KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester – II

17CSU202	**DISCRETE STRUCTURES**	**4H – 4C**

Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Int : **40** Ext : **60** Total: **100**

Scope: It exposes the students to study entities such as sets, relations, graphs, and trees. These entities act as very fundamental representations useful in a broad spectrum of applications across the length and breadth of computer science.

Objective: This course provides a deep knowledge to the learners to develop and analyze algorithms as well as enable them to think about and solve problems in new ways. By the completion of the course students should be able to express ideas using mathematical notation and solve problems using the tools of mathematical analysis.

UNIT I

Sets: Introduction, Sets , finite and infinite sets, uncountably infinite sets, functions, relations, properties of binary relations, closure, partial ordering relations, counting , Pigeonhole principle, Permutation and Combination, Mathematical Induction, Principle of inclusion and Exclusion.

UNIT II

Growth of Functions: Asymptotic Notations, Summation formulas and properties,Bounding Summations, approximation byIntegrals

UNIT III

Recurrences: Recurrence relations, generating functions, linear recurrence relations with constant coefficients and their solution, Substitution Method, recurrence trees, Master theorem.

UNIT IV

Graph Theory : Basic terminology, models and types, multigraphs and weighted graphs, graph representation, graph isomorphism, connectivity, Euler and Hamiltonian Paths and circuits, Planar graphs, graph coloring, trees, basic terminology and properties of trees, introduction to Spanning trees

UNIT V

Prepositional Logic: Logical Connectives, Well-formed Formulas, Tautologies, Equivalences, Inference Theory.

SUGGESTED READINGS

TEXT BOOK

Kenneth Rosen. (2006). Discrete Mathematics and Its Applications (6th ed.). New Delhi: McGraw Hill.

REFERENCES

1. Tremblay , J .P. , & Manohar, R. (1997). Discrete Mathematical Structures with Applications to Computer Science. New Delhi: McGraw-Hill Book Company.

2. Coremen, T.H., Leiserson, C.E. , & R. L. Rivest. (2009). Introduction to algorithms, (3rd ed.). New Delhi: Prentice Hall on India.

3. Albertson, M. O.,& Hutchinson, J. P. (1988). Discrete Mathematics with Algorithms . New Delhi: John wiley Publication.

4. Hein, J. L. (2009). Discrete Structures, Logic, and Computability(3rd ed.). New Delhi: Jones and Bartlett Publishers.

5. Hunter, D.J. (2008). Essentials of Discrete Mathematics. New Delhi: Jones and Bartlett Publishers.

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LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME:A.NEERAJAH
SUBJECT NAME: DISCRETE STRUCTURES SUB.CODE:17CSU202
SEMESTER: II CLASS: I B.Sc CS -A

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
UNIT-I			
1	1	Introduction to sets	T2:chapter-2,Pg.No:104-114
2	1	Relations and properties of binary relations	T2:chapter-2,Pg.No:148-155
3	1	Partial ordering- theorems	T2:chapter-2,Pg.No:183-191
4	1	Functions-Definition and basic concepts	T2:chapter-2,Pg.No:192-197
5	1	Counting- Definition and basic concepts	T1: chapter -4 Pg.No:301-311
6	1	Pigeonhole principle	T1: chapter -4 Pg.No:313-318
7	1	Permutation and Combination-Problems	T1: chapter -4 Pg.No:320-326
8	1	Mathematical induction	R1: chapter -5 Pg.No:-172-181
9	1	Principle of inclusion and exclusion	R1: chapter -5 Pg.No:-182-186
10	1	Recapitulation and Discussion of possible questions	T2:chapter-2,Pg.No:104-114
	Total No of Hours Planned For Unit 1=10		
UNIT-II			
1	1	Introduction to growth of functions	T3: chapter -3 Pg.No:44-51
2	1	Big theta and Little oh-Problems	T5: chapter -5 Pg.No:296-304

3	1	Big oh and Big Omega-Problems	T4: chapter -2 Pg.No:102-109
4	1	Summation- Definition and basic concepts	W_1:Staff.ust.edu.cn/ch3
5	1	Properties of Summation-problems	W_1:Staff.ust.edu.cn/ch3
6	1	Bounding Summation with Examples	W_1:Staff.ust.edu.cn/ch3
7	1	Approximation by integrals-Problems	W_1:Staff.ust.edu.cn/ch3
8	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit II=8		
		UNIT-III	
1	1	Recurrence relations-Definition and basic concepts	R1: chapter -6 Pg.No:193-199
2	1	Linear recurrence relation with constant coefficient	T1: chapter -6 Pg.No:413-418
3	1	Solution of Linear recurrence relations with constant coefficient	T1: chapter -6 Pg.No:419-422
4	1	Generation functions-Problems	T1: chapter -6 Pg.No:435-439
5	1	Substitution method- Problems	T3: chapter -4 Pg.No: 88-92
6	1	Recurrence tree-Problems	T3: chapter -4 Pg.No: 88-92
7	1	Master Method-Problems	T3: chapter -4 Pg.No: 93-96
8	1	Master theorem	T3: chapter -4 Pg.No: 96-99
9	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit III=9		
		UNIT-IV	
1	1	Introduction to Graph theory	T1: chapter -8 Pg.No:545-556
2	1	Representation and isomorphism of graphs	T1: chapter -8 Pg.No:557-566
3	1	Connectivity- Definition and theorems	T1: chapter -8 Pg.No:567-575

4	1	Euler's and Hamiltonian paths	T1: chapter -8 Pg.No:577-592
5	1	Planner graph-theorem	T1: chapter -8 Pg.No:603-612
6	1	Graph coloring-Definition and theorems	T1: chapter -8 Pg.No:613-620
7	1	Tree and its Properties	T1: chapter -9 Pg.No:631-640
8	1	Spanning Tree	T1: chapter -9 Pg.No:674-680
9	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit IV=9		
		UNIT-V	
1	1	Introduction to Statement and Notation Logical Connectives	T2: chapter -1 Pg.No:2-6 T6: chapter-1 Pg. No: 2-6
2	1	Well formed formulae	T5: chapter -7 Pg.No:356-358
3	1	Tautologies-Problems	T2: chapter -1 Pg.No:24-25
4	1	Equivalence of formulae-Problems	T5: chapter -7 Pg.No:368-373
5	1	Continuation of Problems	T5: chapter -7 Pg.No:368-373
6	1	Normal forms-Problems	T2: chapter -1 Pg.No:50-60
7	1	Theory of Inference	T2: chapter -1 Pg.No:65-67
8	1	Rules of inference	T2: chapter -1 Pg.No:68-78
9	1	Recapitulation and Discussion of possible questions	
10	1	Discuss on Previous ESE Question Papers	
11	1	Discuss on Previous ESE Question Papers	
12	1	Discuss on Previous ESE Question Papers	
	Total No of Hours Planned for unit V=12		
Total Planned Hours	**48**		

TEXT BOOK

1. Balakrishnan R., and Ramabadran. M., Second Edition, 1994. Modern Algebra , Vikas Publishing House Pvt.Ltd, New Delhi. (For Unit I&II).
2. Herstein.I.N, 2010. Topic in Algebra ,John Wiley &Sons , New Yark.(For Unit III,IV,V)
3. Vasishtha.A.R.,2005 . Modern Algebra, Krishna Prakasam Mandir , Meerut.

REFERENCES

1. Surjeet Singh and Qazi Zameeruddin., 1992. Modern Algebra, Vikash Publishing House.
2. Seymour Lipschutz and Marc Lipson ,2001 . Linear Algebra, 3rd Edition , Mc Graw Hill.
3. Kanti Bhushan Datta., 2009 . Marix and Linear Algebra Aided with MATLAB, Prentice- Hall of india Private Ltd, New Delhi.
4. Dipak Chatterjee., 2005. Abstract Algebra, Prentice- Hall of India Private Ltd, New Delhi.

WEBSITES

W1: Staff.ust.edu.cn/ch3

UNIT-I

SYLLABUS

Introduction, Sets , finite and infinite sets, uncountably infinite sets, functions, relations, properties of binary relations, closure, partial ordering relations, counting , Pigeonhole principle, Permutation and Combination, Mathematical Induction, Principle of inclusion and Exclusion.

UNIT – I

1. Introduction

Set: any collection of objects (individuals)

Naming sets: A, B, C,….

Members of a set: the objects in the set

Naming objects: a, b, c, ….

Notation: Let A be a set of 3 letters a, b, c.
We write A = {a, b, c}
a is a member of **A**, **a** is in **A**, we write **a ∈ A**
d is not a member of **A,** we write **d ∉ A**

Important:
1. {a}≠ a
   {a} - a set consisting of one element a.
   a - the element itself

2. A set can be a member of another set:
   B = { 1, 2, {1}, {2}, {1,2}}

Finite sets: finite number of elements
Infinite sets: infinite number of elements
Cardinality of a finite set **A**: the number of elements in **A**: #A, or |A|

Describing sets:

a. by enumerating the elements of A:
   for finite sets: {red, blue, yellow}, {1,2,3,4,5,6,7,8,9,0}
   for infinite sets we write: {1,2,3,4,5,….}

b. by property, using predicate logic notation
   Let P(x) is a property, D - universe of discourse

   The set of all objects in D, for which P(x) is true, is :

$$A = \{x \mid P(x)\}$$

   we read: A consists of all objects x in D such that P(x) is true

c. by recursive definition, e.g. sequences

Examples:

1. The set of the days of the week:

{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

2. The set of all even numbers :

$\{ x \mid even(x) \}$
$\{ 2,4,6,8,\ldots.\}$

3. The set of all even numbers, greater than 100:

$\{ x \mid even(x) \wedge x > 100\}$
$\{ 102, 104, 106, 108,\ldots.\}$

4. The set of integers defined as follows:
$a_1 = 1$, $a_{n+1} = a_n + 2$ (the odd natural numbers)

Universal set: **U -** the set of all objects under consideration

Empty set: Ø set without elements.

2. Relations between sets

2.1. Equality

Let **A** and **B** be two sets.
We say that **A** is equal to **B**, **A = B** if and only if they **have the same members**.

Example:
A = {2,4,6}, B = {2,4,6} A = B

A = {a, b, c}, B = {c, a, b} A = B

A = {1,2,3}, B = {1,3,5}, A ≠ B

Written in predicate notation:

A = B if and only if $\forall x, x \in A \leftrightarrow x \in B$

2.2. Subsets

The set of all numbers contains the set of all positive numbers. We say that the set of all positive numbers is a **subset** of the set of all numbers.

Definition: A is a subset of **B** if all elements of **A** are in **B.** However **B** may contain elements that are not in **A**

Notation: $\mathbf{A} \subseteq \mathbf{B}$
Formal definition:

$$A \subseteq B \text{ if and only if } \forall x, x \in A \rightarrow x \in B$$

Example: $\mathbf{A = \{2,4,6\}, B = \{1,2,3,4,5,6\}, A \subseteq B}$

Definition: if **A** is a subset of **B, B** is called **a superset** of **A.**

Other definitions and properties:

a. If $A \subseteq B$ and $B \subseteq A$ then $A = B$

If **A** is a subset of **B**, and **B** is a subset of **A**, **A** and **B** are equal.

b. Proper subsets: **A** is a proper subset of **B**, $\mathbf{A \subset B}$**,** if and only if **A** is a subset of **B** and there is at least one element in **B** that is not in **A**.

$$A \subset B \text{ iff } \forall x, x \in A \rightarrow x \in B \land \exists x, x \in B \land x \notin A$$

The above expression reads:
A is a proper subset of B if and only if all x in A are also in B and there is an element x in B that is not in A.

iff means if and only if

The empty set **Ø** is a subset of all sets.
All sets are subsets of the universal set **U**.

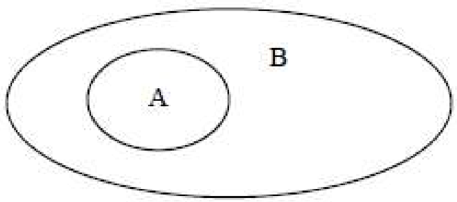
2.3. Disjoint sets

Definition: Two sets A and B are disjoint if and only if they have no common elements

A and **B** are disjoint if and only if $\sim\exists$ **x, (x ∈ A) Λ (x ∈ B)**
i.e. $\forall$ **x, x ∉ A V x ∉ B**

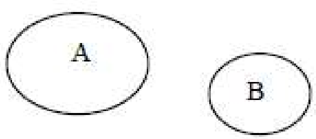
If two sets are not disjoint they have common elements.

Picturing sets: **Venn diagrams** - used to represent relations between sets

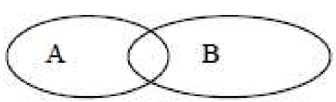


A is a (proper) subset of B

All elements in the set A are also elements in the set B



Disjoint sets



Not disjoint sets

3. Operations on sets

3.1. Intersections

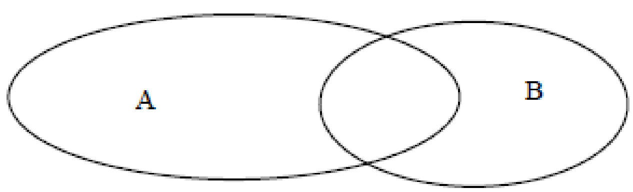
The set of all students at Simpson and the set of all majors in CS have some elements in common - the set of all students in Simpson that are majoring in CS. This set is formed as the intersection of all students in CS and all students at Simpson.

Definition: Let **A** and **B** are two sets. The set of all elements common to **A** and **B** is called the intersection of **A** and **B**

Notation: $\mathbf{A \cap B}$
Formal definition:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Venn diagram:



Example: $\mathbf{A = \{2,4,6\}, B = \{1,2,5,6\}, \quad A \cap B = \{2,6\}}$

Other properties:

$A \cap B \subseteq A, \quad A \cap B \subseteq B$

The intersection of two sets A and B is a subset of A, and a subset of B

$A \cap \varnothing = \varnothing$ The intersection of any set A with the empty set is the empty set
$A \cap U = A$ The intersection of any set A with the universal set is the set A itself.

Intersection corresponds to conjunction in logic.

Let $A = \{x \mid P(x)\}, B = \{x \mid Q(x)\}$

$A \cap B = \{x \mid P(x) \wedge Q(x)\}$

3. 2. Unions

The set of all rational numbers and the set of all irrational numbers taken together form the set of all real numbers - as a **union** of the rational and irrational numbers.

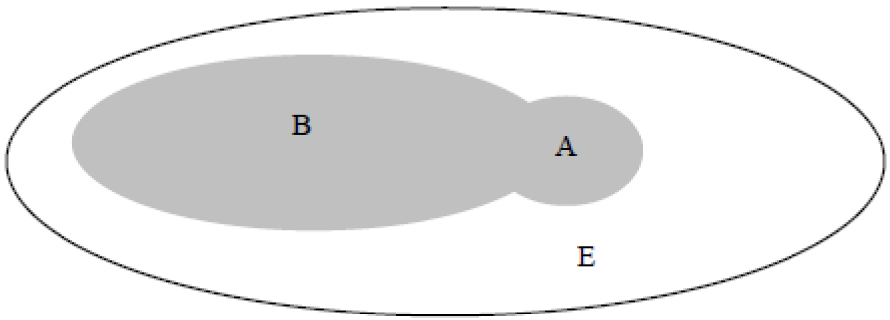
All classes at Simpson consist of students. If we take the elements of all classes, we will get all students - as the union of all classes.

Definition: The union of two sets A and B consists of all elements that are in A combined with all elements that are in B.
(note that an element may belong both to A and B)

Notation: $A \cup B$
Formal definition:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Venn diagram:



Example: $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5,6\}$, $A \cup B = \{1,2,3,4,5,6,8,10\}$
$A \cup B$ contains all elements in A and B without repetitions.

Other properties of unions:

$A \subseteq A \cup B$ $B \subseteq A \cup B$

A is a subset of the union of A and B,
B is a subset of the union of A and B

$A \cup \varnothing = A$ The union of any set A with the empty set is A
$A \cup U = U$ The union of any set A with the universal set E is the universal set.

Union corresponds to disjunction in logic.
Let $A = \{x \mid P(x)\}$, $B = \{x \mid Q(x)\}$
$A \cup B = \{x \mid P(x) \vee Q(x)\}$

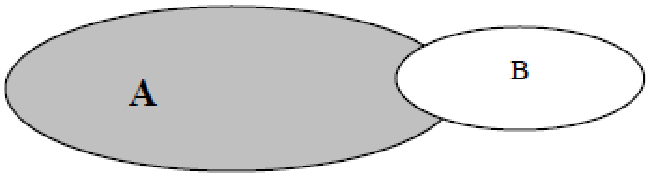
3.3. Differences

Definition: Let A and B be two sets. The set **A - B**, called the difference between A and B, is the set of all elements that are in A and are not in B.

Notation: $\mathbf{A - B}$ **or** $\mathbf{A \setminus B}$
Formal definition:

$$A - B = \{ x \mid (x \in A) \wedge (x \notin B)\}$$

Venn diagram:



Example: A = {2,4,6}, B = {1, 5,6}, A - B = {2,4}

A – Ø = A The difference between A and the empty set is A
A – U = Ø The difference between A and the universal set is the empty set.

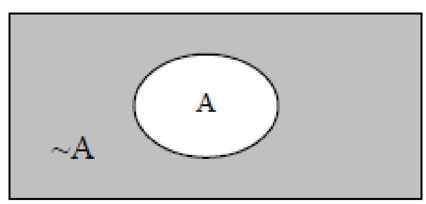
3.4. Complements

Definition: Let **A** be a set. The set of all objects within the universal set that are not in **A**, is called the complement of **A**.

Notation: ~**A**
Formal definition:

$$\sim A = \{x \mid x \notin A\}$$

Venn diagram:



SETS IDENTITIES

Using the operation unions, intersection and complement we can build expressions over sets.

Example:
A - set of all black objects
B - set of all cats
$A \cap B$ - set of all black cats

The set identities are used to manipulate set expressions

$A \cup \sim A = U$	Complementation Law
$A \cap \sim A = \varnothing$	Exclusion Law
$A \cap U = A$ $A \cup \varnothing = A$	Identity Laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\sim(\sim A) = A$	Double Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$\sim(A \cap B) = \sim A \cup \sim B$ $\sim(A \cup B) = \sim A \cap \sim B$	De Morgan's Laws
$A - B = A \cap \sim B$	Alternate representation for set difference

Proof problems for sets

A. Element Proofs

Definitions used in the proofs

Def 1: $A \cup B = \{ x \mid x \in A \ \vee \ x \in B\}$
Def 2: $A \cap B = \{ x \mid x \in A \ \wedge \ x \in B\}$
Def 3: $A - B = \{ x \mid x \in A \ \wedge \ x \notin B\}$
Def 4: $\sim A \quad = \{ x \mid x \notin A \}$

Inference rules often used:

$P \wedge Q \models P, Q$
$P, Q \models P \wedge Q$
$P \models P \vee Q$

How to prove that two sets are equal:
A = B
1) show that $A \subseteq B$, i.e. choose an arbitrary element in A and show that it is in B
2) show that $B \subseteq A$, i.e. choose an arbitrary element in B and show that it is in A

The element was chosen arbitrary, hence any element that is a member of the left set, is also a member of the right set, and vice versa.

Example:
Prove that $A - B = A \cap {\sim}B$

1. Show that $A - B \subseteq A \cap {\sim}B$

Let $x \in A - B$
By Def 3:
$x \in A \wedge x \notin B$ (1)
By (1) $x \in A$ (2)
By (1) $x \notin B$ (3)
By (3) and Def 4: $x \in {\sim}B$ (4)
By (2), (4)
$x \in A \wedge x \in {\sim}B$ (5)
By (5) and Def 2:
$x \in A \cap {\sim}B$

x was an arbitrary element in A – B, therefore $A - B \subseteq A \cap {\sim}B$ (6)

2. Show that $A \cap {\sim}B \subseteq A - B$
Let $x \in A \cap {\sim}B$
By Def 2:
$x \in A \wedge x \in {\sim}B$ (7)
By (7) $x \in A$ (8)
By (7) $x \in {\sim}B$ (9)

By (9) and Def 4: $x \notin B$ (10)
By (8), (10)
$x \in A \ \Lambda \ x \notin B$ (11)
By (11) and Def 3:
$x \in A - B$

x was an arbitrary element in $A \cap \sim B$, therefore $A \cap \sim B \subseteq A - B$ (12)

by (6) and (12):

$$A - B \subseteq A \cap \sim B$$

Q.E.D.

B. Using set identities

Prove that $A \cap (\sim A \cup B) = A \cap B$

Method: Apply the set identities to the expression on the left, until the expression on the right is obtained.

By Distribution Laws: $A \cap (\sim A \cup B) = (A \cap \sim A) \cup (A \cap B)$

By the Exclusion Law $A \cap \sim A = Ø$

Hence $A \cap (\sim A \cup B) = Ø \cup (A \cap B)$

By the Identity Law: $Ø \cup (A \cap B) = A \cap B$

Hence $A \cap (\sim A \cup B) = A \cap B$

1. Set partitions

Two sets are disjoint if they have no elements in common, i.e. their intersection is the empty set.

A and B are disjoint sets iff $A \cap B = \varnothing$

Definition: Consider a set A, and sets $A_1, A_2, \dots A_n$, such that:

a. $A_1 \cup A_2 \cup \dots \cup A_n = A$

b. $A_1, A_2, \dots A_n$, are mutually disjoint, i.e. for all i and j, $A_i \cap A_j = \varnothing$

The set $\{A_1, A_2, \dots A_n\}$ is called a **partition** of A

Example:

1. A = {a, b, c, d, e, f, g}

A_1 = {a, c, d}

A_2 = {b, f}

A_3 = {e, g}

The set {{a, c, d}, {b, f}, {e, g}} is a partition of A.

2. Cartesian product

Consider the identification numbers on license plates: $x_1x_2x_3$ $Y_1Y_2Y_3$
where $x_1x_2x_3$ is a 3-digit number and $Y_1Y_2Y_3$ is a combination of 3 letters

How do we make sure that each license plate would have a different identification number?

The program that assigns numbers uses Cartesian product of sets.

Definition: Let A and B be two sets. The Cartesian product of A and B is defined as the set

$$A \times B = \{(x,y) \mid x \in A \land y \in B\}$$

Example 1:

A = {0, 1, 2, 3}
B = {a, b}

A x B = {(0,a), (0,b), (1,a), (1,b), (2,a), (2,b), (3,a), (3,b)}

Example 2:

A = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
A x A = {(0,0),(0,1), (0,2), …. (0,9),
(1,0),(1,1), (1,2), ……,(1,9),
……
(9,0),(9,1), (9,2), …. (9,9)}

We can consider the result to be the set of all 2-digit numbers.

3. Power sets

Definition: The set of all subsets of a given set A is called power set of A.

Notation 2^A , or $\mathcal{P}(A)$

Example:

A - { a,b,c,d}

$\mathcal{P}$(A) = {∅, {a},{b},{c},{d}
{a,b},{a,c},{a,d},{b,c},{b,d},{c,d}
{a,b,c},{a,b,d},{a,c,d}, {b, c, d}
{a,b,c,d}}

Number of elements in $\mathcal{P}(A)$ is 2^N, where N = number of elements in A
Why 2^N?

<u>Bit notation</u>: For a set A with **n** elements, each subset of A can be represented by a string of length *n* over {0,1}, i.e. a string consisting of 0s and 1s.

For example:

{a,b} = 1 1 0 0
{a,c} = 1 0 1 0
{b,c,d} = 0 1 1 1

The **i**-th element in the string is 1 if the element a_i is in the subset, otherwise it is 0.
Thus the subset {a,b,d} of the set {a,b,c,d} can be represented by the string '1101'

There are 2^n different strings with length **n** over {0,1}(why?), hence the number of the subsets is 2^n .

Set Relations

2. Definition

Let A and B be two sets. A relation **R** from **A** to **B** is any set of pairs (x,y), x ϵ A, y ϵ B, i.e. any subset of A x B.

If **x** and **y** are in relation **R**, we write xRy, or (x,y) ∈ R..

R is a set defined as
R = {(x,y) | x ∈ A, y ∈ B, xRb.}

3. Relations and Cartesian products

Relations between two sets **A** and **B** are sets of pairs of elements of **A** and **B**.
The Cartesian product **A x B** consists of all pairs of elements of **A** and **B.**

Thus relations between two sets are subsets of the Cartesian product of the sets.

Example:

Let A = {1, 3, 4, 5}
B = {2, 7, 8}

The relation R1 :"less than" from set A to set B is defined by the following set:

R1 = {(1, 2), (1, 7), (1, 8), (3, 7), (3, 8), (4, 7), (4, 8), (5, 7), (5, 8)}

This set is a subset of the Cartesian product of A and B:

A x B = **{(1,2),(1,7),(1,8),**
(3,2), **(3,7), (3,8),**
(4,2), **(4,7), (4,8)**,
(5,2),**(5,7),(5,8)}**
(the members of R1 are in boldface)

The relation R2: "greater than" from set A to set B is defined by the set:

R2 = {(3, 2), (4, 2), (5, 2)}

It is also a subset of A x B.

The relation R3 "equal to" from A to B is the empty set, since no element in A is equal to an element in B.

7. Domains and ranges

Let R be a relation from X to Y,
the **domain of R** is the set of all elements in X that occur in at least one pair of the relation,
the **range of R** is the set of all elements in Y that occur in at least one pair of the relation.

In the above example, the domain of **R: choose(x,y)** is the set of students {Ann, Tom , Paul}, and the range is the set of food items: {spaghetti, fish, pie, cake}.

The domain and the range are easily found using the matrix or the graph representations of the relation.

1. **Definition**

Let A and B be two sets. A relation **R** from **A** to **B** is any set of pairs (x,y), $x \in A$, $y \in B$, i.e. any subset of A x B.
The empty set is a subset of the Cartesian product – the empty relation

2. **How to write relations**

a. as set of pairs

A = {1,2,3}, {B = 4,5,6}
R = {{1,4), (1,5), (1,6), (2,4), (2,6), (3,6)}

b. using predicates

A = {1,2,3}, {B = 4,5,6}
R = {(x,y) | $x \in A$, $y \in B$, y is a multiple of x}

3. **Graph and matrix representation**

A = {1,2,3}, {B = 4,5,6}
R = {{1,4), (1,5), (1,6), (2,4), (2,6), (3,6)}

1. Set operations and relations

Relations are sets. All set operations are applicable to relations

Examples:

Let A= {3, 5, 6, 7}
B ={4, 5, 9}

Consider two relations R and S from A to B:

$R = \{(x,y) \mid x \in A, y \in B, x < y\}$

If $(x,y) \in R$ we write xRy

R is a finite set and we can write down explicitly its elements:
R= {(3,4),(3,5),(3,9),(5,9), (6,9),(7,9)}

$S = \{(x,y) \mid x \in A, y \in B, |x-y| = 2\}$

If $(x,y) \in S$ we write xSy

S is a finite set and we can write down explicitly its elements:
S = {(3,5), (6,4), (7,5), (7,9)}

For R and S the universal set is A x B:
{(3,4), (3, 5), (3, 9),
(5, 4), (5, 5), (5, 9),
(6, 4), (6, 5), (6, 9),
(7, 4), (7, 5), (7, 9)}

a) intersection of R and S:

$R \cap S = \{(x,y) \mid xRy \wedge xSy\}$ $R \cap S = \{(3,5),(7,9)\}$

b) union of R and S:

$R \cup S = \{(x,y) \mid xRy \vee xSy\}$

$R \cup S = \{ (3,4),(3,5),(3,9),(5,9), (6,9),(7,9), (6,4), (7,5) \}$

c) complementation:

$\sim R = \{(x,y) \mid \sim(xRy)\}$

$\sim R = U - R$

The universal set for R is the Cartesian product A x B

A = { 3,5,6,7}

B = {4,5,9}

U = A x B = {(3,4), (3,5), (3,9), (5,4), (5,5), (5,9), (6,4), (6,5), (6,9), (7,4), (7,5), (7,9)}

R = {(3,4), (3,5), (3,9), (5,9), (6,9), (7,9)}

U - R = {(5,4), (5,5), (6,4), (7,4), (7,5)}

Note that for any two sets A and B, $A - B = A \cap \sim B$

d) difference R - S, S - R:

$R - S = \{(x,y) \mid xRy \ \Lambda \sim(xSy)\}$

R - S = {(3,4),(3,9),(5,9), (6,9)}

2. Inverse relation

Let R: A→B be a relation from A to B. The inverse relation $R^{-1} : B \to A$ is defined as in the following way:

$$R^{-1} : B \to A \ \{(y,x) \mid (x,y) \in R\}$$

Thus $xRy \equiv yR^{-1} x$

Examples:

a. Let A = {1,2,3}, B = {1,4,9}

Let R: B→A be the set {(1,1), (1,4), (2,2), (2,4), (3,3)}

$R^{-1} : B \to A$ is the relation {(1,1), (4,1), (2,2), (4,2), (3,3)}

b. Let A = {1,2,3}, R: A→A be the relation {(1,2), (1,3), (2,3)}

R^{-1} is the relation: {(2,1), (3,1), (3,2)}

3. Composition of relations

Let X, Y and Z be three sets, R be a relation from X to Y, S be a relation from Y to Z.

A composition of R and S is a relation from X to Z :

$S \circ R = \{(x,z) | x \in X, z \in Z, \exists y \in Y, \text{ such that } xRy, \text{ and } ySz\}$

Note that the operation is right-associative, i.e. we first apply R and then S

Example 1:
Let X, Y, and Z be the sets:
X: {1,3,5}
Y: {2,4,8}
Z : {2,3,6}

Let R : X → Y, and S : Y → Z, be the relation "less than":

R = {(1,2),(1,4),(1,8),(3,4),(3,8),(5,8)}
S = {(2,3),(2,6),(4,6)}

$S \circ R$: {(1,3), (1,6), (3,6)}

The element (1,3) is formed by combining (1,2) from R and (2,3) from S
The element (1,6) is formed by combining (1,2) from R and (2,6) from S
Note, that (1,6) can also be obtained by combining (1,4) from R and (4,6) from S.
The element (3,6) is formed by combining (3,4) from R and (4,6) from S

4. Identity relation

Identity relation on a set A is defined in the following way:

$I = \{(x,x) | x \in A\}$

Example:

Let A = {a, b, c}, I = {(a,a), (b,b), (c,c)}

5. Problems:

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, $C = \{x, y, z\}$

a. Let $R = \{(1,a), (2,b), (3,a)\}$ and $S = \{(a,y),(a,z),(b,x),(b,z)\}$

Find $S \circ R$

b. Let $R = \{(1,a), (2,b), (3,a)\}$ and $S = \{(a,y),(a,z)\}$

Find $S \circ R$

a. Let $R = \{(1,a), (2,b)\}$ and $S = \{(a,y), (b, y), (b,z)\}$
Find $S \circ R$

b. Let $R = \{(1,a), (2,b), (3,a)\}$ and $S = \{(a,y),(a,z),(b,x),(b,z)\}$
Find R^{-1} , S^{-1} and $R^{-1} \circ S^{-1}$

Solutions

Let A = {1, 2, 3}, B = {a, b}, C = {x, y, z}

a. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}

Find S ° R

Solution: {(1,y), (1, z), (2,x),(2,z), (3,y), (3, z)}

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z)}

Find S ° R Solution: {(1,y), (1, z), (3,y), (3, z)}

c. Let R = {(1,a), (2,b)} and S = {(a,y), (b, y), (b,z)}
Find $S \circ R$

Solution: {(1,y), (2,y), (2, z)}

d. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}
Find R^{-1}, S^{-1} and $R^{-1} \circ S^{-1}$

Solution:
R^{-1} = {(a,1), (b,2), (a,3)}
S^{-1} = {(y,a),(z,a),(x,b),(z,b)}

$R^{-1} \circ S^{-1}$ = {(y,1), (y,3), (x,2), (z,1), (z,3), (z,2)}

Definitions:
Let R be a binary relation on a set A.

1. R is **reflexive,** iff for all $x \in A$, $(x,x) \in R$, i.e. **xRx** is true.

2. R is **symmetric**, iff for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$

   i.e **xRy → yRx** is true

3. R is **transitive** iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y,z) \in R$, then $(x, z) \in R$

   i.e. **(xRy Λ yRz) → xRz** is true

A. Reflexive relations

Let R be a binary relation on a set A.

R is **reflexive,** iff for all $x \in A$, $(x,x) \in R$, i.e. **xRx** is true.

1. Examples:

1. Equality is a reflexive relation
   for any object x: $x = x$ is true.
2. "less then" (defined on the set of real numbers) is **not a reflexive relation**.
   for any number x: $x < x$ is **not** true
3. " less then or equal to" (defined on the set of real numbers) **is a reflexive relation**
   for any number x $x \le x$ is true

4. Reflexive and irreflexive relations

Compare the three examples below:

1. A = {1,2,3,4}, R1 = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4)}
2. A = {1,2,3,4}, R2 = {(1,2), (2,3), (3,4), (4,1)}
3. A = {1,2,3,4}, R3= {(1,1), (1,2), (3,4), (4,4)}

R1 is a reflexive relation. R2 ? R3 ?

Definition: Let R be a binary relation on a set A.
R is **irreflexive** iff for all $x \in A$, $(x,x) \notin R$

Definition: Let R be a binary relation on a set A.
R is **neither reflexive, nor irreflexive** iff
there is $x \in A$, such that $(x, x) \in R$. and there is $y \in A$ such that $(y, y) \notin R$

Thus R2 is irreflexive, R3 is neither reflexive nor irreflexive.

reflexive:	for all x: xRx
irreflexive:	for no x: xRx
neither:	for some x: xRx is true, for some y: yRy is false

B. Symmetric relations

R is **symmetric**, iff for all x, y ∈ A, if (x, y) ∈ R, then (y, x) ∈ R

i.e **xRy → yRx** is true

This means: if two elements **x** and **y** are in relation R, then **y** and **x** are also in R, i.e. if xRy is true, yRx is also true.

1. Examples:

1. equality is a symmetric relation: if a = b then b = a
2. "less than" is not a symmetric relation : if a < b is true then b < a is false
3. "sister" on the set of females is symmetric
4. "sister" on the set of all human beings is not symmetric

4. Symmetric and anti-symmetric relations

Compare the relations:

1. A = {1,2,3,4}, R1 = {(1,1), (1,2), (2,1), (2,3), (3,2), (4,4)}
2. A = {1,2,3,4}, R2 = {(1,1), (1,2), (2,3), (4,4)}
3. A = {1,2,3,4}, R3 = {(1,1), (1,2), (2,1) , (2,3), (4,4)}

Definition: Let R be a binary relation on a set A.
R is **anti-symmetric** if for all x, y ∈ A, x ≠ y, **if (x, y) ∈ R, then (y, x) ∉ R.**

Definition: R is **neither symmetric nor anti-symmetric** iff it is not symmetric and not anti-symmetric.

symmetric: xRy → yRx for all x and y
anti-symmetric: xRy and yRx → x = y
neither: for some x and y: xRy, and yRx
for others xRy is true, yRx is not true

C. Transitive relations

Let R be a binary relation on a set A.
R is **transitive** iff for all x, y, z $\in$ A, if (x, y) $\in$ R and (y,z) $\in$ R , then (x, z) $\in$ R

i.e. **(xRy Λ yRz) $\rightarrow$ xRz** is true

1. Examples:

1. Equality is a transitive relation a = b, b = c, hence a = c
2. "less than" is a transitive relation a < b, b < c, hence a < c
3. mother_of(x,y) is <u>not</u> a transitive relation
4. sister(x,y) is a transitive relation
5. brother (x,y) is a transitive relation.
6. A = {1,2,3,4} R = {(1,1), (1,2), (1,3), (2,3), (4,3)} - transitive
7. A = {1,2,3,4} R = {(1,1), (1,2), (1,3), (2,3), (3,4)} - not transitive

Equivalence Relations, Partial Orders

1. Equivalence relations

Definition: A relation R is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

Examples:

Let **m** and **n** be integers and let **d** be a positive integer. The notation

$$m \equiv n \pmod d$$

is read "m is congruent to n modulo d".

The meaning is: the integer division of **d** into **m** gives the same remainder as the integer division of **d** into **n**.

Consider the relation

$$R=\{(x,y) \mid x \bmod 3 = y \bmod 3\}$$

4 mod 3 = 1, 7 mod 3 = 1, hence $(4,7) \in R$

The relation is reflexive: $x \bmod 3 = x \bmod 3$
symmetric: if $x \bmod 3 = y \bmod 3$, then $y \bmod 3 = x \bmod 3$
transitive: if $x \bmod 3 = y \bmod 3$, and $y \bmod 3 = z \bmod 3$, then $x \bmod 3 = z \bmod 3$

Consider the sets $[x] = \{y \mid yRx\}$

$[0] = \{0,3,6,9,12,\dots\}$
$[1] = \{1,4,7,10,13,\dots\}$
$[2] = \{2,5,8,11,14,\dots\}$
……

From the definition of [x] it follows that

$[0] = [3] = [6] \dots$
$[1] = [4] = \dots$
$[2] = [5] = \dots$

Thus the relation R produces three different sets [0], [1] and [2].
Each number is exactly in one of these sets. Thus {[0], [1], [2]} is a **partition** of the set of non-negative integers.

2. Partial Orders

Definition: Let R be a binary relation defined on a set A. R is a partial order relation iff R is transitive and anti-symmetric

Examples:

1. Let A be a set, and P(A) be the power set of A. The relation 'subset of' on P (A) is a partial order relation
   It is reflexive, anti-symmetric, and transitive

2. Let N be the set of positive integers, and R be a relation defined as follows:
   $(x, y) \in R$ iff y is a multiple of x
   e.g. $(3,12) \in R$, while $(3, 4) \notin R$
   R is a partial order relation. It is reflexive, anti-symmetric, and transitive

Functions

1. Definition: A function **f** from a set **X** to a set **Y** is a subset of the Cartesian product X x Y, $f \subseteq X \times Y$, such that

$$\forall x \in X \quad \exists y \in Y, \text{ such that } (x,y) \in f, \text{ and}$$

$$(x,y1) \in f \ \Lambda\ (x,y2) \in f \ \rightarrow y1 = y2$$

i.e. if $(x,y1) \in f$ and $(x,y2) \in f$, then $y1 = y2$

Thus all elements in X can be found in exactly one pair of **f.**

Notation: Let **f** be a function from **A** to **B.** We write

$$f : A \rightarrow B$$
$$a \in A,\ f(a) = b,\ \ b \in B$$

Examples:
$A = \{1,2,3\}$, $B = \{a,b\}$

$R = \{(1,a),(2,a),(3,b)\}$ is a function

Other definitions:
Let **f** be a function from **A** to **B.**

1. **Domain** of **f:** the set **A**
2. **Range** of **f**: $\{b: b \in B$ and there is an $a \in A,\ f(a) = b\}$
3. **Image** of **a** under **f: f(a)**

Example:

A = {1,2,3}, B = {a,b}

f = {(1,a),(2,a),(3,b)}
domain: {1,2,3},
range: {a,b}
a is image of 1 under f: f(1) = a, f(2) = ….. f(3) = ……

2. Functions with more arguments

Let A = A1 x A2 , and **f** be a function from **A** to **B**

We write: f(a1,a2) = b

If A = A1 x A2 x … x An, we write f(a1,a2,…,an) = b

a1, a2, ..,an: arguments of **f**
b : value of **f**

3. Functions of special interest

a. one-to-one

distinct elements have distinct images

if a1 $\neq$ a2, then f(a1) $\neq$ f(a2)

Example:
A = {1,2,3}, B = {a,b,c,d}

one-to-one function f = {(1,a), (2,c), (3,b)}

b. onto

Every element in **B** is an image of some element in **A**

Example:

A = {1,2,3}, B = {a,b }

onto function f = {(1,a), (2,b), (3,b)}

c. bijection

f is **bijection** iff **f** is a **one-to-one** function and **f** is a **onto** function

Example:

A = {1,2,3}, B = {a,b,c }

bijection f = {(1,a), (2,c), (3,b)}

4. Inverse function

If **f is a bijection, f^{-1}** is a function, also a bijection.

$\mathbf{f}^{-1}$ $=\{(y,x) \mid (x,y) \in f\}$

Example:
$A = \{1,2,3\}$, $B = \{a,b,c\}$

$f = \{(1,a), (2,c), (3,b)\}$
$\mathbf{f}^{-1}$ $=\{(a,1), (b,3),(c,2)\}$

5. Composition of functions

Let **f : A → B, g : B → C** be two functions.
The composition **h = g ° f** is a function from **A** to **C** such that **h(a) = g(f(a)).**

Example: Let $f(x) = x + 1$, $g(x) = x^2$.

The composition $h(x) = f(x) \circ g(x) = f(g(x)) = (x^2) + 1$

The composition $p(x) = g(x) \circ f(x) = g(f(x)) = (x+1)^2$

When **f** is a bijection and $\mathbf{f}^{-1}$ exists, we have:
$f^{-1}(f(a)) = a$, $f(f^{-1}(b)) = b$, $a \in A$, $b \in B$,

Counting Principles

The Multiplication Principle

The Multiplication Principle

Let $m \in \mathbb{N}$. For a procedure of m successive distinct and independent steps with n_1 outcomes possible for the first step, n_2 outcomes possible for the second step, ..., and n_m outcomes possible for the mth step, the total number of possible outcomes is

$$n_1 \cdot n_2 \cdots n_m$$

Addition Principle

The Addition Principle

For a collection of m disjoint sets with n_1 elements in the first, n_2 elements in the second, ..., and n_m elements in the mth, the number of ways to choose one element from the collection is

$$n_1 + n_2 + \cdots + n_m$$

Using the Pigeon-Hole Principle

The **Pigeon-Hole Principle** (see Section 4.6) states that if m objects are to be put in n locations, where $m > n > 0$, then at least one location must receive at least two objects. Thus, to prove that a set of objects has at least two elements with the same property, first count the number of distinct properties of objects in the set, and then count the number of distinct elements. If the total number of elements is larger than the number of distinct properties of objects, then the Pigeon-Hole Principle implies that at least two of the elements have the same property. The next example is an illustration of this type of argument.

Example 9. A local bank requires customers to choose a four-digit code to use with an ATM card. The code must consist of two letters in the first two positions and two digits in the other two positions. The bank has 75,000 customers. Show that at least two customers choose the same four-digit code.

Solution. First, use the Multiplication Principle to calculate the number of distinct codes possible:

$$\begin{aligned}(\#\text{ Four-symbol codes}) &= (\#\text{ Choices of letter 1}) \cdot (\#\text{ Choices for letter 2}) \\ &\quad \cdot (\#\text{ Choices for digit 1}) \cdot (\#\text{ Choices for digit 2}) \\ &= 26 \cdot 26 \cdot 10 \cdot 10 \\ &= 67{,}600\end{aligned}$$

Now, apply the Pigeon-Hole Principle. Since there are 75,000 customers and only 67,600 codes, the Pigeon-Hole Principle implies that at least two of the customers choose the same code. ■

Example 10. Suppose a group of vacationers is split into 159 teams. How many leagues must be formed if a league should contain at most 8 teams? 10 teams? 12 teams?

Solution. The Generalized Pigeon-Hole Principle tells us that the answers are

$$\left\lceil \frac{159}{8} \right\rceil = 20 \quad \left\lceil \frac{159}{10} \right\rceil = 16 \quad \left\lceil \frac{159}{12} \right\rceil = 14$$

It remains for the organizers to determine which size of a league is most manageable. ■

Permutations and Combinations

Definition 1. Let $n, r \in \mathbb{N}$. A **permutation** of an n-element set is a linear ordering of the n elements of the set. For $n \geq r \geq 0$ an ***r*-permutation** of an n-element set is a linear ordering of r elements of the set.

Example 1. List all permutation of the elements a, b, and c.

Solution. The permutations are abc, acb, bac, bca, cab, and cba. ■

Let ***P(n, r)*** denote the number of r-permutations of an n-element set. We define $P(n, 0) = 1$ for all $n \in \mathbb{N}$.

Example 2.

(a) How many ways can eight different books be arranged on a shelf?
(b) How many ways can four of eight different books be arranged on a shelf?
(c) How many ways can eight different books be arranged on two shelves so that each shelf contains four books?

Solution.

(a) The answer is the number of ordered ways of arranging the books on the shelf. That is,

$$P(8, 8) = 8! = 40{,}320$$

(b) The number of ways to arrange four of the eight books is

$$P(8, 4) = 1680$$

(c) The answer is the product of the number of ways to put four books on one shelf and the number of ways to put the remaining books on the second shelf. The number of ways to arrange four books on the first shelf is $P(8, 4)$, and the four remaining books can be arranged in $P(4, 4)$ ways on the second shelf. Therefore, the total number of arrangements will be

$$\begin{aligned}(\text{\# Arrangements of books on two shelves}) &= (\text{\# Arrangements on first shelf}) \\ &\quad \cdot (\text{\# Arrangements on second shelf}) \\ &= P(8, 4) \cdot P(4, 4) \\ &= (8!/4!) \cdot (4!/0!) \\ &= 8! \\ &= 40{,}320\end{aligned}$$

Combinations

Definition 2. Let $n, r \in \mathbb{N}$ such that $n \geq r \geq 0$. An unordered selection of r elements from an n element set is called a **combination.**

Example 4. List all the combinations of the set $\{a, b, c\}$.

Solution. The combinations will be of sizes 0, 1, 2, and 3. All combinations are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. ■

Example 5. How many different poker hands are there?

Solution. This answer is just the number of ways of choosing five cards from the 52-card deck:

$$C(52, 5) = \frac{52!}{47!\,5!} = 2{,}598{,}960$$ ■

Example 8. An examination consists of 20 questions, of which the student must answer any 12.

(a) How many different ways can a student choose questions to answer?
(b) The 20-question exam is split into three parts. There are 6 questions in the first part, 10 in the second part, and 4 in the third part. A student must choose three from the first part, eight from the second part, and one from the third part. How many ways can a student choose questions to answer?

Solution.

(a) The answer is just the number of different 12-element subsets of a 20-element set, or $C(20, 12) = 125{,}970$.

(b) By the Multiplication Principle, the answer will be the product of the number of ways to make choices in each category:

$$\begin{aligned}(\#\text{ Possible choices}) &= (\#\text{ Choices for part 1}) \cdot (\#\text{ Choices for part 2}) \\ &\quad \cdot (\#\text{ Choices for part 3}) \\ &= C(6,3) \cdot C(10,8) \cdot C(4,1) \\ &= 3600\end{aligned}$$

■

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Sum of the First n Integers

Use mathematical induction to prove that

$$1+2+\cdots+n=\frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution To construct a proof by induction, you must first identify the property $P(n)$. In this case, $P(n)$ is the equation

$$\boxed{1+2+\cdots+n=\frac{n(n+1)}{2}.} \qquad \leftarrow \text{the property } (P(n))$$

[To see that $P(n)$ is a sentence, note that its subject is "the sum of the integers from 1 to n" and its verb is "equals."]

In the basis step of the proof, you must show that the property is true for $n = 1$, or, in other words that $P(1)$ is true. Now $P(1)$ is obtained by substituting 1 in place of n in $P(n)$. The left-hand side of $P(1)$ is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus $P(1)$ is

$$\boxed{1=\frac{1(1+1)}{2}.} \qquad \leftarrow \text{basis } (P(1))$$

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2}=\frac{1\cdot 2}{2}=1,$$

which equals the left-hand side.

In the inductive step, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. *[This assumption is the inductive hypothesis.]* You must then show that $P(k+1)$ is true. What are $P(k)$ and $P(k+1)$? $P(k)$ is obtained by substituting k for every n in $P(n)$. Thus $P(k)$ is

$$\boxed{1+2+\cdots+k=\frac{k(k+1)}{2}.} \qquad \leftarrow \text{inductive hypothesis } (P(k))$$

Similarly, $P(k+1)$ is obtained by substituting the quantity $(k+1)$ for every n that appears in $P(n)$. Thus $P(k+1)$ is

$$1+2+\cdots+(k+1)=\frac{(k+1)((k+1)+1)}{2},$$

or, equivalently,

$$\boxed{1+2+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.} \quad \leftarrow \text{to show } (P(k+1))$$

Theorem 5.2.2 Sum of the First n Integers

For all integers $n \geq 1$,

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Proof (by mathematical induction):

Let the property $P(n)$ be the equation

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}. \quad \leftarrow P(n)$$

Show that P(1) is true:

To establish $P(1)$, we must show that

$$1=\frac{1(1+1)}{2} \quad \leftarrow \quad P(1)$$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2}=\frac{2}{2}=1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side and the right-hand side of $P(k+1)$ are equal to the same quantity and thus are equal to each other.]

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1 + 2 + 3 + \cdots + (k+1) \\
&= 1 + 2 + 3 + \cdots + k + (k+1) && \text{by making the next-to-last term explicit} \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by substitution from the inductive hypothesis} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+1}{2} && \text{by algebra.}
\end{aligned}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+1}{2}.$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true *[as was to be shown]*.

Possible Questions

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define disjoint sets with example.
2. Define Union of sets with example.
3. If A = {1, 2, 3, 4, 5}, B = {3, 7, 9} then find $A \cup B$, A\B.
4. Define Equivalence relation.
5. If A = {a, b, c} and B = {1, 2} then find $A \times B$ and $B \times A$.
6. Define Injective with example.
7. Define Composition of Function with example.
8. Define Inverse Function with example.
9. Define Permutation.
10. Define Combination.
11. Find the number of three letter words using the given 6 letters without repeating any letters in a given word?
12. State pigeonhole Principle.
13. State Principle of Mathematical Induction.
14. What is Function?
15. Define symmetric and Non symmetric with example.
16. Define Combination with example.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Explain about types of relation with examples.
2. Let R denotes a relation on the set of all ordered pairs of positive integers by (x, y) R (u, v) iff xv=yu. Show that R is a equivalence relations.
3. Write about the types of function with example.
4. In Z, we define aRb iff a-b is a multiple of m. Is R is an equivalence relation?
5. Let A={1,2,3} and f,g,h and s be functions from A to A given by
   f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) } ;
   h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find $f \circ g$, $g \circ f$, $f \circ h \circ g$, $g \circ s$, $s \circ s$, $f \circ s$.
6. If $f : A \to B$ and $g : B \to C$ br the one – one function the prove that $g \circ f : A \to C$ is also 1-1.
7. Prove that $1^2+2^2+3^2+.... +n^2= n(n+1)(2n+1)/6$ by Principle of Mathematical induction.
8. State and prove Pigeonhole Principle.
9. From the 7 men and 4 women a committee of 6 to be formed can this be done when the committee contains i) Exactly 2 women
   ii) At least 2 women

10. How many permutations of the letters A B C D E F G contain (i) the string BCD, (ii) the string CFGA, (iii) the String BA and GF (iv) the string ABC and DE (v) the string ABC and CDE.
11. i) Assuming that repetitions are not permitted, how many four – digit numbers can be formed from the six digits 1, 2, 3, 5, 7, 8?
    (ii) How many of these numbers are less than 4000?

    (iii) How many of the numbers in part (i) are even?

    (iv) How many of the numbers in part (i) are multiples of 5?
12. Prove that sum of first n odd integers is n^2 by induction method.

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Subject: DISCRETE STRUCTURES | **Subject Code: 17CSU202**
Class : I - B.Sc. Computer Science | **Semester : II**

Unit I

Sets

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then RoS = ------	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(1,5),(3,2),(2,5)}
If f(x) = x+2 and g(x) = x^2 –1 then(gof)(x) = ------	x^2 +4x+4	x^2 +4x-3	x^2 -4x+4	x^2 +4x+3	x^2 +4x+3
A relation R in a set X is ---------- if for every x∈X,(x,x)∉R	transitive	symmetric	irreflexive	reflexive	irreflexive
Suppose in RxR, the ordered pairs (x-2, 2y+1) and (y-1, x+2) are equal. The values of x and y are	2,3	3,2	2,-3	3,-2	3,2
A relation R on a set is said to be an equivalence relation if it is ------	Reflexive	Symmetric	Reflexive,Symmetric, Transitive	Transitive	Reflexive,Symmetric, Transitive
Let f : R→R where R is a set of real numbers.Then f(x) = -2x is a -----	One-to-one	Onto	into	bijection	bijection
A mapping f : x→y is called ------ if distinct elements of x are mapped into distinct elements	one-to-one	Onto	into	many to one	one-to-one
If the relation R and S are both reflexive then R ∨ S is ---------	symmetric	reflexive	transitive	not reflexive	reflexive
A One – to –one function is also known as ---------	injective	surjective	bijective	objective	injective
A On to function is also known as ---------	injective	surjective	bijective	objective	surjective
A One – to –one and onto function is also known as ---------	injective	surjective	bijective	objective	bijective
Let f : x→y , g : y→x be the functions then g is equal to f^{-1} only if --------	fog = Iy	gof = I_x	gof=I_y	fog=I_x	gof = I_x
In N, define aRb if a+b = 7. This is symmetric when --------	b+a =7	a+a =7	b+c =7	a + c = 7	b+a =7
If the relation is ---------- relation if aRb,bRa →a = b -------	symmetric	reflexive	Antisymmetric	not reflexive	Antisymmetric
f : R→R, g : R→R defined by f(x) = 4x-1 and g(x) = cos x..The value of fog is ------	4cosx –1	4cosx	4cosx +1	1/4cosx	4cosx –1
Let f : N→N be a function such that f(x) = 5 ,x∈N then the f(x) is called---------function	identity	inverse	equal	constant	constant
A binary relation R in a set X is said to be symmetric if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb⇒bRa
A binary relation R in a set X is said to be reflexive if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRa
A binary relation R in a set X is said to be antisymmetric if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb,bRa⇒a=b
A binary relation R in a set X is said to be transitive if ---	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb,bRc⇒aRc
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then SoS = ------	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(4,5),(3,3),(1,1)}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the domain of R = ---------	{1,3}	{2,3}	{2,4}	{1,4}	{2,4}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the range of R = ---------	{1,3}	{3,1}	{2,4}	{1,4}	{3,1}
In a relation matrix all the diagonal elements are one then it satisfies -------	symmetric	antisymmetric	transitive	reflexive	reflexive
In a relation matrix A=(aij) a_{ij} =a_{ji} then it satisfies ------- relation	symmetric	reflexive	transitive	antisymmetric	symmetric
An ordered arrangement of r - element of a set containning n - distinct element is called an ---------	r permutation of n elements	r - combination of n elements	n permutation of r elements	n combination of r elements	r permutation of n elements
The r - permutation of n elements is denoted by -----------	P (r, n)	P(n,r)	c(r, n)	c(n, r)	P(n,r)
The r - permutation of n elements is denoted by P (n, r) where -------	r ≤ n	r = n	r ≥ n	r > n	r ≤ n
An unordered pair of r elements of a set containing n distinct elements is called an-------	r permutation of n elements	r - combination of n elements	n permutation of r elements	n combination of r elements	r - combination of n elements
The number of different permutations of the word BANANA is -------	720	60	120	360	60
The number of way a person roundtrip by bus from A to C by way of B will be ------	12	48	144	264	144
How many 10 digits numbers can be written by using the digits 1 and 2 ?	C (10, 9) + C (9, 2)	1024	C(10, 2)	10!	1024
The number of ways to arrange th a letters of the word CHEESE are ------	120	240	720	6	120
r - combination of n elements is denoted by ------	P (r, n)	P(n,r)	C(r, n)	C(n, r)	C(n, r)
The value of C(n,n) is ----------	0	1	n	n-1	1
C (n, n-r) = ------	C(n, r)	C(n-1, r)	C(n-1, r-1)	C(n, r-1)	C(n, r)
C (n, r) + C (n, r-1) = ------	C(n, r)	C (n+1, r-1)	C (n+1, r)	C(n, r+1)	C (n+1, r)
The number of arranging different crcular arrangement of n objects = ----------	n!	(n+1)!	(n -1)!	0!	(n -1)!
The number of ways of arranging n beads in the form of a necklace = ----------	(n-1)!	(n-1)!/2	n!	n!/2	(n-1)!/2
The value of C(10, 6) + C(9, 5) + C(8, 4) + C(8, 3) is ----------	C(10, 7)	C(9,7)	C(8, 5)	C(11, 5)	C(11, 5)
The value of C(10, 8) + C(10,7) is -------------	990	165	45	120	165
The number of different words can be formed out of the letters of the word VARANASI, is---------	64	120	40320	720	720
The number of ways can a party of 7 persons arrange themselves around a circular table-----	6!	7!	5!	7	6!
The sum of entries in the fourth row of Pascal's triangle is ----------	8	4	10	16	8
The number of wors can be formed out of the letters of the word PECULIAR beginning with P and ending with R is ------	100	120	720	150	720
The value of P(n,n) = -------------	1	0	n	n-1	n
The value of P(10, 3) is -----------	120	720	60	45	720
If P (10, r) is 720, then the value of r is --------	2	3	4	5	3
In how many ways 5 children out of a class of 20 line for a picture?	P (20, 4)	P(20, 5)	P (5, 20)	P(5, 5)	P(20, 5)
The value of C(n, r) is ----------	an integer	a fraction	an integer or a fraction	a rational number less than 1	an integer
The value of P(n, r) / r! is ---------	r	C(n, r)	n /r	nr	c(n,r)

UNIT-II

SYLLABUS

Asymptotic Notations, Summation formulas and properties,Bounding Summations, approximation by Integrals

Growth of Functions

- We will use something called *big-O notation* (and some siblings described later) to describe how a function grows.
  - What we're trying to capture here is how the function grows.
  - … without capturing so many details that our analysis would depend on processor speed, etc.
  - … without worrying about what happens for small inputs: they should always be fast.
- For functions $f(x)$ and $g(x)$, we will say that "$f(x)$ is $O(g(x))$" [pronounced "$f(x)$ is big-oh of $g(x)$"] if there are positive constants C and k such that

$$|f(x)|\le C|g(x)| \text{ for all } x>k.$$

  - The big-O notation will give us a order-of-magnitude kind of way to describe a function's growth (as we will see in the next examples).
  - Roughly speaking, the k lets us only worry about big values (or input sizes when we apply to algorithms), and C lets us ignore a factor difference (one, two, or ten steps in a loop).
  - I might also say "$f(x)$ is in $O(g(x))$", then thinking of $O(g(x))$ as the set of all functions with that property.
- *Example:* The function $f(x)=2x^3+10x$ is $O(x^3)$.

  Proof: To satisfy the definition of big-O, we just have to find values for C and k that meet the condition.

  Let $C=12$ and $k=2$. Then for $x>k$,

$$|2x^3+10x|=2x^3+10x<2x^3+10x^3=|12x^3|.\blacksquare$$

- Note: there's nothing that says we have to find the *best* C and k. Any will do.

- Also notice that the absolute value doesn't usually do much: since we're worried about running times, negative values don't usually come up. We can just demand that x is big enough that the function is definitely positive and then remove the $|\cdots|$.

- Now it sounds too easy to put a function in a big-O class. But…
- *Example:* The function $f(x)=2x^3+10x$ is **not in** $O(x^2)$.

  Proof: Now we must show that no C and k exist to meet the condition from the definition.

  For any candidate C and k, we can take $x>k$ and $x>0$ and we would have to satisfy

  $$|2x^3+10x| \; 2x^3+10x \; 2x^3 \; x < C|x^2| < Cx^2 < Cx^2 < C/2$$

  So no such C and k can exist to let the inequality hold for large x.■

- *Example:* The function $f(x)=2x^3+10x$ is $O(x^4)$.

  Proof idea: For large x, we know that $x^4>x^3$. We could easily repeat the $O(x^3)$ proof above, applying that inequality in a final step.

- *Example:* The function $f(x)=5x^2-10000x+7$ is $O(x^2)$.

  Proof: We have to be a little more careful about negative values here because of the "$-10000x$" term, but as long as we take $k\geq 2000$, we won't have any negative values since the $5x^2$ term is larger there.

  Let $C=12$ and $k=2000$. Then for $x>k$,

  $$|5x^2-10000x+7|=5x^2-10000x+7<5x^2+7x^2=|12x^2|.$$

- It probably wouldn't take many more proofs to convince you that x^n is always in $O(x^n)$ but never in $O(x^{n-1})$.
  - We can actually do better than that…
- The big-O operates kind of like a $\leq$ for growth rates.

Big-O Results

- *Theorem:* Any degree-n polynomial, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is in $O(x^n)$.

  Proof: As before, we can assume that $x>1$ and then,

  $$|f(x)|=|a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0|\le|a_n|x^n+|a_{n-1}|x^{n-1}+\cdots+|a_1|x+|a_0|=x^n(|a_n|+|a_{n-1}|/x+\cdots+|a_1|/x^{n-1}+|a_0|/x^n)\le x^n(|a_n|+|a_{n-1}|+\cdots+|a_1|+|a_0|).$$

  Now, if we let $C=\sum|a_i|$ and $k=1$, we have satisfied the definition for $O(x^n)$.■

- *Theorem:* If we have two functions $f_1(x)$ and $f_2(x)$ both $O(g(x))$, then $f_1(x)+f_2(x)$ is also $O(g(x))$.

  Proof: From the definition of big-O, we know that there are C_1 and k_1 that make $|f_1(x)|\le C|h(x)|$ for $x>k_1$, and similar C_2 and k_2 for $f_2(x)$.

  Let $C=C_1+C_2$ and $k=\max(k_1,k_2)$. Then for $x>k$,

  $$|f_1(x)+f_2(x)|\le|f_1(x)|+|f_2(x)|\le C_1|g(x)|+C_2|g(x)|=C|g(x)|.$$

  Thus, $f_1(x)+f_2(x)$ is $O(g(x))$.■

- The combination of functions under big-O is generally pretty sensible…
  - *Theorem:* If for large enough x, we have $f(x)\le g(x)$, then $f(x)$ is $O(g(x))$.
    - Sometimes the big-O proof is even easier.
  - *Theorem:* If we have two functions $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$, then $f(x)+g(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.
    - When adding, the bigger one wins.
  - *Theorem:* If we have three functions f,g,h where $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, then $f(x)$ is $O(h(x))$.
    - Approximately: if h is bigger than g and g is bigger than f, then h is bigger than f.
  - *Corollary:* Given $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$ **and** $g_1(x)$ is $O(g_2(x))$ then $f_1(x)+f_2(x)$ is $O(g_2(x))$.
    - That is, if we have two functions we know a big-O bound for, and we add them together, we can ignore the smaller one in the big-O.
  - *Theorem:* If we have two functions $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$, then $f(x)g(x)$ is $O(g_1(x)g_2(x))$.
    - Multiplication happens in the obvious way.

  - *Theorem:* Any constant value is is $O(1)$.
    - Aside: You will often hear a constant running time algorithm described as $O(1)$.
  - *Corollary:* Given $f(x)$ which is $O(g(x))$ and a constant a, we know that $af(x)$ is $O(g(x))$.
    - That is, if we have a function multiplied by a constant, we can ignore the constant in the big-O.
- All of that means that it's usually pretty easy to guess a good big-O category for a function.
  - $f(x)=2^x+x^2$ is in $O(\max(|2^x|,|x^2|))=O(2^x)$, since 2^x is larger than x^2 for large x.
  - $f(x)=\frac{1}{100}x^{12}+100x^{11}-87$ is in $O(x^{12})$.
    - Directly from the theorem about polynomials.
    - For small x, the $100x^{11}$ is the largest, but as x grows, the $\frac{1}{100}x^{12}$ term takes over.
  - $f(x)=14x2^x+x$ is in $O(x2^x)$.
- What is a good big-O bound for $17x^4-12x^2+\log_2 x$?
  - We can start with the obvious:

$$17x^4-12x^2+\log_2 x \text{ is in } O(17x^4-12x^2+\log_2 x).$$

  - From the above, we know we can ignore smaller-order terms:

$$17x^4-12x^2+\log_2 x \text{ is in } O(17x^4).$$

  - And we can ignore leading constants:

$$17x^4-12x^2+\log_2 x \text{ is in } O(x^4).$$

- The "ignore smaller-order terms and leading constants" trick is very useful and comes up a lot.

Big-Ω

- As mentioned earlier, big-O feels like $\le$ for growth rates.
  - … then there must be $\ge$ and $=$ versions.
- We will say that a function $f(x)$ is $\Omega(g(x))$ ("big-omega of $g(x)$") if there are positive constants C and k such that when $x>k$,

$$|f(x)| \geq C|g(x)|.$$

- This is the same as the big-O definition, but with a $\geq$ instead of a $\leq$.

- *Example:* The function $3x^2+19x$ is $\Omega(x^2)$.

  Proof: If we let $C=3$ and $k=1$ then for $x>k$,

  $$|3x^2+19x| \geq 3x^2+19x \geq 3|x^2|.$$

  From the definition, we have that $3x^2+19x$ is $\Omega(x^2)$. ■

- As you can guess, the proofs of big-Ω are going to look just about like the big-O ones.
  - We have to be more careful with negative values: in the big-O proofs, we could just say that the absolute value was bigger and ignore it. Now we need smaller values, so can't be so quick.
  - But the basic ideas are all the same.
- *Theorem:* $f(x)$ is $O(g(x))$ iff $g(x)$ is $\Omega(f(x))$.

  Proof: First assume we have $f(x)$ in $O(g(x))$. Then there are positive C and k so that when $x>k$, we know $|f(x)| \leq C|g(x)|$. Then for $x>k$, we have $|g(x)| \geq \frac{1}{C}|f(x)|$ and we can use k and $\frac{1}{C}$ as constants for the definition of big-Ω.

  Similarly, if we assume that $g(x)$ is $\Omega(f(x))$, we have positive C and k so that when $x>k$, we have $|g(x)| \geq C|f(x)|$. As above we then have for $x>k$, $|f(x)| \leq \frac{1}{C}|g(x)|$. ■

Big-Θ

- We will say that a function $f(x)$ is $\Theta(g(x))$ ("big-theta of $g(x)$") if $f(x)$ is **both** $O(g(x))$ and $\Omega(g(x))$.
  - For a function that is $\Theta(g(x))$, we will say that that function "is order $g(x)$."
- *Example:* The function 2^x+x^2 is order 2^x.

  Proof: To show that 2^x+x^2 is $O(2^x)$, we can take $C=2$ and $k=4$. Then for $x>k$,

  $$|2^x+x^2| = 2^x+x^2 \leq 2 \cdot 2^x.$$

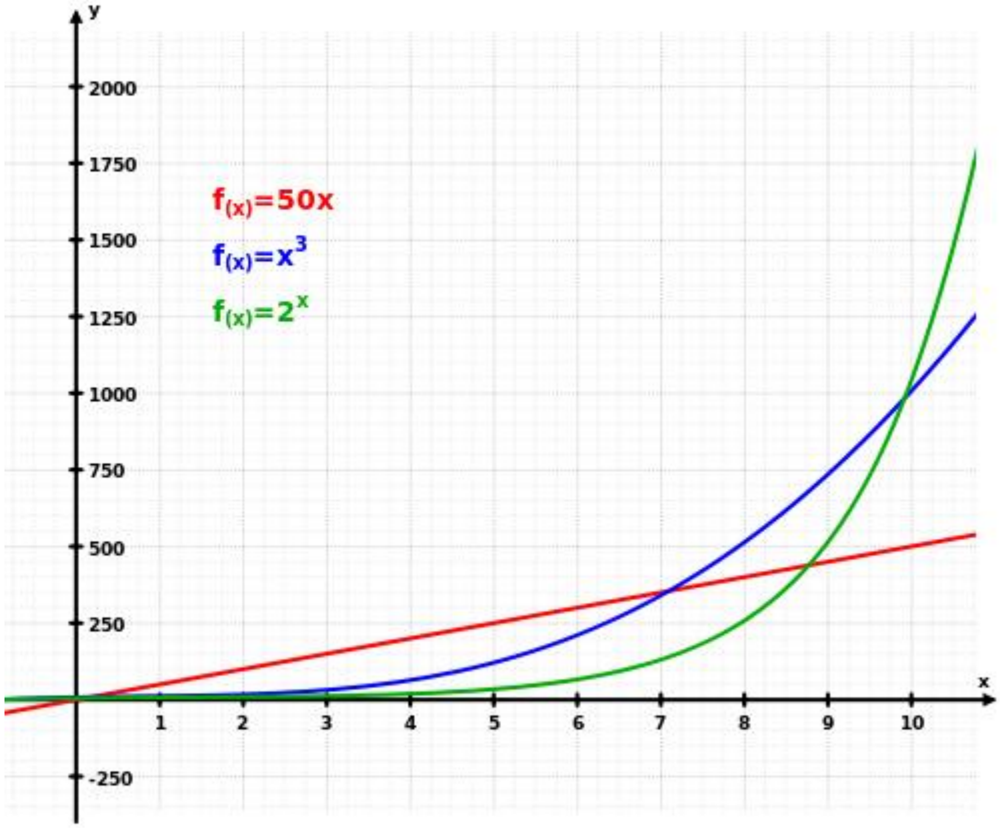
  To show that 2^x+x^2 is $\Omega(2^x)$, we can use $C=1$ and $k=1$. For $x>k$,

$$|2^x+x^2|=2^x+x^2\geq 2^x.$$

Thus, 2^x+x^2 is $\Theta(2^x)$. ■

- The above theorem gives another way to show big-Θ: if we can show that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, then $f(x)$ is $\Theta(g(x))$.
- *Theorem:* Any degree-n polynomial with $a_n\neq 0$, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with $a_n>0$ is in $\Theta(x^n)$.
- A few results on big-Θ…
    - *Theorem:* If we have two functions $f_1(x)$ which is $\Theta(g_1(x))$ and $f_2(x)$ which is $\Theta(g_2(x))$, and $g_2(x)$ is $O(g_1(x))$, then $f_1(x)+f_2(x)$ is $\Theta(g_1(x)))$.
        - That is, when adding two functions together, the bigger one "wins".
    - *Theorem:* If we have two functions $f_1(x)$ which is $\Theta(g(x))$ and $f_2(x)$ which is $O(g(x))$, then $f(x)+g(x)$ is $\Theta(g(x)))$.
    - *Theorem:* for a positive constant a, a function $af(x)$ is $\Theta(g(x))$ iff $f(x)$ is $\Theta(g(x))$.
        - That is, leading constants don't matter.
    - *Corollary:* Any degree-n polynomial, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with $a_n>0$ is in $\Theta(x^n)$.

- What functions have a "higher" big-Θ than others is usually fairly obvious from a graph, but "I looked at a graph" isn't very much of a proof.



Source: *[Wikipedia Exponential.svg]*

- The big-O notation sets up a hierarchy of function growth rates. Here are some of the important "categories":

$$n! \; 2^n \; n^3 \; n^2 \; n\log n \; n \; \sqrt{n}=n^{1/2} \; \log n \; 1$$

  - Each function here is big-O of ones above it, but not below.
  - e.g. $n\log n$ is $O(n^2)$, but n^2 is **not** $O(n\log n)$.
  - So in some important way, n^2 grows faster than $n\log n$.
  - Where we are headed: we will be able to look at an algorithm and say that one that takes $O(n\log n)$ steps is faster than one that takes $O(n^2)$ steps (for large input).

Asymptotic Notation

9.7.1 Little Oh

Definition 9.7.1. For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is *asymptotically smaller* than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

For example, $1000x^{1.9} = o(x^2)$, because $1000x^{1.9}/x^2 = 1000/x^{0.1}$ and since $x^{0.1}$ goes to infinity with x and 1000 is constant, we have $\lim_{x\to\infty} 1000x^{1.9}/x^2 = 0$. This argument generalizes directly to yield

9.7.2 Big Oh

Big Oh is the most frequently used asymptotic notation. It is used to give an upper bound on the growth of a function, such as the running time of an algorithm.

Definition 9.7.5. Given nonnegative functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = O(g)$$

iff

$$\limsup_{x\to\infty} f(x)/g(x) < \infty.$$

This definition[12] makes it clear that

Definition 9.7.12. Given functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = \Omega(g)$$

iff there exists a constant $c > 0$ and an x_0 such that for all $x \geq x_0$, we have $f(x) \geq c|g(x)|$.

In other words, $f(x) = \Omega(g(x))$ means that $f(x)$ is greater than or equal to $g(x)$, except that we are willing to ignore a constant factor and to allow exceptions for small x.

If all this sounds a lot like big-Oh, only in reverse, that's because big-Omega is the opposite of big-Oh. More precisely,

Little Omega

There is also a symbol called little-omega, analogous to little-oh, to denote that one function grows strictly faster than another function.

Definition 9.7.14. For functions $f, g : \mathbb{R} \to \mathbb{R}$ with f nonnegative, we say that

$$f(x) = \omega(g(x))$$

iff

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$

In other words,

$$f(x) = \omega(g(x))$$

iff

$$g(x) = o(f(x)).$$

Definition 9.7.15.

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } g = O(f).$$

The statement $f = \Theta(g)$ can be paraphrased intuitively as "f and g are equal to within a constant factor." Indeed, by Theorem 9.7.13, we know that

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } f = \Omega(g).$$

Example: $n^2 + n = O(n^3)$

Proof:

- Here, we have $f(n) = n^2 + n$, and $g(n) = n^3$
- Notice that if $n \geq 1$, $n \leq n^3$ is clear.
- Also, notice that if $n \geq 1$, $n^2 \leq n^3$ is clear.
- **Side Note:** In general, if $a \leq b$, then $n^a \leq n^b$ whenever $n \geq 1$. This fact is used often in these types of proofs.
- Therefore,

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- We have just shown that

$$n^2 + n \leq 2n^3 \text{ for all } n \geq 1$$

- Thus, we have shown that $n^2 + n = O(n^3)$ (by definition of Big-O, with $n_0 = 1$, and $c = 2$.)

Example: $n^3 + 4n^2 = \Omega(n^2)$

Proof:

- Here, we have $f(n) = n^3 + 4n^2$, and $g(n) = n^2$
- It is not too hard to see that if $n \geq 0$,

$$n^3 \leq n^3 + 4n^2$$

- We have already seen that if $n \geq 1$,

$$n^2 \leq n^3$$

- Thus when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 4n^2$$

- Therefore,

$$1n^2 \leq n^3 + 4n^2 \text{ for all } n \geq 1$$

- Thus, we have shown that $n^3 + 4n^2 = \Omega(n^2)$ (by definition of Big-Ω, with $n_0 = 1$, and $c = 1$.)

Example: $n^2 + 5n + 7 = \Theta(n^2)$

Proof:

- When $n \geq 1$,

$$n^2 + 5n + 7 \leq n^2 + 5n^2 + 7n^2 \leq 13n^2$$

- When $n \geq 0$,

$$n^2 \leq n^2 + 5n + 7$$

- Thus, when $n \geq 1$

$$1n^2 \leq n^2 + 5n + 7 \leq 13n^2$$

Thus, we have shown that $n^2 + 5n + 7 = \Theta(n^2)$ (by definition of Big-Θ, with $n_0 = 1$, $c_1 = 1$, and $c_2 = 13$.)

Show that $\frac{1}{2}n^2 + 3n = \Theta(n^2)$

Proof:

- Notice that if $n \geq 1$,

$$\frac{1}{2}n^2 + 3n \leq \frac{1}{2}n^2 + 3n^2 = \frac{7}{2}n^2$$

- Thus,

$$\frac{1}{2}n^2 + 3n = O(n^2)$$

- Also, when $n \geq 0$,

$$\frac{1}{2}n^2 \leq \frac{1}{2}n^2 + 3n$$

- So

$$\frac{1}{2}n^2 + 3n = \Omega(n^2)$$

- Since $\frac{1}{2}n^2 + 3n = O(n^2)$ and $\frac{1}{2}n^2 + 3n = \Omega(n^2)$,

$$\frac{1}{2}n^2 + 3n = \Theta(n^2)$$

Show that $(n \log n - 2n + 13) = \Omega(n \log n)$

Proof: We need to show that there exist positive constants c and n_0 such that

$$0 \le c\,n \log n \le n \log n - 2n + 13 \text{ for all } n \ge n_0.$$

Since $n \log n - 2n \le n \log n - 2n + 13,$

we will instead show that

$$c\,n \log n \le n \log n - 2n,$$

which is equivalent to

$$c \le 1 - \frac{2}{\log n}, \text{ when } n > 1.$$

If $n \ge 8$, then $2/(\log n) \le 2/3$, and picking $c = 1/3$ suffices. Thus if $c = 1/3$ and $n_0 = 8$, then for all $n \ge n_0$, we have

$$0 \le c\,n \log n \le n \log n - 2n \le n \log n - 2n + 13.$$

Thus $(n \log n - 2n + 13) = \Omega(n \log n)$.

Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Proof:

- We need to find positive constants c_1, c_2, and n_0 such that

$$0 \leq c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2 \text{ for all } n \geq n_0$$

- Dividing by n^2, we get

$$0 \leq c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$$

- $c_1 \leq \frac{1}{2} - \frac{3}{n}$ holds for $n \geq 10$ and $c_1 = 1/5$
- $\frac{1}{2} - \frac{3}{n} \leq c_2$ holds for $n \geq 10$ and $c_2 = 1$.
- Thus, if $c_1 = 1/5$, $c_2 = 1$, and $n_0 = 10$, then for all $n \geq n_0$,

$$0 \leq c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2 \text{ for all } n \geq n_0.$$

Thus we have shown that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.

Summary of the Notation

- $f(n) = O(g(n)) \Rightarrow f \preceq g$
- $f(n) = \Omega(g(n)) \Rightarrow f \succeq g$
- $f(n) = \Theta(g(n)) \Rightarrow f \approx g$
- It is important to remember that a Big-O bound is only an *upper bound*. So an algorithm that is $O(n^2)$ might not ever take that much time. It may actually run in $O(n)$ time.
- Conversely, an Ω bound is only a *lower bound*. So an algorithm that is $\Omega(n \log n)$ might actually be $\Theta(2^n)$.
- Unlike the other bounds, a Θ-bound is precise. So, if an algorithm is $\Theta(n^2)$, it runs in quadratic time.

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define Big oh.
2. Define Big omega.

3. Define little oh.
4. Define big theta.
5. Prove that the function f(x)=$2x^3$+10x is O(x^3).
6. Prove that the function $3x^2$+19x is Ω(x^2).
7. Prove that $n^2 + 5n + 7 = \Theta(n^2)$.
8. Define Arithmetic series.
9. Define Geometric series.
10. Define Harmonic series.
11. Evaluate $\sum_{k=1}^{9}(5k+8)$
12. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{(2n+1)^2}{5n^2+2n+1}$

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Show that (n log n - 2n + 13) = Ω (n log n).
2. Show that if we have two functions $f_1(x)$ and $f_2(x)$ both O(g(x)), then $f_1(x)+f_2(x)$ is also O(g(x)).
3. Prove that f(x) is O(g(x)) iff g(x) is Ω(f(x)).
4. Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.
5. Prove that the function f(x)=$5x^2$−10000x+7 is O(x^2).
6. Prove that $\sum_{i=n}^{m+1}(ax_i + by_i) = a\sum_{i=n}^{m} x_i + b\sum_{i=n}^{m} y_i$
7. Prove that $\sum_{k=1}^{\infty}\Omega(f(\mathrm{k})) = \Omega(\sum_{k=1}^{\infty}f(\mathrm{k}))$.
8. i) prove that the arithmetic series $\sum_{k=1}^{n} k$ evaluates to $\frac{1}{2}n(n+1)$.

   ii) prove that the geometric series $\sum_{k=0}^{n} 3^k$ is O(3^n).
9. i) Evaluate the sum $\sum_{k=1}^{8}(5k^2+8k+1)$

   ii) Evaluate the sum $\sum_{k=7}^{12}(k+1)$
10. If $\sum_{k=1}^{n} k^4 = \frac{4n\,(n+1)(2n+1)(3n^2+3n-1)}{A}$ then find A?
11. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k+9}{n}$
12. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(k\frac{7}{n}\right)^2$
13. The integral $\int_0^5 x^2dx$ is computed as the limit of the sum $\sum_{k=1}^{n}\frac{A}{n}\left(k\frac{A}{n}\right)^2$. What value of A must appear in the sum ?

UNIT-III

SYLLABUS

Recurrence relations, generating functions, linear recurrence relations with constant coefficients and their solution, Substitution Method, recurrence trees, Master theorem.

Solving the Recurrence

Claim 10.1.1. $T_n = 2^n - 1$ *satisfies the recurrence:*

$$\begin{aligned} T_1 &= 1 \\ T_n &= 2T_{n-1} + 1 \qquad (\text{for } n \geq 2). \end{aligned}$$

Proof. The proof is by induction on n. The induction hypothesis is that $T_n = 2^n - 1$. This is true for $n = 1$ because $T_1 = 1 = 2^1 - 1$. Now assume that $T_{n-1} = 2^{n-1} - 1$ in order to prove that $T_n = 2^n - 1$, where $n \geq 2$:

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 1. \end{aligned}$$

Linear Recurrences

In general, a *homogeneous linear recurrence* has the form

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)$$

where $a_1, a_2, \ldots, a_d$ and d are constants. The *order* of the recurrence is d. Commonly, the value of the function f is also specified at a few points; these are called *boundary conditions*. For example, the Fibonacci recurrence has order $d = 2$ with coefficients $a_1 = a_2 = 1$ and $g(n) = 0$. The boundary conditions are $f(0) = 1$ and $f(1) = 1$. The word "homogeneous" sounds scary, but effectively means "the simpler kind". We'll consider linear recurrences with a more complicated form later.

Theorem 10.3.1. *If $f(n)$ and $g(n)$ are both solutions to a homogeneous linear recurrence, then $h(n) = sf(n) + tg(n)$ is also a solution for all $s, t \in \mathbb{R}$.*

Proof.

$$\begin{aligned}
h(n) &= sf(n) + tg(n) \\
&= s\,(a_1 f(n-1) + \ldots + a_d f(n-d)) + t\,(a_1 g(n-1) + \ldots + a_d g(n-d)) \\
&= a_1(sf(n-1) + tg(n-1)) + \ldots + a_d(sf(n-d) + tg(n-d)) \\
&= a_1 h(n-1) + \ldots + a_d h(n-d)
\end{aligned}$$

Solving First-Order Recurrences Using Back Substitution

Theorem 2. (Solution of First-Order Recurrence Relations) The solution of

$$T(n) = \begin{cases} cT(n-1) + f(n) & \text{for } n \geq k \\ f(k) & \text{for } n = k \end{cases}$$

where c is a constant and f is a nonzero function of n for $n \geq k$ is

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Motivation for the Proof. First, use back substitution to decide what the general form of the solution might be, and then prove by induction that this is the solution:

$$\begin{aligned} T(n) &= cT(n-1) + f(n) \\ &= c(cT(n-2) + f(n-1)) + f(n) \\ &= c^2T(n-2) + cf(n-1) + f(n) \\ &= c^2(cT(n-3) + f(n-2)) + cf(n-1) + f(n) \\ &= c^3T(n-3) + c^2f(n-2) + cf(n-1) + f(n) \end{aligned}$$

Using back substitution one more time gives

$$\begin{aligned} T(n) &= c^3\,[cT(n-4) + f(n-3)] + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + c^3f(n-3) + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + \sum_{l=n-3}^{n} c^{n-l} f(l) \end{aligned}$$

If back substitution is continued until the argument of T is k—that is, for $n-k$ steps—then the expression for $T(n)$ becomes

$$\begin{aligned} T(n) &= c^{n-k}T(n-(n-k)) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \\ &= c^{n-k}T(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \end{aligned}$$

Since $T(k) = f(k)$, replace the reference to T on the right-hand side of the equation, getting

$$\begin{aligned} T(n) &= c^{n-k} f(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \\ &= \sum_{l=n-k}^{n} c^{n-l} f(l) \end{aligned}$$

Proof. By induction, show that

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Let $n_0 = k$. Let $\mathcal{T} = \{n \in \mathbb{N} : n \geq k \text{ and } T(n) \text{ is a solution}\}$.

(Base step) First, show that

$$\sum_{l=k}^{n} c^{n-l} f(l)$$

is a solution for $n = k$ so that $k \in \mathcal{T}$.

$$\sum_{l=k}^{k} c^{k-l} f(l) = c^{k-k} f(k) = f(k) = T(k)$$

(Inductive step) Now, assume that $T(n)$ is given by this expression for $n \geq n_0$, that is, $T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$. Now prove that $T(n+1)$ is also given by this expression: In this case, prove that $T(n+1) = \sum_{l=k}^{n+1} c^{n-l} f(l)$.

$$\begin{aligned} T(n+1) &= cT(n) + f(n+1) \quad \text{(Definition of recurrence relation)} \\ &= c\sum_{l=k}^{n} c^{n-l} f(l) + f(n+1) \quad \text{(Inductive hypothesis)} \\ &= \sum_{l=k}^{n} c^{n-l+1} f(l) + f(n+1) \\ &= \sum_{l=k}^{n+1} c^{n+1-l} f(l) \end{aligned}$$

This proves $n + 1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \{n \in \mathbb{N} : n \geq k\}$. ■

Example 1. Solve

$$T(n) = \begin{cases} T(n-1) + n^2 & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = n^2$ for $n \geq 0$, $c = 1$, and $k = 0$. Since $T(0) = f(0)$, by Corollary 1 the solution is

$$T(n) = \sum_{l=1}^{n} l^2 = \frac{1}{6} \cdot (2n+1) \cdot n \cdot (n+1)$$

See Theorem 9(b) in Section 7.10 for a derivation of this formula. ■

Example 2. Solve

$$T(n) = \begin{cases} 3T(n-1) + 4 & \text{for } n \geq 1 \\ 4 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = 4$ for $n \geq 0$, $c = 3$, and $k = 0$. By Corollary 2, the solution is

$$T(n) = 4 \cdot \frac{3^{n+1} - 1}{3 - 1} = 2 \cdot (3^{n+1} - 1)$$ ■

Rules for Solving Second-Order Recurrence Relations

Solving Second-Order Homogeneous Recurrence Relations with Constant Coefficients Using the Complementary Equation with Distinct Real Roots

$$H(n) + AH(n-1) + BH(n-2) = 0,$$
$$H(n_1) = D, \text{ and } H(n_2) = E.$$

STEP 1: Assume $f(n) = c^n$ is a solution, and substitute for $H(n)$, yielding the characteristic equation

$$c^2 + Ac + B = 0$$

STEP 2: Find the roots of the characteristic equation: c_1 and c_2. Use the quadratic formula if the equation does not factor. If $c_1 \neq c_2$, then the general solution is

$$S(n) = Ac_1^n + Bc_2^n$$

STEP 3: Use the initial conditions to form the system of equations

$$H(n_1) = D = Ac_1^{n_1} + Bc_2^{n_2}$$
$$H(n_2) = E = Ac_1^{n_2} + Bc_2^{n_2}$$

STEP 4: Solve the system of equations found in step 3, getting A_0 and B_0 as the two solutions. Form the particular solution

$$H(n) = A_0 c_1{}^n + B_0 c_2{}^n$$

Example 1. Solve the recurrence relation $a_n - 6a_{n-1} - 7a_{n-2} = 0$ for $n \geq 5$ where $a_3 = 344$ and $a_4 = 2400$.

Solution. Form the characteristic equation and then factor it:

$$c^2 - 6c - 7 = 0$$
$$c = 7, -1$$

Form the general solution of the recurrence relation $a_n = A7^n + B(-1)^n$, and solve the system of equations determined by the boundary values $a_3 = 344$ and $a_4 = 2400$ to get the particular solution:

$$a_3 = A7^3 + B(-1)^3$$
$$a_4 = A7^4 + B(-1)^4$$

Now, substituting 344 and 2400 for a_3 and a_4 gives

$$344 = 343A - B$$
$$2400 = 2401A + B$$

Adding the two equations gives

$$2744 = 2744A$$
$$1 = A$$

It follows that $B = -1$. Therefore, $a_n = 7^n + (-1)^{n+1}$ for $n \geq 3$ is the particular solution. ■

Substitution Method

- Guess the form of solution and use induction to find constants
- Determine upper bound on the recurrence

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

Guess the solution as: $T_n = O(n \lg n)$
Now, prove that $T_n \leq cn \lg n$ for some $c > 0$
Assume that the bound holds for $\lfloor \frac{n}{2} \rfloor$
Substituting into the recurrence

$$\begin{aligned} T_n &\leq 2(c\left\lfloor \frac{n}{2} \right\rfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor)) + n \\ &\leq cn \lg\left(\frac{n}{2}\right) + n \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\leq cn \lg n \quad \forall c \geq 1 \end{aligned}$$

Boundary condition: Let the only bound be $T_1 = 1$

$$\nexists c \mid T_1 \leq c 1 \lg 1 = 0$$

Problem overcome by the fact that asymptotic notation requires us to prove

$$T_n \leq cn \lg n \text{ for } n \geq n_0$$

Include T_2 and T_3 as boundary conditions for the proof

$$T_2 = 4 \quad T_3 = 5$$

Choose c such that $T_2 \leq c 2 \lg 2$ and $T_3 \leq c 3 \lg 3$
True for any $c \geq 2$

- If a recurrence is similar to a known recurrence, it is reasonable to guess a similar solution

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

  If n is large, difference between $T_{\lfloor \frac{n}{2} \rfloor}$ and $T_{\lfloor \frac{n}{2} \rfloor + 17}$ is relatively small

- Prove upper and lower bounds on a recurrence and reduce the range of uncertainty.
  Start with a lower bound of $T_n = \Omega(n)$ and an initial upper bound of $T_n = O(n^2)$. Gradually lower the upper bound and raise the lower bound to get asymptotically tight solution of $T_n = \Theta(n \lg n)$

- Recursion trees
  - Recurrence

$$T_n = 2T_{\frac{n}{2}} + n^2$$

  Assume n to be an exact power of 2.

$$\begin{aligned}
T_n &= n^2 + 2T_{\frac{n}{2}} \\
&= n^2 + 2\left(\left(\frac{n}{2}\right)^2 + 2T_{\frac{n}{4}}\right) \\
&= n^2 + \frac{n^2}{2} + 4\left(\left(\frac{n}{4}\right)^2 + 2T_{\frac{n}{8}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + 8\left(\left(\frac{n}{8}\right)^2 + 2T_{\frac{n}{16}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{8} + \cdots \\
&= n^2(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) \\
&= \Theta(n^2)
\end{aligned}$$

  The values above decrease geometrically by a constant factor.

– Recurrence

$$T_n = T_{\frac{n}{3}} + T_{\frac{2n}{3}} + n$$

Longest path from root to a leaf

$$n \rightarrow \left(\frac{2}{3}\right) n \rightarrow \left(\frac{2}{3}\right)^2 n \rightarrow \cdots 1$$

$\left(\frac{2}{3}\right)^k n = 1$ when $k = \log_{\frac{3}{2}} n$, k being the height of the tree
Upper bound to the solution to the recurrence – $n \log_{\frac{3}{2}} n$, or $O(n \log n)$

The Master Method

- Suitable for recurrences of the form

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where $a \geq 1$ and $b > 1$ are constants, and
$f(n)$ is an asymptotically positive function

- For mergesort, $a = 2$, $b = 2$, and $f(n) = \Theta(n)$
- Master Theorem

Theorem 2 *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let T_n be defined on the nonnegative integers by the recurrence*

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where we interpret $\frac{n}{b}$ to mean either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then T_n can be bounded asymptotically as follows

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T_n = \Theta(n^{\log_b a})$*
2. *If $f(n) = \Theta(n^{\log_b a})$, then $T_n = \Theta(n^{\log_b a} \lg n)$*
3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T_n = \Theta(f(n))$*

- In all three cases, compare $f(n)$ with $n^{\log_b a}$
- Solution determined by the larger of the two
  * Case 1: $n^{\log_b a} > f(n)$
    Solution $T_n = \Theta(n^{\log_b a})$
  * Case 2: $n^{\log_b a} \approx f(n)$
    Multiply by a logarithmic factor
    Solution $T_n = \Theta(n^{\log_b a} \lg n) = \Theta(f(n) \lg n)$
  * Case 3: $f(n) > n^{\log_b a}$
    Solution $T_n = \Theta(f(n))$
- In case 1, $f(n)$ must be asymptotically smaller than $n^{\log_b a}$ by a factor of n^{ϵ} for some constant $\epsilon > 0$
- In case 3, f_n must be polynomially larger than $n^{\log_b a}$ and satisfy the "regularity" condition that $af(\frac{n}{b}) \leq cf(n)$

- Using the master method
  - Recurrence
$$T_n = 9T_{\frac{n}{3}} + n$$
$a = 9,\ b = 3,\ f(n) = n$
$n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
$f(n) = O(n^{\log_3 9-\epsilon})$, where $\epsilon = 1$
Apply case 1 of master theorem and conclude $T_n = \Theta(n^2)$
  - Recurrence
$$T_n = T_{\frac{2n}{3}} + 1$$
$a = 1,\ b = \frac{3}{2},\ f(n) = 1$
$n^{\log_b a} = n^{\log_{\frac{3}{2}} 1} = n^0 = 1$
$f(n) = \Theta(n^{\log_b a}) = \Theta(1)$
Apply case 2 of master theorem and conclude $T_n = \Theta(\lg n)$
  - Recurrence
$$T_n = 3T_{\frac{n}{4}} + n\lg n$$
$a = 3,\ b = 4,\ f(n) = n\lg n$
$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$
$f(n) = \Omega(n^{\log_4 3+\epsilon})$, where $\epsilon \approx 0.2$
Apply case 3, if regularity condition holds for $f(n)$
For large n, $af(\frac{n}{b}) = 3\frac{n}{4}\lg(\frac{n}{4}) \leq \frac{3}{4}n\lg n = cf(n)$ for $c = \frac{3}{4}$
Therefore, $T_n = \Theta(n\lg n)$

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define recurrence relation.
2. Define non homogeneous recurrence relation.
3. Define characteristics equation
4. Solve $a_n - 4a_{n-1} = 0$ for n ≥ 2 with $a_0 = 1, a_1 = 1$.
5. If the sequence $a_n = 3.2^n, n \geq 1$ then find the corresponding recurrence relation.
6. State Master theorem.
7. Define Strassen's algorithm.
8. Define recurrence tree.
9. Write the methods for solving recurrence
10. Define divide-and-conquer algorithms.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Solve the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$ with the initial condition $a_0 = a_1 = 1$.
2. Solve the recurrence relation defined by $S_0 = 100$ and $S_k = (1.08)S_{k-1}$ $for\ k \geq 1$.
3. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10$, $a_1 = 41, \ldots$ using generating function.
4. Solve the recurrence relation $a_n - 6a_{n-1} - 7a_{n-2} = 0$ for $n \geq 5$ where $a_3 = 344$ and $a_4 = 2400$.
5. Solve $T(n) = 2\,T\left(\frac{n}{2}\right) + n$ using substitution method.
6. Solve $T(n) = 8\,T\left(\frac{n}{2}\right) + n^2$ ($T(1) = 1$) by recurrence tree method
7. Solve $T(n) = 7\,T\left(\frac{n}{2}\right) + \Theta\,(\mathrm{n}^2)$ by iterative method
8. State and prove Master Theorem.
9. Solve $T(n) = 2\,T(\sqrt{n}\,) + \log n$
10. If $T(n) = aT\left(\frac{n}{b}\right) + n^c, a \geq 1, b \geq 1, c \geq 0$ then prove that $T(n) =$

$$\begin{cases} \Theta\left(n^{\log_b a}\right) & a > b^c \\ \Theta(n^c \log_b n) & a = b^c \\ \Theta(n^c) & a < b^c \end{cases}$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Coimbatore –641 021

Subject: DISCRETE STRUCTURES **Subject Code: 17CSU202**

Class : I - B.Sc. Computer Science **Semester : II**

Unit III

Recurrences

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The procedure for finding the terms of a sequence in a recursive manner is called ………………………..	reflexive relation	recurrence relation	transitive relation	linear relation	recurrence relation
An equation or inequality that describes a function in terms of its value on smaller inputs known as ……………………..	non linear relation	linear relation	symmetric relation	recurrence relation	recurrence relation
Recurrence can be solved to derive the …………….. time	generating	running	starting	terminating	running
In recurrence tree, T_n = …………………………	2Tn/2 - n2	2Tn/2 * n2	2Tn/2 + n2	3Tn/2 + n2	2Tn/2 + n2
A ……………………….. recurrence has the form f (n)= a1f (n – 1)+ a2f (n – 2)+…….+adf(n –d)	non homogeneous linear	non linear	linear	homogeneous linear	homogeneous linear
If f (n) and g(n) are both solutions to a ………………….. recurrence, then h(n)= sf(n) + tg(n) is also a solution for all s, t ϵ R.	non linear	linear	homogeneous linear	non homogeneous linear	homogeneous linear
Generating functions can be used to find a solution to any ………………………..	homogeneous recurrence	non homogeneous linear recurrence	linear recurrence	non linearrecurrence	linear recurrence
The generating function for choosing elements from a union of disjoint sets is the ………………….. of the generating functions for choosing from each set.	product	difference	equal	sum	product
Recurrence relation is a formula that relates two or more successive terms in a ………………………………	values	series	sequence	variables	sequence
Any recurrence relation is accompanied by ……………………………… which specifies the first term of the sequence.	zero condition	initial condition	boundary condition	final condition	initial condition
The purpose of solving a recurrence relation is to find a formula for the general term of the sequence given by that …………………………………	symmetric relation	transitive relation	recurrence relation	reflexive relation	recurrence relation
Solving …………………….. is used in computer science to assess the running time of recursive algorithms.	transitive relation	recurrence relations	reflexive relation	symmetric relation	recurrence relations
Linear homogeneous recurrence relations with…………..coefficients and their sequences.	non zero	constant	varied	zero	constant
A recurrence relation is homogeneous if …………………………….	h(n) = 1	h(n) = 0	h(n) = x	h(n) = x+y	h(n) = 0
Methods for solving recurrences is/are …………………….	both Substitution method and Iteration method	direct method	Iteration method	Substitution method	both Substitution method and Iteration method
Recursion-tree method and Master method are ……………………………. method	constant	Substitution	Iteration	direct	Iteration
Sometimes recurrences can be reduced to simpler ones by changing ……………………….	variables	values	series	constants	variables
A………………………………………….. can be used to visualize the iteration procedure.	fibonacci series	Generating functions	power series	recurrence tree	recurrence tree
The classical Tower of Hanoi problem gives us the recurrence T(n) = 2T(n − 1) + 1 with base case ………………….	T(1) = 0	T(0) = 1	T(0) = 0	T(1) = 1	T(0) = 0
A common class of recurrences arises in the context of recursive backtracking algorithms and counting problems is called …………………………	non homogeneous recurrence	linear recurrences	non linear recurrences	homogeneous recurrence	linear recurrences
A recurrence T(n) = f(n)T(n − 1) + g(n) is called a ……………………….. linear recurrence.	higher order	third order	first order	second order	first order
A recurrence in which T(n) is expressed in terms of a sum of constant multiples of T(k) for certain values k < n is called a ……………………………………..coefficient recurrence.	varied	constant	different	zero	constant
The idea of a Recursion Tree is to expand T (n) to a tree with the ……………………………..total cost.	zero	same	unit	different	same
Recurrences can be used to represent the runtime of ……………………………………	Generating functions	recursive functions	homogeneous functions	linear functions	recursive functions.
The pattern in recurrence tree method is typically ……………………………..	constant series	fibonacci series	a arithmetic or geometric series.	taylor series	a arithmetic or geometric series
In linear recurrence each term of a sequence is a…………………………….. of earlier terms in the sequence.	linear function	non linear functions	recursive functions	Generating functions	linear function
Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal ………………….	fibonacci series	power series	taylor series	constant series	power series
…………………….. can be used for solving a variety of counting problems.	Generating functions	non linear functions	linear functions	homogeneous functions	Generating functions
Generating functions can be used for solving …………………….	homogeneous functions	recurrence relations	non linear functions	linear functions	recurrence relations
………………………….. can be used for proving some of the combinatorial identities	linear functions	homogeneous functions	Generating functions	non linear functions	Generating functions
Generating functions can be used for finding asymptotic formulae for terms of ……………….	relations	sequences	functions	seires	sequences
If the recurrence equations is Fn = Fn-1 + Fn-2 with initial values a1 = a2 = 1 then it is ………………….	Pell number	Fibonacci number	Padovan sequence	Lucas number	Fibonacci number
If the recurrence equations is Fn = Fn-1 + Fn-2 with initial values a1 = 1, a2 = 3 then it is ………………….	Fibonacci number	Padovan sequence	Lucas number	Pell number	Lucas number
If the recurrence equations is Fn = Fn-2 + Fn-3 with initial values a1 = a2 = a3 = 1 then it is ………………….	Padovan sequence	Pell number	Fibonacci number	Lucas number	Padovan sequence
If the recurrence equations is Fn = 2Fn-1 + Fn-2 with initial values a1 = 0, a2 = 1 then it is ………………….	Lucas number	Padovan sequence	Fibonacci number	Pell number	Pell number
The recurrence for Fibonacci numbers Fn = Fn-1 + Fn-2 is a linear homogeneous recurrence relation of degree …………………………..	three	two	four	one	two

UNIT-IV

SYLLABUS

Basic terminology, models and types, multigraphs and weighted graphs, graph representation, graph isomorphism, connectivity, Euler and Hamiltonian Paths and circuits, Planar graphs, graph coloring, trees, basic terminology and properties of trees, introduction to Spanning trees

UNIT – IV

Graphs

Basic Concepts

Definition 8.1.1. **[Pseudograph, Vertex set and Edge set]** A **pseudograph** or a **general graph** G is a pair (V, E) where V is a nonempty set and E is a multiset of unordered pairs of points of V. The set V is called the **vertex set** and its elements are called **vertices**. The set E is called the **edge set** and its elements are called **edges**.

Example 8.1.2. $G = \Big([4], \{\{1,1\},\{1,2\},\{2,2\},\{3,4\},\{3,4\}\}\Big)$ is a pseudograph.

Discussion 8.1.3. A pseudograph can be represented in picture in the following way.

1. Put different points on the paper for vertices and label them.
2. If $\{u, v\}$ appears in E some k times, draw k distinct lines joining the points u and v.
3. A loop at u is drawn if $\{u, u\} \in E$.

Example 8.1.4. A picture for the pseudograph in Example 8.1.2 is given in Figure 8.1.

Definition 8.1.5. **[Loop, End vertex and Incident vertex/edge]**

1. An edge $\{u, v\}$ is sometimes denoted uv. An edge uu is called a **loop**. The vertices u and v are called the **end vertices** of the edge uv. Let e be an edge. We say 'e is **incident** on u' to mean that 'u is an end vertex of e'.

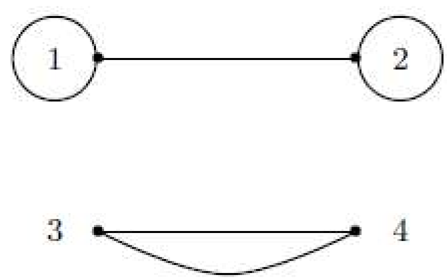


Figure 8.1: A pseudograph

2. **[Multigraph and simple graph]** A **multigraph** is a pseudograph without loops. A multigraph is a **simple graph** if no edge appears twice.[1]
3. Henceforth, all graphs in this book are simple with a finite vertex set, unless stated otherwise.

4. We use $\boldsymbol{V(G)}$ (or simply $\boldsymbol{V}$) and $\boldsymbol{E(G)}$ (or simply $\boldsymbol{E}$) to denote the vertex set and the edge set of G, respectively. The number $|V(G)|$ is the **order** of the graph G. Sometimes it is denoted $|\boldsymbol{G}|$. By $\|\boldsymbol{G}\|$ we denote the number of edges of G. A graph with n vertices and m edges is called a $\boldsymbol{(n, m)}$ **graph**. The $(1, 0)$ graph is the **trivial graph**.

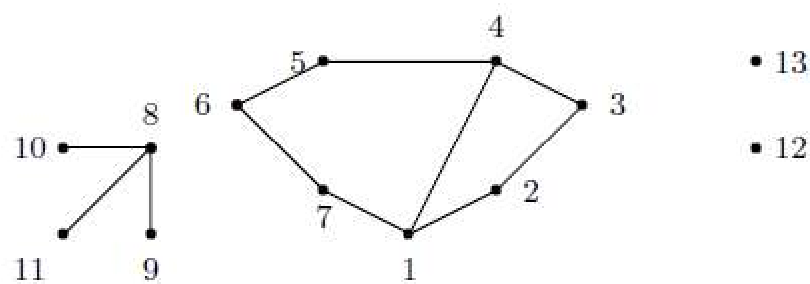
5. **[Neighbor and independent set]** If uv is an edge in G, then we say 'u and v are **adjacent** in G' or 'u is a **neighbor** of v'. We write $u \sim v$ to denote that 'u is adjacent to v'. Two edges e_1 and e_2 are **adjacent** if they have a common end vertex. A set of vertices or edges is **independent** if no two of them are adjacent.

6. **[Isolated and pendant vertex]** If $v \in V(G)$, by $\boldsymbol{N(v)}$ or $\boldsymbol{N_G(v)}$, we denote the set of neighbors of v in G and $|N(v)|$ is called the **degree** of v. It is usually denoted by $\boldsymbol{d_G(v)}$ or $\boldsymbol{d(v)}$. A vertex of degree 0 is called **isolated**. A vertex of degree one is called a **pendant** vertex.

Example 8.1.7. Consider the graph G in Figure 8.2. The vertex 12 is an isolated vertex. We have $N(1) = \{2, 4, 7\}$, $d(1) = 3$. The set $\{9, 10, 11, 2, 4, 7\}$ is an independent vertex set. The set $\{\{1, 2\}, \{8, 10\}, \{4, 5\}\}$ is an independent edge set. The vertices 1 and 6 are not adjacent.

Definition 8.1.8. [Complete graph, path graph, cycle graph and bipartite graph] Let $G = (V, E)$ be a graph on n vertices, say $V = \{v_1, \ldots, v_n\}$. Then, G is said to be a

1. **complete** graph, denoted K_n, if each pair of vertices in G are adjacent.
2. **path graph**, denoted $\boldsymbol{P_n}$, if $E = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$.



3. **cycle graph**, denoted C_n, if $E = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.

4. **complete bipartite graph**, denoted $K_{r,s}$ and $E = \{v_i v_j \mid 1 \leq i \leq r, r+1 \leq j \leq n\}$ with $r+s=n$.

Lemma 8.1.10. **[Hand shaking lemma]** *In any graph G, $\sum\limits_{v \in V} d(v) = 2|E|$. Thus, the number of vertices of odd degree is even.*

Proof. Each edge contributes 2 to the sum $\sum\limits_{v \in V} d(v)$. Hence, $\sum\limits_{v \in V} d(v) = 2|E|$. Note that

$$2|E| = \sum_{v \in V} d(v) = \sum_{d(v) \text{ is odd}} d(v) + \sum_{d(v) \text{ is even}} d(v)$$

is even. So, $\sum\limits_{d(v) \text{ is odd}} d(v)$ is even. Hence, the number of vertices of odd degree is even. ■

Proposition 8.1.12. *In a graph G with $n = |G| \geq 2$, there are two vertices of equal degree.*

Proof. If G has two or more isolated vertices, we are done. So, suppose G has exactly one isolated vertex. Then, the remaining $n-1$ vertices have degree between 1 and $n-2$ and hence by PHP, the result follows. If G has no isolated vertex then G has n vertices whose degree lie between 1 and $n-1$. Now, again apply PHP to get the required result. ■

Example 8.1.13. The graph in Figure 8.5 is called the **Petersen graph**. We shall use it as an example in many places.

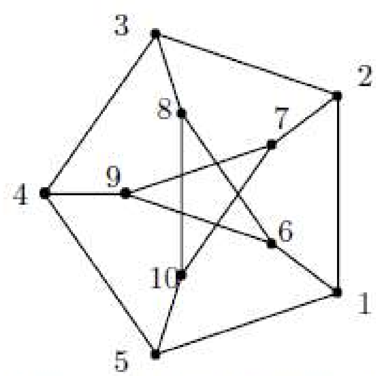


Figure 8.5: Petersen graphs

Definition 8.1.15. **[Regular graph, cubic graph]** The minimum degree of a vertex in G is denoted $\delta(G)$ and the maximum degree of a vertex in G is denoted $\Delta(G)$. A graph G is called **k-regular** if $d(v) = k$ for all $v \in V(G)$. A 3-regular graph is called **cubic**.

Definition 8.1.18. **[Subgraph, induced subgraph, spanning subgraph and k-factor]** A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $U \subseteq V(G)$, then the subgraph **induced** by U is denoted by $\langle U \rangle = (U, E)$, where the edge set $E = \{uv \in E(G) \mid u, v \in U\}$. A subgraph H of G is a **spanning subgraph** if $V(G) = V(H)$. A k-regular spanning subgraph is called a **k-factor**.

Example 8.1.22. Consider the graph G in Figure 8.2. Let H_2 be the graph with $V(H_2) = \{6, 7, 8, 9, 10, 12\}$ and $E(H_2) = \big\{\{6,7\}, \{8,10\}\big\}$. Consider the edge $e = \{8, 9\}$. Then, $H_2 + e$ is the induced subgraph $\langle\{6, 7, 8, 9, 10, 12\}\rangle$ and $H_2 - 8 = \langle\{6, 7, 9, 10, 12\}\rangle$.

Definition 8.1.23. [Complement graph] The **complement** $\overline{G}$ of a graph G is defined as $(V(G), E)$, where $E = \{uv \mid u \neq v, uv \notin E(G)\}$.

Example 8.1.24. 1. See the graphs in Figure 8.6.

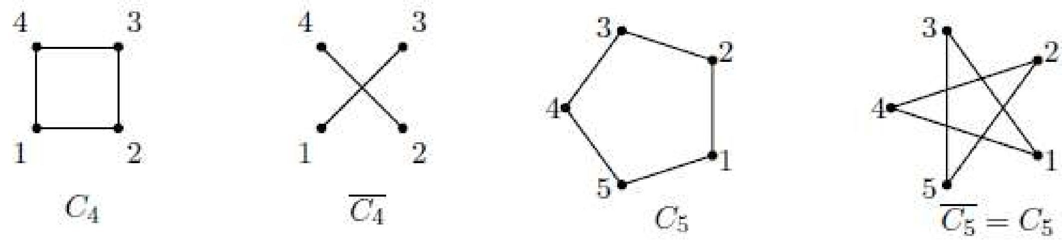
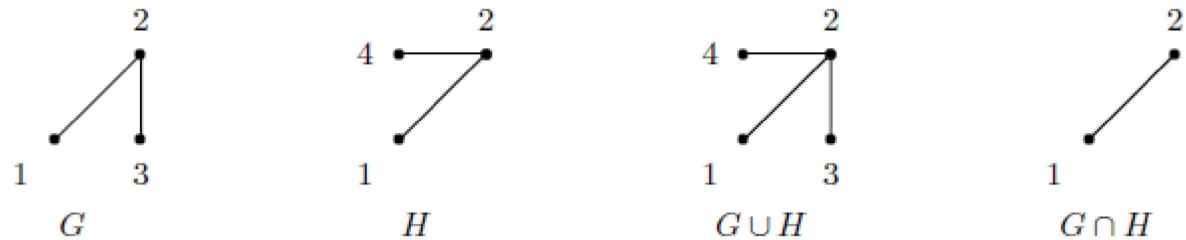


Figure 8.6: Complement graphs

2. The complement of K_3 contains 3 isolated points.
3. For any graph G, $\|G\| + \|\overline{G}\| = C(|G|, 2)$.
4. In any graph G of order n, $d_G(v) + d_{\overline{G}}(v) = n - 1$. Thus, $\Delta(G) + \Delta(\overline{G}) \geq n - 1$.

Definition 8.1.26. [Intersection, union and disjoint union] The **intersection** of two graphs G and H, denoted $G \cap H$, is defined as $(V(G) \cap V(H), E(G) \cap E(H))$. The **union** of two graphs G and H, denoted $G \cup H$, is defined as $(V(G) \cup V(H), E(G) \cup E(H))$. A **disjoint union** of two graphs is the union while treating the vertex sets as disjoint sets.

Example 8.1.27. Two graphs G and H are shown below. The graphs $G \cup H$ and $G \cap H$ are also shown below.



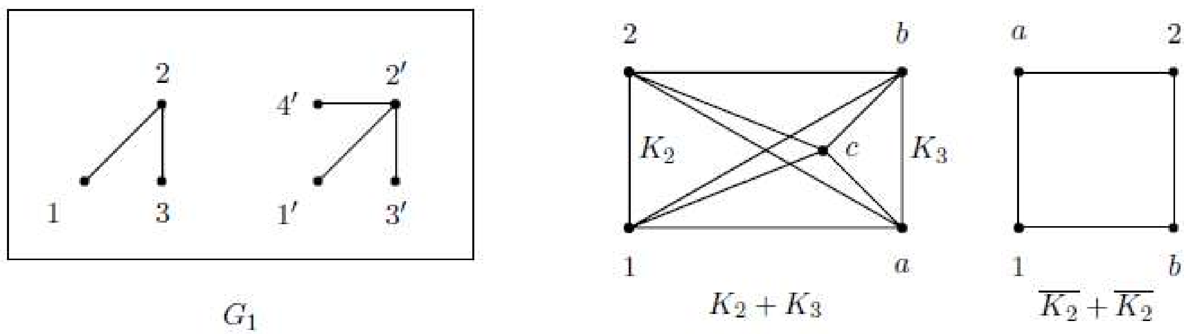


Figure 8.7: Disjoint union and join of graphs

Definition 8.1.28. **[Join of two graphs]** If $V(G) \cap V(G') = \emptyset$, then the **join** $G + G'$ is defined as $G \cup G' + \{vv' : v \in V, v' \in V'\}$. The first '+' means the join of two graphs and the second '+' means adding a set of edges to a given graph.

Connectedness

Definition 8.2.1. **[Walk, trail, path, cycle, circuit, length and internal vertex]** An ***u*-*v* walk** in G is a finite sequence of vertices $[u = v_1, v_2, \cdots, v_k = v]$ such that $v_i v_{i+1} \in E$, for all $i = 1, \cdots, k-1$. The **length** of a walk is the number of edges on it. A walk is called a **trail** if edges on the walk are not repeated. A v-u walk is a called a **path** if the vertices involved are all distinct, except that v and u may be the same. A path can have length 0. A walk (trail, path) is called **closed** if $u = v$. A closed path is called a **cycle/circuit**. Thus, in a simple graph a cycle has length at least 3. A cycle (walk, path) of length k is also written as a k-cycle (k-walk, k-path). If P is an u-v path with $u \neq v$, then we sometimes call u and v as the **end vertices of *P*** and the remaining vertices on P as the **internal vertices**.

Proposition 8.2.3 (Technique). *Let G be a graph and $u, v \in V(G)$, $u \neq v$. Let $W = [u = u_1, \ldots, u_k = v]$ be a walk. Then, W contains an u-v-path.*

Proof. If no vertex on W repeats, then W is itself a path. So, let $u_i = u_j$ for some $i < j$. Now, consider the walk $W_1 = [u_1, \ldots, u_{i-1}, u_j, u_{j+1}, \ldots u_k]$. This is also an u-v walk but of shorter length. Thus, using induction on the length of the walk, the desired result follows. ■

Proposition 8.2.10. *Every graph G containing a cycle satisfies $g(G) \leq 2\,\mathsf{diam}(G) + 1$.*

Proof. Let $C = [v_1, v_2, \ldots, v_k, v_1]$ be the shortest cycle and $\mathsf{diam}(G) = r$. If $k \geq 2r + 2$, then consider the path $P = [v_1, v_2, \ldots, v_{r+2}]$. Since the length of P is $r + 1$ and $\mathsf{diam}(G) = r$, there is a v_{r+2}-v_1 path R of length at most r. Note that P and R are different v_1-v_{r+2} paths. By Proposition 8.2.9, the closed walk $P \cup R$ of length at most $2r + 1$ contains a cycle. Hence, the length of this cycle is at most $2r+1$, a contradiction to C having the smallest length $k \geq 2r+2$. ■

Definition 8.2.11. [Chord, chordal and acyclic graphs] Let $C = [v_1, \ldots, v_k = v_1]$ be a cycle. An edge v_iv_j is called a **chord** of C if it is not an edge of C. A graph is called **chordal** if each cycle of length at least 4 has a chord. A graph is **acyclic** if it has no cycles.

Definition 8.2.14. 1. **[Maximal and minimal graph]** A graph G is said to be **maximal** with respect to a property P if G has property P and no proper supergraph of G has the property P. We similarly define the term **minimal**.

Proposition 8.2.17. *If $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$.*

Proof. Let $[v_1, \cdots, v_k]$ be a longest path in G. As $d(v_k) \geq 2$, v_k is adjacent to some vertex $v \neq v_{k-1}$. If v is not on the path, then we have a path that is longer than $[v_1, \cdots, v_k]$ path. A contradiction. Let i be the smallest positive integer such that v_i is adjacent to v_k. Thus,

$$\delta(G) \leq d(v_k) \leq |\{v_i, v_{i+1}, \cdots, v_{k-1}\}|.$$

Hence, the cycle $C = [v_i, v_{i+1}, \cdots, v_k, v_i]$ has length at least $\delta(G)+1$ and the length of the path $P = [v_i, v_{i+1}, \cdots, v_k]$ is at least $\delta(G)$. ■

Definition 8.2.18. [Edge density] The **edge density**, denoted $\varepsilon(G)$, is defined to be the number $\frac{|E(G)|}{|V(G)|}$. Observe that $\varepsilon(G)$ is also a graph invariant.

Isomorphism in Graphs

Definition 8.3.1. [Isomorphic graphs] Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be **isomorphic** if there is a bijection $f : V \to V'$ such that $u \sim v$ is G if and only if $f(u) \sim f(v)$ in G', for each $u, v \in V$. In other words, an isomorphism is a bijection between the vertex sets which preserves adjacency. We write $G \cong G'$ to mean that G is isomorphic to G'.

Example 8.3.2. Consider the graphs in Figure 8.9. Then, note that

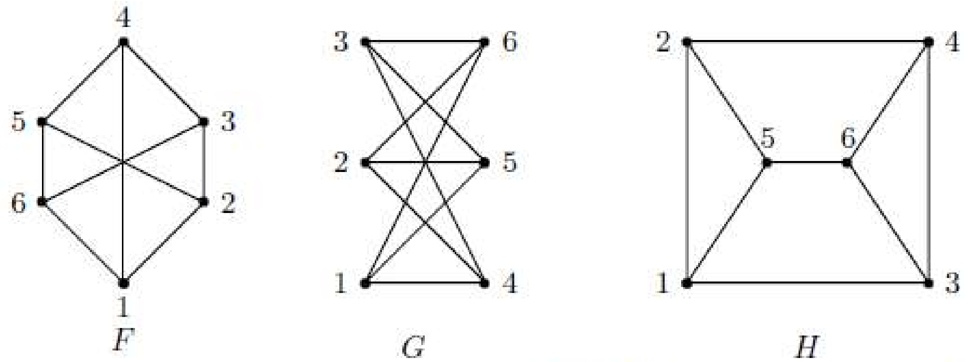


Figure 8.9: F is isomorphic to G but F is not isomorphic to H

1. the graph F is not isomorphic to H as the **independence number**, denoted $\alpha(F)$, of F (the maximum size of an independent vertex set) is 3 whereas $\alpha(H) = 2$. Alternately, H has a 3-cycle, whereas F does not.
2. the graph F is isomorphic to G as the map $f : V(F) \to V(G)$ defined by $f(1) = 1$, $f(2) = 5$, $f(3) = 3$, $f(4) = 4$, $f(5) = 2$ and $f(6) = 6$ gives an isomorphism.

Definition 8.3.5. **[Self-complementary]** A graph G is called **self-complementary** if $G \cong \overline{G}$.

Example 8.3.6. 1. Note that the cycle $C_5 = [0, 1, 2, 3, 4, 0]$ is self complimentary. An isomorphism from G to $\overline{G}$ is described by $f(i) = 2i \pmod 5$.

2. If $|G| = n$ and $G \cong \overline{G}$ then $\|G\| = n(n-1)/4$. Thus, $n = 4k$ or $n = 4k + 1$.

Definition 8.3.8. A **graph invariant** is a function which assigns the same value (output) to isomorphic graphs.

Example 8.3.9. Observe that some of the graph invariants are: $|G|$, $\|G\|$, $\Delta(G)$, $\delta(G)$, the multiset $\{d(v) : v \in V(G)\}$, $\omega(G)$ and $\alpha(G)$.

Definition 8.3.12. An isomorphism of G to G is called an **automorphism**.

Example 8.3.13.
1. Identity map is always an automorphism on any graph.
2. Any permutation in S_n is an automorphism of K_n.
3. There are only two automorphisms of a path P_8.

Trees

Definition 8.4.1. **[Tree and forest]** A connected acyclic graph is called a **tree**. A **forest** is a graph whose components are trees.

Proposition 8.4.2. *Let T be a tree and $u, v \in V(T)$. Then, there is a unique u-v-path in T.*

Proof. On the contrary, assume that there are two u-v-paths in T. Then, by Proposition 8.2.9, T has a cycle, a contradiction. ■

Proposition 8.4.3. *Let G be a graph with the property that 'between each pair of vertices there is a unique path'. Then, G is a tree.*

Proof. Clearly, G is connected. If G has a cycle $[v_1, v_2, \cdots, v_k = v_1]$, then $[v_1, v_2, \ldots, v_{k-1}]$ and $[v_1, v_{k-1}]$ are two v_1-v_{k-1} paths. A contradiction. ■

Definition 8.4.4. **[Cut vertex]** Let G be a connected graph. A vertex v of G is called a **cut vertex** if $G - v$ is disconnected. Thus, $G - v$ is connected if and only if v is not a cut vertex.

Proposition 8.4.5. *Let G be a connected graph with $|G| \geq 2$. If $v \in V(G)$ with $d(v) = 1$, then $G - v$ is connected. That is, a vertex of degree 1 is never a cut vertex.*

Proof. Let $u, w \in V(G - v)$, $u \neq w$. As G is connected, there is an u-w path P in G. The vertex v cannot be an internal vertex of P, as each internal vertex has degree at least 2. Hence, the path P is available in $G - v$. So, $G - v$ is connected. ■

Proposition 8.4.6 (Technique). *Let G be a connected graph with $|G| \geq 2$ and let $v \in V(G)$. If $G - v$ is connected, then either $d(v) = 1$ or v is on a cycle.*

Proof. Assume that $G - v$ is connected. If $d_G(v) = 1$, then there is nothing to show. So, assume that $d(v) \geq 2$. We need to show that v is on a cycle in G.

Let u and w be two distinct neighbors of v in G. As $G - v$ is connected there is a path, say $[u = u_1, \ldots, u_k = w]$, in $G - v$. Then, $[u = u_1, \ldots, u_k = w, v, u]$ is a cycle in G containing v. ■

Definition 8.4.8. **[Cut edge]** Let G be a graph. An edge e in G is called a **cut edge** or a **bridge** if $G - e$ has more connected components than that of G.

Proposition 8.4.9 (Technique). *Let G be connected and $e = [u, v]$ be a cut edge. Then, $G - e$ has two components, one containing u and the other containing v.*

Proof. If $G - e$ is not disconnected, then by definition, e cannot be a cut edge. So, $G - e$ has at least two components. Let G_u (respectively, G_v) be the component containing the vertex u (respectively, v). We claim that these are the only components.

Let $w \in V(G)$. Then, G is a connected graph and hence there is a path, say P, from w to u. Moreover, either P contains v as its internal vertex or P doesn't contain v. In the first case, $w \in V(G_v)$ and in the latter case, $w \in V(G_u)$. Thus, every vertex of G is either in $V(G_v)$ or in $V(G_u)$ and hence the required result follows. ■

Proposition 8.4.10 (Technique). *Let G be a graph and e be an edge. Then, e is a cut edge if and only if e is not on a cycle.*

Proof. Suppose that $e = [u, v]$ is a cut edge of G. Let F be the component of G that contains e. Then, by Proposition 8.4.9, $F - e$ has two components, namely, F_u that contains u and F_v that contains v.

Let if possible, $C = [u, v = v_1, \ldots, v_k = u]$ be a cycle containing $e = [u, v]$. Then, $[v = v_1, \ldots, v_k = u]$ is an u-v path in $F - e$. Hence, $F - e$ is still connected. A contradiction. Hence, e cannot be on any cycle.

Conversely, let $e = [u, v]$ be an edge which is not on any cycle. Now, suppose that F is the component of G that contains e. We need to show that $F - e$ is disconnected.

Let if possible, there is an u-v-path, say $[u = u_1, \ldots, u_k = v]$, in $F - e$. Then, $[v, u = u_1, \ldots, u_k = v]$ is a cycle containing e. A contradiction to e not lying on any cycle.

Hence, e is a cut edge of F. Consequently, e is a cut edge of G. ■

Theorem 8.4.12. *Let G be a graph with $V(G) = [n]$. Then, the following are equivalent.*

1. *G is a tree.*
2. *G is a minimal connected graph on n vertices.*
3. *G is a maximal acyclic graph on n vertices.*

Proof. (a)⇒(b). Suppose that G is a tree. If it is not a minimal connected graph on n vertices, then there is an edge $[u, v]$ such that $G - [u, v]$ is connected. But then, by Theorem 8.4.10, $[u, v]$ is on a cycle in G. A contradiction.

(b)$\Rightarrow$(c). Suppose G is a minimal connected graph on n vertices. If G has a cycle, say Γ, then select an edge $e \in \Gamma$. Thus, by Theorem 8.4.10, $G - e$ is still connected graph on n vertices, a contradiction to the fact that G is a minimal connected graph on n vertices. Hence, G is acyclic. Since G is connected, for any new edge e, the graph $G + e$ contains a cycle and hence, G is maximal acyclic graph.

(c)$\Rightarrow$(a). Suppose G is maximal acyclic graph on n vertices. If G is not connected, let G_1 and G_2 be two components of G. Select $v_1 \in G_1$ and $v_2 \in G_2$ and note that $G + [v_1, v_2]$ is acyclic graph on n vertices. This contradicts that G is a maximal acyclic graph on n vertices. Thus, G is connected and acyclic and hence is a tree. ■

Proposition 8.4.15. *Let T be a tree on n vertices. Then, T has $n-1$ edges.*

Proof. We proceed by induction. Take a tree on $n \geq 2$ vertices and delete an edge e. Then, we get two subtrees T_1, T_2 of order n_1, n_2, respectively, where $n_1 + n_2 = n$. So, $E(T) = E(T_1) \cup E(T_2) \cup \{e\}$. By induction hypothesis $\|T\| = \|T_1\| + \|T_2\| + 1 = n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 = n - 1$. ■

Proposition 8.4.16. *Let G be a connected graph with n vertices and $n-1$ edges. Then, G is acyclic.*

Proof. On the contrary, assume that G has a cycle, say Γ. Now, select an edge $e \in \Gamma$ and note that $G - e$ is connected. We go on selecting edges from G that lie on cycles and keep removing them, until we get an acyclic graph H. Since the edges that are being removed lie on some cycle, the graph H is still connected. So, by definition, H is a tree on n vertices. Thus, by Proposition 8.4.15, $|E(H)| = n - 1$. But, in the above argument, we have deleted at least one edge and hence, $|E(G)| \geq n$. This gives a contradiction to $|E(G)| = n - 1$. ■

Proposition 8.4.17. *Let G be an acyclic graph with n vertices and $n-1$ edges. Then, G is connected.*

Proof. Let if possible, G be disconnected with components $G_1, \ldots, G_k, k \geq 2$. As G is acyclic, by definition, each G_i is a tree on, say $n_i \geq 1$ vertices, with $\sum i = 1^k n_i = n$. Thus, $\|G\| = \sum\limits_{i=1}^{k} (n_i - 1) = n - k < n - 1 = \|G\|$, as $k \geq 2$. A contradiction. ■

Connectivity

Proposition 8.5.1. *Let G be a connected graph on vertex set $[n]$. Then, its vertices can be labeled in such a way that the induced subgraph on the set $[i]$ is connected for $1 \leq i \leq n$.*

Proof. If $n = 1$, there is nothing to prove. Assume that the statement is true if $n < k$ and let G be a connected graph on the vertex set $[k]$. If G is a tree, pick any pendant vertex and label it k. If G has a cycle, pick a vertex on a cycle and label it k. In both the case $G - k$ is connected. Now, use the induction hypothesis to get the required result. ∎

Definition 8.5.2. **[Separating set]** Let G be a graph. Then, a set $X \subseteq V(G) \cup E(G)$ is called a **separating set** if $G - X$ has more connected components than that of G.

Definition 8.5.4. **[Vertex connectivity]** A graph G is said to be **k-connected** if $|G| > k$ and G is connected even after deletion of any $k - 1$ vertices. The **vertex connectivity $\kappa(G)$** of a non trivial graph G is the largest k such that G is k-connected. Convention: $\kappa(K_1) = 0$.

Example 8.5.5.

1. Each connected graph of order more than one is 1-connected.
2. A 2-connected graph is also a 1-connected graph.
3. For a disconnected graph, $\kappa(G) = 0$ and for $n > 1$, $\kappa(K_n) = n - 1$.
4. The graph G in Figure 8.11 is 2-connected but not 3-connected. Thus, $\kappa(G) = 2$.

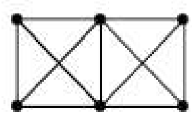


Figure 8.11: graph with vertex connectivity 2

5. The Petersen graph is 3-connected.

Definition 8.5.6. [**Edge connectivity**] A graph G is called **l-edge connected** if $|G| > 1$ and $G - F$ is connected for every $F \subseteq E(G)$ with $|F| < l$. The greatest integer l such that G is l-edge connected is the **edge connectivity** of G, denoted $\boldsymbol{\lambda(G)}$. Convention: $\lambda(K_1) = 0$.

Example 8.5.7.

1. Note that $\lambda(P_n) = 1, \lambda(C_n) = 2$ and $\lambda(K_n) = n - 1$, whenever $n > 1$.
2. Let T be a tree on n vertices. Then, $\lambda(T) = 1$.
3. For the graph G in Figure 8.11, $\lambda(G) = 3$.
4. For the Petersen graph G, $\lambda(G) = 3$.

Theorem 8.5.9. [**H. Whitney,** 1932] *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. If G is disconnected or $|G| = 1$, then we have nothing to prove. So, let G be connected graph and $|G| \geq 2$. Then, there is a vertex v with $d(v) = \delta(G)$. If we delete all edges incident on v, then the graph is disconnected. Thus, $\delta(G) \geq \lambda(G)$.

Suppose that $\lambda(G) = 1$ and $G - uv$ is disconnected with components C_u and C_v. If $|C_u| = |C_v| = 1$, then $G = K_2$ and $\kappa(G) = 1$. If $|C_u| > 1$, then we delete u to see that $\kappa(G) = 1$.

If $\lambda(G) = k \geq 2$, then there is a set of edges, say $e_1, \ldots, e_k$, whose removal disconnects G. Notice that $G - \{e_1, \ldots, e_{k-1}\}$ is a connected graph with a bridge, say $e_k = uv$. For each of $e_1, \ldots, e_{k-1}$ select an end vertex other than u or v. Deletion of these vertices from G results in a graph H with uv as a bridge of a connected component. Note that $\kappa(H) \leq 1$. Hence, $\kappa(G) \leq \lambda(G)$. ∎

Eulerian Graphs

Definition 8.6.1. [Eulerian graph] An **Eulerian tour** in a graph G is a closed walk $[v_0, v_1, \ldots, v_k, v_0]$ such that each edge of the graph appears exactly once in the walk. The graph G is said to be **Eulerian** if it has an Eulerian tour.

Theorem 8.6.2. **[Euler,** 1736] *A connected graph G is Eulerian if and only if $d(v)$ is even, for each $v \in V(G)$.*

Proof. Let G have an Eulerian tour, say $[v_0, v_1, \ldots, v_k, v_0]$. Then, $d(v) = 2r$, if $v \neq v_0$ and v appears r times in the tour. Also, $d(v_0) = 2(r-1)$, if v_0 appears r times in the tour. Hence, $d(v)$ is always even.

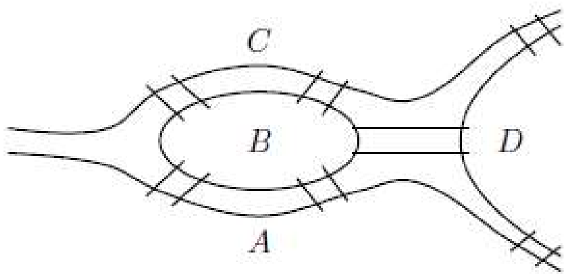


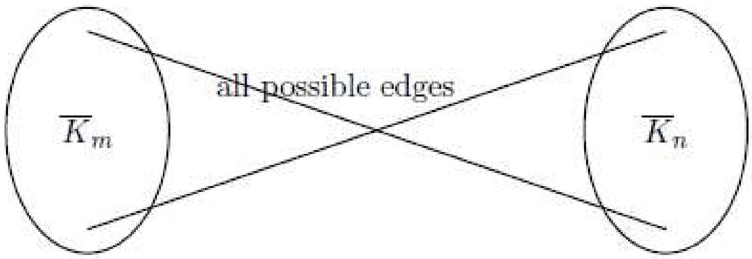
Figure 8.12: Königsberg bridge problem

Conversely, let G be a connected graph with each vertex having even degree. Let $W = v_0 v_1 \cdots v_k$ be a longest walk in G without repeating any edge in it. As v_k has an even degree it follows that $v_k = v_0$, otherwise W can be extended. If W is not an Eulerian tour then there exists an edge, say $e' = v_i w$, with $w \neq v_{i-1}, v_{i+1}$. In this case, $w v_i \cdots v_k (= v_0) v_1 \cdots v_{i-1} v_i$ is a longer walk, a contradiction. Thus, there is no edge lying outside W and hence W is an Eulerian tour. ∎

Proposition 8.6.3. *Let G be a connected graph with exactly two vertices of odd degree. Then, there is an Eulerian walk starting at one of those vertices and ending at the other.*

Proof. Let x and y be the two vertices of odd degree and let v be a symbol such that $v \notin V(G)$. Then, the graph H with $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{xv, yv\}$ has each vertex of even degree and hence by Theorem 8.6.2, H is Eulerian. Let $\Gamma = [v, v_1 = x, \ldots, v_k = y, v]$ be an Eulerian tour. Then, $\Gamma - v$ is an Eulerian walk with the required properties. ∎

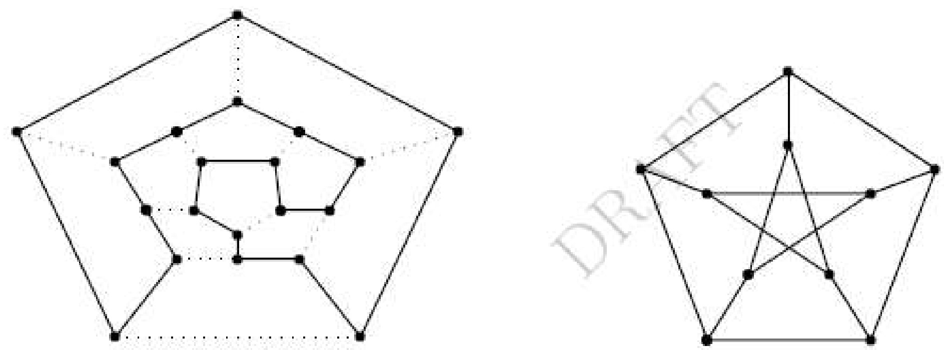
Definition 8.6.8. [**bipartite graph**] A graph $G = (V, E)$ is said to be **bipartite** if $V = V_1 \cup V_2$ such that $|V_1|, |V_2| \geq 1$, $V_1 \cap V_2 = \emptyset$ and $\|\langle V_1 \rangle\| = 0 = \|\langle V_2 \rangle\|$. The complete bipartite graph $K_{m,n}$ is shown below. Notice that $K_{m,n} = \overline{K}_m + \overline{K}_n$.



Hamiltonian Graphs

Definition 8.7.1. [**Hamiltonian**] A cycle in G is said to be **Hamiltonian** if it contains all vertices of G. If G has a Hamiltonian cycle, then G is called a **Hamiltonian** graph. Finding a nice characterization of a Hamiltonian graph is an unsolved problem.

Example 8.7.2. 1. For each positive integer $n \geq 3$, the cycle C_n is Hamiltonian.



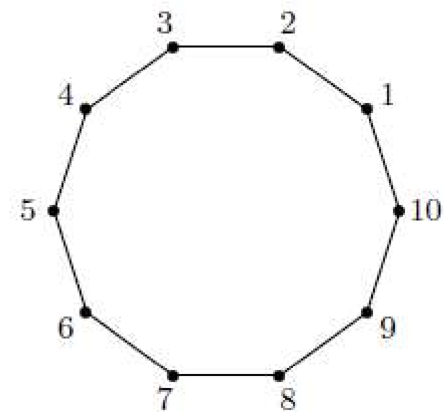
The dodecahedron graph   The Petersen graph

Figure 8.13: A Hamiltonian and a non-Hamiltonian graph

2. The graphs corresponding to all platonic solids are Hamiltonian.
3. The Petersen graph is a non-Hamiltonian Graph (the proof appears below).

Proposition 8.7.3. *The Petersen graph is not Hamiltonian.*

Proof. Suppose that the Petersen graph, say G, is Hamiltonian. Also, each vertex of G has degree 3 and hence, $G = C_{10} + M$, where M is a set of 5 chords in which each vertex appears as an endpoint. We assume that $C_{10} = [1, 2, \ldots, 10, 1]$. Now, consider the vertices 1, 2 and 3.


3
2
4
1
5
10
6
9
7
8

Since, $g(G) = 5$, the vertex 1 can only be adjacent to one of the vertices 5, 6 or 7. Hence, if 1 is adjacent to 5, then the third vertex that is adjacent to 10 creates cycles of length 3 or 4. Similarly, if 1 is adjacent to 7, then there is no choice for the third vertex that can be adjacent to 2. So, let 1 be adjacent to 6. Then, 2 must be adjacent to 8. In this case, note that there is no choice for the third vertex that can be adjacent to 3. Thus, the Petersen graph is non-Hamiltonian. ■

Theorem 8.7.4. *Let G be a Hamiltonian graph. Then, for $S \subseteq V(G)$ with $S \neq \emptyset$, the graph $G - S$ has at most $|S|$ components.*

Proof. Note that by removing k vertices from a cycle, one can create at most k connected components. Hence, the required result follows. ■

Theorem 8.7.5. **[Dirac,** 1952] *Let G be a graph with $|G| = n \geq 3$ and $d(v) \geq n/2$, for each $v \in V(G)$. Then, G is Hamiltonian.*

Proof. Let is possible, G be disconnected. Then, G has a component, say H, with $|V(H)| = k \leq n/2$. Hence, $d(v) \leq k - 1 < n/2$, for each $v \in V(H)$. A contradiction to $d(v) \geq n/2$, for each $v \in V(G)$. Now, let $P = [v_1, v_2, \cdots, v_k]$ be a longest path in G. Since P is the longest path, all neighbors of v_1 and v_k are in P.

We claim that there exists an i such that $v_1 \sim v_i$ and $v_{i-1} \sim v_k$. Otherwise, for each $v_i \sim v_1$, we must have $v_{i-1} \nsim v_k$. Then, $|N(v_k)| \leq k-1-|N(v_1)|$. Hence, $|N(v_1)|+|N(v_k)| \leq k-1 < n$, a contradiction to $d(v) \geq n/2$, for each $v \in V(G)$. So, the claim is valid and hence, we have a cycle $\tilde{P} := v_1 v_i v_{i+1} \cdots v_k v_{i-1} \cdots v_1$ of length k.

We now prove that $\tilde{P}$ gives a Hamiltonian cycle. Suppose not. Then, there exists $v \in V(G)$ such that v is outside P and v is adjacent to some v_j. Note that in this case, P cannot be the path of longest length, a contradiction. Thus, the required result follows. ■

Definition 8.7.8. [**closure of a graph**] The **closure** of a graph G, denoted $C(G)$, is obtained by repeatedly choosing pairs of nonadjacent vertices u, v such that $d(u) + d(v) \geq n$ and adding edges between them.

Proposition 8.7.9. *The closure of G is unique.*

Proof. Let K be a closure obtained by adding edges $e_1 = u_1v_1, \ldots, e_k = u_kv_k$ sequentially and F be a closure obtained by adding edges $f_1 = x_1y_1, \ldots, f_r = x_ry_r$ sequentially. Let e_i be the first edge in the e-sequence which does not appear in the f-sequence. Put $H = G + e_1 + \cdots + e_{i-1}$. Then, $e_i = u_iv_i$ implies that $e_i \notin E(H)$ and $d_H(u_i)+d_H(v_i) \geq n$. Also, H is a subgraph of F and hence, $d_F(u_i) + d_F(v_i) \geq n$. Moreover, $e_i = u_i v_i \notin F$ as e_i does not appear in the f-sequence. Thus, F cannot be a closure and therefore the required result follows. ■

EXERCISE **8.7.10.** *Let G be a graph on $n \geq 3$ vertices.*

1. *If G has a cut vertex, then prove that $C(G) \neq K_n$.*
2. *Then, prove a generalization of Dirac's theorem: If the closure $C(G) \cong K_n$, then G is Hamiltonian.*

Theorem 8.7.11. *Let $d_1 \leq \cdots \leq d_n$ be the vertex degrees of G. Suppose that, for each $k < n/2$ with $d_k \leq k$, the condition $d_{n-k} \geq n-k$ holds. Then, prove that G is Hamiltonian.*

Proof. We show that under the above condition $H = C(G) \cong K_n$. On the contrary, assume that there exist a pair of vertices $u, v \in V(G)$ such that $uv \notin E(G)$ and $d_H(u) + d_H(v) \leq n-1$. Among the above pairs, choose a pair $u, v \in V(G)$ such that $uv \notin E(H)$ and $d_H(u) + d_H(v)$ is maximum. Assume that $d_H(v) \geq d_H(u) = k$ (say). Clearly, $k < n/2$.

Now, let $S_v = \{x \in V(H) \mid xv \notin E(H), x \neq v\}$ and $S_u = \{w \in V(H) \mid wu \notin E(H), w \neq u\}$. Therefore, the assumption that $d_H(u) + d_H(v)$ is the maximum among each pair of vertices u, v with $uv \notin E(H)$ and $d_H(u) + d_H(v) \leq n-1$ implies that $|S_v| = n - 1 - d_H(v) \geq d_H(u) = k$ and for each $x \in S_v$, $d_H(x) \leq d_H(u) = k$. So, there are at least k vertices in H (elements of S_v) with degrees at most k.

Also, for any $w \in S_u$, note that the choice of the pair u, v implies that $d_H(w) \leq d_H(v) \leq n - 1 - d_H(u) = n - 1 - k < n - k$. Moreover, $|S_u| = n - 1 - k$. Further, the condition $d_H(u) + d_H(v) \leq n-1$, $d_H(v) \geq d_H(u) = k$ and $u \notin S_u$ implies that $d_H(u) \leq n - 1 - d_H(v) \leq n - 1 - k < n - k$. So, there are $n - k$ vertices in H with degrees less than $n - k$.

Therefore, if $d'_1 \leq \cdots \leq d'_n$ are the vertex degrees of H, then we observe that there exists a $k < n/2$ for which $d'_k \leq k$ and $d'_{n-k} < n - k$. As $k < n/2$ and $d_i \leq d'_i$, we get a contradiction. ■

Bipartite Graphs

Definition 8.8.1. [**2-colorable graphs**] A graph is 2-colorable if it's vertices can be colored with two colors in a way that adjacent vertices get different colors.

Lemma 8.8.2. *Let P and Q be two v-w-paths in G such that length of P is odd and length of Q is even. Then, G contains an odd cycle.*

Proof. Suppose P, Q have an inner vertex x (a vertex other than v, w) in common. Then, one of $P(v, x), P(x, w)$ has odd length and the other is even, say, $P(v, x)$ is odd. If the lengths of $Q(v, x)$ and $Q(x, w)$ are both odd then we consider the x-w-paths $P(x, w)$ and $Q(x, w)$, otherwise we consider the paths $P(v, x)$ and $Q(v, x)$.

In view of the above argument, we may assume that P, Q have no inner vertex in common. In that case it is clear that $P \cup Q$ is an odd cycle. ■

Theorem 8.8.3. *Let G be a connected graph with at least two vertices. Then, the following statements are equivalent.*

1. *G is 2 colorable.*
2. *G is bipartite.*
3. *G does not have an odd cycle.*

Proof. Part 1 $\Rightarrow$ Part 2. Let G be 2-colorable. Let V_1 be the set of red vertices and V_2 be the set of blue vertices. Clearly, G is bipartite with partition V_1, V_2.

Part 2 $\Rightarrow$ Part 1. Color the vertices in V_1 with red color and that of V_2 with blue color to get the required 2 colorability of G.

Part 2 $\Rightarrow$ Part 3. Let G be bipartite with partition V_1, V_2. Let $v_0 \in V_1$ and suppose $\Gamma = v_0 v_1 v_2 \cdots v_k = v_0$ is a cycle. It follows that $v_1, v_3, v_5 \cdots \in V_2$. Since, $v_k \in V_1$, we see that k is even. Thus, Γ has an even length.

Part 3 $\Rightarrow$ Part 2. Suppose that G does not have an odd cycle. Pick any vertex v. Define $V_1 = \{w \mid \text{there is a path of even length from } v \text{ to } w\}$ and $V_2 = \{w \mid \text{there is a path of odd length from } v \text{ to } w\}$. Clearly, $v \in V_1$. Also, G does not have an odd cycle implies that $V_1 \cap V_2 = \emptyset$. As G is connected each w is either in V_1 or in V_2.

Let $x \in V_1$. Then, there is an even path $P(v, x)$ from v to x. If $xy \in E(G)$, then we have a v-y-walk of odd length. Deleting all cycles from this walk, we have an odd v-y-path. Thus, $y \in V_2$. Similarly, if $x \in V_2$ and $xy \in E$, then $y \in V_1$. Thus, G is bipartite with parts V_1, V_2. ■

Planar Graphs

Definition 8.12.1. **[Embedding, Planar graph]** A graph is said to be **embedded** on a surface S when it is drawn on S so that no two edges intersect. A graph is said to be **planar** if it can be embedded on the plane. A **plane graph** is a graph which is embedded on the plane.

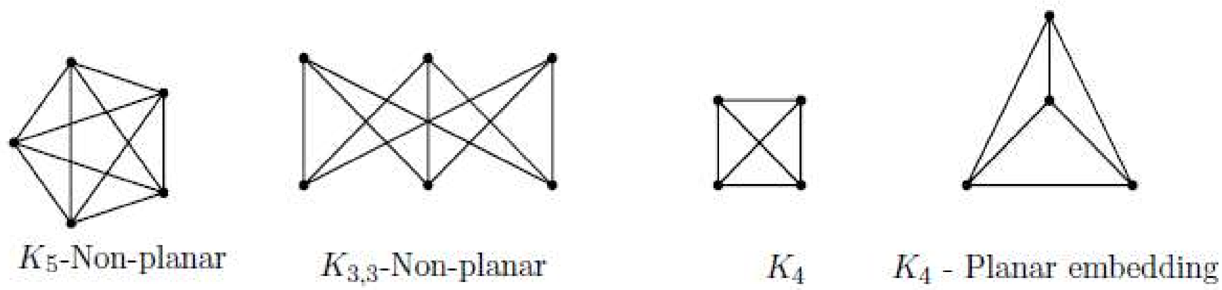
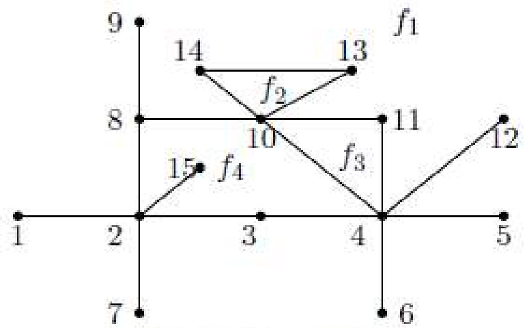


Figure 8.15: Planar and non-planar graphs

Example 8.12.2.

1. A tree is embed-able on a plane and when it is embedded we have only one face, the exterior face.
2. Any cycle C_n, $n \geq 3$ is planar and any plane representation of C_n has two faces.
3. The planar embedding of K_4 is given in Figure 8.15.
4. Draw a planar embedding of $K_{2,3}$.
5. Draw a planar embedding of the three dimensional cube.

Definition 8.12.3. **[Face of a planar embedding]** Consider a planar embedding of a graph G. The regions on the plane defined by this embedding are called **faces/regions** of G. The unbounded face/region is called the **exterior face** (see Figure 8.16).



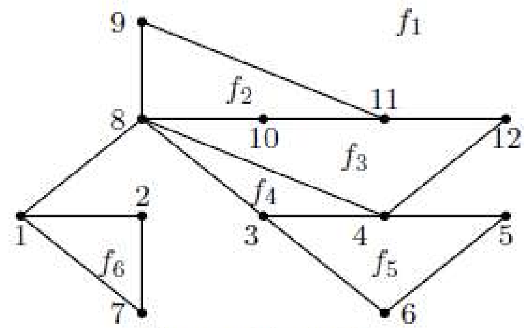


Figure 8.16: Planar graphs with labeled faces to understand the Euler's theorem

Theorem 8.12.4. [Euler formula] *Let G be a connected plane graph with f as the number of faces. Then,*

$$|G| - \|G\| + f = 2. \tag{8.3}$$

Proof. We use induction on f. Let $f = 1$. Then, G cannot have a subgraph isomorphic to a cycle. For if, G has a subgraph isomorphic to a cycle then in any planar embedding of G, $f \geq 2$. Therefore, G is a tree and hence $|G| - \|G\| + f = n - (n-1) + 1 = 2$.

So, assume that Equation (8.3) is true for all plane connected graphs having $2 \leq f < n$. Now, let G be a connected planar graph with $f = n$. Now, choose an edge that is not a cut-edge, say e. Then, $G - e$ is still a connected graph. Also, the edge e is incident with two separate faces and hence it's removal will combine the two faces and thus $G - e$ has only $n - 1$ faces. Thus,

$$|G| - \|G\| + f = |G - e| - (\|G - e\| + 1) + n = |G - e| - \|G - e\| + (n-1) = 2$$

using the induction hypothesis. Hence, the required result follows. ■

Lemma 8.12.5. *Let G be a plane bridgeless graph with $\|G\| \geq 2$. Then, $2\|G\| \geq 3f$. Further, if G has no cycle of length 3 then, $2\|G\| \geq 4f$.*

Proof. For each edge put two dots on either side of the edge. The total number of dots is $2\|G\|$. If G has a cycle then each face has at least three edges. So, the total number of dots is at least $3f$. Further, if G does not have a cycle of length 3, then $2\|G\| \geq 4f$. ∎

Theorem 8.12.6. *The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.*

Proof. If K_5 is planar, then consider a plane representation of it. By Equation (8.3), $f = 7$. But, by Lemma 8.12.5, one has $20 = 2\|G\| \geq 3f = 21$, a contradiction.

If $K_{3,3}$ is planar, then consider a plane representation of it. Note that it does not have a C_3. Also, by Euler's formula, $f = 5$. Hence, by Lemma 8.12.5, one has $18 = 2\|G\| \geq 4f = 20$, a contradiction. ∎

Definition 8.12.7. [**Subdivision, homeomorphic**] Let G be a graph. Then, a **subdivision** of an edge uv in G is obtained by replacing the edge by two edges uw and wv, where w is a new vertex. Two graphs are said to be **homeomorphic** if they can be obtained from the same graph by a sequence of subdivisions.

Definition 8.12.12. [**Maximal planar**] A graph is called **maximal planar** if it is planar and addition of any more edges results in a non-planar graph. A maximal plane graph is necessarily connected.

Proposition 8.12.13. *If G is a maximal planar graph with m edges and $n \geq 3$ vertices, then every face is a triangle and $m = 3n - 6$.*

Proof. Suppose there is a face, say f, described by the cycle $[u_1, \ldots, u_k, u_1]$, $k \geq 4$. Then, we can take a curve joining the vertices u_1 and u_3 lying totally inside the region f, so that $G + u_1u_3$ is planar. This contradicts the fact that G is maximal planar. Thus, each face is a triangle. It follows that $2m = 3f$. As $n - m + f = 2$, we have $2m = 3f = 3(2 - n + m)$ or $m = 3n - 6$. ∎

Vertex Coloring

Definition 8.13.1. [**k-colorable**] A graph G is said to be **k-colorable** if the vertices can be assigned k colors in such a way that adjacent vertices get different colors. The **chromatic number** of G, denoted $\chi(G)$, is the minimum k such that G is k-colorable.

Theorem 8.13.2. *For every graph G, $\chi(G) \leq \Delta(G) + 1$.*

Proof. If $|G| = 1$, the statement is trivial. Assume that the result is true for $|G| = n$ and let G be a graph on $n+1$ vertices. Let $H = G - 1$. As H is $(\Delta(G)+1)$-colorable and $d(1) \leq \Delta(G)$, the vertex 1 can be given a color other than its neighbors. ∎

Theorem 8.13.4. [**5-color Theorem**] *Every Planar graph is 5-colorable.*

Proof. Let G be a minimal planar graph on $n \geq 6$ vertices and m edges, such that G is not 5-colorable. Then, by Proposition 8.12.13, $m \leq 3n - 6$. So, $n\delta(G) \leq 2m \leq 6n - 12$ and hence, $\delta(G) \leq 2m/n \leq 5$. Let v be a vertex of degree 5. Note that by the minimality of G, $G - v$ is 5-colorable. If neighbors of v use at most 4 colors, then v can be colored with the 5-th color to get a 5-coloring of G. Else, take a planar embedding in which the neighbors $v_1, \ldots, v_5$ of v appear in clockwise order.

Let $H = G[V_i \cup V_j]$ be the graph spanned by the vertices colored i or j. If v_i and v_j are in different connected components of H, then we can swap colors i and j in a component that contains v_i, so that the vertices $v_1, \ldots, v_5$ use only 4 colors. Thus, as above, in this case the graph G is 5-colorable. Otherwise, there is a $1,3$-colored path between v_1 and v_3 and similarly, a $2,4$-colored path between v_2 and v_4. But this is not possible as the graph G is planar. Hence, every planar graph is 5-colorable. ∎

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

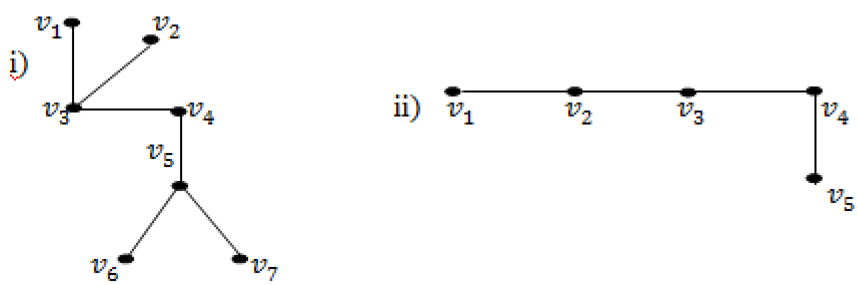
1. Define graph
2. Define simple graph
3. Define directed graph.
4. How many vertices does a regular graph of degree 4 with 10 edges have?
5. Define regular graph.

6. Define euler's graph
7. Define Hamiltonian path
8. Define tree.
9. Define spanning tree.
10. Define planar graph
11. Define isomorphic graph.
12. Define chromatic number
13. Define coloring.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. State and prove handshaking lemma.
2. Define graph. Explain the various types of graph with an example.
3. Prove that the number of vertices of odd degree in a graph is always even.
4. Describe about konigsberg bridge problem.
5. If G is connected simple planar graph with $n(\geq 3)$ vertices and e edges the $e \leq 3n - 6$.
6. Define i) Proper coloring graph ii) Chromatic Number iii) Independent set
7. State and prove polyhedron formula.
8. Find the eccentricity of all vertices, center, radius and diameter of the following graph.



9. Prove that the number of pendent vertices of a tree is equal to $\frac{n+1}{2}$
10. Show that if a fully binary tree has i internal vertices then it has i+1 terminal vertices and (2i+1) total vertices.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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Subject: DISCRETE STRUCTURES **Subject Code: 17CSU202**

Class : I - B.Sc. Computer Science **Semester : II**

Unit II

Graph Theory

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If X and Y be the sets. Then the set (X - Y) union (Y-X) union (X intersection Y) is equal to?	X U Y	X^c U Y^c	X ∩Y	$X^c \cap Y^c$	X U Y
If G is an undirected planar graph on n vertices with e edges then ?	e ≤ n	e ≤ 2n	e ≤ 3n	e > n	e ≤ 2n
The number of circuits that can be created by adding an edge between any two vertices in a tree is ?	Two	Exactly one	More than one	At least two	Exactly one
In a tree between every pair of vertices there is ?	Exactly one path	A self loop	Two circuits	*n* number of paths	Exactly one path
A graph is a collection of.... ?	Row and columns	Vertices and edges	Equations	lines	Vertices and edges
The degree of any vertex of graph is .... ?	The number of edges incident with vertex	Number of vertex in a graph	Number of vertices adjacent to that vertex	Number of edges in a graph	The number of edges incident with vertex
If for some positive integer k, degree of vertex d(v)=k for every vertex v of the graph G, then G is called... ?	K graph	K-regular graph	Empty graph	Trivial graph	K-regular graph
A graph with no edges is known as empty graph. Empty graph is also known as... ?	Trivial graph	Regular graph	Bipartite graph	cycle graph	Trivial graph
Length of the walk of a graph is .... ?	The number of vertices in walk W	The number of edges in walk W	Total number of edges in a graph	Total number of vertices in a graph	The number of edges in walk W
If the origin and terminus of a walk are same, the walk is known as... ?	open	closed	path	neither open nor closed	closed
A graph G is called a ..... if it is a connected acyclic graph ?	Cyclic graph	Regular graph	Tree	path	Tree
The complete graph K, has... different spanning trees?	n^{n-2}	n*n	n^2	n	n^{n-2}
A continuous non - intersecting curve in the plane whose origin and terminus coincide ?	Planar	Jordan	Hamiltanion	unique	Jordan
A path in graph G, which contains every vertex of G once and only once ?	Eular tour	Hamiltanion Path	Eular trail	Hamiltanion Tour	Hamiltanion Path
A tree having a main node, which has no predecessor is.... ?	Spanning Tree	Rooted Tree	Weighted Tree	forest	Rooted Tree
Diameter of a graph is denoted by diam(G) is defined by.... ?	max (e(v) : v belongs to V)	max(d(u,v))	both max (e(v) : v belongs to V) and max(d(u,v))	min (d(u,v))	both max (e(v) : v belongs to V) and max(d(u,v))
A vertex of a graph is called even or odd depending upon ?	number of edges	number of vertices	degree	eccentricity	degree
An edge having the same vertex as both its end vertices is called ____________.	graph	tree	self-loop	node	self-loop
The maximum number of edges in a simple graph with n vertices is _________.	n	(n-2)/2	(n-1)/2	n+1	(n-1)/2
A vertex of degree zero is called an -------------------------	null vertex	isolated vertex	null graph	pendant vertex	null vertex
Vertices with which a walk begins or ends are called its__________.	isolated vertex	null vertex	pendant vertex	terminal vertices	terminal vertices
A graph with no vertices is a ___________.	null graph	trivial	Empty graph	parallel	null graph
A ____________ is connected graph without circuit	graph	directed graph	undirected graph	tree	tree
The sum of the degrees of all vertices of a graph is equal to __________ the number of edges.	twice	thrice	same	any	twice
A node with no children is called_____________.	siblings	node	leaf	tree	leaf
A graph is ___________ if it has no parallel edges or self-loops	simple	directed	adjacent	self-loop	simple
A graph in which some edges are directed and some are undirected is called_______________.	mixed graph	regular graph	complete graph	simple graph	mixed graph
Every graph is its own______________.	mixed graph	sub graph	simple graph	complete graph	sub graph
__________ is also called cycle.	circuit	walk	path	closed walk	circuit
If no vertex appears more than once in an open walk then it is called a__________.	closed walk	circuit	walk	path	path
The number of edges in a path is called the __________of the path.	length	walk	same	circuit	length
A simple graph G with n vertices is said to be a_____________ if the degree of every vertex is n-1.	regular graph	complete graph	simple graph	null graph	complete graph
A walk is also called___________.	chain	edge	vertex	graph	chain
. A ____________ is a closed , non intersecting walk.	closed walk	circuit	walk	path	circuit
The total number of degrees of an isolated node is __________.	0	2	1	3	0
A tree is an ____________graph.	cyclic	directed	acyclic	disconnected	acyclic
A __________ is a graph whose components are all trees.	tree	graph	forest	walk	forest
. A _______________ consists of set of vertices and edges such that each edge is incident with vertices.	graph	path	forest	walk	graph
A vertex having no edge incident on it is called_______________.	end vertex	pendant vertex	isolated vertex	null graph	isolated vertex
A graph is said to be ___________ if there exists at least one path between every pair of vertices in G.	connected	disconnected	null graph	hamiltanion	connected
A tree with n vertices has _____ edges	n	n-1	n-2	n+1	n-1
A graph in which all nodes are of equal degrees is known as…………………	regular graph	complete graph	simple graph	null graph	regular graph

UNIT-V

SYLLABUS

Logical Connectives, Well-formed Formulas, Tautologies, Equivalences, Inference Theory.

UNIT – V

Propositions. Compound Statements. Truth Tables

Statements (Propositions): Sentences that claim certain things, either true or false

Notation: A, B, …P, Q, R, …., p, q, r, etc.

Examples of statements: Today is Monday. This book is expensive
If a number is smaller than 0 then it is positive.

Examples of sentences that **are not statements**: Close the door! What is the time?

Propositional variables: A, B, C, …, P., Q, R, … Stand for statements. May have true or false value.

Propositional constants:

T – true
F - false

Basic logical connectives: NOT, AND, OR
Other logical connectives can be represented by means of the basic connectives

Logical connectives	pronounced	Symbol in Logic
Negation	NOT	¬, ~, ‘
Conjunction	AND	Λ
Disjunction	OR	V
Conditional	if then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$
Exclusive or	Exclusive or	$\oplus$

Truth tables - Define formally the meaning of the logical operators.

The abbreviation **iff** means **if and only if**

a. Negation (NOT, ~, ¬, ‘)

P	~P	
T	F	~P is true if and only if P is false
F	T	

b. Conjunction (AND, Λ, &&)

P	Q	P Λ Q	
T	**T**	**T**	P Λ Q is true iff both P and Q are true. In all other cases P Λ Q is false
T	F	F	
F	**T**	F	
F	F	F	

c. Disjunction / Inclusive OR (OR, V, ||)

P	Q	P V Q	
T	**T**	**T**	P V Q is true iff P is true or Q is true or both are true.
T	F	**T**	P V Q is false iff both P and Q are false
F	**T**	**T**	
F	F	F	

d. Conditional , known also as implication (→)

P	Q	P→ Q	
T	**T**	**T**	The implication P→ Q is false iff P is true however Q is false.
T	F	F	In all other cases the implication is true
F	**T**	**T**	
F	F	**T**	

e. Biconditional (↔)

P	Q	P ↔ Q	
T	**T**	**T**	P↔ Q is true iff P and Q have same values - both are true or both are false.
T	F	F	If P and Q have different values, the biconditional is false.
F	**T**	F	
F	F	**T**	

f. Exclusive OR (⊕)

P	Q	P ⊕ Q	
T	**T**	F	P⊕ Q is true iff P and Q have different values
T	F	**T**	We say: "P or Q but not both"
F	**T**	**T**	
F	F	F	

Precedence of the logical connectives:

Connectives within parentheses, innermost parentheses first

¬	**negation**
Λ	**conjunction**
V	**disjunction**
→	**conditional**
↔, ⊕	**biconditional, exclusive OR**

Compound Statements: Logical expressions that consist of propositional variables and logical connectives. They may contain also propositional constants.

Evaluating compound statements : by building their truth tables

Example: ¬ P V Q

P	Q	¬ P	¬ P V Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(P V Q) Λ ¬ (P Λ Q)

P	Q	P V Q A	P Λ Q B	¬ (P Λ Q) ¬ B	(P V Q) Λ ¬ (P Λ Q) A Λ ¬B
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

(the letters A and B are used as shortcuts)

1. Tautologies and Contradictions

A propositional expression is a **tautology** if and only if for all possible assignments of truth values to its variables its truth value is **T**

Example: P V ¬ P is a tautology

P	¬ P	P V ¬ P
T	F	T
F	T	T

A propositional expression is a **contradiction** if and only if for all possible assignments of truth values to its variables its truth value is **F**

Example: P Λ ¬ P is a contradiction

P	¬ P	P Λ ¬ P
T	F	F
F	T	F

Usage of tautologies and contradictions - in proving the validity of arguments; for rewriting expressions using only the basic connectives.

Definition: Two propositional expressions P and Q are logically equivalent, if and only if P ↔ Q is a tautology. We write P ≡ Q or P ⇔ Q.

Note that the symbols ≡ and ⇔ are **not logical connectives**

Exercise:

a) Show that P → Q ↔ ¬ P V Q is a tautology, i.e. P → Q ≡ ¬ P V Q

P	Q	¬ P	¬ P V Q	P → Q	P → Q ↔ ¬ P V Q
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

2. Logical equivalences

Similarly to standard algebra, there are **laws** to manipulate logical expressions, given as logical equivalences.

1. Commutative laws	$P \lor Q \equiv Q \lor P$	
	$P \land Q \equiv Q \land P$	
2. Associative laws	$(P \lor Q) \lor R \equiv P \lor (Q \lor R)$	
	$(P \land Q) \land R \equiv P \land (Q \land R)$	
3. Distributive laws:	$(P \lor Q) \land (P \lor R) \equiv P \lor (Q \land R)$	
	$(P \land Q) \lor (P \land R) \equiv P \land (Q \lor R)$	
4. Identity	$P \lor F \equiv P$	
	$P \land T \equiv P$	
5. Complement properties	$P \lor \neg P \equiv T$	(excluded middle)
	$P \land \neg P \equiv F$	(contradiction)
6. Double negation	$\neg(\neg P) \equiv P$	
7. Idempotency (consumption)	$P \lor P \equiv P$	
	$P \land P \equiv P$	
8. De Morgan's Laws	$\neg(P \lor Q) \equiv \neg P \land \neg Q$	
	$\neg(P \land Q) \equiv \neg P \lor \neg Q$	
9. Universal bound laws (Domination)	$P \lor T \equiv T$	
	$P \land F \equiv F$	
10. Absorption Laws	$P \lor (P \land Q) \equiv P$	
	$P \land (P \lor Q) \equiv P$	
11. Negation of T and F:	$\neg T \equiv F$	
	$\neg F \equiv T$	

1. Truth table of the conditional statement

P	Q	$P \to Q$
T	T	T
T	**F**	**F**
F	T	T
F	F	T

P is called **antecedent**
Q is called **consequent**
Meaning of the conditional statement: The truth of P implies (leads to) the truth of Q

Note that when P is false the conditional statement is true no matter what the value of Q is. We say that in this case the conditional statement is **true by default or vacuously true.**

2. Representing the implication by means of disjunction

$$P \to Q \equiv \neg P \vee Q$$

P	Q	$\neg$ P	**$P \to Q$**	**$\neg P \vee Q$**
T	T	F	T	T
T	F	F	**F**	**F**
F	T	T	T	T
F	F	T	T	T

Same truth tables

Usage:

1. To rewrite "OR" statements as conditional statements and vice versa (for better understanding)
2. To find the negation of a conditional statement using De Morgan's Laws

3. Rephrasing "or" sentences as "if-then" sentences and vice versa

Consider the sentence:
(1) "The book can be found in the library or in the bookstore".

Let

A = The book can be found in the library
B = The book can be found in the bookstore

Logical form of (1): **A V B**

Rewrite A V B as a conditional statement

In order to do this we need to use the commutative laws, the equivalence $\neg(\neg P) \equiv P$, and the equivalence $P \rightarrow Q \equiv \neg P \vee Q$

Thus we have:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$

The last expression $\neg A \rightarrow B$ is translated into English as
**"If the book cannot be found in the library,
it can be found in the bookstore"**.

Here the statement "The book cannot be found in the library" is represented by $\neg A$

There is still one more conditional statement to consider.
$A \vee B \equiv B \vee A$ (commutative laws)

Then, following the same pattern we have:

$B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

The English sentence is: **"If the book cannot be found in the bookstore, it can be found in the library.**

We have shown that:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$
$A \vee B \equiv B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

Thus the sentence **"The book can be found in the library or in the bookstore"** can be rephrased as:
"If the book cannot be found in the library, it can be found in the bookstore".
"If the book cannot be found in the bookstore, it can be found in the library.

4. **Negation of conditional statements**

Positive: The sun shines
Negative: The sun does not shine

Positive: " If the temperature is 250°F then the compound is boiling "
Negative: ?
In order to find the negation, we use De Morgan's Laws.

Let
P = the temperature is 250°F
Q = the compound is boiling

Positive: $P \rightarrow Q \equiv \neg P \vee Q$
Negative: $\neg (P \rightarrow Q) \equiv \neg(\neg P \vee Q) \equiv \neg(\neg P) \wedge \neg Q \equiv P \wedge \neg Q$

Negative: The temperature is 250°F however the compound is not boiling

IMPORTANT TO KNOW:

The negation of a disjunction is a conjunction.
The negation of a conjunction is a disjunction

The **negation of a conditional statement is a conjunction**, not another if-then statement

Question: Which logical connective when negated will result in a conditional statement?

5. Necessary and sufficient conditions

Definition:

"P is a **sufficient condition** for Q" means : **if P then Q, $P \rightarrow Q$**
"P is a **necessary condition** for Q" means: **if not P then not Q, $\sim P \rightarrow \sim Q$**
The statement **$\sim P \rightarrow \sim Q$ is equivalent to $Q \rightarrow P$**

Hence given the statement **$P \rightarrow Q$,**
P is a sufficient condition for Q, and Q is a necessary condition for P.

Examples:

If n is divisible by 6 then n is divisible by 2.
The sufficient condition to be divisible by 2 is to be divisible by 6.
The necessary condition to be divisible by 6 is to be divisible by 2

If n is odd then n is an integer.
The sufficient condition to be an integer to be odd.
The necessary condition to be odd is to be an integer.

If and only if - the biconditional

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q$ is true whenever P and Q have same values. Otherwise it is false.

This means that **both P → Q and Q → P have to be true**

P	Q	**P → Q**	**Q → P**	P↔Q
T	**T**	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	F
F	**F**	**T**	**T**	**T**

Contrapositive

Definition: The expression ~Q → ~P is called **contrapositive** of P → Q

The conditional statement P → Q and its contrapositive ~Q → ~P **are equivalent**.
The proof is done by comparing the truth tables

The truth table for P → Q and ¬ Q → ¬ P is:

P	Q	¬ P	¬ Q	P→ Q	¬ Q → ¬ P
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We can also prove the equivalence by using the disjunctive representation:

$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to \neg P$

Converse and inverse

Definition: The converse of P → Q is the expression Q → P

Definition: The inverse of P → Q is the expression ~P → ~Q

Neither the converse nor the inverse are equivalent to the original implication. Compare the truth tables and you will see the difference.

P	Q	¬ P	¬ Q	P→ Q	Q→ P	**¬ P → ¬ Q**
T	T	F	F	T	T	T
T	F	F	T	**F**	**T**	**T**
F	T	T	F	**T**	**F**	**F**
F	F	T	T	T	T	T

Valid and Invalid Arguments.

Definition: An argument is a sequence of statements, ending in a conclusion. All the statements but the final one (the conclusion) are called premises(or assumptions, hypotheses)

Verbal form of an argument:

(1) If Socrates is a human being then Socrates is mortal.
(2) Socrates is a human being

Therefore (3) Socrates is mortal

Another way to write the above argument:

$$P \rightarrow Q$$

$$P$$

$$\therefore \quad Q$$

2. Testing an argument for its validity

Three ways to test an argument for validity:

A. Critical rows

1. Identify the assumptions and the conclusion and assign variables to them.
2. Construct a truth table showing all possible truth values of the assumptions and the conclusion.
3. Find the **critical rows - rows in which all assumptions are true**
4. For each critical row determine whether the conclusion is also true.
   a. If the conclusion is **true in all critical rows**, then **the argument is valid**
   b. If there is at least one row **where the assumptions are true, but the conclusion is false**, then the argument is **invalid**

B. Using tautologies

The argument is true if the conclusion is true whenever the assumptions are true.
This means: If all assumptions are true, then the conclusion is true.
"All assumptions" means the conjunction of all the assumptions.

Thus, let A1, A2, … An be the assumptions, and B - the conclusion.

For the argument to be valid, the statement
If (A1 Λ A2 Λ… Λ An) then B must be **a tautology -** true for all assignments of values to its variables, i.e. its column in the truth table must contain only **T**

i.e.

$(A1 \wedge A2 \wedge \ldots \wedge An) \to B \equiv \mathbf{T}$

C. Using contradictions

If the argument is valid, then we have $(A1 \wedge A2 \wedge \ldots \wedge An) \to B \equiv \mathbf{T}$
This means that the negation of $(A1 \wedge A2 \wedge \ldots \wedge An) \to B$ should be a contradiction - containing only **F** in its truth table

In order to find the negation we have first to represent the conditional statement as a disjunction and then to apply the laws of De Morgan

$(A1 \wedge A2 \wedge \ldots \wedge An) \to B \equiv \sim(A1 \wedge A2 \wedge \ldots \wedge An) \vee B \equiv$

$\sim A1 \vee \sim A2 \vee \ldots \vee \sim An \vee B.$

The negation is:

$\sim((A1 \wedge A2 \wedge \ldots \wedge An) \to B) \equiv \sim(\sim A1 \vee \sim A2 \vee \ldots \vee \sim An \vee B)$

$\equiv A1 \wedge A2 \wedge \ldots \wedge An \wedge \sim B$

The argument is valid if $A1 \wedge A2 \wedge \ldots \wedge An \wedge \sim B \equiv F$

There are two ways to show that a logical form is a tautology or a contradiction:

a. by constructing the truth table
b. by logical transformations applying the logical equivalences (logical identities)

Examples:

1. Consider the argument:

$$P \to Q$$
$$P$$
$$\therefore \quad Q$$

Testing its validity:

a. by examining the truth table:

P	Q	P→ Q
T	**T**	**T**
T	F	F
F	T	T
F	F	T

b. By showing that the statement 'If all premises then the conclusion" is a tautology: The premises are P and P→ Q. The statement to be considered is:

$(P \wedge (P \to Q)) \to Q$

We shall show that it is a tautology by using the following identity laws:

(1) $P \to Q \equiv \sim P \vee Q$

(2) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ commutative laws

$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

(3) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$ distributive law

(4) $P \wedge \sim P \equiv F$

(5) $P \vee \sim P \equiv T$

(6) $P \vee F \equiv P$

(7) $P \vee T \equiv T$

(8) $P \wedge T \equiv P$

(9) $P \wedge F \equiv F$

(10) $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ De Morgan's Laws

		$(P \wedge (P \to Q)) \to Q$
by (1)	$\equiv$	$\sim(P \wedge (P \to Q)) \vee Q$
by (10)	$\equiv$	$(\sim P \vee \sim(P \to Q)) \vee Q$
by (1)	$\equiv$	$(\sim P \vee \sim(\sim P \vee Q)) \vee Q$
by (10)	$\equiv$	$(\sim P \vee (P \wedge \sim Q)) \vee Q$
by (3)	$\equiv$	$((\sim P \vee P) \wedge (\sim P \vee \sim Q)) \vee Q$
by (5)	$\equiv$	$(\mathbf{T} \wedge (\sim P \vee \sim Q)) \vee Q$
by (8)	$\equiv$	$(\sim P \vee \sim Q) \vee Q$
by (2)	$\equiv$	$\sim P \vee (\sim Q \vee Q)$
by (5)	$\equiv$	$\sim P \vee \mathbf{T}$
by(7)	$\equiv$	$\mathbf{T}$

2. Consider the argument

$$P \rightarrow Q$$
$$Q$$
$$\therefore \quad P$$

We shall show that this argument is invalid by examining the truth tables of the assumptions and the conclusion. The critical rows are in boldface.

P	Q	P→ Q	
T	**T**	**T**	
T	F	F	
F	**T**	**T**	**here the assumptions are true, however the conclusion is false**
F	F	T	

Exercise:

Show the validity of the argument:

1. P V Q (premise)
2. ~Q (premise)

Therefore P (conclusion)

a. by using critical rows
b. by contradiction using logical identities

Solution:

a. by critical rows

conclusion		**Premises**		
P	**Q**	**P V Q**	**~Q**	
T	T	T	F	
T	F	**T**	**T**	**Critical row**
F	T	T	F	
F	F	F	T	

b. By contradiction using identities

$((P \vee Q) \wedge \sim Q) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee (Q \wedge \sim Q)) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee \mathbf{F}) \wedge \sim P \equiv$

$(P \wedge \sim Q) \wedge \sim P \equiv$

$P \wedge \sim P \wedge \sim Q \equiv \mathbf{F} \wedge \sim Q \equiv \mathbf{F}$

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define conjunction.
2. Define disjunction.
3. Construct the truth table for $\neg(P \wedge Q)$.
4. Construct the truth table for $\neg(P) \vee \neg(Q)$.
5. Define tautology
6. Define contradiction.
7. Prove that without using truth table $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is a tautology.
8. Define disjunctive normal form.
9. Define conjunctive normal form.
10. Define PCNF and PDNF.
11. Find PDNF for $\neg P \vee Q$.
12. Prove that $P \rightarrow (Q \vee R) \leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$.
13. Demonstrate that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$ and P.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. i) Construct the truth table for $(P \leftrightarrow R) \wedge (\neg Q \rightarrow S)$
   ii) Obtain PDNF of $(\neg((P \vee Q) \wedge R)) \wedge (P \vee R))$
2. Obtain PCNF and PDNF of $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$
3. i) Prove that $(\neg Q \wedge P) \wedge Q$ is contradiction.
   ii) Show that the following implication without constructing truth table.

   $$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$$
4. i) Verify that a proposition $P \vee \neg(P \wedge Q)$ is a tautology. ii) Prove that $P \rightarrow (Q \vee R) \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$
5. Define disjunctive normal form and conjunctive normal form. Also obtain disjunctive normal form of $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$
6. i) Prove that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

ii) Show that (x) M(x) follows logically from the premises $(x)(H(x) \rightarrow M(x))$ and (x)H(x).

7. Find the minterm normal form of $(\urcorner((P \vee Q) \wedge R)) \wedge (P \vee R)$
8. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \urcorner H$, $\urcorner H \rightarrow (A \wedge \urcorner B)$ and $(A \wedge \urcorner B) \rightarrow (R \vee S)$.
9. Prove that $(P \vee Q) \wedge \urcorner(\urcorner P \wedge (\urcorner Q \vee \urcorner R) \vee (\urcorner P \wedge \urcorner Q) \vee (\urcorner P \wedge \urcorner R))$ is a tautology.
10. Show that the following premises are Inconsistent.
    i) If Jack misses many classes through illness, he fails in school.

    ii) If jack fails in school, then he is uneducated.

    iii) If jack reads a lot of books, then he is not uneducated.

    iv) Jack misses many classes through illness and reads a lot of books.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
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Subject: DISCRETE STRUCTURES **Subject Code: 17CSU202**
Class : I - B.Sc. Computer Science **Semester : II**

Unit V
Prepositional Logic

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Let p be “He is tall” and let q “He is handsome”. Then the statement “It is false that he is short or handsome” is:	p^q	~(~pv q)	~pv q	p v q	~(~pv q)
The proposition p^ (~ p v q) is……………………….	A tautulogy	a contradiction	Logically equivalent to p ^q	an assumption	Logically equivalent to p ^ q
Which of the following is/are tautology?	a v b → b ^ c	a ^ b → b v c	a v b → (b → c)	a v b → b v c	a ^ b → b v c
Identify the valid conclusion from the premises Pv Q, Q → R, P → M, ˥M	P ^ (R v R)	P ^ (P ^ R)	R ^ (P v Q)	Q ^ (P v R)	Q ^ (P v R)
Let a, b, c, d be propositions. Assume that the equivalence a ↔ (b v ˥b) and b ↔ c hold. Then truth value of the formula (a ^ b) → ((a ^ c) v d) is always	TRUE	FALSE	Same as the truth value of a	Same as the truth value of b	TRUE
Which of the following is a declarative statement?	It's right	He says	Two may not be an even integer	I love you	He says
P → (Q → R) is equivalent to	(P ^ Q) → R	(P v Q) → R	(P v Q) → ˥R	(P v Q) → P	(P ^ Q) → R
If F1, F2 and F3 are propositional formulae such that F1 ^ F2 → F3 and F1 ^ F2→F3 are both tautologies, then which of the following is TRUE?	Both F1 and F2 are tautologies	The conjuction F1 ^ F2 is not satisfiable	Neither is tautologies	F1v F2 is tautology	Both F1 and F2 are tautologies
Consider two well-formed formulas in propositional logic F1 : P →˥P F2 : (P →˥P) v (˥P →), then	F1 is satisfiable, F2 is unsatisfiable	F1 is unsatisfiable, F2 is satisfiable	F1 is unsatisfiable, F2 is valid	F1 & F2 are both satisfiable	F1 is unsatisfiable, F2 is valid
What can we correctly say about proposition P1 : (p v ˥q) ^ (q →r) v (r v p)	P1 is tautology	P1 is satisfiable	If p is true and q is false and r is false, the P1 is true	If p as true and q is true and r is false, then P1 is true	If p is true and q is false and r is false, the P1 is true
(P v Q) ^ (P → R)^ (Q →S) is equivalent to	S ^ R	S → R	S v R	S U R	S v R
In propositional logic , which of the following is equivalent to p → q?	~p → q	~p v q	~p v~ q	p →q	~p v q
˥(P → Q) is equivalent to	P ^ ˥Q	P ^ Q	˥P v Q	˥P ^ Q	P ^ ˥Q
(P v Q) ^ (P→R) ^ (Q → R) is equivalent to	P	Q	R	True=T	R
How many rows would be in the truth table for the following compound proposition: (p ∨ q) ∧ ¬(q ∧ t) ∨ (r → s)	32	34	27	25	32
Which of the following statement is the negation of the statement,“2 is even and –3 is negative”?	2 is even and –3 is not negative.	2 is odd and –3 is not negative.	2 is even or –3 is not negative.	2 is odd or –3 is not negative.	2 is odd or –3 is not negative.
p→q is logically equivalent to	~ q→p	~ p→q	~ p ^ q	~ p v q	~ p v q
Which of the following is not a well formed formula?	"for all x [P(x)→f (x)^ x]	for all x1,x2,x3 { x1 = x2 ^ x2 = x3 Þ x1 = x3}	~ (p →q)→q	[T v P(a, b)]→zQ(z)	for all x1,x2,x3 { x1 = x2 ^ x2 = x3 Þ x1 = x3}
[~ q ^ (p→q)]→~ p is,	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
An and statement is true if, and only if, both components are ……………..	TRUE	FALSE	not true	neither true nor false	TRUE
If P : It is hot & Q : It is humid,then what does P ^ (~ Q):mean?	It is not hot and it is not humid	It is hot and it is humid	It is hot and it is not humid	It is not hot and it is not humid	It is hot and it is not humid
An or statement is false if, and only if, both components are ………………..	TRUE	FALSE	not true	neither true nor false	FALSE
Two statement forms are logically equivalent if, and only if they always have………………………….	not same truth values	the same truth values	the different truth values	the same false values	the same truth values
A tautology is a statement that is always ………………….	TRUE	FALSE	not true	neither true nor false	TRUE
A contradiction is a statement that is always ………………….	FALSE	TRUE	not true	neither true nor false	FALSE
The statement (p^q) Þ p is a…………………	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
In propositional logic which one of the following is equivalent to p→q ?	p→q	p→q	p v q	p v~q	~p v q
Which of the following proposition is a tautology?	(p v q)→p	p v (q→p)	p v(p→q)	(p v q)→q	p v(p→q)
Which one is the contrapositive of q → p ?	~p → ~ q	p → ~ q	~p → q	p → q	~p → ~ q
The statement form pv(~p) is a……………………..	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
Let p and q be statements given by “p →q". Then q is called	hypothesis	conclusion	TRUE	FALSE	conclusion
The statement form p^(~p) is a……………………..	contradiction	Unsatisfiable	Tautology	Invalid	contradiction
If p and q are statement variables, the conditional of q by p is given by ……………..	~p → ~ q	p → ~ q	~p → q	p → q	p → q
Let p and q be statements given by “p →q". Then p is called………………	hypothesis	conclusion	TRUE	FALSE	hypothesis
The statement (p → r) ∧ (q → r) is equivalent to…………	p ∨ q →~ r	p ∨ q → r	p ∨~ q → r	~p ∨ q → r	p ∨ q → r
The Negation of a Conditional Statement p →q is given by ………………..	p ∧ ~q	~p ∧ ~q	p V ~q	p ∧ q	p ∧ ~q
Given statement variables p and q, the biconditional of p and q is given by …………	p«~q	p→q	~p«q	p«q	p«q
The inverse of “if p then q” is …………………………	if ~p then ~q	if ~p then ~q	if ~p then ~q	if ~p then ~q	if ~p then ~q
“R is a………………………. condition for S” means “if R then S .”	valid	inevitable	sufficient	necessary	sufficient
A conditional statement and its contrapositive are …………………	A tautulogy	a contradiction	Logically equivalent	an assumption	Logically equivalent
A rule of inference is a form of argument that is ……………	valid	a contradiction	an assumption	A tautulogy	valid

Reg no-------------

(17ITU202/17CSU202/17CTU202/17CAU202)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF CS/CT/CA/IT

Second Semester

I Internal Test - Jan'2018

Discrete Structures

Date: 20-01-2018 **Time: 2 Hours**

Class: I-Bsc IT,CT,CS,BCA **Maximum Marks:50**

PART-A(20X1=20 Marks)

Answer all the Questions:

1. Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the range of R = ______
   (a) {1,3} (b){1,5} (c){2,4} (d){1,4}
2. A One – to –one and onto function is also known as ________
   (a) injective (b) surjective (c) bijective d) objective
3. Let f : R→R where R is a set of real numbers. Then f(x) = -2x is a _____
   (a) One-to-one (b) Onto (c) into (d) bijection
4. A binary relation R in a set X is said to be reflexive if _____
   (a) aRa (b) aRb→bRa
   (c) aRb,bRc→aRc (d) aRb,bRa→a=b
5. A mapping f : x→y is called ____________if distinct elements of x are mapped into distinct elements
   (a) one-one (b) onto (c) into (d) many-one
6. The r - permutation of n elements is denoted by ______
   (a) P (r, n) (b) P(n,r) (c) c(r, n) (d) c(n, r)
7. If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then RoS = ______
   (a) {(4,2),(3,2),(1,4)} (b) {(1,5),(3,2),(2,5)}
   (c) {(1,2),(2,2)} (d) {(4,5),(3,3),(1,1)}
8. If the relation R and S are both reflexive then R Ú S is _______
   (a) transitive (b) not reflexive
   (c) reflexive (d) symmetric
9. Let f : x→y , g : y→ x be the functions then g is equal to f^{-1} if and only if ______
   (a) fog = I_y (b)gof = I_y (c) fog = I_x (d) gof = I_x
10. The number of ways can a party of 7 persons arrange themselves around a circular table______
    (a) 6! (b) 7! (c) 0 (d) 3!
11. The value of C(n,n) is ________
    (a) 0 (b) -1 (c) 2 (d) 1
12. Let f : N→N be a function such that f(x) = 5 ,for every x in N then the f(x) is called________function.
    (a) constant (b) identity (c) unit (d) zero
13. In N, define aRb if a+b = 7. This is symmetric when ______
    (a) b+a=7 (b) a=b (c) ab=7 (d) a+a=7
14. The number of different permutations of the word BANANA is _________
    (a) 720 (b) 60 (c)120 (d) 360
15. The value of C(10, 8) + C(10,7) is _____________
    (a) 990 (b) 165 (c) 45 (d) 120
16. The sum of entries in the fourth row of Pascal's triangle is ________
    (a) 10 (b) 4 (c)10 (d) 16
17. The growth of ___________is directly related to the complexity of algorithms.
    (a) Functions (b) relations (c) parameters (d) polynomials
18. How many 10 digits numbers can be written by using the digits 1 and 2 ?
    (a) C (10, 9) + C (9, 2) (b) 1024 (c)C(10, 2) (d) 10!
19. A binary relation R in a set X is said to be antisymmetric if ____________
    (a) aRa (b) aRb→bRa
    (c) aRb,bRc→aRc (d) aRb,bRa→a=b
20. If log n = log $_2$ n then it is ______________
    (a) Binary logarithm (b) composition
    (c)exponentiation (d) relation

PART-B (3X2=6 Marks)

Answer all the Questions:

21. Prove that Commutative property under intersection.
22. Define symmetric and Non symmetric withexample.
23. Define Big oh.

PART-C (3X8=24 Marks)

Answer all the Questions:

24. (a) Prove that the associative property under union.

    (OR)

    (b) Prove that $1^2+2^2+3^2+\dots+n^2= n(n+1)(2n+1)/6$ by Principle of Mathematical induction.

25. (a) In Z, we define aRb iff a-b is a multiple of m. Is R is an equivalence relation?

    (OR)

    (b) Explain about properties of relation.

26. (a) Let A={1,2,3} and f,g,h and s be functions from A to A given by

    f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) } ;

    h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find

    $f \circ g$, $g \circ f$, $f \circ h \circ g$, $g \circ s$, $s \circ s$, $f \circ s$.

    (OR)

    (b) Show that if we have two functions $f_1(x)$ and $f_2(x)$ both $O(g(x))$, then $f_1(x)+f_2(x)$ is also $O(g(x))$.