KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Enable | Enlighten | Enrich
(Deemed to be University)
(Under Section 3 of UGC Act 1956)

Subject: REAL ANALYSIS -I **Subject Code: 15MMU501**

L	T	P	C
5	0	0	5

PO: After the completion of this course, the learners get a clear knowledge in the foundational concepts of analysis which is the motivating tool in the study of pure Mathematics. The learner understands the functional relationships between the variables which have more applications in expressing the laws of Physics, Chemistry, and Mechanics etc.

PLO: To introduce the different concepts of Real and Complex number systems and to provide a strong base in the analysis part of Mathematics such as ordered pairs ,Cartesian product of two sets ,open balls and open sets, Completeness sequences etc.

UNIT I

The Real and Complex number systems the field axioms, the order axioms –integers –the unique Factorization theorem for integers –Rational numbers –Irrational numbers –Upper bounds, maximum Elements, least upper bound –the completeness axiom –some properties of the supremum –properties of the integers deduced from the completeness axiom- The Archimedean property of the real number system .

UNIT II

Basic notions of a set theory: Notations –ordered pairs –Cartesian product of two sets – Relations and functions – further terminology concerning functions –one –one functions and inverse –composite functions –sequences –similar sets-finite and infinite sets –countable and uncountable sets –uncountability of the real number system.

UNIT III

Elements of point set topology: Euclidean space R^n –open balls and open sets in R^n. The structure of open Sets in R^n –closed sets and adherent points –The Bolzano –Weierstrass theorem–the Cantor intersection Theorem.

UNIT IV

Covering –Lindelof covering theorem –the Heine Borel covering theorem –Compactness in R^n – Metric Spaces –point set topology in metric spaces –compact subsets of a metric space – Boundary of a set.

UNIT V

Convergent sequences in a metric space –Cauchy sequences –Completeness sequences – complete metric Spaces. Limit of a function –Continuous functions –continuity of composite functions. Continuous complex valued and vector valued functions.

TEXT BOOK

1.Apostol.T.M.,1990. Mathematical Analysis, Second edition, Narosa Publishing Company, Chennai.

REFERENCES

1. Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

2. Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.
3. Royden .H.L ., 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.
4. Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .
5. Sterling. K. Berberian, 2004. A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

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Subject Name: REAL ANALYSIS **Subject Code: 15MMU501**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
UNIT-I			
1	1	Field axioms and order axioms	T1 : Chap 1, Pg.No : 1-4
2	1	Concepts about integers, unique Factorization theorem	T1 : Chap 1, Pg.No : 4-6
3	1	Continuation of unique Factorization theorem	T1 : Chap 1, Pg.No : 4-6
4	1	Rational numbers	T1 : Chap 1, Pg.No : 6-7
5	1	Irrational numbers	T1 : Chap 1, Pg.No : 6-7
6	1	Theorems for irrational numbers	T1 : Chap 1, Pg.No : 6-7
7	1	Upper bounds, Completeness axiom	T1 : Chap 1, Pg.No : 8-9, 36
8	1	Some properties of the Supremum	T1 : Chap 1, Pg.No :36
9	1	Properties of the integers deduced from the completeness axiom-	T1 : Chap 1, Pg.No : 10
10	1	The Archimedean property,	T1 : Chap 1, Pg.No : 10-1
11	1	Absolute values	T1 : Chap 1, Pg.No : 11-2
12	1	The triangle inequality	T1 : Chap 1, Pg.No : 12-3
13	1	Cauchy-Schwarz inequality	T1 : Chap 1, Pg.No : 13-4
14	1	Plus and minus infinity and the extended real number system.	R1 : Chap 9, Pg.No:40
15	1	Recapitulation and Discussion of possible questions	
Total	**15 Hrs**		

Text Book:
T1: Apostol.T.M.,1990. Mathematical Analysis, Second edition, Narosa Publishing Company, Chennai. **Reference Book:**
R1: Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

UNIT-II

1	1	Basic notions of a set theory	R2 : Chap 9 Pg.No :245-246
2	1	Cartesian product of two sets	R2 : Chap 9 Pg.No :247-254
3	1	Theorems for Cartesian product of two sets	R2 : Chap 9 Pg.No :247-254
4	1	Relations and function	R2 : Chap 9 Pg.No :254-267
5	1	Examples of Relation and function	R2 : Chap 9 Pg.No :254-267
6	1	One –one functions and inverse and composite functions	R2 : Chap 9 Pg.No :267-269
7	1	Examples of 1-1, inverse and composite function	R2 : Chap 9 Pg.No :267-269
8	1	Sequence and similar sets	R2 : Chap 9 Pg.No :269-277
9	1	Theorems for Sequence and similar sets	R2 : Chap 9 Pg.No :269-277
10	1	Finite and infinite sets	R4: Chap 2, Pg.No :24-26
11	1	Countable and uncountable sets	R4: Chap 2, Pg.No :26-29
12	1	Theorems for Countable and uncountable sets	R4: Chap 2, Pg.No :26-29
13	1	Set algebra	R4: Chap 2, Pg.No :29-30
14	1	Recapitulation and discussion of possible questions on unit II	
Total	**14 Hrs**		

Reference Book:
R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.
R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

UNIT-III

1	1	Elements of point set topology	R2 : Chap 10 Pg.No :280-281
2	1	Euclidean space R^n	R2 : Chap 10 Pg.No :281-283
3	1	Basic Definitions with examples	R2 : Chap 10 Pg.No :281-283
4	1	Open balls and open sets	R2 : Chap 10 Pg.No :283-287
5	1	Theorems for open Sets	R2 : Chap 10 Pg.No :283-287
6	1	The Structure of open Sets in R^n	R2 : Chap 10 Pg.No :287-288
7	1	Closed sets and adherent points	R2 : Chap 10

			Pg.No :288-290
8		Theorems for Closed sets	R2 : Chap 10 Pg.No :288-290
9	1	Constructions from Sample data	R2 : Chap 10 Pg.No :290-300
10	1	The Bolzano –Weierstrass theorem	R4: Chap 2 Pg.No :40-42
11	1	Continuation of the Bolzano –Weierstrass theorem	R4: Chap 2 Pg.No :40-42
12	1	Cantor intersection theorem	R4: Chap 2 Pg.No : 45-47
13		Continuation of Cantor intersection theorem	R4: Chap 2 Pg.No : 45-47
14	1	Recapitulation and discussion of possible questions on unit III	
Total	**14 Hrs**		

Reference Book:

R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

UNIT-IV

1	1	Covering	R2 : Chap 11 Pg.No :302-304
2	1	Lindelof covering theorem	R2 : Chap 11 Pg.No :304-312
3	1	Continuation of Lindelof covering theorem	R2 : Chap 11 Pg.No :304-312
4	1	Continuation of Lindelof covering theorem	
5	1	Lemma for Heine Borel covering theorem	R2 : Chap 11 Pg.No :304-312
6	1	Heine Borel covering theorem	R2 : Chap 11 Pg.No :312-317
7	1	Continuous of Heine Borel covering theorem	R2 : Chap 11 Pg.No :312-317
8	1	Compactness in R^n	R2 : Chap 11 Pg.No :317-321
9	1	Theorems for Compactness in R^n	R2 : Chap 11 Pg.No :317-321
10	1	Metric Spaces	R2 : Chap 11 Pg.No :321-323
11	1	Point set topology in metric spaces	R2 : Chap 11 Pg.No :323-325
12	1	Compact subsets of a metric space	R4: Chap 2, Pg.No :30-36

13	1	Theorem for Compact subsets of a metric space	R4: Chap 2, Pg.No :30-36
14	1	Boundary of a set	R4: Chap 3, Pg.No :52-57
15	1	Recapitulation and discussion of possible questions on unit IV	
Total	**15 Hrs**		

Reference Book:

R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

UNIT-V

1	1	Convergent sequences in a metric space	T1 : Chap 4, Pg.No :70-72
2	1	Cauchy sequences	T1 : Chap 4, Pg.No :72-74
3	1	Theorem for Cauchy's theorem	T1 : Chap 4, Pg.No :72-74
4	1	Completeness sequences	T1 : Chap 4, Pg.No :74
5	1	complete metric Spaces	T1 : Chap 4, Pg.No :74-76
6	1	Limit of a function	R3 : Chap 3, Pg.No :82-84
7	1	Problems for limit of a function	
8	1	Problems for limit of a function	R5 : Chap 4, Pg.No :102
9	1	Continuous function	T1 : Chap 4, Pg.No :70-72
10	1	Theorems for Continuous function	T1 : Chap 4, Pg.No :70-72
11	1	Continuity of composite function	R4: Chap 4, Pg.No:82-84
12	1	Problems of Composition function	
13	1	Continuous complex valued and vector valued functions.	R4: Chap 1, Pg.No :12-16
14	1	Examples for Real and Complex valued functions	R4: Chap 1, Pg.No :12-16
15	1	Discussion of previous ESE question papers	
16	1	Discussion of previous ESE question papers	
17	1	Discussion of previous ESE question papers	
Total	**17Hrs**		

Text Book:

T1: Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

Reference Book:

R3: Kenneth Hoffman., Ray Kunze., (2003). Linear Algebra, Second edition, Prentice Hall of

India Pvt Ltd, New Delhi.
R4: Royden .H.L ., 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.
R5: Sterling. K. Berberian, 2004. A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

Total no. of Hours for the Course: 75 hours

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DEPARTMENT OF MATHEMATICS

Subject: Real Analysis-I	**Semester: V**	**L**	**T**	**P**	**C**
Subject Code: 15MMU501	**Class: III-B.Sc Mathematics-A**	**5**	**0**	**0**	**5**

UNIT I

The Real and Complex number systems the field axioms, the order axioms –integers –the unique Factorization theorem for integers –Rational numbers –Irrational numbers –Upper bounds, maximum Elements, least upper bound –the completeness axiom –some properties of the supremum –properties of the integers deduced from the completeness axiom- The Archimedean property of the real number system .

Text Book:

T1: Apostol.T.M.,1990. Mathematical Analysis, Second edition, Narosa Publishing Company, Chennai. **Reference Book:**

R1: Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

THE REAL AND COMPLEX NUMBER SYSTEMS

THE FIELD AXIOMS

Along with the set **R** of real numbers we assume the existence of two operations, called *addition* and *multiplication*, such that for every pair of real numbers x and y the *sum* $x + y$ and the *product* xy are real numbers uniquely determined by x and y satisfying the following axioms. (In the axioms that appear below, x, y, z represent arbitrary real numbers unless something is said to the contrary.)

Axiom 1. $x + y = y + x,\ xy = yx$ (*commutative laws*).

Axiom 2. $x + (y + z) = (x + y) + z,\ x(yz) = (xy)z$ (*associative laws*).

Axiom 3. $x(y + z) = xy + xz$ (*distributive law*).

Axiom 4. *Given any two real numbers x and y, there exists a real number z such that $x + z = y$. This z is denoted by $y - x$; the number $x - x$ is denoted by 0. (It can be proved that 0 is independent of x.) We write $-x$ for $0 - x$ and call $-x$ the negative of x.*

Axiom 5. *There exists at least one real number $x \neq 0$. If x and y are two real numbers with $x \neq 0$, then there exists a real number z such that $xz = y$. This z is denoted by y/x; the number x/x is denoted by 1 and can be shown to be independent of x. We write x^{-1} for $1/x$ if $x \neq 0$ and call x^{-1} the reciprocal of x.*

THE ORDER AXIOMS

We also assume the existence of a relation $<$ which establishes an ordering among the real numbers and which satisfies the following axioms:

Axiom 6. *Exactly one of the relations $x = y$, $x < y$, $x > y$ holds.*

NOTE. $x > y$ means the same as $y < x$.

Axiom 7. *If $x < y$, then for every z we have $x + z < y + z$.*

Axiom 8. *If $x > 0$ and $y > 0$, then $xy > 0$.*

Axiom 9. *If $x > y$ and $y > z$, then $x > z$.*

NOTE. A real number x is called *positive* if $x > 0$, and *negative* if $x < 0$. We denote by $\mathbf{R}^+$ the set of all positive real numbers, and by $\mathbf{R}^-$ the set of all negative real numbers.

From these axioms we can derive the usual rules for operating with inequalities. For example, if we have $x < y$, then $xz < yz$ if z is positive, whereas $xz > yz$ if z is negative. Also, if $x > y$ and $z > w$ where both y and w are positive, then $xz > yw$.

Theorem

Given real numbers a and b such that

$$a \leq b + \varepsilon \quad \text{for every } \varepsilon > 0. \tag{1}$$

Then $a \leq b$.

Proof. If $b < a$, then inequality (1) is violated for $\varepsilon = (a - b)/2$ because

$$b + \varepsilon = b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a.$$

Therefore, by Axiom 6 we must have $a \le b$.

INTEGERS

Definition

A set of real numbers is called an inductive set if it has the following two properties:

a) *The number 1 is in the set.*

b) *For every x in the set, the number $x + 1$ is also in the set.*

For example, **R** is an inductive set. So is the set $\mathbf{R}^+$. Now we shall define the positive integers to be those real numbers which belong to every inductive set.

Definition

A real number is called a positive integer if it belongs to every inductive set. The set of positive integers is denoted by $\mathbf{Z}^+$.

THE UNIQUE FACTORIZATION THEOREM FOR INTEGERS

If n and d are integers and if $n = cd$ for some integer c, we say d is a *divisor* of n, or n is a *multiple* of d, and we write $d|n$ (read: d divides n). An integer n is called a *prime* if $n > 1$ and if the only positive divisors of n are 1 and n. If $n > 1$ and n is not prime, then n is called *composite*. The integer 1 is neither prime nor composite.

This section derives some elementary results on factorization of integers, culminating in the *unique factorization theorem*, also called *the fundamental theorem of arithmetic*.

The fundamental theorem states that (1) every integer $n > 1$ can be represented as a product of prime factors, and (2) this factorization can be done in only one way, apart from the order of the factors. It is easy to prove part (1).

Theorem

Every integer $n > 1$ is either a prime or a product of primes.

Proof. We use induction on n. The theorem holds trivially for $n = 2$. Assume it is true for every integer k with $1 < k < n$. If n is not prime it has a positive divisor d with $1 < d < n$. Hence $n = cd$, where $1 < c < n$. Since both c and d are $<n$, each is a prime or a product of primes; hence n is a product of primes.

Before proving part (2), uniqueness of the factorization, we introduce some further concepts.

If $d|a$ and $d|b$ we say d is a *common divisor* of a and b. The next theorem shows that every pair of integers a and b has a common divisor which is a linear combination of a and b.

Theorem

Every pair of integers a and b has a common divisor d of the form

$$d = ax + by$$

where x and y are integers. Moreover, every common divisor of a and b divides this d.

Proof. First assume that $a \geq 0$, $b \geq 0$ and use induction on $n = a + b$. If $n = 0$ then $a = b = 0$, and we can take $d = 0$ with $x = y = 0$. Assume, then, that the theorem has been proved for $0, 1, 2, \ldots, n - 1$. By symmetry, we can assume $a \geq b$. If $b = 0$ take $d = a$, $x = 1$, $y = 0$. If $b \geq 1$ we can apply the induction hypothesis to $a - b$ and b, since their sum is $a = n - b \leq n - 1$. Hence there is a common divisor d of $a - b$ and b of the form $d = (a - b)x + by$. This d also divides $(a - b) + b = a$, so d is a common divisor of a and b and we have $d = ax + (y - x)b$, a linear combination of a and b. To complete the proof we need to show that every common divisor divides d. Since a common divisor divides a and b, it also divides the linear combination $ax + (y - x)b = d$. This completes the proof if $a \geq 0$ and $b \geq 0$. If one or both of a and b is negative, apply the result just proved to $|a|$ and $|b|$.

Theorem

(Euclid's Lemma). *If $a|bc$ and $(a, b) = 1$, then $a|c$.*

Proof. Since $(a, b) = 1$ we can write $1 = ax + by$. Therefore $c = acx + bcy$. But $a|acx$ and $a|bcy$, so $a|c$.

Theorem

If a prime p divides ab, then $p|a$ or $p|b$. More generally, if a prime p divides a product $a_1 \cdots a_k$, then p divides at least one of the factors.

Proof. Assume $p|ab$ and that p does not divide a. If we prove that $(p, a) = 1$, then Euclid's Lemma implies $p|b$. Let $d = (p, a)$. Then $d|p$ so $d = 1$ or $d = p$. We cannot have $d = p$ because $d|a$ but p does not divide a. Hence $d = 1$.

Theorem

(Unique factorization theorem). *Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.*

Proof. We use induction on n. The theorem is true for $n = 2$. Assume, then, that it is true for all integers greater than 1 and less than n. If n is prime there is nothing more to prove. Therefore assume that n is composite and that n has two factorizations into prime factors, say

$$n = p_1 p_2 \cdots p_s = q_1 q_2 \cdots q_t. \tag{2}$$

We wish to show that $s = t$ and that each p equals some q. Since p_1 divides the product $q_1 q_2 \cdots q_t$, it divides at least one factor. Relabel the q's if necessary so that $p_1 | q_1$. Then $p_1 = q_1$ since both p_1 and q_1 are primes. In (2) we cancel p_1 on both sides to obtain

$$\frac{n}{p_1} = p_2 \cdots p_s = q_2 \cdots q_t.$$

Since n is composite, $1 < n/p_1 < n$; so by the induction hypothesis the two factorizations of n/p_1 are identical, apart from the order of the factors. Therefore the same is true in (2) and the proof is complete.

RATIONAL NUMBERS

Quotients of integers a/b (where $b \neq 0$) are called *rational numbers*. For example, $1/2$, $-7/5$, and 6 are rational numbers. The set of rational numbers, which we denote by $\mathbf{Q}$, contains $\mathbf{Z}$ as a subset. The reader should note that all the field axioms and the order axioms are satisfied by $\mathbf{Q}$.

IRRATIONAL NUMBERS

Real numbers that are not rational are called *irrational*. For example, the numbers $\sqrt{2}$, e, π and e^{π} are irrational.

Theorem

If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

Proof. Suppose first that n contains no square factor >1. We assume that $\sqrt{n}$ is rational and obtain a contradiction. Let $\sqrt{n} = a/b$, where a and b are integers having no factor in common. Then $nb^2 = a^2$ and, since the left side of this equation is a multiple of n, so too is a^2. However, if a^2 is a multiple of n, a itself must be a multiple of n, since n has no square factors >1. (This is easily seen by examining the factorization of a into its prime factors.) This means that $a = cn$, where c is some integer. Then the equation $nb^2 = a^2$ becomes $nb^2 = c^2n^2$, or $b^2 = nc^2$. The same argument shows that b must also be a multiple of n. Thus a and b are both multiples of n, which contradicts the fact that they have no factor in common. This completes the proof if n has no square factor >1.

Theorem

If $e^x = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots$, then the number e is irrational.

Proof. We shall prove that e^{-1} is irrational. The series for e^{-1} is an alternating series with terms which decrease steadily in absolute value. In such an alternating series the error made by stopping at the nth term has the algebraic sign of the first neglected term and is less in absolute value than the first neglected term. Hence, if $s_n = \sum_{k=0}^{n} (-1)^k/k!$, we have the inequality

$$0 < e^{-1} - s_{2k-1} < \frac{1}{(2k)!},$$

from which we obtain

$$0 < (2k-1)!\,(e^{-1} - s_{2k-1}) < \frac{1}{2k} \leq \frac{1}{2}, \tag{3}$$

for any integer $k \geq 1$. Now $(2k-1)!\,s_{2k-1}$ is always an integer. If e^{-1} were rational, then we could choose k so large that $(2k-1)!\,e^{-1}$ would also be an integer. Because of (3) the difference of these two integers would be a number between 0 and $\frac{1}{2}$, which is impossible. Thus e^{-1} cannot be rational, and hence e cannot be rational.

UPPER BOUNDS, MAXIMUM ELEMENT, LEAST UPPER BOUND (SUPREMUM)

Definition

Let S be a set of real numbers. If there is a real number b such that $x \le b$ for every x in S, then b is called an upper bound for S and we say that S is bounded above by b.

We say *an* upper bound because every number greater than b will also be an upper bound. If an upper bound b is also a member of S, then b is called the *largest member* or the *maximum element* of S. There can be at most one such b. If it exists, we write

$$b = \max S.$$

A set with no upper bound is said to be *unbounded above*.

Definitions of the terms *lower bound, bounded below, smallest member* (or *minimum element*) can be similarly formulated. If S has a minimum element we denote it by $\min S$.

Examples

1. The set $\mathbf{R}^+ = (0, +\infty)$ is unbounded above. It has no upper bounds and no maximum element. It is bounded below by 0 but has no minimum element.
2. The closed interval $S = [0, 1]$ is bounded above by 1 and is bounded below by 0. In fact, $\max S = 1$ and $\min S = 0$.
3. The half-open interval $S = [0, 1)$ is bounded above by 1 but it has no maximum element. Its minimum element is 0.

Definition

Let S be a set of real numbers bounded above. A real number b is *called a least upper bound for S if it has the following two properties:*

a) *b is an upper bound for S.*

b) *No number less than b is an upper bound for S.*

Examples. If $S = [0, 1]$ the maximum element 1 is also a least upper bound for S. If $S = [0, 1)$ the number 1 is a least upper bound for S, even though S has no maximum element.

It is an easy exercise to prove that a set cannot have two different least upper bounds. Therefore, if there is a least upper bound for S, there is *only* one and we can speak of *the* least upper bound.

It is common practice to refer to the least upper bound of a set by the more concise term *supremum*, abbreviated *sup*. We shall adopt this convention and write

$$b = \sup S$$

to indicate that b is the supremum of S. If S has a maximum element, then $\max S = \sup S$.

The *greatest lower bound*, or *infimum* of S, denoted by $\inf S$, is defined in an analogous fashion.

THE COMPLETENESS AXIOM

Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number b such that $b = \sup S$.

As a consequence of this axiom it follows that every nonempty set of real numbers which is bounded below has an infimum.

Theorem

(Approximation property). *Let S be a nonempty set of real numbers with a supremum, say* $b = \sup S$. *Then for every* $a < b$ *there is some x in S such that*

$$a < x \leq b.$$

Proof. First of all, $x \leq b$ for all x in S. If we had $x \leq a$ for every x in S, then a would be an upper bound for S smaller than the least upper bound. Therefore $x > a$ for at least one x in S.

Theorem

(Additive property). *Given nonempty subsets A and B of $\mathbf{R}$, let C denote the set*

$$C = \{x + y : x \in A, \quad y \in B\}.$$

If each of A and B has a supremum, then C has a supremum and

$$\sup C = \sup A + \sup B.$$

Proof. Let $a = \sup A$, $b = \sup B$. If $z \in C$ then $z = x + y$, where $x \in A$, $y \in B$, so $z = x + y \le a + b$. Hence $a + b$ is an upper bound for C, so C has a supremum, say $c = \sup C$, and $c \le a + b$. We show next that $a + b \le c$. Choose any $\varepsilon > 0$. By Theorem 1.14 there is an x in A and a y in B such that

$$a - \varepsilon < x \qquad \text{and} \qquad b - \varepsilon < y.$$

Adding these inequalities we find

$$a + b - 2\varepsilon < x + y \le c.$$

Thus, $a + b < c + 2\varepsilon$ for every $\varepsilon > 0$ so, by Theorem 1.1, $a + b \le c$.

Theorem

(Comparison property). *Given nonempty subsets S and T of $\mathbf{R}$ such that $s \le t$ for every s in S and t in T. If T has a supremum then S has a supremum and*

$$\sup S \le \sup T.$$

PROPERTIES OF THE INTEGERS DEDUCED FROM THE COMPLETENESS AXIOM

Theorem

The set $\mathbf{Z}^+$ of positive integers 1, 2, 3, . . . is unbounded above.

Proof. If $\mathbf{Z}^+$ were bounded above then $\mathbf{Z}^+$ would have a supremum, say $a = \sup \mathbf{Z}^+$. By Theorem 1.14 we would have $a - 1 < n$ for some n in $\mathbf{Z}^+$. Then $n + 1 > a$ for this n. Since $n + 1 \in \mathbf{Z}^+$ this contradicts the fact that $a = \sup \mathbf{Z}^+$.

Theorem

THE ARCHIMEDEAN PROPERTY OF THE REAL NUMBER SYSTEM

For every real x there is a positive integer n such that $n > x$.

Theorem

If $x > 0$ and if y is an arbitrary real number, there is a positive integer n such that $nx > y$.

RATIONAL NUMBERS WITH FINITE DECIMAL REPRESENTATION

A real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

where a_0 is a nonnegative integer and $a_1, \ldots, a_n$ are integers satisfying $0 \le a_i \le 9$, is usually written more briefly as follows:

$$r = a_0 . a_1 a_2 \cdots a_n.$$

This is said to be a *finite decimal representation* of r. For example,

$$\frac{1}{2} = \frac{5}{10} = 0.5, \qquad \frac{1}{50} = \frac{2}{10^2} = 0.02, \qquad \frac{29}{4} = 7 + \frac{2}{10} + \frac{5}{10^2} = 7.25.$$

FINITE DECIMAL APPROXIMATIONS TO REAL NUMBERS

Theorem

Assume $x \ge 0$. Then for every integer $n \ge 1$ there is a finite decimal $r_n = a_0 . a_1 a_2 \cdots a_n$ such that

$$r_n \le x < r_n + \frac{1}{10^n}.$$

Proof. Let S be the set of all nonnegative integers $\le x$. Then S is nonempty, since $0 \in S$, and S is bounded above by x. Therefore S has a supremum, say $a_0 = \sup S$. It is easily verified that $a_0 \in S$, so a_0 is a nonnegative integer. We call a_0 the *greatest integer* in x, and we write $a_0 = [x]$. Clearly, we have

$$a_0 \le x < a_0 + 1.$$

Now let $a_1 = [10x - 10a_0]$, the greatest integer in $10x - 10a_0$. Since $0 \le 10x - 10a_0 = 10(x - a_0) < 10$, we have $0 \le a_1 \le 9$ and

$$a_1 \le 10x - 10a_0 < a_1 + 1.$$

In other words, a_1 is the largest integer satisfying the inequalities

$$a_0 + \frac{a_1}{10} \le x < a_0 + \frac{a_1 + 1}{10}.$$

More generally, having chosen $a_1, \ldots, a_{n-1}$ with $0 \le a_i \le 9$, let a_n be the largest integer satisfying the inequalities

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \le x < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10^n}. \tag{4}$$

Then $0 \le a_n \le 9$ and we have

$$r_n \le x < r_n + \frac{1}{10^n},$$

where $r_n = a_0 . a_1 a_2 \cdots a_n$. This completes the proof. It is easy to verify that x is actually the supremum of the set of rational numbers $r_1, r_2, \ldots$.

ABSOLUTE VALUES AND THE TRIANGLE INEQUALITY

the absolute value of x, denoted by $|x|$, is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Theorem

For arbitrary real x and y we have

$$|x + y| \leq |x| + |y| \qquad (\textit{the triangle inequality}).$$

Proof. We have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Addition gives us $-(|x| + |y|) \leq x + y \leq |x| + |y|$, and from Theorem 1.21 we conclude that $|x + y| \leq |x| + |y|$. This proves the theorem.

The triangle inequality is often used in other forms. For example, if we take $x = a - c$ and $y = c - b$ in Theorem 1.22 we find

$$|a - b| \leq |a - c| + |c - b|.$$

Also, from Theorem 1.22 we have $|x| \geq |x + y| - |y|$. Taking $x = a + b$, $y = -b$, we obtain

$$|a + b| \geq |a| - |b|.$$

Interchanging a and b we also find $|a + b| \geq |b| - |a| = -(|a| - |b|)$, and hence

$$|a + b| \geq \big||a| - |b|\big|.$$

By induction we can also prove the generalizations

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

and

$$|x_1 + x_2 + \cdots + x_n| \geq |x_1| - |x_2| - \cdots - |x_n|.$$

THE CAUCHY-SCHWARZ INEQUALITY

Theorem

(Cauchy–Schwarz inequality). *If* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *are* *arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

Moreover, if some $a_i \neq 0$ *equality holds if and only if there is a real* x *such that* $a_k x + b_k = 0$ *for each* $k = 1, 2, \ldots, n$.

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{k=1}^{n} (a_k x + b_k)^2 \ge 0$$

for every real x, with equality if and only if each term is zero. This inequality can be written in the form

$$Ax^2 + 2Bx + C \ge 0,$$

where

$$A = \sum_{k=1}^{n} a_k^2, \qquad B = \sum_{k=1}^{n} a_k b_k, \qquad C = \sum_{k=1}^{n} b_k^2.$$

If $A > 0$, put $x = -B/A$ to obtain $B^2 - AC \le 0$, which is the desired inequality. If $A = 0$, the proof is trivial.

NOTE. In vector notation the Cauchy–Schwarz inequality takes the form

$$(\mathbf{a}\cdot\mathbf{b})^2 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2,$$

where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ are two n-dimensional vectors,

$$\mathbf{a}\cdot\mathbf{b} = \sum_{k=1}^{n} a_k b_k,$$

is their dot product, and $\|\mathbf{a}\| = (\mathbf{a}\cdot\mathbf{a})^{1/2}$ is the length of $\mathbf{a}$.

Possible Questions

PART-B (8 Mark)

1. Prove that the set Z^+ are positive integers is unbounded above.
2. Prove that every pair of integers a & b has a common divisor of the form d=ax+by where x & y are integers
3. State and prove Cauchy Schwarz inequality.
4. Prove that $\sqrt{2}$ is irrational
5. State and prove the Triangle Inequality.
6. State and prove Archimedean property.
7. Prove that every integer n > 1 can be represented as a product of prime factors in only one way apart from the order of the factors.
8. State and prove Unique Factorization Theorem
9. Prove that every integer n > 1 is either prime or product of primes.
10. Prove that if $a \geq 0$ then we have the inequality $|x| \leq a$ if and only if $-a \leq x \leq a$.

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Pollachi Main Road, Eachanari (Po),
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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Real Analysis-I **Subject Code: 15MMU501**

UNIT-I

<table>
<tr><th>Question</th><th>Option-1</th><th>Option-2</th><th>Option-3</th><th>Option-4</th><th>Answer</th></tr>
<tr><td>x(y + z) = xy + xz is --------------- law</td><td>commutative</td><td>associative</td><td>distributive</td><td>closure</td><td>distributive</td></tr>
<tr><td>If x < y, then for every z we have -----------</td><td>(x + z) < (y + z)</td><td>(x + z) > (y + z)</td><td>(x + z) = (y + z)</td><td>x + z = 0</td><td>(x + z) < (y + z)</td></tr>
<tr><td>If x > 0 and y > 0, then ----------------</td><td>xy less than equal to 0</td><td>xy > 0</td><td>xy greater than equal to 0</td><td>xy < 0</td><td>xy > 0</td></tr>
<tr><td>If x > y and y > z, then -----------</td><td>x < y</td><td>x = z</td><td>x > z</td><td>x < z</td><td>x > z</td></tr>
<tr><td>If a less than equal to b + $ for every $ > 0, then -----------</td><td>a < b</td><td>a > b</td><td>a greater than equal to b</td><td>a less than equal to b</td><td>a less than equal to b</td></tr>
<tr><td>The set of all points between a and b is called ------------</td><td>integer</td><td>interval</td><td>elements</td><td>set</td><td>interval</td></tr>
<tr><td>The set {x: a < x < b} is --------------</td><td>(a, b)</td><td>[a, b]</td><td>(a, b]</td><td>[a, b)</td><td>(a, b)</td></tr>
<tr><td>A real number is called a positive integer if it belongs to ----------</td><td>interval</td><td>open interval</td><td>closed interval</td><td>inductive set</td><td>inductive set</td></tr>
<tr><td>If d is a divisor of n, then ---------------</td><td>n = c</td><td>n < cd</td><td>n > cd</td><td>n = cd</td><td>n = cd</td></tr>
<tr><td>If a|bc and (a, b) =1, then ------------------</td><td>a|c</td><td>a|b</td><td>b|a</td><td>c|a</td><td>a|c</td></tr>
<tr><td>If a|bc and (a, b) =1, then a|c is ---------------</td><td>Unique factorisation theorem</td><td>additive property</td><td>approximation property</td><td>Euclid's lemma</td><td>Euclid's lemma</td></tr>
<tr><td>Rational numbers is of the form -----------------</td><td>pq</td><td>p + q</td><td>p/q</td><td>p - q</td><td>p/q</td></tr>
<tr><td>The constant value e is ---------------</td><td>rational</td><td>irrational</td><td>prime</td><td>composite</td><td>irrational</td></tr>
<tr><td>An integer n is called ----------- if the only possible divisors of n are 1 and n</td><td>rational</td><td>irrational</td><td>prime</td><td>composite</td><td>prime</td></tr>
<tr><td>If d|a and d|b, then d is called ----------</td><td>LCM</td><td>common divisor</td><td>prime</td><td>function</td><td>common divisor</td></tr>
<tr><td>If (a, b) = 1, then a and b are called ---------------</td><td>twin prime</td><td>common factor</td><td>LCM</td><td>relatively prime</td><td>relatively prime</td></tr>
<tr><td>If an upper bound 'b' of a set S is also a member of S then 'b' is called ---------------</td><td>rational</td><td>irrational</td><td>maximum element</td><td>minimum element</td><td>maximum element</td></tr>
<tr><td>If an lower bound 'b' of a set S is also a member of S then 'b' is called ---------------</td><td>rational</td><td>irrational</td><td>maximum element</td><td>minimum element</td><td>minimum element</td></tr>
<tr><td>A set with no upper bound is called ------------</td><td>bounded above</td><td>bounded below</td><td>prime</td><td>function</td><td>bounded above</td></tr>
<tr><td>A set with no lower bound is called ------------</td><td>bounded above</td><td>bounded below</td><td>prime</td><td>function</td><td>bounded below</td></tr>
<tr><td>The least upper bound is called -----------</td><td>bounded above</td><td>bounded below</td><td>supremum</td><td>infimum</td><td>supremum</td></tr>
<tr><td>The greatest lower bound is called -----------</td><td>bounded above</td><td>bounded below</td><td>supremum</td><td>infimum</td><td>infimum</td></tr>
<tr><td>The supremum of {3, 4} is ----------</td><td>3</td><td>4</td><td>(3, 4)</td><td>[3, 4]</td><td>4</td></tr>
<tr><td>Every finite set of numbers is ----------</td><td>bounded</td><td>unbounded</td><td>prime</td><td>bounded above</td><td>bounded</td></tr>
<tr><td>A set S of real numbers which is bounded above and bounded below is called --------</td><td>bounded set</td><td>inductive set</td><td>super set</td><td>subset</td><td>bounded set</td></tr>
<tr><td>The set N of natural numbers is ----------</td><td>bounded</td><td>not bounded</td><td>irrational</td><td>rational</td><td>not bounded</td></tr>
<tr><td>The completeness axiom is ------------</td><td>b = sup S</td><td>S = sup b</td><td>b = inf S</td><td>S = inf b</td><td>b = sup S</td></tr>
<tr><td>The infimum of {3, 4} is ------------</td><td>3</td><td>4</td><td>(3, 4)</td><td>[3, 4]</td><td>3</td></tr>
<tr><td>Sup C = Sup A + Sup B is called -------------- property</td><td>approximation</td><td>additive</td><td>archimedean</td><td>comparison</td><td>additive</td></tr>
<tr><td>For any real x, there is a positive integer n such that -----------</td><td>n > x</td><td>n < x</td><td>n = x</td><td>n = 0</td><td>n > x</td></tr>
<tr><td>If x > 0 and if y is an arbitrary real number, there is a positive number n such that nx > y is -------------- property</td><td>approximation</td><td>additive</td><td>archimedean</td><td>comparison</td><td>archimedean</td></tr>
<tr><td>The set of positive integers is -------------</td><td>bounded above</td><td>bounded below</td><td>unbounded above</td><td>unbounded below</td><td>unbounded above</td></tr>
<tr><td>The absolute value of x is denoted by ---------------</td><td>|x|</td><td>||x||</td><td>x < 0</td><td>x > 0</td><td>|x|</td></tr>
<tr><td>If x < 0 then ---------------</td><td>|x| = x</td><td>||x|| = |x|</td><td>||x|| = -x</td><td>|x| = -x</td><td>|x| = -x</td></tr>
<tr><td>If S = [0, 1) then sup S = ----------------</td><td>0</td><td>1</td><td>(0, 1)</td><td>[0,1]</td><td>1</td></tr>
<tr><td>Triangle inequality is -------------------</td><td>|a| + |b| greater equal to |a + b|</td><td>|a| > |a + b|</td><td>|b| > |a + b|</td><td>|a + b| less than equal to |a | + |b|</td><td>|a + b| less than equal to |a | + |b|</td></tr>
<tr><td>|x + y| greater than equal to -----------------</td><td>|x| + |y|</td><td>|x| |y|</td><td>|x| - |y|</td><td>| |x| - |y| |</td><td>| |x| - |y| |</td></tr>
<tr><td>Set of real numbers S is bounded above implies S has a ----------</td><td>supremum</td><td>infimum</td><td>additive property</td><td>comparison property</td><td>supremum</td></tr>
<tr><td>In { (3n +2) / (2n + 1) such that n is in N}, the greatest lower bound is --------------</td><td>5 divided by 3</td><td>8 divided by 5</td><td>11 divided by 47</td><td>3 divided by 2</td><td>3 divided by 2</td></tr>
<tr><td>In Cauchy-Schwarz inequality, the equality holds iff ----------</td><td>akx = 0</td><td>akx + bkx = 0</td><td>akx + bk = 0</td><td>bk = 0</td><td>akx + bk = 0</td></tr>
<tr><td>The linear combination of gcd (117,213) =3 can be written ----------</td><td>11*213 + (-20)*117</td><td>10*213 + (-20)*117</td><td>11*117 + (-20)*213</td><td>20*213 + (-25)*117</td><td>11*213 + (-20)*117</td></tr>
<tr><td>The smallest 3 digit prime number is ------------------</td><td>104</td><td>103</td><td>102</td><td>101</td><td>101</td></tr>
<tr><td>The total number of primes less than 50 is ________________</td><td>14</td><td>15</td><td>16</td><td>18</td><td>15</td></tr>
<tr><td>Given integers a and b, there is a common divisor d then d = ______________</td><td>ax+by</td><td>ax*by</td><td>ax/by</td><td>by/ax</td><td>ax+by</td></tr>
<tr><td>If d is gcd of a and b then d is ____________________</td><td>0</td><td>>0</td><td><0</td><td>less than aor equal to 0</td><td>>0</td></tr>
<tr><td>Given integers a and b, number of gcd is ________________</td><td>4</td><td>3</td><td>2</td><td>1</td><td>1</td></tr>
<tr><td>a and b are said to be realtively prime if gcd is ____________</td><td>4</td><td>3</td><td>2</td><td>1</td><td>1</td></tr>
<tr><td>Suppose A is a finite subset of natural numbers and let B ={(a,b): a,b in A}. Then B contains__________________</td><td>only negative pair</td><td>some negative pair</td><td>0 only</td><td>only positive pair</td><td>only positive integers</td></tr>
<tr><td>a and b are said to be not realtively prime if gcd is __________</td><td>not equal to 4</td><td>not equal to 3</td><td>not equal to 2</td><td>not equal to 1</td><td>not equal to 1</td></tr>
</table>

if d is gcd of 2 and 5th Fermat number then d is ________________	1	2	3	4	1
If a|bc and (a,b)=1 then ________________	a|b	a|c	b|a	b|c	a|c
the number ___________ is a neither prime nor composite.	1	2	3	4	1
Let A be the set of even prime numbers. Then number of elements in A is ________________	1	2	3	4	1
The gcd of 8 and 5 is________________	1	2	3	4	1
The total number of primes upto 100 is ___________	21	22	23	25	25
The consecutive integers will always be _______________	even numbers	odd numbers	relatively prime numbers	not relatively prime numbers	relatively prime numbers
The product of two prime numbers will always be __________________	even number	odd number	neither prime nor composite	composite	composite
Let A be the set of all prime numbers. Then number of elements in A is ________________	countable	uncountable	finite	empty	countable
If A is the set of even prime numbers and B is the set of odd prime numbers. Then ________________	A is a subset of B	B is a subset of A	A and B are disjoint	A and B are not disjoint	A and B are disjoint
If a prime p does not divide a then (a,p) is ________________	1	2	3	4	1
If a prime p divides abcdef then p divides __________________	a or b or ….or f	a and b and … and f	a only	b only	a or b or ….or f
The prime power factorization of 7007 is ______________	(7^3)*11*3	(7^2)*11*3	7*11*3	7*(11^2)*3	(7^2)*11*3
Every integer n>1 can be expressed as a product of prime power factors in ____________	only one way	in two different ways	in three different ways	in more than three different ways	only one way
If n = P1* P2*.......*Ps = Q1*Q2*.....*Qt then _______________	s<t	t<s	t=s	t+s=0	t=s

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Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: Real Analysis-I	**Semester: V**	**L T P C**
Subject Code: 15MMU501	**Class: III-B.Sc Mathematics-A**	**5 0 0 5**

UNIT II

Basic notions of a set theory: Notations –ordered pairs –Cartesian product of two sets –Relations and functions – further terminology concerning functions –one –one functions and inverse –composite functions –sequences –similar sets-finite and infinite sets –countable and uncountable sets –uncountability of the real number system.

Reference Book:

R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

SOME BASIC NOTIONS OF SET THEORY

NOTATIONS

Sets will usually be denoted by capital letters:

$$A, B, C, \ldots, X, Y, Z,$$

and *elements* by lower-case letters: $a, b, c, \ldots, x, y, z$. We write $x \in S$ to mean "x is an element of S," or "x belongs to S." If x does not belong to S, we write $x \notin S$. We sometimes designate sets by displaying the elements in braces; for example, the set of positive even integers less than 10 is denoted by $\{2, 4, 6, 8\}$. If S is the collection of all x which satisfy a property P, we indicate this briefly by writing $S = \{x : x \text{ satisfies } P\}$.

From a given set we can form new sets, called *subsets* of the given set. For example, the set consisting of all positive integers less than 10 which are divisible by 4, namely, $\{4, 8\}$, is a subset of the set of even integers less than 10. In general, we say that a set A is a subset of B, and we write $A \subseteq B$ whenever every element of A also belongs to B. The statement $A \subseteq B$ does not rule out the possibility that $B \subseteq A$. In fact, we have both $A \subseteq B$ and $B \subseteq A$ if, and only if, A and B have the same elements. In this case we shall call the sets A and B equal and we write $A = B$. If A and B are not equal, we write $A \neq B$. If $A \subseteq B$ but $A \neq B$, then we say that A is a *proper subset* of B.

It is convenient to consider the possibility of a set which contains no elements whatever; this set is called the *empty set* and we agree to call it a subset of every set. The reader may find it helpful to picture a set as a box containing certain objects, its elements. The empty set is then an empty box. We denote the empty set by the symbol $\emptyset$.

ORDERED PAIRS

Suppose we have a set consisting of two elements a and b; that is, the set $\{a, b\}$. By our definition of equality this set is the same as the set $\{b, a\}$, since no question of order is involved. However, it is also necessary to consider sets of two elements in which order *is* important. For example, in analytic geometry of the plane, the coordinates (x, y) of a point represent an *ordered pair* of numbers. The *point* $(3, 4)$ is different from the point $(4, 3)$, whereas the *set* $\{3, 4\}$ is the same as the set $\{4, 3\}$. When we wish to consider a set of two elements a and b as being *ordered*, we shall enclose the elements in parentheses: (a, b). Then a is called the first element and b the second. It is possible to give a purely set-theoretic definition of the concept of an ordered pair of objects (a, b). One such definition is the following:

Definition

$(a, b) = \{\{a\}, \{a, b\}\}$.

This definition states that (a, b) is a set containing two elements, $\{a\}$ and $\{a, b\}$. Using this definition, we can prove the following theorem:

Theorem

$(a, b) = (c, d)$ *if, and only if,* $a = c$ *and* $b = d$.

CARTESIAN PRODUCT OF TWO SETS

Definition

Given two sets A and B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of A and B, and is denoted by $A \times B$.

Example. If **R** denotes the set of all real numbers, then $\mathbf{R} \times \mathbf{R}$ is the set of all complex numbers.

RELATIONS AND FUNCTIONS

Let x and y denote real numbers, so that the ordered pair (x, y) can be thought of as representing the rectangular coordinates of a point in the xy-plane (or a complex number). We frequently encounter such expressions as

$$xy = 1, \qquad x^2 + y^2 = 1, \qquad x^2 + y^2 \le 1, \qquad x < y. \tag{a}$$

Definition

Any set of ordered pairs is called a relation.

If S is a relation, the set of all elements x that occur as first members of pairs (x, y) in S is called the *domain* of S, denoted by $\mathscr{D}(S)$. The set of second members y is called the *range* of S, denoted by $\mathscr{R}(S)$.

Definition

A function F is a set of ordered pairs (x, y), no two of which have the same first member. That is, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

The definition of function requires that for every x in the domain of F there is exactly one y such that $(x, y) \in F$. It is customary to call y the *value of F at x* and to write

$$y = F(x)$$

instead of $(x, y) \in F$ to indicate that the pair (x, y) is in the set F.

FURTHER TERMINOLOGY CONCERNING FUNCTIONS

When the domain $\mathscr{D}(F)$ is a subset of **R**, then F is called a *function of one real variable*. If $\mathscr{D}(F)$ is a subset of **C**, the complex number system, then F is called a *function of a complex variable*.

If $\mathscr{D}(F)$ is a subset of a cartesian product $A \times B$, then F is called a *function of two variables*. In this case we denote the function values by $F(a, b)$ instead of $F((a, b))$. A function of two real variables is one whose domain is a subset of $\mathbf{R} \times \mathbf{R}$.

If S is a subset of $\mathscr{D}(F)$, we say that F is *defined on S*. In this case, the set of $F(x)$ such that $x \in S$ is called the *image of S under F* and is denoted by $F(S)$. If T is any set which contains $F(S)$, then F is also called a *mapping from S to T*. This is often denoted by writing

$$F : S \to T.$$

If $F(S) = T$, the mapping is said to be *onto T*. A mapping of S into itself is sometimes called a *transformation*.

ONE-TO-ONE FUNCTIONS AND INVERSES

Definition

Let F be a function defined on S. We say F is one-to-one on S if, and only if, for every x and y in S,

$$F(x) = F(y) \quad \textit{implies} \quad x = y.$$

Definition

Given a relation S, the new relation $\check{S}$ defined by

$$\check{S} = \{(a, b) : (b, a) \in S\}$$

is called the converse of S.

Thus an ordered pair (a, b) belongs to $\check{S}$ if, and only if, the pair (b, a), with elements interchanged, belongs to S. When S is a *plane relation*, this simply means that the graph of $\check{S}$ is the reflection of the graph of S with respect to the line $y = x$. In the relation defined by $x < y$, the converse relation is defined by $y < x$.

Theorem

If the function F is one-to-one on its domain, then $\check{F}$ is also a function.

Proof. To show that $\check{F}$ is a function, we must show that if $(x, y) \in \check{F}$ and $(x, z) \in \check{F}$, then $y = z$. But $(x, y) \in \check{F}$ means that $(y, x) \in F$; that is, $x = F(y)$. Similarly, $(x, z) \in \check{F}$ means that $x = F(z)$. Thus $F(y) = F(z)$ and, since we are assuming that F is one-to-one, this implies $y = z$. Hence, $\check{F}$ is a function.

NOTE. The same argument shows that if F is one-to-one on a subset S of $\mathscr{D}(F)$, then the restriction of F to S has an inverse.

COMPOSITE FUNCTIONS

Definition

Given two functions F and G such that $\mathscr{R}(F) \subseteq \mathscr{D}(G)$, we can form a new function, the composite $G \circ F$ of G and F, defined as follows: for every x in the domain of F, $(G \circ F)(x) = G[F(x)]$.

Since $\mathscr{R}(F) \subseteq \mathscr{D}(G)$, the element $F(x)$ is in the domain of G, and therefore it makes sense to consider $G[F(x)]$. In general, it is not true that $G \circ F = F \circ G$. In fact, $F \circ G$ may be meaningless unless the range of G is contained in the domain of F. However, the associative law,

$$H \circ (G \circ F) = (H \circ G) \circ F,$$

SEQUENCES

Definition

By a finite sequence of n terms we shall understand a function F whose domain is the set of numbers $\{1, 2, \ldots, n\}$.

The range of F is the set $\{F(1), F(2), F(3), \ldots, F(n)\}$, customarily written $\{F_1, F_2, F_3, \ldots, F_n\}$. The elements of the range are called *terms* of the sequence and, of course, they may be arbitrary objects of any kind.

Let $s = \{s_n\}$ be an infinite sequence, and let k be a function whose domain is the set of positive integers and whose range is a subset of the positive integers. Assume that k is "order-preserving," that is, assume that

$$k(m) < k(n), \qquad \text{if } m < n.$$

Then the composite function $s \circ k$ is defined for all integers $n \geq 1$, and for every such n we have

$$(s \circ k)(n) = s_{k(n)}.$$

Such a composite function is said to be a *subsequence* of s. Again, for brevity, we often use the notation $\{s_{k(n)}\}$ or $\{s_{k_n}\}$ to denote the subsequence of $\{s_n\}$ whose nth term is $s_{k(n)}$.

Example. Let $s = \{1/n\}$ and let k be defined by $k(n) = 2^n$. Then $s \circ k = \{1/2^n\}$.

SIMILAR (EQUINUMEROUS) SETS

Definition

Two sets A and B are called similar, or equinumerous, and we write $A \sim B$, if and only if there exists a one-to-one function F whose domain is the set A and whose range is the set B.

FINITE AND INFINITE SETS

A set S is called *finite* and is said to contain n elements if

$$S \sim \{1, 2, \ldots, n\}.$$

The integer n is called the *cardinal number* of S. It is an easy exercise to prove that if $\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}$ then $m = n$. Therefore, the cardinal number of a finite set is well defined. The empty set is also considered finite. Its cardinal number is defined to be 0.

Sets which are not finite are called *infinite sets*. The chief difference between the two is that an infinite set must be similar to some proper subset of itself, whereas a finite set cannot be similar to any proper subset of itself. (See Exercise 2.13.) For example, the set $\mathbf{Z}^+$ of all positive integers is similar to the proper subset $\{2, 4, 8, 16, \ldots\}$ consisting of powers of 2. The one-to-one function F which makes them similar is defined by $F(x) = 2^x$ for each x in $\mathbf{Z}^+$.

COUNTABLE AND UNCOUNTABLE SETS

A set S is said to be *countably infinite* if it is equinumerous with the set of all positive integers; that is, if

$$S \sim \{1, 2, 3, \ldots\}.$$

In this case there is a function f which establishes a one-to-one correspondence between the positive integers and the elements of S; hence the set S can be displayed as follows:

$$S = \{f(1), f(2), f(3), \ldots\}.$$

Definition

A set S is called countable if it is either finite or countably infinite.
A set which is not countable is called uncountable.

The words *denumerable* and *nondenumerable* are sometimes used in place of *countable* and *uncountable*.

Theorem

Every subset of a countable set is countable.

Proof. Let S be the given countable set and assume $A \subseteq S$. If A is finite, there is nothing to prove, so we can assume that A is infinite (which means S is also infinite). Let $s = \{s_n\}$ be an infinite sequence of distinct terms such that

$$S = \{s_1, s_2, \ldots\}.$$

Define a function on the positive integers as follows:

Let $k(1)$ be the smallest positive integer m such that $s_m \in A$. Assuming that $k(1), k(2), \ldots, k(n-1)$ have been defined, let $k(n)$ be the smallest positive integer $m > k(n-1)$ such that $s_m \in A$. Then k is order-preserving: $m > n$ implies $k(m) > k(n)$. Form the composite function $s \circ k$. The domain of $s \circ k$ is the set of positive integers and the range of $s \circ k$ is A. Furthermore, $s \circ k$ is one-to-one, since

$$s[k(n)] = s[k(m)],$$

implies

$$s_{k(n)} = s_{k(m)},$$

which implies $k(n) = k(m)$, and this implies $n = m$. This proves the theorem.

UNCOUNTABILITY OF THE REAL NUMBER SYSTEM

Theorem

The set of all real numbers is uncountable.

Proof. It suffices to show that the set of x satisfying $0 < x < 1$ is uncountable. If the real numbers in this interval were countable, there would be a sequence $s = \{s_n\}$ whose terms would constitute the whole interval. We shall show that this is impossible by constructing, in the interval, a real number which is not a term of this sequence. Write each s_n as an infinite decimal:

$$s_n = 0.u_{n,1}u_{n,2}u_{n,3}\cdots,$$

where each $u_{n,i}$ is $0, 1, \ldots,$ or 9. Consider the real number y which has the decimal expansion

$$y = 0.v_1v_2v_3\ldots,$$

where

$$v_n = \begin{cases} 1, & \text{if } u_{n,n} \neq 1, \\ 2, & \text{if } u_{n,n} = 1. \end{cases}$$

Then no term of the sequence $\{s_n\}$ can be equal to y, since y differs from s_1 in the first decimal place, differs from s_2 in the second decimal place, ..., from s_n in the nth decimal place. (A situation like $s_n = 0.1999\ldots$ and $y = 0.2000\ldots$ cannot occur here because of the way the v_n are chosen.) Since $0 < y < 1$, the theorem is proved.

Theorem

Let $\mathbf{Z}^+$ denote the set of all positive integers. Then the cartesian product $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

Proof. Define a function f on $\mathbf{Z}^+ \times \mathbf{Z}^+$ as follows:

$$f(m, n) = 2^m 3^n, \qquad \text{if } (m, n) \in \mathbf{Z}^+ \times \mathbf{Z}^+.$$

Then f is one-to-one on $\mathbf{Z}^+ \times \mathbf{Z}^+$ and the range of f is a subset of $\mathbf{Z}^+$.

COUNTABLE COLLECTIONS OF COUNTABLE SETS

Definition

If F is a collection of sets such that every two distinct sets in F are disjoint, then F is said to be a collection of disjoint sets.

Theorem

If F is a countable collection of disjoint sets, say $F = \{A_1, A_2, \ldots\}$, such that each set A_n is countable, then the union $\bigcup_{k=1}^{\infty} A_k$ is also countable.

Proof. Let $A_n = \{a_{1,n}, a_{2,n}, a_{3,n} \ldots\}$, $n = 1, 2, \ldots$, and let $S = \bigcup_{k=1}^{\infty} A_k$. Then every element x of S is in at least one of the sets in F and hence $x = a_{m,n}$ for some pair of integers (m, n). The pair (m, n) is uniquely determined by x, since F is a collection of disjoint sets. Hence the function f defined by $f(x) = (m, n)$ if $x = a_{m,n}$, $x \in S$, has domain S. The range $f(S)$ is a subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$ (where $\mathbf{Z}^+$ is the set of positive integers) and hence is countable. But f is one-to-one and therefore $S \sim f(S)$, which means that S is also countable.

Theorem

If $F = \{A_1, A_2, \ldots\}$ is a countable collection of sets, let $G = \{B_1, B_2, \ldots\}$, where $B_1 = A_1$ and, for $n > 1$,

$$B_n = A_n - \bigcup_{k=1}^{n-1} A_k.$$

Then G is a collection of disjoint sets, and we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k.$$

Proof. Each set B_n is constructed so that it has no elements in common with the earlier sets $B_1, B_2, \ldots, B_{n-1}$. Hence G is a collection of disjoint sets. Let $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. We shall show that $A = B$. First of all, if $x \in A$, then $x \in A_k$ for some k. If n is the smallest such k, then $x \in A_n$ but $x \notin \bigcup_{k=1}^{n-1} A_k$, which means that $x \in B_n$, and therefore $x \in B$. Hence $A \subseteq B$. Conversely, if $x \in B$, then $x \in B_n$ for some n, and therefore $x \in A_n$ for this same n. Thus $x \in A$ and this proves that $B \subseteq A$.

Theorem

If F is a countable collection of countable sets, then the union of all sets in F is also a countable set.

Example 1. The set $\mathbf{Q}$ of all rational numbers is a countable set.

Proof. Let A_n denote the set of all positive rational numbers having denominator n. The set of all positive rational numbers is equal to $\bigcup_{k=1}^{\infty} A_k$. From this it follows that $\mathbf{Q}$ is countable, since each A_n is countable.

Example 2. The set S of intervals with rational endpoints is a countable set.

Proof. Let $\{x_1, x_2, \ldots\}$ denote the set of rational numbers and let A_n be the set of all intervals whose left endpoint is x_n and whose right endpoint is rational. Then A_n is countable and $S = \bigcup_{k=1}^{\infty} A_k$.

Possible Questions

PART-B (8 Mark)

1. Show that the Cartesian product Z^+ x Z^+ where Z^+ is a set of positive integers is a countable set.
2. If F = {A_1, A_2, … } is a countable collection of sets and G = {B_1, B_2, … } when $B_1 = A_1$

   for n > 1, $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$, then prove that G is a collection of disjoint sets, and we

   have $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$.
3. Let F be the collection of sets, then for any set B, prove that

   i) $B - \bigcup_{A\epsilon F} A = \bigcap_{A\epsilon F} (B - A)$

   ii) $B - \bigcap_{A\epsilon F} A = \bigcup_{A\epsilon F} (B - A)$
4. Prove that the set of all real number is countable.
5. Prove that (a,b) = (c,d) iff a=c & b=d.
6. Two functions F and G are equal iff

   i) D(F) = D(G) [F,G have the same domain.]

   ii) F(x) = G(x) $\forall$ x in D(F).
7. Prove that the set of rational is countable.
8. Prove that every subset of countable set is countable.
9. Prove that the set of real numbers is uncountable.
10. Show that the Cartesian product Z^+ x Z^+ where Z^+ is a set of positive integers is a countable set.

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Real Analysis-I **Subject Code: 15MMU501**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
Let S be the set of of all 26 letters in the alphabet and let A be the set of letters in the word "trivial". Then the number of elements in S-A is ___________ is	19	20	21	22	20
Let A={1,2}. Then A X A = _________________	{(1,1),(2,2)}	{(1,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}	{(1,1),(2,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}
Let A={1,2} and B={a,b,c}. Then number of elements in A X B =	2	3	2*2*2	2*3	2*3
Suppose n(A)=a and n(B)=b. Then number of elements in A X B is	a	b	ab	a+b	ab
Let A={1,2} and B={a,b,c}. Then which of the following element does not belongs to A X B =	(1,a)	(3,c)	(c,2)	(1,c)	(c,2)
Let F be a function and (x,y) in F and (x,z) in F. Then we must have	x=y	y=z	z=x	x=x	y=z
Let f:A→B then which of the followin is always true?	range of f is not equal to B	range of f is a subset of B	range of f is containing B	range of f is proper subset of B	range of f is a subset of B
If the number of elements in a set S is 5. Then the number of elements of the power set P(S) is ____________	5	6	16	32	32
If range of f is equal to codomain set, then f is _______________	into	onto	one-one	many to one	onto
The inverse relationof f is a function only if f is __________________	into	onto	one-one	bijection	bijection
The Inverse function is always _____________	into	onto	one-one	bijection	bijection
If A and B contains n elements then number bijection between A and B is _____________	n!	n	n+1	n-1	n!
Let f be a function from A to B. Then we call f as a sequence only if A is a __________________	set of positive integers	set of all real numbers	set of all rationals	set of irrationals	set of positive integers
Two sets A and B are said to be similar iff there is a function f exists such that f is __________________	into	one-one	onto	bijection	bijection
If two sets A={1,2,…,m} and B={1,2,..,n} are smilar then ________________	m<n	n<m	n=m	n>0	n=m
Which of the following is an example for countable?	set of real numbers	set of all irrationals	set of all rationals	(0,1)	set of all rationals
Number of elements in the set of all real numbers is __________________	finite	countably infinite	10000000000	uncountable	uncountable
The union of elements A and B is the set of elements belongs to _____________	either A or B	neither A not B	both A and B	A and not in B	either A or B
The set of elements belongs A and not in B is ________________	B	A	B-A	A-B	A-B
The set of elements belongs B and not in A is_____________	B	A	B-A	A-B	B-A
Countable union of countable set is_______________	uncountable	countable	finite	countably infinite	countable
The set of elements in N X N is _______________	uncountable	countable	finite	countably infinite	countable
The set of elements in Z X R is_________________	uncountable	countable	finite	countably infinite	uncountable
The set of elements in R x R is ________________	uncountable	countable	finite	countably infinite	uncountable
The set of sequences consists of only 1 and 0 is ______________	uncountable	countbale	finite	countably infinite	uncountable
Every subset of a countable set is ______________	uncountable	countable	finite	countably infinite	countable
Every subset of a finite set is _______________	uncountable	countable	finite	countably infinite	finite
Fibonnaci numbers is an example for ________________	uncountable set	countable set	finite set	infinte set	countable
Suppose A and B is countable then A X B is __________________	uncountable	countable	finite	infinite	countable
A X B is similar to ____________	A	B	A XA	A X B	A X B
The set of all even integers is _______________	uncountable	countable	finite	infinite	countable
The set A = {x/x in (0,1]} is ______________	uncountable	countable	finite	countably infinite	uncountable
The set K = {1,2,…..,100000} is _______________	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y implies __________________	f(x) is not equal to f(y)	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)
Suppose f is a one to one function. Then f(x)=f(y) implies ______________	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatble then B is ________________	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x)=x. Then f is _______________	onto	one to one	bijection	neither one to one nor onto	bijection
Constant function is an example for ____________	onto	one to one	many to one	bijection	many to one
Stricly increasing function is _______________	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is _____________	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, then *g* (2) _____________	8	9	13	17	13
A = {x/ x ≠ x }represents ____________	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A is _____________	n!	2n	2^n	n	2^n

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DEPARTMENT OF MATHEMATICS

Subject: Real Analysis-I	**Semester: V**	**L**	**T**	**P**	**C**
Subject Code: 15MMU501	**Class: III-B.Sc Mathematics-A**	**5**	**0**	**0**	**5**

UNIT III

Elements of point set topology: Euclidean space R^n –open balls and open sets in R^n. The structure of open Sets in R^n –closed sets and adherent points –The Bolzano –Weierstrass theorem–the Cantor intersection Theorem.

Reference Book:

R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

ELEMENTS OF POINT SET TOPOLOGY

EUCLIDEAN SPACE $\mathbf{R}^n$

A point in two-dimensional space is an ordered pair of real numbers (x_1, x_2). Similarly, a point in three-dimensional space is an ordered triple of real numbers (x_1, x_2, x_3). It is just as easy to consider an ordered n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ and to refer to this as a point in n-dimensional space.

Definition

Let $n > 0$ be an integer. An ordered set of n real numbers $(x_1, x_2, \ldots, x_n)$ is called an n-dimensional point or a vector with n components. Points or vectors will usually be denoted by single bold-face letters; for example,

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \quad \text{or} \quad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

The number x_k is called the kth *coordinate of the point* $\mathbf{x}$ *or the* kth *component of the vector* $\mathbf{x}$. *The set of all n-dimensional points is called n-dimensional Euclidean space or simply n-space, and is denoted by* $\mathbf{R}^n$.

Definition

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be in $\mathbf{R}^n$. We define:

a) *Equality:*

$$\mathbf{x} = \mathbf{y} \text{ if, and only if, } x_1 = y_1, \ldots, x_n = y_n.$$

b) *Sum:*

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n).$$

c) *Multiplication by real numbers (scalars):*

$$a\mathbf{x} = (ax_1, \ldots, ax_n) \qquad (a \text{ real}).$$

d) *Difference:*

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y}.$$

e) *Zero vector or origin:*

$$\mathbf{0} = (0, \ldots, 0).$$

f) *Inner product or dot product:*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k.$$

g) *Norm or length:*

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}.$$

The norm $\|\mathbf{x} - \mathbf{y}\|$ is called the *distance* between $\mathbf{x}$ and $\mathbf{y}$.

NOTE. In the terminology of linear algebra, $\mathbf{R}^n$ is an example of a *linear space*.

Theorem

Let $\mathbf{x}$ and $\mathbf{y}$ denote points in $\mathbf{R}^n$. Then we have:

a) $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0$ *if, and only if,* $\mathbf{x} = \mathbf{0}$.

b) $\|a\mathbf{x}\| = |a|\,\|\mathbf{x}\|$ *for every real* a.

c) $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$.

d) $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$ (*Cauchy–Schwarz inequality*).

e) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (*triangle inequality*).

Proof. Statements (a), (b) and (c) are immediate from the definition, and the Cauchy–Schwarz inequality was proved in Theorem 1.23. Statement (e) follows from (d) because

$$\|\mathbf{x} + \mathbf{y}\|^2 = \sum_{k=1}^{n} (x_k + y_k)^2 = \sum_{k=1}^{n} (x_k^2 + 2x_k y_k + y_k^2)$$
$$= \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

NOTE. Sometimes the triangle inequality is written in the form

$$\|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|.$$

This follows from (e) by replacing $\mathbf{x}$ by $\mathbf{x} - \mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y} - \mathbf{z}$. We also have

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|.$$

Definition

The unit coordinate vector $\mathbf{u}_k$ *in* $\mathbf{R}^n$ *is the vector whose* kth *component is* 1 *and whose remaining components are zero. Thus,*

$$\mathbf{u}_1 = (1, 0, \ldots, 0), \qquad \mathbf{u}_2 = (0, 1, 0, \ldots, 0), \; \ldots, \; \mathbf{u}_n = (0, 0, \ldots, 0, 1).$$

If $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x} = x_1\mathbf{u}_1 + \cdots + x_n\mathbf{u}_n$ and $x_1 = \mathbf{x}\cdot\mathbf{u}_1$, $x_2 = \mathbf{x}\cdot\mathbf{u}_2, \ldots, x_n = \mathbf{x}\cdot\mathbf{u}_n$. The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are also called *basis vectors*.

OPEN BALLS AND OPEN SETS IN $\mathbf{R}^n$

Let $\mathbf{a}$ be a given point in $\mathbf{R}^n$ and let r be a given positive number. The set of all points $\mathbf{x}$ in $\mathbf{R}^n$ such that

$$\|\mathbf{x} - \mathbf{a}\| < r,$$

is called an open *n-ball* of radius r and center $\mathbf{a}$. We denote this set by $B(\mathbf{a})$ or by $B(\mathbf{a}; r)$.

The ball $B(\mathbf{a}; r)$ consists of all points whose distance from $\mathbf{a}$ is less than r. In $\mathbf{R}^1$ this is simply an open interval with center at $\mathbf{a}$. In $\mathbf{R}^2$ it is a circular disk, and in $\mathbf{R}^3$ it is a spherical solid with center at $\mathbf{a}$ and radius r.

Definition of an interior point. *Let S be a subset of $\mathbf{R}^n$, and assume that $\mathbf{a} \in S$. Then $\mathbf{a}$ is called an interior point of S if there is an open n-ball with center at $\mathbf{a}$, all of whose points belong to S.*

In other words, every interior point $\mathbf{a}$ of S can be surrounded by an n-ball $B(\mathbf{a}) \subseteq S$. The set of all interior points of S is called the *interior* of S and is denoted by int S. Any set containing a ball with center $\mathbf{a}$ is sometimes called a *neighborhood* of $\mathbf{a}$.

Definition of an open set. *A set S in $\mathbf{R}^n$ is called open if all its points are interior points.*

NOTE. A set S is open if and only if $S = \text{int } S$.

Theorem

The union of any collection of open sets is an open set.

Proof. Let F be a collection of open sets and let S denote their union, $S = \bigcup_{A \in F} A$. Assume $\mathbf{x} \in S$. Then $\mathbf{x}$ must belong to at least one of the sets in F, say $\mathbf{x} \in A$. Since A is open, there exists an open n-ball $B(\mathbf{x}) \subseteq A$. But $A \subseteq S$, so $B(\mathbf{x}) \subseteq S$ and hence $\mathbf{x}$ is an interior point of S. Since every point of S is an interior point, S is open.

Theorem

The intersection of a finite collection of open sets is open.

Proof. Let $S = \bigcap_{k=1}^{m} A_k$ where each A_k is open. Assume $\mathbf{x} \in S$. (If S is empty, there is nothing to prove.) Then $\mathbf{x} \in A_k$ for every $k = 1, 2, \ldots, m$, and hence there is an open n-ball $B(\mathbf{x}; r_k) \subseteq A_k$. Let r be the smallest of the positive numbers $r_1, r_2, \ldots, r_m$. Then $\mathbf{x} \in B(\mathbf{x}; r) \subseteq S$. That is, $\mathbf{x}$ is an interior point, so S is open.

Definition of component interval. *Let S be an open subset of $\mathbf{R}^1$. An open interval I (which may be finite or infinite) is called a component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$.*

In other words, a component interval of S is not a proper subset of any other open interval contained in S.

Theorem

Every point of a nonempty open set S belongs to one and only one component interval of S.

Proof. Assume $x \in S$. Then x is contained in some open interval I with $I \subseteq S$. There are many such intervals but the "largest" of these will be the desired component interval. We leave it to the reader to verify that this largest interval is $I_x = (a(x), b(x))$, where

$$a(x) = \inf \{a : (a, x) \subseteq S\}, \qquad b(x) = \sup \{b : (x, b) \subseteq S\}.$$

Here $a(x)$ might be $-\infty$ and $b(x)$ might be $+\infty$. Clearly, there is no open interval J such that $I_x \subseteq J \subseteq S$, so I_x is a component interval of S containing x. If J_x is another component interval of S containing x, then the union $I_x \cup J_x$ is an open interval contained in S and containing both I_x and J_x. Hence, by the definition of component interval, it follows that $I_x \cup J_x = I_x$ and $I_x \cup J_x = J_x$, so $I_x = J_x$.

Theorem

(Representation theorem for open sets on the real line). *Every nonempty open set S in $\mathbf{R}^1$ is the union of a countable collection of disjoint component intervals of S.*

Proof. If $x \in S$, let I_x denote the component interval of S containing x. The union of all such intervals I_x is clearly S. If two of them, I_x and I_y, have a point in common, then their union $I_x \cup I_y$ is an open interval contained in S and containing both I_x and I_y. Hence $I_x \cup I_y = I_x$ and $I_x \cup I_y = I_y$ so $I_x = I_y$. Therefore the intervals I_x form a disjoint collection.

It remains to show that they form a countable collection. For this purpose, let $\{x_1, x_2, x_3, \dots\}$ denote the countable set of rational numbers. In each component interval I_x there will be infinitely many x_n, but among these there will be exactly one with *smallest index n*. We then define a function F by means of the equation $F(I_x) = n$, if x_n is the rational number in I_x with smallest index n. This function F is one-to-one since $F(I_x) = F(I_y) = n$ implies that I_x and I_y have x_n in common and this implies $I_x = I_y$. Therefore F establishes a one-to-one correspondence between the intervals I_x and a subset of the positive integers. This completes the proof.

CLOSED SETS

Definition of a closed set. *A set S in* $\mathbf{R}^n$ *is called closed if its complement* $\mathbf{R}^n - S$ *is open.*

Examples. A closed interval $[a, b]$ in $\mathbf{R}^1$ is a closed set. The cartesian product

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

of n one-dimensional closed intervals is a closed set in $\mathbf{R}^n$ called an *n-dimensional closed interval* $[\mathbf{a}, \mathbf{b}]$.

Theorem

The union of a finite collection of closed sets is closed, and the intersection of an arbitrary collection of closed sets is closed.

Theorem

If A is open and B is closed, then $A - B$ *is open and* $B - A$ *is closed.*

Proof. We simply note that $A - B = A \cap (\mathbf{R}^n - B)$, the intersection of two open sets, and that $B - A = B \cap (\mathbf{R}^n - A)$, the intersection of two closed sets.

ADHERENT POINTS. ACCUMULATION POINTS

Closed sets can also be described in terms of adherent points and accumulation points.

Definition of an adherent point. *Let S be a subset of* $\mathbf{R}^n$, *and* $\mathbf{x}$ *a point in* $\mathbf{R}^n$, $\mathbf{x}$ *not necessarily in S. Then* $\mathbf{x}$ *is said to be adherent to S if every n-ball* $B(\mathbf{x})$ *contains at least one point of S.*

Examples

1. If $\mathbf{x} \in S$, then $\mathbf{x}$ adheres to S for the trivial reason that every n-ball $B(\mathbf{x})$ contains $\mathbf{x}$.
2. If S is a subset of $\mathbf{R}$ which is bounded above, then sup S is adherent to S.

Some points adhere to S because every ball $B(\mathbf{x})$ contains points of S *distinct* from $\mathbf{x}$. These are called accumulation points.

Definition of an accumulation point. *If* $S \subseteq \mathbf{R}^n$ *and* $\mathbf{x} \in \mathbf{R}^n$, *then* $\mathbf{x}$ *is called an accumulation point of S if every n-ball* $B(\mathbf{x})$ *contains at least one point of S distinct from* $\mathbf{x}$.

In other words, $\mathbf{x}$ is an accumulation point of S if, and only if, $\mathbf{x}$ adheres to $S - \{\mathbf{x}\}$. If $\mathbf{x} \in S$ but $\mathbf{x}$ is not an accumulation point of S, then $\mathbf{x}$ is called an *isolated point* of S.

Examples

1. The set of numbers of the form $1/n$, $n = 1, 2, 3, \ldots$, has 0 as an accumulation point.
2. The set of rational numbers has every real number as an accumulation point.
3. Every point of the closed interval $[a, b]$ is an accumulation point of the set of numbers in the open interval (a, b).

Theorem

If $\mathbf{x}$ is an accumulation point of S, then every n-ball $B(\mathbf{x})$ contains infinitely many points of S.

Proof. Assume the contrary; that is, suppose an n-ball $B(\mathbf{x})$ exists which contains only a finite number of points of S distinct from $\mathbf{x}$, say $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. If r denotes the smallest of the positive numbers

$$\|\mathbf{x} - \mathbf{a}_1\|, \quad \|\mathbf{x} - \mathbf{a}_2\|, \quad \ldots, \quad \|\mathbf{x} - \mathbf{a}_m\|,$$

then $B(\mathbf{x}; r/2)$ will be an n-ball about $\mathbf{x}$ which contains no points of S distinct from $\mathbf{x}$. This is a contradiction.

CLOSED SETS AND ADHERENT POINTS

A closed set was defined to be the complement of an open set. The next theorem describes closed sets in another way.

Theorem

A set S in $\mathbf{R}^n$ is closed if, and only if, it contains all its adherent points.

Proof. Assume S is closed and let $\mathbf{x}$ be adherent to S. We wish to prove that $\mathbf{x} \in S$. We assume $\mathbf{x} \notin S$ and obtain a contradiction. If $\mathbf{x} \notin S$ then $\mathbf{x} \in \mathbf{R}^n - S$ and, since $\mathbf{R}^n - S$ is open, some n-ball $B(\mathbf{x})$ lies in $\mathbf{R}^n - S$. Thus $B(\mathbf{x})$ contains no points of S, contradicting the fact that $\mathbf{x}$ adheres to S.

To prove the converse, we assume S contains all its adherent points and show that S is closed. Assume $\mathbf{x} \in \mathbf{R}^n - S$. Then $\mathbf{x} \notin S$, so $\mathbf{x}$ does not adhere to S. Hence some ball $B(\mathbf{x})$ does not intersect S, so $B(\mathbf{x}) \subseteq \mathbf{R}^n - S$. Therefore $\mathbf{R}^n - S$ is open, and hence S is closed.

Definition of closure. *The set of all adherent points of a set S is called the closure of S and is denoted by $\overline{S}$.*

Theorem

A set S in $\mathbf{R}^n$ is closed if, and only if, it contains all its accumulation points.

THE BOLZANO–WEIERSTRASS THEOREM

Definition of a bounded set. *A set S in $\mathbf{R}^n$ is said to be bounded if it lies entirely within an n-ball $B(\mathbf{a}; r)$ for some $r > 0$ and some $\mathbf{a}$ in $\mathbf{R}^n$.*

Theorem

(Bolzano–Weierstrass). *If a bounded set S in $\mathbf{R}^n$ contains infinitely many points, then there is at least one point in $\mathbf{R}^n$ which is an accumulation point of S.*

Proof. To help fix the ideas we give the proof first for $\mathbf{R}^1$. Since S is bounded, it lies in some interval $[-a, a]$. At least one of the subintervals $[-a, 0]$ or $[0, a]$ contains an infinite subset of S. Call one such subinterval $[a_1, b_1]$. Bisect $[a_1, b_1]$ and obtain a subinterval $[a_2, b_2]$ containing an infinite subset of S, and continue this process. In this way a countable collection of intervals is obtained, the nth interval $[a_n, b_n]$ being of length $b_n - a_n = a/2^{n-1}$. Clearly, the sup of the left endpoints a_n and the inf of the right endpoints b_n must be equal, say to x. [Why are they equal?] The point x will be an accumulation point of S because, if r is any positive number, the interval $[a_n, b_n]$ will be contained in $B(x; r)$ as soon as n is large enough so that $b_n - a_n < r/2$. The interval $B(x; r)$ contains a point of S distinct from x and hence x is an accumulation point of S. This proves the theorem for $\mathbf{R}^1$. (Observe that the accumulation point x may or may not belong to S.)

Next we give a proof for $\mathbf{R}^n$, $n > 1$, by an extension of the ideas used in treating $\mathbf{R}^1$. (The reader may find it helpful to visualize the proof in $\mathbf{R}^2$ by referring to Fig. 3.1.)

Since S is bounded, S lies in some n-ball $B(\mathbf{0}; a)$, $a > 0$, and therefore within the n-dimensional interval J_1 defined by the inequalities

$$-a \le x_k \le a \qquad (k = 1, 2, \ldots, n).$$

Here J_1 denotes the cartesian product

$$J_1 = I_1^{(1)} \times I_2^{(1)} \times \cdots \times I_n^{(1)};$$

that is, the set of points $(x_1, \ldots, x_n)$, where $x_k \in I_k^{(1)}$ and where each $I_k^{(1)}$ is a one-dimensional interval $-a \le x_k \le a$. Each interval $I_k^{(1)}$ can be bisected to

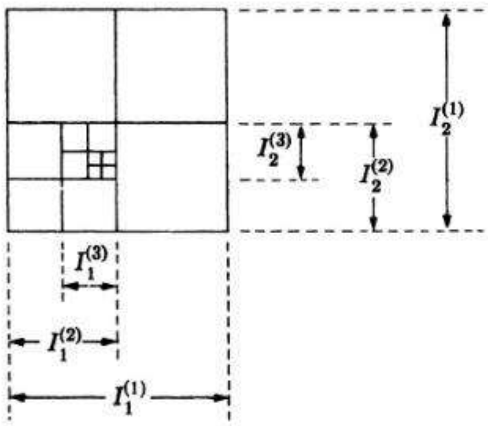

Figure 3.1

form two subintervals $I_{k,1}^{(1)}$ and $I_{k,2}^{(1)}$, defined by the inequalities

$$I_{k,1}^{(1)}: -a \le x_k \le 0; \qquad I_{k,2}^{(1)}: 0 \le x_k \le a.$$

Next, we consider all possible cartesian products of the form

$$I_{1,k_1}^{(1)} \times I_{2,k_2}^{(1)} \times \cdots \times I_{n,k_n}^{(1)}, \tag{a}$$

where each $k_i = 1$ or 2. There are exactly 2^n such products and, of course, each such product is an n-dimensional interval. The union of these 2^n intervals is the original interval J_1, which contains S; and hence at least one of the 2^n intervals in (a) must contain infinitely many points of S. One of these we denote by J_2, which can then be expressed as

$$J_2 = I_1^{(2)} \times I_2^{(2)} \times \cdots \times I_n^{(2)},$$

where each $I_k^{(2)}$ is one of the subintervals of $I_k^{(1)}$ of length a. We now proceed with J_2 as we did with J_1, bisecting each interval $I_k^{(2)}$ and arriving at an n-dimensional interval J_3 containing an infinite subset of S. If we continue the process, we obtain a countable collection of n-dimensional intervals $J_1, J_2, J_3, \ldots$, where the mth interval J_m has the property that it contains an infinite subset of S and can be expressed in the form

$$J_m = I_1^{(m)} \times I_2^{(m)} \times \cdots \times I_n^{(m)}, \qquad \text{where } I_k^{(m)} \subseteq I_k^{(1)}.$$

Writing

$$I_k^{(m)} = [a_k^{(m)}, b_k^{(m)}],$$

we have

$$b_k^{(m)} - a_k^{(m)} = \frac{a}{2^{m-2}} \qquad (k = 1, 2, \cdots, n).$$

For each fixed k, the sup of all left endpoints $a_k^{(m)}$, $(m = 1, 2, \ldots)$, must therefore be equal to the inf of all right endpoints $b_k^{(m)}$, $(m = 1, 2, \ldots)$, and their common value we denote by t_k. We now assert that the point $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ is an accumulation point of S. To see this, take any n-ball $B(\mathbf{t}; r)$. The point $\mathbf{t}$, of course, belongs to each of the intervals $J_1, J_2, \ldots$ constructed above, and when m is such that $a/2^{m-2} < r/2$, this neighborhood will include J_m. But since J_m contains infinitely many points of S, so will $B(\mathbf{t}; r)$, which proves that $\mathbf{t}$ is indeed an accumulation point of S.

THE CANTOR INTERSECTION THEOREM

Theorem

Let $\{Q_1, Q_2, \ldots\}$ be a countable collection of nonempty sets in $\mathbf{R}^n$ such that:

i) $Q_{k+1} \subseteq Q_k \qquad (k = 1, 2, 3, \ldots)$.

ii) *Each set Q_k is closed and Q_1 is bounded.*

Then the intersection $\bigcap_{k=1}^{\infty} Q_k$ is closed and nonempty.

Proof. Let $S = \bigcap_{k=1}^{\infty} Q_k$. Then S is closed because of Theorem 3.13. To show that S is nonempty, we exhibit a point $\mathbf{x}$ in S. We can assume that each Q_k contains infinitely many points; otherwise the proof is trivial. Now form a collection of distinct points $A = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$, where $\mathbf{x}_k \in Q_k$. Since A is an infinite set contained in the bounded set Q_1, it has an accumulation point, say $\mathbf{x}$. We shall show that $\mathbf{x} \in S$ by verifying that $\mathbf{x} \in Q_k$ for each k. It will suffice to show that $\mathbf{x}$ is an accumulation point of each Q_k, since they are all closed sets. But every neighborhood of $\mathbf{x}$ contains infinitely many points of A, and since all except (possibly) a finite number of the points of A belong to Q_k, this neighborhood also contains infinitely many points of Q_k. Therefore $\mathbf{x}$ is an accumulation point of Q_k and the theorem is proved.

Possible Questions

PART-B (8 Mark)

1. Prove that intersection of a finite collection of open set is open.
2. If A is open and B is closed prove that A – B is open and B – A is closed.
3. Prove that union of any collection of open sets is an open set.
4. Prove that a set S is closed if and only if S = $\bar{S}$.
5. Let (X, d) be a metric space, x $\in X, and\ \delta > 0$. Prove that then B(x, δ) is open and C(x, δ) is closed.
6. If A is open and B is closed prove that A – B is open and B – A is closed.
7. Prove that if (S, d) be a metric space subspace of (M, d) and let X be a subset of S. Then X is open in S iff X=A∩S for some set a which is open in M.
8. Prove that a set S in R^n is closed if and only if it contains all its adherent points.
9. Prove that every point of a non empty open set s belongs to one and only one component interval of S.

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(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Real Analysis-I **Subject Code: 15MMU501**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
Let X in n dimensional space. Then we must have __________	norm of x is 0	norm of x is 1	norm of x is greater than 0	norm of x is greater than or equal to 0	norm of x is greater than or equal to 0
A set S is closed if ______________	complement of S is open	complement of S is closed	complement of S is nether open nor closed	complement of S is both open and closed	complement of S is open
A set S is open if ________________	complement of S is closed	complement of S is closed	complement of S is nether open nor closed	complement of S is both open and closed	complement of S is closed
Let a is said to be adherent point of S if every every n ball contains ________________	center a only	atleast center a	atleast one point otherthan center a	atmost one point otherthan a	atleast center a
Suppose X is an accumulation point then S is ________________	finite set	empty set	infinite	countable set	infinte set
The set of all accumulation points of (0,1] is ______________	[0,1]	(0,1)	(0,1]	[0,1]	[0,1]
The set of limit points of {1,2,…,10} is _______________	empty set	{0,1,2,…,10}	[0,1]	[0,1]	empty set
Number of limit points of a finite is ______________	0	1	2	3	0
Number of limit points of set of all integers is ________________	0	1	2	3	0
Number of limit points of set of all rationals is ______________	0	1	2	2	0
The number of limit points of a set {1/n: where n=1, 2, 3, ... } is ______________	1	2	3	4	1
The limit point of the set {1/n: where n=1, 2, 3, ... } is ____________	0	1	2	3	0
Every finite set is _____________	an open set	a closed set	neither open nor closed	both open and closed set	a closed set
Number of isloated points of [a,b] where a and b in R is ___________	1	2	3	0	0
The collection of all intervals of the form 1/n<x<2/n, n=1,2,… is an open covering of _____________	(1,2)	(0,1)	R	Z	(1,2)
The set of all interior points of A=[a,b] is _______________	(a,b)	(a,b]	[a,b)	[a,b]	[a,b]
Which of the following is not a limit point of [0,1) ?	1	2	0	1/2	2
If A = closure of A, then A is ____________	open	closed	both open and closed	neither open nor closed	closed
The set of all interior of a finite set Ais ______________	empty set	A	R	Z	empty set
The set of all interior points of set of all rational is ________________	empty set	A	R	Z	empty set
Let A = {1, 1/2, 1/3, ...} then the set of all interior points of A is ________________	empty set	A	R	Z	empty set
In a discrete metric space, for any set A , the set of interior poits of A=________________	empty set	A	R	Z	A
With usual metric in R, let A=[0.1]. The set of all interior points of A is ________________	empty set	A	R	Z	A
If A is open then Interior of A= ________________	empty set	A	R	Z	A
If A = interiot of A,then A is ____________	closed	open	both open and closed	neither open nor closed	open
For any closed set the closure of A is ________________	empty set	A	R	Z	A
In R with usual metric, (0,1] is ________________	closed	open	both open and closed	neither open nor closed	neither open nor closed
In R with usual metric, (0,1) is ________________	closed	open	both open and closed	neither open nor closed	open
In R with usual metric, [0,1] is ______________	closed	open	both open and closed	neither open nor closed	closed
In R with usual metric, [a,b) is ________________	closed	open	both open and closed	neither open nor closed	neither open nor closed
The set of all integers is _______________	closed	open	both open and closed	neither open nor closed	closed
In R with usual metric every singleton set is ________________	not closed	not open	both open and closed	neither open nor closed	closed
Every subset of a discrete metric space is ________________	not closed	not open	both open and closed	neither open nor closed	both open and closed
In any metric (M,d) , the empry set is __________	not closed	not open	both open and closed	neither open nor closed	both open and closed
In any metric space (M,d), M is ________________	not closed	not open	both open and closed	neither open nor closed	both open and closed
In any metric space the union of a finite number of closed sets is______________	closed	open	both open and closed	neither open nor closed	closed
In any metric space the arbitrary intersection of closed sets is ___________	closed	open	both open and closed	neither open nor closed	closed
In any metric space the arbitrary union of open sets is ________________	closed	open	both open and closed	neither open nor closed	open
In any metric, the intersection of finite number of open sets is ____________	closed	open	both open and closed	neither open nor closed	open
In a discrete metric space the union of arbitrary number of closed sets is ________________	not closed	not open	both open and closed	neither open nor closed	both open and closed
In a discrete metric space the interection of arbitrary open sets is ________________	not closed	not open	both open and closed	neither open nor closed	both open and closed
Number of limit points of a subset of discrete metric space ____________	0	1	2	3	0
Closure of R is _______________	R	Z	Q	N	R
A is closed iff A contains _______________	all adherent points of A	some of adherent points of A	no adherent points of A	only finite number of adhrent points of A	all adhrent points of A
A is closed iff A contains _______________	all accumulation points of A	some of accumulation points of A	no accumulation points of A	only finite number of accumulation points of A	all accumulation points of A
x is an accumulation point of A then every n ball of x contains ________________	atleast one point other than x	atmost one point other than x	exactly one point other than x	no point other than x	atleast one point other than x
Every n ball of x contains more than one point other than x is called ________________	adherent point	accumulation point	isolated point	not an adherent point	accumulation point

If every point of A is an interior point then A is called ____________	closed	open	both open and closed	neither open nor closed	open
Let (M,d) be a discrete metric space with 5 elements. Then the number of open subsets of M is ____________	32	5	10	5!	32
Let(M,d) be a discrete metric space with 5 elements. Then the number of closed sets is ____________	32	5	10	120	32
Which of the following is not a bounded set?	R	[a,b]	[a,b)	(a,b]	R
The set of all even prime numbers is ________________	unbounded	bounded	infinite	uncountable	bounded
The set of all even prime numbers is __________________	closed	open	both open and closed	neither open nor closed	closed
The set of all odd prime numbers is __________________	bounded	unbounded	open	finite	unbounded
The set of all prime numbers is ______________	bounded	unbounded	open	finite	unbounded
The set of all prime numbers is __________	uncountable	countable	finite	bounded	countable
If A is open and B is closed then B-A ______________	open	closed	both open and closed	neither open nor closed	closed
If A is open and B is closed then A-B is ______________	open	closed	both open and closed	neither open nor closed	open

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Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: Real Analysis-I	**Semester: V**	**L**	**T**	**P**	**C**
Subject Code: 15MMU501	**Class: III-B.Sc Mathematics-A**	**5**	**0**	**0**	**5**

UNIT IV

Covering –Lindelof covering theorem –the Heine Borel covering theorem –Compactness in R^n –Metric Spaces –point set topology in metric spaces –compact subsets of a metric space –Boundary of a set.

Reference Book:

R2: Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4: Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

THE LINDELÖF COVERING THEOREM

Theorem

Definition of a covering. *A collection F of sets is said to be a covering of a given set S if $S \subseteq \bigcup_{A \in F} A$. The collection F is also said to cover S. If F is a collection of open sets, then F is called an open covering of S.*

Examples

1. The collection of all intervals of the form $1/n < x < 2/n$, $(n = 2, 3, 4, \ldots)$, is an open covering of the interval $0 < x < 1$. This is an example of a countable covering.
2. The real line $\mathbf{R}^1$ is covered by the collection of all open intervals (a, b). This covering is not countable. However, it contains a countable covering of $\mathbf{R}^1$, namely, all intervals of the form $(n, n + 2)$, where n runs through the integers.

Let $G = \{A_1, A_2, \ldots\}$ denote the countable collection of all n-balls having rational radii and centers at points with rational coordinates. Assume $\mathbf{x} \in \mathbf{R}^n$ and let S be an open set in $\mathbf{R}^n$ which contains $\mathbf{x}$. Then at least one of the n-balls in G contains $\mathbf{x}$ and is contained in S. That is, we have

$$\mathbf{x} \in A_k \subseteq S \qquad \text{for some } A_k \text{ in } G.$$

Proof. The collection G is countable because of Theorem 2.27. If $\mathbf{x} \in \mathbf{R}^n$ and if S is an open set containing $\mathbf{x}$, then there is an n-ball $B(\mathbf{x}; r) \subseteq S$. We shall find a point $\mathbf{y}$ in S with rational coordinates that is "near" $\mathbf{x}$ and, using this point as center, will then find a neighborhood in G which lies within $B(\mathbf{x}; r)$ and which contains $\mathbf{x}$. Write

$$\mathbf{x} = (x_1, x_2, \ldots, x_n),$$

and let y_k be a rational number such that $|y_k - x_k| < r/(4n)$ for each $k = 1, 2, \ldots, n$. Then

$$\|\mathbf{y} - \mathbf{x}\| \le |y_1 - x_1| + \cdots + |y_n - x_n| < \frac{r}{4}.$$

Next, let q be a rational number such that $r/4 < q < r/2$. Then $\mathbf{x} \in B(\mathbf{y}; q)$ and $B(\mathbf{y}; q) \subseteq B(\mathbf{x}; r) \subseteq S$. But $B(\mathbf{y}; q) \in G$ and hence the theorem is proved.

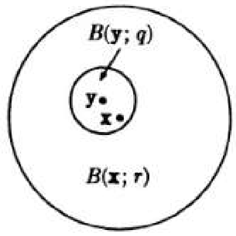

Figure 3.2

Theorem

(Lindelöf covering theorem). *Assume $A \subseteq \mathbf{R}^n$ and let F be an open covering of A. Then there is a countable subcollection of F which also covers A.*

Proof. Let $G = \{A_1, A_2, \ldots\}$ denote the countable collection of all n-balls having rational centers and rational radii. This set G will be used to help us extract a countable subcollection of F which covers A.

Assume $\mathbf{x} \in A$. Then there is an open set S in F such that $\mathbf{x} \in S$. By Theorem 3.27 there is an n-ball A_k in G such that $\mathbf{x} \in A_k \subseteq S$. There are, of course, infinitely many such A_k corresponding to each S, but we choose only one of these, for example, the one of smallest index, say $m = m(\mathbf{x})$. Then we have $\mathbf{x} \in A_{m(\mathbf{x})} \subseteq S$. The set of all n-balls $A_{m(\mathbf{x})}$ obtained as $\mathbf{x}$ varies over all elements of A is a countable collection of open sets which covers A. To get a countable subcollection of F which covers A, we simply correlate to each set $A_{k(\mathbf{x})}$ one of the sets S of F which contained $A_{k(\mathbf{x})}$. This completes the proof.

THE HEINE–BOREL COVERING THEOREM

Theorem

(Heine–Borel). *Let F be an open covering of a closed and bounded set A in* $\mathbf{R}^n$. *Then a finite subcollection of F also covers A.*

Proof. A countable subcollection of F, say $\{I_1, I_2, \dots\}$, covers A, by Theorem 3.28. Consider, for $m \geq 1$, the finite union

$$S_m = \bigcup_{k=1}^{m} I_k.$$

This is open, since it is the union of open sets. We shall show that for some value of m the union S_m covers A.

For this purpose we consider the complement $\mathbf{R}^n - S_m$, which is closed. Define a countable collection of sets $\{Q_1, Q_2, \dots\}$ as follows: $Q_1 = A$, and for $m > 1$,

$$Q_m = A \cap (\mathbf{R}^n - S_m).$$

That is, Q_m consists of those points of A which lie outside of S_m. If we can show that for some value of m the set Q_m is empty, then we will have shown that for this m no point of A lies outside S_m; in other words, we will have shown that some S_m covers A.

Observe the following properties of the sets Q_m: Each set Q_m is closed, since it is the intersection of the closed set A and the closed set $\mathbf{R}^n - S_m$. The sets Q_m are decreasing, since the S_m are increasing; that is, $Q_{m+1} \subseteq Q_m$. The sets Q_m, being subsets of A, are all bounded. Therefore, if no set Q_m is empty, we can apply the Cantor intersection theorem to conclude that the intersection $\bigcap_{k=1}^{\infty} Q_k$ is also not empty. This means that there is some point in A which is in all the sets Q_m, or, what is the same thing, outside all the sets S_m. But this is impossible, since $A \subseteq \bigcup_{k=1}^{\infty} S_k$. Therefore some Q_m must be empty, and this completes the proof.

COMPACTNESS IN $\mathbf{R}^n$

Definition of a compact set. *A set S in* $\mathbf{R}^n$ *is said to be compact if, and only if, every open covering of S contains a finite subcover, that is, a finite subcollection which also covers S.*

The Heine–Borel theorem states that every closed and bounded set in $\mathbf{R}^n$ is compact. Now we prove the converse result.

Theorem

Let S be a subset of $\mathbf{R}^n$. *Then the following three statements are* *equivalent:*

a) *S is compact.*

b) *S is closed and bounded.*

c) *Every infinite subset of S has an accumulation point in S.*

Proof. As noted above, (b) implies (a). If we prove that (a) implies (b), that (b) implies (c) and that (c) implies (b), this will establish the equivalence of all three statements.

Assume (a) holds. We shall prove first that S is bounded. Choose a point $\mathbf{p}$ in S. The collection of n-balls $B(\mathbf{p}; k)$, $k = 1, 2, \ldots$, is an open covering of S. By compactness a finite subcollection also covers S and hence S is bounded.

Next we prove that S is closed. Suppose S is not closed. Then there is an accumulation point $\mathbf{y}$ of S such that $\mathbf{y} \notin S$. If $\mathbf{x} \in S$, let $r_{\mathbf{x}} = \|\mathbf{x} - \mathbf{y}\|/2$. Each $r_{\mathbf{x}}$ is positive since $\mathbf{y} \notin S$ and the collection $\{B(\mathbf{x}; r_{\mathbf{x}}) : \mathbf{x} \in S\}$ is an open covering of S. By compactness, a finite number of these neighborhoods cover S, say

$$S \subseteq \bigcup_{k=1}^{p} B(\mathbf{x}_k; r_k).$$

Let r denote the smallest of the radii $r_1, r_2, \ldots, r_p$. Then it is easy to prove that the ball $B(\mathbf{y}; r)$ has no points in common with any of the balls $B(\mathbf{x}_k; r_k)$. In fact, if $\mathbf{x} \in B(\mathbf{y}; r)$, then $\|\mathbf{x} - \mathbf{y}\| < r \le r_k$, and by the triangle inequality we have $\|\mathbf{y} - \mathbf{x}_k\| \le \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_k\|$, so

$$\|\mathbf{x} - \mathbf{x}_k\| \ge \|\mathbf{y} - \mathbf{x}_k\| - \|\mathbf{x} - \mathbf{y}\| = 2r_k - \|\mathbf{x} - \mathbf{y}\| > r_k.$$

Hence $\mathbf{x} \notin B(\mathbf{x}_k; r_k)$. Therefore $B(\mathbf{y}; r) \cap S$ is empty, contradicting the fact that $\mathbf{y}$ is an accumulation point of S. This contradiction shows that S is closed and hence (a) implies (b).

Assume (b) holds. In this case the proof of (c) is immediate, because if T is an infinite subset of S then T is bounded (since S is bounded), and hence by the Bolzano–Weierstrass theorem T has an accumulation point $\mathbf{x}$, say. Now $\mathbf{x}$ is also an accumulation point of S and hence $\mathbf{x} \in S$, since S is closed. Therefore (b) implies (c).

Assume (c) holds. We shall prove (b). If S is unbounded, then for every $m > 0$ there exists a point $\mathbf{x}_m$ in S with $\|\mathbf{x}_m\| > m$. The collection $T = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is an infinite subset of S and hence, by (c), T has an accumulation point $\mathbf{y}$ in S. But for $m > 1 + \|\mathbf{y}\|$ we have

$$\|\mathbf{x}_m - \mathbf{y}\| \ge \|\mathbf{x}_m\| - \|\mathbf{y}\| > m - \|\mathbf{y}\| > 1,$$

contradicting the fact that $\mathbf{y}$ is an accumulation point of T. This proves that S is bounded.

To complete the proof we must show that S is closed. Let $\mathbf{x}$ be an accumulation point of S. Since every neighborhood of $\mathbf{x}$ contains infinitely many points of S, we can consider the neighborhoods $B(\mathbf{x}; 1/k)$, where $k = 1, 2, \ldots$, and obtain a countable set of distinct points, say $T = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$, contained in S, such that $\mathbf{x}_k \in B(\mathbf{x}; 1/k)$. The point $\mathbf{x}$ is also an accumulation point of T. Since T is an infinite subset of S, part (c) of the theorem tells us that T must have an accumulation point in S. The theorem will then be proved if we show that $\mathbf{x}$ is the only accumulation point of T.

To do this, suppose that $\mathbf{y} \neq \mathbf{x}$. Then by the triangle inequality we have

$$\|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}_k\| + \|\mathbf{x}_k - \mathbf{x}\| < \|\mathbf{y} - \mathbf{x}_k\| + 1/k, \qquad \text{if } \mathbf{x}_k \in T.$$

If k_0 is taken so large that $1/k < \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|$ whenever $k \geq k_0$, the last inequality leads to $\frac{1}{2}\|\mathbf{y} - \mathbf{x}\| < \|\mathbf{y} - \mathbf{x}_k\|$. This shows that $\mathbf{x}_k \notin B(\mathbf{y}; r)$ when $k \geq k_0$, if $r = \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|$. Hence $\mathbf{y}$ cannot be an accumulation point of T. This completes the proof that (c) implies (b).

METRIC SPACES

Definition of a metric space. *A metric space is a nonempty set M of objects (called points) together with a function d from $M \times M$ to $\mathbf{R}$ (called the metric of the space) satisfying the following four properties for all points x, y, z in M:*

1. $d(x, x) = 0$.
2. $d(x, y) > 0$ *if* $x \neq y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, y) \leq d(x, z) + d(z, y)$.

The nonnegative number $d(x, y)$ is to be thought of as the distance from x to y. In these terms the intuitive meaning of properties 1 through 4 is clear. Property 4 is called the *triangle inequality*.

M and the metric d play a role in the definition of a metric space.

Examples

1. $M = \mathbf{R}^n$; $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. This is called the *Euclidean metric*. Whenever we refer to Euclidean space $\mathbf{R}^n$, it will be understood that the metric is the Euclidean metric unless another metric is specifically mentioned.
2. $M = \mathbf{C}$, the complex plane; $d(z_1, z_2) = |z_1 - z_2|$. As a metric space, $\mathbf{C}$ is indistinguishable from Euclidean space $\mathbf{R}^2$ because it has the same points and the same metric.
3. M any nonempty set; $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$. This is called the *discrete metric*, and (M, d) is called a *discrete metric space*.
4. If (M, d) is a metric space and if S is any nonempty subset of M, then (S, d) is also a metric space with the same metric or, more precisely, with the restriction of d to $S \times S$ as metric. This is sometimes called the *relative metric* induced by d on S, and S is called a *metric subspace* of M. For example, the rational numbers $\mathbf{Q}$ with the metric $d(x, y) = |x - y|$ form a metric subspace of $\mathbf{R}$.
5. $M = \mathbf{R}^2$; $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + 4(x_2 - y_2)^2}$, where $\mathbf{x} = (x_1, x_2)$ and $y = (y_1, y_2)$. The metric space (M, d) is not a metric subspace of Euclidean space $\mathbf{R}^2$ because the metric is different.
6. $M = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$, the unit circle in $\mathbf{R}^2$; $d(\mathbf{x}, \mathbf{y}) =$ the length of the smaller arc joining the two points $\mathbf{x}$ and $\mathbf{y}$ on the unit circle.
7. $M = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$, the unit sphere in $\mathbf{R}^3$; $d(\mathbf{x}, \mathbf{y}) =$ the length of the smaller arc along the great circle joining the two points $\mathbf{x}$ and $\mathbf{y}$.
8. $M = \mathbf{R}^n$; $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$.
9. $M = \mathbf{R}^n$; $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

POINT SET TOPOLOGY IN METRIC SPACES

The basic notions of point set topology can be extended to an arbitrary metric space (M, d).

If $a \in M$, the *ball* $B(a; r)$ with center a and radius $r > 0$ is defined to be the set of all x in M such that

$$d(x, a) < r.$$

Sometimes we denote this ball by $B_M(a; r)$ to emphasize the fact that its points come from M. If S is a metric subspace of M, the ball $B_S(a; r)$ is the intersection of S with the ball $B_M(a; r)$.

Examples. In Euclidean space $\mathbf{R}^1$ the ball $B(0; 1)$ is the open interval $(-1, 1)$. In the metric subspace $S = [0, 1]$ the ball $B_S(0; 1)$ is the half-open interval $[0, 1)$.

NOTE. The geometric appearance of a ball in $\mathbf{R}^n$ need not be "spherical" if the metric is not the Euclidean metric. (See Exercise 3.27.)

If $S \subseteq M$, a point a in S is called an *interior point* of S if some ball $B_M(a; r)$ lies entirely in S. The *interior*, int S, is the set of interior points of S. A set S is called *open* in M if all its points are interior points; it is called *closed* in M if $M - S$ is open in M.

Examples.

1. Every ball $B_M(a; r)$ in a metric space M is open in M.
2. In a discrete metric space M *every* subset S is open. In fact, if $x \in S$, the ball $B(x; \frac{1}{2})$ consists entirely of points of S (since it contains only x), so S is open. Therefore every subset of M is also closed!
3. In the metric subspace $S = [0, 1]$ of Euclidean space $\mathbf{R}^1$, every interval of the form $[0, x)$ or $(x, 1]$, where $0 < x < 1$, is an open set in S. These sets are not open in $\mathbf{R}^1$.

Theorem

Let (S, d) be a metric subspace of (M, d), and let X be a subset of S. Then X is open in S if, and only if,

$$X = A \cap S$$

for some set A which is open in M.

Proof. Assume A is open in M and let $X = A \cap S$. If $x \in X$, then $x \in A$ so $B_M(x; r) \subseteq A$ for some $r > 0$. Hence $B_S(x; r) = B_M(x; r) \cap S \subseteq A \cap S = X$ so X is open in S.

Conversely, assume X is open in S. We will show that $X = A \cap S$ for some open set A in M. For every x in X there is a ball $B_S(x; r_x)$ contained in X. Now $B_S(x; r_x) = B_M(x; r_x) \cap S$, so if we let

$$A = \bigcup_{x \in X} B_M(x; r_x),$$

then A is open in M and it is easy to verify that $A \cap S = X$.

Theorem

Let (S, d) be a metric subspace of (M, d) and let Y be a subset of S. Then Y is closed in S if, and only if, $Y = B \cap S$ for some set B which is closed in M.

Proof. If $Y = B \cap S$, where B is closed in M, then $B = M - A$ where A is open in M so $Y = S \cap B = S \cap (M - A) = S - A$; hence Y is closed in S.

Conversely, if Y is closed in S, let $X = S - Y$. Then X is open in S so $X = A \cap S$, where A is open in M and

$$Y = S - X = S - (A \cap S) = S - A = S \cap (M - A) = S \cap B,$$

where $B = M - A$ is closed in M. This completes the proof.

If $S \subseteq M$, a point x in M is called an *adherent point* of S if every ball $B_M(x; r)$ contains at least one point of S. If x adheres to $S - \{x\}$ then x is called an *accumulation point* of S. The *closure* $\bar{S}$ of S is the set of all adherent points of S, and the *derived set* S' is the set of all accumulation points of S. Thus, $\bar{S} = S \cup S'$.

The following theorems are valid in every metric space (M, d) and are proved exactly as they were for Euclidean space $\mathbf{R}^n$. In the proofs, the Euclidean distance $\|\mathbf{x} - \mathbf{y}\|$ need only be replaced by the metric $d(x, y)$.

Theorem

a) *The union of any collection of open sets is open, and the intersection of a finite collection of open sets is open.*

b) *The union of a finite collection of closed sets is closed, and the intersection of any collection of closed sets is closed.*

Theorem

If A is open and B is closed, then $A - B$ is open and $B - A$ is closed.

Theorem

For any subset S of M the following statements are equivalent:

a) *S is closed in M.*

b) *S contains all its adherent points.*

c) *S contains all its accumulation points.*

d) *$S = \bar{S}$.*

Example. Let $M = \mathbf{Q}$, the set of rational numbers, with the Euclidean metric of $\mathbf{R}^1$. Let S consist of all rational numbers in the open interval (a, b), where both a and b are irrational. Then S is a closed subset of $\mathbf{Q}$.

COMPACT SUBSETS OF A METRIC SPACE

Let (M, d) be a metric space and let S be a subset of M. A collection F of open subsets of M is said to be an *open covering* of S if $S \subseteq \bigcup_{A \in F} A$.

A subset S of M is called *compact* if every open covering of S contains a finite subcover. S is called *bounded* if $S \subseteq B(a; r)$ for some $r > 0$ and some a in M.

Theorem

Let S be a compact subset of a metric space M. Then:

i) *S is closed and bounded.*

ii) *Every infinite subset of S has an accumulation point in S.*

Proof. To prove (i) we refer to the proof of Theorem 3.31 and use that part of the argument which showed that (a) implies (b). The only change is that the Euclidean distance $\|\mathbf{x} - \mathbf{y}\|$ is to be replaced throughout by the metric $d(x, y)$.

To prove (ii) we argue by contradiction. Let T be an infinite subset of S and assume that no point of S is an accumulation point of T. Then for each point x in S there is a ball $B(x)$ which contains no point of T (if $x \notin T$) or exactly one point of T (x itself, if $x \in T$). As x runs through S, the union of these balls $B(x)$ is an open covering of S. Since S is compact, a finite subcollection covers S and hence also covers T. But this is a contradiction because T is an infinite set and each ball contains at most one point of T.

Theorem

Let X be a closed subset of a compact metric space M. Then X is compact.

Proof. Let F be an open covering of X, say $X \subseteq \bigcup_{A \in F} A$. We will show that a finite number of the sets A cover X. Since X is closed its complement $M - X$ is open, so $F \cup \{(M - X)\}$ is an open covering of M. But M is compact, so this covering contains a finite subcover which we can assume includes $M - X$. Therefore

$$M \subseteq A_1 \cup \cdots \cup A_p \cup (M - X).$$

This subcover also covers X and, since $M - X$ contains no points of X, we can delete the set $M - X$ from the subcover and still cover X. Thus $X \subseteq A_1 \cup \cdots \cup A_p$ so X is compact.

BOUNDARY OF A SET

Definition

Let S be a subset of a metric space M. A point x in M is called a boundary point of S if every ball $B_M(x; r)$ contains at least one point of S and at least one point of $M - S$. The set of all boundary points of S is called the boundary of S and is denoted by ∂S.

The reader can easily verify that

$$\partial S = \bar{S} \cap \overline{M - S}.$$

This formula shows that ∂S is closed in M.

Example In $\mathbf{R}^n$, the boundary of a ball $B(\mathbf{a}; r)$ is the set of points $\mathbf{x}$ such that $\|\mathbf{x} - \mathbf{a}\| = r$. In $\mathbf{R}^1$, the boundary of the set of rational numbers is all of $\mathbf{R}^1$.

Possible Questions

PART-B (8 Mark)

1. Let S be a compact subset of a metric space M. then prove that
   i) S is closed and bounded
   ii) Every infinite subset of S has an accumulation point of S.
2. Prove that every convergence sequence is a Cauchy sequence.
3. Prove that in a metric space (S,d) a sequence $\{x_n\}$converges to p if and only if every subsequence $\{x_{k(n)}\}$ converges to p.
4. Let X be a closed subset of a compact metric space M. Then prove that X is compact.
5. Prove that a sequence $\{x_n\}$ in the metric space (S, d) can converge to atmost one point in S..
6. Let f and g be complex valued functions defined on a subset A of a metric space (S, d). Let p be an accumulation point of A, and assume that lim f(x) = a and lim g(x) = b. Then prove that lim f(x) g(x) = ab
7. In a metric space (S, d) assume that $x_n \to p$ and let T = $\{x_1, x_2, \ldots\}$ (ie) the range of the sequence $\{x_n\}$ then prove that i) T is bounded and ii) p is an adherent point of T.
8. State and prove the Heine – Borel theorem.
9. Prove that closed intervals in $\mathbb{R}$ are compact.
10. Assume p is an accumulation point of A and assume b$\in$T then prove that $\lim_{x\to p} f(x)$ = p if and only if $\lim_{n\to\infty} f(xn)$ = b for every sequence $\{x_n\}$ of points in which converges to p.

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Real Analysis-I **Subject Code: 15MMU501**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
With usual metric, R is ______________	closed	compact	not compact	bounded	not compact
If every open covering of S contains a finite subcover the A is called is ____________	closed	compact	open	unbounded	compact
With usual metric, (0,1) is __________________	closed	compact	not comapct	unbounded	not compact
With usual metric, [0,1) is ___________________	closed	compact	not compact	unbounded	not compact
With usual metric, (0,1] is __________________	closed	compact	not compact	unbounded	not compact
With usual metric [0, ∞) is _______________	closed	compact	not compact	bounded	not compact
With usual metric [a,b] is __________	compact	not compact	unbounded	open	comapct
Any compact set is _____________	closed	open	both open and closed	neither open nor closed	closed
Any compact set is ____________	bounded	unbounded	both open and closed	neither open nor closed	bounded
A closed sub set of a compact metric space is ___________	not comapct	unbounded	open	compact	compact
Every finite metric space is __________	not comapct	unbounded	open	compact	compact
If A and B are compact subsets of a metric space then the union of A nd B is ________________	not comapct	unbounded	open	compact	compact
Any closed interval [a,b] is _____________	comapct	unbounded	not compact	open	comapct
A set S in n dimensional space is compact iff ______________	S is open only	S is closed only	S is closed and bounded	S is either open or closed	S is closed and bounded
Every infinite subset of S has an accumulation point in S is ______________	S is open only	S is closed only	S is closed and bounded	S is either open or closed	S is closed and bounded
Every infinite subset of S has an accumulation point in S is _______________	compact	unbounded	not comapct	open	compact
The collection of all open intervals is an open covering of R. The open covering is ____________	uncountable covering	countable covering	finite covering	countably infinite covering	uncountable covering
The collection of all open intervals (n,n+1), n in Z is an open covering of R. The open covering is ___________	uncountable covering	countable covering	finite covering	unbounded covering	countable covering
Which of the following is not true?	[a,b] is compact	[1,100] is compact	(1,100) is compact	[0,1] X [0,1]X [0,1] is compact	(1,100) is compact
Let A be a subset of a compact set B. Then A is compact if A is ______________	bounded	closed	neither closed nor bounded	both closed and bounded	both closed and bounded
If S is compact and T is compact then which of the following statement is true?	S-T is compact	T-S is compact	union of S and T is compact	intersection of S and T is compact	intersection of S and T is compact
The intersection of any collection of compact subsets is ______________	compact	not compact	unbounded	open	compact
The union of finite number of compact subset is ________________	compact	not compact	unbounded	open	compact
The union of any collection of compact subsets is _________________	compact	not compact	both bounded and closed	closed	not compact
Let A = {1/n: n in N}. A is not a compact since A is ___________	not bounded	closed	bounded	not closed	not closed
Let A = {0,1, 1/2, 1/4,1/8, ...}. A is not compact since A is ________________	bounded	closed	bounded and closed	neither bounded nor closed	bounded and closed
Let (M, d) be a metric space then d(x, y) > 0 if _____________	x<y	y<x	x=y		x is not equal to y
Let (M,d) be a metric space. Then value of d(x,y) is________________________	0	1	either 0 or 1	0 and 1	either 0 or 1
Every n ball in a metric space is _________________	open	closed	unbounded	neither open nor closed	open
In discrete metric space, every subset is __________________	open	unbounded	neither closed nor bounded	bounded and not closed	open
Let (M, d) be a metric space then D(x, y) = d(x, y)/1+d(x, y) then D(x, y) is ____________	greater than 0	less than 0	equal to 0	greater than or equal to 0	greater than or equal to 0

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Subject: Real Analysis-I	**Semester: V**	**L**	**T**	**P**	**C**
Subject Code: 15MMU501	**Class: III-B.Sc Mathematics-A**	**5**	**0**	**0**	**5**

UNIT V

Convergent sequences in a metric space –Cauchy sequences –Completeness sequences –complete metric Spaces. Limit of a function –Continuous functions –continuity of composite functions. Continuous complex valued and vector valued functions.

Text Book:

T1: Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

Reference Book:

R3: Kenneth Hoffman., Ray Kunze., (2003). Linear Algebra, Second edition, Prentice Hall of
India Pvt Ltd, New Delhi.

R4: Royden .H.L ., 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.

R5: Sterling. K. Berberian, 2004. A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here d denotes the distance in X.)

In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ [see Theorem 3.2(*b*)], and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge*.

As examples, consider the following sequences of complex numbers (that is, $X = R^2$):

(*a*) If $s_n = 1/n$, then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.
(*b*) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.
(*c*) If $s_n = 1 + [(-1)^n/n]$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.
(*d*) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.
(*e*) If $s_n = 1$ $(n = 1, 2, 3, \ldots)$, then $\{s_n\}$ converges to 1, is bounded, and has finite range.

We now summarize some important properties of convergent sequences in metric spaces.

Theorem *Let $\{p_n\}$ be a sequence in a metric space X.*

(*a*) *$\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.*
(*b*) *If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.*
(*c*) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*
(*d*) *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Proof (*a*) Suppose $p_n \to p$ and let V be a neighborhood of p. For some $\varepsilon > 0$, the conditions $d(q, p) < \varepsilon, q \in X$ imply $q \in V$. Corresponding to this ε, there exists N such that $n \geq N$ implies $d(p_n, p) < \varepsilon$. Thus $n \geq N$ implies $p_n \in V$.

Conversely, suppose every neighborhood of p contains all but finitely many of the p_n. Fix $\varepsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \varepsilon$. By assumption, there exists N (corresponding to this V) such that $p_n \in V$ if $n \geq N$. Thus $d(p_n, p) < \varepsilon$ if $n \geq N$; hence $p_n \to p$.

(*b*) Let $\varepsilon > 0$ be given. There exist integers N, N' such that

$$n \geq N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2},$$

$$n \geq N' \quad \text{implies} \quad d(p_n, p') < \frac{\varepsilon}{2}.$$

Hence if $n \geq \max(N, N')$, we have

$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon.$$

Since ε was arbitrary, we conclude that $d(p, p') = 0$.

(*c*) Suppose $p_n \to p$. There is an integer N such that $n > N$ implies $d(p_n, p) < 1$. Put

$$r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}.$$

Then $d(p_n, p) \leq r$ for $n = 1, 2, 3, \ldots$.

(*d*) For each positive integer n, there is a point $p_n \in E$ such that $d(p_n, p) < 1/n$. Given $\varepsilon > 0$, choose N so that $N\varepsilon > 1$. If $n > N$, it follows that $d(p_n, p) < \varepsilon$. Hence $p_n \to p$.

This completes the proof.

SUBSEQUENCES

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p. We leave the details of the proof to the reader.

3.6 Theorem

(*a*) *If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.*

(*b*) *Every bounded sequence in R^k contains a convergent subsequence.*

Proof

(*a*) Let E be the range of $\{p_n\}$. If E is finite then there is a $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \cdots$, such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The subsequence $\{p_{n_i}\}$ so obtained converges evidently to p.

If E is infinite, Theorem 2.37 shows that E has a limit point $p \in X$. Choose n_1 so that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{i-1}$, we see from Theorem 2.20 that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p.

(*b*) This follows from (*a*), since Theorem 2.41 implies that every bounded subset of R^k lies in a compact subset of R^k.

CAUCHY SEQUENCES

3.8 Definition A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

In our discussion of Cauchy sequences, as well as in other situations which will arise later, the following geometric concept will be useful.

3.9 Definition Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the *diameter* of E.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \ldots$, it is clear from the two preceding definitions that *$\{p_n\}$ is a Cauchy sequence if and only if*

$$\lim_{N \to \infty} \operatorname{diam} E_N = 0.$$

Theorem

(*a*) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(*b*) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.*

(*c*) *In R^k, every Cauchy sequence converges.*

Proof

(*a*) If $p_n \to p$ and if $\varepsilon > 0$, there is an integer N such that $d(p, p_n) < \varepsilon$ for all $n \geq N$. Hence

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\varepsilon$$

as soon as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

(*b*) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N = 1, 2, 3, \ldots$, let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \ldots$. Then

$$\lim_{N \to \infty} \operatorname{diam} \bar{E}_N = 0, \tag{3}$$

by Definition 3.9 and Theorem 3.10(*a*). Being a closed subset of the compact space X, each $\bar{E}_N$ is compact (Theorem 2.35). Also $E_N \supset E_{N+1}$, so that $\bar{E}_N \supset \bar{E}_{N+1}$.

Theorem 3.10(*b*) shows now that there is a unique $p \in X$ which lies in every $\bar{E}_N$.

Let $\varepsilon > 0$ be given. By (3) there is an integer N_0 such that $\operatorname{diam} \bar{E}_N < \varepsilon$ if $N \geq N_0$. Since $p \in \bar{E}_N$, it follows that $d(p, q) < \varepsilon$ for every $q \in \bar{E}_N$, hence for every $q \in E_N$. In other words, $d(p, p_n) < \varepsilon$ if $n \geq N_0$. This says precisely that $p_n \to p$.

(*c*) Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in R^k. Define E_N as in (*b*), with $\mathbf{x}_i$ in place of p_i. For some N, $\operatorname{diam} E_N < 1$. The range of $\{\mathbf{x}_n\}$ is the union of E_N and the finite set $\{\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$. Hence $\{\mathbf{x}_n\}$ is bounded. Since every bounded subset of R^k has compact closure in R^k (Theorem 2.41), (*c*) follows from (*b*).

3.12 Definition A metric space in which every Cauchy sequence converges is said to be *complete*.

CONTINUOUS FUNCTIONS

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at* p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

4.8 Theorem *A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.*

Proof Suppose f is continuous on X and V is an open set in Y. We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if $d_Y(f(p), y) < \varepsilon$; and since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$ and $\varepsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \varepsilon$. Then V is open; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $x \in f^{-1}(V)$as soon as $d_X(p, x) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$.

This completes the proof.

Corollary *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*

4.9 Theorem *Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X.*

CONTINUITY AND COMPACTNESS

4.13 Definition A mapping **f** of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

4.14 Theorem *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, Theorem 4.8 shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}). \tag{12}$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, (12) implies that

$$f(X) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}. \tag{13}$$

This completes the proof.

Note: We have used the relation $f(f^{-1}(E)) \subset E$, valid for $E \subset Y$. If $E \subset X$, then $f^{-1}(f(E)) \supset E$; equality need not hold in either case.

We shall now deduce some consequences of Theorem 4.14.

4.15 Theorem *If* **f** *is a continuous mapping of a compact metric space X into* R^k, *then* $\mathbf{f}(X)$ *is closed and bounded. Thus,* **f** *is bounded.*

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

(15) $$d_Y(f(p), f(q)) < \varepsilon$$

for all p and q in X for which $d_X(p, q) < \delta$.

4.19 Theorem *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Proof Let $\varepsilon > 0$ be given. Since f is continuous, we can associate to each point $p \in X$ a positive number $\phi(p)$ such that

(16) $$q \in X, d_X(p, q) < \phi(p) \quad \text{implies} \quad d_Y(f(p), f(q)) < \frac{\varepsilon}{2}.$$

Let $J(p)$ be the set of all $q \in X$ for which

(17) $$d_X(p, q) < \tfrac{1}{2}\phi(p).$$

Since $p \in J(p)$, the collection of all sets $J(p)$ is an open cover of X; and since X is compact, there is a finite set of points $p_1, \ldots, p_n$ in X, such that

(18) $$X \subset J(p_1) \cup \cdots \cup J(p_n).$$

We put

(19) $$\delta = \tfrac{1}{2} \min\,[\phi(p_1), \ldots, \phi(p_n)].$$

Then $\delta > 0$. (This is one point where the finiteness of the covering, inherent in the definition of compactness, is essential. The minimum of a finite set of positive numbers is positive, whereas the inf of an infinite set of positive numbers may very well be 0.)

Now let q and p be points of X, such that $d_X(p, q) < \delta$. By (18), there is an integer m, $1 \le m \le n$, such that $p \in J(p_m)$; hence

(20) $$d_X(p, p_m) < \tfrac{1}{2}\phi(p_m),$$

and we also have

$$d_X(q, p_m) \le d_X(p, q) + d_X(p, p_m) < \delta + \tfrac{1}{2}\phi(p_m) \le \phi(p_m).$$

Finally, (16) shows that therefore

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)) < \varepsilon.$$

This completes the proof.

CONTINUITY AND CONNECTEDNESS

4.22 Theorem *If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.*

Proof Assume, on the contrary, that $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y. Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \bar{A}$ (the closure of A), we have $G \subset f^{-1}(\bar{A})$; the latter set is closed, since f is continuous; hence $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B$ is empty, we conclude that $\bar{G} \cap H$ is empty.

The same argument shows that $G \cap \bar{H}$ is empty. Thus G and H are separated. This is impossible if E is connected.

Possible Questions

PART-B (8 Mark)

1. Let S be a compact subset of a metric space M. then prove that
   i) S is closed and bounded
   ii) Every infinite subset of S has an accumulation point of S.
2. Prove that every convergence sequence is a Cauchy sequence.
3. Prove that in a metric space (S,d) a sequence $\{x_n\}$converges to p if and only if every subsequence $\{x_{k(n)}\}$ converges to p.
4. Let X be a closed subset of a compact metric space M. Then prove that X is compact.
5. Prove that a sequence $\{x_n\}$ in the metric space (S, d) can converge to atmost one point in S..
6. Let f and g be complex valued functions defined on a subset A of a metric space (S, d). Let p be an accumulation point of A, and assume that lim f(x) = a and lim g(x) = b. Then prove that lim f(x) g(x) = ab
7. In a metric space (S, d) assume that $x_n \rightarrow$ p and let T = $\{x_1, x_2, \ldots\}$ (ie) the range of the sequence $\{x_n\}$ then prove that i) T is bounded and ii) p is an adherent point of T.
8. State and prove the Heine – Borel theorem.
9. Prove that closed intervals in $\mathbb{R}$ are compact.
10. Assume p is an accumulation point of A and assume b$\in$T then prove that $\lim_{x\to p} f(x)$ = p if and only if $\lim_{n\to\infty} f(xn)$ = b for every sequence $\{x_n\}$ of points in which converges to p.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Real Analysis-I **Subject Code: 15MMU501**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
A sequence of {Xn} with $X_n > X_{n+1}$ for all ni is ______________	increasing sequence	decreasing sequence	bounded sequence	constant sequence	decresing sequence
If an increasing sequence is bounded above then ____________	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 1	sequence converges to 0	sequence converges to sup of its range
If {an } decreasing sequence is bounded below then __________	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 2	sequence converges to 1	sequence converges to inf of its range
The limit of a function $F(x) = (x^2)+2x$ as $x\to 3$ is _________________	13	12	14	15	15
The limit of a function $F(x) = 5x$ as $x\to\infty$ is _________________	exists	exists ans is 1	does not exists	exists and is 0	does not exists
The limit of a function $F(x) = 1/x$ as $x\to\infty$ is _________________	0	-1	2	-2	0
The limit of a function $F(x) = 1/x$ as $x\to 1$ is _________________	0	2	-1	1	1
Fibonacci sequence is ______________________	an increasing sequence	a decresing sequence	constant sequence	bounded sequence	an incresing sequence
A sequence in a metric space (S,d) can converge _______________	at least one point	more than two point	atmost one point	more than three point	atmost one point
Suppose a sequence in a metric space (S,d) converges to both a and b. Then we must have ________________________	a<b	a>b	a-b=1	a=b	a=b
In a metric space (S,d), a sequence converges to p. Then range of the sequence is _____________________	bounded	unbounded	finite	infinite	bounded
The range of a constant sequence is ___________________	infinite	countably infinite	uncountable	singlton set	singleton set
Suppose in a metric space (S,d), a sequence converges to p. Then the point p is ________________________	an adherent point of S	an accumulation point of S	an isolated point of S	not an adherent point of S	an adherent point of S
Suppose in a metric space (S,d) , a sequence converges to p and the rnage of the sequence is infinite. Then p is _________________	an adherent point of S	an accumulation point of S	an isolated point of S	not an accumulation point of S	an accumulation point of S
Suppose in a metric space, a sequence converges. Then _______________________	every sequence in a metric space converges	every subsequence of convergent sequence converges	some subsequence of convergent sequence converges	some sequence in a metric space converges	every subsequence of convergent sequence converges
A sequence is said to be bounded if if its range is _________________________	unbounded	bounded	countable	uncountable	bounded
If {Xn} with Xn = 1/n then therange of the sequence _____________	finite	{1}	{}	infinite	infinite
If {Xn} with Xn = 1/n then therange of the sequence _____________	unbounded	bounded	{}	{1,0}	bounded
If {Xn} with Xn = 1/n then the sequence _______________	converges	diverges	oscilates	converges to 1	converges
If {Xn} with Xn = n^2 then the sequence _______________	converges	diverges	oscilates	converges to 2	diverges
If {Xn} with Xn = n^2 then the range of the sequence _____________	unbounded	bounded	{}	{0.1}	unbounded
If {Xn} with Xn = n^2 then the range of the sequence _____________	finite	{1}	{}	infinite	infinite
If {Xn} with Xn = i^n then the sequence _______________	converges	diverges	oscilates	converges to 0	diverges
If {Xn} with Xn = i^n then the range of the sequence _____________	unbounded	bounded	{}	{0,1}	bounded
If {Xn} with Xn = i^n then the range of the sequence _____________	finite	infinite	{}	{0,1}	finite
If {Xn} with Xn = 1 then the sequence _______________	converges	diverges	oscilates	converges to 0	converges
If {Xn} with Xn = i^n then the range of the sequence ______________	{}	{1}	{1,0}	{1,2,3}	{1}
If {Xn} with Xn = i^n then the range of the sequence _____________	boundea	unbounded	{1,0}	{0}	bounded
If $X_n\to X$ and $Y_n\to Y$ then X_n+Y_n is converge to _____________	xy	x+y	x/y	x-y	x+y
If $X_n\to X$ and $Y_n\to Y$ then X_nY_n is converge to _____________	xy	x+y	x/y	x-y	xy
If $X_n\to X$ then C Xn converges to ______________	c	x	c/x	cx	cx
If $X_n\to X$ then 1/Xn converges to 1/X if ______________	x=0	x=1	x=2		x is not equal to y
In Euclidean metric space every cauchy sequence is _____________	convergent	divergent	oscilates	convergent to 0	converges
Every convergent sequence is a _________________	constant seqeunce	cauchy sequence	increasing sequence	decreasing sequence	cauchy sequence
A metric space is called complete if ________________	every cauchy sequence converges	some cauchy sequence converges	no cauchy sequence converges	every cauchy seqeunce diverges	every cauchy sequence converges
Any discrete metric space is _______________	complete	not complete	bounded	unbounded	complete
A subset A of a complete metric space S is complete if A is _____________	open	closed	both open and closed	not closed	closed
The set of all rationals is ________________	open	closed	both open and closed	not complete	not complete
Every compact metric space is _______________	open	complete	not complete	not bounded	complete
$\sqrt{(n + 1)} - \sqrt{(n - 1)}$ converges to ________________________	0	1	2	3	0
Let X be a space with the discrete metric. Let x ∈ X. Show that B(x, 1 2) = B(x, 1 2) = _________________	{12}	{1}	{2}	{x}	{x}

Which of the following is complete susbset of [0,1]?	(0,1.234)	(0,0.234)	[0.0.234]	(0,0.345)	[0,0.234]

Reg no-------------

(15MMU501)

KARPAGAM UNIVERSITY

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Fifth Semester

I Internal Test - July'2017

Real Analysis

Date: -07-2017 **Time: 2 Hours**

Class: III-Bsc Mathematics **Maximum Marks:50**

PART-A(20X1=20 Marks)

Answer all the Questions:

1. The number e is ----------------
   a. Rational b. Irrational c. Prime d. Composite
2. The supremum for {3, 4} is -------------
   a. 3 b. 4 c. (3, 4) d. [3, 4]
3. The inequality $|x + y| \geq$ --------------is holds.
   a. $|x| + |y|$ b. $|x| - |y|$ c. $|xy|$ d. $||x| - |y||$
4. The coordinates (x, y) of a point represent an --------------- of numbers
   a. Function b. Relation c. Ordered pair d. Set
5. One-one function is also called -------------
   a. Injective b. Bijective c. Transformation d. Codomain
6. Similar sets are also called as -------------- set
   a. Denumerable b. Countable c. Finite d. Equinumerous
7. The set of rational numbers is ---------------
   a. Uncountable b. Finite c. Countable d. Complete
8. The set of points between a and b is called ----------------
   a. Integer b. Interval c. Element d. Set
9. If (a, b) =1 then a and b are called --------------
   a.Prime b. Common factor c. LCM d. Relatively prime
10. The completeness axiom is ----------------
    a. b = sup S b. S = sup b c. b = inf S d. S = inf b
11. If S is a relation, the set of all elements that occur as first members in S is called --------
    a. Function b. Codomain c. Domain d. Range
12. If f(x) = f(y) implies x = y then f is a -------------- function
    a. One-one b. Onto c. Into d. Inverse
13. If d is a divisor of n, then ----------------
    a. n = c b. n < cd c. n > cd d. n = cd
14. If a|bc and (a, b) = 1 then -----------
    a. a|c b. a|b c. c|a d. b|a
15. The greatest lower bound is called ----------
    a. Unbounded above b. Unbounded below
    c. Supremum d. Infimum
16. A set with no upper bound is called ----------
    a. Unbounded above b. Unbounded below
    c. Prime d. Open set
17. Every finite set of numbers is ----------
    a. Bounded b. Unbounded c. Prime d. Bounded above
18. Sup C = Sup A + Sup B is called ------------ property
    a. Approximation b. Additive
    c. Archimedean d. Comparison
19. A x B denotes the ----------- of the sets A and B
    a. Product b. Polar form c. Cartesian product d. Complement
20. Uncountable sets are also called --------------- set
    a. Denumerable b. Non-denumerable
    c. Similar d. Equal

PART-B (3X10=30 Marks)

Answer all the Questions:

21. (a) Prove that every integer n > 1 can be represented as a product of prime factors in only one way apart from the order of the factors.

    (OR)

    (b) Prove that the set of real numbers is uncountable.

22. (a) Prove that e is irrational.

(OR)

(b) State and prove Archimedean property.

23. (a) State and prove Cauchy-Schwarz inequality..

(OR)

(b) Prove that $\sqrt{n}$ is irrational.

Reg. No ....

(15MMU501)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Karpagam University

COIMATORE – 21

Department of Mathematics

Fifth Semester

II Internal Test – August'2017

Real Analysis - I

Class: III B.Sc Mathematics — Maximum: 50 Marks

Date: 07.08.2017 (AN) — Time: 2 Hours

PART – A (20 x 1 = 20 Marks)

Answer all the questions

1. The inverse relation of f is a function only if f is ----------.
a) into b) onto c) one-to-one d) bijection
2. If two sets A={1,2,...,m} and B={1,2,...,n} are similar then ----------.
a) m<n b) n<m c) n=m d) n>0
3. Countable union of countable set is ------------.
a) uncountable b) countable c) finite d) countably infinite
4. If a set A has n element, then the total number of subsets of A is -------.
a) n! b) 2n c) 2^n d) n
5. Suppose f is a one-to-one function. Then x not equal y implies ------.
a) f(x)≠f(y) b) f(x)=F(y) c) f(x)>f(y) d) f(x)<f(y)
6. Let f be a function defined on A and itself such that f(x)=x. Then f is -----.
a) onto b) one-to-one c) bijection d) neither one-to-one nor onto
7. The set of all even integers is -------------.
a) uncountable b) countable c) finite d) infinite
8. The set of sequences consists of only 1 and 0 is------
a) uncountable b) countable c) finite d) infinite
9. Let f:A→B then which of the following is always true?
a) range of f is not equal to B b) range of f is a subset of B
c) range of f is containing B d) range of f is proper subset of B
10. Fibonnaci numbers is an example for --------.
a) uncountable b) countable c) finite d) infinite
11. Let X be in n dimensional space. Then we must have ---------.
a) norm of x is 0 b) norm of x is 1
c) norm of x is greater than 0 d) norm of x is greater than or equal to 0
12. Every finite set is -----------.
a) an open set b) a closed set
c) neither open nor closed d)both open and closed set
13. The set of all interior points of a set of all rational is ---------.
a) empty set b) A c) R d) Z
14. If A is open and B is closed the B-A is ---------.
a) open b) closed c) both open and closed d) neither open nor closed
15. Every n ball of x contains more than one point than x is called --------.
a) adherent point b) accumulation point
c) isolated point d) not an adherent point
16. Which of the following is not a limit point of [0,1)?
a) 1 b) 2 c) 0 d) 1/2
17. x is an accumulation point of A then every n ball of x contains ---------.
a) atleast one point other than x b) atmost one point other than x
c) exactly one point other than x d) no point other than x
18. The set of prime numbers is --------------.
a) open b) finite c) bounded d) unbounded
19. Which of the following is not a bounded set?
a) R b) [a,b] c) [a,b) d) (a,b]
20. A set is closed if ----------------------------.
a) complement of S is closed
b) complement of S is open
c) complement of S is neither open nor closed
d) complement of S is both open and closed

PART – B (3 x 10 = 30 Marks)

Answer all the questions

21. a) Prove that the set of all real numbers $\mathbb{R}$ is uncountable.

(OR)

b) If $\mathbb{F} = \{A_1, A_2, \dots\}$ is a countable collection of sets and $\mathbb{G} = \{B_1, B_2, \dots\}$ when $B_1 = A_1$ for $n > 1$, $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$, then prove that G is a collection of disjoint sets, and we have $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$.

22. a) Prove that the Cartesian product $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable where $\mathbb{Z}^+$ denote the set of all positive integers.

(OR)

b) Determine whether d(x, y) defined on $\mathbb{R}$ by $d(x, y) = (x-y)^2$ is a metric space or not.

23. a) i) State and prove Triangle inequality.

ii) Define Metric space with example.

(OR)

b) Prove that the union of any collection of open set is open.

Reg. No ………………

(15MMU501)

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education

Coimbatore – 21

DEPARTMENT OF MATHEMATICS

Fifth Semester

Model Examination – September'2017

Real Analysis - I

Date: .09.2017( ) **Time: 3 Hours**

Class: III B.Sc Mathematics **Maximum: 60 Marks**

PART – A (20 x 1 = 20 Marks)

Answer all the questions

1. If d is a divisor of n, then ----------------

a) n = c b) n < cd c) n > cd d) **n = cd**

2. If a|bc and (a, b) = 1 then -----------

a) **a|c** b) a|b c) c|a d) b|a

3. The greatest lower bound is called ----------

a) unbounded above b) unbounded below

c) supremum d) **infimum**

4. If x < 0 then -----------

a) |x| = x b) ||x|| = |x| c) ||x|| = -x d) **|x| = -x**

5. If (x, y) Є F and (x, z) Є F, then ------------

a) x = z b) x = y c) xy = z d) **y = z**

6. If m < n then K(n) < K(m) implies that K is a ------------

a) sequence b) subsequence

c) **order preserving** d) equinumerous

7. The set of those elements which belongs to either A or b or both is called -------.

a) complement b) **union** c) intersection d) disjoint

8. The set S of intervals with rational end points is ------------ set.

a) uncountable b) finite c) **countable** d) disjoint

9. ||x – y|| ≤ -----------

a) ||y – x|| b) **||x|| - ||y||** c) ||x|| + ||y|| d) ||xy||

10. In R^3, the open ball B(a; r) is -------------

a) open interval b) closed interval

c) **spherical solid** d) circular disc

11. Every singleton set is ------------

a) **bounded** b) unbounded c) open d) closed

12. A collection F of sets is said to be a covering of a given set S if --------------

a) S ⊆ UA b) A ⊆ US c) **S = UA** d) A = US

13. d(x, y) = ------------

a) d(xy) b) **d(y, x)** c) d(x + y) d) d(x – y)

14. The metric (R^n, d) is called ------------ metric

a) **Euclidean** b) complex c) discrete d) bounded

15. An increasing sequence which is bounded above will converge to its ----------

a) **supremum** b) infimum c) fixed point d) adherent point

16. Every closed set is -------------

a) **convergent** b) divergent c) Euclidean d) complete

17. A real valued continuous function f is said to be two valued on S if ---------

a) **$f(S) \subseteq \{0, 1\}$** b) $f(S) \subseteq (0, 1)$

c) $f(S) \subseteq [0, 1]$ d) $f(S) = \{0, 1\}$

18. If f: $[0, 1] \rightarrow S$ such that $f(0) = a$ and $f(1) = b$, then a set S in R^n is called --------

a) arcwise connected b) jointly connected

c) **simply connected** d) eventually connected

19. A set in R^n is called a ---------- if it is the union of an open connected set with some or all its boundary points.

a) component b) path c) **region** d) interval

20. The function $f(x) = x^2$ where x belongs to R^1 and $A = (0, 1]$ is ------------ on A.

a) continuous b) **uniformly continuous**

c) not continuous d) analytic

PART – B (5 x 8 = 40 marks)

Answer all the questions

21. a) State and prove Cauchy Schwarz inequality.

(OR)

b) Prove that $\sqrt{2}$ is irrational.

22. a) Prove that the set of real numbers is uncountable.

(OR)

b) Show that the Cartesian product $Z^+ \times Z^+$ where Z^+ is a set of positive integers is a countable set.

23. a) Prove that union of any collection of open sets is an open set.

(OR)

b) Prove that a set S is closed if and only if $S = \overline{S}$.

24. a) Prove that closed intervals in $\Re$ are compact.

(OR)

b) State and prove the Heine – Borel theorem.

25. a) Prove that every arcwise connected set S in R^n is connected.

(OR)

b) Let f: $S \rightarrow M$ be a function from a metric space S to another metric space M. Let X be a connected subset of S. If f is continuous on X, then prove that f(X) is a connected subset of M.