s \cdot xy)$$

and tenth conditions are also straightforward.

$$1 \cdot xy = 1x1y = xy$$

The whole plane, the set $\Re^2$, is a vector space if the operations '+' and '·' have their usual meaning.

$$x_1y_1 + x_2y_2 = x_1 + x_2y_1 + y_2 \qquad r \cdot xy = rxry$$

We shall check just two of the conditions, the closure conditions.

For (1) observe that the result of the vector sum

$$x_1y_1 + x_2y_2 = x_1 + x_2y_1 + y_2$$

is a column array with two real entries, and so is a member of the plane $\Re^2$. In contrast with the prior example, here there is no restriction on the vectors that we must check.

Condition (6) is similar. The vector

$$r \cdot xy = rxry$$

has two real entries, and so is a member of $\Re^2$.

In a similar way, each $\Re^n$ is a vector space with the usual operations of vector addition and scalar multiplication. (In $\Re^1$, we usually do not write the members as column vectors, i.e., we usually do not write '(π)'. Instead we just write 'π'.)

PlaneThruOriginSubsp gives a subset of $\Re^2$ that is a vector space. For contrast, consider the set of two-tall columns with entries that are integers, under the same operations of component-wise addition and scalar multiplication. This is a subset of $\Re^2$ but it is not a vector space: it is not closed under scalar multiplication, that is, it does not satisfy condition (6). For instance, on the left below is a vector with integer entries, and a scalar.

$$0.5 \cdot [r]43 = [r]21.5$$

On the right is a column vector that is not a member of the set, since its entries are not all integers.

The one-element set

$$[r]0000$$

is a vector space under the operations

$$[r]0000 + [r]0000 = [r]0000 \qquad r \cdot [r]0000 = [r]0000$$

that it inherits from $\Re^4$.

A vector space must have at least one element, its zero vector. Thus a one-element vector space is the smallest possible.

A one-element vector space is a trivial space.

The examples so far involve sets of column vectors with the usual operations. But vector spaces need not be collections of column vectors, or even of row vectors. Below are some other types of vector spaces. The term ‘vector space’ does not mean ‘collection of columns of reals’. It means something more like ‘collection in which any linear combination is sensible’.

Consider ${}_3 = a_0 + a_1x + a_2x^2 + a_3x^3 a_0, \ldots, a_3 \in \Re$, the set of polynomials of degree three or less (in this book, we’ll take constant polynomials, including the zero polynomial, to be of degree zero). It is a vector space under the operations

$$\begin{aligned}(a_0 + a_1x + a_2x^2 + a_3x^3) &+ (b_0 + b_1x + b_2x^2 + b_3x^3)\\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\end{aligned}$$

and

$$r \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (ra_0) + (ra_1)x + (ra_2)x^2 + (ra_3)x^3$$

(the verification is easy). This vector space is worthy of attention because these are the polynomial operations familiar from high school algebra. For instance, $3 \cdot (1 - 2x + 3x^2 - 4x^3) - 2 \cdot (2 - 3x + x^2 - (1/2)x^3) = -1 + 7x^2 - 11x^3$.

Although this space is not a subset of any $\Re^n$, there is a sense in which we can think of $_3$ as "the same" as $\Re^4$. If we identify these two space's elements in this way

$$a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{corresponds to} \quad a_0a_1a_2a_3$$

then the operations also correspond. Here is an example of corresponding additions.

$$\begin{array}{rr} & 1 - 2x + 0x^2 + 1x^3 \\ + & 2 + 3x + 7x^2 - 4x^3 \\ \hline & 3 + 1x + 7x^2 - 3x^3 \end{array} \quad \text{corresponds to} \quad [r]1 - 201 + [r]237 - 4 = [r]317 - 3$$

Things we are thinking of as "the same" add to "the same" sum. Chapter Three makes precise this idea of vector space correspondence. For now we shall just leave it as an intuition.

In general we write $_n$ for the vector space of polynomials of degree n or less $a_0 + a_1x + a_2x^2 + \cdots + a_nx^na_0, \ldots, a_n \in \Re$, under the operations of the usual polynomial addition and scalar multiplication. We will often use these spaces as examples.

The set $_{22}$ of 22 matrices with real number entries is a vector space under the natural entry-by-entry operations.

$$abcd + wxyz = a + wb + xc + yd + z \qquad r \cdot abcd = rarbrcrd$$

As in the prior example, we can think of this space as "the same" as $\Re^4$.

We write $_{nm}$ for the vector space of nm matrices under the natural operations of matrix addition and scalar multiplication. As with the polynomial spaces, we will often use these as examples.

The set $ff\Re$ of all real-valued functions of one natural number variable is a vector space under the operations

$$(f_1 + f_2)\,(n) = f_1(n) + f_2(n) \qquad (r \cdot f)\,(n) = r\,f(n)$$

so that if, for example, $f_1(n) = n^2 + 2\sin(n)$ and $f_2(n) = -\sin(n) + 0.5$ then $(f_1 + 2f_2)\,(n) = n^2 + 1$.

We can view this space as a generalization of ex:RealVecSpacesinstead of 2-tall vectors, these functions are like infinitely-tall vectors.

n	$f(n) = n^2 + 1$
0	1
1	2
2	5
3	10
$\vdots$	$\vdots$

corresponds to [r]1251010

Addition and scalar multiplication are component-wise, as in ex:RealVecSpaces. (We can formalize "infinitely-tall" by saying that it means an infinite sequence, or that it means a function from to $\Re$.)

Possible Questions

8 marks

1. State and prove fundamental theorem of homomorphism of linear transformations
2. State and prove rank theorem
3. Let V be a vector space over F. Let A and B be subspaces of V. Prove that $\frac{A+B}{A} \approxeq \frac{B}{A\cap B}$
4. Let V and W be two vector spaces. Suppose $T : V \to W$ is a linear transformation. Then prove the following
   (a) $T(0) = 0$
   (b) $T(-v) = -T(v)$ for all $v \in V$
5. If V is a vector space with a finite spanning set, then prove that every basis for V contains the same number of vectors.
6. State and prove fundamental theorem of homomorphism on vector spaces
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defned by $T(a_1, a_2, a_3) = (3a_1 + a_2, a_1 + a_3, a_1 - a_3)$. Find the matrix representation of T w.r.t the standard basis for both domain and range
8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defned by $T(x, y) = (x - y, x + y)$. Find the matrix representation of T w.r.t the standard basis for domain and $\{(1, 1), (1. - 1)\}$ for range

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Ring theory and Linear Algebra I **Subject Code: 16MMU302**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
Let ℝ and ℝ' be ring. A function F:R→R' is called ___________	homomorphism	Bisections	monomorphic	onto	homomorphism
If F is one-one, then F is called ___________________	homomorphism	Bisections	monomorphic	onto	monomorphic
If F is onto then F is called _____________________	endomorphism	epimorphism	monomorphic	One-one	epimorphism
A homomorphism of ring onto itself is called an ____________	endomorphism	epimorphism	One-one	onto	endomorphism
If F: R→R' defined by f(a)=0 all aϵR is obviously a homomorphism and f is called___________ homomorphism	mono	Trivial	Non trivial	Identity	Trivial
Let R be any ring, the identity function f: R→R is _______________	Homomorphism	Principal ideal ring	Maximal ideal	constant	Homomorphism
Let R and R' be ring and f:R→R' be a Homomorphism then f(0)=________________	0	F'(0)	F"(0)	0'	0'
Let R and R' be ring and f: R→R' be a Homomorphism then f(-a)=_____________________	f(a)	- f(a)	0	1	- f(a)
If is a commutative ring then f(R) is ______________ring	Associative	Principal ideal	Commutative	maximal	commutative
Two elements (a,b) and (c,d) ϵS are said to be equivalent iff _____________	ab=dc	ad=bc	ac=bd	ab=ba	ad=bc
The equivalence class containing (a,b) is denoted by ______________	$\frac{b}{a}$	$\frac{a}{b}$	$\frac{a}{ab}$	$\frac{b}{ab}$	$\frac{a}{b}$
The map F:D→R given by f(a) $\frac{a}{1}$ is an isomorphism of D _______f(D)	Onto	Into	One-one	One-onto	Onto
The field of quantity F of an integral domain D is the ____________ field containing D	Smallest	Largest	Finite	Infinite	Smallest
A ring is a set R equipped with ___________ operations	Under addition	Under multiplication	Two binary	Either A or B	Two binary
The additive identity, the additive inverse of each element, and the multiplication are ___________	complex	Unique	Real	All the above	Real
If 0=1 in a ring R, then R has only one element, and the ___________	Ideal	Field	Unity	Zero ring	Zero ring
The study of ring originated from the theory of _____________	polynomial rings and algebraic integers	Complex numbers and real function	Polynomial rings and complex number	Polynomial ring and real function	polynomial rings and algebraic integers
The term "Zahlring" is__________	ideal	Field	Number ring	Zero ring	Number ring
Ring could mean ______________	Association	Mapping	analysing	All the above	Association
The Zero ring has number of elements ______	Only one	Two or more	Infinite	All the above	Only one
R is the set of even integers under the usual operations of addition and multiplication. R is a _____________	Commutative rings	Unit element	Commutative rings but has no unit element.	Commutative ring with unit element.	Commutative rings but has no unit element.
If R is a commutative ring, then a≠0 ϵ R is said to be a zero-divisor if there exists a,bϵR, b≠0, such that _______	ab=0.	a/b≠0	a=0	b=0	ab=0
If R is a commutative ring, then a≠0 ϵ R is said to be a ___________if there exists a, bϵR, b≠0, such that ab=0.	ideal	zero-divisor	Integral domain	Field	zero-divisor
rng-_________	Ring without i	Ring with i	Ideal	field	Ring without i
Set of even integers with usual + and – is a ________	Ring	Not an ring	Ideal	Not an ideal	Not an ring
Ring that satisfy commutative for multiplication are called ______________	ideal	Field	Integral domain	Commutative ring	Commutative ring
In ring, multiplication does not have to have an inverse. A commutative ring such that every non zero elements has multiplicative inverse is called __________________	ideal	Field	Integral domain	Commutative ring	Field
Ring that satisfy commutative for _______________ are called field	Addition	Multiplication	Both binary operation	None of these	Multiplication
In ring, multiplication does not have to have an inverse. A commutative ring such that every non zero elements has ______________ inverse is called Field.	Addition	Multiplication	Both binary operation	None of these	Multiplication
The ___________ holds for any commuting pair of elements.	Binomial formula	Binomial function	Linear function	Linear formula	Binomial formula
The rational, real, and complex numbers are commutative ring of a types called __________	ideal	Field	Integral domain	Commutative ring	Field
The set of natural numbers ℕ with the usual operation is ___________	Ring	Not a ring	Group	Cyclic group	Not a ring
___________ in a ring is analogous to that of Normal subgroup in a group.	Ideal	Field	Integral domain	Commutative ring	Ideal
A nonempty subset I of R is said to be left ideal in R if, for any x,y in I and r in R,____________	x+y and xy are in R	x+y and xy are in I	x+y and xy are in both I and R	None of these	x+y and xy are in I
Let R be ring. I is said to left ideal is	RI subset I	IR subset of I	Both a and b	None of these	RI subset I
If x is in R, then Rx and xR are left and right ideals respectively; They are called _____________	Principal left ideal	Principal right ideal	Principal ideal	All the above	Principal ideal
A ring is said to be ___________if it is non zero and it has no proper non-zero two sided ideals	Group	Real	Complex	Simple	Simple
A proper ideal P of R is called a______________	Prime ideal	Proper ring	Proper set	All the above	Prime ideal

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DEPARTMENT OF MATHEMATICS

Subject: Ring theory & Linear algebra I	**Semester: III**	**L T P C**
Subject Code: 16MMU302	**Class: II-B.Sc. Mathematics**	**6 2 0 6**

UNIT V

Isomorphism: Isomorphism theorems, invertibility and isomorphisms, change of coordinate matrix

TEXT BOOK

1. Fraleigh.J.B., (2004). A First Course in Abstract Algebra , Seventh Edition , Pearson Education Ltd, Singapore.

REFERENCES

1. Joseph A. Gallian., (1999). Contemporary Abstract Algebra, Fourth Edition, Narosa Publishing House, New Delhi.

2. Kumaresan S., (1999). Linear Algebra- A Geometric Approach, Prentice Hall of India, New Delhi.

The set of polynomials with real coefficients

$$a_0 + a_1 x + \cdots + a_n x^n n \in \text{ and } a_0, \ldots, a_n \in \Re$$

makes a vector space when given the natural '+'

$$(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and '·'.

$$r \cdot (a_0 + a_1 x + \ldots a_n x^n) = (ra_0) + (ra_1)x + \ldots (ra_n)x^n$$

This space differs from the space $_3$ of ex:PolySpaceThree. This space contains not just degree three polynomials, but degree thirty polynomials and degree three hundred polynomials, too. Each individual polynomial of course is of a finite degree, but the set has no single bound on the degree of all of its members.

We can think of this example, like the prior one, in terms of infinite-tuples. For instance, we can think of $1 + 3x + 5x^2$ as corresponding to $(1, 3, 5, 0, 0, \ldots)$. However, this space differs from the one in ex:FcnsNToRIsVecSp. Here, each member of the set has a finite degree, that is, under the correspondence there is no element from this space matching $(1, 2, 5, 10, \ \ldots)$. Vectors in this space correspond to infinite-tuples that end in zeroes.

The set $ff\Re\Re$ of all real-valued functions of one real variable is a vector space under these.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \qquad (r \cdot f)(x) = r\, f(x)$$

The difference between this and ex:FcnsNToRIsVecSp is the domain of the functions.

The set $F = \{a \cos\theta + b \sin\theta a, b \in \Re\}$ of real-valued functions of the real variable θ is a vector space under the operations

$$(a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos\theta + b_2 \sin\theta) = (a_1 + a_2)\cos\theta + (b_1 + b_2)\sin\theta$$

and

$$r \cdot (a \cos\theta + b \sin\theta) = (ra) \cos\theta + (rb) \sin\theta$$

inherited from the space in the prior example. (We can think of F as "the same" as $\Re^2$ in that $a \cos\theta + b \sin\theta$ corresponds to the vector with components a and b.)

The set

$$f \Re\Re \frac{d^2 f}{dx^2} + f = 0$$

is a vector space under the, by now natural, interpretation.

$$(f + g)(x) = f(x) + g(x) \qquad (r \cdot f)(x) = r\, f(x)$$

In particular, notice that closure is a consequence

$$\frac{d^2(f+g)}{dx^2} + (f+g) = (\frac{d^2 f}{dx^2} + f) + (\frac{d^2 g}{dx^2} + g)$$

and

$$\frac{d^2(rf)}{dx^2} + (rf) = r(\frac{d^2 f}{dx^2} + f)$$

of basic Calculus. This turns out to equal the space from the prior exam-plefunctions satisfying this differential equation have the form $a \cos\theta + b \sin\theta$but this description suggests an extension to solutions sets of other differential equations.

The set of solutions of a homogeneous linear system in n variables is a vector space under the operations inherited from $\Re^n$. For example, for closure under addition consider a typical equation in that system $c_1x_1 + \cdots + c_nx_n = 0$ and suppose that both these vectors

$$\vec{v} = v_1 v_1 v_n \qquad \vec{w} = w_1 w_1 w_n$$

satisfy the equation. Then their sum $\vec{v} + \vec{w}$ also satisfies that equation: $c_1(v_1+w_1)+\cdots+c_n(v_n+w_n) = (c_1v_1+\cdots+c_nv_n)+(c_1w_1+\cdots+c_nw_n) = 0$. The checks of the other vector space conditions are just as routine.

We often omit the multiplication symbol '$\cdot$' between the scalar and the vector. We distinguish the multiplication in c_1v_1 from that in $r\vec{v}$ by context, since if both multiplicands are real numbers then it must be real-real multiplication while if one is a vector then it must be scalar-vector multiplication.

ex:HomoSlnMakesVS has brought us full circle since it is one of our motivating examples. Now, with some feel for the kinds of structures that satisfy the definition of a vector space, we can reflect on that definition. For example, why specify in the definition the condition that $1 \cdot \vec{v} = \vec{v}$ but not a condition that $0 \cdot \vec{v} =$?

One answer is that this is just a definitionit gives the rules and you need to follow those rules to continue.

Another answer is perhaps more satisfying. People in this area have worked to develop the right balance of power and generality. This definition is shaped so that it contains the conditions needed to prove all of the interesting and important properties of spaces of linear combinations. As we proceed, we shall derive all of the properties natural to collections of linear combinations from the conditions given in the definition.

The next result is an example. We do not need to include these properties in the definition of vector space because they follow from the properties already listed there.

In any vector space V, for any $\vec{v} \in V$ and $r \in \Re$, we have (1) $0 \cdot \vec{v} =$, (2) $(-1 \cdot \vec{v}) + \vec{v} =$, and (3) $r\cdot =$.

For (1) note that $\vec{v} = (1 + 0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} =$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ &= +0 \cdot \vec{v} \\ &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1 + 1) \cdot \vec{v} = 0 \cdot \vec{v} =$. For (3), $r\cdot = r \cdot (0\cdot) = (r \cdot 0)\cdot =$ will do.

The second item shows that we can write the additive inverse of $\vec{v}$ as '$-\vec{v}$' without worrying about any confusion with $(-1) \cdot \vec{v}$.

A recap: our study in Chapter One of Gaussian reduction led us to consider collections of linear combinations. So in this chapter we have defined a vector space to be a structure in which we can form such combinations, subject to simple conditions on the addition and scalar multiplication operations. In a phrase: vector spaces are the right context in which to study linearity.

From the fact that it forms a whole chapter, and especially because that chapter is the first one, a reader could suppose that our purpose in this book is the study of linear systems. The truth is that we will not so much use vector spaces in the study of linear systems as we instead have linear systems start us on the study of vector spaces. The wide variety of examples from this subsection shows that the study of vector spaces is interesting and important in its own right. Linear systems won't go away. But from now on our primary objects of study will be vector spaces.

Name the zero vector for each of these vector spaces. The space of degree three polynomials under the natural operations. The space of 24 matrices. The space $f[0..1]\Re f$ is continuous. The space of real-valued functions of one natural number variable. $0 + 0x + 0x^2 + 0x^3$ $[r]0000$ 0000 The constant function $f(x) = 0$ The constant function $f(n) = 0$

Find the additive inverse, in the vector space, of the vector. In $_3$, the vector $-3 - 2x + x^2$. In the space 2,

$$[r]1 - 103.$$

In $ae^x + be^{-x}a, b \in \Re$, the space of functions of the real variable x under the natural operations, the vector $3e^x - 2e^{-x}$. $3 + 2x - x^2$ $[r] - 1 + 1$ $0 - 3$ $-3e^x + 2e^{-x}$

For each, list three elements and then show it is a vector space. The set of linear polynomials $_1 = a_0 + a_1xa_0, a_1 \in \Re$ under the usual polynomial addition and scalar multiplication operations. The set of linear polynomials $a_0 + a_1xa_0 - 2a_1 = 0$, under the usual polynomial addition and scalar multiplication operations. *Hint.* Use PlaneThruOriginSubsp as a guide. Most of the ten conditions are just verifications. Three elements are: $1+2x$, $2-1x$, and x. (Of course, many answers are possible.)

The verification is just like ex:RealVecSpaces. We first do conditions 1-5 from def:VecSpace, having to do with addition. For closure under addition, condition (1), note that where $a + bx, c + dx \in_1$ we have that $(a + bx) + (c + dx) = (a + c) + (b + d)x$ is a linear polynomial with real coefficients and so is an element of $_1$. Condition (2) is verified with: where $a+bx, c+dx \in_1$ then $(a+bx)+(c+dx) = (a+c)+(b+d)x$, while in the other order they are $(c+dx)+(a+bx) = (c+a)+(d+b)x$, and both $a+c = c+a$ and $b+d = d+b$ as these are real numbers. Condition (3) is similar: suppose $a+bx, c+dx, e+fx \in$ then $((a+bx)+(c+dx))+(e+fx) = (a + c + e) + (b + d + f)x$ while $(a + bx) + ((c + dx) + (e + fx)) = (a + c + e) + (b + d + f)x$, and the two are equal (that is, real number addition is associative so $(a+c)+e = a+(c+e)$ and $(b+d)+f = b+(d+f)$). For condition (4) observe that the linear polynomial $0 + 0x \in_1$ has the property that $(a+bx)+(0+0x) = a+bx$ and $(0+0x)+(a+bx) = a+bx$. For the last condition in this paragraph, condition (5), note that for any $a + bx \in_1$ the additive inverse is $-a - bx \in_1$ since $(a + bx) + (-a - bx) = (-a - bx) + (a + bx) = 0 + 0x$.

We next also check conditions (6)-(10), involving scalar multiplication. For (6), the condition that the space be closed under scalar multiplication, suppose that r is a real number and $a + bx$ is an element of $_1$, and then $r(a + bx) = (ra) + (rb)x$ is an element of $_1$ because it is a linear polynomial with real number coefficients. Condition (7) holds because $(r + s)(a + bx) = r(a + bx) + s(a + bx)$ is true from the distributive property for real number multiplication. Condition (8) is similar: $r((a + bx) + (c + dx)) = r((a + c) + (b + d)x) = r(a+c)+r(b+d)x = (ra+rc)+(rb+rd)x = r(a+bx)+r(c+dx)$. For (9) we have $(rs)(a + bx) = (rsa) + (rsb)x = r(sa + sbx) = r(s(a + bx))$. Finally, condition (10) is $1(a + bx) = (1a) + (1b)x = a + bx$. Call the set P. In the prior item in this exercise there was no restriction on the coefficients but here we are restricting attention to those linear polynomials where $a_0 - 2a_1 = 0$, that is, where the constant term minus twice the coefficient of the linear term is zero. Thus, three typical elements of P are $2 + 1x$, $6 + 3x$, and $-4 - 2x$.

For condition (1) we must show that if we add two linear polynomials that satisfy the restriction then we get a linear polynomial also satisfying the restriction: here that argument is that if $a + bx, c + dx \in P$ then $(a+bx)+(c+dx) = (a+c)+(b+d)x$ is an element of P because $(a+c)-2(b+d) = (a-2b)+(c-2d) = 0+0 = 0$. We can verify condition (2) with: where $a+bx, c+dx \in_1$ then $(a+bx)+(c+dx) = (a+c)+(b+d)x$, while in the other order they are $(c+dx)+(a+bx) = (c+a)+(d+b)x$, and both $a+c = c+a$ and $b+d = d+b$ as these are real numbers. (That is, this condition is not affected by the restriction and the verification is the same as the verification in the first item of this exercise). Condition (3) is also not affected by the extra restriction: suppose that $a+bx, c+dx, e+fx \in$ then $((a+bx)+(c+dx))+(e+fx) = (a+c+e)+(b+d+f)x$ while $(a+bx)+((c+dx)+(e+fx)) = (a+c+e)+(b+d+f)x$, and the two are equal. For condition (4) observe that the linear polynomial satisfies the restriction $0+0x \in P$ because its constant term minus twice the coefficient of its linear term is zero, and then the verification from the first item of this question applies: $0+0x \in_1$ has the property that $(a+bx)+(0+0x) = a+bx$ and $(0+0x)+(a+bx) = a+bx$. To check condition (5), note that for any $a+bx \in P$ the additive inverse is $-a-bx$ since it is an element of P (because $a+bx \in P$ we know that $a-2b = 0$ and multiplying both sides by -1 gives that $-a+2b = 0$), and as in the first item it acts as the additive inverse $(a+bx)+(-a-bx) = (-a-bx)+(a+bx) = 0+0x$.

We must also check conditions (6)-(10), those for scalar multiplication. For (6), the condition that the space be closed under scalar multiplication, suppose that r is a real number and $a+bx \in P$ (so that $a-2b = 0$), then $r(a+bx) = (ra)+(rb)x$ is an element of P because it is a linear polynomial with real number coefficients satisfying that $(ra)-2(rb) = r(a-2b) = 0$. Condition (7) holds for the same reason that it holds in the first item of this exercise, because $(r + s)(a + bx) = r(a + bx) + s(a + bx)$ is true from the distributive property for real number multiplication. Condition (8) is also unchanged from the first item: $r((a + bx) + (c + dx)) = r((a + c) + (b + d)x) = r(a + c) + r(b + d)x = (ra + rc) + (rb + rd)x = r(a + bx) + r(c + dx)$. So is (9): $(rs)(a + bx) = (rsa) + (rsb)x = r(sa + sbx) = r(s(a + bx))$. Finally, so is condition (10): $1(a + bx) = (1a) + (1b)x = a + bx$.

For each, list three elements and then show it is a vector space. The set of 2 matrices with real entries under the usual matrix operations. The set of 2 matrices with real entries where the 2, 1 entry is zero, under the usual matrix operations. Use ex:RealVecSpaces as a guide. (*Comment.* Because many of the conditions are quite easy to check, sometimes a person can feel that they must have missed something. Keep in mind that easy to do, or routine, is different from not necessary to do.) Here are three elements.

$$1234,\ -1 - 2 - 3 - 4,\ 0000$$

For (1), the sum of 2 real matrices is a 2 real matrix. For (2) we consider the sum of two matrices

$$abcd + efgh = a + eb + fc + gd + h$$

and apply commutativity of real number addition

$$= e + af + bg + ch + d = efgh + abcd$$

to verify that the addition of the matrices is commutative. The verification for condition (3), associativity of matrix addition, is similar to the prior verification:

$$\big(abcd + efgh\big) + ijkl = (a + e) + i(b + f) + j(c + g) + k(d + h) + l$$

while

$$abcd + \big(efgh + ijkl\big) = a + (e + i)b + (f + j)c + (g + k)d + (h + l)$$

and the two are the same entry-by-entry because real number addition is associative. For (4), the zero element of this space is the 2 matrix of zeroes. Condition (5) holds because for any 2 matrix A the additive inverse is the matrix whose entries are the negative of A's, the matrix $-1 \cdot A$.

Condition (6) holds because a scalar multiple of a 2 matrix is a 2 matrix. For condition (7) we have this.

$$= (r + s)a(r + s)b$$

(r+s)c (r+s)d
= ra+sa rb+sb
rc+sc rd+sd = r a b
c d + s a b
c d
$(r+s)a$
bc
d
= (r+s)a (r+s)b
(r+s)c (r+s)d
= ra+sa rb+sb
rc+sc rd+sd = r a b
c d + s a b
c d

Condition (8) goes the same way.

$$+ef$$

g h) = r a+e b+f
c+g d+h = ra+re rb+rf
rc+rg rd+rh

= r a b

c d + r e f

g h = r (a b

c d + e f

g h)

r(*a*

bc

d

+ e f

g h) = r a+e b+f

c+g d+h = ra+re rb+rf

rc+rg rd+rh

= r a b

c d + r e f

g h = r (a b

c d + e f

g h)

For (9) we have this.

$$(rs)abcd = rsarsbrscrsd = rsasbscsd = r(sabcd)$$

Condition (10) is just as easy.

$$1abcd = 1 \cdot a1 \cdot b1 \cdot c1 \cdot d = sasbscsd$$

This differs from the prior item in this exercise only in that we are restricting to the set T of matrices with a zero in the second row and first column. Here are three elements of T.

$$1204,\ -1 - 20 - 4,\ 0000$$

Some of the verifications for this item are the same as for the first item in this exercise, and below we'll just do the ones that are different.

For (1), the sum of 2 real matrices with a zero in the 2, 1 entry is also a 2 real matrix with a zero in the 2, 1 entry.

$$ab0d + ef0ha + eb + f0d + h$$

The verification for condition (2) given in the prior item works in this item also. The same holds for condition (3). For (4), note that the 2 matrix of zeroes is an element of T. Condition (5) holds because for any 2 matrix A the additive inverse is the matrix $-1 \cdot A$ and so the additive inverse of a matrix with a zero in the 2, 1 entry is also a matrix with a zero in the 2, 1 entry.

Condition 6 holds because a scalar multiple of a 2 matrix with a zero in the 2, 1 entry is a 2 matrix with a zero in the 2, 1 entry. Condition (7)'s verification is the same as in the prior item. So are condition (8)'s, (9)'s, and (10)'s.

For each, list three elements and then show it is a vector space. The set of three-component row vectors with their usual operations. The set

$$xyzw \in \Re^4 x + y - z + w = 0$$

under the operations inherited from $\Re^4$. Three elements are 123, 213, and 000.

We must check conditions (1)-(10) in def:VecSpace. Conditions (1)-(5) concern addition. For condition (1) recall that the sum of two three-component row vectors

$$abc + def = a + db + ec + f$$

is also a three-component row vector (all of the letters $a, \ldots, f$ represent real numbers). Verification of (2) is routine

a

b

$c + d$

e

$f = a + d$

$b + e$

$c + f$

$= d + a$

$e + b$

$f + c = d$

e

$f + a$

b

c

(the second equality holds because the three entries are real numbers and real number addition commutes). Condition (3)'s verification is similar.

$(a$

b

$c + d$

e

$f) + g$

h

$i = (a + d) + g$

$(b + e) + h$

$(c + f) + i$

$= a + (d + g)$

$b + (e + h)$

$c + (f + i) = a$

b

$c + (d$

e

$f + g$

h

$i)$

For (4), observe that the three-component row vector 000 is the additive identity: $abc + 000 = abc$. To verify condition (5), assume we are given the element abc of the set and note that $-a - b - c$ is also in the set and has the desired property: $abc + -a - b - c = 000$.

Conditions (6)-(10) involve scalar multiplication. To verify (6), that the space is closed under the scalar multiplication operation that was given, note that $rabc = rarbrc$ is a three-component row vector with real entries. For (7) we compute.

$(r + s)a$

$$b$$

$$c = (r + s)a$$

$$(r + s)b$$

$$(r + s)c = ra + sa$$

$$rb + sb$$

$$rc + sc$$

$$= ra$$

$$rb$$

$$rc + sa$$

$$sb$$

$$sc = ra$$

$$b$$

$$c + sa$$

$$b$$

$$c$$

Condition (8) is very similar.

$r(a$

$$b$$

$$c + d$$

$$e$$

$$f)$$

$= ra + d \quad b + ec + f = r(a + d) \quad r(b + e)r(c + f)$

$$= ra + rd$$

$$rb + re$$

$$rc + rf = ra$$

$$rb$$

$$rc + rd$$

$$re$$

$$rf$$

$$= ra$$

$$b$$

$$c + rd$$

$$e$$

$$f$$

So is the computation for condition (9).

$$(rs)abc = rsarsbrsc = rsasbsc = r(sabc)$$

Condition (10) is just as routine $1abc = 1 \cdot a1 \cdot b1 \cdot c = abc$. Call the set L. Closure of addition, condition (1), involves checking that if the summands are members of L then the sum

$$abcd + efgh = a + eb + fc + gd + h$$

is also a member of L, which is true because it satisfies the criteria for membership in L: $(a+e)+(b+f)-(c+g)+(d+h) = (a+b-c+d)+(e+f-g+h) = 0+0$. The verifications for conditions (2), (3), and (5) are similar to the ones in the first part of this exercise. For condition (4) note that the vector of zeroes is a member of L because its first component plus its second, minus its third, and plus its fourth, totals to zero.

Condition (6), closure of scalar multiplication, is similar: where the vector is an element of L,

$$rabcd = rarbrcrd$$

is also an element of L because $ra+rb-rc+rd = r(a+b-c+d) = r\cdot 0 = 0$. The verification for conditions (7), (8), (9), and (10) are as in the prior item of this exercise.

Show that each of these is not a vector space. (*Hint.* Check closure by listing two members of each set and trying some operations on them.) Under the operations inherited from $\Re^3$, this set

$$xyz \in \Re^3 x + y + z = 1$$

Under the operations inherited from $\Re^3$, this set

$$xyz \in \Re^3 x^2 + y^2 + z^2 = 1$$

Under the usual matrix operations,

$$a1bca, b, c \in \Re$$

Under the usual polynomial operations,

$$a_0 + a_1 x + a_2 x^2 a_0, a_1, a_2 \in \Re^+$$

where $\Re^+$ is the set of reals greater than zero Under the inherited operations,

$$xy \in \Re^2 x + 3y = 4 \text{ and } 2x - y = 3 \text{ and } 6x + 4y = 10$$

In each item the set is called Q. For some items, there are other correct ways to show that Q is not a vector space. It is not closed under addition; it fails to meet condition (1).

$$[r]100, [r]010 \in Q \qquad [r]110 \notin Q$$

It is not closed under addition.

$$[r]100, [r]010 \in Q \qquad [r]110 \notin Q$$

It is not closed under addition.

$$[r]0100,\ [r]1100 \in Q \qquad [r]1200 \notin Q$$

It is not closed under scalar multiplication.

$$1 + 1x + 1x^2 \in Q \qquad -1 \cdot (1 + 1x + 1x^2) \notin Q$$

It is empty, violating condition (4).

Define addition and scalar multiplication operations to make the complex numbers a vector space over $\Re$. The usual operations $(v_0 + v_1 i) + (w_0 + w_1 i) = (v_0 + w_0) + (v_1 + w_1)i$ and $r(v_0 + v_1 i) = (rv_0) + (rv_1)i$ suffice. The check is easy.

Is the set of rational numbers a vector space over $\Re$ under the usual addition and scalar multiplication operations? No, it is not closed under scalar multiplication since, e.g., $\pi \cdot (1)$ is not a rational number.

Show that the set of linear combinations of the variables x, y, z is a vector space under the natural addition and scalar multiplication operations. The natural operations are $(v_1 x + v_2 y + v_3 z) + (w_1 x + w_2 y + w_3 z) = (v_1 + w_1)x + (v_2 + w_2)y + (v_3 + w_3)z$ and $r \cdot (v_1 x + v_2 y + v_3 z) = (rv_1)x + (rv_2)y + (rv_3)z$. The check that this is a vector space is easy; use ex:RealVecSpaces as a guide.

Prove that this is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$x_1 y_1 + x_2 y_2 = x_1 - x_2 y_1 - y_2 \qquad r \cdot xy = rxry$$

The '+' operation is not commutative (that is, condition (2) is not met); producing two members of the set witnessing this assertion is easy.

Prove or disprove that $\Re^3$ is a vector space under these operations. $x_1y_1z_1 + x_2y_2z_2 = 000$ and $rxyz = rxryrz$ $x_1y_1z_1 + x_2y_2z_2 = 000$ and $rxyz = 000$ It is not a vector space.

$$(1+1)\cdot[r]100 \neq [r]100 + [r]100$$

It is not a vector space.

$$1\cdot[r]100 \neq [r]100$$

For each, decide if it is a vector space; the intended operations are the natural ones. The diagonal 2 matrices

$$a00ba, b \in \Re$$

This set of 2 matrices

$$xx + yx + yyx, y \in \Re$$

This set

$$xyzw \in \Re^4 x + y + w = 1$$

The set of functions $f\Re\Re df/dx + 2f = 0$ The set of functions $f\Re\Re df/dx + 2f = 1$ For each "yes" answer, you must give a check of all the conditions given in the definition of a vector space. For each "no" answer, give a specific example of the failure of one of the conditions. Yes. Yes. No, this set is not closed under the natural addition operation. The vector of all 1/4's is a member of this set but when added to itself the result, the vector of all 1/2's, is a nonmember. Yes. No, $f(x) = e^{-2x} + (1/2)$ is in the set but $2\cdot f$ is not (that is, condition (6) fails).

Prove or disprove that this is a vector space: the real-valued functions f of one real variable such that $f(7) = 0$. It is a vector space. Most conditions of the definition of vector space are routine; we here check only closure. For addition, $(f_1 + f_2)(7) = f_1(7) + f_2(7) = 0 + 0 = 0$. For scalar multiplication, $(r\cdot f)(7) = rf(7) = r0 = 0$.

Show that the set $\Re^+$ of positive reals is a vector space when we interpret '$x+y$' to mean the product of x and y (so that $2+3$ is 6), and we interpret '$r \cdot x$' as the r-th power of x. We check def:VecSpace.

First, closure under '+' holds because the product of two positive reals is a positive real. The second condition is satisfied because real multiplication commutes. Similarly, as real multiplication associates, the third checks. For the fourth condition, observe that multiplying a number by $1 \in \Re^+$ won't change the number. Fifth, any positive real has a reciprocal that is a positive real.

The sixth, closure under '$\cdot$', holds because any power of a positive real is a positive real. The seventh condition is just the rule that v^{r+s} equals the product of v^r and v^s. The eight condition says that $(vw)^r = v^r w^r$. The ninth condition asserts that $(v^r)^s = v^{rs}$. The final condition says that $v^1 = v$.

Is $(x, y)x, y \in \Re$ a vector space under these operations? $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$ $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, 0)$ No: $1 \cdot (0, 1) + 1 \cdot (0, 1) \neq (1 + 1) \cdot (0, 1)$. No; the same calculation as the prior answer shows a condition in the definition of a vector space that is violated. Another example of a violation of the conditions for a vector space is that $1 \cdot (0, 1) \neq (0, 1)$.

Prove or disprove that this is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial. It is not a vector space since it is not closed under addition, as $(x^2)+(1+x-x^2)$ is not in the set.

At this point "the same" is only an intuition, but nonetheless for each vector space identify the k for which the space is "the same" as $\Re^k$. The 23 matrices under the usual operations The nm matrices (under their usual operations) This set of 2 matrices

$$a0bca, b, c \in \Re$$

This set of 2 matrices

$$a0bca + b + c = 0$$

6 *nm* 3 To see that the answer is 2, rewrite it as

$$a0b - a - ba, b \in \Re$$

so that there are two parameters.

Using $\vec{+}$ to represent vector addition and $\vec{\cdot}$ for scalar multiplication, restate the definition of vector space. A vector space (over $\Re$) consists of a set V along with two operations '$\vec{+}$' and '$\vec{\cdot}$' subject to these conditions. Where $\vec{v}, \vec{w} \in V$, (1) their vector sum $\vec{v} \vec{+} \vec{w}$ is an element of V. If $\vec{u}, \vec{v}, \vec{w} \in V$ then (2) $\vec{v} \vec{+} \vec{w} = \vec{w} \vec{+} \vec{v}$ and (3) $(\vec{v} \vec{+} \vec{w}) \vec{+} \vec{u} = \vec{v} \vec{+} (\vec{w} \vec{+} \vec{u})$. (4) There is a zero vector $\in V$ such that $\vec{v}\vec{+} = \vec{v}$ for all $\vec{v} \in V$. (5) Each $\vec{v} \in V$ has an additive inverse $\vec{w} \in V$ such that $\vec{w} \vec{+} \vec{v} =$. If r, s are scalars, that is, members of $\Re$), and $\vec{v}, \vec{w} \in V$ then (6) each scalar multiple $r \cdot \vec{v}$ is in V. If $r, s \in \Re$ and $\vec{v}, \vec{w} \in V$ then (7) $(r + s) \cdot \vec{v} = r \cdot \vec{v} \vec{+} s \cdot \vec{v}$, and (8) $r \vec{\cdot} (\vec{v} + \vec{w}) = r \vec{\cdot} \vec{v} + r \vec{\cdot} \vec{w}$, and (9) $(rs) \vec{\cdot} \vec{v} = r \vec{\cdot} (s \vec{\cdot} \vec{v})$, and (10) $1 \vec{\cdot} \vec{v} = \vec{v}$.

Prove these. For any $\vec{v} \in V$, if $\vec{w} \in V$ is an additive inverse of $\vec{v}$, then $\vec{v}$ is an additive inverse of $\vec{w}$. So a vector is an additive inverse of any additive inverse of itself. Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ implies that $\vec{s} = \vec{t}$. Let V be a vector space, let $\vec{v} \in V$, and assume that $\vec{w} \in V$ is an additive inverse of $\vec{v}$ so that $\vec{w} + \vec{v} =$. Because addition is commutative, $= \vec{w} + \vec{v} = \vec{v} + \vec{w}$, so therefore $\vec{v}$ is also the additive inverse of $\vec{w}$. Let V be a vector space and suppose $\vec{v}, \vec{s}, \vec{t} \in V$. The additive inverse of $\vec{v}$ is $-\vec{v}$ so $\vec{v} + \vec{s} = \vec{v} + \vec{t}$ gives that $-\vec{v} + \vec{v} + \vec{s} = -\vec{v} + \vec{v} + \vec{t}$, which says that $+\vec{s} = +\vec{t}$ and so $\vec{s} = \vec{t}$.

The definition of vector spaces does not explicitly say that $+\vec{v} = \vec{v}$ (it instead says that $\vec{v}+ = \vec{v}$). Show that it must nonetheless hold in any vector space. Addition is commutative, so in any vector space, for any vector $\vec{v}$ we have that $\vec{v} = \vec{v}+ = +\vec{v}$.

Prove or disprove that this is a vector space: the set of all matrices, under the usual operations. It is not a vector space since addition of two matrices of unequal sizes is not defined, and thus the set fails to satisfy the closure condition.

In a vector space every element has an additive inverse. Can some elements have two or more? Each element of a vector space has one and only one additive inverse.

For, let V be a vector space and suppose that $\vec{v} \in V$. If $\vec{w}_1, \vec{w}_2 \in V$ are both additive inverses of $\vec{v}$ then consider $\vec{w}_1 + \vec{v} + \vec{w}_2$. On the one hand, we have that it equals $\vec{w}_1 + (\vec{v} + \vec{w}_2) = \vec{w}_1 + = \vec{w}_1$. On the other hand we have that it equals $(\vec{w}_1 + \vec{v}) + \vec{w}_2 = +\vec{w}_2 = \vec{w}_2$. Therefore, $\vec{w}_1 = \vec{w}_2$.

Prove that every point, line, or plane thru the origin in $\Re^3$ is a vector space under the inherited operations. What if it doesn't contain the origin? Every such set has the form $r \cdot \vec{v} + s \cdot \vec{w} r, s \in \Re$ where either or both of $\vec{v}, \vec{w}$ may be . With the inherited operations, closure of addition $(r_1\vec{v} + s_1\vec{w}) + (r_2\vec{v} + s_2\vec{w}) = (r_1 + r_2)\vec{v} + (s_1 + s_2)\vec{w}$ and scalar multiplication $c(r\vec{v} + s\vec{w}) = (cr)\vec{v} + (cs)\vec{w}$ are easy. The other conditions are also routine. No such set can be a vector space under the inherited operations because it does not have a zero element.

Using the idea of a vector space we can easily reprove that the solution set of a homogeneous linear system has either one element or infinitely many elements. Assume that $\vec{v} \in V$ is not . Prove that $r \cdot \vec{v} =$ if and only if $r = 0$. Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$. Prove that any nontrivial vector space is infinite. Use the fact that a nonempty solution set of a homogeneous linear system is a vector space to draw the conclusion. Assume that $\vec{v} \in V$ is not . One direction of the if and only if is clear: if $r = 0$ then $r \cdot \vec{v} =$. For the other way, let r be a nonzero scalar. If $r\vec{v} =$ then $(1/r) \cdot r\vec{v} = (1/r) \cdot$ shows that $\vec{v} =$, contrary to the assumption. Where r_1, r_2 are scalars, $r_1\vec{v} = r_2\vec{v}$ holds if and only if $(r_1 - r_2)\vec{v} =$. By the prior item, then $r_1 - r_2 = 0$. A nontrivial space has a vector $\vec{v} \neq$. Consider the set $k \cdot \vec{v} k \in \Re$. By the prior item this set is infinite. The solution set is either trivial, or nontrivial. In the second case, it is infinite.

Is this a vector space under the natural operations: the real-valued functions of one real variable that are differentiable? Yes. A theorem of first semester calculus says that a sum of differentiable functions is differentiable and that $(f + g)' = f' + g'$, and that a multiple of a differentiable function is differentiable and that $(r \cdot f)' = r\,f'$.

A vector space over the complex numbers has the same definition as a vector space over the reals except that scalars are drawn from instead of from $\mathfrak{R}$. Show that each of these is a vector space over the complex numbers. (Recall how complex numbers add and multiply: $(a_0 + a_1 i) + (b_0 + b_1 i) = (a_0 + b_0) + (a_1 + b_1)i$ and $(a_0 + a_1 i)(b_0 + b_1 i) = (a_0 b_0 - a_1 b_1) + (a_0 b_1 + a_1 b_0)i$.) The set of degree two polynomials with complex coefficients This set

$$0ab0a, b \in \text{ and } a + b = 0 + 0i$$

The check is routine. Note that '1' is $1 + 0i$ and the zero elements are these. $(0 + 0i) + (0 + 0i)x + (0 + 0i)x^2$ $0 + 0i0 + 0i$
$0 + 0i0 + 0i$

Name a property shared by all of the $\mathfrak{R}^n$'s but not listed as a requirement for a vector space. Notably absent from the definition of a vector space is a distance measure.

Prove that for any four vectors $\vec{v}_1, \ldots, \vec{v}_4 \in V$ we can associate their sum in any way without changing the result.

$$\begin{aligned}((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 = (\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 &= (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\ = \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4) &= \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4))\end{aligned}$$

This allows us to write '$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4$' without ambiguity. Prove that any two ways of associating a sum of any number of vectors give the same sum. (*Hint.* Use induction on the number of vectors.) A small rearrangement does the trick.

$$\begin{aligned}(\vec{v}_1 + (\vec{v}_2 + \vec{v}_3)) + \vec{v}_4 &= ((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \vec{v}_4 \\ &= (\vec{v}_1 + \vec{v}_2) + (\vec{v}_3 + \vec{v}_4) \\ &= \vec{v}_1 + (\vec{v}_2 + (\vec{v}_3 + \vec{v}_4)) \\ &= \vec{v}_1 + ((\vec{v}_2 + \vec{v}_3) + \vec{v}_4)\end{aligned}$$

Each equality above follows from the associativity of three vectors that is given as a condition in the definition of a vector space. For instance, the second '=' applies the rule $(\vec{w}_1 + \vec{w}_2) + \vec{w}_3 = \vec{w}_1 + (\vec{w}_2 + \vec{w}_3)$ by taking $\vec{w}_1$ to be $\vec{v}_1 + \vec{v}_2$, taking $\vec{w}_2$ to be $\vec{v}_3$, and taking $\vec{w}_3$ to be $\vec{v}_4$. The base case for induction is the three vector case. This case $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$ is one of the conditions in the definition of a vector space.

For the inductive step, assume that any two sums of three vectors, any two sums of four vectors, $\ldots$, any two sums of k vectors are equal no matter how we parenthesize the sums. We will show that any sum of $k + 1$ vectors equals this one $((\cdots((\vec{v}_1 + \vec{v}_2) + \vec{v}_3) + \cdots) + \vec{v}_k) + \vec{v}_{k+1}$.

Any parenthesized sum has an outermost '+'. Assume that it lies between $\vec{v}_m$ and $\vec{v}_{m+1}$ so the sum looks like this.

$$(\cdots\ \vec{v}_1 \cdots \vec{v}_m\ \cdots) + (\cdots\ \vec{v}_{m+1} \cdots \vec{v}_{k+1}\ \cdots)$$

The second half involves fewer than $k + 1$ additions, so by the inductive hypothesis we can re-parenthesize it so that it reads left to right from the inside out, and in particular, so that its outermost '+' occurs right before $\vec{v}_{k+1}$.

$$= (\cdots\ \vec{v}_1\ \cdots\ \vec{v}_m\ \cdots) + ((\cdots(\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots + \vec{v}_k) + \vec{v}_{k+1})$$

Apply the associativity of the sum of three things

$$= ((\cdots\ \vec{v}_1\ \cdots\ \vec{v}_m\ \cdots) + (\cdots\ (\vec{v}_{m+1} + \vec{v}_{m+2}) + \cdots\ \vec{v}_k)) + \vec{v}_{k+1}$$

and finish by applying the inductive hypothesis inside these outermost parenthesis.

ex:ColsIntEntNotVS gives a subset of $\mathfrak{R}^2$ that is not a vector space, under the obvious operations, because while it is closed under addition, it is not closed under scalar multiplication. Consider the set of vectors in the plane whose components have the same sign or are 0. Show that this set is closed under scalar multiplication but not addition. Let $\vec{v}$ be a member of $\mathfrak{R}^2$ with components v_1 and v_2. We can abbreviate the condition that both components have the same sign or are 0 by $v_1 v_2 \geq 0$.

To show the set is closed under scalar multiplication, observe that the components of $r\vec{v}$ satisfy $(rv_1)(rv_2) = r^2(v_1v_2)$ and $r^2 \geq 0$ so $r^2v_1v_2 \geq 0$.

To show the set is not closed under addition we need only produce one example. The vector with components -1 and 0, when added to the vector with components 0 and 1 makes a vector with mixed-sign components of -1 and 1.

Possible Questions
8 marks

1. State and prove two properties of inverse of a linear map

2. Prove that inverse of a linear map is linear

3. If A is an algebra with unit element over F then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V

4. If V is finite dimensional over F and if is singular then prove that there exists an in $A(V)$ such that $ST = TS = 0$

5. If A is an algebra with unit element over F then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V

6. Let T be a linear transformation from V to W. Prove that the image of V under T is a subspace of W

7. If $\{v_1, v_2, \cdots, v_m\}$ and $\{u_1, u_2, \cdots, u_m\}$ are both bases of a vector space V over a field F, then $m = n$

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defned by $T(a_1, a_2, a_3) = (3a_1 + a_2, a_1 + a_3, a_1 - a_3)$. Find the matrix representation of T w.r.t the standard basis for both domain and range

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defned by $T(x, y) = (x - y, x + y)$. Find the matrix representation of T w.r.t the standard basis for domain and $\{(1, 1), (1. - 1)\}$ for range

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Ring theory and Linear Algebra I **Subject Code: 16MMU302**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
_____________that satisfies all of the ring axioms but the associativity and the existence of a multiplicative identity.	Associative ring	Commutative ring	Non ideal	Nonassociative ring	Nonassociative ring
Ring not requiring multiplicative identity is______________	Proper ideal	Field	Pseudo-ring	Integral domain	Pseudo-ring
rng=	Proper ideal	Field	Pseudo-ring	Integral domain	Pseudo-ring
A nonzero ring with no nonzero zero divisors ____________	Field	Ideal	Domain	Pseudo-ring	Domain
A division ring is a ring such that every non zero element is a	simple	Complex	Proper	unit	Unit
A commutative division ring example of a	Field	Ideal	Domain	Pseudo-ring	Field
The unique identity of the additive group (ℝ,+) is denoted by 0 and it is called _____ element of the ring	Non zero	Zero	Trivial	Non trivial	Zero
{0} with two binary operation +and · is called ______________ring	null	empty	Singleton	Boolean	Null
A ring R is called a Boolean ring if _____________for all aϵℝ	a^2 #NAME?	$\underline{a}$ a^2	$\frac{a}{a^2}$	a=0	a^2 #NAME?
A ring R is called an _________________ if a^2 #NAME?	Integral domain	Boolean ring	Pseudo-ring	All the above	Boolean ring
The example for Boolean ring	(p(s), Δ, ∩)	(p(s), ∩, Δ)	(p(s), ∪,∩)	(p(s), ∩, ∪)	(p(s), Δ, ∩)
A ring is said to be commutative ring if ___________________	ab=ba	a^2 #NAME?	b^2 #NAME?	ab=ab	ab=ba
The set of all integers, rationales, and reals are all ________rings.	Non commutative	identity	Commutative	Boolean	Commutative
__________is both addition identity and multiplication identity for zero ring	1	0	-1	2	0
In a ring with identity the identity element is a ____________	Equal	More than one	Different	unique	Unique
Consider a ring (R,+,-)with identity then then unit of R is __________	Zero element	All non-zero elements	Identity element	Invers element	All non-zero elements
the unit of ring (M_2(R)) is	Singular matrix	Non-singular matrix	Idempotent	Identity matrix	
The unit of (Z,+,-) with identity 1 is _________	1 and -1	0 and 1	0 and -1	-1 and -2	1 and -1
Identify the zero divisors of (Z_{10},+,-)	2,3,4,6	2,4,6,8	1,2,3,4	2,8,16,18	2,3,4,6
ℤ□ is an integral domain iff n is ____________	composite	Prime	Real	Neither prime nor composite	Prime
The characteristic of ring (Z□,+,-) is ________	1	0	N	n-1	N
Δ ∩ The characteristic of ring (p(s),	1	2	3	4	2
The characteristic of Boolean ring is	1	2	3	4	2
The characteristic of ring (M_2(R))	1	2	3	4	2
The characteristic of integral domain is	0	prime	Either 0 or prime	Neither 0 nor prime	Either 0 or prime
The characteristic of any field is	0	prime	Either 0 or prime	Neither 0 nor prime	
Choose the correct idempotent matrix	0 0 0 0 0 0	0 1 0 0 0 1	0 1 0 1 0 0	0 0 0 1 0 0	0 0 0 0 0 0
If F is a field its only ideals are ___________	0 and F itself	0	F itself	None of these	0 and F itself
If F is a commutative ring then aR=Ra is an ideal and is called ____________	Principal ideal ring	Ideal ring	Maximal ring	Commutative Ring	Principal ideal ring
Which of the following is a division ring usual addition and a multiplication	ø	ℝ	ℂ	All the above	All the above
The only idempotent element of an integral domain are	0	1	0 and 1	Neither 0 nor 1	0 and 1
R is set of integers, 0, positive and negative; + and. is the usual addition and multiplication of integers R is a ______________	Commutative rings	Unit element	Commutative rings but has no unit element.	Commutative ring with unit element.	Commutative rings with unit element.
A field is a	Commutative ring	unit	Division ring	Commutative division ring	Commutative division ring
A finite integral domain is a	finite	field	Ideal	unit	Field
The ring of integers is thus of characteristic ___	0	1	2	3	0
An integral domain D is said to be of finite characteristic if there exists a positive integers m such that __________for all aϵD	ma=1	≠ ma	ma=0	≠ ma	ma=0
The ring of integers is an	Integral domain	field	Ideal	unit	Integral domain
If p is a prime number then J□, then the ring of integers mod p is a	Integral domain	field	Ideal	unit	Field
Set of even integers with usual + and · is a	unit	field	rng	ring	Rng
__________defined the concept of ring of integers of number field	Fraenkel	Noether	Hilbert	Richard Dedekind	Richard Dedekind
Unital ring, unitary ring, ring with unit, ring with identity, or ring with 1 is defined as	Ring with multiplicative identity	Ring with multiplicative inverse	Both a and b	Neither a or b	Ring with multiplicative identity
Ring ____________________________________ is rng or pseudo-ring.	requiring multiplicative identity	Not requiring multiplication identity	requiring multiplication inverse	Not requiring multiplication inverse	Not requiring multiplication identity
Ring with n elements is	Z	R	Z_n	Q	Z_n
Which of the following is not a field?	Z	R	Z_n	Q	Z

Reg. No..................

16MMU302

Karpagam Academy of Higher Education
Karpagam University
Coimbatore-21
Department of Mathematics
Third Semester- I Internal test
Ring theory and Linear Algebra I

Date: Time: 2 hours
Class: II B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. If $U(\mathbb{Z})$ is the set of all units of $(\mathbb{Z}, +, \cdot)$, then $U(\mathbb{Z}) =$

   a. $\{1\}$ b. $\{1, -1\}$
   c. $\{-1, 0, 1\}$ d. $\{-1\}$

2. If every nonzero element of R is a unit, then R is called a

   a. division ring b. ring
   c. integral domain d. field

3. If $1 = 0$, then the ring consists of

   a. countable elements b. uncountable elements
   c. two elements d. one element

4. A commutative division ring is called a

   a. an ideal b. subring
   c. integral domain d. field

5. Any finite integral domain is a

   a. an ideal b. subring
   c. principal ideal d. field

6. Which of the following is an idempotent element in $M_2(\mathbb{Z})$?

   a. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ d. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

7. Suppose that $a^2 = a$ in a ring R. Then $(1 - a)^2$,

   a. a b. $1 - a$
   c. 1 d. 0.

8. Number of units of a ring $(\mathbb{Q}, +, \cdot)$ is

   a. countable b. uncountable
   c. 2 d. 1

9. Let R be a ring with identity element 1. Then $U(R)$, the set of all units forms ——- under multiplication
   a. an abelian b. group
   c. an ideal d. field

10. If u is a unit and is idempotent, then $u =$

    a. 0 b. 1
    c. either 0 or 1 d. neither 0 nor 1

11. Let R be a commutative ring. Then the ——— of subrings of R is a subring of R.

    a.intersection of any collection b.intersection of finite collection c.union of any collection d. union of finite collection

12. In an integral domain idempotent elements are

a. 0 and -1 b. 0 and 1
c. either 0 or 1 d. neither 0 nor 1

13. $M_2(\mathbb{R})$ is a

a. commutative ring b. non commutative fing
c. ring without identity d. ring without zero divisors

14. Let R be a ring without identity. For $a \in R$

a. $a \notin (a)$ b. $a \in (a)$
c. $1 \in (a)$ d.either b nor c

15. $\mathbb{Z}_n$ is an integral domain iff n is

a. an integer b. prime
c. composite d. neither b nor c

16. Which of the following is a Boolean ring?

a. $\mathbb{Z}_1$ b. $\mathbb{Z}_2$
c. $\mathbb{Z}_3$ d.$\mathbb{Z}_4$

17. $(\mathscr{P}(S), \Delta, \cap)$ Characteristic of

a. 1 b. 0
c. 2 d.3

18. ——- is a subring of every ring.

a. $\{1\}$ b. $\{0\}$
c. R d.$\mathbb{Z}$

19. The prime ideals of $\mathbb{Z}$ are

a. $(0), (1), (2), (3), \cdots$ b. $(1), (2), (3), \cdots$
c. $(1), (3), \cdots$ d.$(0), (2), (3), \cdots$

20. Ideals of $\mathbb{Z}$ are

a. $(0), (1), (2), (3), \cdots$ b. $(1), (2), (3), \cdots$
c. $(1), (3), (5), \cdots$ d.$(0), (2), (3), \cdots$

Part B-($3 \times 2 = 6$ **marks)**

21. Define Boolean ring and give an example

22. Define principal ideal generated by an elenment

23. If U is an ideal of R and $1 \in U$ then prove that $U = R$

Part C-($3 \times 8 = 24$ **marks)**

24. a) Prove that $\mathbb{Z}_{\ltimes}$ is a field iff n is prime

OR

b) Prove that the characteristic of an integral domain is either prime or 0

25. a) Classify the ring $(S, \oplus_{10}, \odot_{10})$ where $S = \{0, 2, 4, 6, 8\}$. What is the unity of the ring. Is it a ring with or without zero divisors?

OR

b) If two operations $*$ and $\bigcirc$ on the set of integers $\mathbb{Z}$ are defined by

$$a * b = a + b + 1$$

and

$$a \bigcirc b = a + b + ab$$

for all $a, b \in \mathbb{Z}$. Show that $(\mathbb{Z}, *, \bigcirc)$ is a commutative ring. What is the zero of the ring? Is it ring with unity?

26. a) Prove that any field is an integral domain. Justify the converse by an example.

OR

b) Prove that the set of all units in R is a group under multiplication

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Karpagam University
Coimbatore-21
Department of Mathematics
Third Semester- II Internal test
Ring theory and Linear Algebra I

Date: Time: 2 hours
Class: II B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A (20 × 1 = 20 marks)

1. If $f : R \rightarrow R^{'}$ is a homomorphism then

   a. $f(a + b) = f(a) + f(b)$ b. $f(a \cdot b) = f(a) \cdot f(b)$
   c. neither a nor b d. both a and b

2. f is called monomorphism if f is

   a. 1-1 b. homomorphism
   c. neither a nor b d. both a and b

3. f is called isomorphism if f is

   a. 1-1 b. homomorphism
   c. neither a nor b d. both a and b

4. Which of the following is homomorphism?

   a. $f(a) = 0$ b. $f(a) = a$
   c. neither a nor b d. both a and b

5. Let $a \in Kernal$ of f. Then $f(a) =$

   a. a b. 1
   c. 0 d. -1

6. *Kernal* of f =

   a. $f^{-1}(\{0\})$ b. $f^{-1}(\{-1\})$
   c. $f^{-1}(\{1\})$ d. $f^{-1}(\{2\})$

7. Any homomorphism of a field to itself is

   a. 1-1 b. many to one
   c. either a or b d. neither a nor b.

8. Let f be a homomorphism from a commutative ring with identity to a field. Then *Kernal* of f is

   a. an ideal b. a maximal ideal
   c. neither a nor b d. both a and b

9. Let V be a vector space over a field F. Then $\alpha \in F$ and $u \in V$ imples

   a. $\alpha u \in V$ b.$\alpha u \notin V$
   c. neither a nor b d. both a and b

10. Scalar multiplication is a function from —— to ——

    a. F, V b. $F \times V, V$
    c. neither a nor b d. both a and b

11. The elements of F are called

    a. vectors b. scalars
    c. neither a nor b d. both a and b

12. The elements of V are called

a. vectors b. scalars
c. neither a nor b d. both a and b

13. $\alpha\mathbf{0}$=

a. 0 b. $\mathbf{0}$
c. neither a nor b d. both a and b

14. $0v$=

a. 0 b. $\mathbf{0}$
c. neither a nor b d. both a and b

15. αv=$\mathbf{0}$ implies

a. $\alpha = 0$ b. v =$\mathbf{0}$
c. either a or b d. both a and b

16. $W = \{(a, 0, 0) : a \in \mathbb{R}\}$ is a subspace of

a. $\mathbb{R}$ b. $\mathbb{R}^2$
c. $\mathbb{R}^3$ d.$\mathbb{R}^4$

17. Let A and B are subspaces of V. Which of the following is a subspace of V?

a. $A + B$ b. $A \cap B$
c. either a or b d. both a and b

18. Let A and B are subspaces of V. Then $A \cup B$ is a subspace of V if

a. $A \subset B$ b. $B \subset A$
c. either a or b d. both a and b

19. Let A and B are subspaces of V. Which of the following is a subspace of V containing A and B?

a. $A + B$ b. $A \cap B$
c. either a or b d. both a and b

20. Let A and B are subspaces of V. Then V is called direct sum of A and B if

a. $A + B = V$ b. $A \cap B$ =$\mathbf{0}$
c. either a or b d. both a and b

Part B-($3 \times 2 = 6$ **marks)**

21. Define homomorphism and give an example

22. Define vector space

23. Show that union of two subspaces need not be a subspace

Part C-($3 \times 8 = 24$ **marks)**

24. a) State and prove isomorphism theorem I

OR

b) Prove that any integral domain D can be embedded in a field F and every element of F can be expressed as a quotient of two elements of D

25. a) State and prove two properties of homomorphism

OR

b) Let $f : R \to R'$ be a homomorphism and K be the kernal of f. Prove that K is an ideal of R

26. a) Prove that $\mathbb{R} \times \mathbb{R}$ is a vector space over $\mathbb{R}$

OR

b) Let W be a nonemepty subset of a vector space V. State and prove the necessary and sufficient condition for W to be a subspace of V

Reg. No..................

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Karpagam Academy of Higher Education
Karpagam University
Coimbatore-21
Department of Mathematics
Third Semester- III Internal test
Ring theory and Linear Algebra I

Date: Time: 2 hours
Class: II B.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. Let V and W be vector spaces. A function $T : V \to W$ is called a linear transformation if for any vectors u, v in V and scalar α,

   a. $T(u + v) = T(u) + T(v)$ b. $(\alpha u) = \alpha T(u)$
   c. neither a nor b d. both a and b

2. The inverse images of **0** is called the

   a. Kernal of T b. homomorphism
   c. neither a nor b d. both a and b

3. T is called isomorphism if T is

   a. 1-1 b. homomorphism
   c. neither a nor b d. both a and b

4. Which of the following is linear transformation?

   a. $T(u) =\mathbf{0}$ b. $T(u) = u$
   c. neither a nor b d. both a and b

5. Let $a \in Kernal$ of T. Then $T(u) =$

   a. u b. v
   c. **0** d. -1

6. *Kernal* of T =

   a. $T^{-1}(\{0\})$ b. $T^{-1}(\{-1\})$
   c. $T^{-1}(\{1\})$ d. $T^{-1}(\{2\})$

7. A set of vectors $\{v_1, \cdots, v_n\}$ is said to be linearly independent if $\alpha_1 v_1 + \cdots \alpha_n v_n = 0$ implies

   a. some $\alpha_i = 0$ b. $\alpha_i = 0$
   c. either a or b d. neither a nor b.

8. Let $S = \{(6, 2, 1), (-1, 3, 2)\}$ and S be linearly independent. Suppose $\alpha(6, 2, 1) + \beta(-1, 3, 2) = (0, 0, 0)$. Then

   a. $\alpha = 0$ b. $\beta = 0$
   c. neither a nor b d. both a and b

9. Let V be a vector space and S be a subset of V. Suppose S ia a basis for V. Then

   a. $L(S) = V$ b.S is linearly independent
   c. neither a nor b d. both a and b

10. Consider the vector space $\mathbb{R}^2$. Then basis for $\mathbb{R}^2$ is .

    a. $\{e_1, e_2\}$ b. $\{e_1, e_2, e_3\}$
    c. neither a nor b d. both a and b

11. Which of the following is a basis for $\mathbb{C}$?

    a. $\{1, 0\}$ b. $\{1, i\}$
    c. neither a nor b d. both a and b

12. Which of the following is a basis for $\mathbb{R}$?

a. $\{1\}$ b. $\{0\}$
c. neither a nor b d. both a and b

13. Which of the following is a basis for $\mathbb{R}^3$?

a. $\{(1,1,1),(1,-1,1),(1,1,-1)\}$ b. $\{e_1,e_2,e_3\}$
c. neither a nor b d. both a and b

14. Let $S = \{v_1,\cdots,v_n\}$ be a basis for V. Then every subset of V contains more than n elements is linearly

a. dependent b. independent
c. neither a nor b d. both a and b

15. Let $S_1 = \{v_1,\cdots,v_n\}$ and $S_2 = \{v_1,\cdots,v_m\}$ are bases for V. Then

a. $m \leq n$ b. $n \leq m$
c. either a or b d. both a and b

16. dim $\mathbb{R}$ =

a. 1 b. 0
c. n d. 4

17. dim $\mathbb{C}$ =

a. 1 b. 0
c. n d. 4

18. dim $M_2(\mathbb{R})$ =

a. 1 b. 0
c. n d. 4

19. dim $V_n(\mathbb{R})$ =

a. 1 b. 0
c. n d. 4

20. The dimension of a vector space V, is number of elements in

a. V b. basis for V
c. either a or b d. both a and b

Part B-($3 \times 2 = 6$ **marks)**

21. Define basis for a vector space

22. Write the standard basis for $\mathbb{R}^n$

23. Determine, whether $S = \{(0,0,0),(1,5,6),(6,2,1)\}$ is a basis of $\mathbb{R}^3$ or not?

Part C-($3 \times 8 = 24$ **marks)**

24. a) Let $S = \{(6,2,1),(-1,3,2)\}$. Determine, if S is linearly independent or dependent?

OR

b) Let $S = \{(1,0,0),(0,4,0),(0,0,-6),(1,5,-3)\}$. Determine, if S is linearly independent or dependent?

25. a) State and prove two properties of linear transformation

OR

b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defned by $T(a_1,a_2,a_3) = (3a_1+a_2, a_1+a_3, a_1-a_3)$. Find the matrix representation of T w.r.t the standard basis for both domain and range

26. a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defned by $T(x, y) = (x - y, x + y)$. Find the matrix representation of T w.r.t the standard basis for domain and $\{(1, 1), (1. - 1)\}$ for range

OR

b) State and prove fundamental theorem of linear transformation