KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: COMPLEX ANALYSIS -I **Subject Code: 15MMU502**

L	T	P	C
5	0	0	5

PO: This course will enhance the learner to understand the important concepts such as complex number system, complex plane analyticity of a function, function of complex variables etc which plays a crucial role in the application of two dimensional problems in Science.

PLO: To enable the students to learn various aspects complex number system, complex function and complex integration

UNIT I

Complex number system:Complex number-Field of a complex numbers-Conjugation –Absolute value of a complex number.

Complex plane: Complex number by points-n_{th} root of a complex number-Angle between two rays-Elementary transformation- Stereographic projection.

UNIT II

Analytic functions: Limit of a function –continuity –differentiability – Analytical function defined in a region –necessary conditions for differentiability –sufficient conditions for differentiability – Cauchy-Riemann equation in polar coordinates –Definition of entire function.

UNIT III

Power Series: Absolute convergence –circle of convergence –Analyticity of the sum of a power series-Uniqueness of representation of a function by a power series- Elementary functions : Exponential, Logarithmic, Trigonometric and Hyperbolic functions. Harmonic functions: Definition and determination.

UNIT IV

Bilinear transformation-Circles and Inverse points-Transformation mappings $w=Z^2$,$w=Z^{1/2}$,$w=e^Z$, w = sin Z,and w=cos Z -Conformal mapping-isogonal mapping.

UNIT V

Complex integration: Simple rectifiable oriented curves –Integration of complex functions- Definite integral-Interior and Exterior of a closed curve-Simply connected region-Cauchy‟s fundamental theorem-Cauchy‟s formula for higher derivatives- Morera‟s theorem.

TEXT BOOK

1.Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2

REFERENCES

1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi

2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.

4. Ponnusamy.S., 2004. Foundations of Complex Analysis, Narosa Publishing House, Chennai.

5. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.

6. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore 641 021
DEPARTMENT OF MATHEMATICS
Lecture Plan

Subject Name: COMPLEX ANALYSIS-I **Subject Code: 15MMU502**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
UNIT-I			
1.	1	Complex number system- Introduction	T1.Ch 1: pg:1-2
2.	1	Field of Complex numbers, Field of Real numbers and its problems	R1:Ch 1:1.1:pg:1-4
3.	1	Conjugation, Theorems on Conjugation related examples	T1.Ch 1: pg:3-4
4.	1	Absolute value of Complex number and its inequalities related examples	R1.Ch 1:1.4: pg:5-7
5.	1	Continuaion of Absolute value of Complex number and its inequalities related examples	R1.Ch 1:1.4: pg:7-9
6.	1	Complex plane: complex number by points and n^{th} roots of a complex number problems	T1.Ch 2: pg:9-12
7.	1	Continuaion of Complex plane: complex number by points and n^{th} roots of a complex number problems	T1.Ch 2: pg:12-14
8.	1	Angle between two rays, Equations of straight lines and circle examples	R2: Ch 1.6: 12-15
9.	1	Continuaion of Angle between two rays, Equations of straight lines and circle examples	R2: Ch 1.6: 15-17
10.	1	Elementary Transformation	R2: Ch 1.6:18-20
11.	1	Continuaion of Elementary Transformation	R2: Ch 1.6:20-22
12	1	Continuaion of Elementary Transformation	R2: Ch 1.6:22-24
13	1	Infinity and Extended complex plane and its examples	T1.Ch 2: pg:20-23

14	1	Stereographic projection related problems	R3.Ch 1:1.8:pg:15-17
15	1	Recapitulation and Discussion of possible questions	
Total	**15Hours**		

Text book
T1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .
References
R1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi
R2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.
R3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.

UNIT-II

1.	1	Analytic functions: complex functions: definition and examples	T1.Ch 4: pg:33-40
2.	1	Limit and continuity of a function: definition and examples	R4.Ch 2: 2.1:pg:83-87
3.	1	Limit and continuity of a function: definition and examples	T1.Ch 4: pg:40-46
4.	1	Uniform continuity and Differentiability of a function and its examples	T1.Ch 4: pg:46-48
5.	1	Uniform continuity and Differentiability of a function and its examples	R4.Ch 2: 2.1:pg:88-92
6.	1	Analytical function defined in a regionrelated examples	T1.Ch 4: pg:50-51
7.	1	Necessary conditions for differentiability	T1.Ch 4: pg:51-54
8.	1	Sufficient conditions for differentiability	T1.Ch 4: pg:54-55
9	1	Cauchy-Riemann equation in polar coordinates related examples	R6: Ch 1:1.9:pg:29-31
10	1	Cauchy-Riemann equation in polar coordinates related examples	R6: Ch 1:1.9:pg:31-33
11	1	Related concepts on Cauchy-Riemann equation in	T1.Ch 4: pg:57-58

		polar coordinates	
12	1	Entire functions and its problems	R5: Ch 2: 2.9:pg:61-63
13	1	Continuation of Entire functions and its problems	R5: Ch 2: 2.9:pg:63-65
14	1	Continuation of Entire functions and its problems	R5: Ch 2: 2.9:pg:65-67
15	1	Recapitulation and Discussion of possible questions	
Total	**15 Hours**		

Text book

T1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

References

R4. Ponnusamy.S., 2004. Foundations of Complex Analysis, Narosa Publishing House, Chennai.
R5. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.
R6. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

UNIT-III

1	1	Power Series :Introduction	T1.Ch 6: pg:71-74
2	1	Absolute convergence of power series and Circle of convergence -Theorems and its examples	R6: Ch 1:1.7.1:pg:18-22
3	1	Absolute convergence of power series and Circle of convergence -Theorems and its examples	T1.Ch 6: pg:81-84
4	1	Uniform convergence of Power Series related examples	R5: Ch 5:5.6:pg:221-223
5	1	Uniform convergence of Power Series related examples	R5: Ch 5:5.6:pg:223-225
6	1	Analyticity of the sum of power series problems	T1.Ch 6: pg:84-87
7	1	Uniqueness of Representation of a function by power series	R2:Chap 4:4.2:pg:101-103
8	1	Uniqueness of Representation of a function by power series	R2:Chap 4:4.2:pg:103-105

9	1	Elementary functions : Exponential functions Theorems and its examples	T1.Ch 6: pg:87-88
10	1	Elementary functions : Exponential functions Theorems and its examples	T1.Ch 6: pg:88- 89
11	1	Logarithmic functions and its Problems	T1.Ch 6: pg:90-92
12	1	Trigonometric functions and its Problems	R6: Ch 1:1.8:pg:25-27
13	1	Hyperbolic functions-related examples	R6: Ch 1:1.8:pg:28-29
14	1	Hyperbolic functions-related examples	R6: Ch 1:1.8:pg:29-31
15	1	Recapitulation and Discussion of possible questions	
Total	**15Hours**		

Textbook:

T1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

References

R2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

R5. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.

R6. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

UNIT-IV

1	1	Harmonic functions and Conjugate Harmonic functions	T1.Ch 6: pg:93-96
2	1	Determination and problems	T1.Ch 6: pg:97-99
3	1	Continuation of problems on determination	T1.Ch 6: pg:99-100
4	1	Continuation of problems on determination	T1.Ch 6: pg:100-102
5	1	Bilinear transformation-Theorem related problems	T1.Ch 7: pg:103-104
6	1	Bilinear transformation-Theorem related problems	T1.Ch 7: pg:105-106
7	1	Bilinear transformation-Theorem related	T1.Ch 7: pg:106-107

		problems	
8	1	Circles and inverse points related problems	T1.Ch 7: pg:113-116
9	1	Transformation mappings $w = z^2$ and $w = z^{1/2}$	R2:Ch 5: pg:118-119
10	1	Transformation mappings $w = z^2$ and $w = z^{1/2}$	R2:Ch 5: pg:119-121
11	1	Transformation mappings $w = e^z$	T1.Ch 7: pg:117-118
12	1	Transformation mappings w = sin z and w=cos z	R2:Ch 5: pg:124-126
13	1	Conformal Mapping-Theorem related examples	T1.Ch 7: pg:120-122
14	1	Conformal Mapping-Theorem related examples	T1.Ch 7: pg:123-124
15	1	Recapitulation and Discussion of possible questions	
Total	**15Hours**		

Textbook:
T1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

References
R2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

UNIT-V

1	1	Complex Integration: Simple rectifiable oriented curve and its Theorems	R1.Ch 4:4.1: pg:104-105
2	1	Complex Integration: Simple rectifiable oriented curve and its Theorems	R1.Ch 4:4.1: pg:106-107
3	1	Complex Integration: Simple rectifiable oriented curve and its Theorems	R1.Ch 4:4.1: pg:107-109
4	1	Integration of complex function- Definite Integral - Theorem and its problems	R3.Ch 3:3.1: pg:194-196

5	1	Integration of complex function- Definite Integral - Theorem and its problems	R3.Ch 3:3.1: pg:196-198
6	1	Interior and exterior of a closed curve, simply connected region related examples	R6.Ch 3:3.1: pg:94-96
7	1	G and related examples	R3.Ch 3:3.4: pg:220-223
8	1	G and related examples	R3.Ch 3:3.4: pg:223-225
9	1	G related problems	R3.Ch 3:3.1: pg:226-229
10	1	G related problems	R3.Ch 3:3.1: pg:229-231
11	1	and related examples	R6.Ch 3:3.7: pg:121-123
12	1	Recapitulation and Discussion of possible questions	
13	1	Discussion on Previous ESE Question Papers	
14	1	Discussion on Previous ESE Question Papers	
15	1	Discussion on Previous ESE Question Papers	
Total	**15 Hours**		

REFERENCES

R1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi
R3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.
R6. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

Total no. of Hours for the Course: 75 hours

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT OF MATHEMATICS

Subject: COMPLEX ANALYSIS-I	**Semester: V**	**L T P C**
Subject Code: 15MMU502	**Class: I-B.Sc Mathematics**	**5 0 0 5**

UNIT I

Complex number system:Complex number-Field of a complex numbers-Conjugation –Absolute value of a complex number.
Complex plane: Complex number by points-nth root of a complex number-Angle between two rays-Elementary transformation- Stereographic projection.

TEXT BOOK

1.Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2

REFERENCES

1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi

2. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.

3. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.

DEFINITION

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

Terminology

The notations $a + ib$ and $a + bi$ are used interchangeably.

The real number a in $z = a+ ib$ is called the **real part** of z; the real number b is called the **imaginary part** of z.The real and imaginary parts of a complex number z are abbreviated Re(z) and Im(z), respectively.F or example, if $z = 4 - 9i$, then Re(z) = 4 and Im(z) = −9.A real constant multiple of the imaginary unit is called a **pure imaginary number**.F or example, $z = 6i$ is a pure imaginary number.Tw o complex numbers are **equal** if their corresponding real and imaginary parts are equal.

Complex numbers $z1 = a1 + ib1$ and $z2 = a2 + ib2$ are **equal**, $z1 = z2$, if $a1 = a2$ and $b1 = b2$.

In terms of the symbols Re(z) and Im(z), Definition 1.2 states that $z1 = z2$ if Re($z1$) = Re($z2$) and Im($z1$) = Im($z2$).

The totality of complex numbers or the set of complex numbers is usually denoted by the symbol **C**.Because *any* real number a can be written as $z = a + 0i$, we see that the set **R** of real numbers is a subset of **C**.

Arithmetic Operations Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows.

Addition: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Subtraction: $z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

Multiplication:
$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= a_1a_2 - b_1b_2 + i(b_1a_2 + a_1b_2)$$

Division:
$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}, \ a_2 \neq 0, \text{ or } b_2 \neq 0$$
$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$$

The familiar commutative, associative, and distributive laws hold for complex numbers:

Commutative laws:
$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$

Associative laws:
$$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

In view of these laws, there is no need to memorize the definitions of addition, subtraction, and multiplication.

Addition, Subtraction, and Multiplication

(*i*) *To add (subtract) two complex numbers, simply add (subtract) the corresponding real and imaginary parts.*

(*ii*) *To multiply two complex numbers, use the distributive law and the fact that* $i^2 = -1$.

EXAMPLE 1 Addition and Multiplication

If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find (**a**) $z_1 + z_2$ and (**b**) $z_1 z_2$.

Solution (**a**) By adding real and imaginary parts, the sum of the two complex numbers z_1 and z_2 is

$$z_1 + z_2 = (2 + 4i) + (-3 + 8i) = (2 - 3) + (4 + 8)i = -1 + 12i.$$

(**b**) By the distributive law and $i^2 = -1$, the product of z_1 and z_2 is

$$\begin{aligned} z_1 z_2 = (2 + 4i)(-3 + 8i) &= (2 + 4i)(-3) + (2 + 4i)(8i) \\ &= -6 - 12i + 16i + 32i^2 \\ &= (-6 - 32) + (16 - 12)i = -38 + 4i. \end{aligned}$$

Zero and Unity The **zero** in the complex number system is the number $0 + 0i$ and the **unity** is $1 + 0i$. The zero and unity are denoted by 0 and 1, respectively. The zero is the **additive identity** in the complex number system since, for any complex number $z = a + ib$, we have $z + 0 = z$. To see this, we use the definition of addition:

$$z + 0 = (a + ib) + (0 + 0i) = a + 0 + i(b + 0) = a + ib = z.$$

Similarly, the unity is the **multiplicative identity** of the system since, for any complex number z, we have $z \cdot 1 = z \cdot (1 + 0i) = z$.

There is also no need to memorize the definition of division, but before discussing why this is so, we need to introduce another concept.

Conjugate If z is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of z and is denoted by the symbol $\bar{z}$. In other words, if $z = a + ib$, then its conjugate is $\bar{z} = a - ib$. For example, if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$; if $z = -5 - i$, then $\bar{z} = -5 + i$. If z is a real number, say, $z = 7$, then $\bar{z} = 7$. From the definitions of addition and subtraction of complex numbers, it is readily shown that the conjugate of a sum and difference of two complex numbers is the sum and difference of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2. \tag{1}$$

Moreover, we have the following three additional properties:

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad \bar{\bar{z}} = z. \tag{2}$$

Of course, the conjugate of any finite sum (product) of complex numbers is the sum (product) of the conjugates.

The definitions of addition and multiplication show that the sum and product of a complex number z with its conjugate $\bar{z}$ is a real number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a \tag{3}$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2. \tag{4}$$

The difference of a complex number z with its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib. \tag{5}$$

Since $a = \text{Re}(z)$ and $b = \text{Im}(z)$, (3) and (5) yield two useful formulas:

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \qquad \text{and} \qquad \text{Im}(z) = \frac{z - \bar{z}}{2i}. \tag{6}$$

However, (4) is *the* important relationship in this discussion because it enables us to approach division in a practical manner.

Division

To divide z_1 by z_2, multiply the numerator and denominator of z_1/z_2 by the conjugate of z_2. That is,

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} \tag{7}$$

and then use the fact that $z_2\bar{z}_2$ is the sum of the squares of the real and imaginary parts of z_2.

Inverses In the complex number system, every number z has a unique **additive inverse**. As in the real number system, the additive inverse of $z = a + ib$ is its *negative*, $-z$, where $-z = -a - ib$. For any complex number z, we have $z + (-z) = 0$. Similarly, every *nonzero* complex number z has a **multiplicative inverse**. In symbols, for $z \neq 0$ there exists one and only one nonzero complex number z^{-1} such that $zz^{-1} = 1$. The multiplicative inverse z^{-1} is the same as the **reciprocal** $1/z$.

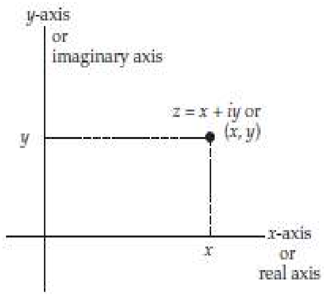


Figure 1.1 z-plane

Complex Plane Because of the correspondence between a complex number $z = x + iy$ and one and only one point (x, y) in a coordinate plane, we shall use the terms *complex number* and *point* interchangeably. The coordinate plane illustrated in Figure 1.1 is called the **complex plane** or simply the **z-plane**. The horizontal or x-axis is called the **real axis** because each point on that axis represents a real number. The vertical or y-axis is called the **imaginary axis** because a point on that axis represents a pure imaginary number.

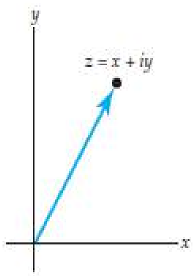


Figure 1.2 z as a position vector

Vectors In other courses you have undoubtedly seen that the numbers in an ordered pair of real numbers can be interpreted as the components of a vector. Thus, a complex number $z = x + iy$ can also be viewed as a two-dimensional position **vector**, that is, a vector whose initial point is the origin and whose terminal point is the point (x, y). See Figure 1.2. This vector interpretation prompts us to define the length of the vector z as the distance $\sqrt{x^2 + y^2}$ from the origin to the point (x, y). This length is given a special name.

Definition 1.3 Modulus

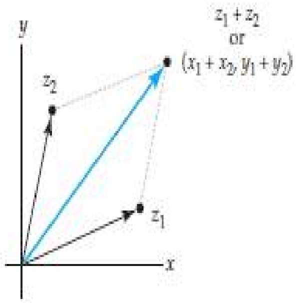
The **modulus** of a complex number $z = x + iy$, is the real number

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

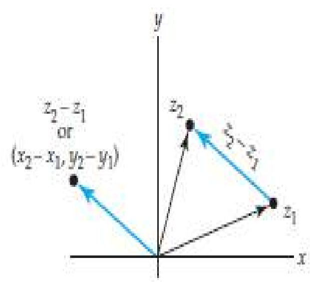
The modulus $|z|$ of a complex number z is also called the **absolute value** of z. We shall use both words *modulus* and *absolute value* throughout this text.

EXAMPLE 1 Modulus of a Complex Number

If $z = 2 - 3i$, then from (1) we find the modulus of the number to be $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. If $z = -9i$, then (1) gives $|-9i| = \sqrt{(-9)^2} = 9$.



(a) Vector sum



(b) Vector difference

Figure 1.3 Sum and difference of vectors

Distance Again The addition of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ given in Section 1.1, when stated in terms of ordered pairs:

$$(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2)$$

is simply the component definition of vector addition. The vector interpretation of the sum $z_1 + z_2$ is the vector shown in Figure 1.3(a) as the main diagonal of a parallelogram whose initial point is the origin and terminal point is $(x_1 + x_2,\ y_1 + y_2)$. The difference $z_2 - z_1$ can be drawn either starting from the terminal point of z_1 and ending at the terminal point of z_2, or as a position vector whose initial point is the origin and terminal point is $(x_2 - x_1,\ y_2 - y_1)$. See Figure 1.3(b). In the case $z = z_2 - z_1$, it follows from (1) and Figure 1.3(b) that the **distance between two points** $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is the same as the distance between the origin and the point $(x_2 - x_1,\ y_2 - y_1)$; that is, $|z| = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)|$ or

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{5}$$

When $z_1 = 0$, we see again that the modulus $|z_2|$ represents the distance between the origin and the point z_2.

EXAMPLE 2 Set of Points in the Complex Plane

Describe the set of points z in the complex plane that satisfy $|z| = |z - i|$.

Solution We can interpret the given equation as equality of distances: The distance from a point z to the origin equals the distance from z to the point

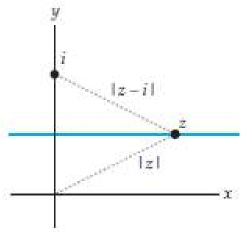


Figure 1.4 Horizontal line is the set of points satisfying $|z| = |z - i|$.

i. Geometrically, it seems plausible from Figure 1.4 that the set of points z lie on a horizontal line. To establish this analytically, we use (1) and (5) to write $|z| = |z - i|$ as:

$$\begin{aligned}\sqrt{x^2 + y^2} &= \sqrt{x^2 + (y-1)^2}\\ x^2 + y^2 &= x^2 + (y-1)^2\\ x^2 + y^2 &= x^2 + y^2 - 2y + 1.\end{aligned}$$

The last equation yields $y = \frac{1}{2}$. Since the equality is true for arbitrary x, $y = \frac{1}{2}$ is an equation of the horizontal line shown in color in Figure 1.4. Complex numbers satisfying $|z| = |z - i|$ can then be written as $z = x + \frac{1}{2}i$.

Inequalities In the Remarks at the end of the last section we pointed out that no order relation can be defined on the system of complex numbers. However, since $|z|$ is a real number, we can compare the absolute values of two complex numbers. For example, if $z_1 = 3 + 4i$ and $z_2 = 5 - i$, then $|z_1| = \sqrt{25} = 5$ and $|z_2| = \sqrt{26}$ and, consequently, $|z_1| < |z_2|$. In view of (1), a geometric interpretation of the last inequality is simple: *The point* $(3, 4)$ *is closer to the origin than the point* $(5, -1)$.

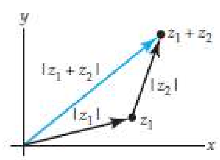


Figure 1.5 Triangle with vector sides

Now consider the triangle given in Figure 1.5 with vertices at the origin, z_1, and $z_1 + z_2$. We know from geometry that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be longer than the sum of the lengths of the remaining two sides. In symbols we can express this observation by the inequality

This inequality can be derived using the properties of complex numbers in Section 1.1. See Problem 50 in Exercises 1.2.

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{6}$$

The result in (6) is known as the **triangle inequality**. Now from the identity $z_1 = z_1 + z_2 + (-z_2)$, (6) gives

$$|z_1| = |z_1 + z_2 + (-z_2)| \le |z_1 + z_2| + |-z_2|.$$

Since $|z_2| = |-z_2|$ (see Problem 47 in Exercises 1.2), solving the last result for $|z_1 + z_2|$ yields another important inequality:

$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{7}$$

But because $z_1 + z_2 = z_2 + z_1$, (7) can be written in the alternative form $|z_1 + z_2| = |z_2 + z_1| \ge |z_2| - |z_1| = -(|z_1| - |z_2|)$ and so combined with the last result implies

$$|z_1 + z_2| \ge \big||z_1| - |z_2|\big|. \tag{8}$$

It also follows from (6) by replacing z_2 by $-z_2$ that $|z_1 + (-z_2)| \le |z_1| + |(-z_2)| = |z_1| + |z_2|$. This result is the same as

$$|z_1 - z_2| \le |z_1| + |z_2|. \tag{9}$$

1.3 Polar Form of Complex Numbers

Recall from calculus that a point P in the plane whose rectangular coordinates are (x, y) can also be described in terms of **polar coordinates**. The polar coordinate system, invented by Isaac Newton, consists of point O called the **pole** and the horizontal half-line emanating from the pole called the **polar axis**. If r is a directed distance from the pole to P and θ is an angle of inclination (in radians) measured from the polar axis to the line OP, then the point can be described by the ordered pair (r, θ), called the polar coordinates of P. See Figure 1.6.

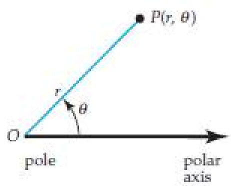


Figure 1.6 Polar coordinates

Polar Form Suppose, as shown in Figure 1.7, that a polar coordinate system is superimposed on the complex plane with the polar axis coinciding with the positive x-axis and the pole O at the origin. Then x, y, r and θ are related by $x = r\cos\theta$, $y = r\sin\theta$. These equations enable us to express a nonzero complex number $z = x + iy$ as $z = (r\cos\theta) + i(r\sin\theta)$ or

$$z = r(\cos\theta + i\sin\theta). \tag{1}$$

We say that (1) is the **polar form** or **polar representation** of the complex number z. Again, from Figure 1.7 we see that the coordinate r can be interpreted as the distance from the origin to the point (x, y). In other words, we shall adopt the convention that *r is never negative*† so that we can take r to

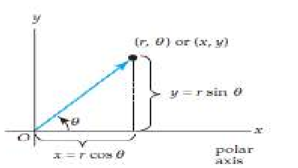


Figure 1.7 Polar coordinates in the complex plane

Be careful using $\tan^{-1}(y/x)$

be the modulus of z, that is, $r = |z|$. The angle θ of inclination of the vector z, which *will always be measured in radians* from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise. The angle θ is called an **argument** of z and is denoted by $\theta = \arg(z)$. An argument θ of a complex number must satisfy the equations $\cos\theta = x/r$ and $\sin\theta = y/r$. An argument of a complex number z is not unique since $\cos\theta$ and $\sin\theta$ are 2π-periodic; in other words, if θ_0 is an argument of z, then necessarily the angles $\theta_0 \pm 2\pi$, $\theta_0 \pm 4\pi, \ldots$ are also arguments of z. In practice we use $\tan\theta = y/x$ to find θ. However, because $\tan\theta$ is π-periodic, some care must be exercised in using the last equation. A calculator will give only angles satisfying $-\pi/2 < \tan^{-1}(y/x) < \pi/2$, that is, angles in the first and fourth quadrants. We have to choose θ consistent with the quadrant in which z is located; this may require adding or subtracting π to $\tan^{-1}(y/x)$ when appropriate. The following example illustrates how this is done.

EXAMPLE 1 A Complex Number in Polar Form

Express $-\sqrt{3} - i$ in polar form.

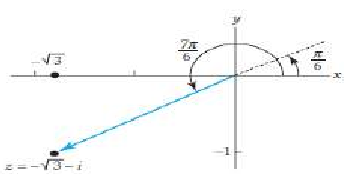


Figure 1.8 $\arg\left(-\sqrt{3} - i\right)$

Solution With $x = -\sqrt{3}$ and $y = -1$ we obtain $r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$. Now $y/x = -1/(-\sqrt{3}) = 1/\sqrt{3}$, and so a calculator gives $\tan^{-1}(1/\sqrt{3}) = \pi/6$, which is an angle whose terminal side is in the first quadrant. But since the point $(-\sqrt{3}, -1)$ lies in the third quadrant, we take the solution of $\tan\theta = -1/(-\sqrt{3}) = 1/\sqrt{3}$ to be $\theta = \arg(z) = \pi/6 + \pi = 7\pi/6$. See Figure 1.8. It follows from (1) that a polar form of the number is

$$z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right). \tag{2}$$

Principal Argument The symbol $\arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta \leq \pi$ is called the **principal value** of $\arg(z)$ or the **principal argument** of z. The principal argument of z is *unique* and is represented by the symbol $\operatorname{Arg}(z)$, that is,

$$-\pi < \operatorname{Arg}(z) \leq \pi.$$

For example, if $z = i$, we see in Figure 1.9 that some values of $\arg(i)$ are $\pi/2$, $5\pi/2$, $-3\pi/2$, and so on, but $\operatorname{Arg}(i) = \pi/2$. Similarly, we see from Figure 1.10 that the argument of $-\sqrt{3} - i$ that lies in the interval $(-\pi, \pi)$, the principal argument of z, is $\operatorname{Arg}(z) = \pi/6 - \pi = -5\pi/6$. Using $\operatorname{Arg}(z)$ we can express the complex number in (2) in the alternative polar form:

$$z = 2\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right].$$

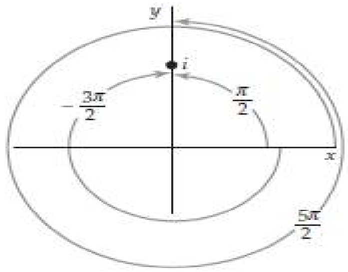


Figure 1.9 Some arguments of i

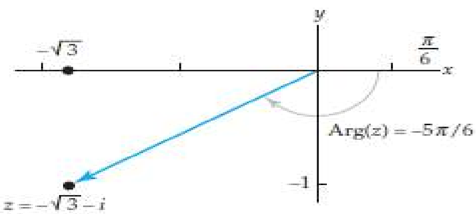


Figure 1.10 Principal argument $z=-\sqrt{3}-i$

In general, arg(z) and Arg(z) are related by

$$\arg(z) = \operatorname{Arg}(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{3}$$

For example, $\arg(i) = \frac{\pi}{2} + 2n\pi$. For the choices $n = 0$ and $n = -1$, (3) gives $\arg(i) = \operatorname{Arg}(i) = \pi/2$ and $\arg(i) = -3\pi/2$, respectively.

Multiplication and Division The polar form of a complex number is especially convenient when multiplying or dividing two complex numbers. Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then

$$z_1 z_2 = r_1 r_2 [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \tag{4}$$

and, for $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]. \tag{5}$$

From the addition formulas‡ for the cosine and sine, (4) and (5) can be rewritten as

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \tag{6}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \tag{7}$$

Inspection of the expressions in (6) and (7) and Figure 1.11 shows that the lengths of the two vectors $z_1 z_2$ and z_1/z_2 are the product of the lengths of z_1 and z_2 and the quotient of the lengths of z_1 and z_2, respectively. See (3) of Section 1.2. Moreover, the arguments of $z_1 z_2$ and z_1/z_2 are given by

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2). \tag{8}$$

de Moivre's Formula When $z = \cos\theta + i\sin\theta$, we have $|z| = r = 1$, and so (9) yields

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{10}$$

This last result is known as **de Moivre's formula** and is useful in deriving certain trigonometric identities involving $\cos n\theta$ and $\sin n\theta$.

EXAMPLE 4 de Moivre's Formula

From (10), with $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$ and $\sin\theta = 1/2$:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3 = \cos 3\theta + i\sin 3\theta = \cos\left(3\cdot\frac{\pi}{6}\right) + i\sin\left(3\cdot\frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$

Roots Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$ are polar forms of the complex numbers z and w. Then, in view of (9) of Section 1.3, the equation $w^n = z$ becomes

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta). \tag{1}$$

From (1), we can conclude that

$$\rho^n = r \tag{2}$$

and
$$\cos n\phi + i\sin n\phi = \cos\theta + i\sin\theta. \tag{3}$$

See Problem 47 in Exercises 1.3.

From (2), we define $\rho = \sqrt[n]{r}$ to be the unique positive nth root of the positive real number r. From (3), the definition of equality of two complex numbers implies that

$$\cos n\phi = \cos\theta \quad \text{and} \quad \sin n\phi = \sin\theta.$$

These equalities, in turn, indicate that the arguments θ and ϕ are related by $n\phi = \theta + 2k\pi$, where k is an integer. Thus,

$$\phi = \frac{\theta + 2k\pi}{n}.$$

As k takes on the successive integer values $k = 0, 1, 2, \ldots, n-1$ we obtain n *distinct* nth roots of z; these roots have the same modulus $\sqrt[n]{r}$ but different arguments. Notice that for $k \geq n$ we obtain the same roots because the sine and cosine are 2π-periodic. To see why this is so, suppose $k = n + m$, where $m = 0, 1, 2, \ldots$. Then

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

and
$$\sin\phi = \sin\left(\frac{\theta + 2m\pi}{n}\right), \quad \cos\phi = \cos\left(\frac{\theta + 2m\pi}{n}\right).$$

We summarize this result. The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \tag{4}$$

where $k = 0, 1, 2, \ldots, n-1$.

EXAMPLE 1 Cube Roots of a Complex Number

Find the three cube roots of $z = i$.

Solution Keep in mind that we are basically solving the equation $w^3 = i$. Now with $r = 1$, $\theta = \arg(i) = \pi/2$, a polar form of the given number is given by $z = \cos(\pi/2) + i\sin(\pi/2)$. From (4), with $n = 3$, we then obtain

$$w_k = \sqrt[3]{1}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right], \quad k = 0, 1, 2.$$

Hence the three roots are,

$$k=0,\quad w_0=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$
$$k=1,\quad w_1=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$
$$k=2,\quad w_2=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i.$$

Principal nth Root On page 17 we pointed out that the symbol arg(z) really stands for a *set* of arguments for a complex number z. Stated another way, for a given complex number $z\neq 0$, arg(z) is *infinite-valued.* In like manner, $z^{1/n}$ is n-valued; that is, the symbol $z^{1/n}$ represents the set of n nth roots w_k of z. The *unique* root of a complex number z (obtained by using the principal value of arg(z) with $k=0$) is naturally referred to as the **principal nth root** of w. In Example 1, since Arg(i) $=\pi/2$, we see that $w_0=\frac{1}{2}\sqrt{3}+\frac{1}{2}i$ is the principal cube root of i. The choice of Arg(z) and $k=0$ guarantees us that when z is a positive real number r, the principal nth root is $\sqrt[n]{r}$.

$\sqrt{4}=2$ and $\sqrt[3]{27}=3$ are the principal square root of 4 and the principal cube root of 27, respectively.

Since the roots given by (4) have the same modulus, the n nth roots of a nonzero complex number z lie on a circle of radius $\sqrt[n]{r}$ centered at the origin in the complex plane. Moreover, since the difference between the arguments of any two successive roots w_k and w_{k+1} is $2\pi/n$, the n nth roots of z are equally spaced on this circle, beginning with the root whose argument is θ/n. Figure 1.12 shows the three cube roots of i obtained in Example 1 spaced at equal angular intervals of $2\pi/3$ on the circumference of a unit circle beginning with the root w_0 whose argument is $\pi/6$.

As the next example shows, the roots of a complex number do not have to be "nice" numbers as in Example 1.

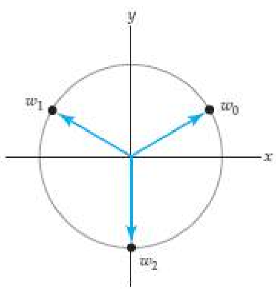


Figure 1.12 Three cube roots of i

EXAMPLE 2 Fourth Roots of a Complex Number

Find the four fourth roots of $z=1+i$.

Solution In this case, $r=\sqrt{2}$ and $\theta=\arg(z)=\pi/4$. From (4) with $n=4$, we obtain

$$w_k=\sqrt[4]{2}\left[\cos\left(\frac{\pi/4+2k\pi}{4}\right)+i\sin\left(\frac{\pi/4+2k\pi}{4}\right)\right],\quad k=0,\ 1,\ 2,\ 3.$$

With the aid of a calculator we find

$$k=0,\quad w_0=\sqrt[4]{2}\left[\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right]\approx 1.1664+0.2320i$$
$$k=1,\quad w_1=\sqrt[4]{2}\left[\cos\frac{9\pi}{16}+i\sin\frac{9\pi}{16}\right]\approx -0.2320+1.1664i$$
$$k=2,\quad w_2=\sqrt[4]{2}\left[\cos\frac{17\pi}{16}+i\sin\frac{17\pi}{16}\right]\approx -1.1664-0.2320i$$
$$k=3,\quad w_3=\sqrt[4]{2}\left[\cos\frac{25\pi}{16}+i\sin\frac{25\pi}{16}\right]\approx 0.2320-1.1664i.$$

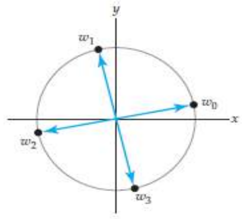


Figure 1.13 Four fourth roots of $1 + i$

As shown in Figure 1.13, the four roots lie on a circle centered at the origin of radius $r = \sqrt[4]{2} \approx 1.19$ and are spaced at equal angular intervals of $2\pi/4 = \pi/2$ radians, beginning with the root whose argument is $\pi/16$.

STEREOGRAPHIC PROJECTION

Since we have a notion of distance (*i.e.*, $d(z, w) = |z - w|$) in $\mathbb{C}$ we may view $\mathbb{C}$ as a metric space. It is clear that this space is *complete* in the sense that any Cauchy sequence converges; to see this note that since $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$ for any $z \in \mathbb{C}$ it follows that if $z_j = x_j + iy_j$, $j = 1, 2, \ldots$ is a Cauchy sequence in $\mathbb{C}$, then x_j, $j = 1, 2, \ldots$ and y_j, $j = 1, 2, \ldots$ are Cauchy sequences in $\mathbb{R}$. Furthermore, if $x_j \to x \in \mathbb{R}$ and $y_j \to y \in \mathbb{R}$ as $j \to \infty$, then $x_j + iy_j \to x + iy \in \mathbb{C}$ as $j \to \infty$. Thus the completeness of $\mathbb{C}$ follows from that of $\mathbb{R}$.

From the point of view of topology, it would be even better if $\mathbb{C}$ were *compact*, *i.e.*, any open cover of $\mathbb{C}$ should have a finite subcover. This is not true, however, as can be seen by considering the open cover of $\mathbb{C}$ consisting of all open balls $|z| < R$ centered at 0, which obviously has no finite subcover. One can make $\mathbb{C}$ compact without changing its topology by adding (at least) one 'ideal' point and modifying the metric. This *one-point compactification* of the complex plane is very important in the theory of functions of a complex variable and we will give a very enlightening geometric interpretation of it in this section.

Imagine $\mathbb{C}$ as the x_1x_2-plane in $\mathbb{R}^3$ and let S_2 be the unit sphere; it will intersect $\mathbb{C}$ along the unit circle. Call the point $(0, 0, 1)$ on the sphere the *North pole* N (so that $(0, 0, -1)$ is the South pole). We can map $\mathbb{C}$ in a one-to-one fashion onto $S_2 \setminus \{N\}$ by mapping $z \in \mathbb{C}$ onto the point $(x_1, x_2, x_3) \in S_2$ such that the straight line connecting z with N goes through (x_1, x_2, x_3). This map is called *stereographic projection* and has many interesting properties, as we shall see. In this connection S_2 is called the *Riemann sphere*.

It is nearly obvious that this stereographic projection is a bi-continuous map, using the topology induced by the metric of $\mathbb{R}^3$. To make absolutely sure, let us find the mapping explicitly. The line through N and $z = x + iy \in \mathbb{C}$ is $(x_1, x_2, x_3) = (0, 0, 1) + t(x, y, -1)$. The intersection with S_2 is given by t satisfying $t^2(x^2 + y^2) + (1 - t)^2 = 1$ which gives $t = 0$, *i.e.*, N, and the more interesting $t = 2/(x^2 + y^2 + 1)$.

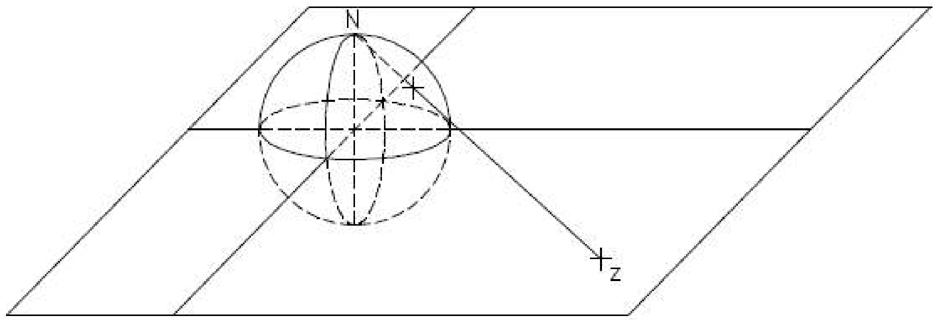


FIGURE 1. Stereographic projection

We therefore get

$$\begin{cases} x_1 = \dfrac{2\,\mathrm{Re}\,z}{|z|^2+1} \\ x_2 = \dfrac{2\,\mathrm{Im}\,z}{|z|^2+1} \\ x_3 = \dfrac{|z|^2-1}{|z|^2+1} \end{cases} .$$

Since $|z|^2 + 1 = 2/(1 - x_3)$ by the third equation the inverse is easily seen to be given by

$$z = \frac{x_1 + ix_2}{1 - x_3} .$$

It is clear that these maps are both continuous (note that $(x_1, x_2, x_3) \in S_2 \setminus \{N\}$ so $x_3 \neq 1$). We may now introduce a new metric in $\mathbb{C}$ by setting the distance between points in $\mathbb{C}$ equal to the Euclidean distance between their image points on S_2.

EXERCISE 1.12. Show that this metric is given by

$$d(z, w) = 2\frac{|z - w|}{(|z|^2 + 1)^{1/2}(|w|^2 + 1)^{1/2}}.$$

Also show that the distance between the image of z and N is $\frac{2}{(|z|^2+1)^{1/2}}$.

In view of Exercise 1.12 we may now add to $\mathbb{C}$ an 'ideal' point ∞, the image of which under stereographic projection is N. This new set is called the *extended complex plane* and we denote it by $\mathbb{C}^*$. Using the metric of Exercise 1.12 in $\mathbb{C}^*$ the extended plane becomes homeomorphic to the Riemann sphere with the topology of Euclidean distance. Since S_2 is compact, so is the extended plane; we have compactified the plane. For the statement of the next theorem, note that a circle in S^2 is the intersection of S^2 by a non-tangential plane, and any such (non-empty) intersection is a circle.

UNIT-I

1. If Z_1 and Z_2 are any two complex numbers ,then prove that $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$.

ii) Show that every complex numbers z whose absolute value is 1, can be expressed in the form z = (1+it)/(1-it),t is a real number.

2. Explain the Stereographic projection of a complex plane.

3. i) Show that the argument of the product of two complex numbers is the sum of the arguments of the complex numbers.

ii) Show that, if $|Z| < ½$ then $|(1+i) z^3+iz| < ¾$

4. Show that stereographic projection maps circles on the Riemann sphere onto circles or Straight lines in the complex plane.

5. i) If Z_1 and Z_2 are any two complex numbers ,then prove that $(\overline{Z_1 Z_2}) = \overline{Z_1}.\overline{Z_2}$.

ii) If Z_1 and Z_2 are any two complex numbers, then $arg\left(\frac{Z_1}{Z_2}\right) = argZ_1 - argZ_2$

6. If Z, Z_1 and Z_2 are any three complex numbers, then prove that

i) $-|Z| \leq Re Z \leq |Z|$, $-|Z| \leq Im Z \leq |Z|$,

ii) $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$

iii) $|Z_1 - Z_2| \geq ||Z_1| - |Z_2||$

7. If Z_1 and Z_2 are the images in the complex plane of two diametrically opposite points on the Riemann sphere,show that $Z_1 \overline{Z_2} = -1$

8. Explain the transformation w=1/z.

9. Explain about the transformation w=az.

10. If z,z_1,z_2 are any three complex number then prove that i) $|z| = |\overline{z}|$ ii) $z\overline{z} = |z|^2$

iii) $|z_1 z_2| = |z_1||z_2|$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMPLEX ANLAYSIS-I **Subject Code: 15MMU502**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
The multiplicative identity of complex number is	(0,1)	(1,0)	(0,0)	(0,1)	(1,0)
The inverse of (α,β) under addition is	(-α,β)	(-α,-β)	(α,β)	(α,-β)	(-α,-β)
$\lvert Z_1 . Z_2\rvert$=	$\lVert z_1\rVert \lVert z_2\rVert$	$\lvert z_1\rvert \lVert z_2\rVert$	$\lvert z_1\rvert \lvert z_2\rvert$	$\lvert z_1\rvert + \lvert z_2\rvert$	$\lvert z_1\rvert \lvert z_2\rvert$
The value of i^2 is	1	-1	0	i	-1
If Z_1 and Z_2 are any two complex numbers ,then	$\arg(Z_1Z_2) = \arg(Z_1)+\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)-\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)/\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)*\arg(Z_2)$	$\arg(Z_1Z_2) = \arg(Z_1)/\arg(Z_2)$
The Equation of the unit sphere is	$x^2+y^2+z^2=1$	$x^2+y^2+z^2=2$	$x^2-y^2+z^2=1$	$x^2-y^2-z^2=1$	$x^2+y^2+z^2=1$
The element (1,0) is the -------	Additive identity	Multiplicative identity	identity	unique	Multiplicative identity
The element (0,0) is the ----------	Additive identity	Multiplicative identity	identity	unique	Additive identity
If $\lvert Z_1\rvert = \lvert Z_2\rvert$ and $\arg(Z_1)= \arg(Z_2)$ then -----	$Z_1 \neq Z_2$	$Z_1 < Z_2$	$Z_1 > Z_2$	$Z_1 = Z_2$	$Z_1 = Z_2$
The Equation of the unit circle whose centre is the origin is	$\lvert Z\rvert =1$	$\lvert Z-a\rvert =1$	$\lvert Z\rvert =0$	$\lvert Z\rvert \neq 1$	$\lvert Z\rvert =1$
The complex plane containing all the finite complex numbers and infinity is called the	infinite complex plane	extended complex plane	complex plane	finite complex plane	extended complex plane
The inversion w = 1/z maps the region $\lvert z\rvert < 1$ into the region	$\lvert w\rvert <1$	$\lvert w\rvert >1$	$\lvert w\rvert =1$	$\lvert w\rvert \leq 1$	$\lvert w\rvert >1$
The square of real number is ---------	Non negative	Non positive	Negative	absolute value	absolute value
The absolute value of z = x+iy is	√x	√y	$\sqrt{x^2-y^2}$	$\sqrt{x^2+y^2}$	$\sqrt{x^2+y^2}$
If Z_1 and Z_2 are any two complex numbers ,then	$\lvert Z_1 +Z_2\rvert$£$\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 +Z_2\rvert=\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 +Z_2\rvert$³$\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 +Z_2\rvert\neq\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 +Z_2\rvert$£$\lvert Z_1\rvert+\lvert Z_2\rvert$
The mapping W=1/Z is called an	Linear transformation	Translation	Inversion	Rotation	Inversion
The polar form of x+iy is	r(cos q +isinq)	r(cos q -isinq)	cos q +isinq	r(cos q - sinq)	r(cos q +isinq)
If Z_1 and Z_2 are any two complex numbers ,then	$\lvert Z_1 -Z_2\rvert$£$\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 -Z_2\rvert=\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 -Z_2\rvert$³$\lvert\,\lvert Z_1\rvert-\lvert Z_2\rvert\,\rvert$	$\lvert Z_1 -Z_2\rvert\neq\lvert Z_1\rvert+\lvert Z_2\rvert$	$\lvert Z_1 -Z_2\rvert$³$\lvert\,\lvert Z_1\rvert-\lvert Z_2\rvert\,\rvert$
The complex plane containing all the finite complex numbers is called the	infinite complex plane	extended complex plane	complex plane	finite complex plane	finite complex plane
The conjugation of 5+i3 is	5	3	5+i3	5-i3	5-i3
If Z_1 and Z_2 are any two complex numbers ,then	arg(Z1/Z2) = $\arg(Z_1)+\arg(Z_2)$	$\arg(Z_1/Z_2) = \arg(Z_1)-\arg(Z_2)$	$\arg(Z_1/Z_2) = \arg(Z_1)/\arg(Z_2)$	$\arg(Z_1/Z_2) = \arg(Z_1)*\arg(Z_2)$	$\arg(Z_1/Z_2) = \arg(Z_1)-\arg(Z_2)$
The mapping W=Z+b ,b is a complex number, is called the	Linear transformation	Translation	Inversion	Rotation	Translation
All the complex numbers except infinity are called	Complex numbers	Complex plane	finite complex numbers	infinite complex numbers	finite complex numbers
If x= rcosθ , y = rsinθ then for z we get	z= rcosθ+ r sinθ	z= rsinθ + ircosθ	z= rcosθ+irsinθ	z= rcosθ-irsinθ	z= rcosθ+irsinθ
The angle made by the vector (x,y)measured in the anticlockwise direction is	mod of z	norm of z	argument of z	0	argument of z
The argument θ is ------------- as it can take infinite values	unique	not unique	finite	infinite	not unique
From x= rcosθ and y = rsinθ weget θ =	$\sin^{-1} y/x$	$\cos^{-1}y/x$	$\tan^{-1}y/x$	$\cot^{-1}y/x$	$\tan^{-1}y/x$
$\arg \bar{z}$	arg z	- arg z	arg(-z)	arg 1/z	- arg z
The argument of the product of two complex numbers is---- of the complex number	The sum of the arguments	the argument of the sum	the argument of the division	the product of the arguments	the argument of the sum
arg ((z1. Z2)					
arg (z1 . z2)	$\arg z_1+ \arg z_2$	$\arg z_1 . \arg z_2$	$\arg z_1/\arg z_2$	$\arg(z_1 +z_2)$	$\arg z_1 . \arg z_2$
The cross ratio of the form…..	$(z_1-z_2)(z_2-z_4)/(z_1-z_4)(z_2-z_3)$	$(z_1-z3)(z_2-z_4)/(z_1-z_4)(z_2-z_3)$	$(z_1-z_2)(z_2-z_4)/(z_1-z_4)$	$(z_1-z_2)/(z_1-z_4)(z_2-z_3)$	$(z_1-z3)(z_2-z_4)/(z_1-z_4)(z_2-z_3)$
If z = -1 + i , then z-1 = ……..	-1+i	-1-i	(-1)/2 + i 1/2	(-1)/2 - i 1/2	-1-i
The stereographic projection of the complex point z = (√2 ,1) is	(1/√2, 1/√2, 0)	(0, √2, 1)	(1/√2, 1/2 , 1/2)	(0, 0,1)	(1/√2, 1/2 , 1/2)
The inversion w = 1/z maps the region $\lvert z\rvert >1$ into the region	$\lvert w\rvert <1$	$\lvert w\rvert >1$	$\lvert w\rvert =1$	$\lvert w\rvert \leq 1$	$\lvert w\rvert <1$
Under the transformation w = az there are ------ fixed points	one	two	zero	∞	two
According to De Moivre's theorem $(\cos θ +i\sin θ)^n$ = ……………….	$\cos^n θ + i\sin^n θ$	cosn θ+isinn θ	ncos θ+insin θ	1	cosn θ+isinn θ
The transformation w = az=b , where a, b are complex constants ,is a composition of …… tranformations	Rotation and Homothetic	Translation and Rotation 1/2	Rotation , Homothetic and Translation	Homothetic and Translation	Rotation , Homothetic and Translation
The fixed points for w = (2z-1) / (z+3) are	0,∞		2/3	-1/2+i(√3/2) , -1/2-i(√3/2)	-1/2+i(√3/2) , -1/2-i(√3/2)
The equation $z\bar{z} + \bar{a}z + a\bar{z} + c =0$, where c is real and a is complex , is a equation of a	Line	Ray	Ellipse	circle	circle

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT OF MATHEMATICS

Subject:COMPLEX ANALYSISI-I	**Semester: V**	**L T P C**
Subject Code: 15MMU502	**Class: III-B.Sc Mathematics**	**5 0 0 5**

UNIT II

Analytic functions: Limit of a function –continuity –differentiability – Analytical function defined in a region –necessary conditions for differentiability –sufficient conditions for differentiability –Cauchy-Riemann equation in polar coordinates –Definition of entire function.

TEXT BOOK

1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

REFERENCES

1. Ponnusamy.S., 2004. Foundations of Complex Analysis, Narosa Publishing House, Chennai.
2. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.
3. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

Limits

Real Limits The description of a real limit given in the section introduction is only an intuitive definition of this concept. In order to give the rigorous definition of a real limit, we must precisely state what is meant by the phrases "arbitrarily close to" and "sufficiently close to." The first thing to

recognize is that a precise statement of these terms should involve the use of absolute values since $|a-b|$ measures the distance between two points on the real number line. On the real line, the points x and x_0 are close if $|x-x_0|$ is a small positive number. Similarly, the points $f(x)$ and L are close if $|f(x)-L|$ is a small positive number. In mathematics, it is customary to let the Greek letters ε and δ represent small positive real numbers. Hence, the expression "$f(x)$ can be made arbitrarily close to L" can be made precise by stating that for any real number $\varepsilon > 0$, x can be chosen so that $|f(x)-L| < \varepsilon$. In our intuitive definition we require that $|f(x)-L| < \varepsilon$ whenever values of x are "sufficiently close to, but not equal to, x_0." This means that there is some distance $\delta > 0$ with the property that if x is within distance δ of x_0 and $x \neq x_0$, then $|f(x)-L| < \varepsilon$. In other words, if $0 < |x-x_0| < \delta$, then $|f(x)-L| < \varepsilon$. The real number δ is not unique and, in general, depends on the choice of ε, the function f, and the point x_0. In summary, we have the following precise definition of the real limit:

Limit of a Real Function $f(x)$

The limit of f as x tends x_0 exists and is equal to L

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x)-L| < \varepsilon$ (1)

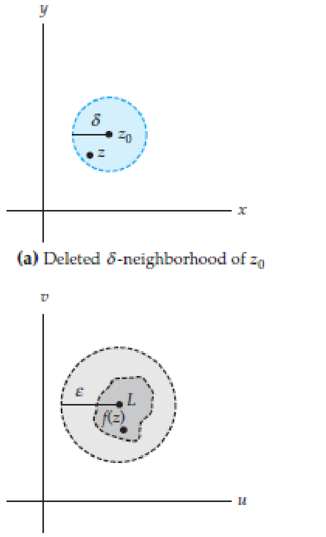
whenever $0 < |x-x_0| < \delta$.

The geometric interpretation of (1) is shown in Figure 2.50. In this figure we see that the graph of the function $y = f(x)$ over the interval $(x_0-\delta,\ x_0+\delta)$, excluding the point x_0, lies between the lines $y = L-\varepsilon$ and $y = L+\varepsilon$ shown dashed in Figure 2.50. In the terminology of mappings, the interval $(x_0-\delta,\ x_0+\delta)$, excluding the point $x = x_0$, shown in color on the x-axis is mapped onto the set shown in black in the interval $(L-\varepsilon,\ L+\varepsilon)$ on the y-axis. For the limit to exist, the relationship exhibited in Figure 2.50 must exist for any choice of $\varepsilon > 0$. We also see in Figure 2.50 that if a smaller ε is chosen, then a smaller δ may be needed.

Complex Limits A complex limit is, in essence, the same as a real limit except that it is based on a notion of "close" in the complex plane. Because the distance in the complex plane between two points z_1 and z_2 is given by the modulus of the difference of z_1 and z_2, the precise definition of a complex limit will involve $|z_2-z_1|$. For example, the phrase "$f(z)$ can be made arbitrarily close to the complex number L," can be stated precisely as: for every $\varepsilon > 0$, z can be chosen so that $|f(z)-L| < \varepsilon$. Since the modulus of a complex number is a *real* number, both ε and δ still represent small positive *real* numbers in the following definition of a complex limit. The complex analogue of (1) is:

Definition 2.8 Limit of a Complex Function

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The **limit of f as z tends to z_0 exists and is equal to L**, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



(a) Deleted δ-neighborhood of z_0

(b) ε-neighborhood of L

Figure 2.51 The geometric meaning of a complex limit

Because a complex function f has no graph, we rely on the concept of complex mappings to gain a geometric understanding of Definition 2.8. Recall from Section 1.5 that the set of points w in the complex plane satisfying $|w - L| < \varepsilon$ is called a neighborhood of L, and that this set consists of all points in the complex plane lying within, but not on, a circle of radius ε centered at the point L. Also recall from Section 1.5 that the set of points satisfying the inequalities $0 < |z - z_0| < \delta$ is called a deleted neighborhood of z_0 and consists of all points in the neighborhood $|z - z_0| < \delta$ excluding the point z_0. By Definition 2.8, if $\lim_{z \to z_0} f(z) = L$ and if ε is any positive number, then there is a deleted neighborhood of z_0 of radius δ with the property that for every point z in this deleted neighborhood, $f(z)$ is in the ε neighborhood of L. That is, f maps the deleted neighborhood $0 < |z - z_0| < \delta$ in the z-plane into the neighborhood $|w - L| < \varepsilon$ in the w-plane. In Figure 2.51(a), the deleted neighborhood of z_0 shown in color is mapped onto the set shown in dark gray in Figure 2.51(b). As required by Definition 2.8, the image lies within the ε-neighborhood of L shown in light gray in Figure 2.51(b).

Complex and real limits have many common properties, but there is at least one very important difference. For real functions, $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{x \to x_0^+} f(x) = L$ and $\lim_{x \to x_0^-} f(x) = L$. That is, there are two directions from which x can approach x_0 on the real line, from the right (denoted by $x \to x_0^+$) or from the left (denoted by $x \to x_0^-$). The real limit exists if and only if these two one-sided limits have the same value. For example, consider the real function defined by:

$$f(x) = \begin{cases} x^2, & x < 0 \\ x - 1, & x \geq 0 \end{cases}.$$

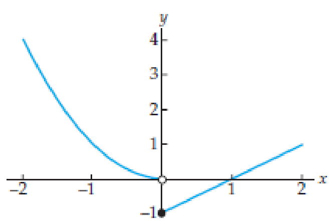


Figure 2.52 The limit of f does not exist as x approaches 0.

The limit of f as x approaches to 0 does not exist since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$, but $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x - 1) = -1$. See Figure 2.52.

For limits of complex functions, z is allowed to approach z_0 from *any* direction in the complex plane, that is, along any curve or path through z_0. See Figure 2.53. In order that $\lim_{z \to z_0} f(z)$ exists and equals L, we require that $f(z)$ approach the same complex number L along every possible curve through z_0. Put in a negative way:

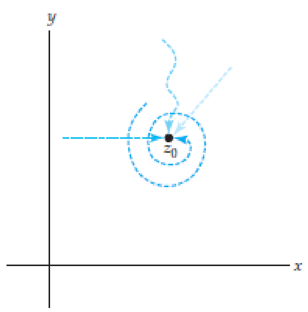


Figure 2.53 Different ways to approach z_0 in a limit

Furthermore, for $z_0 = 1 - i$ we have:

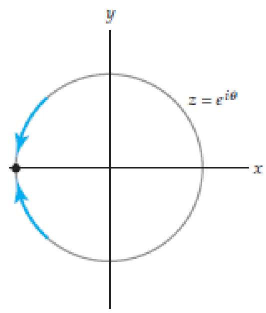
$$f(z_0) = f(1-i) = (1-i)^2 - i(1-i) + 2 = 1 - 3i.$$

Since $\lim_{z \to z_0} f(z) = f(z_0)$, we conclude that $f(z) = z^2 - iz + 2$ is continuous at the point $z_0 = 1 - i$.

As Example 5 indicates, the continuity of complex polynomial and rational functions is easily determined using Theorem 2.2 and the limits in (15) and (16). More complicated functions, however, often require other techniques.

EXAMPLE 6 Discontinuity of Principal Square Root Function

Show that the principal square root function $f(z) = z^{1/2}$ defined by (7) of Section 2.4 is discontinuous at the point $z_0 = -1$.



re 2.54 Figure for Example 6

Solution We show that $f(z) = z^{1/2}$ is discontinuous at $z_0 = -1$ by demonstrating that the limit $\lim_{z\to z_0} f(z) = \lim_{z\to -1} z^{1/2}$ does not exist. In order to do so, we present two ways of letting z approach -1 that yield different values of this limit. Before we begin, recall from (7) of Section 2.4 that the principal square root function is defined by $z^{1/2} = \sqrt{|z|}e^{i\text{Arg}(z)/2}$. Now consider z approaching -1 along the quarter of the unit circle lying in the second quadrant. See Figure 2.54. That is, consider the points $|z| = 1$, $\pi/2 < \arg(z) < \pi$. In exponential form, this approach can be described as $z = e^{i\theta}$, $\pi/2 < \theta < \pi$, with θ approaching π. Thus, by setting $|z| = 1$ and letting $\text{Arg}(z) = \theta$ approach π, we obtain:

$$\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|}e^{i\text{Arg}(z)/2} = \lim_{\theta\to\pi} \sqrt{1}e^{i\theta/2}.$$

However, since $e^{i\theta/2} = \cos(\theta/2) + i\sin(\theta/2)$, this simplifies to:

$$\lim_{z\to -1} z^{1/2} = \lim_{\theta\to\pi}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i. \quad (18)$$

Next, we let z approach -1 along the quarter of the unit circle lying in the third quadrant. Again refer to Figure 2.54. Along this curve we have the points $z = e^{i\theta}$, $-\pi < \theta < -\pi/2$, with θ approaching $-\pi$. By setting $|z| = 1$ and letting $\text{Arg}(z) = \theta$ approach $-\pi$ we find:

$$\lim_{z\to -1} z^{1/2} = \lim_{z\to -1} \sqrt{|z|}e^{i\text{Arg}(z)/2} = \lim_{\theta\to-\pi} e^{i\theta/2} = \lim_{\theta\to-\pi}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = -i. \quad (19)$$

Because the complex values in (18) and (19) do not agree, we conclude that $\lim_{z\to -1} z^{1/2}$ does not exist. Therefore, the principal square root function $f(z) = z^{1/2}$ is discontinuous at the point $z_0 = -1$.

Properties of Continuous Functions Because the concept of continuity is defined using the complex limit, various properties of complex limits can be translated into statements about continuity. Consider Theorem 2.1, which describes the connection between the complex limit of $f(z) = u(x,\ y)+iv(x,\ y)$ and the real limits of u and v. Using the following definition of continuity for real functions $F(x,\ y)$, we can restate this theorem about limits as a theorem about continuity.

Continuity of a Real Function $F(x, y)$

A function F is continuous at a point $(x_0,\ y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} F(x,\ y) = F(x_0,\ y_0). \tag{20}$$

Again, this definition of continuity is analogous to (17). From (20) and Theorem 2.1, we obtain the following result.

Theorem 2.3 Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x,\ y) + iv(x,\ y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point $(x_0,\ y_0)$.

Proof Assume that the complex function f is continuous at z_0. Then from Definition 2.9 we have:

$$\lim_{z\to z_0} f(z) = f(z_0) = u(x_0,\ y_0) + iv(x_0,\ y_0). \tag{21}$$

By Theorem 2.1, this implies that:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,\ y) = u(x_0,\ y_0) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,\ y) = v(x_0,\ y_0). \tag{22}$$

Therefore, from (20), both u and v are continuous at $(x_0,\ y_0)$. Conversely, if u and v are continuous at $(x_0,\ y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} u(x,\ y) = u(x_0, y_0) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,\ y) = v(x_0,\ y_0).$$

The Derivative Suppose $z = x+iy$ and $z_0 = x_0+iy_0$; then the change in z_0 is the difference $\Delta z = z - z_0$ or $\Delta z = x - x_0 + i(y - y_0) = \Delta x + i\Delta y$. If a complex function $w = f(z)$ is defined at z and z_0, then the corresponding change in the function is the difference $\Delta w = f(z_0 + \Delta z) - f(z_0)$. The **derivative** of the function f is defined in terms of a limit of the difference quotient $\Delta w/\Delta z$ as $\Delta z \to 0$.

Definition 3.1 Derivative of Complex Function

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1}$$

provided this limit exists.

If the limit in (1) exists, then the function f is said to be **differentiable** at z_0. Two other symbols denoting the derivative of $w = f(z)$ are w' and dw/dz. If the latter notation is used, then the value of a derivative at a specified point z_0 is written $\left.\dfrac{dw}{dz}\right|_{z=z_0}$.

EXAMPLE 1 Using Definition 3.1

Use Definition 3.1 to find the derivative of $f(z) = z^2 - 5z$.

Solution Because we are going to compute the derivative of f at any point, we replace z_0 in (1) by the symbol z. First,

$$f(z + \Delta z) = (z + \Delta z)^2 - 5(z + \Delta z) = z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z.$$

Second,

$$\begin{aligned} f(z + \Delta z) - f(z) &= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z - (z^2 - 5z) \\ &= 2z\Delta z + (\Delta z)^2 - 5\Delta z. \end{aligned}$$

Then, finally, (1) gives

$$f'(z) = \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z - 5).$$

The limit is $f'(z) = 2z - 5$.

Rules of Differentiation The familiar rules of differentiation in the calculus of real variables carry over to the calculus of complex variables. If f and g are differentiable at a point z, and c is a complex constant, then (1) can be used to show:

Differentiation Rules

Constant Rules: $\frac{d}{dz}c = 0$ and $\frac{d}{dz}cf(z) = cf'(z)$ (2)

Sum Rule: $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$ (3)

Product Rule: $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$ (4)

Quotient Rule: $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ (5)

Chain Rule: $\frac{d}{dz}f(g(z)) = f'(g(z))\,g'(z).$ (6)

The **power rule** for differentiation of powers of z is also valid:

$$\frac{d}{dz}z^n = nz^{n-1}, \quad n \text{ an integer.} \tag{7}$$

Combining (7) with (6) gives the **power rule for functions**:

$$\frac{d}{dz}[g(z)]^n = n[g(z)]^{n-1}g'(z), \quad n \text{ an integer.} \tag{8}$$

EXAMPLE 2 Using the Rules of Differentiation

Differentiate:

(a) $f(z) = 3z^4 - 5z^3 + 2z$ **(b)** $f(z) = \dfrac{z^2}{4z+1}$ **(c)** $f(z) = \left(iz^2 + 3z\right)^5$

Solution

(a) Using the power rule (7), the sum rule (3), along with (2), we obtain

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 \cdot 1 = 12z^3 - 15z^2 + 2.$$

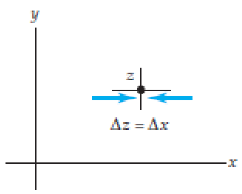
(b) From the quotient rule (5),

$$f'(z) = \frac{(4z+1) \cdot 2z - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}.$$

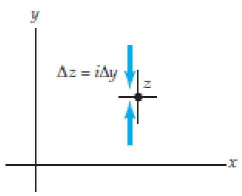
(c) In the power rule for functions (8) we identify $n = 5$, $g(z) = iz^2 + 3z$, and $g'(z) = 2iz + 3$, so that

$$f'(z) = 5(iz^2 + 3z)^4 (2iz + 3).$$

For a complex function f to be differentiable at a point z_0, we know from the preceding chapter that the limit $\lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ must exist and equal the same complex number from any direction; that is, the limit must exist regardless how Δz approaches 0. This means that in complex analysis, the requirement of differentiability of a function $f(z)$ at a point z_0 is a far greater demand than in real calculus of functions $f(x)$ where we can approach a real number x_0 on the number line from only two directions. If a complex function is made up by specifying its real and imaginary parts u and v, such as $f(z) = x + 4iy$, there is a good chance that it is not differentiable.



(a) $\Delta z \to 0$ along a line parallel to x-axis



(b) $\Delta z \to 0$ along a line parallel to y-axis

Figure 3.1 Approaching z along a horizontal line and then along a vertical line

EXAMPLE 3 A Function That Is Nowhere Differentiable

Show that the function $f(z) = x + 4iy$ is not differentiable at any point z.

Solution Let z be any point in the complex plane. With $\Delta z = \Delta x + i\Delta y$,

$$f(z + \Delta z) - f(z) = (x + \Delta x) + 4i(y + \Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

and so
$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}. \tag{9}$$

Now, as shown in Figure 3.1(a), if we let $\Delta z \to 0$ along a line parallel to the x-axis, then $\Delta y = 0$ and $\Delta z = \Delta x$ and

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1. \tag{10}$$

Analytic Functions Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called **analytic functions**.

Definition 3.2 Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

A function f **is analytic in a domain** D if it is analytic at every point in D. The phrase "analytic *on* a domain D" is also used. Although we shall not use these terms in this text, a function f that is analytic throughout a domain D is called **holomorphic** or **regular**.

Very Important ☞ You should reread Definition 3.2 carefully. Analyticity at a point is *not* the same as differentiability at a point. Analyticity *at a point is a neighborhood property*; in other words, analyticity is a property that is defined over an open set. It is left as an exercise to show that the function $f(z) = |z|^2$ is differentiable at $z = 0$ but is not differentiable anywhere else. Even though $f(z) = |z|^2$ is differentiable at $z = 0$, it is not analytic at that point because there exists no neighborhood of $z = 0$ *throughout which* f is differentiable; hence the function $f(z) = |z|^2$ is nowhere analytic. See Problem 19 in Exercises 3.1.

In contrast, the simple polynomial $f(z) = z^2$ is differentiable at every point z in the complex plane. Hence, $f(z) = z^2$ is analytic everywhere.

Entire Functions A function that is analytic at every point z in the complex plane is said to be an **entire function**. In view of differentiation rules (2), (3), (7), and (5), we can conclude that polynomial functions are differentiable at every point z in the complex plane and rational functions are analytic throughout any domain D that contains no points at which the denominator is zero. The following theorem summarizes these results.

Theorem 3.1 Polynomial and Rational Functions

(*i*) A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a nonnegative integer, is an entire function.

(*ii*) A rational function $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Singular Points Since the rational function $f(z) = 4z/\left(z^2 - 2z + 2\right)$ is discontinuous at $1+i$ and $1-i$, f fails to be analytic at these points. Thus by (*ii*) of Theorem 3.1, f is not analytic in any domain containing one or both of these points. In general, a point z at which a complex function $w = f(z)$ fails to be analytic is called a **singular point** of f. We will discuss singular points in greater depth in Chapter 6.

$\pm i$ are zeros of the denominator f f.

If the functions f and g are analytic in a domain D, it can be proved that:

Analyticity of Sum, Product, and Quotient

The sum $f(z) + g(z)$, difference $f(z) - g(z)$, and product $f(z)g(z)$ are analytic. The quotient $f(z)/g(z)$ is analytic provided $g(z) \neq 0$ in D.

An Alternative Definition of $f'(z)$ Sometimes it is convenient to define the derivative of a function f using an alternative form of the difference quotient $\Delta w/\Delta z$. Since $\Delta z = z - z_0$, then $z = z_0 + \Delta z$, and so (1) can be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{12}$$

In contrast to what we did in Example 1, if we wish to compute f' at a general point z using (12), then we replace z_0 by the symbol z *after* the limit is computed. See Problems 7–10 in Exercises 3.1.

As in real analysis, if a function f is differentiable at a point, the function is necessarily continuous at the point. We use the form of the derivative given in (12) to prove the last statement.

Theorem 3.2 Differentiability Implies Continuity

If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

Proof The limits $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ and $\lim_{z \to z_0} (z - z_0)$ exist and equal $f'(z_0)$ and 0, respectively. Hence by Theorem 2.2(*iii*) of Section 2.6, we can write the following limit of a product as the product of the limits:

$$\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot(z-z_0)$$
$$= \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\lim_{z\to z_0}(z-z_0) = f'(z_0)\cdot 0 = 0.$$

From $\lim_{z\to z_0}(f(z)-f(z_0)) = 0$ we conclude that $\lim_{z\to z_0} f(z) = f(z_0)$. In view of Definition 2.9, f is continuous at z_0.

Of course the converse of Theorem 3.2 is not true; continuity of a function f at a point does not guarantee that f is differentiable at the point. It follows from Theorem 2.3 that the simple function $f(z) = x + 4iy$ is continuous everywhere because the real and imaginary parts of f, $u(x, y) = x$ and $v(x, y) = 4y$ are continuous at any point (x, y). Yet we saw in Example 3 that $f(z) = x + 4iy$ is not differentiable at any point z.

A Necessary Condition for Analyticity In the next theorem we see that if a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point z, then the functions u and v must satisfy a pair of equations that relate their first-order partial derivatives.

Theorem 3.4 Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

Proof The derivative of f at z is given by

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}. \tag{2}$$

By writing $f(z) = u(x, y) + iv(x, y)$ and $\Delta z = \Delta x + i\Delta y$, (2) becomes

$$f'(z) = \lim_{\Delta z\to 0}\frac{u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)-u(x,y)-iv(x,y)}{\Delta x + i\Delta y}. \tag{3}$$

Since the limit (2) is assumed to exist, Δz can approach zero from any convenient direction. In particular, if we choose to let $\Delta z \to 0$ along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$. We can then write (3) as

$$\begin{aligned} f'(z) &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x, y)-u(x,y)+i\left[v(x+\Delta x, y)-v(x,y)\right]}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x, y)-u(x,y)}{\Delta x} + i\lim_{\Delta x\to 0}\frac{v(x+\Delta x, y)-v(x,y)}{\Delta x}. \end{aligned} \tag{4}$$

The existence of $f'(z)$ implies that each limit in (4) exists. These limits are the definitions of the first-order partial derivatives with respect to x of u and v, respectively. Hence, we have shown two things: both $\partial u/\partial x$ and $\partial v/\partial x$ exist at the point z, and that the derivative of f is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{5}$$

We now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$, (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}. \tag{6}$$

In this case (6) shows us that $\partial u/\partial y$ and $\partial v/\partial y$ exist at z and that

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{7}$$

By equating the real and imaginary parts of (5) and (7) we obtain the pair of equations in (1).

Because Theorem 3.4 states that the Cauchy-Riemann equations (1) hold at z as a *necessary* consequence of f being differentiable at z, we cannot use the theorem to help us determine where f is differentiable. *But* it is important to realize that Theorem 3.4 can tell us where a function f does not possess a derivative. If the equations in (1) are *not* satisfied at a point z, then f cannot be differentiable at z. We have already seen in Example 3 of Section 3.1 that $f(z) = x + 4iy$ is not differentiable at any point z. If we identify $u = x$ and $v = 4y$, then $\partial u/\partial x = 1$, $\partial v/\partial y = 4$, $\partial u/\partial y = 0$, and $\partial v/\partial x = 0$. In view of

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 4$$

the two equations in (1) cannot be simultaneously satisfied at any point z. In other words, f is nowhere differentiable.

It also follows from Theorem 3.4 that if a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic throughout a domain D, then the real functions u and v satisfy the Cauchy-Riemann equations (1) at every point in D.

A Sufficient Condition for Analyticity By themselves, the Cauchy-Riemann equations do not ensure analyticity of a function $f(z) = u(x, y) + iv(x, y)$ at a point $z = x + iy$. It is possible for the Cauchy-Riemann equations to be satisfied at z and yet $f(z)$ may not be differentiable at z, or $f(z)$ may be differentiable at z but nowhere else. In either case, f is not analytic at z. See Problem 35 in Exercises 3.2. However, when we add the condition of continuity to u and v *and* to the four partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$, it can be shown that the Cauchy-Riemann equations are not only necessary but also sufficient to guarantee analyticity of $f(z) = u(x, y) + iv(x, y)$ at z. The proof is long and complicated and so we state only the result.

Theorem 3.5 Criterion for Analyticity

Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations (1) at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Possible questions

1. Treating f(z) as a function of x &y and x &y is a function of z & $\bar{z}$ show that

i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log|f'(z)| = 0.$

2. State and prove C-R equations in polar coordinates.

3. Prove that the necessary condition for a function to be differentiable at a point is the Continuity of the function at the point.

4. Show that in a compact set every continuous function is uniformly continuous.

5. Suppose f(z) is a function differentiable in a region D and the mapping w=f(z) is one to one and the inverse mapping is z=φ (w).If z_0 is a point in D such that $f'(z_0)\neq 0$, then

i) The inverse function φ (w) is differential at w_0, where $w_0= f(z_0)$ and

ii) $\varphi'(w_0) = 1/f'(z_0)$

6. Prove that an analytic function f (z) and the C-R equations can be put in the condensed form $\frac{\partial f}{\partial \bar{z}} = 0.$

7. Suppose f(z)= u(x,y)+iv(x,y) is a single valued function defined in a neighbourhood of $z_0=x_0+iy_0$.Then the necessary condition for the differentiability of f(z) at z_0 is the existence of the partial derivatives u_x, u_y, v_x, v_y at (x_0,y_0),which satisfy the relations $u_x=v_y, u_y=-v_x$.

8. Show that the single valued continuous function $f(z) = z^{1/2} = r^{\frac{1}{2}}(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta)$, $r > 0, 0< \theta < 2\pi$ is analytic, Find $f'(z)$.

9. Derive the C-R equations in polar coordinates.

10. Prove that if f(z) is continuous at $z=z_0$ and if for any M>0 there exists a d such that $|f(z)|>M$ for all z in the disc $|z-z_0|<d$ then 1/f(z) is continuous at $z=z_0$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMPLEX ANALYSIS-I **Subject Code: 15MMU502**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
If f(z) of f has only one value it is called ----------- valued function.	single	multi	double	many	single
If $\vert f(z) \vert$ < M for all z in s , then f(z) is said to -------------- in S	multi valued	continuous	bounded	analytic	bounded
The limit of a function is -----------	unique	does not exist	different	multivalued	unique
If f(z) = 2iz is defined then $\lim_{z\to 1} f(z) =$	2	2i	-2	i	2i
If $\vert f(z) - f(z_0) \vert < \epsilon$ for all z in S with $\vert z - z_0 \vert < \delta$ then f(z) is	bounded	continuous	unique	does not exist	continuous
If f (z) and g(z) are continuous at z_0 then f(z) ± g(z) is	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
If f (z) and g(z) are continuous at z_0 then f(z) /g(z) is	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
In a compact set every continuous function is	bounded in s	uniformly continuous in s	unique	does not exist	bounded in s
If $\vert f(z_1) - f(z_2) \vert < \epsilon$ for all z_1, z_2 S with $\vert z_1 - z_2 \vert < \delta$ then f(z) is	bounded in s	uniformly continuous in s	unique	does not exist	uniformly continuous in s
If a function is differentiable at all points in some neighbourhood of a point , then the function is said to be ---- at that point	bounded	analytic	differentiable	compact	analytic
A function which is analytic everywhere in the finite plane is called an ------------------ function.	single	multi	entire	continuous	entire
f(z) is a function differentiable at z0 , then f(z) is	Continuous at z_0	compact at z	Continuous at z	differentiable at z	Continuous at z_0
A ---- point of a function is a point at which the function ceases to be analytic	non singular	Singular	entire	continuous	Singular
f(z) = $\vert z \vert^2$ is ---------------- everywhere	analytic	not analytic	continuous	exist	not analytic
d/dz [f(z)+g(z)]	$f^1(z)$	$f^1(z) + g^1(z)$	$f^1(z) - g^1(z)$	$f^1(z) . g^1(z)$	$f^1(z) + g^1(z)$
d/dz [f(z).g(z)] =	$f^1(z)$	$f^1(z) + g^1(z)$	$f^1(z) - g^1(z)$	$f^1(z) . g^1(z)$	$f^1(z) . g^1(z)$
d/dz [cf(z)] =	$cf^1(z)$	$f^1(z)$	$f^1(z)+c$	$f^1(z) /c$	$cf^1(z)$
d/dz [fg(z)] =	$f^1(g(z)). g^1(z)$	$f^1(z) . g^1(z)$	$f^1(z) - g^1(z)$	$f^1(z) - g^1(z)$	$f^1(g(z)). g^1(z)$
The quotient of two polynomials is called a	Exponential function	logarithmic function	Continuous function	rational function	rational function
If f(z) and g(z) are continuous at z_0 then f(z)/g(z), g(z)≠0 is	Continuous at z_0	differentiable at z_0	Continuous at z	differentiable at z	Continuous at z_0
If f(1/z) is analytic at 0 then f(z) is	Analytic at ∞	Continuous at ∞	Differentiable at ∞	Differentiable at 0	Analytic at ∞
The cartesian coordinates of C-R equations are	$u_x=v_y$ and $u_y= -v_x$	$u_x=v_y$ and $u_y= -v_x$	$u_x=v_y$ and $u_x= -v_x$	$u_x=1$ and $u_y= -v_x$	$u_x=v_y$ and $u_y= -v_x$
A function of complex variable is sometimes called a	complex variable	variable	complex function	constant	complex function
If the product of the slopes is -1, then the curves cut each other ------	diagonally	orthogonally	at the origin	at the point 1	orthogonally
The function that is multiple valued is	$f(z) = z^2$	$f(z) = e^z$	f(z) = 1/z	$f(z) = z^{1/2}$	$f(z) = z^{1/2}$
logz is a ------------- valued function	single	multi	double	three	multi
If $\lim_{z\to\infty} f(z) = A$ then $\lim_{z\to 0} f(\frac{1}{z})$	0	A	1/A	∞	A
If $\lim_{z\to 0} f(z) = A$ then $\lim_{z\to\infty} f(\frac{1}{z})$	0	A	1/A	∞	A
If f(z) = $1/z^2$ then $\lim_{z\to\infty} f(z) =$	0	2	1	-1	0
If $f(z_0) = \infty$, the function f(z) is …………… at $z = z_0$	continuous	not continuous	differentiable	bounded	not continuous
The function f(z) = Re z/$\vert z \vert$, when z ≠0 ; f(z) = 0 when f(z) = 0 is …………….	continuous	not continuous	differentiable	bounded	not continuous
The function $\vert z \vert^2$ is ……………. at that point.	continuous	analytic	not analytic	bounded	not analytic
If f(z) = u +iv is analytic , then u(x,y) and v(x,y) are …………….. Functions	harmonic	analytic	continuous	bounded	harmonic
The function f(z) = log z,then u(r,θ) = …… v(r,θ) = ………..	log θ , log r	r, log θ	log r , θ	r,θ	log r , θ
If f(z) = 1/z then $\lim_{z\to\infty} f(z) =$	∞	-1	0	1	0
A continuous function f(z) defined on a set D is uniformly continuous when	D is bounded	D is closed	D is compact	D is open	D is compact

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT OF MATHEMATICS

Subject: COMPLEX ANALYSIS-I	**Semester: V**	**L T P C**
Subject Code: 15MMU502	**Class: III-B.Sc Mathematics**	**5 0 0 5**

UNIT III

Power Series: Absolute convergence –circle of convergence –Analyticity of the sum of a power series-Uniqueness of representation of a function by a power series- Elementary functions : Exponential, Logarithmic, Trigonometric and Hyperbolic functions. Harmonic functions: Definition and determination.

TEXT BOOK

1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

REFERENCES

1. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.
2. Vasishtha A.R ., 2005. Complex Analysis, Krishna Prakashan Media Pvt. Ltd., Meerut.
3. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan (printers & publishers) pvt Ltd, Madras.

Power Series The notion of a power series is important in the study of analytic functions. An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \tag{11}$$

where the coefficients a_k are complex constants, is called a **power series** in $z-z_0$. The power series (11) is said to be **centered at z_0**; the complex point z_0 is referred to as the **center** of the series. In (11) it is also convenient to define $(z-z_0)^0 = 1$ even when $z = z_0$.

Circle of Convergence Every complex power series (11) has a **radius of convergence**. Analogous to the concept of an interval of convergence for real power series, a complex power series (11) has a **circle of convergence**, which is the circle centered at z_0 of largest radius $R > 0$ for which (11) converges at every point *within* the circle $|z-z_0| = R$. A power series converges absolutely at all points z within its circle of convergence, that is, for all z satisfying $|z-z_0| < R$, and diverges at all points z exterior to the circle, that is, for all z satisfying $|z-z_0| > R$. The radius of convergence can be:

(i) $R = 0$ (in which case (11) converges only at its center $z = z_0$),

(ii) R a finite positive number (in which case (11) converges at all interior points of the circle $|z-z_0| = R$), or

(iii) $R = \infty$ (in which case (11) converges for all z).

A power series may converge at some, all, or at none of the points *on* the actual circle of convergence. See Figure 6.3 and the next example.

EXAMPLE 5 Circle of Convergence

Consider the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test (9),

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1}|z| = |z|.$$

Thus the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$. Note that on the circle of convergence $|z| = 1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the well-known divergent harmonic series. Bear in mind *this does not say* that the series diverges on the circle of convergence. In fact, at $z = -1$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is the convergent alternating harmonic series. Indeed, it can be shown that the series converges at all points on the circle $|z| = 1$ *except* at $z = 1$.

It should be clear from Theorem 6.4 and Example 5 that for a power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$, the limit (9) depends only on the coefficients a_k. Thus, if

$$(i)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \neq 0, \text{ the radius of convergence is } R = \frac{1}{L}; \tag{12}$$

$$(ii)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0, \text{ the radius of convergence is } R = \infty; \tag{13}$$

$$(iii)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty, \text{ the radius of convergence is } R = 0. \tag{14}$$

Similar conclusions can be made for the root test (10) by utilizing

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}. \tag{15}$$

For example, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0$, then $R = 1/L$.

EXAMPLE 6 Radius of Convergence

Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z - 1 - i)^k$. With the identification $a_n = (-1)^{n+1}/n!$ we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

EXAMPLE 7 Radius of Convergence

Consider the power series $\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^{k}(z-2i)^{k}$. With $a_n = \left(\frac{6n+1}{2n+5}\right)^{n}$, the root test in the form (15) gives

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{6n+1}{2n+5} = 3.$$

By reasoning similar to that leading to (12), we conclude that the radius of convergence of the series is $R = \frac{1}{3}$. The circle of convergence is $|z-2i| = \frac{1}{3}$; the power series converges absolutely for $|z-2i| < \frac{1}{3}$.

Harmonic Functions A solution $\phi(x, y)$ of Laplace's equation (1) in a domain D of the plane is given a special name.

Definition 3.3 Harmonic Functions

A real-valued function ϕ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Theorem 3.7 Harmonic Functions

Suppose the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

Proof Assume $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D and that u and v have continuous second-order partial derivatives in D.† Since f is analytic, the Cauchy-Riemann equations are satisfied at every point z. Differentiating both sides of $\partial u/\partial x = \partial v/\partial y$ with respect to x and differentiating both sides of $\partial u/\partial y = -\partial v/\partial x$ with respect to y give, respectively,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \tag{2}$$

With the assumption of continuity, the mixed partials $\partial^2 v/\partial x\partial y$ and $\partial^2 v/\partial y\partial x$ are equal. Hence, by adding the two equations in (2) we see that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{or} \qquad \nabla^2 u = 0.$$

This shows that $u(x, y)$ is harmonic.

Now differentiating both sides of $\partial u/\partial x = \partial v/\partial y$ with respect to y and differentiating both sides of $\partial u/\partial y = -\partial v/\partial x$ with respect to x, give, in turn, $\partial^2 u/\partial y\partial x = \partial^2 v/\partial y^2$ and $\partial^2 u/\partial x\partial y = -\partial^2 v/\partial^2 x$. Subtracting the last two equations yields $\nabla^2 v = 0$. ✎

the harmonic conjugate of u is $v(x, y) = 3x^2y - y^3 + 5x + C$.

In Example 2, by combining u and its harmonic conjugate v as $u(x, y) + iv(x, y)$, the resulting complex function

$$f(z) = x^3 - 3xy^2 - 5y + i(3x^2 - y^3 + 5x + C)$$

is an analytic function throughout the domain D consisting, in this case, of the (**a**) Verify that the function $u(x,\ y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(**b**) Find the harmonic conjugate function of u.

Solution

(**a**) From the partial derivatives

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \frac{\partial^2 u}{\partial x^2} = 6x,\ \frac{\partial u}{\partial y} = -6xy - 5,\ \frac{\partial^2 u}{\partial y^2} = -6x$$

we see that u satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

(**b**) Since the conjugate harmonic function v must satisfy the Cauchy-Riemann equations $\partial v/\partial y = \partial u/\partial x$ and $\partial v/\partial x = -\partial u/\partial y$, we must have

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \text{and} \quad \frac{\partial v}{\partial x} = 6xy + 5. \tag{3}$$

Partial integration of the first equation in (3) with respect to the variable y gives $v(x,\ y) = 3x^2y - y^3 + h(x)$. The partial derivative with respect to x of this last equation is

$$\frac{\partial v}{\partial x} = 6xy + h'(x).$$

When this result is substituted into the second equation in (3) we obtain $h'(x) = 5$, and so $h(x) = 5x + C$, where C is a real constant. Therefore, the harmonic conjugate of u is $v(x,\ y) = 3x^2y - y^3 + 5x + C$.

Exponential Function and its Derivative We begin by repeating the definition of the complex exponential function given in Section 2.1.

Definition 4.1 Complex Exponential Function

The function e^z defined by

$$e^z = e^x \cos y + ie^x \sin y \tag{1}$$

is called the **complex exponential function**.

One reason why it is natural to call this function the *exponential* function was pointed out in Section 2.1. Namely, the function defined by (1) agrees with the real exponential function when z is real. That is, if z is real, then $z = x + 0i$, and Definition 4.1 gives:

$$e^{x+0i} = e^x\,(\cos 0 + i \sin 0) = e^x(1 + i \cdot 0) = e^x. \tag{2}$$

The complex exponential function also shares important differential properties of the real exponential function. Recall that two important properties of the real exponential function are that e^x is differentiable everywhere and that $\dfrac{d}{dx}e^x = e^x$ for all x. The complex exponential function e^z has similar properties.

Theorem 4.1 Analyticity of e^z

The exponential function e^z is entire and its derivative is given by:

$$\frac{d}{dz}e^z = e^z. \tag{3}$$

Proof In order to establish that e^z is entire, we use the criterion for analyticity given in Theorem 3.5. We first note that the real and imaginary parts, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$, of e^z are continuous real functions and have continuous first-order partial derivatives for all (x, y). In addition, the Cauchy-Riemann equations in u and v are easily verified:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Therefore, the exponential function e^z is entire by Theorem 3.5. By (9) of Section 3.2, the derivative of an analytic function f is given by $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and so the derivative of e^z is:

$$\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z.$$

Using the fact that the real and imaginary parts of an analytic function are harmonic conjugates, we can also show that the *only* entire function f that agrees with the real exponential function e^x for real input and that satisfies the differential equation $f'(z) = f(z)$ is the complex exponential function e^z defined by (1). See Problem 50 in Exercises 4.1.

EXAMPLE 1 Derivatives of Exponential Functions

Find the derivative of each of the following functions:

(a) $iz^4 \left(z^2 - e^z\right)$ and **(b)** $e^{z^2-(1+i)z+3}$.

Solution (a) Using (3) and the product rule (4) in Section 3.1:

$$\begin{aligned}\frac{d}{dz}\left[iz^4 \left(z^2 - e^z\right)\right] &= iz^4 \left(2z - e^z\right) + 4iz^3 \left(z^2 - e^z\right)\\ &= 6iz^5 - iz^4e^z - 4iz^3e^z.\end{aligned}$$

(b) Using (3) and the chain rule (6) in Section 3.1:

$$\frac{d}{dz}\left[e^{z^2-(1+i)z+3}\right] = e^{z^2-(1+i)z+3} \cdot (2z - 1 - i).$$

Modulus, Argument, and Conjugate The modulus, argument, and conjugate of the exponential function are easily determined from (1). If we express the complex number $w = e^z$ in polar form:

$$w = e^x \cos y + ie^x \sin y = r(\cos\theta + i\sin\theta),$$

then we see that $r = e^x$ and $\theta = y + 2n\pi$, for $n = 0, \pm1, \pm2, \ldots$. Because r is the modulus and θ is an argument of w, we have:

$$|e^z| = e^x \tag{4}$$

and

$$\arg(e^z) = y + 2n\pi,\ n = 0,\ \pm1, \pm2, \ldots . \tag{5}$$

We know from calculus that $e^x > 0$ for all real x, and so it follows from (4) that $|e^z| > 0$. This implies that $e^z \neq 0$ for all complex z. Put another way, the point $w = 0$ is not in the range of the complex function $w = e^z$. Equation (4) does not, however, rule out the possibility that e^z is a negative real number. In fact, you should verify that if, say, $z = \pi i$, then $e^{\pi i}$ is real and $e^{\pi i} < 0$.

A formula for the conjugate of the complex exponential e^z is found using properties of the real cosine and sine functions. Since the real cosine function is even, we have $\cos y = \cos(-y)$ for all y, and since the real sine function is odd, we have $-\sin y = \sin(-y)$ for all y, and so:

$$\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\bar{z}}.$$

Therefore, for all complex z, we have shown:

$$\overline{e^z} = e^{\bar{z}}. \tag{6}$$

Complex Logarithmic Function

In real analysis, the natural logarithm function $\ln x$ is often defined as an inverse function of the real exponential function e^x. From this point on, we will use the alternative notation $\log_e x$ to represent the real exponential function. Because the real exponential function is one-to-one on its domain $\mathbf{R}$, there is no ambiguity involved in defining this inverse function. The situation is very different in complex analysis because the complex exponential function e^z is not a one-to-one function on its domain $\mathbf{C}$. In fact, given a fixed nonzero complex number z, the equation $e^w = z$ has infinitely many solutions. For example, you should verify that $\frac{1}{2}\pi i$, $\frac{5}{2}\pi i$, and $-\frac{3}{2}\pi i$ are all solutions to the equation $e^w = i$. To see why the equation $e^w = z$ has infinitely many solutions, in general, suppose that $w = u + iv$ is a solution of $e^w = z$. Then we must have $|e^w| = |z|$ and $\arg(e^w) = \arg(z)$. From (4) and (5), it follows that $e^u = |z|$ and $v = \arg(z)$, or, equivalently, $u = \log_e |z|$ and $v = \arg(z)$. Therefore, given a nonzero complex number z we have shown that:

$$\text{If } e^w = z, \text{ then } w = \log_e |z| + i\arg(z). \tag{10}$$

Because there are infinitely many arguments of z, (10) gives infinitely many solutions w to the equation $e^w = z$. The set of values given by (10) defines a multiple-valued function $w = G(z)$, as described in Section 2.4, which is called the complex logarithm of z and denoted by $\ln z$. The following definition summarizes this discussion.

The multiple-valued function $\ln z$ defined by:

$$\ln z = \log_e |z| + i \arg(z) \tag{11}$$

is called the **complex logarithm**.

Hereafter, the notation $\ln z$ will always be used to denote the multiple-valued *complex* logarithm. By switching to exponential notation $z = re^{i\theta}$ in (11), we obtain the following alternative description of the complex logarithm:

$$\ln z = \log_e r + i(\theta + 2n\pi), \ n = 0, \ \pm 1, \ \pm 2, \ldots. \tag{12}$$

From (10) we see that the complex logarithm can be used to find all solutions to the exponential equation $e^w = z$ when z is a nonzero complex number.

Theorem 4.3 Algebraic Properties of $\ln z$

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

(ii) $\ln\left(\dfrac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

(iii) $\ln z_1^n = n \ln z_1$.

Proof of (i) By Definition 4.2,

$$\begin{aligned} \ln z_1 + \ln z_2 &= \log_e |z_1| + i \arg(z_1) + \log_e |z_2| + i \arg(z_2) \\ &= \log_e |z_1| + \log_e |z_2| + i(\arg(z_1) + \arg(z_2)). \end{aligned} \tag{13}$$

Because the real logarithm has the property $\log_e a + \log_e b = \log_e(ab)$ for $a > 0$ and $b > 0$, we can write $\log_e |z_1 z_2| = \log_e |z_1| + \log_e |z_2|$. Moreover, from (8) of Section 1.3, we have $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$. Therefore, (13) can be rewritten as:

$$\ln z_1 + \ln z_2 = \log_e |z_1 z_2| + i \arg(z_1 z_2) = \ln(z_1 z_2).$$

Proofs of Theorems 4.3(ii) and 4.3(iii) are similar. See Problems 53 and 54 in Exercises 4.1.

The complex function $\mathrm{Ln}\, z$ defined by:

$$\mathrm{Ln}\, z = \log_e |z| + i\mathrm{Arg}(z)$$

is called the **principal value of the complex logarithm**.

EXAMPLE 4 Principal Value of the Complex Logarithm

Compute the principal value of the complex logarithm $\operatorname{Ln} z$ for

(a) $z = i$ **(b)** $z = 1 + i$ **(c)** $z = -2$

Solution In each part we apply (14) of Definition 4.3.

(a) For $z = i$, we have $|z| = 1$ and $\operatorname{Arg}(z) = \pi/2$, and so:

$$\operatorname{Ln} i = \log_e 1 + \frac{\pi}{2}i.$$

However, since $\log_e 1 = 0$, this simplifies to:

$$\operatorname{Ln} i = \frac{\pi}{2}i.$$

(b) For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\operatorname{Arg}(z) = \pi/4$, and so:

$$\operatorname{Ln}(1+i) = \log_e \sqrt{2} + \frac{\pi}{4}i.$$

Because $\log_e \sqrt{2} = \frac{1}{2}\log_e 2$, this can also be written as:

$$\operatorname{Ln}(1+i) = \frac{1}{2}\log_e 2 + \frac{\pi}{4}i \approx 0.3466 + 0.7854i.$$

(c) For $z = -2$, we have $|z| = 2$ and $\operatorname{Arg}(z) = \pi$, and so:

$$\operatorname{Ln}(-2) = \log_e 2 + \pi i \approx 0.6931 + 3.1416i.$$

Ln z as an Inverse Function Because $\operatorname{Ln} z$ is *one* of the values of the complex logarithm $\ln z$, it follows from (10) that:

$$e^{\operatorname{Ln} z} = z \text{ for all } z \neq 0. \tag{16}$$

This suggests that the logarithmic function $\operatorname{Ln} z$ is an inverse function of exponential function e^z. Because the complex exponential function is not one-to-one on its domain, this statement is not completely accurate. Rather, the relationship between these functions is similar to the relationship between the squaring function z^2 and the principal square root function $z^{1/2} = \sqrt{|z|}e^{i\operatorname{Arg}(z)/2}$ defined by (7) in Section 2.4. The exponential function must first be restricted to a domain on which it is one-to-one in order to have a well-defined inverse function. In Problem 52 in Exercises 4.1, you will be asked to show that e^z is a one-to-one function on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, shown in Figure 4.1.

We now show that if the domain of e^z is restricted to the fundamental region, then the principal value of the complex logarithm $\operatorname{Ln} z$ is its inverse function. To justify this claim, consider a point $z = x + iy$ in the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$. From (4) and (5), we have that $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$, n an integer. Thus, y is an argument of e^z. Since z is in the fundamental region, we also have $-\pi < y \leq \pi$, and from this it follows that y is the principal argument of e^z. That is, $\operatorname{Arg}(e^z) = y$. In addition, for the real logarithm we have $\log_e e^x = x$, and so from Definition 4.3 we obtain:

$$\begin{aligned} \operatorname{Ln} e^z &= \log_e |e^z| + i\operatorname{Arg}(e^z) \\ &= \log_e e^x + iy \\ &= x + iy. \end{aligned}$$

Thus, we have shown that:

$$\operatorname{Ln} e^z = z \text{ if } -\infty < x < \infty \quad \text{and} \quad -\pi < y \leq \pi. \tag{17}$$

From (16) and (17), we conclude that $\operatorname{Ln} z$ is the inverse function of e^z defined on the fundamental region. The following summarizes the relationship between these functions.

Complex Trigonometric Functions

If x is a real variable, then it follows from Definition 4.1 that:

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x. \tag{1}$$

By adding these equations and simplifying, we obtain an equation that relates the real cosine function with the complex exponential function:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{2}$$

In a similar manner, if we subtract the two equations in (1), then we obtain an expression for the real sine function:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{3}$$

The formulas for the *real* cosine and sine functions given in (2) and (3) can be used to define the *complex* sine and cosine functions. Namely, we define these complex trigonometric functions by replacing the real variable x with the complex variable z in (2) and (3). This discussion is summarized in the following definition.

Definition 4.6 Complex Sine and Cosine Functions

The complex **sine** and **cosine** functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{4}$$

It follows from (2) and (3) that the complex sine and cosine functions defined by (4) agree with the real sine and cosine functions for real input. Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \text{and} \quad \csc z = \frac{1}{\sin z}. \tag{5}$$

These functions also agree with their real counterparts for real input.

EXAMPLE 1 Values of Complex Trigonometric Functions

In each part, express the value of the given trigonometric function in the form $a + ib$.

(a) $\cos i$ **(b)** $\sin(2+i)$ **(c)** $\tan(\pi - 2i)$

Solution For each expression we apply the appropriate formula from (4) or (5) and simplify.

(a) By (4),

$$\cos i = \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \frac{e^{-1} + e}{2} \approx 1.5431.$$

(b) By (4),

$$\begin{aligned}\sin(2+i) &= \frac{e^{i(2+i)} - e^{-i(2+i)}}{2i}\\ &= \frac{e^{-1+2i} - e^{1-2i}}{2i}\\ &= \frac{e^{-1}(\cos 2 + i\sin 2) - e(\cos(-2) + i\sin(-2))}{2i}\\ &\approx \frac{0.9781 + 2.8062i}{2i}\\ &\approx 1.4031 - 0.4891i.\end{aligned}$$

(c) By the first entry in (5) together with (4) we have:

$$\begin{aligned}\tan(\pi - 2i) &= \frac{(e^{i(\pi-2i)} - e^{-i(\pi-2i)})/2i}{(e^{i(\pi-2i)} + e^{-i(\pi-2i)})/2} = \frac{e^{i(\pi-2i)} - e^{-i(\pi-2i)}}{(e^{i(\pi-2i)} + e^{-i(\pi-2i)})\, i}\\ &= -\frac{e^2 - e^{-2}}{e^2 + e^{-2}}i \approx -0.9640i.\end{aligned}$$

Complex Hyperbolic Functions

The *real* hyperbolic sine and hyperbolic cosine functions are defined using the *real* exponential function as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The *complex* hyperbolic sine and cosine functions are defined in an analogous manner using the *complex* exponential function.

Definition 4.7 Complex Hyperbolic Sine and Cosine

The complex **hyperbolic sine** and **hyperbolic cosine** functions are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{25}$$

Since the complex exponential function agrees with the real exponential function for real input, it follows from (25) that the complex hyperbolic functions agree with the real hyperbolic functions for real input. However, unlike the real hyperbolic functions whose graphs are shown in Figure 4.11, the complex hyperbolic functions are periodic and have infinitely many zeros. See Problem 50 in Exercises 4.3.

The complex hyperbolic tangent, cotangent, secant, and cosecant are defined in terms of $\sinh z$ and $\cosh z$:

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \text{and } \operatorname{csch} z = \frac{1}{\sinh z}. \tag{26}$$

Observe that the hyperbolic sine and cosine functions are entire because the functions e^z and e^{-z} are entire. Moreover, from the chain rule (6) in Section 3.1, we have:

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z + e^{-z}}{2}$$

or

$$\frac{d}{dz}\sinh z = \cosh z.$$

A similar computation for $\cosh z$ yields

$$\frac{d}{dz}\cosh z = \sinh z.$$

POSSIBLE QUESTIONS

1. i). Find the radius of convergence of the power series f(z) = $\sum_0^\infty \frac{z^n}{2^n(1+in^2)}$

   ii) State and prove Euler's relation.

2. State and prove Abel's theorem.

3. Prove that the sum of a convergent power series in z is analytic in the interior of its circle of convergence.

4. Find the domain of convergence of

   i) $\sum_1^\infty \left(\frac{iz-1}{2+i}\right)^n$ ii) $\sum_1^\infty \left(\frac{z+i}{2i}\right)^n$ iii) $\sum_1^\infty \left(\frac{1}{1+z2}\right)^n$

5. State and prove Uniqueness theorem

6. Find the radii of convergence of the following power series

   i) $\sum \frac{n^k}{n^n} z^n$ ii) $\sum \frac{n!}{n^n} z^n$

7. If a power series in z is convergent at z =z_1 , then it converges absolutely in the circular Open disc | Z |< | Z_1|.

8. i) Define circle of convergence.

   ii) Prove that a power series is divergent in the exterior of its circle of convergence.

9. Explain about an exponential function.

10. Find the radii of convergence of the following power series

   i) $\sum \frac{(2)^n}{n!} z^n$ ii) $\sum \frac{2+in}{2^n} z^n$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMPLEX ANALYSIS-I **Subject Code: 15MMU502**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
The power series of the form $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$ Is said to be a series about	$z = 0$	$z = -a$	$z = a$	$z = \infty$	$z = a$
The power series $a_0 + a_1 z + a_2 z^2 + \ldots$ converges absolutely in the open disc ……..	$\|z\| = R$	$\|z\| > R$	$\|z\| < R$	$\|z\| = 0$	$\|z\| < R$
The circle of the convergence of the series $a_0 + a_1 z + a_2 z^2 + \ldots$	$\|z\| > R$	$\|z\| < R$	$\|z\| = 0$	$\|z\| = R$	$\|z\| = R$
The circle of the convergence of the series $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$	$\|z-a\| > R$	$\|z-a\| < R$	$\|z-a\| = 0$	$\|z-a\| = R$	$\|z-a\| = R$
A power series … in the exterior of its circle of convergence	absolutely convergent	converges	diverges	uniformly convergent	diverges
If $R = 0$ the series is divergent in the extended plane except at	$z = 0$	$z = 1$	$z = \infty$	$z = -1$	$z = 0$
The sequence $\{z_n\}$ is bounded if there exists a constant M such that ------ for all n.	$\|z_n\| = M$	$\|z_n\| \leq M$	$\|zn\| \geq M$	$\|z_n\| > M$	$\|z_n\| \leq M$
For all finite $z = h + ik$, $\|e^z\| = \ldots$	e^{h+k}	e^{h+ik}	e^h	e^k	e^h
Euler's relation $e^{x+iy} =$	$e^x(\cos y + i\sin y)$	$e^x(\sin y + i\cos y)$	$e^y(\cos x + i\sin x)$	$e^y(\sin x + i\cos x)$	$e^x(\cos y + i\sin y)$
The polar form $r(\cos\theta + i\sin\theta)$ of a complex numbers in exponential form as	re^{θ}	$e^{i\theta}$	$re^{i\theta}$	$1/re^{i\theta}$	$re^{i\theta}$
e^z is not defined at	$z = \infty$	$z = 0$	$z = 1$	$z = -1$	$z = \infty$
The inverse function of the exponential function is the ……	Trignometric functions	hyberbolic functions	harmonic functions	Logarithmic functions	Logarithmic functions
Logarithamic function log z = --------------------- $n = 0, \pm 1, \pm 2$	$\log r + i\theta + n(2\pi i)$	$\log 1/r + ie^{i\theta} + n(2\pi i)$	$\log r + ie^{i\theta} + n(2\pi i)$	$\log r + i\theta + n2\pi$	$\log r + i\theta + n(2\pi i)$
$\frac{d}{dz}(\log z) =$	z	$\frac{1}{z}$	$-z$	e^z	$\frac{1}{z}$
siniz	sinz	sinhz	isinz	isinhz	isinhz
cosiz	cosz	icosz	icoshz	coshz	coshz
tanz and secz are analytic in a bounded region in which	$\tan z \neq 0$	$\sec z \neq 0$	$\sin z \neq 0$	$\cos z \neq 0$	$\sin z \neq 0$
cot z and cosecz are analytic in a bounded region in which	$\cot z \neq 0$	$\mathrm{cosec} z \neq 0$	$\sin z \neq 0$	$\cos z \neq 0$	$\cos z \neq 0$
$\cosh^2 z - \sinh^2 z =$	0	1	-1	∞	1
singular points of logz are	$z = 0$ and $z = \infty$	$z = 1$ and $z = 0$	$z = 0$ and $z = -1$	$z = 1$ and $z = \infty$	$z = 0$ and $z = \infty$
Principle value of logz is obtained when n =	0	-1	1	2	0
The logarithmic function is a ------ valued function	single	multiple	two	zero	multiple
In a complex field $z = x + iy$ then $\theta = \ldots$	$\sin^{-1}(y/x)$	$\cos^{-1}(y/x)$	$\tan^{-1}(y/x)$	$\cot^{-1}(y/x)$	$\tan^{-1}(y/x)$
The sum f(z) of a powerseries is analytic in …………	$\|z\| > R$	$\|z\| < R$	$\|z\| \leq R$	$\|z\| = R$	$\|z\| < R$
A power series …………… is the interior of the circle of convergence	converges	diverges	uniformly converges	converges absolutely	converges absolutely
The radius of convergence of the series $\sum (2+in)/2^n . z^n$ …………	2	0	∞	1	2
$\sin(\frac{1}{2}\pi - z) = \ldots$	sinz	cosz	tanz	cosecz	cosz
If u+iv is analytic then v+iu is…………	analytic	not analytic	continuous	conjugate	not analytic
coshiz	cosz	cosiz	sinz	coshiz	cosz
a^z is a …………… valued function	single	double	multiple	triple	multiple
The function $a^z =$	$e^{z\log a}$	$e^{\log a}$	$e^{a\log z}$	$e^{-z\log a}$	$e^{z\log a}$
The radius of convergence of the series $\sum n^2 . z^n$ …………	1	0	2	n	1
$\cos(z_1 + z_2) =$	$\cos z_1 \cos z_2 - \sin z_1 \sin z_2$	$\cos z_1 \sin z_2 - \sin z_1 \cos z_2$	$\cos z_1 \cos z_2 + \sin z_1 \sin z_2$	$\sin z_1 \cos z_2 - \cos z_1 \sin z_2$	$\cos z_1 \cos z_2 - \sin z_1 \sin z_2$
The radius of convergence of the series $\sum n^n . z^n$ …………	1	0	2	n	0

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT OF MATHEMATICS

Subject COMPLEX ANALYSIS-I	**Semester: V**	**L T P C**
Subject Code: 15MMU502	**Class: III-B.Sc Mathematics**	**5 0 0 5**

UNIT IV

Bilinear transformation-Circles and Inverse points-Transformation mappings w=Z2 ,w=Z1/2,w=eZ, w = sin Z,and w=cos Z -Conformal mapping-isogonal mapping.

TEXT BOOK

1. Duraipandian.P., Lakshmi Duraipandian.,1997.Complex analysis,Emerald publishers, Chennai-2 .

REFERENCES

1. Arumugam.S., Thangapandi Isaac., and A.Somasundaram., 2002. Complex Analysis, SCITECH Publications Pvt. Ltd,Chennai.
2. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.

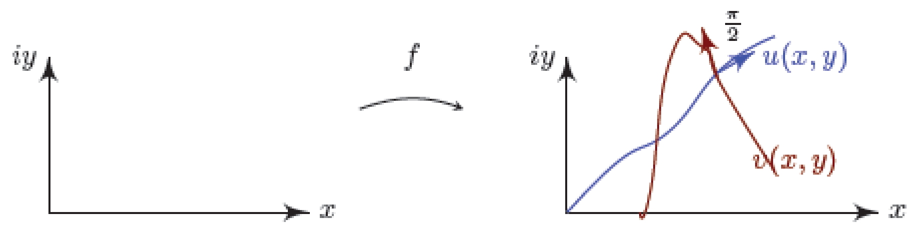
Conformal Mappings

Theorem 2.12

If $f = u + iv$ is holomorphic defined in an open connected domain $\Omega \subset \mathbb{C}$, then the level sets of u and v

$$\text{(a) } u(x, y) = C, \quad \text{(b) } v(x, y) = K$$

where C, K are constants, is orthogonal at every point where $f'(z) \neq 0$ and u, v intersect.



The proof used the Cauchy-Riemann equations. We look into this; it turns out if we map curves by a holomorphic f, such that $f'(z) \neq 0$ at the point of intersection, then the angle between the curves is always preserved.

Let us consider a smooth curve $\gamma \subset \mathbb{C}$ parameterised by $z(t) = x(t) + iy(t), t \in [a, b]$. For each $t_0 \in [a, b]$, there is the direction vector

$$\begin{aligned} L_{t_0} &= \{z(t_0) + tz'(t_0), t \in \mathbb{R}\} \\ &= \{x(t_0) + tx'(t_0) + i(y(t_0) + ty'(t_0))\} \end{aligned}$$

Consider now two curves, γ_1, γ_2 parameterised by $z_1(t), z_2(t)$. Let $t \in [0, 1]$ and assume that at $t = 0$, $z_1(0) = z_2(0)$. We define the angle between the curves γ_1 and γ_2 to be $\arg(z_2'(0) - z_1'(0))$.

Definition (Conformal). We say that a complex function f is *conformal* is an open set $\Omega \subset \mathbb{C}$ if it is holomorphic in Ω and $f'(z) \neq 0, \forall z \in \Omega$.

E.g $f(z) = z^2$ is conformal in $\mathbb{C} \backslash \{0\}$.

Möbius transformation

Definition. A Möbius transformation (also called a bilinear transformation) is a map

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, ad-bc \neq 0$$

Remark. If $ad - bc = 0 \implies \frac{a}{c} = \frac{b}{d} = \text{constant} \implies f(z) = \text{constant}$.

If f is holomorphic in $\mathbb{C}\backslash\{-d/c\}$,

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0 \qquad (\dagger)$$

and defined for $z \neq -d/c$.

Theorem 6.2

The inverse of a Möbius tranformation is a Möbius tranformation.

Proof. We want to find $g(w)$ such that $g(f(z) = z$. Then take

$$g(w) = \frac{dw-b}{-cw+a}, \text{ where } f(z) = \frac{az+b}{cz+d} \qquad \blacksquare$$

Let

$$f_1(z) = \frac{a_1z+b_1}{c_1z+d_1}, \quad f_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$$

Then

$$f_1 \circ f_2(z) = f_1(f_2(z)) = \frac{Az+B}{Cz+D}$$

Here $A = a_1a_2 + b_1c_2, B = a_1b_2 + b_1b_2, C = c_1a_1 + d_1c_2, D = c_1b_2 + d_1d_2$. Then

$$AD - BC = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0$$

So the composition is also a Möbius transformation. We can write this as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

For the inverse mapping, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Special Möbius tranformations

(M1) $z \mapsto az$. In this case, $b = 0, c = 0, d = 1$.

- If $|a| = 1, a = e^{i\theta}, z = re^{i\phi} \implies az = re^{i(\phi+\theta)}$. This is the rotation anticlockwise.
- If $a = |a|e^{i\theta}, |a| \neq 1 \implies az = |a| \cdot re^{i(\phi+\theta)}$. This is a rotation & dilation.

(M2) $z \mapsto z + b$ $(a = d = 1, c = 0)$. This is a translation by b

(M3) $z \mapsto \frac{1}{z}$ $(a = d = 0, b = c = 1)$.

Note that (M1), (M2), (M3) transformations map circles onto circles.

Theorem 6.3

Every Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ is a composition of transformations of the type (M1), (M2) and (M3)

Proof.

- Case 1: Let $c = 0$ and $d \neq 0$. We wish to take $f(z) = \frac{az+b}{d} = g_2 \circ g_1(z)$. Let $g_1(z) = \frac{a}{d}z$ (M1) and $g_2(z) = z + \frac{b}{d}$ (M2). Then

$$f(z) = g_2(g_1(z)) = \frac{a}{d}z + \frac{b}{d} = \frac{az+b}{d}$$

- Case 2: $c \neq 0$, we want to take $f(z) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z)$. Let $g_1(z) = cz$ (M1), $g_2(z) = z + d$ (M2), $g_3(z) = \frac{1}{z}$ (M3), so that

$$g_3(g_2(g_1(z))) = \frac{1}{cz+d}$$

Then

$$f_1 \circ f_2(z) = f_1(f_2(z)) = \frac{Az + B}{Cz + D}$$

Here $A = a_1a_2 + b_1c_2, B = a_1b_2 + b_1b_2, C = c_1a_1 + d_1c_2, D = c_1b_2 + d_1d_2$. Then

$$AD - BC = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0$$

So the composition is also a Möbius transformation. We can write this as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

For the inverse mapping, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Special Möbius tranformations

(M1) $z \mapsto az$. In this case, $b = 0, c = 0, d = 1$.

- If $|a| = 1, a = e^{i\theta}, z = re^{i\phi} \implies az = re^{i(\phi+\theta)}$. This is the rotation anticlockwise.
- If $a = |a|e^{i\theta}, |a| \neq 1 \implies az = |a| \cdot re^{i(\phi+\theta)}$. This is a rotation & dilation.

(M2) $z \mapsto z + b$ $(a = d = 1, c = 0)$. This is a translation by b

(M3) $z \mapsto \frac{1}{z}$ $(a = d = 0, b = c = 1)$.

Note that (M1), (M2), (M3) transformations map circles onto circles.

Theorem 6.3

Every Möbius transformation $f(z) = \dfrac{az + b}{cz + d}$ is a composition of transformations of the type (M1), (M2) and (M3)

Proof.

- Case 1: Let $c = 0$ and $d \neq 0$. We wish to take $f(z) = \frac{az+b}{d} = g_2 \circ g_1(z)$. Let $g_1(z) = \frac{a}{d}z$ (M1) and $g_2(z) = z + \frac{b}{d}$ (M2). Then

$$f(z) = g_2(g_1(z)) = \frac{a}{d}z + \frac{b}{d} = \frac{az + b}{d}$$

- Case 2: $c \neq 0$, we want to take $f(z) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z)$. Let $g_1(z) = cz$ (M1), $g_2(z) = z + d$ (M2), $g_3(z) = \frac{1}{z}$ (M3), so that

$$g_3(g_2(g_1(z))) = \frac{1}{cz + d}$$

Now let $g_4(z) = \underbrace{\frac{1}{c}(bc - ad)}_{\text{constant}} z$, which is (M1), so that

$$g_4(g_3(g_2(g_1(z)))) = \frac{bc - ad}{c} \cdot \frac{1}{cz + d}$$

Finally take $g_5(z) = z + \frac{a}{c}$, so

$$\begin{aligned} g_5(g_4(g_3(g_2(g_1(z))))) &= \frac{bc - ad}{c} \cdot \frac{cz + d}{+} \frac{a}{d} \\ &= \frac{1}{c} \frac{bc - ad + a(cz + d)}{cz + d} \\ &= \frac{acz + bc}{c(cz + d)} = \frac{az + b}{cz + d} \end{aligned}$$ ∎

Corollary 6.4. *Möbius transformation maps circles into circles and interior points into interior points. (Here we mean that straight lines are also circles whose radius equal infinity!)*

Let $H = \{z = x + iy \in \mathbb{C} \text{ s.t. } \operatorname{Im} z = y > 0\}$. A remarkable surprising fact is that the unbounded set H is conformally equivalent to the unit disc. Moreover, an explicit formula is

$$w = f(z) = \frac{i - z}{i + z}, \quad g(w) = i\frac{1 - w}{1 + w} = f^{-1}$$

Theorem 6.5

Let $\mathbb{D} = \{z : |z| < 1\}$. Then the map $f : H \to \mathbb{D}$ is a conformal map with inverse $g : \mathbb{D} \to H$.

Proof. We have $f(z) = \frac{i-z}{i+z}$. If $\operatorname{Im} z > 0 \implies z \neq -i \implies f$ is holomorphic and $f'(z) \neq 0$, $\forall z : \operatorname{Im} z \geq 0$. Since, using (†):

$$f'(z) = \frac{ad - bc}{(cz + d)^2}$$

with $a = -1, b = i, c = 1, d = i$.

Then

$$f'(z) = \frac{-i - i}{(z + i)^2} = \frac{-2i}{(z + i)^2} \neq 0$$

Hence f is conformal. Similarly with g.

Now if we write $z = x + iy$, we find

$$|f(z)|^2 = \left|\frac{i - z}{i + z}\right|^2 = \left|\frac{-x - iy + i}{x + iy + i}\right|^2 = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2} < 1$$

Similarly writing $w = u + iv$, we see that $|w| < 1 \iff u^2 + v^2 < 1$. Now

$$\begin{aligned}\operatorname{Im} g(z) = \operatorname{Re}\left(\frac{1-w}{1+w}\right) &= \operatorname{Re}\left(\frac{1-u-iv}{1+u+iv}\right) \\ &= \operatorname{Re}\frac{(1-u-iv)(1+u-iv)}{(1+u)^2+v^2} \\ &= \frac{(1-u^2)-v^2}{(1+u)^2+v^2} > 0\end{aligned}$$

So $g : \mathbb{D} \to H$.

Finally,

$$f \circ f(w) = f(g(w)) = \frac{i - i\frac{1-w}{1+w}}{i + i\frac{1-w}{1+w}} = \frac{i(1+w) - i(1-w)}{i(1+w) + i(1-w)} = \frac{2wi}{2i} = w$$

Similarly $g \circ f(z) = z$. ∎

Clearly f is holomorphic on $\mathbb{C}\backslash\{-i\}$. In particular, f is continuous on the boundary of H, $\partial H = \{z = x + i0\}$.

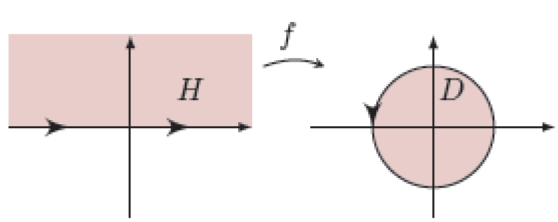
$$|f(z)|_{z=x+i0} = \left|\frac{i-x}{i+x}\right| = 1$$

Now

$$f(z) = \frac{i-x}{i+x} = \frac{1-x^2}{1+x^2} + \frac{2x}{1+x^2}i$$

Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$, then

$$\begin{aligned}\frac{i-x}{i+x} &= \frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{2\tan\theta}{1+\tan^2\theta} \\ &= \cos 2\theta + i\sin 2\theta = e^{2i\theta}\end{aligned}$$



Here, f maps $\pm\infty \mapsto -1$, $0 \mapsto +1$.

Theorem 6.6: Cross-Ratios Möbius Transformation

If $w = f(x)$ is a Möbius is a transformation that maps the distinct points (z_1, z_2, z_3) into the distinct points (w_1, w_2, w_3) respectively, then

$$\frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} \quad \forall z$$

Proof. Consider the Möbius transformation

$$g(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

It maps $z_1, z_2, z_3 \mapsto 0, 1, \infty$ respectively. Similarly

$$h(w) = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$

maps w_1, w_2, w_3 to $0, 1, \infty$. So $h^{-1} \circ g$ maps z_1, z_2, z_3 into w_1, w_2, w_3. ∎

Example 6.7. Find a Möbius transformation $w = f(z)$ such that f maps the points $1, i, -1 \mapsto -1, 0, 1$.

We have $z_1 = 1, z_2 = i, z_3 = -1; w_1 = -1, w_2 = 0, w_3 = 1$. So

$$\frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = \frac{w+1}{w-1} \cdot \frac{0-1}{0+1}$$

and

$$\frac{i+1}{i-1} = -\frac{1+i}{1-i} = -\frac{(i+i)^2}{2} = -\frac{2i}{2} = -i$$

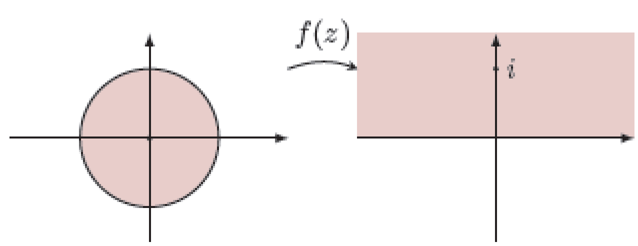
Then

$$\frac{z-1}{z+1} i = \frac{w+1}{w-i} \implies i(w-1)(z-1) = (w+1)(z+1)$$

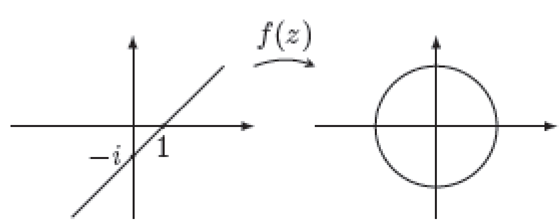
$$\implies w((i(z-1) - (z+1)) = i(z-1) + (z+1)$$

$$\implies w = \frac{iz - i + z + 1}{iz - i - z - 1} = \frac{z(1+i) + 1 - i}{iz(1+i) - (i+1)} = \frac{z + \frac{1-i}{i+1}}{iz - 1} = \frac{z - i}{iz - 1}$$

Let $z = 0 \implies w = \dfrac{0 - i}{0 - 1} = i$.



Example 6.8. Find a Möbius transformation $w = f(z)$ that maps the points $z_1 = -i, z_2 = 1, z_3 = \infty$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$, i.e.



We calculate

$$\lim_{z_3\to\infty}\frac{z+i}{z-z_3}\cdot\frac{1-z_3}{1+i}=\lim_{t\to 0}\frac{z+i}{z-1/t}\cdot\frac{1-1/t}{1+i}$$
$$=\lim_{t\to 0}\frac{z+i}{tz-1}\cdot\frac{t-1}{1+i}$$
$$=\frac{z+i}{-1}\cdot\frac{-1}{1+i}=\frac{z+i}{1+i}$$

Thus

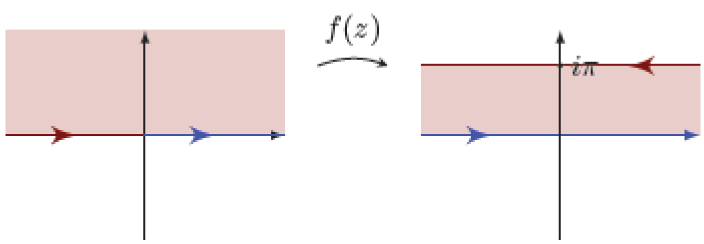
$$\frac{z+i}{1+i}=\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}=\frac{w-1}{w+1}\cdot\frac{i+1}{i-1}$$

and so

$$(z+i)(i-1)(w+1)=(w-1)(i+1)^2$$
$$\implies w=\frac{z(i-1)+(i-1)}{z(1-i)+(1+3i)}$$

Example 6.9. $w=f(x)=\log z$ with the negative imaginary axis as the branch cut. This is a conformal mapping that takes the upper half-plane to the strip

$$\{w=u+iv,u\in\mathbb{R},0<v<\pi\}$$



If $x\in(-\infty,0)\implies\log(z)=\ln x+i\pi$. If $x\in\in(0,\infty)\implies\log(z)=\ln x$.

We already considered a theorem saying that

$$g:\mathbb{D}\to H=\{w:\operatorname{Im}w>0\}$$

Then

$$g(w)=i\frac{1-w}{1+w}$$

Corollary 6.10. *Let* $w=F(z)=\log(i\frac{1-z}{1+z})$ *and* $G(w)=\frac{i-e^w}{i+e^w}$ *Both* F *and* G *are conformal and they are inverses to each other.* F *maps formally the unit disc to* $\{w:w=u+iv,i\mathbb{R},0<v<\pi\}$.

Example 6.11 (Joukowski's aerofoils).

$$\frac{w-2}{2+2}=\left(\frac{z-1}{z+1}\right)^2$$

It can be simplified $w = z + \frac{1}{z} = \frac{z^2+1}{z}$, so

$$\frac{w-2}{w+2} = \frac{\frac{z^2+1}{z} - 2}{\frac{z^2+1}{z} + 2} = \frac{z^2 - 2z + 1}{z^2 + 2z + 1} \cdot \frac{z}{z} = \left(\frac{z-1}{z+1}\right)^2$$

This transformation maps a circle with the centre $-1/4 + i/2$.

Theorem 6.12: Invariance of harmonicity on conformal maps

Let $f : \Omega_1 \to \Omega_2$ be conformal and let ϕ be a harmonic on Ω_2 ($\Delta\phi = 0$). We can consider ϕ as a real part of a holomorphic function. Then $\phi \circ f$ is harmonic on Ω_1.

Proof. Let $g = \phi + i\psi$ be holomorphic and let $w = f(z) = u + iv$.

$$g(w) = g \circ f(z) = \phi(u(x,y), v(x,y)) + i\psi(u(x,y), v(x,y))$$

Since $g \circ f$ is holomorphic, $\phi(u(x,y), v(x,y))$ is harmonic, and so

$$\Delta_{x,y}\phi(u(x,y), v(x,y)) = 0 \qquad \blacksquare$$

6.2 * Dirichlet Problem in a strip *

The Dirichlet problem in an open set Ω consists of solving the following equation

$$\begin{cases} \Delta u & = 0 \text{ in } \Omega \\ u & = f \text{ on } \partial\Omega \end{cases}$$

Consider now

$$\Omega = \{z : z = x + iy, x \in \mathbb{R}, 0 < y < \pi\}$$

Here $u(x,0) = f_0(x), u(x,\pi) = f_1(x)$. We assume that f_0 and f_1 are continuous and vanish at infinity, $\lim_{|x|\to\infty} f_j(x) = 0, j = 0, 1$.

We introduce

$$F(w) = \frac{1}{\pi}\log\left(i\frac{1-w}{1+w}\right), \quad G(z) = 0\frac{i - e^z}{i + e^z}$$

F maps the unit disc on the strip Ω and we can define

$$\tilde{f}_1(\theta) = f_1(F(e^{i\theta}) - i\pi), \quad -\pi < \theta < 0$$
$$\tilde{f}_0(\theta) = f_0(F(e^{i\theta}), \quad 0 < \theta < \pi$$

and

$$\tilde{f}(\theta) = \begin{cases} \tilde{f}_1(\theta) & -\pi < \theta < 0 \\ \tilde{f}_0(\theta) & 0 < \theta < \pi \end{cases}$$

We reduce the problem on the strip to the problem on $\mathbb{D} = \{z : |z| < 1\}$,

$$\begin{cases} \Delta \tilde{u} = 0 \\ \tilde{u}|_{\partial \mathbb{D}} = \tilde{f} \end{cases}$$

The solution of this problem can be written via the Poisson integral:

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\frac{1 - r^2}{1 - 2\cos(\theta - \phi) + r^2}}_{\text{Poisson Kernel}} \tilde{f}(\phi)\, d\phi$$

We derive this by considering

$$p_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

writing $\omega = re^{i\theta}$

$$= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n = \frac{1}{1-\omega} + \frac{\overline{\omega}}{1 - \overline{\omega}}$$

$$= \frac{1 - \overline{\omega} + (1-\omega)\overline{\omega}}{(1-\omega)(1-\overline{\omega})} = \frac{1 - |\omega|^2}{|1 - \omega|^2}$$

$$= \frac{1 - r^2}{1 - 2\cos\theta r + r^2}$$

6.3 * Univalent functions *

Felina. We finally look at some univalent functions (non-examinable).

> **Definition.** A single valued function f is called *univalent* in $\Omega \subset \mathbb{C}$ if it never takes the same value twice, that is $f(z_1) \neq f(z_2)$ for all z_1, z_2 such that $z_1 \neq z_2$.

Remark. For a holomorphic function, the condition $f'(z) \neq 0$ is equivalent to univalence.

Consider a class S of univalent functions on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ such that $f(0) = 0, f'(0) + 1$. For such $f \in S$ we have $f(z) = z + a_2 z^2 + \dots$. The leading example of a function from class S is the *Koebe function*,

$$K(z) = \frac{z}{(1-z)^2} = z(1 + z + z^2 + \dots)^2 = z + 2z^2 + 3z^3 = \dots$$

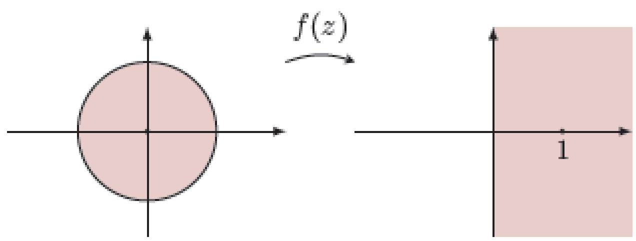
The Koebe function maps the unit disc $\mathbb{D}$ on the $\Omega = \mathbb{C}\backslash(-\infty, -1/4)$, since

$$\begin{aligned} K(z) &= \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} \\ &= \frac{1}{4}\left(\frac{(1+z)^2-(1-z)^2}{(1-z)^2}\right) \quad = \frac{1}{4}\left(\frac{1+2z+z^2-1+2z-z^2}{(1-z)^2}\right) \\ &= \frac{z}{(1-z)^2} \end{aligned}$$

And the conformal mapping

$$w = \frac{1+z}{1-z}$$

maps $\mathbb{D}$ onto $\mathrm{Re}\, w > 0$ $[z = i \mapsto i, z = 1 \mapsto \infty, z = 0 \mapsto 1]$



The square of this maps onto $\mathbb{C}\backslash(-\infty, 0)$, so $K(z)$ maps $\mathbb{D}$ onto $\mathbb{C}\backslash(-\infty, -1/4)$.

> **Examples 6.13.** Univalent functions:
>
> (i) The identity $f(z) = z$.
>
> (ii) $f(z) = z(1-z)^{-1}$ the maps $\mathbb{D}$ onto $\mathrm{Re} w > 1/2$.
>
> (iii) $f(z) = z(1-z^2)^{-1}$ maps $\mathbb{D}$ onto $\Omega\backslash\{(-\infty, -1/2) \cup (1/2, +\infty)\}$.

Closely related to S is the class Σ of functions

$$g(z) = z + b_0 + b_{-1}z^{-1} + b_{-2}z^{-2} + \dots$$

holomorphic and univalent in $\{z : |z| > 1\}$.

Theorem 6.14: Area

Let $g \in \Sigma$. Then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

Proof. Let us recall Green's Theorem:

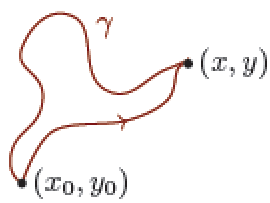
$$\oint_\gamma P \,\mathrm{d}u + Q \,\mathrm{d}v = \iint_\Omega \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u \,\mathrm{d}v$$

Harmonic Conjugate

In a simply connected domain Ω, every harmonic function u has a harmonic conjugate v defined the the line integral

$$v(z) = v(x, y) = \int_\gamma \left(-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y \right)$$

where the path of integration γ is a curve starting at a fixed base-point $(x_0, y_0) \in \Omega$ with end point $(x, y) \in \Omega$. The integral is independent of the path:



Let us assume that γ is a closed curve and show that

$$\int_\gamma \left(-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y \right) = 0$$

Green's formula is

$$\oint P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y$$

Let $P = -\frac{\partial u}{\partial y}$, $Q = \frac{\partial u}{\partial x}$, then

$$0 = \iint_\Omega \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \mathrm{d}x\,\mathrm{d}y = \oint_\gamma \left(-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y \right)$$

Now

$$\begin{aligned} v(x, y) &= \int_\gamma \left(-\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y \right) \\ &= \int_0^x -\frac{\partial u}{\partial y}\,\mathrm{d}x + \int_0^y \frac{\partial u}{\partial x}\,\mathrm{d}y \end{aligned}$$

So

$$\left. \begin{aligned} v'_x &= -\tfrac{\partial u}{\partial y};\; v''_{xx} = -\tfrac{\partial^2 u}{\partial x \partial y} \\ v'_y &= -\tfrac{\partial u}{\partial x};\; v''_{yy} = -\tfrac{\partial^2 u}{\partial x \partial y} \end{aligned} \right\} v''_{xx} + v''_{yy} = 0$$

PROPOSITION 1.21. *A Möbius transform different from the identity has either one or two fixpoints, as a map defined on the extended plane.*

EXERCISE 1.22. Find the fixed points of the linear transformations

$$w = \frac{z}{2z-1}\ , \quad w = \frac{2z}{3z-1}\ , \quad w = \frac{3z-4}{z-1}\ , \quad w = \frac{z}{2-z}\ .$$

In particular, a Möbius transform that leaves three distinct points invariant is the identity. It also follows that there can be at most one Möbius transform that takes three given, distinct points into three specified, distinct points. Because, if there were two, say f and g, then $f^{-1} \circ g$ would be a transform different from the identity and leaving the given points invariant. Conversely, we will prove that there actually always exists a Möbius transform that takes the given points into the specified ones. To see this, define the *cross ratio* of four distinct points z_0, z_1, z_2, z_3 in $\mathbb{C}^*$ by

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} / \frac{z_1 - z_2}{z_1 - z_3}$$

when all the points are finite. If one of them is ∞, the cross ratio is defined as the appropriate limit of the expression above. The following proposition follows by inspection.

PROPOSITION 1.23. *Suppose z_1, z_2, z_3 are distinct points in $\mathbb{C}^*$. The unique Möbius transform taking these points to $1, 0, \infty$ in order is $z \mapsto (z, z_1, z_2, z_3)$.*

It is now clear that to find the unique Möbius transform taking the distinct points z_1, z_2, z_3 into the distinct points w_1, w_2, w_3 in order, one simply has to solve for w in $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$.

EXERCISE 1.24. Find the Möbius transformation that carries 0, i, $-i$ in order into 1, -1, 0.

EXERCISE 1.25. Show that any Möbius transformation which leaves $\mathbb{R} \cup \{\infty\}$ invariant may be written with real coefficients.

EXERCISE 1.26. Show that the map $z \mapsto \frac{z-1}{z+1}$ maps the right half-plane (*i.e.*, the set $\operatorname{Re} z > 0$) onto the interior of the unit circle.

Two points z and z^* are said to be *symmetric with respect to* $\mathbb{R}$ if $z^* = \overline{z}$. If T is a Möbius transform that maps $\mathbb{R} \cup \{\infty\}$ onto itself, then according to Exercise 1.25 one may write T with real coefficients. It follows that Tz and $T(z^*)$ are symmetric with respect to the real axis if and only if z and z^* are. To generalize the concept of symmetry with respect to the real axis to symmetry with respect to any circle in the extended plane we make the following definition.

DEFINITION 1.27. Let Γ be a circle in $\mathbb{C}^*$. Two points z and z^* are said to be *symmetric with respect to* Γ if there is a Möbius transform T which maps Γ onto the real axis for which $T(z^*) = \overline{Tz}$.

By the reasoning just before the definition it is clear that this is a genuine extension of the notion of conjugate points and that z and z^* are symmetric with respect to Γ precisely if $T(z^*) = \overline{Tz}$ for *any* Möbius transform T that takes Γ to the real axis. For, if T and S both take Γ onto the real axis and $T(z^*) = \overline{Tz}$, then $U = ST^{-1}$ maps the real axis onto itself so that $S(z^*) = UT(z^*) = U(\overline{Tz}) = \overline{UTz} = \overline{Sz}$. There is therefore for every z precisely one point z^* so that z, z^* are symmetric with respect to Γ. A similar calculation proves the next theorem.

THEOREM 1.28. *Suppose S is a Möbius transform that takes the circle $\Gamma \in \mathbb{C}^*$ onto the circle $\Gamma' \in \mathbb{C}^*$. Then the points z and z^* are symmetric with respect to Γ if and only Sz and $S(z^*)$ are symmetric with respect to Γ'.*

PROOF. If T maps Γ onto the real axis, then $U = TS^{-1}$ maps Γ' onto the real axis. But $US(z^*) = T(z^*)$ and $USz = Tz$ so that $US(z^*) = \overline{USz}$ if and only if $T(z^*) = \overline{Tz}$. The theorem follows. □

In short, Theorem 1.28 says that symmetry is preserved by Möbius transforms. The next theorem allows us to calculate the symmetric point to any given z and circle.

THEOREM 1.29. *If Γ is a straight line, then z and z^* are symmetric with respect to Γ precisely if they are each others mirror image in Γ. If Γ is a genuine circle with center a and radius R, then a and ∞ are symmetric with respect to Γ. If z is finite and $\neq a$, then z and z^* are symmetric precisely if $(z^* - a)\overline{(z - a)} = R^2$.*

PROOF. If Γ is a straight line it is mapped onto the real axis by a translation or a rotation and these transformations obviously preserve mirror images.

If Γ is a circle with center a and radius R the map $z \mapsto i\frac{z-a-R}{z-a+R}$ takes Γ onto the real axis (since $a + R \mapsto 0$, $a - R \mapsto \infty$ and $a - iR \mapsto 1$). Now a and ∞ are mapped onto $-i$ and i respectively, so they are a symmetric pair. If z has neither of these values a simple calculation shows that z and z^* are mapped onto conjugate points precisely if

In particular the fact that the center of a circle and ∞ are symmetric with respect to the circle are often very helpful in trying to find maps that take a given circle into another.

EXERCISE 1.30. Find the Möbius transform which carries the circle $|z| = 2$ into $|z + 1| = 1$, the point -2 into the origin, and the origin into i.

EXERCISE 1.31. Find all Möbius transforms that leave the circle $|z| = R$ invariant. Which of these leave the *interior* of the circle invariant?

EXERCISE 1.32. Suppose a Möbius transform maps a pair of concentric circles onto a pair of concentric circles. Is the ratio of the radii invariant under the map?

EXERCISE 1.33. Find all circles that are orthogonal to $|z| = 1$ and $|z - 1| = 4$.

We will end this section by discussing *conjugacy classes* of Möbius transforms.

DEFINITION 1.34. Two Möbius transforms S and T are called *conjugate* if there is a Möbius transform U such that $S = U^{-1}TU$.

Conjugacy is obviously an equivalence relation, *i.e.*, if we write $S \sim T$ when S is conjugate to T, then we have:

(1) $S \sim S$ for any Möbius transform S. (reflexive)
(2) If $S \sim T$, then $T \sim S$ (symmetric)
(3) If $S \sim T$ and $T \sim W$, then $S \sim W$. (transitive)

It follows that the set of all Möbius transforms is split into *equivalence classes* such that every transform belongs to exactly one equivalence class and is equivalent to all the transforms in the same class, but to no others.

EXERCISE 1.35. Prove the three properties above and the statement about equivalence classes. What are the elements of the equivalence class that contains the identity transform?

The concept of conjugacy has importance in the theory of (discrete) *dynamical systems*. This is the study of sequences generated by the iterates of some map, *i.e.*, if S is a map of some set M into itself, one studies sequences of the form $z, Sz, S^2z, \dots$ where $z \in M$. This sequence is called the (forward) *orbit* of z under the map S. One is particularly interested in what happens 'in the long run', *e.g.*, for which z's the sequence has a limit (and what the limit then is), for which z's the sequence is periodic and for which z's there seems to be

On the other hand, if λ is neither positive nor of absolute value 1 there is no disk which is invariant under T_λ. Show this as an exercise! The transforms in the conjugacy class of T_1 are called *parabolic*, those in the conjugacy class of T_λ for some $\lambda > 0$ but $\neq 1$ are called *hyperbolic* and those in the conjugacy class of T_λ for some $\lambda \neq 1$ with $|\lambda| = 1$ are called *elliptic*. The reason for these names will be clear from the result of Exercise 1.37. The remaining Möbius transforms are called *loxodromic*. This is because they are conjugate to a T_λ for which the sequence of iterates $z, T_\lambda z, T_\lambda^2 z, \ldots$ lie on a logarithmic spiral, which under stereographic projection becomes a curve known as a loxodrome.

EXERCISE 1.37. Suppose that the coefficients of the transformation

$$Sz = \frac{az+b}{cz+d}$$

are normalized by $ad-bc = 1$. Show that S is elliptic if $0 \leq (a+d)^2 < 4$, parabolic if $(a+d)^2 = 4$, hyperbolic if $(a+d)^2 > 4$ and loxodromic in all other cases. *Hint:* The determinant and the trace $a+d$ of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invariant under conjugation by an invertible matrix.

EXERCISE 1.38. Show that a linear transformation which satisfies $S^n = S$ for some integer n is necessarily elliptic.

EXERCISE 1.39. If S is hyperbolic or loxodromic, show that $S^n z$ converges to a fixpoint as $n \to \infty$, the same for all z which are not equal to the other fixpoint. The exceptional fixpoint is called *repelling*, the other one *attractive*. What happens when $n \to -\infty$? What happens in the parabolic and elliptic cases?

EXERCISE 1.40. Find all linear transformations that are rotations of the Riemann sphere.
Hint: The *antipodal point* to a point on the unit sphere is obtained by multiplication by -1. Use the fact that an antipodal pair is mapped onto an antipodal pair by a rotation.

POSSIBLE QUESTIONS

1. Show that a function f(z)=u(x,y)+iv(x.y) defined in a region D is analytic in it iff u(x,y) and v(x,y) are conjugate harmonic functions.

2.Prove that the cross ratio is preserved by a Bilinear transformation.

3. Show that a bilinear transformation maps straight lines and circles into straight lines and circles.

4. Find the analytic function f(z)= u + iv given that u-$v = e^x(\cos y - \sin y)$.

5. Find the analytic function f(z) if its real part is u(x,y)= $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

6. i) Prove that under a bilinear transformation no two points in z plane go to the same point in w plane.

   ii) Prove that the cross ratio is preserved by a bilinear transformation.

7. If f(z) = u + iv and u-v = e^x(cos y – sin y), find f(z) in terms of z.

8. Show that the transformation w=z^2 transform the families of lines x=h and y=k into co focal parabolas having w=0 as the common focus.

9.Show that the function u(x,y) =sin x coshy is harmonic .Find its harmonic conjugate v(x,y) and the analytic function f(z)=u + iv

10. Prove that the bilinear transformation which transforms z_1,z_2,z_3 into w_1,w_2,w_3 is

    $(w-w_1)(w_2-w_3) / (w-w_3)(w_2-w_1)= (z-z_1)(z_2-z_3) / (z-z_3)(z_2-z_1)$.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMPLEX ANALYSIS-I **Subject Code: 15MMU502**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
The Laplace equation of the form	$U_{xx}+U_{yy}=0$	$U_{xx}-U_{yy}=0$	$V_{xx}+U_{yy}=0$	$V_{xx}+Vx_{y}=0$	$U_{xx}+U_{yy}=0$
If $U=x^2-y^2$ then Uyy = ?	3	1	0	2	2
If $u(x,y)=e^x cosy$ then find u_x= ?	$e^x cosx$	$e^x cosy$	cosy	e^x	$e^x cosy$
The second order partial derivatives exist, continuous and satisfies the laplace equation is called functions	Analytic	Continuous	differentiable	harmonic	harmonic
If $U=x^2-y^2$ then U_{xx} = ?	3	2	0	1	2
The fixed point's transformation is also known as …… points transformation	mobius	invariant	constant	bilinear	constant
The bilinear transformation of the form W=	az+b/cz+d	az+b/c+d	az+b	az+b/c	az+b/cz+d
A function which is ………. in region which is not close may or may not be bounded in it.	Analytic	differentiable	continuous	bounded	Analytic
The function 1/(1+z) is analytic at infinity because the function 1/(1+1/z) is	Analytic at 0	continuous at 0	differentiable at 0	analytic at 1	Analytic at 0
If a function is differentiable at a points then the function is said to be ….	analytic at that point	continuous at that point	differentiabe at that point	not differentiable at that point	differentiabe at that point
The Laplace equation of the format	$U_{xx}+U_{yy}=0$	$U_{xx}-U_{yy}=0$	$V_{xx}+U_{yy}=0$	$V_{xx}+V_{yy}=0$	$U_{xx}+U_{yy}=0$
The bilinear transformation is also known as ……… transformation	non mobius	linear	mobius	non linear	mobius
The equations $u_x=v_y$ and $u_y= -v_x$ are	Polar equation	Euler equation	C - R equation	coordinates	C - R equation
If u or v is not harmonic , then u+iv is ………	analytic	not analytic	conjugate harmonic	diffrentiable	not analytic
If f(z) =u(x,y) + iv(x,y) is analytic in domain d iff u(x,y) and v(x,y) are	harmonic	conjugate harmonic	differentiable	continuous	conjugate harmonic
In a two dimensional flow the stream function is $tan^{-1}y/x$ then the velocity potential is	$1/2log(x^2 + y^2)$	$Sin^{-1} y/x$	$log(x^2 + y^2)$	$cos^{-1} y/x$	$1/2log(x^2 + y^2)$
By Milne – Thomson method if u (x,y) = $x^2 - y^2$ then f(z) =	Z^2	2x+2y	x+y	z	Z^2
The function f(z) = $z^{1/2}$ is …………. Valued function	single	multi	double	triple	double
The transformation w = z^2 maps the -------------- onto the straight lines	parabola	hyperbola	ellipse	rectangular hyperbola	rectangular hyperbola
If f(z) = u+iv is an analytic function then -if(z) =	u-iv	v+iu	u+v	v+i(-u)	v+i(-u)
The value of m such that $2x - x^2 + my^2$ may be harmonic is ----	1	2	0	3	1
If f(z) = u+iv is an analytic function then(1 -i)f(z) =	(u+v)+i(v-u)	(u+v)-i(v-u)	(u-v)+i(v-u)	(u+v)+i(v+u)	(u+v)+i(v-u)
If f(z) = u+iv is an analytic function then(1+i)f(z) =	(u+v)+i(v-u)	(u+v)-i(v-u)	(u-v)+i(u+v)	(u+v)+i(v+u)	(u-v)+i(u+v)
Harmonic functions in polar coordinates are	$U_{rr} + 1/r\, u_r + 1/r^2 u_{\theta\theta}$	$U_{rr} + r\, u_r + 1/r^2 u_{\theta\theta}$	$U_{rr} + 1/r^2 u_{\theta\theta}$	$U_{rr} + 1/r\, u_r + 1/r^2 u_{\theta\theta}$	$U_{rr} + 1/r\, u_r + 1/r^2 u_{\theta\theta}$
The function ----------- is called zhukosky's function	1/z	z+1/z	z	sinz	z+1/z
If w = u+iv under w = z+1/z then u = ….	u = (r + 1/r)cosθ	u = (r - 1/r)cosθ	u = (r + 1/r)sinθ	u = r cosθ	u = (r + 1/r)cosθ
If w = u+iv under w = z+1/z then v = ….	v = (r + 1/r)cosθ	v = rsinθ	v= (r - 1/r)sinθ	v = r cosθ	v= (r - 1/r)sinθ
A circle whose centre is origin goes onto an …… whose centre is the origin under the zhukosky's transformation.	parabola	hyperbola	ellipse	rectangular hyperbola	ellipse
A ray emanating from the origin goes onto a ….. Whose centre is the origin under the zhukosky's transformation	parabola	hyperbola	ellipse	rectangular hyperbola	hyperbola
The principle value of log z are ….	logr	logr+iθ	log1/r	logr-iθ	logr+iθ
. The partial derivatives are all ----- in domain D	analytic	not analytic	does not exists	continuous	analytic
w = cos z is a ------ function	analytic	continuous	not analytic	limit	analytic
. f(z) = xy + iy is --------	analytic	continuous	analytic anywhere	limit	continuous
. The function f(z) = |z| is differentiable -----	on real part	on imaginary part	at the origin	at the point 2	at the origin
If f(z) has the derivative only at the origin, it is ------	analytic everywhere	not analytic nowhere	analytic nowhere	continuous nowhere	not analytic nowhere
f(z) = 1/z is a ------ function	differentiable	continuous	analytic	not analytic	analytic
An analytic function with constant real part is ------	constant	real	imaginary	not analytic	constant
An analytic function with constant imaginary part is ------	constant	real	imaginary	not analytic	constant
An analytic function with constant modulus part is ------	constant	real	imaginary	not analytic	constant
Both real part and imaginary part of any analytic function satisfies ------	wave equation	polynomial equation	del operator	laplace's equation	laplace's equation

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

DEPARTMENT OF MATHEMATICS

Subject: COMPLEX ANALYSIS-I **Semester: V** **L T P C**
Subject Code: 15MMU502 **Class: III-B.Sc Mathematics** **5 0 0 5**

UNIT V

Complex integration: Simple rectifiable oriented curves –Integration of complex functions- Definite integral-Interior and Exterior of a closed curve-Simply connected region-Cauchy‟s fundamental theorem-Cauchy‟s formula for higher derivatives- Morera‟s theorem.

REFERENCES

1. Lars V.Ahlfors.,1979. Complex Analysis, Third edition, Mc-Graw Hill Book Company,New Delhi

2. Choudhary.B., 2003. The Elements of Complex Analysis ,New Age International Pvt.Ltd ,New Delhi.

3. Narayanan .S., T.K Manichavachagam Pillay, 1992. Complex Analysis. S.Viswanathan

(printers & publishers) pvt Ltd, Madras

Definite Integral It is likely that you have retained at least two associations from your study of elementary calculus: the derivative with slope, and the definite integral with area. But as the derivative $f'(x)$ of a real function $y = f(x)$ has other uses besides finding slopes of tangent lines, so too the value of a definite integral $\int_a^b f(x)\,dx$ need not be area "under a curve." Recall, if $F(x)$ is an antiderivative of a continuous function f, that is, F is a function for which $F'(x) = f(x)$, then the definite integral of f on the interval $[a, b]$ is the number

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a). \tag{1}$$

For example, $\int_{-1}^2 x^2 dx = \frac{1}{3}x^3\Big|_{-1}^2 = \frac{8}{3} - \left(-\frac{1}{3}\right) = 3$. Bear in mind that the **fundamental theorem of calculus**, just given in (1), is a method of *evaluating* $\int_a^b f(x)\,dx$; it is not the *definition* of $\int_a^b f(x)\,dx$.

In the discussion that follows we present the definitions of two types of **real integrals**. We begin with the five steps leading to the definition of the definite (or Riemann) integral of a function f; we follow it with the definition of line integrals in the Cartesian plane. Both definitions rest on the limit concept.

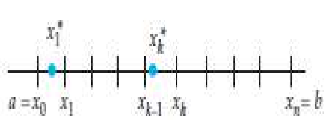


Figure 5.1 Partition of $[a, b]$ with x_k^* in each subinterval $[x_{k-1}, x_k]$

Steps Leading to the Definition of the Definite Integral

1. *Let f be a function of a single variable x defined at all points in a closed interval $[a, b]$.*
2. *Let P be a partition:*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

of $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$. See Figure 5.1.

3. *Let $\|P\|$ be the* **norm** *of the partition P of $[a, b]$, that is, the length of the longest subinterval.*

4. *Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$ of $[a, b]$. See Figure 5.1.*

5. *Form n products $f(x_k^*)\Delta x_k$, $k = 1, 2, \ldots, n$, and then sum these products:*

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

Definition 5.1 Definite Integral

The **definite integral** of f on $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k. \tag{2}$$

Whenever the limit in (2) exists we say that f is **integrable** on the interval $[a, b]$ or that the definite integral of f **exists**. It can be proved that if f is continuous on $[a, b]$, then the integral defined in (2) exists.

The notion of the definite integral $\int_a^b f(x)\,dx$, that is, *integration of a real function $f(x)$ over an interval on the x-axis from $x = a$ to $x = b$* can be generalized to *integration of a real multivariable function $G(x, y)$ on a curve C from point A to point B* in the Cartesian plane. To this end we need to introduce some terminology about curves.

Orientation of a Curve In definite integration we normally assume that the interval of integration is $a \leq x \leq b$ and the symbol $\int_a^b f(x)\,dx$ indicates that we are integrating in the positive direction on the x-axis. Integration in the opposite direction, from $x = b$ to $x = a$, results in the negative of the original integral:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{14}$$

Line integrals possess a property similar to (14), but first we have to introduce the notion of **orientation** of the path C. If C is not a closed curve, then we say the **positive direction** on C, or that C has **positive orientation**, if we traverse C from its initial point A to its terminal point B. In other words, if $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ are parametric equations for C, then the positive direction on C corresponds to increasing values of the parameter t. If C is traversed in the sense opposite to that of the positive orientation, then C is said to have **negative orientation.** If C has an orientation (positive or negative), then the **opposite curve**, the curve with the opposite orientation, will be denoted by the symbol $-C$. In Figure 5.8 if we assume that A and B are the initial and terminal points of the curve C, respectively, then the arrows on curve C indicate that we are traversing the curve from its initial point to its terminal point, and so C has positive orientation. The curve to the right of C that is labeled $-C$ then has negative orientation. Finally, if $-C$

denotes the curve having the opposite orientation of C, then the analogue of (14) for line integrals is

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy, \tag{15}$$

or, equivalently

$$\int_{-C} P\,dx + Q\,dy + \int_C P\,dx + Q\,dy = 0. \tag{16}$$

For example, in part (a) of Example 1 we saw that $\int_C xy^2\,dx = -64$; we conclude from (15) that $\int_{-C} xy^2\,dx = 64$.

It is important to be aware that a line integral is independent of the parametrization of the curve C, provided C is given the same orientation by all sets of parametric equations defining the curve. See Problem 33 in

Curves Revisited Suppose the continuous real-valued functions $x = x(t)$, $y = y(t)$, $a \le t \le b$, are parametric equations of a curve C in the complex plane. If we use these equations as the real and imaginary parts in $z = x + iy$, we saw in Section 2.2 that we can describe the points z on C by means of a complex-valued function of a real variable t called a **parametrization** of C:

$$z(t) = x(t) + iy(t), \quad a \le t \le b. \tag{1}$$

For example, the parametric equations $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}, 0 \le t \le 2\pi$. See (6)–(10) in Section 2.2.

The point $z(a) = x(a) + iy(a)$ or $A = (x(a),\ y(a))$ is called the **initial point** of C and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ is its **terminal point**. We also saw in Section 2.7 that $z(t) = x(t) + iy(t)$ could also be interpreted as a two-dimensional vector function. Consequently, $z(a)$ and $z(b)$ can be interpreted as position vectors. As t varies from $t = a$ to $t = b$ we can envision the curve C being traced out by the moving arrowhead of $z(t)$. See Figure 5.15.

Contours The notions of curves in the complex plane that are smooth, piecewise smooth, simple, closed, and simple closed are easily formulated in terms of the vector function (1). Suppose the derivative of (1) is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \le t \le b$. As shown Figure 5.16, since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent; put yet another way, a smooth curve can have no sharp corners or cusps. See Figure 5.17. A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together. A curve C in the complex plane is said to be a **simple** if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$, except possibly for $t = a$ and $t = b$. C is a **closed curve** if $z(a) = z(b)$. C is a **simple closed curve** if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$ and $z(a) = z(b)$. In complex analysis, a piecewise smooth curve C is called a **contour** or **path**.

Just as we did in the preceding section, we define the **positive direction** on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. It is also said that the curve C has **positive orientation**. In the case of a simple closed curve C, the positive direction roughly corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left. For example, the circle $z(t) = e^{it}$, $0 \le t \le 2\pi$, has positive orientation. See Figure 5.18. The **negative direction** on a contour C is the direction opposite the positive direction. If C has an orientation, the **opposite curve**, that is, a curve with opposite orientation, is denoted by $-C$. On a simple closed curve, the negative direction corresponds to the clockwise direction.

Complex Integral An integral of a function f of a complex variable z

Evaluation of Contour Integrals To facilitate the discussion on how to evaluate a contour integral $\int_C f(z)\,dz$, let us write (2) in an abbreviated form. If we use $u+iv$ for f, $\Delta x+i\Delta y$ for Δz, lim for $\lim_{||P||\to 0}$, $\sum$ for $\sum_{k=1}^{n}$ and then suppress all subscripts, (2) becomes

$$\begin{aligned}\int_C f(z)dz &= \lim \sum (u+iv)(\Delta x + i\Delta y)\\ &= \lim \left[\sum (u\Delta x - v\Delta y) + i\sum (v\Delta x + u\Delta y)\right].\end{aligned}$$

The interpretation of the last line is

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \tag{9}$$

See Definition 5.2. In other words, the real and imaginary parts of a contour integral $\int_C f(z)\,dz$ are a pair of real line integrals $\int_C u\,dx - v\,dy$ and $\int_C v\,dx + u\,dy$. Now if $x = x(t)$, $y = y(t)$, $a \le t \le b$ are parametric equations of C, then $dx = x'(t)\,dt$, $dy = y'(t)\,dt$. By replacing the symbols x, y, dx,

Properties The following properties of contour integrals are analogous to the properties of real line integrals as well as the properties listed in (5)–(8).

Theorem 5.2 Properties of Contour Integrals

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then

(i) $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, k a complex constant.

(ii) $\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

The four parts of Theorem 5.2 also hold if C is a piecewise smooth curve in D.

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C u(x,y)\,dx - v(x,y)\,dy + i\oint_C v(x,y)\,dx + u(x,y)\,dy\\ &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA.\end{aligned} \tag{3}$$

Because f is analytic in D, the real functions u and v satisfy the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, at every point in

D. Expressed yet another way, a simply connected domain has no "holes" in it. The entire complex plane is an example of a simply connected domain; the annulus defined by $1 < |z| < 2$ is not simply connected. (Why?) A domain that is not simply connected is called a **multiply connected domain**; that is, a multiply connected domain has "holes" in it. Note in Figure 5.27 that if the curve C_2 enclosing the "hole" were shrunk to a point, the curve would have to leave D eventually. We call a domain with one "hole" **doubly connected**, a domain with two "holes" **triply connected**, and so on. The open disk defined by $|z| < 2$ is a simply connected domain; the open circular annulus defined by $1 < |z| < 2$ is a doubly connected domain.

Cauchy's Theorem In 1825 the French mathematician Louis-Augustin Cauchy proved one the most important theorems in complex analysis.

Cauchy's Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. (1)

Cauchy's Proof of (1) The proof of this theorem is an immediate consequence of Green's theorem in the plane and the Cauchy-Riemann equations. Recall from calculus that if C is a positively oriented, piecewise smooth, simple closed curve forming the boundary of a region R within D, and if the real-valued functions $P(x, y)$ and $Q(x, y)$ along with their first-order partial derivatives are continuous on a domain that contains C and R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{2}$$

Now in the statement (1) we have assumed that f' is continuous throughout the domain D. As a consequence, the real and imaginary parts of $f(z) = u+iv$ and their first partial derivatives are continuous throughout D. By (9) of Section 5.2 we write $\oint_C f(z)\,dz$ in terms of real line integrals and apply Green's theorem (2) to each line integral:

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C u(x,y)\,dx - v(x,y)\,dy + i\oint_C v(x,y)\,dx + u(x,y)\,dy \\ &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA.\end{aligned} \tag{3}$$

Because f is analytic in D, the real functions u and v satisfy the Cauchy-

D. Using the Cauchy-Riemann equations to replace $\partial u/\partial y$ and $\partial u/\partial x$ in (3) shows that

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dA + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dA$$
$$= \iint_R (0)\ dA + i\iint_R (0)\ dA = 0.$$

This completes the proof.

In 1883 the French mathematician Edouard Goursat proved that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the **Cauchy-Goursat theorem**. As one might expect, with fewer hypotheses, the proof of this version of Cauchy's theorem is more complicated than the one just presented. A form of the proof devised by Goursat is outlined in Appendix II.

Theorem 5.4 Cauchy-Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$.

Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can be stated in the slightly more practical manner:

If f is analytic at all points within and on a simple closed contour C, then $\oint_C f(z)\,dz = 0$. (4)

EXAMPLE 1 Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C e^z\,dz$, where the contour C is shown in Figure 5.28.

Solution The function $f(z) = e^z$ is entire and consequently is analytic at all points within and on the simple closed contour C. It follows from the form of the Cauchy-Goursat theorem given in (4) that $\oint_C e^z\,dz = 0$.

The point of Example 1 is that $\oint_C e^z\,dz = 0$ for *any* simple closed contour in the complex plane. Indeed, it follows that for any simple closed contour C and any entire function f, such as $f(z) = \sin z$, $f(z) = \cos z$, and $p(z) = a_n z^n + a_{n-1}z^n + \cdots + a_1 z + a_0$, $n = 0, 1, 2, \ldots$,that

$$\oint_C \sin z\,dz = 0, \quad \oint_C \cos z\,dz = 0, \quad \oint_C p(z)\,dz = 0,$$

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0, & n \neq 1. \end{cases} \tag{6}$$

The fact that the integral in (6) is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem. When n is zero or a negative integer, $1/(z-z_0)^n$ is a *polynomial* and therefore entire. Theorem 5.4 and the discussion following Example 1 then indicates that $\oint_C dz/(z-z_0)^n = 0$. It is left as an exercise to show that the integral is still zero when n is a positive integer different from 1. See Problem 24 in Exercises 5.3.

Analyticity of the function f at all points within and on a simple closed contour C is *sufficient* to guarantee that $\oint_C f(z)\,dz = 0$. However, the result in (6) emphasizes that analyticity is not *necessary*; in other words, it can happen that $\oint_C f(z)\,dz = 0$ without f being analytic within C. For instance, if C in Example 2 is the circle $|z| = 1$, then (6), with the identifications $n = 2$ and $z_0 = 0$, immediately gives $\oint_C \frac{dz}{z^2} = 0$. Note that $f(z) = 1/z^2$ is not analytic at $z = 0$ within C.

EXAMPLE 4 Applying Formula (6)

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is circle $|z-2| = 2$.

Solution Since the denominator factors as $z^2 + 2z - 3 = (z-1)(z+3)$ the integrand fails to be analytic at $z = 1$ and $z = -3$. Of these two points, only $z = 1$ lies within the contour C, which is a circle centered at $z = 2$ of radius $r = 2$. Now by partial fractions

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3}$$

and so

$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz = 3\oint_C \frac{1}{z-1}\,dz + 2\oint_C \frac{1}{z+3}\,dz. \tag{7}$$

In view of the result given in (6), the first integral in (7) has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem.

$|z-i| = 1$, which from (10) of Section 2.2 can be parametrized by $z = i + e^{it}$, $0 \leq t \leq 2\pi$. From $z - i = e^{it}$ and $dz = ie^{it}dt$ we obtain

$$\oint_C \frac{dz}{z-i} = \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}dt = i\int_0^{2\pi} dt = 2\pi i.$$

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3} dz$, where C is circle $|z-2|=2$.

Solution Since the denominator factors as $z^2+2z-3=(z-1)(z+3)$ the integrand fails to be analytic at $z=1$ and $z=-3$. Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. Now by partial fractions

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3}$$

and so
$$\oint_C \frac{5z+7}{z^2+2z-3}\, dz = 3\oint_C \frac{1}{z-1}\, dz + 2\oint_C \frac{1}{z+3}\, dz. \qquad (7)$$

In view of the result given in (6), the first integral in (7) has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem.

Hence, (7) becomes

$$\oint_C \frac{5z+7}{z^2+2z-3}\, dz = 3(2\pi i) + 2(0) = 6\pi i.$$

5.5.1 Cauchy's Two Integral Formulas

First Formula If f is analytic in a simply connected domain D and z_0 is any point in D, the quotient $f(z)/(z-z_0)$ is not defined at z_0 and hence is *not* analytic in D. Therefore, we *cannot* conclude that the integral of $f(z)/(z-z_0)$ around a simple closed contour C that contains z_0 is zero by the Cauchy-Goursat theorem. Indeed, as we shall now see, the integral of $f(z)/(z-z_0)$ around C has the value $2\pi i f(z_0)$. The first of two remarkable formulas is known simply as the **Cauchy integral formula**.

Theorem 5.9 Cauchy's Integral Formula

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz. \tag{1}$$

Proof Let D be a simply connected domain, C a simple closed contour in D, and z_0 an interior point of C. In addition, let C_1 be a circle centered at z_0 with radius small enough so that C_1 lies within the interior of C. By the principle of deformation of contours, (5) of Section 5.3, we can write

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz. \tag{2}$$

We wish to show that the value of the integral on the right is $2\pi i f(z_0)$. To this end we add and subtract the constant $f(z_0)$ in the numerator of the integrand,

$$\begin{aligned}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz &= \oint_{C_1} \frac{f(z_0)-f(z_0)+f(z)}{z-z_0}\,dz.\\ &= f(z_0)\oint_{C_1}\frac{1}{z-z_0}\,dz + \oint_{C_1}\frac{f(z)-f(z_0)}{z-z_0}\,dz. \end{aligned} \tag{3}$$

From (6) of Section 5.3 we know that

$$\oint_{C_1} \frac{1}{z-z_0}\,dz = 2\pi i \tag{4}$$

and so (3) becomes

$$\oint_{C_1} \frac{f(z)}{z-z_0}\,dz = 2\pi i\,f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \qquad (5)$$

Since f is continuous at z_0, we know that for any arbitrarily small $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. In particular, if we choose the circle C_1 to be $|z - z_0| = \frac{1}{2}\delta < \delta$, then by the *ML*-inequality (Theorem 5.3) the absolute value of the integral on the right side of the equality in (5) satisfies

$$\left| \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le \frac{\varepsilon}{\delta/2} 2\pi \left(\frac{\delta}{2}\right) = 2\pi\varepsilon.$$

In other words, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle C_1 to be sufficiently small. This can happen only if the integral is 0. Thus (5) is $\oint_{C_1} \frac{f(z)}{z-z_0}\,dz = 2\pi i\,f(z_0)$. The theorem is proved by dividing both sides of the last result by $2\pi i$. ✎

Because the symbol z represents a point on the contour C, (1) indicates that *the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.*

Cauchy's integral formula (1) can be used to evaluate contour integrals. Since we often work problems without a simply connected domain explicitly defined, a more practical restatement of Theorem 5.9 is:

If f is analytic at all points within and on a simple closed contour C, and z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$.

EXAMPLE 1 Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$, where C is the circle $|z| = 2$.

Solution First, we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula (1) we obtain

$$\oint_C \frac{z^2-4z+4}{z+i}\,dz = 2\pi i f(-i) = 2\pi i(3+4i) = \pi(-8+6i).$$

Second Formula We shall now build on Theorem 5.9 by using it to prove that the values of the derivatives $f^{(n)}(z_0), n = 1, 2, 3, \ldots$ of an analytic function are also given by a integral formula. This second integral formula is similar to (1) and is known by the name **Cauchy's integral formula for derivatives.**

Theorem 5.10 Cauchy's Integral Formula for Derivatives

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{6}$$

Partial Proof We will prove (6) only for the case $n = 1$. The remainder of the proof can be completed using the principle of mathematical induction. We begin with the definition of the derivative and (1):

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\ &= \lim_{\Delta z\to 0}\frac{1}{2\pi i\,\Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz - \oint_C \frac{f(z)}{z-z_0}\,dz\right] \\ &= \lim_{\Delta z\to 0}\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz. \end{aligned}$$

Before continuing, let us set out some preliminaries. Continuity of f on the contour C guarantees that f is bounded (see page 124 of Section 2.6), that is, there exists a real number M such that $|f(z)| \le M$ for all points z on C.

In addition, let L be the length of C and let δ denote the shortest distance between points on C and the point z_0. Thus for all points z on C we have

$$|z-z_0| \geq \delta \qquad \text{or} \qquad \frac{1}{|z-z_0|^2} \leq \frac{1}{\delta^2}.$$

Furthermore, if we choose $|\Delta z| \leq \frac{1}{2}\delta$, then by (10) of Section 1.2,

$$|z-z_0-\Delta z| \geq \big|\,|z-z_0| - |\Delta z|\,\big| \geq \delta - |\Delta z| \geq \tfrac{1}{2}\delta$$

and so,
$$\frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{\delta}.$$

Now,

$$\left|\oint_C \frac{f(z)}{(z-z_0)^2}\,dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz\right|$$
$$= \left|\oint_C \frac{-\Delta z\, f(z)}{(z-z_0-\Delta z)(z-z_0)^2}\,dz\right| \leq \frac{2ML\,|\Delta z|}{\delta^3}.$$

Because the last expression approaches zero as $\Delta z \to 0$, we have shown that

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz,$$

which is (6) for $n = 1$.

Like (1), formula (6) can be used to evaluate integrals.

EXAMPLE 3 Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where C is the circle $|z| = 1$.

Solution Inspection of the integrand shows that it is not analytic at $z = 0$ and $z = -2i$, but only $z = 0$ lies within the closed contour. By writing the integrand as

$$\frac{z+1}{z^4+2iz^3} = \frac{\dfrac{z+1}{z+2i}}{z^3}$$

we can identify, $z_0 = 0$, $n = 2$, and $f(z) = (z+1)/(z+2i)$. The quotient rule gives $f''(z) = (2-4i)/(z+2i)^3$and so $f''(0) = (2i-1)/4i$. Hence from (6) we find

$$\oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2!}f''(0) = -\frac{\pi}{4}+\frac{\pi}{2}i.$$

EXAMPLE 4 Using Cauchy's Integral Formula for Derivatives

Evaluate $\int_C \frac{z^3+3}{z(z-i)^2}\,dz$, where C is the figure-eight contour shown in Figure 5.45.

Solution Although C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2 as indicated in Figure 5.45. Since the arrows on C_1 flow clockwise or in the negative direction, the opposite curve $-C_1$ has positive orientation. Hence, we write

$$\int_C \frac{z^3+3}{z(z-i)^2}\,dz = \int_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \int_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\oint_{-C_1} \frac{\dfrac{z^3+3}{(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{(z-i)^2}\,dz = -I_1 + I_2,$$

and we are in a position to use both formulas (1) and (6).

To evaluate I_1 we identify $z_0 = 0$, $f(z) = (z^3+3)/(z-i)^2$, and $f(0) = -3$. By (1) it follows that

$$I_1 = \oint_{-C_1} \frac{\dfrac{z^3+3}{(z-i)^2}}{z}\,dz = 2\pi i\, f(0) = 2\pi i(-3) = -6\pi i.$$

To evaluate I_2 we now identify $z_0 = i$, $n = 1$, $f(z) = (z^3+3)/z$, $f'(z) = (2z^3-3)/z^2$, and $f'(i) = 3+2i$. From (6) we obtain

$$I_2 = \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f'(i) = 2\pi i(3+2i) = -4\pi + 6\pi i.$$

Finally, we get

$$\int_C \frac{z^3+3}{z(z-i)^2}\,dz = -I_1 + I_2 = 6\pi i + (-4\pi + 6\pi i) = -4\pi + 12\pi i.$$

Cauchy's Inequality We begin with an inequality derived from the Cauchy integral formula for derivatives.

Theorem 5.12 Cauchy's Inequality

Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \leq M$ for all points z on C, then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{r^n}. \tag{7}$$

Proof From the hypothesis,

$$\left| \frac{f(z)}{(z - z_0)^{n+1}} \right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{M}{r^{n+1}}.$$

Thus from (6) and the *ML*-inequality (Theorem 5.3), we have

$$\left| f^{(n)}(z_0) \right| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}.$$

The number M in Theorem 5.12 depends on the circle $|z - z_0| = r$. But notice in (7) that if $n = 0$, then $M \geq | f(z_0) |$ for *any* circle C centered at z_0 as long as C lies within D. In other words, an upper bound M of $|f(z)|$ on C cannot be smaller than $| f(z_0) |$.

is analytic in view of Theorem 5.11. Since $f(z) = F'(z)$, we see that f is analytic in D.

An alternative proof of this last result is outlined in Problem 31 in Exercises 5.5.

We could go on at length stating more and more results whose proofs rest on a foundation of theory that includes the Cauchy-Goursat theorem and the Cauchy integral formulas. But we shall stop after one more theorem.

In Section 2.6 we saw that if a function f is continuous on a closed and bounded region R, then f is bounded; that is, there is some constant M such that $|f(z)| \leq M$ for z in R. If the boundary of R is a simple closed curve C, then the next theorem, which we present without proof, tells us that $|f(z)|$ assumes its maximum value at some point z on the boundary C.

Theorem 5.16 Maximum Modulus Theorem

Suppose that f is analytic and nonconstant on a closed region R bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.

If the stipulation that $f(z) \neq 0$ for all z in R is added to the hypotheses of Theorem 5.16, then the modulus $|f(z)|$ also attains its *minimum* on C. See Problems 27 and 33 in Exercises 5.5.

EXAMPLE 5 Maximum Modulus

Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.

Solution From (2) of Section 1.2 we know that $|z|^2 = z\bar{z}$. By replacing the symbol z by $2z + 5i$ we have

$$+\,5i|^2 = (2z+5i)\overline{(2z+5i)} = (2z+5i)(2\bar{z}-5i) = 4z\bar{z} - 10i(z-\bar{z}) + 25. \tag{8}$$

But from (6) of Section 1.1, $\bar{z} - z = 2i\,\mathrm{Im}(z)$, and so (8) is

$$|2z+5i|^2 = 4|z|^2 + 20\,\mathrm{Im}(z) + 25. \tag{9}$$

Theorem 5.15 Morera's Theorem

If f is continuous in a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed contour C in D, then f is analytic in D.

Proof By the hypotheses of continuity of f and $\oint_C f(z)\,dz = 0$ for every closed contour C in D, we conclude that $\int_C f(z)\,dz$ is independent of the path. In the proof of (7) of Section 5.4 we then saw that the function F defined by $F(z) = \int_{z_0}^{z} f(s)\,ds$ (where s denotes a complex variable, z_0 is a fixed point in D, and z represents any point in D) is an antiderivative of f; that is, $F'(z) = f(z)$. Hence, F is analytic in D. In addition, $F'(z)$ is analytic in view of Theorem 5.11. Since $f(z) = F'(z)$, we see that f is analytic in D. ✎

An alternative proof of this last result is outlined in Problem 31 in Exercises 5.5.

We could go on at length stating more and more results whose proofs rest on a foundation of theory that includes the Cauchy-Goursat theorem and the Cauchy integral formulas. But we shall stop after one more theorem.

In Section 2.6 we saw that if a function f is continuous on a closed and bounded region R, then f is bounded; that is, there is some constant M such that $|f(z)| \le M$ for z in R. If the boundary of R is a simple closed curve C, then the next theorem, which we present without proof, tells us that $|f(z)|$ assumes its maximum value at some point z on the boundary C.

POSSIBLE QUESTIONS

1. State and prove Cauchy's Integral formula for nth derivative.

2. State and prove Morera's theorem

3. State and prove Cauchy's formula for first derivative.

4. State and prove Goursat's Lemma.

5. If C is an simple arc of length L and f(z) is a continuous function defined on C and if on C, max| f(z) |$\leq M$ then | $\left|\int_c f(z)\,dz\right| \leq ML$.

6. i). Find the integral of f(z) = z^2 along the parabolic arc y= x^2 from (0,0) to (1,1).

   ii) Prove that $\int_C 1/(z-a)\ dz = 2\prod i$, where C is a positively oriented circle whose radius is r and centre is z=a.

7.State and prove Cauchy's integral formula

8. i) Evaluate the integrals $\int_c \frac{1}{z^2-1} dz$ along C,the positively oriented circle |z|=2.

   ii) State and prove Cauchy's fundamental theorem.

9. State and prove Morera's theorem.

10. Using Cauchy's integral formula, evaluate$\int_c \frac{z+4}{z^2+2z+5} dz$ where C is |z+1+i|=2.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMPLEX ANALYSIS-I **Subject Code: 15MMU502**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
The equation z = cost+isint, 0≤t≤π represents a ……………	arc	simple arc	closed arc	curve	simple arc
The unit circle z=cost+isint, 0£t£2P are	positively oriented circle	negatively oriented circle	circle	unit circle	positively oriented circle
The unit circle z = cos(-t) +isin(-t), 0≤t≤2π are ……..	positively oriented circle	negatively oriented circle	circle	unit circle	negatively oriented circle
It the region lies to the left of a person when he travels along C, then the curve C is called a ………..	postively oriented simple closed curve	negatively oriented simple closed curve	open curve	simple closed curve	postively oriented simple closed curve
The simple closed rectifible curve is abbreviated as……	curve	scr curve	scro curve	arc	scr curve
In cauchy's fundamental theorem, ò f(z) dz=…	1	2	0	4	0
The simple closed rectifiable positively oriented curve is abbreviated as …..	curve	scr curve	scro curve	arc	scro curve
The simple arc is also known as ….	multiple	Jordan	double	multiple	Jordan
The derivative of an analytic function is also …	analytic	continuous	derivative	bounded	continuous
The integral ò f(z) dz=F(b)-F(a) is called a	integral	indefinite	definite	derivative	derivative
The poles of an analytic function are ……	essential	removable	pole	isolated	isolated
If C is a positively oriented circle then ò 1/(z-a) dz=	2P	2Pi	0	P	2Pi
When the order of the pole is 2,the pole is said to be ……pole	double	simple	multiple	triple	multiple
The limit point of zero's of an analytic function is a ……point of the function	singular	nonsingular	poles	zeros	singular
A region which has only one hole is an …..region	origin	set	annular	moment	annular
A region which is not simply connected is called …	connected	compact	multiply-connected	region.	compact
The integrals along scr curves are called….	complex integrals	integrals	contour integrals	partial integrals	contour integrals
If f(z) is a continuous function defined on a simple rectifiable curve then ∫f(z) dz =	∫ (u dx- vdy) + i∫(udy-vdx)	∫ (u dx- vdy) - i∫(udy-vdx)	∫ (u dx- vdy) +∫(udy-vdx)	∫ (u dx+ vdy) + i∫(udy+vdx)	∫ (u dx- vdy) + i∫(udy-vdx)
∫ [f_1(z) +f_2(z)]dz on C is	∫ f_1(z)dz + ∫f_2(z)dz	∫ f_1(z)dz - ∫f_2(z)dz	∫ f_1(z)dz . ∫f_2(z)dz	∫ f_1(z)dz / ∫f_2(z)dz	∫ f_1(z)dz + ∫f_2(z)dz
If f(z) is analytic in a simply connected domain , then the values of the integrals of f(z) along all paths in the region joining ------ fixed points are the same	one	two	three	multiple	two
The bounded region of C is called ……………	interior	exterior	interior nor exterior	interior and exterior	interior
A region D is said to be …………… for every closed curve in D , Ci is contained in D	connected	simply - connected	disconnected	disjoint	simply - connected
When A is fixed and B(z) moves in D, the integral	single - valued	double -valued	multi - valued	zero	single valued
cauchy's integral formula ………………..	1/2πi		1/2π	1/2πi	1/2πi
cauchy's integral formula for first derivatives ………………..	1/2πi	1/2π		1/2π	1/2πi
The function $(z-i)^2$ have a zero i of order…..	2	1	0	3	2
………… of an analytic function are isolated	zeros	poles	residues	points	zers
If f(z) = $(z – a)^m[a_0 + a_1(z-a)………]$,$a_0 \neq 0$, then z =a is a zero of order …..	m	1	2	0	m
If C is an arc in D , joining a fixed point z_0 and the arbitrary point z then d/dz	0	1	f(z)	c	f(z)
A function analytic in D has ………….. of all orders in D	derivatives	points	curves	zeros	derivatives
The path independent integral can be written as ………	$\int_0^a f(z)dz$	$\int_a^b f(z)dz$	-	$\int_0^b f(z)dz$	$\int_a^b f(z)dz$
A curve is said to be piece-wise smooth if C is not smooth at a …….. number of points in it.	finite	infinite	zero	one	finite

Reg No----------------------

[15MMU502]

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
FIFTH SEMESTER
I INTERNAL TEST-Jul'17
COMPLEX ANALYSIS-I

Date : .07.2017 **Time: 2 Hours**
Class : III B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)

Answer all the questions

1. The element (1,0) is the -------
   a. Additive identity b. Multiplicative identity
   c. identity d. unique
2. If $|Z_1| = |Z_2|$ and $\arg(Z_1)= \arg(Z_2)$ then -----
   a. $Z_1 \neq Z_2$ b. $Z_1 < Z_2$ c. $Z_1 > Z_2$ d. $Z_1 = Z_2$
3. The Equation of the unit circle whose centre is the origin is -
   a. $|Z| =1$ b. $|Z-a| =1$ c. $|Z| =0$ d. $|Z|\neq1$
4. The complex plane containing all the finite complex numbers and infinity is called the -----------
   a. infinite complex plane b. extended complex plane
   c. complex plane d. finite complex plane

5. The square of real number is ---------

   a. Non negative b. Non positive
   c. Negative d. absolute value

6. The polar form of x+iy is -----------

   a. $r(\cos\theta + i\sin\theta)$ b. $r(\cos\theta - i\sin\theta)$
   c. $\cos\theta + i\sin\theta$ d. $r(\cos\theta - \sin\theta)$

7. The conjugation of 5+i3 is -----------

   a. 5 b .3 c. 5+i3 d. 5-i3

8. The stereographic projection of the complex point z = (√2 ,1) is

   a. (1/√2, 1/√2, 0) b. (0, √2, 1) c. (1/√2, 1/2 , 1/2) d. (0, 0,1)

9. According to De Moivre's theorem $(\cos\theta + i\sin\theta)^n$ = ………

   a. $\cos^n\theta + i\sin^n\theta$ b. $\cos n\theta + i\sin n\theta$
   c. $n\cos\theta + in\sin\theta$ d. 1

10. The value of i^2 is -------

    a. 1 b. -1 c. 0 d. i

11. The square of real number is ---------

    a. Non negative b. Non positive

    c. Negative d. absolute value

12. The absolute value of z = x+iy is

    a. $\sqrt{x}$ b. $\sqrt{y}$ c. $\sqrt{x^2-y^2}$ d. $\sqrt{x^2+y^2}$

13. Under the transformation w = az there are ------ fixed points

    a. one b. two c. zero d. ∞

14. From $x= r\cos\theta$ and $y = r\sin\theta$ weget θ =

    a. $\sin^{-1} y/x$ b. $\cos^{-1}y/x$ c. $\tan^{-1}y/x$ d. $\cot^{-1}y/x$

15.A function analytic in D has ………….. of all orders in D

    a.derivatives b.points c. curves d.zeros

16. A ……………. of a function is a point at which the function ceases to be analytic

a. singular point b. analytic point

c. pole point d. essential point

17. Point z=a is a singular point of f(z) if f(z) is not …………… at z= a.

a. continuous b. not continuous

c. differentiable d. bounded

18. ………… of an analytic function are isolated.

a. zeros b. poles c. residues d. points

19. A bounded entire function is ………………………..

a.analytic b. function c.Constant d. zero

20. If f(z) is analytic inside and on an scr curve C,then the maximum always occurs at a ...…point

a. interior b. boundary c. singular d. analytic

PART-B(3X10=30 Marks)

ANSWER ALL THE QUESTIONS

21)a) i) If Z_1 and Z_2 are any two complex numbers ,then prove that

$\overline{(Z_1 Z_2)} = \overline{Z_1} . \overline{Z_2}$.

ii) If Z1 and Z_2 are any two complex numbers, then $arg\left(\frac{Z_1}{Z_2}\right) = argZ_1 - argZ_2$

(OR)

b) Explain the Stereographic projection of a complex plane.

22)a) i) If Z_1 and Z_2 are any two complex numbers ,then prove that $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$.

ii) Show that the argument of the product of two complex numbers is the sum of the arguments of the complex numbers.

(OR)

b) If Z, Z_1 and Z_2 are any three complex numbers ,then prove that

i) $-|Z| \leq Re\, Z \leq |Z|$, $-|Z| \leq Im\, Z \leq |Z|$,

ii) $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$

iii) $|Z_1 - Z_2| \geq ||Z_1| - |Z_2||$

23) a) Derive the C-R equations in polar coordinates.

(OR)

b) State and prove the sufficient condition for differentiability.

Reg No----------------------
[15MMU502]

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
FIFTH SEMESTER
II INTERNAL TEST-Aug'17
COMPLEX ANALYSIS-I

Date : .08.2017 **Time: 2 Hours**
Class : III B.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
Answer all the questions

1. The functions of the form, $P_n(Z)= a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n$, $a_n\neq 0$ is called a…………………
   a)polynomial of degree n b) polynomial of degree 5
   c)polynomial of degree 2n d)polynomial of degree n-1
2. The power series of the form $a_0 + a_1(z-a) + a_2(z-a)^2+\ldots$ is said to be a series about ………………….
   a)z = 0 b)z = -a c)z = a d)z = ∞
3. If R = 0 the series is divergent in the extended plane except at …
   a)z = 0 b)z =1 c)z = ∞ d)z = -1
4. The inverse function of the exponential function is the ………..
   a)Trignometric functions b) hyberbolic functions
   c)harmonic functions d)Logarithmic functions
5. If D is a simply-connected region, f(z) is analytic and a is a point in D and C is the largest circle whose interior lies in D then the power series is… and its sum is f(z)
   a)Convergent b)uniformly convergent
   c)divergent d)absolutely convergent
6. The function $e^z - 1$ has a zero z = 0 of order………..
   a)two b)one c)three d)zero
7. The whole series is absolutely convergent if both the positive and negative series are……………….
   a)absolutely convergent b)uniformly convergent
   c)Convergent d)divergent
8. The logarithmic series is valid when $|z|$ ……………
   a)< 1 b) equal to 1 c)> 1 d)> 0
9. Maclaurin's series expansion of the function cosz is valid in …
   a)|z| > 1 b)|z|= 0 c)|z| < ∞ d)|z| < 1
10. Suppose f(z) is not identically zero and analytic in a region D. In any closed bounded region D, f(z) has ……………….. number of zeros.
    a)infinite b)only a finite c)many d)finite and many
11. Suppose f(z) is analytic in a region D and z_n , n= 1,2,3,…., in D are the zeros of f(z), where the sequence $\{z_n\}$ converges to a limit z = a in D then f(z)…. in D.
    a)Constant b)vanishes identically
    c)bounded d)analytic
12. Maclaurin's series expansion of the function $e^{(-z)}$ is valid in …
    a)|z|= 0 b)|z| < 1 c)|z| < ∞ d)|z| > 1
13. If f(z) can be expanded as a series of non- negative integral powers which is convergent for all z in C_i then the series is called ……….for f(z) about z=a.
    a)Power series b)Taylor's series
    c)Laurent's series d)convergent series
14. If C is the largest circle with z=a as its center such that f(z) is analytic in C_i but is not …………… somewhere on C.
    a)analytic b)singular c)zero d)constant
15. ………………….. is the expansion of f(z) about z = 0.
    a)convergent series b)Taylor's series
    c)Power series d)Maclaurin's series
16. The binomial series is valid when |z| = ----------
    a)< 1 b)equal to 1 c)> 1 d)> 0
17. Maclaurin's series expansion of the function coshz is valid in ………………………
    a)|z| < 0 b)|z| < ∞ c)|z|= 0 d)|z| <1

18. The function f(z) = |z| is differentiable -----

a)on real part b)on imaginary part
c)at the origin d)at the point 2

19. The expansion of f(z) = e^z is valid in the ……………….plane.

a)partial complex b)entire real
c)entire complex d)partial real

20. The region of validity for Taylor's series about z = 0 of the function e^z is ………………

a)|z|= 0 b)|z| < 1 c)|z| < ∞ d)|z| > 1

PART-B (3 x 10 = 30 Marks)

Answer All the Questions:

21. a) Treating f(z) as a function of x &y and x &y is a function of z & $\bar{z}$ show that

i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log|f'(z)| = 0.$

(OR)

b) Prove that an analytic function f (z) and the C-R equations can be put in the condensed form $\frac{\partial f}{\partial \bar{z}} = 0.$

22. a) State and prove Uniqueness theorem

(OR)

b) i)Define circle of convergence.

ii) Prove that a power series is divergent in the exterior of its circle of convergence.

23. a) State and prove Abel's theorem.

(OR)

b) Find the radii of convergence of the following power series.

i) $\sum \frac{n^k}{n^n} z^n$ ii) $\sum \frac{n!}{n^n} z^n$

Reg. No ----------------

(15MMU502)

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education

COIMBATORE-21

Fifth Semester

Model Examination

Department of Mathematics

Complex Analysis-I

Date : .09.17(AN) **Time : 3 Hours**

Class : III-B.Sc Mathematics(A &B) **Maximum: 60 Marks**

PART-A (20x 1 =20 Marks)

ANSWER ALL THE QUESTIONS

1. The complex plane containing all the finite complex numbers and infinity is called the
   a) Infinite complex plane b) extended complex plane
   c) Finite complex plane d) interior complex plane
2. If Zn ………………… is bounded in the finite plane, then it is a constant
   a) simple function b)analytic function
   c)complex function d)entire function
3. Every function analytic in the extended plane is a ………
   a)Constant b)zero c)non zero d)analytic
4. Every polynomial in z of degree equal to or greater than 1, has at least…………… zero
   a)two b)three c)one d)four
5. If f(z) can be expanded as a series of non- negative integral powers which is convergent for all z in C_i then the series is called ……….for f(z) about z=a.
   a)Power series b)Taylor's series
   c)Laurent's series d)convergent series
6. If C is the largest circle with z=a as its center such that f(z) is analytic in C_i but is not …………… somewhere on C.
   a)analytic b)singular c)zero d)constant
7. ……….. is the expansion of f(z) about z = 0.
   a)convergent series b)Taylor's series
   c)Power series d)Maclaurin's series
8. The binomial series is valid when $|z|$ ……………
   a)< 1 b)equal to 1 c)> 1 d)> 0
9. If z = a is ……….. of f(z), then f(z) is not bounded in every deleted neighbourhood of z=a
   a)a pole b)a removable singularity
   c)an essential singularity d) isolated singularity
10. The entire function f(z) = e^z is not defined at z=∞ and z=∞ is the only ……………. Point
    a)singular b)analytic c)pole d)essential
11. The point ………… is a singular point of all the trigonometric function and hyperbolic functions because they are function of e^z
    a)z=0 b)z =1 c)z=∞ d)z=a
12. If z = a is an isolated singular point of a function f(z), then the singularity is called…….. according as the Laurent's series about z=a of f(z), valid in a deleted neighbourhood of z=a has an infinite number of negative powers
    a)a pole b)an essential singularity
    c) isolated singularity d)a removable singularity
13. Principle value of logz is obtained when n = ……….
    a)0 b)-1 c)1 d)2
14. If a function h(z) is analytic at z=a and h(a) not equal to zero then the residue of function f(z)=h(z)/(z-a) at z=a is ……
    a)h(z) b)h(a) c)f(a) d)z(a)
15. From x= rcosθ and y = rsinθ weget θ = ………
    a)$\sin^{-1} y/x$ b)$\cos^{-1}y/x$ c)$\tan^{-1}y/x$ d)$\cot^{-1}y/x$

16. The complex integration along the scro curve used in evaluating the definite integral is called…………………

a) differentiation b)contour differentiation

c)contour integration d)integration

17. A function which has no singularities in a region other than a finite number of poles is said to be ……………… in that region

a)analytic b) meromorphic

c)isomorphic d)morphic

18. Find the number of roots of the function f(z) = $z^8 - 5z^5 - 2z+1$ which lie inside the unit circle C:|z| = 1 ………………….

a)2 b)4 c)6 d)5

19. n (P,f) denotes the number of ……………. of f(z) in C.

a)ones b)constants c)zeros d)poles

20. Find the number of zeros of the function f(z) = $z^6+z^3 -6z +9$ which lie inside the unit circle C:|z| = 1 ……………….

a)1 b)2 c)0 d)4

PART-B (5 x 8 = 40 Marks)

Answer All the Questions

21. a) i) If Z_1 and Z_2 are any two complex numbers ,then prove that $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$.

ii) Show that every complex numbers z whose absolute value is 1, can be expressed in

the form z = (1+it)/(1-it),t is a real number.

(OR)

b) Explain the Stereographic projection of a complex plane.

22. a) Treating f(z) as a function of x &y and x &y is a function of z & $\bar{z}$ show that

i) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log|f'(z)| = 0.$

(OR)

b)State and prove C-R equations in polar coordinates.

23. a) i). Find the radius of convergence of the power series f(z) = $\sum_0^\infty \frac{z^n}{2^n(1+in^2)}$

ii) State and prove Euler's relation.

(OR)

b) State and prove Abel's theorem.

24. a) Show that a function f(z)=u(x,y)+iv(x.y) defined in a region D is analytic in it iff u(x,y) and v(x,y) are conjugate harmonic functions.

(OR)

b) Prove that the cross ratio is preserved by a Bilinear transformation.

25. a). State and prove Cauchy's Integral formula for nth derivative.

(OR)

b) State and prove Morera's theorem