KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT Of MATHEMATICS

SUBJECT :REAL ANALYSIS **SUBJECT CODE :17MMP102**

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4	0	0	4

PO: After the completion of this course, the learner will be enriched with the concept of analysis which is the motivating tool in other area such as applied mathematics.

PLO: To understand the Riemann – Stieltjes Integral, Infinite series, infinite products, Sequences of functions, the Lebesgue integral, Implicit functions and extremum problems to have a sound knowledge in Measure Theory.

UNIT I

The Riemann – Stieltjes Integral:

Introduction – Basic Definitions – Linear Properties – Integration by parts – Change of variable in a Riemann – Stieltjes Integral – Reduction to a Riemann Integral – Step functions as integrators – Reduction of a Riemann – Stieltjes Integral to a finite sum – Monotonically increasing – Additive and linear properties – Riemann condition – Comparison theorems – Integrators of bounded variation – Sufficient condition for Riemann Stieltjes integral.

UNIT II

Infinite series and infinite products:

Introduction – Basic definitions – Ratio test and root test – Dirichlet test and Able's test Rearrangement of series – Riemann's theorem on conditionally convergent series – Sub series - Double sequences – Double series – Multiplication of series – Cesaro summability.

UNIT III

Sequences of functions:

Basic definitions – Uniform convergence and continuity - Uniform convergence of infinite series of functions – Uniform convergence and Riemann – Stieltjes integration – Non uniformly convergent sequence – Uniform convergence and differentiation – Sufficient condition for uniform convergence of a series.

UNIT IV

The Lebesgue integral:

Introduction- The class of Lebesgue – integrable functions on a general interval- Basic properties of the Lebesgue integral- Lebesgue integration and sets of measure zero- The Levi monotone convergence theorem- The Lebesgue dominated convergence theorem-

Applications of Lebesgue dominated convergence theorem- Lebesgue integrals on unbounded intervals as limit of integrals on bounded intervals- Improper Riemann integrals- Measurable functions.

UNIT V

Implicit functions and extremum problems:

Introduction – Functions with non zero Jacobian determinant – Inverse function theorem – Implicit function theorem – Extrema of real valued functions of one variable and several variables.

SUGGESTED READINGS

TEXT BOOK

1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

REFERENCES

1. Tom .M. Apostol., (2002). Mathematical Analysis, Second edition, Narosa Publishing House, New Delhi.

2. Balli. N.P., (1981). Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

3. Gupta.S.L. and Gupta.N.R.,(2003).Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd, Singapore.

4. Royden .H.L., (2002). Real Analysis, Third edition, Prentice hall of India,New Delhi.

5. Sterling. K. Berberian., (2015).A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

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DEPARTMENT OF MATHEMATICS
Lecture Plan

Subject: Real Analysis **Class: I M.Sc Mathematics**
Sub. Code: 17MMP102 **Name of the faculty: S.KOHILA**

S. No.	Lecture Duration (Hours)	Topics to be covered	Support Materials
UNIT – I			
1.	1	Introduction on Riemann-Stieltjes integral	T1: Chap: 6, Pg. No : 120-124
2.	1	Some basic definitions on Riemann-Stieltjes integral	T1: Chap: 6, Pg. No : 124-128
3.	1	Linear properties of Riemann-Stieltjes integral	T1: Chap: 6, Pg. No : 128-129
4.	1	Continuation on linear properties of Riemann-Stieltjes integral	T1: Chap: 6, Pg. No : 129-131
5.	1	Integration by parts	T1: Chap: 6, Pg. No : 134
6.	1	Change of variable in Riemann-Stieltjes integral	T1: Chap: 6, Pg. No : 132-133
7.	1	Reduction to Riemann integral	R1: Chap: 7, Pg. No : 145-146
8.	1	Step function as integrators	R1: Chap: 7, Pg. No : 147-148
9.	1	Reduction of a Riemann-Stieltjes integral to a finite sum	R1: Chap: 7, Pg. No : 148-149
10.	1	Monotonically increasing on Riemann integral	R1: Chap: 7, Pg. No : 150-152
11.	1	Additive and linear properties on Riemann	R1: Chap: 7, Pg. No : 153

		integral	
12.	1	Riemann condition	R1: Chap: 7, Pg. No : 154
13.	1	Comparison theorems & Integrators of bounded variation	R1: Chap: 7, Pg. No : 155-156
14.	1	Sufficient condition for Riemann stieltjes integral	R1: Chap: 7, Pg. No : 156-158
15.	1	Recapitulation and discussion of important questions on unit I	R1: Chap: 7, Pg. No : 159
Total	**15 Hours**		

T1. Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.

UNIT – II

1.	1	Introduction on infinite series & infinite products	R3: Chap: 5, Pg. No : 5.1-5.3
2.	1	Some basic definitions on infinite series and infinite products	R1: Chap: 8, Pg. No : 183-191
3.	1	Ratio test and root test	R1: Chap: 8, Pg. No : 193-194
4.	1	Drichlet test and	R3: Chap: 6, Pg. No : 6.1-6.4
5.	1	Able's test	R1: Chap: 8, Pg. No : 196
6.	1	Rearrangement of Series	R1: Chap: 8, Pg. No : 197
7.	1	Riemann's Theorem on conditionally convergent series	R1: Chap: 8, Pg. No : 197-199
8.	1	Sub series on conditionally convergent series Double sequences on	R1: Chap: 8, Pg. No : 199-200

9.	1	conditionally convergent series	R1: Chap: 8, Pg. No :200-202
10.	1	Double series on conditionally convergent series	R1: Chap: 8, Pg. No :203-205
11.	1	Problems on double series on conditionally convergent series	R1: Chap: 8, Pg. No :205-206
12.	1	Multiplication of series	R1: Chap: 8, Pg. No :206-208
13	1	conditionally convergent series	R1: Chap: 8, Pg. No :209-210
14	1	Cesaro Summability	R1: Chap: 8, Pg. No :11-212
15	1	Recapitulation and discussion of important questions on unit II	
Total	**15 Hours**		

R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.

R3. Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

UNIT –III

1.	1	Basic definitions on sequences	R5: Chap:3, Pg. No :33-36
2.	1	Basic definitions on of functions	R5: Chap:3, Pg. No :39-41
3.	1	Uniform convergence on sequences of functions	T1: Chap:7, Pg. No :143-146
4.	1	Continuity of uniform convergence on sequences of functions	T1: Chap:7, Pg. No :147-148
5.	1	Uniform convergence of infinite series	R2: Chap:9, Pg. No :533-534

6.	1	Problems on infinite series of functions	T1: Chap:7, Pg. No :152-153
7.	1	Riemann-Stieltjes integration	T1: Chap:7, Pg. No :152-154
8.	1	Non-uniformly convergent sequence	R1: Chap:9, Pg. No :228-229
9.	1	Non-uniformly convergent sequence.	R1: Chap:9, Pg. No :230-231
10.	1	Problems on non-uniformly convergent sequence	
11	1	Problems on non-uniformly convergent sequence	R1: Chap:9, Pg. No :231
12	1	Problems on non-uniformly convergent sequence .	R1: Chap:9, Pg. No :232
13	1	Necessary condition for uniform convergence of a series	R1: Chap:9, Pg. No :235
14	1	Sufficient condition for uniform convergence of a series	R1: Chap:9, Pg. No :235
15	1	Recapitulation and discussion of important questions on unit III	
Total	**15 Hours**		

T1. Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.
R5. Sterling. K. Berberian ., 2004.A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

UNIT-IV

1.	1	Introduction on Lebesgue integral	R4:Chap:4,Pg.No:75-77
2.	1	The class of Lebesgue – integrable functions	R1:Chap:10,Pg.No:254-

		on a general interval	256
3.	1	Basic properties of a Lebesgue integral	R4:Chap:4,Pg.No:85-88
4.	1	Lebesgue integration and sets of measure zero	R1:Chap:10,Pg.No:264-265
5.	1	The Levi monotone convergence theorem	R1:Chap:10,Pg.No:266-268
6.	1	The Lebesgue dominated convergence theorem	R1:Chap:10,Pg.No:268-270
7.	1	Applications of Lebesgue dominated convergence theorem	R1:Chap:10,Pg.No:270-272
8.	1	Continuation of applications of Lebesgue dominated convergence theorem	R1:Chap:10,Pg.No:272-274
9.	1	Lebesgue integrals on unbounded interval as limit of integrals on bounded intervals	R1:Chap:10,Pg.No:274-275
10.	1	Improper Riemann integrals	R1:Chap:10,Pg.No:276-279
11.	1	Measurable functions	R1:Chap:10,Pg.No:279-280
12.	1	Problems on Improper Riemann integrals	R1:Chap:10,Pg.No:278
13	1	Problems on Improper Riemann integrals	R1:Chap:10,Pg.No:282
14	1	Problems on Measurable functions	R1:Chap:10,Pg.No:279-280
15		Recapitulation and discussion of important questions on unit IV	
Total	**15 Hours**		
R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.			
UNIT – V			

1.	1	Introduction on implicit functions	R1:Chap:13,Pg.No:367-368
2.	1	Functions with non-zero Jacobian determinant	R1:Chap:13,Pg.No:368-371
3.	1	Inverse function theorem	R1:Chap:13,Pg.No:372-373
4.	1	Implicit function theorem	R1:Chap:13,Pg.No:373-375
5.	1	Extrema of real valued functions of one variable	R1:Chap:13,Pg.No:375-376
6.	1	Extrema of real valued functions of several variables-Theorems	R1:Chap:13,Pg.No:376-379
7.	1	Extrema of real valued functions of several variables -Theorems	R1:Chap:13.Pg No : 380-381
8.	1	Problems on Extrema of real valued functions of several variables	R1:Chap:13.Pg No : 380
9.	1	Problems on Inverse function theorem	R1:Chap:13.Pg No : 380
10.	1	Problems on Inverse function theorem	R1:Chap:13.Pg No : 376
11	1	Problems on Inverse function theorem	R1:Chap:13.Pg No : 378
12	1	Recapitulation and discussion of important questions on unit V	
13	1	Discussion of previous ESE question papers.	
14	1	Discussion of previous ESE question papers.	
15	1	Discussion of previous ESE question papers.	
Total	**15 Hours**		
R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.			

Text book and References:

T1. Rudin. W.,1976 .Principles of mathematical Analysis, Mcgraw hill, Newyork .

R1. Tom .M. Apostol .,2002. Mathematical Analysis, Second edition, Narosa Publishing House,New Delhi.

R2. Balli. N.P., 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

R3. Gupta . S.L ., and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.

R4. Royden .H.L ., 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.

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DEPARTMENT Of MATHEMATICS

Subject : REAL ANALYSIS	**SEMESTER: I**	**L T P C**
SUBJECT CODE: 17MMP102	**CLASS : I M.Sc**	**4 0 0 4**

UNIT I

The Riemann – Stieltjes Integral:

Introduction – Basic Definitions – Linear Properties – Integration by parts – Change of variable in a Riemann – Stieltjes Integral – Reduction to a Riemann Integral – Step functions as integrators – Reduction of a Riemann – Stieltjes Integral to a finite sum – Monotonically increasing – Additive and linear properties – Riemann condition – Comparison theorems – Integrators of bounded variation – Sufficient condition for Riemann Stieltjes integral.

TEXT BOOK

1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

REFERENCES

1. Tom .M. Apostol., (2002). Mathematical Analysis, Second edition, Narosa Publishing House, New Delhi.
2. Balli. N.P., (1981). Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.
3. Royden .H.L., (2002). Real Analysis, Third edition, Prentice hall of India,New Delhi.

1.1.The Riemann-Stieltjes Integral.

Definitions:

Let [a, b] be a given interval. Then a set P = {x0 , x1, …, xn-1 , xn } of [a, b] such that a= x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b is said to be a Partition of [a, b]. The set of all partitions of [a, b] is denoted by P([a, b]). The intervals [x0 , x1], [x0 , x1], [x1 , x2], …, [xn-1 , xn] are called the subintervals of [a, b]. Write Δxi = xi – xi-1 is called the length of the interval [xi-1, xi] (i = 1, …, n) and max |Δxi | is called the norm of the partition P and is denoted by ||P|| or Q is called the refinement or finer of the partition P.⊂(P). A partition Q of [a, b] such that P ⊂ Q is called the refinement or finer of the partition P.⊂(P). A partition Q of [a, b] such that P Suppose f is a bounded real valued function defined on [a, b] and 2 P([a, b]). ThεMi = sup f(x) , m i = inf f(x) (xi-1 ≤ x ≤ xi) for each P

Suppose f is a bounded real valued function defined on [a, b] and P([a, b]). Thenε

Mi = sup f(x) , m i = inf f(x) (xi-1 ≤ x ≤ xi) for each P n n m i Δxi are called the Upper and Lower Riemann sumsΣ M i Δxi and L(P, f) = ΣU(P, f) = i =1 i =1 or Upper and Lower Darboux sums of f on [a, b] with respect to the partition P .

Further write - b b ∫ f dx = inf U(P, f) and ∫ f dx = sup L(P, f) a - a where the inf and the sup are taken over all partitions P of [a, b] are called the Upper and Lower Riemann integrals of f over [a, b], respectively.

If the upper and lower Riemann integrals are R[a, b] and we denoteεequal, we say that f is Riemann-integrable on [a, b] and we write f the common value of these integrals by b ∫ f d(x), a - b b b i.e., ∫ f dx = ∫ f dx = ∫ f dx .

R is bounded function then the upper and lower Riemann→1.1.1. Lemma . If f : [a, b] integrals of f are bounded. Since f is bounded, there exist two numbers m and M such that m ≤ f(x) ≤ M (a ≤ x ≤ b). Hence, for every partition P of [a, b] we have M≤ Mi ≤ mi ≤m miΔxiΣ ≤mΔxi Σ ⇒ MiΔxiΣ ≤ MΔxiΣ ≤ , i = 1, 2, 3, …, n. m(b –a) ≤ L(P,f) ≤ U(P,f) ≤ M(b-a),⇒ so that the numbers L(P,f) and U(P,f) form a bounded set. Therefore by the definition of lower and upper Riemann integrals this shows that the upper and lower integrals are defined for every bounded function f are bounded also. The question of their equality, and hence the question of the integrability of f,

R is bounded function, P is any partition of [a, b] and P* is the→

1.1.2. Lemma. If f : [a, b] refinement of P, then L(P, f) ≤ L(P*, f) and U(P*, f) ≤ U(P, f). R is bounded function and P1, P2 are any two partitions of [a, b]→

1.1.3. Lemma. R is bounded function and P1, P2 are any two partitions of [a, b]→If f : [a, b]

L(P1, f) ≤ U(P2, f) and L(P2, f) ≤ U(P1, f).

R are bounded functions and P is any partition of [a, b] then→1.1.4.

Lemma. If f, g : [a, b] (i) L(P, f + g) ≥ L(P, f) + L(P, g) (ii) U(P, f + g) ≤ U(P, f) + U(P, g). R is bounded function .

Theorem. If f : [a, b] - b b ∫ f dx ≥ ∫ f dx a - a εR is bounded function then for →1.1.2. Theorem (Darboux). If f : [a, b] δ> 0 there exists > 0 such that - b b U(P, f) < and L(P, f)ε∫ f + ε> ∫ f dx - a - a R is bounded function is Riemann Integrable if the oscillatory→

1.1.3. Theorem. If f : [a, b] sum < (P, f) = U(P,f) – L(P,f)ω, i.e. ε < ε , for ε > 0 and any partition P of [a, b]. R is Riemann Integrable.

1.1.4.Theorem. Every continuous function f : [a, b] R is Riemann Integrable.→1.1.5. Theorem. Every monotone function f : [a, b] Students you studied the properties given above and other properties of Riemann Integrals in previous classes therefore we are not interested to investigate these here. However we shall immediately consider a more general situation. be a monotonically increasingα R is bounded function and →

1.1.2 Definition. Let f : [a, b] function on [a, b]. Let P = {x0 , x1, …, xn-1 , xn } such that a= x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b be any Partition of [a, b] . We write (xi-1), i = 1, 2, 3, … , n.α(xi) - αi =αΔ is bounded on [a, b],α(b) are finite therefore α(a) and αBy the definition of monotone function i ≥ 0, i = 1, 2, 3, …, n.αΔ is monotonically increasing function then clearly αalso since P([a, b]). We defineεLet Mi = sup f(x) , mi = inf f(x) (xi-1 ≤ x ≤ xi) for each P n n i,α mi ΔΣ) = αi, and L(P, f, α Mi ΔΣ) = αU(P, f, i=1 i=1 are called the Upper and Lower Riemann Stieltjes sums respectively. Further we define -b b),α = sup L(P, f, α) and ∫ f dα = inf U(P, f, α∫ f d a -a where the inf and the sup are taken over all partitions P of [a, b], are called the Upper and Lower Riemann Stieltjes integrals of f over [a, b], respectively.

If the upper and lower Riemann Stieltjes integrals are equal, we say that f is Riemann Stieltjes integrable on [a, b]

Lower Riemann Stieltjes integrals of f over [a, b], respectively. If the upper and lower Riemann Stieltjes integrals are equal, we say that f is Riemann Stieltjes integrable on [a, b]

∫ f d (x)α or ∫ f(x) dα∫ f d

overαThis is the Rientatm-Stieltjes integral (or simply the Slielljes integral of f with respect to (x) = x we see that the Riemann integral is the special case of the Riemannα[a,b]. If we put (x) = x

we see that the Riemann integral is the special case of the Riemanns

If f: [a, b] be a monotonically increasing functionα R is bounded function

. Lemma If f: [a, b] on [a, b]. Let P be any Partition of [a, b] .Then the upper and lower Riemann-Stietjes integrals of be a monotonically increasing function on [a, b]. Let P be any Partition of [a, b] .Then the upper and lower Riemann-Stietjes integrals of are bounded.αf with respect to

Proof. Since f is bounded, there exist two numbers m and M such that m ≤ f(x) ≤ M (a ≤ x ≤ b). Hence, for every partition P of [a, b] we have M≤ Mi ≤ mi ≤m iαmiΔΣ ≤i αmΔΣ ⇒ iαMiΔΣ ≤ iαMΔΣ ≤ , i = 1, 2, 3, …, n. (a)],α(b) – α) ≤ M[α) ≤ U(P, f, α(a)] ≤ L(P, f, α(b) – α m[⇒) form a bounded set.

Therefore by the definition ofα) and U(P, f, αso that the numbers L(P, f, lower and upper Riemann-Stietjes integrals this shows that the upper and lower integrals are defined for every bounded function f are bounded also. 1.1.6.

Lemma. If P* is a refinement of the partition P of [a, b], then).α) ≤ U(P, f , α) and U(P*, f , α) ≤ L(P*, f , αL(P, f ,

Proof. Let P = {x0 , x1, …, xn-1 , xn } such that a = x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b be any Partition of [a, b] and P* the refinement of P contains just one point X* more than P such that xi-1< x*

where x i-1 and xi are two consecutive points of P.

Let mi, mi , mi′ are the infimum of f(x) in′′ mi≤[xi-1 , xi], [xi-1 , x*] and [x* , xi] respectively then clearly mi mi ≤ and mi ′ . Therefore′′) = miα) - L(P,f, αL(P*,f , (xi-1)] + miα(x*) - α [′ (xi-1)]α(xi) - α(x*)] - mi[α(xi) - α [′′ = mi (xi-1)] + miα(x*) - α[′ (xi-1)]α(x*) - α(x*) + α(xi) - α(x*)] - mi[α(xi) - α[′′ = (mi (xi-1)] + (miα(x*) - α- mi)[′ 0.≥(x*)] α(xi) - α- mi)[′′).

If P* contains k points more than P then byb repeating theα L(P*, f , ≤) αHence L(P, f ,) is analogousα) ≤ U(P, f , α)same process we arrive at the same result.

as $n \to \infty$, we have

$$\int_0^{10} f(x)\,d\alpha(x) = 50 + 55 = 105.$$

E.2. Properties

Theorem E.4. *Let c_1, c_2 be two constants in $\mathbb{R}$.*

(1) *If $f, g \in R(\alpha)$ on $[a,b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$, and*

$$\int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha.$$

(2) *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a,b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a,b]$, and*

$$\int_a^b f\,d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b f\,d\beta.$$

(3) *If $c \in [a,b]$, then*

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Definition E.5. If $a < b$, we define

$$\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha.$$

Theorem E.6. *If $f \in R(\alpha)$ and α has a continuous derivative on $[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

E.3.1. Integration by parts.

Theorem E.7 (Integration by parts). *If $f \in R(\alpha)$ on $[a,b]$, then $\alpha \in R(f)$ on $[a,b]$, and*

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\,df(x).$$

Example E.8. As in Example E.3, $f(x) = x$, and $\alpha(x) = x + [x]$. Then

$$\begin{aligned}\int_0^{10} f(x)\,d\alpha(x) &= f(10)\alpha(10) - f(0)\alpha(0) - \int_0^{10} \alpha(x)\,df(x) \\ &= 10 \times 20 - 0 \times 0 - \int_0^{10} (x + [x])\,dx \\ &= 200 - 50 - \int_0^{10} [x]\,dx = 150 - 45 = 105\end{aligned}$$

E.3.2. Change of variables.

Theorem E.9 (Change of variables). *Suppose that $f \in R(\alpha)$ on $[a,b]$ and g is a strictly increasing continuous function on $[c,d]$ with $a = g(c)$, $b = g(d)$. Let $h = f \circ g$, $\beta = \alpha \circ g$. Then $h \in R(\beta)$ on $[c,d]$ and*

$$\int_a^b f(x)\,d\alpha(x) = \int_c^d f(g(t))\,d\alpha(g(t)) = \int_c^d h(t)\,d\beta(t).$$

Example E.10. Let $y = \sqrt{x}$, we have

$$\begin{aligned}\int_0^4 ([\sqrt{x}] + x^2)\,d\sqrt{x} &= \int_0^2 ([y] + y^4)\,dy = \int_0^2 [y]\,dy + \int_0^2 y^4\,dy \\ &= 1 + \frac{1}{5}y^5\Big|_{y=0}^{2} = \frac{37}{5}\end{aligned}$$

E.3.3. Step functions as α. By Remark C.6 and Theorem E.4(2), we have

$$\int_a^b f(x)\,dF(x) = \int_a^b f(x)\,dF_{ac}(x) + \int_a^b f(x)\,dF_{sc}(x) + \int_a^b f(x)\,dF_d(x) \qquad \text{(E.2)}$$

Remark E.11. If $\alpha \equiv$ constant on $[a, b]$, then $S(P, f, \alpha) = 0$ for all partition P, and

$$\int_a^b f(x)\, d\alpha(x) = 0.$$

Example E.15. (1) Consider

$$f(x) = 1 \quad \text{for } x \in [-1, 1], \qquad \text{and} \qquad \alpha(x) = -I_{\{0\}},$$

then

$$\int_{-1}^{1} f(x)\, d\alpha(x) = f(0)(\alpha(0+) - \alpha(0-)) = 0$$

(2) Consider

$$f(x) = 2I_{\{0\}} + I_{[-1,0)\cup(0,1]} \qquad \text{and} \qquad \alpha(x) = -I_{[0,1]}.$$

Then both of α and f are discontinuous from the left at $x = 0$. This implies that the Riemann-Stieltjes integral $\int_{-1}^{1} f\, d\alpha$ does not exist.

Theorem E.16 (Reduction of a Riemann-Stieltjes Integral to a finite sum). *Let α be a step function on $[a, b]$ with jump*

$$c_k = \alpha(x_k+) - \alpha(x_k-) \quad \text{at } x = x_k.$$

Let f be defined on $[a, b]$ in such a way that not both of f and α are discontinuous from the left or from the right at x_k. Then $\int_a^b f(x)\, d\alpha(x)$ exists and

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k) c_k.$$

Theorem E.16 (Reduction of a Riemann-Stieltjes Integral to a finite sum). *Let α be a step function on $[a, b]$ with jump*

$$c_k = \alpha(x_k+) - \alpha(x_k-) \quad \text{at } x = x_k.$$

Let f be defined on $[a, b]$ in such a way that not both of f and α are discontinuous from the left or from the right at x_k. Then $\int_a^b f(x)\, d\alpha(x)$ exists and

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k) c_k.$$

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Theorem E.16 (Reduction of a Riemann-Stieltjes Integral to a finite sum). *Let α be a step function on $[a, b]$ with jump*

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Let f be defined on $[a, b]$ in such a way that not both of f and α are discontinuous from the left or from the right at x_k. Then $\int_a^b f(x)\, d\alpha(x)$ exists and

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k) c_k.$$

Theorem E.12. *Given $c \in (a, b)$. Define*

$$\alpha(x) = pI_{[a,c)} + rI_{\{c\}} + qI_{(c,b]}$$

(as given in Figure E.1). Suppose at least one of the functions f or α is continuous from the left at c, and at least one is continuous from the right at c. Then $f \in R(\alpha)$ and

$$\int_a^b f(x)\, d\alpha(x) = f(c)(\alpha(c+) - \alpha(c-)) = f(c)(q - p).^1$$

Remark E.13. The integral $\int_a^b f\, d\alpha$ does not exist if both of f and α are discontinuous from the left or from the right at c.

Remark E.14. (1) If $\alpha(x) = pI_{\{a\}} + qI_{(a,b]}$, then

$$\int_a^b f(x)\, d\alpha(x) = f(a)(\alpha(a+) - \alpha(a))$$

Example E.15. (1) Consider

$$f(x) = 1 \quad \text{for } x \in [-1, 1], \qquad \text{and} \qquad \alpha(x) = -I_{\{0\}},$$

then

$$\int_{-1}^{1} f(x)\, d\alpha(x) = f(0)(\alpha(0+) - \alpha(0-)) = 0$$

(2) Consider

$$f(x) = 2I_{\{0\}} + I_{[-1,0)\cup(0,1]} \qquad \text{and} \qquad \alpha(x) = -I_{[0,1]}.$$

Then both of α and f are discontinuous from the left at $x = 0$. This implies that the Riemann-Stieltjes integral $\int_{-1}^{1} f\, d\alpha$ does not exist.

Example E.17. (1) Let

$$f(x) = \begin{cases} 3 & \text{if } x \le 0 \\ 3 - 4x & \text{if } 0 < x < 1 \\ -1 & \text{if } x \ge 1 \end{cases}$$

and

$$\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

Since f is continuous, $\int_{-3}^{3} f(x)\, d\alpha(x)$ exists and

$$\begin{aligned} \int_{-3}^{3} f(x)\, d\alpha(x) &= f(0)(\alpha(0+) - \alpha(0-)) + f(1)(\alpha(1+) - \alpha(1-)) \\ &= 3(2-0) + (-1)(0-2) = 8. \end{aligned}$$

(2) Let $\alpha(x) = 2I_{[0,2)} + 5I_{[2,3)} + 6I_{[3,\infty)}$

$$\begin{aligned} \int_{-5}^{10} e^{-3x}\, d\alpha(x) &= e^{-3\cdot 0}(2-0) + e^{-3\cdot 2}(5-2) + e^{-3\cdot 3}(6-5) \\ &= 2 + 3e^{-6} + e^{-9}. \end{aligned}$$

Example E.18. Suppose F is the Cantor function (see Figure C.1). By integration by parts, we have

$$\int_0^1 x\, dF(x) = xF(x)\big|_{x=0}^{1} - \int_0^1 F(x)\, dx = 1 - \int_0^1 F(x)\, dx.$$

Since $\int_0^1 F(x)\, dx$ is the area of the Cantor function on $[0, 1]$, we get

$$\int_0^1 F(x)\, dx = \frac{1}{2}.$$

Hence,

$$\int_0^1 x\, dF(x) = \frac{1}{2}.$$

E.3.4. Comparison theorem.

Theorem E.19. *Assume that α is an increasing function on $[a,b]$. If $f, g \in R(\alpha)$ on $[a,b]$, and if $f(x) \leq g(x)$ for $x \in [a,b]$, then*

$$\int_a^b f(x)\,d\alpha(x) \leq \int_a^b g(x)\,d\alpha(x).$$

Corollary E.20. *If $g(x) \geq 0$ and α is an increasing function on $[a,b]$, then*

$$\int_a^b f(x)\,d\alpha(x) \geq 0.$$

Theorem E.21. *Assume that α is an increasing function on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then*

(1) $|f| \in R(\alpha)$ *on* $[a,b]$, *and*

$$\left|\int_a^b f(x)\,d\alpha(x)\right| \leq \int_a^b |f(x)|\,d\alpha(x).$$

(2) $f^2 \in R(\alpha)$ *on* $[a,b]$.

Definition E.23. A function $\alpha : [a,b] \longrightarrow \mathbb{R}$ is said to be of bounded variation if there exists a constant M such that

$$\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \leq M$$

for every partition $\{x_0, x_1, \cdots, x_n\}$ of $[a,b]$.

Theorem E.24. *Let α be defined on $[a,b]$, then α is of bounded variation on $[a,b]$, if and only if there exist two increasing functions α_1 and α_2, such that $\alpha = \alpha_1 - \alpha_2$*

Theorem E.25. *If f is continuous on $[a,b]$, and if α is of bounded variation on $[a,b]$, then $f \in R(\alpha)$. Moreover, the function*

$$F(t) = \int_0^t f(x)\, d\alpha(x)$$

has the following properties :

(1) *F is of bounded variation on $[a,b]$.*

(2) *Every continuous point of α is also a continuous point of F.*

POSSIBLE QUESTIONS

UNIT-I PART-B (5 × 6 = 30)

1. Show that Reimann Steiljes integral can be reduced to a finite sum.

2. Prove: If $f \in R(\alpha)$ and $f \in R(\beta)$, then $f \in R(c_1 \ \alpha + c_2 \beta)$.

3. If f ∈ R (α) on [a, b] then α ∈ R (f) on [a,b,] we have

$$\int_a^b f(x)d\, \alpha(x) + \int_a^b \alpha(x)d\, f(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a).$$

4. Explain change of variables in Riemann – Stieltjes integral ?

5. State and prove linearity property in Reimann Steiltjes integrals.

6. Let $f(x)= \alpha(x)= x$ on [0,1] .Then show that $f \epsilon R(\alpha)$ and find the integral.

PART- C (1 × 10 =10)

1. Prove that the modulus of Reimann Steiltjes integrable functions are also Reimann Steiltjes integrable.
2. State and Prove relation between Riemann integral and Reimann Steiltjes integral.
3. Show that every bounded function on a closed interval is Reimann integrable.
4. Prove that a function f is continous iff its bounded variation function is also continous.
5. State and prove necessary and sufficient condition for Reiamann Steiltjes integrability

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Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: REAL ANALYSIS **Subject Code: 17MMP102**

UNIT-I

	Option 1	option 2	Option 3	Option 4	ANSWERS
x(y + z) = xy + xz is --------------- law	commutative	associative	distributive	closure	distributive
If x < y, then for every z we have -----------	(x + z) < (y + z)	(x + z) > (y + z)	(x + z) = (y + z)	x + z = 0	(x + z) < (y + z)
If x > 0 and y > 0, then ----------------	xy less than equal to 0	xy > 0	xy greater than equal to 0	xy < 0	xy > 0
If x > y and y > z, then -----------	x < y	x = z	x > z	x < z	x > z
If a less than equal to b + $ for every $ > 0, then -----------	a < b	a > b	a greater than equal to b	a less than equal to b	a less than equal to b
The set of all points between a and b is called ------------	integer	interval	elements	set	interval
The set {x: a < x < b} is --------------	(a, b)	[a, b]	(a, b]	[a, b)	(a, b)
A real number is called a positive integer if it belongs to -----------	interval	open interval	closed interval	inductive set	inductive set
If d is a divisor of n, then ---------------	n = c	n < cd	n > cd	n = cd	n = cd
If $a \mid bc$ and (a, b) =1, then ------------------	$a \mid c$	$a \mid b$	$b \mid a$	$c \mid a$	$a \mid c$
If $a \mid bc$ and (a, b) =1, then $a \mid c$ is ---------------	Unique factorisation theorem	additive property	approximation property	Euclid's lemma	Euclid's lemma
Rational numbers is of the form -----------------	pq	p + q	p/q	p - q	p/q
e is ---------------	rational	irrational	prime	composite	irrational
An integer n is called ----------- if the only possible divisors of n are 1 and n	rational	irrational	prime	composite	prime
If $d \mid a$ and $d \mid b$, then d is called ----------	LCM	common divisor	prime	function	common divisor
If (a, b) = 1, then a and b are called ---------------	twin prime	common factor	LCM	relatively prime	relatively prime
If an upper bound 'b' of a set S is also a member of S then 'b' is called ---------------	rational	irrational	maximum element	minimum element	maximum element
If an lower bound 'b' of a set S is also a member of S then 'b' is called ---------------	rational	irrational	maximum element	minimum element	minimum element
A set with no upper bound is called ------------	bounded above	bounded below	prime	function	bounded above
A set with no lower bound is called ------------	bounded above	bounded below	prime	function	bounded below
The least upper bound is called -----------	bounded above	bounded below	supremum	infimum	supremum
The greatest lower bound is called -----------	bounded above	bounded below	supremum	infimum	infimum
The supremum of {3, 4} is ----------	3	4	(3, 4)	[3, 4]	4
Every finite set of numbers is ----------	bounded	unbounded	prime	bounded above	bounded
A set S of real numbers which is bounded above and bounded below is called --------	bounded set	inductive set	super set	subset	bounded set
The set N of natural numbers is ----------	bounded	not bounded	irrational	rational	not bounded
The completeness axiom is ------------	b = sup S	S = sup b	b = inf S	S = inf b	b = sup S
The infimum of {3, 4} is ------------	3	4	(3, 4)	[3, 4]	3
Sup C = Sup A + Sup B is called -------------- property	approximation	additive	archimedean	comparison	additive
For any real x, there is a positive integer n such that -----------	n > x	n < x	n = x	n = 0	n > x
If x > 0 and if y is an arbitrary real number, there is a positive number n such that nx > y is -------------- property	approximation	additive	archimedean	comparison	archimedean
The set of positive integers is -------------	bounded above	bounded below	unbounded above	unbounded below	unbounded above
The absolute value of x is denoted by ---------------	$\lvert x\rvert$	$\lvert\lvert x\rvert\rvert$	x < 0	x > 0	$\lvert x\rvert$
If x < 0 then ---------------	$\lvert x\rvert = x$	$\lvert\lvert x\rvert\rvert = \lvert x\rvert$	$\lvert\lvert x\rvert\rvert = -x$	$\lvert x\rvert = -x$	$\lvert x\rvert = -x$
If S = [0, 1) then sup S = ----------------	0	1	(0, 1)	[0,1]	1
Triangle inequality is -------------------	$\lvert a\rvert + \lvert b\rvert$ greater equal to $\lvert a + b\rvert$	$\lvert a\rvert > \lvert a + b\rvert$	$\lvert b\rvert > \lvert a + b\rvert$	$\lvert a + b\rvert$ less than equal to $\lvert a\rvert + \lvert b\rvert$	$\lvert a + b\rvert$ less than equal to $\lvert a\rvert + \lvert b\rvert$
$\lvert x + y\rvert$ greater than equal to -----------------	$\lvert x\rvert + \lvert y\rvert$	$\lvert x\rvert \lvert y\rvert$	$\lvert x\rvert - \lvert y\rvert$	$\lvert \lvert x\rvert - \lvert y\rvert \rvert$	$\lvert \lvert x\rvert - \lvert y\rvert \rvert$
Set of real numbers S is bounded above implies S has a ----------	supremum	infimum	additive property	comparison property	supremum
In { (3n +2) / (2n + 1) such that n is in N}, the greatest lower bound is --------------	5 divided by 3	8 divided by 5	11 divided by 47	3 divided by 2	3 divided by 2
In Cauchy-Schwarz inequality, the equality holds iff ----------	akx = 0	akx + bkx = 0	akx + bk = 0	bk = 0	akx + bk = 0
If a set consists of a finite number of elements is called	infinite set	finite set	cantor set	null set	finite set
A sequence < S n > is said to be monotonically increasing if	S n+1≥ S n	S n+1< S n	S n+1>S n	S n+1 = S n	S n+1≥ S n
If A,B,C are three sets then what is A -(B -C) =	(A -B) U (A ∩ C)	A -(B ∩ C)	(A -B) U C	(A -B) U (A - C)	(A -B) U (A ∩ C)
If S is a non-empty set of real numbers and S is unbounded below ,then	inf S = - ∞	inf S = + ∞	Sup S = - ∞	Sup S = + ∞	inf S = - ∞
If P (A) denotes the power set of A and A is the void set then P {P {P{P(A)}}}} =	0	1	4	16	16
If X ∈ R then	X/∞ = ∞	X/∞ = 0	X/∞ = X	X/∞ = - ∞	X/∞ = 0
If x < 0 then ---------------	X (- ∞) = -∞	X (- ∞) = ∞	X (- ∞) = 0	X (- ∞) = X	X (- ∞) = -∞
If R * is an extended real number system then the least upper bound is	∞	negative infinity	0	no least upper bound	∞
Let f : R → R be a functiondefined as f (x) = x $\lvert x\rvert$ then	f is 1-1 but not onto	neither f is 1-1 nor onto	f is 1-1 both onto	f is onto but not one-one	
The value of (0, ∞) is	∞	0	not defined	can not be determined	0

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UNIT II

Infinite series and infinite products:

Introduction – Basic definitions – Ratio test and root test – Dirichlet test and Able's test Rearrangement of series – Riemann's theorem on conditionally convergent series – Sub series - Double sequences – Double series – Multiplication of series – Cesaro summability.

TEXT BOOK

1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

REFERENCES

1. Gupta.S.L. and Gupta.N.R.,(2003).Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd, Singapore.
2. Sterling. K. Berberian., (2015).A First Course in Real Analysis, Springer Pvt Ltd, New Delhi.

Definition :

Given an infinite sequence , the *n*th partial sum is the sum of the first *n* terms of the sequence, that is,

A series is **convergent** if the sequence of its partial sums tends to a limit; that means that the partial sums become closer and closer to a given number when the number of their terms increases. More precisely, a series converges, if there exists a number such that for any arbitrarily small positive number , there is a (sufficiently large) integer such that for all ,

If the series is convergent, the number (necessarily unique) is called the **sum of the series**.

Any series that is not convergent is said to be divergent.

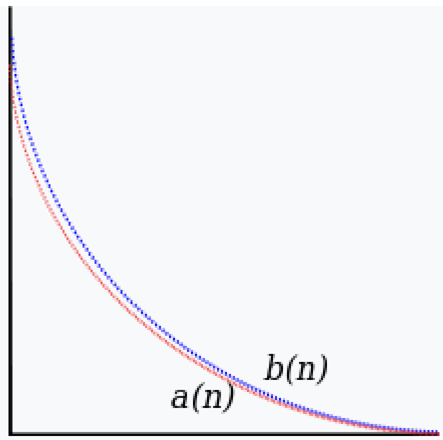
- The reciprocals of the positive integers produce a divergent series (harmonic series):
- Alternating the signs of the reciprocals of positive integers produces a convergent series:
- The reciprocals of prime numbers produce a divergent series (so the set of primes is "large"):
- The reciprocals of triangular numbers produce a convergent series:
- The reciprocals of factorials produce a convergent series (see e):
- The reciprocals of square numbers produce a convergent series (the Basel problem):
- The reciprocals of powers of 2 produce a convergent series (so the set of powers of 2 is "small"):

The reciprocals of powers of any n produce a convergent series:
Alternating the signs of reciprocals of powers of 2 also produces a convergent series:

- Alternating the signs of reciprocals of powers of any n produces a convergent series:

- The reciprocals of Fibonacci numbers produce a convergent series (see ψ):

There are a number of methods of determining whether a series converges or diverges.



If the blue series, , can be proven to converge, then the smaller series, must converge. By contraposition, if the red series, is proven to diverge, then must also diverge.

Comparison test. The terms of the sequence are compared to those of another sequence . If,

for all *n*, , and converges, then so does
However, if,

for all *n*, , and diverges, then so does

Ratio test. Assume that for all *n*, . Suppose that there exists such that

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the ratio test is inconclusive, and the series may converge or diverge.

Root test or ***n*****th root test**. Suppose that the terms of the sequence in question are non-negative. Define r as follows:

where "lim sup" denotes the limit superior (possibly ∞; if the limit exists it is the same value).

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the root test is inconclusive, and the series may converge or diverge.

The ratio test and the root test are both based on comparison with a geometric series, and as such they work in similar situations. In fact, if the ratio test works (meaning that the limit exists and is not equal to 1) then so does the root test; the converse, however, is not true. The root test is therefore more generally applicable, but as a practical matter the limit is often difficult to compute for commonly seen types of series.

Integral test. The series can be compared to an integral to establish

convergence or divergence. Let be a positive and monotone decreasing function. If

then the series converges. But if the integral diverges, then the series does so as well.

Limit comparison test. If , and the limit exists and is not

zero, then converges if and only if converges.

Alternating series test. Also known as the *Leibniz criterion*, the alternating series test states that for an alternating series of the

form , if is monotone decreasing, and has a limit of 0 at infinity, then the series converges.

Cauchy condensation test. If is a positive monotone decreasing

sequence, then converges if and only if converges.

Dirichlet's test

Abel's test

Raabe's test

Conditional and absolute convergence[edit]

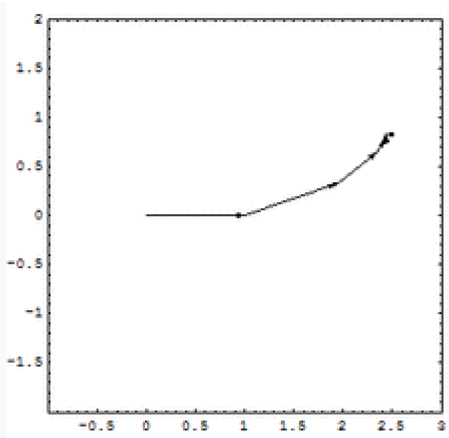


Illustration of the absolute convergence of the power series of Exp[z] around 0 evaluated at z = Exp[$i/3$]. The length of the line is finite.

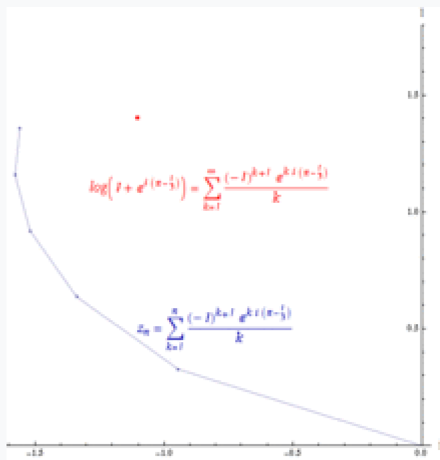


Illustration of the conditional convergence of the power series of log(z+1) around 0 evaluated at z = exp(($\pi - 1/3$)i). The length of the line is infinite.

For any sequence , for all n. Therefore,

This means that if converges, then also converges (but not vice versa).

If the series converges, then the series is absolutely convergent. An absolutely convergent sequence is one in which the length of the line created by joining together all of the increments to the partial sum is finitely long. The power series of the exponential function is absolutely convergent everywhere.

If the series converges but the series diverges, then the

series is conditionally convergent. The path formed by connecting the partial sums of a conditionally convergent series is infinitely long. The power series of the logarithm is conditionally convergent.

The Riemann series theorem states that if a series converges conditionally, it is possible to rearrange the terms of the series in such a way that the series converges to any value, or even diverges.

Uniform convergence[edit]

Main article: uniform convergence

Let be a sequence of functions. The series is said to

converge uniformly to *f* if the sequence of partial sums defined by

converges uniformly to *f*.

There is an analogue of the comparison test for infinite series of functions called the Weierstrass M-test.

Cauchy convergence criterion[edit]

The **Cauchy convergence criterion** states that a series

converges if and only if the sequence of partial sums is

a Cauchy sequence. This means that for every there is a

positive integer such that for all we have

1. Sequences

1.1. Sequences. An infinite sequence of real numbers is an ordered unending list of real numbers. E.g.:

$$1, 2, 3, 4, \ldots$$

We represent a generic sequence as $a_1, a_2, a_3, \ldots$, and its n-th as a_n.

In order to define a sequence we must give enough information to find its n-th term. Two ways of doing this are:

1. With a formula. E.g.:

$$a_n = \frac{1}{n}$$

$$a_n = \frac{1}{10^n}$$

$$a_n = \sqrt{3n - 7}$$

1.2. Limit of a Sequence. We say that a sequence a_n *converges* to a limit L if the difference $|a_n - L|$ can be made as small as we wish by taking n large enough. We write $a_n \to L$, or more formally:

$$\lim_{n\to\infty} a_n = L\,.$$

E.g.:

$$\lim_{n\to\infty} \frac{1}{n} = 0\,.$$

If a sequence does not converge we say that it *diverges*. E.g., the following sequences diverge:

$$n = 1, 2, 3, 4, \cdots \to \qquad \text{diverges (to } +\infty)$$

$$(-1)^n = -1, 1, -1, 1, \cdots \to \qquad \text{diverges}$$

1.3. **Limit Laws for Sequences.** If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then:

$$\lim_{n\to\infty} (a_n + b_n) = A + B$$

$$\lim_{n\to\infty} (a_n - b_n) = A - B$$

$$\lim_{n\to\infty} (a_n\, b_n) = A\, B$$

$$\lim_{n\to\infty} (a_n / b_n) = A/B \qquad \text{(provided } B \neq 0)$$

So, a "complicated" limit such as $L = \lim_{n\to} \frac{1+\frac{1}{n}}{3+\frac{1}{10^n}}$ can be computed by replacing smaller parts of it with their limits $1/n \to 0$, $1/10^n \to 0$: $L = \frac{1+0}{3+0} = \frac{1}{3}$.

1.4. **Squeeze Law.** If $a_n \le c_n \le b_n$, and a_n and b_n have the same limit: $a_n \to L$, $b_n \to L$, then c_n has also the same limit: $c_n \to L$. This can be used to compute limits such as the following one:

$$\lim_{n\to\infty} \frac{\sin n}{n}\,.$$

In this case we have:

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}\,.$$

Since $-1/n \to 0$ and $1/n \to 0$ then $\frac{\sin n}{n} \to 0$ also.

1.5. **Limits of Functions of Sequences.** If $a_n = f(n)$ for some function f and $\lim_{x\to n} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. This basically allows us to replace limits of sequences with limits of functions. In particular this is useful for using L'Hôpital's rule in computing limits of sequences. E.g:

$$\lim_{n\to\infty} \frac{e^n}{n} = \lim_{x\to\infty} \frac{e^x}{x} = \text{(L'Hôpital's rule)} = \lim_{x\to\infty} \frac{e^x}{1} = \infty\,.$$

2.1. Series. An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

---

[1]Proof: We use induction. First note that $0 < a_1 = \sqrt{6} < 3$. By adding 6 and taking square roots we get $\sqrt{6} < \sqrt{6+a_1} < \sqrt{6+3} = 3$, i.e.: $a_1 < a_2 < 3$. Now assume $a_n < a_{n+1} < 3$ for a given $n \geq 1$ (induction hypothesis). Again, by adding 6 and taking square roots we get $\sqrt{6+a_n} < \sqrt{6+a_{n+1}} < 3$, i.e. $a_{n+1} < a_{n+2} < 3$ (induction step). From here we get that $a_n < a_{n+1} < 3$ for every $n \geq 1$, which proves both, a_n is increasing and is bounded by 3.

[2]C.H. Edwards, Jr. & David E. Penney: *Calculus with Analytic Geometry*, 5th edition, Prentice Hall.

where $\{a_n\}$ is a sequence of numbers—sometimes the series starts at $n = 0$ or some other term instead of $n = 1$. Its Nth partial sum is

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N .$$

2.2. Sum of a Series. The sum

$$S = \sum_{n=1}^{\infty} a_n$$

of a series is defined as the limit of its partial sums

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

if it exists—it this case we say that the series converges. For instance, consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Its partial sum is

$$S_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^N} = 1 - \frac{1}{2^N} .$$

Hence, its sum is

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right) = 1 ,$$

i.e.:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 .$$

A series may or may not have a sum. For instance, in the following series:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

the sequence of partial sums $S_N = 1, 0, 1, 0, 1, 0, \ldots$ diverges, and the series has no sum.

2.3. **Telescopic Series.** Telescopic series are series for which all terms of its partial sum can be canceled except the first and last ones. For instance, consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

Its nth term can be rewritten in the following way:

$$a_n = \frac{1}{n\,(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

Hence, its Nth partial sum becomes:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n\,(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}\,. \end{aligned}$$

Hence:

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1\,.$$

2.4. **Geometric series.** A geometric series $\sum_{n=0}^{\infty} a_n$ is a series in which each term is a fixed multiple of the previous one: $a_{n+1} = r\,a_n$, where r is called the *ratio*. A geometric series can be rewritten in this way:

$$\sum_{n=0}^{\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots\,.$$

If $|r| < 1$ its sum is

$$\sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}\,.$$

Note that a is the first term of the series. If $a \neq 0$ and $|r| \geq 1$, the series diverges.

Examples:

$$\sum_{n=0} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2\,.$$

$$\sum_{n=0} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}.$$

Note that in the last example $r = a_{n+1}/a_n = \frac{(-1)^{n+1}/2^{n+1}}{(-1)^n/2^n} = -1/2$.

6.1. **Alternating Series Test.** If an alternating series verifies:

1. a_n it is decreasing: $a_n \geq a_n > 0$ for every n, and
2. the nth term tends to zero: $\lim_{n\to\infty} a_n = 0$,

then the series converges.

So, in this particular case the "reciprocal" of the nth term test holds.

E.g.:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

6.3. **Absolute Convergence.** A general series $\sum a_n$ is said to be *absolutely convergent* if the series of absolute values of its terms $\sum |a_n|$ is convergent.

We have that a series can be:

1. Convergent and absolutely convergent, e.g:

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

2. Convergent but not absolutely convergent—in this case the series is called **conditionally convergent**—, e.g:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

3. Not convergent nor absolutely convergent, e.g:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

However, a series cannot be absolutely convergent and not convergent, because absolute convergence implies convergence:

$$\text{absolute convergent} \implies \text{convergent}$$

Example: Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converge? Answer: Look at the series of absolute values:

$$\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

By comparison test, it converges (the right hand side is a p-series with $p > 1$), hence the given series is absolutely convergent, which implies that it is indeed convergent.

6.4. **Ratio Test.** Suppose that the limit $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists or is infinity. Then

1. If $\rho < 1 \implies \sum a_n$ converges absolutely.
2. If $\rho > 1 \implies \sum a_n$ diverges.
3. If $\rho = 1 \implies$ the ratio test is inconclusive.

As a rule of thumb, for geometric series $\rho = |r|$ (the ratio), and the conclusion of the ratio test is analogous to the one for geometric series, i.e., the series converges for $|r| < 1$ and diverges for $|r| > 1$.

Example: For the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ we have

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \lim_{n\to\infty} \frac{(n+1)^2}{2\,n^2} = \frac{1}{2} < 1\,,$$

hence, it converges absolutely.[3]

6.5. **Root Test.** In some cases in which the ratio test is unable to provide an answer, the root test may help. It says the following: Suppose that the limit $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or is infinity. Then

1. If $\rho < 1 \implies \sum a_n$ converges absolutely.
2. If $\rho > 1 \implies \sum a_n$ diverges.
3. If $\rho = 1 \implies$ the root test is inconclusive.

Example: Consider the following series $\sum_{n=1}^{\infty} \frac{1}{2^{n+\sin n}}$. For this series the ratio test cannot be used, because

$$\frac{a_{n+1}}{a_n} = 2^{-1+\sin n - \sin(n+1)} = 2^{-1-2\sin\frac{1}{2}\cos(n+\frac{1}{2})}$$

which has no limit. However, the root tests shows that the series is absolutely convergent:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{n+\sin n}}} = \lim_{n\to\infty} \frac{1}{2^{1+\sin n/n}} = \frac{1}{2} < 1\,.$$

POSSIBLE QUESTIONS

UNIT-II PART-B (5 × 6 = 30)

1.Prove: If $\sum a_n$ converges absolutely then the subseries $\sum b_n$ is also converges absolutely.Also $|\sum_{n=1}^{\infty} b_n| \le \sum_{n=1}^{\infty} |a_n|$.

2.Suppose $\sum_{n=1}^{\infty} a_n$ convereges absolutely and $\sum_{n=1}^{\infty} a_n$ =A and $\sum_{n=1}^{\infty} b_n$ =B; then show that the product of these two series converges to AB.

3.State and prove Ratio test for convergence of series.

4.State and Prove Riemann theorem on conditionally convergent series .

.5.Using Root test, show that the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots . \, converges$$

PART- C (1 × 10 =10)

1. State and prove uniform convergence in double sequences
2. Let $\sum a_n$ be an absolutely convergent series having sum S then every rearrangement of $\sum a_n$ also converges absolutely has sum S.

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Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: REAL ANALYSIS **Subject Code: 17MMP102**

UNIT II	OPTION1	OPTION2	OPTION3	OPTION4	ANSWERS
The coordinates (x,y) of a point represent an ------------ of numbers	function	relation	ordered pair	set	ordered pair
(a, b) = ----------------	{{a},{b},{a,b}}	{{a},{b}}	{{a},{a,b}}	{{a},{b},{}}	{{a},{a,b}}
(a, b) = (c, d) if and only if -------------	a = c & b = d	a = b & c = d	a = d & c = b	ab = cd	a = c & b = d
AxB denotes the ----------- of the sets A & B	product	cartesian product	polar form	complement	cartesian product
Any set of ordered pairs is called -------------	function	relation	ordered pair	set	relation
If S is a relation, the set of all elements that occur as first members in S is called the -----------	function	codomain	domain	range	domain
If S is a relation, the set of all elements that occur as second members in S is called the -----------	function	codomain	domain	range	range
If (x, y) belongs to F and (x, z) belongs to F, then -------------	x = z	x = y	xy = z	y = z	y = z
A mapping S into itself is called ------------	function	relation	domain	transformation	transformation
If F(x) = F(y) implies x =y is a --------------- function	one-one	onto	into	inverse	one-one
One-one function is also called -----------	injective	bijective	transformation	codomain	injective
S = {(a,b) : (b,a) is in S} is called ---------------	inverse	domain	codomain	converse	converse
The composite functions are denoted by ---------------	GxF	GoF	GF	G + F	GoF
GoF(x) = ------------------	G[F(x)]	F[G(x)]	G(x)	F(x)	G[F(x)]
In general the composite function GoF is ------------	GoF = FoG	GoF is not equal to FoG	GoF < FoG	GoF > FoG	GoF is not equal to FoG
If m < n, then K(m) < K(n) implies that K is -------------	sequence	subsequence	order preserving	equinumerous	order preserving
Similar sets are also called as ------------ set	denumerable	uncountable	finite	equinumerous	equinumerous
If A and B are two sets andif there exists a one-one correspondence between them,then it is called ------------ set	denumerable	uncountable	finite	equinumerous	equinumerous
A set which is equinumerous with the set of all positive integers is called ---------------- set	finite	infinite	countably infinite	countably finite	countably infinite
A set which is either finite or countably infinite is called ------------ set	countable	uncountable	similar	equal	countable
Uncountable sets are also called ------------- set	denumerable	non-denumerable	similar	equal	non-denumerable
Countable sets are also called --------------- set	denumerable	non-denumerable	similar	equal	denumerable
Every subset of a countable set is ------------	countable	uncountable	rational	irrational	countable
The set of all real numbers is ----------------	countable	uncountable	rational	irrational	uncountable
The cartesian product of the set of all positive integers is ----------	countable	uncountable	rational	irrational	countable
The set of those elements which belong either to A or to B or to both is called ---------	complement	intersection	union	disjoint	union
The set of those elements which belong to both A and B is called ------------	complement	intersection	union	disjoint	intersection
Union of sets is --------------	commutative	not commutative	not associative	disjoint	commutative
The complement of A relative to B is denoted by --------------	B - A	B	A	A - B	B - A
If A intersection B is the empty set, then A and B are called --------	commutative	not commutative	not associative	disjoint	disjoint
B - (union A) = -----------------	union (B -A)	B - (intersection A)	intersection (B - A)	{}	intersection (B - A)
B - (intersection A) = -----------------------	union (B -A)	B - (union A)	intersection (B - A)	{}	union (B -A)
Union of countable sets is -----------------	uncountable	infinite	countable	disjoint	countable
The set of all rational numbers is ---------------	uncountable	infinite	countable	disjoint	countable
The set S of intervals with rational end points is ---------- set	uncountable	infinite	countable	disjoint	countable
A relation which is symmetric, reflexive and transitive is called ------------------ relation	equivalence	component	composite	countable	equivalence
Any collection of disjoint intervals of positive length is ------------	equivalence relation	countable set	composite function	uncountable set	countable set
If A similar to B and B similar to C, then --------------	C similar to A	A similar to C	A < C	A = C	A similar to C
If the root of an algebraic equation f(x) = 0, then the real number is called ----------	prime	positive	algebraic	composite	algebraic
For all subsets A and B of S with B contained in A, we have -------	f(A + B) = f(A)	f(A + B) = f(B)	f(A - B) = f(A) - f(B)	f(A - B) = f(A)	f(A - B) = f(A) - f(B)
If f(A U B) = f(A) + f(B), then the function f is called -------------	additive	multiplicative	disjoint	equinumerous	additive
f(A U B) = ----------------------	f(A) + f(B)	f(A) - f(B)	f(A) + f(B) - f(B - A)	f(A) + f(B - A)	f(A) + f(B - A)
The sequence < (-1) n > is	monotonically increasing	monotonically decreasing	either increasing or decreasing	neither monotonically increasing nor monotonically decreasing	neither monotonically increasing nor monotonically decreasing
An unbounded sequence	a limit point	does not have a limit point	may or may not have a limit point	unique limit point	may or may not have a limit point
The series ∑▒〖(-1)〗^(n-1) 1/n	convergent	conditionally convergent	absolutely divergent	need not convergent	conditionally convergent
Every absolutely convergent series is	convergent	conditionally convergent	absolutely divergent	need not convergent	convergent
The sequence { 1/n } is	convergent & bounded	divergent &unbounded	divergent & bounded	convergent & Unbounded	convergent & bounded
If a sequence {an}n=1 to ∞ converges to a real number then the given sequence is	unbounded sequence	convergent	divergent & bounded	bounded	unbounded sequence
Every subsequence has a ----------------	limit pount	convergent	monotonic subsequence	non monotoni sequence	monotonic subsequence
The series 1+ r+ r ^2 +r^3 +………… is oscillatory if	r=1	r= -1	r >1	r < 1	r= -1

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Subject : REAL ANALYSIS	**SEMESTER: I**	**L T P C**
SUBJECT CODE: 17MMP102	**CLASS : I M.Sc**	**4 0 0 4**

UNIT III

Sequences of functions:

Basic definitions – Uniform convergence and continuity - Uniform convergence of infinite series of functions – Uniform convergence and Riemann – Stieltjes integration – Non uniformly convergent sequence – Uniform convergence and differentiation – Sufficient condition for uniform convergence of a series

TEXT BOOK

1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

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Uniform Convergence.

1 Introduction.

In this course we study amongst other things **Fourier series**. The **Fourier series** for a periodic function $f(x)$ with period 2π is defined as the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the coefficients a_k, b_k are defined as

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx,$$

with $k = 0, 1, \ldots$ (note that this means that $b_0 = 0$).

This is an example of a **functional series**, which is a series whose terms are functions:

$$\sum_{k=0}^{\infty} u_k(x).$$

As usual with series, we define the above infinite sum as a limit:

$$\sum_{k=0}^{\infty} u_k(x) = \lim_{N \to \infty} \sum_{k=0}^{N} u_k(x),$$

providing the limit exists. Note that different values of x will, in general, give different limits, if they exist.

In this lecture we shall look at functional series, and functional sequences, and we shall consider first the question of convergence. To deal with this, we consider two types of convergence: **pointwise convergence** and **uniform convergence**. There are three main results: the first one is that **uniform convergence** of a sequence of continuous functions gives us a continuous function as a limit. The second main result is **Weierstrass' Majorant Theorem**, which gives a condition that guarantees that a functional series converges to a continuous function. The third result is that integrals of a sequence of functions which converges uniformly to a limit function $f(x)$ also converge with the limit being the integral of $f(x)$. These results are not only good for your mental health, they are also important tools in our later discussion of Fourier series, and that is the reason for looking at them.

Definition 2.1 *(***Pointwise convergence.***) Suppose* $\{f_n(x) : n = 0, 1, 2, \dots\}$ *is a sequence of functions defined on an interval* I*. We say that* $f_n(x)$ **converges pointwise to the function** $f(x)$ *on the interval* I *if*

$$f_n(x) \to f(x), \quad \text{as } n \to \infty, \quad \text{for each } x \in I.$$

We call the function $f(x)$ **the limit function***.*

Uniform continuity

Definition. A function $f\colon D \to \mathbb{R}$ is *continuous at a point* $d \in D$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$,

$$|x - d| < \delta \Rightarrow |f(x) - f(d)| < \epsilon.$$

The following is an equivalent 'sequential' definition of the same concept.

Definition. A function $f\colon D \to \mathbb{R}$ is *continuous at a point* $d \in D$ if for any sequence $(x_n)_{n\in\mathbb{N}}$ in D with $d = \lim_{n\to\infty} x_n$, the convergence $f(d) = \lim_{n\to\infty} f(x_n)$ holds as well.

Definition. A function $f\colon D \to \mathbb{R}$ is *continuous* if it is continuous at every point of D.

Definition. A function $f\colon D \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in D$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

The cruical point to note here is that for uniform continuity, the number δ may only depend on ϵ. For continuity (at x), it may depend on x as well. Using quantifiers, we express continuity of f as follows:

$$\forall x \in D\, \forall \epsilon > 0\, \exists \delta > 0\, \forall y \in D\colon |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Uniform continuity, on the other hand, means that

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in D\colon |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

It is clear that uniform continuity implies continuity. But continuity does not imply uniform continuity.

Example. Define $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x^2)$. Then f is continuous but not uniformly continuous (see Exercise 1.1).

Example 2.1 $f_n(x) = x - \frac{1}{n}$. *Then* $f_n(x)$ *converges pointwise to* x *for each* $x \in \mathbb{R}$:

$$|f_n(x) - x| = \frac{1}{n} \to 0 \quad as \ n \to \infty.$$

Example 2.2 $f_n(x) = e^{-nx}$ *on* $[1,3]$. *For each* $x \in [1,3]$ *we have* $nx \to \infty$ *as* $n \to \infty$ *and therefore* $f_n(x) \to 0$ *as* $n \to \infty$ *for each* $x \in [1,3]$. *Thus* $f_n(x)$ *converges pointwise to* $f(x) = 0$ *for each* $x \in [1,3]$.

Example 2.3 $f_n(x) = e^{-nx}$ *on* $[0,3]$. *For each* $0 < x \leq 3$ *we have* $nx \to \infty$ *as* $n \to \infty$ *and therefore* $f_n(x) \to 0$ *as* $n \to \infty$ *for each* $0 < x \leq 3$. *However, at* $x = 0$ *we have* $f_n(0) = 1$ *for all* n. *Thus* $f_n(x)$ *converges pointwise to the function* $f(x)$ *defined by* $f(0) = 1$, $f(x) = 0$ *for each* $0 < x \leq 3$. *This is not a continuous function, despite the fact that each function* $f_n(x)$ *is continuous.*

Example 2.4 *Let the sequence f_n be defined as*

$$f_n(x) = \frac{nx}{(nx+1)^3} \quad x \in [0, \infty[.$$

Then $f_n(0) = 0$ and for each fixed $x > 0$

$$\begin{aligned} f_n(x) &= \frac{n^2x}{(nx+1)^3} \\ &= \frac{n^2x}{n^3(x+\frac{1}{n})^3} \\ &= \frac{1}{n}\frac{x}{(x+\frac{1}{n})^3} \to 0 \quad as\ n \to \infty. \end{aligned}$$

So that $f_n(x) \to f(x) = 0$ pointwise on $[0, \infty[$.

Then for $x > -\frac{1}{n}$ we have

$$f_n'(x) = \frac{n^2(1-2nx)}{(nx+1)^4}$$

and we see that for $x > 0$ we have $f_n'(x) \to 0$ as $n \to \infty$ whereas $f_n'(0) = n^2 \to \infty$. Here we see that $f_n' \to f'$ only on for $x > 0$. This shows that differentiability is not always respected by pointwise convergence.

The last two examples then lead us to pose the question: what extra condition (other than just pointwise convergence) can guarantee that the limit function is also continuous or differentiable? The answer to this is given by the concept of **uniform convergence**.

3 Uniform convergence

We define for a real-valued (or complex-valued) function f on a non-empty set I the **supremum norm** of f on the set I:

$$\|f\|_I = \sup_{x \in I} |f(x)|.$$

Note that if f is a bounded function on I then

$$\sup_{x \in I} |f(x)| = \sup\{\, |f(x)| \ : x \in I\}$$

exists, by the so-called **supremum axiom**. Observe that

$$|f(x)| \le \|f\|_I \quad \text{for all } \ x \in I,$$

and that $|f(x)|$ takes on values which are arbitrarily near $\|f\|_I$. In particular $\|f\|_I =$ the largest value of $|f(x)|$ whenever such a value exists (such as when I is a closed, bounded interval and $f(x)$ is a continuous function on I).

Definition. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n\colon D \to \mathbb{R}$ and another function $f\colon D \to \mathbb{R}$.

(i) We say that f_n *converges to* f *pointwise* if for all $x \in D$ and all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon.$$

(ii) We say that f_n *converges to* f *uniformly* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $x \in D$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon.$$

The above definition is equivalent to the following.

Definition. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n\colon D \to \mathbb{R}$ and another function $f\colon D \to \mathbb{R}$.

(i) We say that f_n *converges to* f *pointwise* if

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every fixed $x \in D$.

(ii) We say that f_n *converges to* f *uniformly* if

$$\sup_{x\in D} |f_n(x) - f(x)| \to 0$$

as $n \to \infty$.

Clearly uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

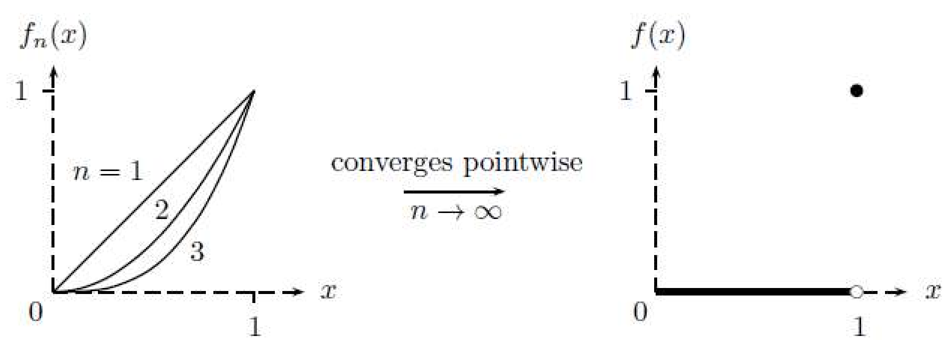
Example. Let $D = [0, 1]$. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n\colon D \to \mathbb{R}$, $x \mapsto x^n$. The pointwise limit is the discontinuous function $f : D \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1. \end{cases}$$

The convergence is not uniform, however, since

$$\sup_{x\in D} |f_n(x) - f(x)| = 1$$

for all $x \in \mathbb{N}$.



5.4. Properties of uniform convergence

In this section we prove that, unlike pointwise convergence, uniform convergence preserves boundedness and continuity. Uniform convergence does not preserve differentiability any better than pointwise convergence. Nevertheless, we give a result that allows us to differentiate a convergent sequence; the key assumption is that the derivatives converge uniformly.

5.4.1. Boundedness. First, we consider the uniform convergence of bounded functions.

Theorem 5.14. Suppose that $f_n : A \to \mathbb{R}$ is bounded on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Then $f : A \to \mathbb{R}$ is bounded on A.

Proof. Taking $\epsilon = 1$ in the definition of the uniform convergence, we find that there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1 \qquad \text{for all } x \in A \text{ if } n > N.$$

Choose some $n > N$. Then, since f_n is bounded, there is a constant $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \qquad \text{for all } x \in A.$$

It follows that

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M_n \qquad \text{for all } x \in A,$$

meaning that f is bounded on A (by $1 + M_n$). □

We do not assume here that all the functions in the sequence are bounded by the same constant. (If they were, the pointwise limit would also be bounded by that constant.) In particular, it follows that if a sequence of bounded functions converges pointwise to an unbounded function, then the convergence is not uniform.

Example 5.15. The sequence of functions $f_n : (0, 1) \to \mathbb{R}$ in Example 5.2, defined by

$$f_n(x) = \frac{n}{nx + 1},$$

cannot converge uniformly on $(0, 1)$, since each f_n is bounded on $(0, 1)$, but their pointwise limit $f(x) = 1/x$ is not. The sequence (f_n) does, however, converge uniformly to f on every interval $[a, 1)$ with $0 < a < 1$. To prove this, we estimate for $a \leq x < 1$ that

$$|f_n(x) - f(x)| = \left| \frac{n}{nx + 1} - \frac{1}{x} \right| = \frac{1}{x(nx + 1)} < \frac{1}{nx^2} \leq \frac{1}{na^2}.$$

Thus, given $\epsilon > 0$ choose $N = 1/(a^2\epsilon)$, and then

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in [a, 1) \text{ if } n > N,$$

5.4.2. Continuity. One of the most important property of uniform convergence is that it preserves continuity. We use an "$\epsilon/3$" argument to get the continuity of the uniform limit f from the continuity of the f_n.

Theorem 5.16. If a sequence (f_n) of continuous functions $f_n : A \to \mathbb{R}$ converges uniformly on $A \subset \mathbb{R}$ to $f : A \to \mathbb{R}$, then f is continuous on A.

Proof. Suppose that $c \in A$ and $\epsilon > 0$ is given. Then, for every $n \in \mathbb{N}$,

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|.$$

By the uniform convergence of (f_n), we can choose $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \qquad \text{for all } x \in A,$$

and for such an n it follows that

$$|f(x) - f(c)| < |f_n(x) - f_n(c)| + \frac{2\epsilon}{3}.$$

(Here we use the fact that f_n is close to f at both x and c, where x is an an arbitrary point in a neighborhood of c; this is where we use the uniform convergence in a crucial way.)

Since f_n is continuous on A, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(c)| < \frac{\epsilon}{3} \qquad \text{if } |x - c| < \delta \text{ and } x \in A,$$

which implies that

$$|f(x) - f(c)| < \epsilon \qquad \text{if } |x - c| < \delta \text{ and } x \in A.$$

This proves that f is continuous. □

This result can be interpreted as justifying an "exchange in the order of limits"

$$\lim_{n\to\infty} \lim_{x\to c} f_n(x) = \lim_{x\to c} \lim_{n\to\infty} f_n(x).$$

Such exchanges of limits always require some sort of condition for their validity — in this case, the uniform convergence of f_n to f is sufficient, but pointwise convergence is not.

It follows from Theorem 5.16 that if a sequence of continuous functions converges pointwise to a discontinuous function, as in Example 5.3, then the convergence is not uniform. The converse is not true, however, and the pointwise limit of a sequence of continuous functions may be continuous even if the convergence is not uniform, as in Example 5.4.

Definition 3.1 *A sequence of functions $f_n(x)$ defined on an set I is said to* **converge uniformly to** $f(x)$ **on** I *if*

$$\|f_n - f\|_I \to 0 \quad as \;\; n \to \infty.$$

We write this as

$$\lim_{n\to\infty} f_n = f \quad \text{uniformly on } I$$

or as

$$f_n \to f \quad \text{uniformly on } I \;\text{ as }\; n \to \infty.$$

Uniform convergence implies **pointwise convergence**, however there are sequences which converge pointwise but not uniformly. Indeed we have

$$|f_n(x) - f(x)| \le \sup_{x\in I} |f_n(x) - f(x)| = \|f_n - f\|_I,$$

so that

$$f_n \to f \;\text{ uniformly on } I \;\text{ as }\; n \to \infty$$

$$\implies \;\; |f_n(x) - f(x)| \to 0 \;\text{ for each }\; x \in I$$

$$\implies \;\; f_n \to f \;\text{ pointwise on }\; I.$$

Example 3.1 $f_n(x) = e^{-nx}$ *on* $[1, 3]$. *We have seen above that $f_n(x)$ converges pointwise to $f(x) = 0$ for each $x \in [1, 3]$. Then we have $|f_n(x) - f(x)| = |f_n(x)|$ and we then have*

$$\begin{aligned}\|f_n - f\| &= \sup_{x\in[1,3]} |f_n(x)| \\ &= \sup_{x\in[1,3]} |e^{-nx}| \\ &= \sup_{x\in[1,3]} e^{-nx} \\ &= e^{-n} \to 0 \;\; as\, n \to \infty.\end{aligned}$$

Thus we have uniform convergence in this case. Note that the last step follows from the observation that e^{-nx} is strictly decreasing for $x \ge 0$ with $n \ge 0$, so that $e^{-n} \ge e^{-nx}$ for all $x \ge 1$.

Theorem 4.3 *Suppose that $\{f_n(x); n = 0, 1, 2, \dots\}$ is a sequence of functions on an interval I and satisfying the following conditions:*

(i) $f_n(x)$ *is* **differentiable** *on I for each* $n = 0, 1, 2, \dots$

(ii) $f_n(x)$*converges* **pointwise** *to $f(x)$ on I*

(iii) $f'_n(x)$ *is continuous for each n and $f'_n \to g$ converges* **uniformly** *on I where $g(x)$ is a continuous function on I.*

Then the limit function $f(x)$ is differentiable and $f'(x) = g(x)$.

Proof: First note that

$$f_n(x) - f_n(a) = \int_a^x f'_n(t)dt$$

for each $f_n(x)$ and for each choice of $x, a \in I$. Because $f_n(x)$ converges pointwise to $f(x)$ for all $x \in I$, the left-hand side converges to $f(x) - f(a)$ as $n \to \infty$. Also, $f'_n \to g$ uniformly on I so by Theorem 4.2 we have that

$$\int_a^x f'_n(t)dt \to \int_a^x g(t)dt,$$

and we then find that

$$f(x) - f(a) = \int_a^x g(t)dt.$$

Now, $g(t)$ is continuous, so that, by the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_a^x g(t)dt = g(x)$$

so that $f(x)$ must be differentiable and $f'(x) = g(x)$.

This result is very useful, as we shall see, in examining the differentiability of functional series.

5 Applications to functional series.

Definition 5.1 *A* **functional series** *is a series*

$$\sum_{k=0}^{\infty} u_k(x)$$

where each term of the series $u_k(x)$ is a function on an interval I.

We can also define **pointwise convergence** for functional series:

Definition 5.2 *The functional series*

$$\sum_{k=0}^{\infty} u_k(x)$$

s **pointwise convergent** *for each $x \in I$ if the limit*

$$\sum_{k=0}^{\infty} u_k(x) = \lim_{N\to\infty} \sum_{k=0}^{N} u_k(x)$$

xists for each $x \in I$.

Thus, we always define a **sequence of partial sums** $S_N(x)$ given as

$$S_N(x) = \sum_{k=0}^{N} u_k(x)$$

o that

$$S_0(x) = u_0(x), \quad S_1(x) = u_0(x) + u_1(x), \quad S_2(x) = u_0(x) + u_1(x) + u$$

nd if

$$\lim_{N\to\infty} S_N(x)$$

xists for x then we say that the series

$$\sum_{k=0}^{\infty} u_k(x) = \lim_{N\to\infty} S_N(x)$$

onverges at x. It converges pointwise on the interval I if

$$\lim_{N\to\infty} S_N(x)$$

xists for each $x \in I$.

Theorem 5.1 *Suppose that the functional series*

$$\sum_{k=0}^{\infty} u_k(x)$$

is defined on an interval I and that there is a **sequence of positive constants** M_k *so that*

$$|u_k(x)| \leq M_k, \ \ k = 0, 1, 2, \ldots$$

for all $x \in I$. If

$$\sum_{k=0}^{\infty} M_k$$

converges, then

$$\sum_{k=0}^{\infty} u_k(x)$$

converges uniformly on I.

Proof: If the conditions are fullfilled then we immediately have, from the **Comparison Theorems for Positive Series**, that, for each $x \in I$, the series

$$\sum_{k=0}^{\infty} |u_k(x)|$$

is convergent, so that

$$\sum_{k=0}^{\infty} u_k(x)$$

is absolutely convergent, and therefore convergent. This means that

$$\sum_{k=0}^{\infty} u_k(x)$$

is **pointwise convergent on** I, and we denote the limit by $S(x)$. We now show that the partial sums

$$S_N(x) = \sum_{k=0}^{N} u_k(x)$$

converges uniformly to $S(x)$ on I under the conditions of the theorem. We have

$$S(x) - S_N(x) = \sum_{k=N+1}^{\infty} u_k(x)$$

(all we do is subtract the first N terms from the series). Then it follows that

$$|S(x) - S_N(x)| \leq \sum_{k=N+1}^{\infty} |u_k(x)| \leq \sum_{k=N+1}^{\infty} M_k$$

for each $x \in I$, since $|u_k(x)| \leq M_k$ for each $x \in I$ according to our assumption. Then

$$\|S - S_N\|_I \leq \sum_{k=N+1}^{\infty} M_k.$$

We also know (by assumption) that $\sum_{k=0}^{\infty} M_k$ converges, so we must have that $\sum_{k=N+1}^{\infty} M_k \to 0$ as $N \to \infty$. Consequently,

$$\|S - S_N\|_I \to 0 \quad \text{as } N \to \infty,$$

and our result is proved.

Corollary 5.1 *If*

(i) the functional series

$$S(x) = \sum_{k=0}^{\infty} u_k(x) \quad \textit{converges uniformly on interval} \quad I,$$

(ii) $u_k(x)$ is a continuous function on I for each $k = 0, 1, 2, \dots,$

then $S(x)$ is continuous on I.

Proof: Because a finite sum of continuous functions is again a continuous function, it follows that the partial sums

$$S_N(x) = \sum_{k=0}^{N} u_k(x)$$

are continuous functions for $N = 0, 1, 2, \dots$. Then by Theorem 4.1, we have that $S(x) = \lim_{N\to\infty} S_N(x)$ is a continuous function.

Example 5.1 *Take the functional series*

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}.$$

We have

$$|u_k(x)| = \left|\frac{\sin kx}{k^2}\right| = \frac{|\sin kx|}{k^2} \leq \frac{1}{k^2}$$

since $|\sin t| \leq 1$ for all real t. We know (standard positive series) that

$$\sum_{1}^{\infty} \frac{1}{k^2}$$

converges (series of the form $\sum 1/k^{\alpha}$ converge for $\alpha > 1$ and diverge for $\alpha \leq 1$). Hence, by Weierstrass' Majorant Theorem,

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$$

converges uniformly for all x, and by Corollary 5.1 this series is a continuous function of x for all $x \in \mathbb{R}$.

Theorem 5.2 *If*

(i) the functional series

$$\sum_{k=0}^{\infty} u_k(x) \quad \textit{converges uniformly on the interval} \quad I$$

(ii) $u_k(x)$ is continuous on I for each $k = 0, 1, 2, \ldots,$

then

$$\int_a^x \left(\sum_{k=0}^{\infty} u_k(t) \right) dt = \sum_{k=0}^{\infty} \left(\int_a^x u_k(t) dt \right)$$

for all $a, x \in I$. In other words, if the series of continuous functions converges uniformly on I, then the integral of the sum is the sum of the integrals of the functions, just as in the case of a finite sum.

Proof: Put $S(t) = \sum_{k=0}^{\infty} u_k(t)$ and $S_N(t) = \sum_{k=0}^{N} u_k(t)$, then we have $S_N \to S$ uniformly on I so that

$$\lim_{N\to\infty} \int_a^x S_N(t)dt = \int_a^x \lim_{N\to\infty} S_N(t)dt = \int_a^x S(t)dt,$$

according to Theorem 4.2. Note that since $S_N(t)$ is a finite sum of functions, we see that

$$\int_a^x S_N(t)dt = \int_a^x \left(\sum_{k=0}^{N} u_k(t)\right) dt$$
$$= \sum_{k=0}^{N} \left(\int_a^x u_k(t)dt\right),$$

and we then find that

$$\int_a^x \left(\sum_{k=0}^{\infty} u_k(t)\right) dt = \sum_{k=0}^{\infty} \left(\int_a^x u_k(t)dt\right).$$

We can also say something about the differentiability of the series $\sum u_k(x)$, using Theorem 4.3 In this case, as in the previous two theorems, we replace $f_n(x)$ by $S_N(t)$ and $f(x)$ by $S(t)$. Thus, we want the following:

- $S_N(x) \to S(x)$ pointwise on I
- $S'_N(x) \to G(x)$ uniformly on I
- $S_N(x)$ is continuously differentiable for each N

and then we may conclude that $S(x)$ is continuously differentiable with $S'(x) = G(x)$. All we need is to formulate these requirements and result as follows:

Theorem 5.3 *Suppose that* $\sum_{k=0} u_k(x)$ *satisfies the following conditions:*

- $\sum_{k=0}^{\infty} u_k(x)$ *converges pointwise on* I
- $\sum_{k=0}^{\infty} u'_k(x)$ *converges uniformly on* I
- $u_k(x)$ *is continuously differentiable for each* k

Then $\sum_{k=0}^{\infty} u_k(x)$ *is continuously differentiable and*

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} u_k(x)\right) = \sum_{k=0}^{\infty} u'_k(x).$$

Proof: We have that each $S'_N(x)$ is continuous on I and

$$S_N(x) \longrightarrow S(x) \quad \text{pointwise on } I$$

$$S'_N \longrightarrow G \quad \text{uniformly on } I.$$

Then by Theorem 4.3, $S(x)$ is differentiable and $S'(x) = G(x)$ on I. In other words:

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} u_k(x)\right) = \sum_{k=0}^{\infty} u'_k(x)$$

Definition 6.1 *We say that the integral*

$$F(x) = \int_a^{\infty} f(x,y)dy$$

converges uniformly *on I if:*

(i) $F(x) = \int_a^\infty f(x,y)dy$ *converges pointwise for each* $x \in I$;

(ii) the family of functions F_R *defined as*

$$F_R(x) = \int_a^R f(x,y)dy$$

converges uniformly to F *on* I. *That is, if*

$$\|F_R - F\|_I \longrightarrow 0 \quad as \ R \to \infty.$$

A test for uniform convergence of integrals is an analogy of the Weierstrass functional series:

Theorem 6.1 *(***M-test***) Suppose*

(i) $f(x,y)$ *is continuous on* $I \times [a,\infty[$

(ii) $|f(x,y)| \le M(y)$ *for all* $x \in I$ *and* $y \in [a,\infty[$

(iii) $\int_a^\infty M(y)dy$ *converges.*

Then

$$F(x) = \int_a^\infty f(x,y)dy$$

converges uniformly on I.

Proof: We have

$$F_R(x) - F(x) = \int_R^\infty f(x,y)dy$$

from which we obtain

$$\begin{aligned} |F_R(x) - F(x)| &\le \int_R^\infty |f(x,y)|dy \\ &\le \int_R^\infty M(y)dy, \end{aligned}$$

by assumption. Consequently,

$$\|F_R - F\|_I \le \int_R^\infty M(y)dy \to 0 \quad \text{as} \ R \to \infty,$$

because

$$\int_a^\infty M(y)dy \quad \text{converges}$$

implies that

$$\int_R^\infty M(y)dy \longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

Thus, $F_R \longrightarrow F$ uniformly on I.

Now we come to proving that if $F_R \to F$ uniformly on an interval I, then F is continuous if each F_R is continuous. here, the problem is to show that $F_R(x)$ is continuous. We have the following result:

Lemma 7.1 *Suppose that* $f(x)$ *is a* **real-valued continuous function** *on the* **closed, bounded interval** $[a, b]$. *Then*

$$\sup_{x\in[a,b]} f(x) = \max\{f(x) : x \in [a, b]\}, \quad \inf_{x\in[a,b]} f(x) = \min\{f(x) : x \in [a, b]\}.$$

That is, the supremum of a continuous function over a closed, bounded interval is equal to its largest value over that interval, and the infimum is the least value of the function over the interval.

Proof: Since $f(x)$ is continuous and the interval is closed, then $f(x)$ has a largest value and a least value on the interval: there exist $x_1, x_2 \in [a, b]$ so that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$, and we now see that

$$\sup_{x\in[a,b]} f(x) = f(x_2), \quad \inf_{x\in[a,b]} f(x) = f(x_1),$$

and the result is proved.

We sketch the proof and refer to any good book on analysis for further details.

Our task is to prove that for each $x \in [c, d]$ we have

$$F_R(x+h) \longrightarrow F_R(x) \quad \text{as} \quad h \to 0.$$

Then we have

$$|F_R(x+h) - F_R(x)| \leq \int_a^R |f(x+h, y) - f(x, y)| dy.$$

Now if f is continuous on $[c, d] \times [a, R]$ it can be shown that for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f(x_0, y_0) - f(x_1, y_1)| < \epsilon$$

whenever $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} < \delta$. That is, whenever the distance between the points (x_0, y_0) and (x_1, y_1) is less than δ. This is called **uniform continuity**. Using this fact, we choose $\epsilon > 0$ (arbitrarily small) and then for each given R we find a $\delta > 0$ so that

$$|f(x+h, y) - f(x, y)| < \frac{\epsilon}{R-a}$$

whenever $|h| < \delta$. From this it follows that

$$|F_R(x+h) - F_R(x)| < \epsilon \quad \text{whenever} \quad |h| < \delta.$$

Note that we have $\epsilon > 0$ arbitrarily small, and for each such choice there is a corresponding δ. From this it follows that

$$|F_R(x+h) - F_R(x)| \longrightarrow 0 \quad \text{as} \quad h \to 0.$$

Hence $F_R(x)$ is continuous on $[c, d]$, for each choice of $R > 0$.

7.3 Cauchy's condition for uniform convergence of series

In this section we record, without proof, a result which is of some interest: **Cauchy's criterion for uniform convergence of functional series**, and we make some comments on some aspects of uniform convergence.

Theorem 7.1 *Suppose that $\{u_k(x)\}$ is a sequence of continuous functions defined on an interval I. Then the series*

$$\sum_{k=0}^{\infty} u_k(x)$$

is uniformly convergent on I if and only if for each choice of $\epsilon > 0$, however small, there exists a (corresponding) integer $N > 0$ so that

$$|u_{k+1}(x) + u_{k+2}(x) + \cdots + u_m(x)| < \epsilon$$

for all $m > k \geq N$ and for all $x \in I$. In particular, we then have, on putting $m = k + 1$,

$$|u_{k+1}(x)| < \epsilon$$

for all $k \geq N$ and all $x \in I$.

The proof of this result requires more mathematical machinery than we have at hand, and can be found in any good textbook on Mathematical Analysis.

As an illustration of the usefulness of this result we look at the series expansion of e^x. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This expansion is true for each $x \in \mathbb{R}$, so we have pointwise convergence of the series. However, we do not have uniform convergence on $\mathbb{R}$. To see this, we apply the last comment in the theorem: we need to find, for a given $\epsilon > 0$, an $N > 0$ so that **for all** $x \in \mathbb{R}$ we have

$$|u_{k+1}(x)| = \frac{|x|^{k+1}}{k!} < \epsilon$$

whenever $k \geq N$. However, if we choose any k we may choose x so that

$$\frac{|x|^{k+1}}{k!}$$

is as large as we like, contradicting the requirement for uniform convergence. Hence we do not have uniform convergence on the whole of $\mathbb{R}$. However, if we only consider $x \in [-a, a]$ for some $a > 0$ then we can prove uniform convergence of the series on this **closed, bounded interval**. This can be done using Weierstrass' Majorant Theorem. This phenomenon occurs often, and then we say that the series **converges uniformly on closed, bounded intervals** or **converges on compact sets**. Another example of this phenomenon occurs in **power series** (which are the first kind of functional series taught in elementary calculus courses). For instance, for the geometric series

$$\sum_{k=0}^{\infty} x^k$$

we have absolute convergence for $|x| < 1$ and divergence for $|x| \geq 1$. For $|x| < 1$ we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = S(x).$$

The corresponding partial sums are

$$S_N(x) = \sum_{k=0}^{N} x^k = \frac{1 - x^{N+1}}{1-x}.$$

The sequence $S_N(x)$ does not converge uniformly to $S(x)$ on the interval $]-1,1[$: we have

$$|S_N(x) - S(x)| = \frac{|x|^{N+1}}{1-x}$$

and as $x \to 1_-$ we see that $|x|^{N+1} \to 1$ and $1/(1-x) \to \infty$, so that we may make $|S_N(x) - S(x)|$ as large as we like, and it then follows that $\|S_N - S\|$ does not exist, so it is impossible for $\|S_N - S\| \to 0$ as $N \to \infty$. However, if we consider the series on the closed, bounded interval $[-a, a]$ with a fixed $0 < a < 1$, we have

$$|S_N(x) - S(x)| \leq \frac{a^{N+1}}{1-a}$$

for all $x \in [-a, a]$, and therefore

$$\|S_N - S\| = \sup_{x \in [-a,a]} |S_N(x) - S(x)| \leq \frac{a^{N+1}}{1-a} \to 0 \text{ as } N \to \infty,$$

because $0 < a < 1$ gives $a^{N+1} \to \infty$ when $N \to \infty$. So we have uniform convergence of the series **on compact subsets of** $]-1,1[$.

POSSIBLE QUESTIONS

UNIT-III PART-B (5 × 6 = 30)

1. Show that the sequence of functions {fn} converges uniformly on E iff $\forall \in > 0$,$\exists$ an integer N such that for every m, n ≥ N ; $|f_n\ (x) - f_m$ (x)|< ϵ.
2. Prove that limit of sequences of functions in $R(\propto)$ which converge uniformly on[a,b] is also in $R(\propto)$.

3. State and prove Weistrass M-test for Uniform convergence of series of functions.

4. Let series $\sum f_n$ (x) = f (x) converges uniformly on S be such that each f_n is continuous at a point x_0 of S then f is also continuous at x_0.

5. State and prove Cauchy criterion for convergence of sequences of functions.

6. If $\lim_{n\to\infty} f_n =$ f and $\lim_{n\to\infty} g_n =$ g on [a,b];define h (x) = $\int_a^x f(t)g(t)dt$ and h_n (x) =$\int_a^b f_n$(t) g_n (t) dt for each x ∈ [a,b] then prove that $h_n \to$ h uniformly on [a,b].

7. Let the sequence { f_n } is a real valued function such that f_n ∈ R (α) on [a,b] for each n ϵ N. If $f_n \to$ f uniformly on [a, b], then f ∈ R (α) on [a,b]".

PART- C(1 × 10 =10)

1. Let { f_n } br sequence of differentiable functions on[a,b].If { f_n } converges to f uniuformly, then show that $f'(x) = \lim_{n\to\infty} f_n'(x)$.
2. State and prove Arzela Theorem.
3. Let the series $\sum f_n\,(x)$ be such that {f_n } is uniformly convergent on S .Suppose{ g_n } be a sequence of real valued function such that $g_{(n+1)} \leq g_n$ (x) and uniformly bounded on S. Then show that the series $\sum f_n\,(x)$ g_n (x) converges uniformly on S.
4. Let α be of bounded variation on [a,b] and each term of the series Σ f_n is Reimann Steiltjes integrable with respect to α on [a,b] and if $\sum_{n=1}^{\infty} f_n\ (x) = f(x)\ (a \leq x \leq b)$ series converges . uniformly on [a,b], then show that $f \in R(\alpha)$ and

$$\int_a^b f\ d \propto = \sum_{n=1}^{\infty} \int_a^b f_n\ d \propto$$

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)
Subject Name: REAL ANALYSIS Subject Code: 17MMP102

UNIT III	OPTION1	OPTION2	OPTION3	OPTION4	ANSWERS
A Sequence of functions is said to boundadly convergent on T is seq	point wise convergent	it contains no limit points	it contains some limit points	it contains infinite limit points	it contains all of its limit points
A set Fis closed if	it contains all of its limit points	uncountable collection of disjoint	countable collection of disjoint	uncountable collection of disjoint	countable collection of disjoint
Every open set of real numbers is the union of	countable collection of disjoint	open intervals	closed intervals	closed intervals	open intervals
composite number n is	a prime number and n >1	Uncountable	infinite	finite	
The union of a finite or collection of countable sets is	countable	a does not belongs to S	a is not lower bound of S	a is not upper bound of S	
An element a is an minimal element of set S,then	a belongs to S	n <x	n=x	n ≠ x	
For every real number x,there is a positive integer n such that	n>x	uncountable subset	proper subset	improper subset	countable subset
Every infinite set has a	countable subset	finite	countable	uncountable	finite
Set of real numbers is bounded above is Sup S	infinite	minimal element only	maximal and minimal	no maximal no minimal	minimal element only
The half interval [0,1) have	maximal element only	finite	countable	uncountable	infinite
Set of real numbers is unbounded above is Sup S	infinite	closed	semi open	semi closed	closed
The arbitrary intersection of closed set is	open	a singleton set	a finite set	not a well defined set	not a well defined set
The set of intelligent student in a class is	a null set	sum of prime numbers	product of prime numbers	prime numbers or a product of prime numbers	prime numbers or a product of prime numbers
Every integer n>1 is	prime numbers	non ordered set	set of irrational numbers	does not satisfies principle induction	ordered set
The set of integer is	ordered set	unbounded below	unbounded above	no maximal element	bounded above
The closed interval S= [0,1] is	bounded above	1	2	empty	1
If S = [0,1) the least upper bound for S is	0	prime number	not a point of closure	non prime number	a point of closure to S
If S is a set of real numbers which is bounded below then inf S is	a point of closure to S	closed set	uncountable set	countable set	closed set
A finite set is	open set	inf E>sup E	inf E=sup E	inf E ≠ sup E	
If E is a nonemptyset then	inf E< sup E	∞	∞ (negative)	no infimum	∞ (negative)
If R is a extended real number system then inf R is	0	1	0	2	-1
The set of negative integers having least upper bound is	-1	closure of E contains non empty opensets	closure of E contains empty opensets	closure of E contains no non empty closedsets	closure of E contains no non empty opensets
The set E is nowhere dense ,if	closure of E contains no non empty opensets	upper bound	maximal element	minimal element	lower bound
The set of natural numbers has	lower bound	0	1	no maximal element	φ
Let S =[0,1) the maximal elementof S is	φ	its complement is closed set	its complement is null set	its complement is semiclosed set	its complement is closed set
If A is a non-empty open set then	its complement is open set	closed set	empty set	unbounded set	open set
the intersection finite collection of open set is	open set	finite set	unbounded set	unbounded set	open set
The set of real number R is	open set	bounded from below	bounded from above	bounded	unbounded
The set of real numbers is	unbounded	closed set	empty set	non empty set	closed set
The intersection of any collection of closed set is	open set	limit point	infinite limit point	finite limit point	limit point
An infinite set must possess a	does not have a limit point	open intervals	open	closed intervals	open
The empty set is	imperfect	open set	uncountable set	countable set	closed
single ton set { x } is	closed	closed	{ 0}	φ	open set
The union of any or collection of open sets is	φ	open	semi open	either open or closed	closed
The derived set of a set is	closed	limit point	largest limit point	no limit point	smallest limit point
Every bounded infinite set has	smallest limit point	countable	finite	infinite	countable
The set of all integers is	uncountable	countable	finite	infinite	countable
The cartesian product of two countable set is	uncountable	E " is null	E " is open	E " is closed interval	E " is closed
Let E " is the set of point of closure of E	E " is closed	semi open	closed intervals	open intervals	open set
Null set	open set	bounded above by 0 & minimal element is 0	bounded below by 1 & no maximal element	bounded above by 1 & maximal element is 1	bounded above by 1 & maximal element is 1
S=(0 , 1] is	bounded above by 1 & maximal element is 1	A - B is non empty set	A - B is closed set	A - B is empty set	A - B is open set
If A is open set and B is closed set then	A - B is open set	may be closed set	may not always be a closed set	open set	may not always be a closed set
The union of an arbitrary family of closed set	closed set	equal sequence	range set of a sequence	null sequence	range set of a sequence
The set of all distict element of a sequence is called	constant sequence	bounded below	bounded	neither bounded above nor bounded below	neither bounded above nor bounded below
$< (-1)^n n >$	bounded above	oscillates finitely	diverges	converges or oscillates finitely	converges or oscillates finitely
A bounded sequence	converges	one limit	many limit	no limit point	more than one limit
A sequence can not converge to	more than one limit	must a member of the sequence	need not be a member of the sequence	not a member of the sequence	need not be a member of the sequence
limit point of a sequence	member of the sequence	no limit point	a limit point	more than two limit point	a limit point
Everu bounded real sequence has	many limit point				

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DEPARTMENT Of MATHEMATICS

Subject : QUANTITATIVE METHODS FOR MANAGEMENT SEMESTER: I L T P C
SUBJECT CODE: 17MBAP105 CLASS : I MBA 4 1 0 4

Unit IV

Transportation problem - Mathematical formulation of Transportation problem - Initial Basic Feasible solution - Optimum solution for non degeneracy and degeneracy models - Unbalanced Transportation problems and Maximization case in Transportation problem. The Assignment problem - Mathematical formulation of Assignment problem – Hungarian method – Unbalanced Assignment problem - Maximization case in Assignment problem.

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TRANSPORTATION AND ASSIGNMENT PROBLEMS

Transportation Problems:

Introduction

Transportation deals with the transportation of a commodity (single product) from 'm' sources (origins or supply or capacity centers) to 'n' destinations (sinks or demand or requirement centers). It is assumed that

(i) Level of supply at each source and the amount of demand at each destination and
(ii) The unit transportation cost of transportation is linear.

It is also assumed that the cost of transportation is linear.

The objective is to determine the amount to be shifted from each sources to each destination such that the total transportation cost is minimum.

Note: The transportation model also can be modified to Account for multiple commodities.

1. Mathematical Formulation of a Transportation problem:

Let us assume that there are m sources and n destinations.

Let a_i be the supply (capacity) at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source *i* to destination j and x_{ij} be the number of units shifted from sources *i* to destination *j*.

Then the transportation problems can be expressed mathematically as

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints

$\sum_{j=1}^{n} x_{ij} = a_i$, $i = 1, 2, 3, \ldots\ldots m$

$\sum_{i=1}^{m} x_{ij} = b_j$, $j = 1, 2, 3, \ldots\ldots N.$

And $x_{ij} \geqq 0$, for all *i* and *j*.

Note 1: The two sets of constraints will be consistent if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(total supply) *(total demand)*

Which is the necessary and sufficient condition for a transportation problems to have a feasible solution. Problems satisfying this condition are **balanced transportation problems.**

Note 2: If $\sum a_i \neq \sum b_j$

Note 3: For any transportation problems, the coefficient of all x_{ij} in the constraints are unity.

Note 4: The objective function and the constraints being all linear, the transportation problems is a special class of linear programming problem. Therefore it can be solved by simplex method. But the number of variables being large, there will be too many calculations. So we can look for some other technique which would be simpler that the usual simplex method.

Standard transportation table:

Transportation problem is explicitly represented by the following transportation table.

Destination

		D_1	D_1	D_1	.....	D_1	.....	D_1	supply
	S_1	C_{11}	C_{12}	C_{13}		C_{1j}		C_{1n}	a_1
Source	S_1	C_{21}	C_{22}	C_{23}		C_{2j}		C_{2n}	a_2
									. .
	S_1	C_{i1}	C_{i2}			C_{ij}		C_{in}	.
	S_1	C_{m1}	C_{m2}			C_{mj}		C_{mn}	a_n
Demand		b_1	b_2	b_3	.....	.....	.....	b_n	$\sum a_i = \sum b_j$

The *mn* squares are called **cells.** The unit transportation cost c_{ij} from the i^{th} source to the j^{th} destination is displayed in the **upper left side of the (i,j)th cell.** Any feasible solution us shown in the table by entering the value of x_{ij} **in the center of the** $(i,j)^{th}$ cell. The various a's and b's are called **rim requirements.** The feasibility of a solution can be verified by summing the values if x_{ij} along the rows and down the columns.

Definition 1: A set of non-negative values x_{ij}, i=1,2……………m; j=1,2………..n. that satisfies the constrains (rim conditions and also the non-negativity restrictions) is called a **feasible solution** to the transportation problems.

Note: A balanced transportation problems will always have a feasible solution.

Definition 2: A feasible solution to a (m x n) transportation problems that contains no more than m + n-1 non-negative allocations is called a **basic feasible solution** (BFS) to the transportation problem.

Definition 3: A basic feasible solution to a (m x n) transportation problem is said to be a **non-degenerate basic feasible solution** if it contains exactly m + n-1 non-negative allocations in independent positions.

Definition 4: A basic feasible solution that contains less than m + n -1 non-negative allocations is said to be a degenerate basic feasible solution.

Definition 5: A feasible solution (not necessarily basic) is said to be an **optimal solution** if it minimize is atmost m + n -1.

Note: The number of non-basic variables in an m x n balanced transportation problem is almost m + n -1.

Note: The number of non-basic variables in an m x n balanced transportation problem is atleast mn – (m + n -1).

II. Methods for finding initial basic feasible solution

The transportation problems has a solution is and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced. If not once has to balance the transportation problems first. The way to doing this is discussed in section 7.4 page 7.40. In this section all the given transportation problems are balanced.

Method I: North west corner rule:

Step I: The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is x_{11} = min {a_1, b1}.

Case (i): If min {a_1, b1} =a_1, then put x_{11}=a_1, decrease b_1 by a_1 and move vertically to the 2nd row (i.e.,) to the cell (2, 1) cross out the first row.

Case (ii): If min {a_1, b1} =b_1, then put x_{11}=b_1, decrease a_1 by b_1 and move horizontally right (i.e.,) to the cell (2, 1) cross out the first column.

Case (iii): If min {a_1, b1} = a_1 =b_1, the put x_{11} = a_1=b_1 and move diagonally to the cell (2, 2) cross out the first row and the first column.

Step 2: Repeat the procedure until all the rim requirements are satisfied.

Method 2: Lest cost method (or) Matrix minima method (or) Lowest cost entry method:

Step 1: Identify the cell with smallest cost and allocate x_{ij} = Min {a_i,b_j}

Case (i): If min {a_i,b_j}= a_i, then put x_{ij} = a, cross out the ith row and decrease b_j by a_i, go to step(2).

Case (ii): If min $\{a_i,b_j\}= b_j$, then put $x_{ij} = b_j$, cross out the jth column and decrease a_j by b_j, go to step(2).

Case (iii): If min $\{a_i,b_j\}=a_i= b_j$, then put $x_{ij}=a_i = b_j$, cross out either ith row and jth column but not both, go to step(2).

Step 2: Repeat step (1) for the resulting reduced transportation table until all the rim requirements are satisfied.

Method 3: Vogel's approximation method (VAM) (or) Unit cost penalty method:

Step 1: Find the difference (penalty) between the smallest and next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step 2: Identify the row (or) column with large penalty. If a tie occurs, break the tie arbitrarily. Choose the sell with smallest cost in that selected row or column and allocate as much as possible to this cell and cross out the satisfied row or column and go to step (3).

Step 3: Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Example 1: Determine basic feasible solution to the following transportation problems using North West Corner Rule:

		Sin k					
		A	**B**	**C**	**D**	**E**	**Supply**
Origin	**P**	**2**	**11**	**10**	**3**	**7**	**4**
	Q	**1**	**4**	**7**	**2**	**1**	**8**
	R	**3**	**9**	**4**	**8**	**12**	**9**
Demand		**3**	**3**	**4**	**5**	**6**	

[MU. BE. Apr 94]

Solution:

Since $a_i = b_j = 21$, the given problem is balanced. ∴ There exists a feasible solution to the transportation problem.

2 **3**	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9

3 3 4 5 6

Following North West Corner rule, the first allocation is made in the cell(1,1)

Here x_{11} = min {a_1, b_1} = min {4,3} = 3

Allocate 3 to the cell(1,1) and decrease 4 by 3 i.e., 4 – 3 = 1

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 **1**	**10**	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here the North West Corner cell is (1,2).

So allocate x_{11}= min {1, 3} = 1 to the cell (1,2) and move vertically to cell (2, 2). The resulting transportation table is

4 **2**	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate x_{22}= min {8, 2} = 2 to the cell (2, 2) and move horizontally to cell (2, 3). The resulting transportation table is

7 **4**	2	1	6
4	8	12	9

4 5 6

Allocate x_{23}= min {6, 4} = 4 and move horizontally to cell (2, 4). The resulting reduced transportation table is

2 **2**	**1**	2
8	12	9
5	6	

Allocate x_{24}= min {2, 5} = 2 and move vertically to cell (3, 4). The resulting reduced transportation table is

8 **3**	**12**	9
3	6	

Allocate x_{34}= min {9, 3} = 3 and move horizontally to cell (3, 5).which is

12 **6**	6
6	

Allocate x_{35}= min {6, 6} = 6

Finally the initial basic feasible solution is as shown in the following table.

2 **3**	11 **1**	10	3	7
1	4 **2**	7 **4**	2 **2**	1
3	9	4	8 **3**	12 **6**

From this table we see that the number of positive independent allocations is equal to

$m + n - 1 = 3 + 5 – 1 = 7$. This ensures that the solution is non degenerate basic feasible.

∴The initial transportation cost = Rs. 2 x 3 + 11 x 1 + 4 x 2 + 7 x 4 + 2 x 2 + 8 x 3 + 12 x 6

= Rs. 153/-

Example 2:

Find the initial basic feasible solution for the following transportation problem by Least Cost Method.

	To				Supply
From	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Demand	20	40	30	10	

[MU. BE. Apr 95,BE. Nov 96]

Solution:

Since $\sum a_i = \sum b_j$ = 100, the given TPP is balanced. There exists a feasible solution to the transportation problem.

1 **20**	2	1	4	30
3	3	2	1	50
4	2	5	9	20
20	40	30	10	

By least cost method, min c_{ij} = c_{11} = c_{13} = c_{24} = 1

Since more than one cell having the same minimum c_{ij}, break the tie.

Let us choose the cell (1,1) and allocate x_{11}= min $\{a_1, b_1\}$ = min {30,20} = 20 and cross out the satisfied column and decrease 30 by 20.

The resulting reduced transportation table is

2	1 **10**	4	10
3	2	1	50

2	5	9	20
40	30	10	

Here min c_{ij} = c_{13} = c_{24} = 1.Choose the cell (1,3) and allocate x_{13}= min {a_1, b_3} = min {10,30} = 10 and cross out the satisfied row.

The resulting reduced transportation table is

3	2	1 **10**	50
2	5	9	20
40	20	10	

Here min c_{ij} = c_{24} = 1

∴ Allocate x_{24}= min {a_2, b_4} = min (50,10) = 10 and cross out the satisfied column.

The resulting transportation is

3	**2** **20**	40
2	**5**	20
40	20	

Here c_{ij} = c_{23} = c_{32} = 2. Choose the cell (2,3) and allocate x_{23}= min {a_2, b_3}=

min (40,20) = 10 and cross out the satisfied column.

The resulting reduced transportation table is

3	20
2 20	20
40	

Here min = c_{ij}= c_{32} = 2. Choose the cell (3,2) and allocate x_{32}= min {a_3, b_2}=

min (20,40) = 20 and cross out the satisfied row.

The resulting reduced transportation table is

3 **20**	20
20	

Finally the initial basic feasible solution is as shown in the following table.

1 **20**	2	1 **10**	4
3	3 **20**	2 **20**	1 **10**
4	2 **20**	5	9

From this table we see that the number of positive independent allocations is equal to

$m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

∴ The initial transportation = Rs. 1 x 20 + 1 x 10 + 3 x 20 + 2 x 20

Cost 1 x 10 + 2 x 20

= Rs. 20 + 10 + 60 + 40 + 10 + 40

= Rs. 180/-

Example 3:

Find the initial basic feasible solution for the following transportation problem by VAM.

Distribution centres

		D_1	D_1	D_1	D_1	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

Solution:

Since $\sum a_i = \sum b_j$ = 100, the given is balanced. ∴ There exists a feasible solution to this problem.

11 **200**	13	17	14	250	(2)
16	18	14	10	300	(4)
21	24	13	10	400	(3)
200	225	275	250		

(5) (5) (1) (0)

First let us find the difference (penalty) between the smallest and next smallest costs in each row and column and write them in brackets against the respective rows and columns.

The largest of these differences is (5) and is associated with the first two columns of the transportation table. We choose the first column arbitrarily.

In this selected column, the cell (1,1) has the minimum unit transportation cost c_{11} = 11.

∴ Allocate x_{11}= min (250,200) = 200 to this cell (1,1) and decrease 250 by 200 and cross out the satisfied column.

The resulting reduced transportation table is

13 50	17	14	50	(1)
18	14	10	300	(4)
24	13	10	400	(3)
225	275	250		
(5)	(1)	(0)		

The row and column differences are now computed for this reduced transportation table. The largest of these is (5) which is associated with the second column. Since c_{12} = 13 is the minimum cost, we allocate x_{12}= min (50,225) = 50 to the cell (1,2) and decrease 225 by 50 and cross out the satisfied row.

Continuing in this manner, the subsequent reduced transportation tables and the differences for the surviving rows and columns are shown below:

18 175	14	10	300	(4)
24	13	10	400	(3)
175	275	250		
(6)	(1)	(0)		

(i)

14	10 125	125	**(4)**
13	10	400	(3)
	250		
(1)	(0)		

(ii)

14	10 125	400
275	125	

(iii)

13 275	275
275	

(iv)

Finally the initial basic feasible solution is as shown in the following table.

11 **200**	13 **50**	17	14
16	18 **175**	14	10 **125**
21	24	13 **275**	10 **125**

From this table we see that the number of positive independent allocation is equal to

$m + n - 1 = 3 + 4 - 1 = 6$. This ensures that the solution is non degenerate basic feasible.

∴ The initial transportation = Rs. 11 x 200 + 13 x 50 + 18 x 175 +

Cost = + 10 x 125 + 13 x 275 + 10 x 125

= Rs. 12075/-

Example 4:

Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using ***(i). North West Corner rule***

(ii). Least Cost method

(iii). Vogel's approximation method.

Solution:

Since $\sum a_i = \sum b_j$ = 100, the given Transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

(i). North West Corner rule: Using this method, the allocation are shown in the tables below:

1 **7**	**2**	**6**	7
0	4	2	12
3	1	5	11
10	10	10	

(i)

0 **3**	4	2	12
3	1	5	11
3	10	10	

(ii)

4 **8**	**2**	9
1	5	11
10	10	

(iii)

1 **1**	5	11
1	10	

5 10	10
10	

(iv) ***(v)***

The starting solution is as shown in the following table

1 **7**	2	6
0 **3**	4 **9**	2
3	1 **1**	5 **10**

∴ The initial transportation cost = Rs. 1 x 7 + 0 x 3 + 4 x 9 + 1 x 1 + 5 x 10

= Rs. 94/-

(ii). **Least Cost method:** Using this method, the allocation are as shown in the table below:

1	2	6 7	7
0 10	4	2	12
3	1	5	11
10	10	10	

(i)

2	6	7
4	2	2
1 (10)	5	11
10	10	

(ii)

6	7
2 (2)	2
5	1
10	

(iii)

6	7
5 (1)	1
8	

(iv)

6 (7)	7
7	

(v)

The starting solution is as shown in the following table:

1	2	6 **7**
0 **10**	4	2 **2**
3	1 **10**	5 **1**

∴ The initial transportation cost = Rs. 6 x 7 + 0 x 10 + 2 x 2 + 1 x 10 + 5 x 1

= Rs. 61/-

(iii). Vogel's approximation Method: Using this method, the allocations are shown in the table below:

1	2	6	7	(1)
0	4	2 / 10	12	(2)
3	1	5	11	(2)
10	10	10		
(1)	(1)	**(3)**		

(i)

1	2	7	(1)
0 / 2	4	2	**(4)**
3	1	11	(2)
10	10		
(1)	(1)		

(ii)

1	2	7	(1)
3	1 / 10	11	**(2)**
8	10		
(2)	(1)		

(iii)

1 / 7	7
3	1
8	

(iv)

3 / 1	1
1	

(v)

The starting solution is as shown in the following table:

1 **7**	2	6
0 **2**	4	2 **10**
3 **1**	1 **10**	5

∴ The initial transportation cost = Rs.1 x 7 + 0 x 2 + 2 x 10 + 3 x 1 + 1 x 10

= Rs. 40/-

Note: For the above problem, the number of positive allocation in independent positions is (m + n -1) (i.e., m + n – 1 = 3 + 3 – 1 = 5). This ensures that the given problem has a non-degenerate basic feasible solution by using all the three methods. This need not be the case in all the problems.

Transportation Algorithm (or) MODI Method (modified distribution method) (Test for optimal solution).

Step 1: Find the initial basic feasible solution of the given problems by Northwest Corner rule (or) Least Cost method or VAM.

Step 2: Check the number of occupied cells. If these are less than m + n – 1, there exists degeneracy and we introduce a very small positive assignment of $\in$ ($\approx$ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to m + n – 1.

Step 3: Find the set of values u_i, v_j (i= 1,2,3,….m; j=1,2,3…n) from the relation $c_{ij} = u_i + v_j$ for each occupied cell (i,j), by starting initially with u_i =0 or v_j =0 preferably for which the corresponding row or column has maximum number of individual allocations.

Step 4: Find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding cell (i,j).

Step 5: Find the cell evaluations $d_{ij} = c_{ij} – (u_i + v_j)$ (d_{ij} = upper left – upper right) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

Step 6: Examine the cell evolutions d_{ij} for all unoccupied cells (i,j) and conclude that

(i) If all $d_{ij}>0$, then the solution under the test is optimal and unique.
(ii) If all $d_{ij}>0$, with atleast one $d_{ij}=0$, then the solution under the test is optimal and an alternative optimal solution exists.
(iii) If atleast one $d_{ij}<0$, then the solution is not optimal. Go to the next step.

Step 7: Form a new B>F>S by giving maximum allocation to the cell for which $d_{ij}$is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell for which d_{ij} is most negative and having its **other corners at some allocated cells.** Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having $-\theta$. Add this minimum allocation to the cells with $+\theta$ and subtract this minimum allocation from the allocation to the cells with $-\theta$.

Step 8: Repeat steps (2) to (6) to test the optimality of this new basic feasible solution.

Step 9: Continue the above procedure till an optimum solution is attained.

Note: The Vogels approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields better initial solution than obtained from other methods in general. This can be justified by the above example (4). So to find the initial solution, give preference to VAM unless otherwise specified.

Example 1: Solve the transportation problem:

	1	2	3	4	Supply
I	**21**	**16**	**25**	**13**	**11**
II	**17**	**18**	**14**	**23**	**13**
III	**32**	**27**	**18**	**41**	**19**
Demand	**6**	**10**	**12**	**15**	

Solution: Since $\sum a_i = \sum b_j$ =43, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is an shown in the following table.

21	16	25	13 **11**	(3)	-	-	-
17 **6**	18 **3**	14	23 **4**	(3)	(3)	(3)	(3)
32	27 **7**	18 **12**	41	(9)	(9)	(9)	(9)

(4)	(2)	(4)	**(10)**
(15)	(9)	(4)	**(18)**
(15)	(9)	(4)	
	(9)	(4)	

That is

21	16	25	13 **11**
17 **6**	18 **3**	14	23 **4**
32	27 **7**	18 **12**	41

From this table, we see that the number of non-negative independent allocations is (m + n – 1) = (3+4-1) =6. Hence the solution is non-degenerate basic feasible.

∴ The initial transportation cost.

= Rs. 13 x 11 + 17 x 6 +18 x 3 +23 x 4 + 27 x 7 + 18 x 12

= Rs. 796/-

To find the optimal solution

Consider the above transportation table. Since m+n-1=6, we apply MODI method,

Now we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij} = u_i + v_j$. As the 2nd row contains maximum number of allocations, we choose $u_2=0$.

Therefore

$C_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$

$C_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow v_2 = 18$

$C_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$

$C_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$

$C_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$

$C_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$

Thus we have the following transportation table:

21	16	25	13 **11**	$u_1 = -10$

17 **6**	18 **3**	14	23 **4**	$u_2 = 0$
32	27 **7**	18 **12**	41	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Now we find $u_i + v_j$ for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding unoccupied cell(i,j).

Then we find the cell evalutions $d_{ij} = c_{ij} - (u_i + v_j)$ (ie., upper left corner – upper right corner) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j).

21 7 14	16 8 8	25 -1 26	13 **11**	$u_1 = -10$
17 **6**	18 **3**	14 9 5	23 **4**	$u_2 = 0$
32 26 6	27 **7**	18 **12**	41 32 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij}>0$, with $d_{32} = 0$, the current solution is optimal and unique.

∴ The optimum allocation schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$, and the optimum (minimum) transportation cost

= Rs. 13x 11 + 17 x 6 + 18 x 3 +23 x 4 + 27 x 7 + 18 x 12

= Rs. 796/-

Example 2:

Obtain on optimum has feasible solution to the following transportation problem:

	To			Available
	7	3	2	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

Solution:

Since $\sum a_i = \sum b_j$ =43, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By Vogel's approximation method, the initial solution is as shown in the following table:

7	3	2 **2**	**(1) (5)**
2	1 **1**	3 **2**	(1) (1) (1)
3 **4**	4	6 **1**	(1) (3) **(3)**
(1)	**(2)**	(1)	
(1)		(1)	
(1)		(3)	

That is

7	3	2 **2**
2	1 **1**	3 **2**
3 **4**	4	6 **1**

From this table we see that the number of non-negative allocation is

$m + n - 1 = (3 + 3 - 1) = 5$.

Hence the solution is non-degenerate basic feasible

∴ The initial transportation cost = Rs. 2 x 2 + 1 x 1 + 3 x 2 + 3 x 4 + 6 x 1

= Rs. 29/-

For optimality: since the number of non – negative independent allocation is m + n -1, we apply MODI method.

Since the third column contains maximum number of allocations, we choose $v_3 = 0$.

Now we determine a set of values u_i and v_j by using the occupied cells and the relation $c_{ij} = u_i + v_j$.

That is

7 -1	3 0	2 **2**	$u_1 = 2$
2	1 **1**	3 **2**	$u_2 = 3$

3 **4**	4	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Now we find $u_i + v_j$ for each unoccupied cell (i, j) and enter at the corresponding unoccupied cell (I,j).

Then we find the cell evaluations $d_{ij} = c_{ij} – (u_i + v_j)$ for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell (i, j).

Thus we get the following table

7 -1 8	3 0 3	2 **2**	$u_1 = 2$
2 0 2	1 **1**	3 **2**	$u_2 = 3$
3 **4**	4 4 0	6 **1**	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since all $d_{ij}>0$, with $d_{32} = 0$, the current solution is optimal and there exists an alternative optimal solution.

∴ The optimum allocation schedule is given by $x_{13} = 2$, $x_{32} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$, and the optimum (minimum) transportation cost

= Rs.2x2+1x1+3x2+3x4+6x1 = Rs.29/-

Example 3: Find the optimal transportation cost of the following matrix using least cost method for finding the critical solution.

		A	B	C	D	E	Available
	P	4	1	2	6	9	100
Factory	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120

Demand	40	50	70	90	90

Solution:

Since $\sum a_i = \sum b_j = 340$, the given transportation problem is balanced.∴ There exists a basic feasible solution to this problem.

By using Least cost method, the initial solution is an shown in the following table:

4	1 **50**	2 **50**	6	9
6 **10**	4	3 **20**	5	7 **90**
5 **30**	2	6	4 **90**	8

∴ The initial transportation cost = Rs. 1 x 50 + 2 x 50 + 6 x 10 + 3 x 20 + 7 x 90

+ 5 x 30 + 4 x 90

= Rs. 1410/-

For optimality: Since the number of non – negative independent allocations is (m + n -1), we apply MODI method:

4 5 -1	1 **50**	2 **50**	6 4 2	9 6 3	u_1 = -1
6 **10**	4 2 2	3 **20**	5 5 0	7 **90**	u_2 = 0
5 **30**	2 1 1	6 2 4	4 **90**	8 6 2	u_3 = -1

$v_1 = 6 \quad v_2 = 2 \quad v_3 = 3 \quad v_4 = 5 \quad v_5 = 7$

Since $d_{11} = -1 < 0$, the current solution is not optimal.

Now let us form a new basic feasible solution by giving maximum allocation to the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here the cell (1,1) having the negative value d_{11}=-1. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $+\theta$ alternatively at the corners. We have

4 $+\theta$	1 **50**	2 **50** $-\theta$	6	9
6 **10** $-\theta$	4	3 **20** $+\theta$	5	7 **90**
5 **30**	2	6	4 **90**	8

From the two cells (1,3), (2,1) having $+\theta$, we find that the minimum of the allocations 50,10 is 10. Add this cells with $+\theta$ and subtract this 10 to the cells with$+\theta$.

Hence the new basic feasible solution is displayed in the following table:

4 **10**	1 **50**	2 **40**	6	9
6	4	3 **30**	5	7 **90**
5 **30**	2	6	4 **90**	8

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent position. So we apply MODI method.

4 **10**	1 **50**	2 **40**	6 3 3	9 6 3	$u_1 = 0$
6 5 1	4 2 2	3 **30**	5 4 1	7 **90**	$u_2 = 1$
5 **30**	2 2 0	6 3 3	4 **90**	8 7 1	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij}>0$, with $d_{32}=0$, the current solution is optimal and there exists an alternative optimal solution.

The optimum allocation schedule is given by $x_{11}=10$, $x_{12}=50$, $x_{13}=40$, $x_{23}=30$, $x_{25}=90$, $x_{31}=30$, $x_{34}=90$ and the optimum (minimum) transportation cost.

= Rs. 4 x 10 + 1 x 50 + 2 x 40 + 3 x 30 + 7 x 90 + 5 x 30 + 4 x 90.

= Rs. 1400/-

Degeneracy in Transportation Problems

In transportation problems, whenever the number of non-negative independent allocations in less than m + n – 1, the transportation problems is said to be **degenerate** one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero)to one or more empty cells of the transportation table(generally minimum cost cells if possible), so that the total number of occupied cells becomes (m + n – 1) at independent positions. WE denote this small amount by $\in$ (epsilon) satisfying the following conditions:

(i) $0 < \in < x_{ij}$, for all $x_{ij} > 0$

(ii) $X_{ij} \pm \in = x_{ij}$, for all $x_{ij} > 0$

The cells containing ∈ are then treated like other occupied cells and the problems is solved in the usual way. The∈'s are kept till the optimum solution is attained. Then we let each ∈→0.

Example 1: find the non-degenerate basic feasible solution for the following transportation problems using

- **(i) North west corner rule**
- **(ii) Least cost method**
- **(iii) Vogel's approximation method.**

		To			**supply**
	10	**20**	**5**	**7**	**10**
	13	**9**	**12**	**8**	**20**
From	**4**	**5**	**7**	**9**	**30**
	14	**7**	**1**	**0**	**40**
	3	**12**	**5**	**19**	**50**
Demand	**60**	**60**	**20**	**10**	

Solution: Since $\sum a_i = \sum b_j = 150$, the given transportation problems is balanced.

∴ There exists a basic feasible solution to this problem.

(i) The starting solution by NWC rule is an shown in the following table.

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0

3	12 **20**	5 **20**	19 **10**

Since the number of non-negative allocations at independent positions is 7 which is less than (m + n – 1) = (5 + 4 – 1) = 8, this basic feasible solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity ∈ to the unoccupied cell (5,1) so that the number of occupied cells becomes (m+n-1) (m + n – 1). Hence the non-degenerate basic feasible solution is an shown in the following table.

10 **10**	20	5	7
13 **20**	9	12	8
4 **30**	5	7	9
14	7 **40**	1	0
3 ∈	12 **20**	5 **20**	19 **10**

The initial transportation cost = Rs.

10x10+13x20+4x30+7x40+3x∈+20x20+5x20+19x10

= Rs.(1290 = 3∈)

= Rs. 1290/- as ∈→ 0.

(ii) Least cost method: Using this method the starting solution is an shown in the following table:

10	20 **10**	5	7
13	9 **20**	12	8
4 **10**	5 **20**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

Since the number of non-negative allocations at independent positions is (m + n – 1) =8, the solution is non-degenerate basic feasible.

The initial transportation cost = Rs.20x10+9x20+4x10+5x20+7x10+1x20+0x10+3x50

= Rs. 760/-

(iii) Vogel's approximation method: The starting solution by this method is an shown in the following table:

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9

14	7 **10**	1 **20**	0 **10**
3 **50**	12	5	19

Since the number of non-negative allocations is 7 which is less than (m + n – 1) = (5+4-1)=8., this basic solution is a degenerate one.

To resolve this degeneracy, we allocate a very small quantity ∈ to the unoccupied cell(5,2) so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is an shown in the following table.

10 **10**	20	5	7
13	9 **20**	12	8
4	5 **30**	7	9
14	7 **10**	1 **20**	0 **10**
3 **50**	12 ∈	5	19

∴ The initial transportation cost

= Rs. 10 x 10 + 9 x 20 + 5 x 30 + 7 x 10 + 1 x 20 + 0 x 10 + 3 x 50 + 12 x ∈

= Rs. (670 +12∈)

= Rs.670/- = as ∈→0.

Example 2: Solve the following transportation problems using vogel's method.

		A	B	C	D	E	F	Available
	1	**9**	**12**	**9**	**6**	**9**	**10**	**5**
Factory	**2**	**7**	**3**	**7**	**7**	**5**	**5**	**6**
	3	**6**	**5**	**9**	**11**	**3**	**11**	**2**
	4	**6**	**8**	**11**	**2**	**2**	**10**	**9**
Requirement		**4**	**4**	**6**	**2**	**4**	**2**	

Solution: Since $\sum a_i = \sum b_j = 22$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem. By Vogel's approximation method, the initial solution is as shown in the following table:

9	12	9 **5**	6	9	10
7	3 **4**	7	7	5	5 **2**
6 **1**	5 ∈	9 **1**	11	3	11
6 **3**	8	11	2 **2**	2 **4**	10

Since the number of non-negative allocations is 8 which is less than (m + n – 1)=(4+6-1)=9, this basic solution is degenerate one.

To resolve degeneracy, we allocate a very small quantity ∈ to the cell (3,2), so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is as shown in the following table.

9	12	9 **5**	6	9	10
7	3 **4**	7	7	5	5 **2**

6 **1**	5 ∈	9 **1**	11	3	11
6 **3**	8	11	2 **2**	2 **4**	10

The initial transportation cost = Rs.9 x 5 + 3 x 4 + 5 x 2 + 6 x 1 + 5 x ∈ + 9 x 1

+ 6 x 3 + 2 x 2 + 2 x 4

=Rs.(112+5∈) = Rs.112/-, ∈→0.

To find the optimal solution

Now the number of non-negative allocations at independent positions is (m + n – 1). We apply the MODI method.

9 6 3	12 5 7	9 **5**	6 2 4	9 2 7	10 7 3	u_1 = 0
7 4 3	3 **4**	7 7 0	7 0 7	5 0 5	5 **2**	u_2 = -2
6 **1**	5 ∈	9 **1**	11 2 9	3 2 1	11 7 4	u_3 = 0
6 **3**	8 5 3	11 9 2	2 **2**	2 **4**	10 7 3	u_4 = 0
v_1 = 6	v_2 = 5	v_3 = 9	v_4 = 2	v_5 = 2	v_6 = 7	

Since all d_{ij}>0 with d_{23}=0, the solution under the test is optimal and an alternative optimal solution is also exists.

∴ The optimum allocation schedule is given by x_{14}=5, x_{22}=4,x_{26}=2,x_{31}=1,x_{32}=∈ ,x_{33}=1,x_{41}=3,x_{44}=2,x_{45}=4 and the optimum(minimum) transportation cost is

=Rs.9x5+3x4+5x2+6x1+5x∈+9x1+6x3+2x2+2x4

=Rs. (112+5∈)

=Rs. 112 as ∈→0.

Example 3: Solve the following transportation problem to minimize the total cost of transportation.

	To				Supply
	1	**2**	**3**	**4**	**6**
From	**4**	**3**	**2**	**0**	**8**
	0	**2**	**2**	**1**	**10**
Demand	**4**	**6**	**8**	**6**	

Solution: Since $\sum a_i = \sum b_j = 24$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

1	2 **6**	3	4
4	3	2 **2**	0 **6**
0 **4**	2	2 **6**	1

4

Since the number of non-negative allocations is 5, which is less than (m + n – 1)=(3+4-1)=6, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity ∈ to the cell (1,4), so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is given in the following table

1	2 **6**	3	4
4	3	2 **2**	0 **6**

0 **4**	2 $\in$	2 **6**	1

$\therefore$ The initial transportation cost = Rs. 2 x 6 + 2 x 2 + 0 x 6 + 0 x 4 + 2 x $\in$ + 2 x 6

= Rs. (28 + 2$\in$)

= Rs. 28/-, as $\in \longrightarrow 0$.

To find the optimum solution:

Now the number of non-negative allocations at independent positions is (m + n – 1). We apply MODI method.

1 0 1	2 **6**	3 2 1	4 0 4	$u_1 = 0$
4 0 4	3 2 1	2 **2**	0 **6**	$u_2 = 0$
0 **4**	2 $\in$	2 **6**	1 0 1	$u_3 = 0$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

Since all $d_{ij} > 0$ the solution under the test is optimal and unique.

$\therefore$ The optimal allocation schedule is given by $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = \in$, $x_{33} = 6$ and the optimum (minimum) transportation cost

= Rs. 2 x 6 + 2 x 2 + 0 x 6 + 0 x 4 + 2 x $\in$ + 2 x 6

= Rs. (28 + 2$\in$) = Rs. 28, as $\in \longrightarrow 0$.

Example 5:

Solve the following transportation problem to minimize the total cost of transportation.

Destination

	1	2	3	4	supply
1	14	56	48	27	70
Origin 2	82	35	21	81	47
3	99	31	71	63	93
Demand	70	35	45	60	210

Solution:

Since $\sum a_i = \sum b_j = 210$, the given transportation problem is balanced. ∴ There exists a basic feasible solution to this problem.

By using Vogel's approximation method, the initial solution is as shown in the following table:

14 **70**	56	48	27
82	35	21 **45**	81 **2**
99	31 **35**	71	63 **58**

Since the number of non-negative allocations is 5, which is less than (m + n – 1) = (3+4-1) = 6, this basic feasible solution is degenerate.

To resolve degeneracy, we allocate a very small quantity ∈ to the cell (1,4). So that the number of occupied cells becomes ($m + n – 1$). Hence the non-degenerate basic feasible solution is given in the following table.

14 **70**	56	48	27 ∈
82	35	21 **45**	81 **2**
99	31 **35**	71	63 **58**

To find the optimum solution:

Now the number of non-negative allocations at independent positions is (m + n – 1) =6. We apply MODI method.

14 **70**	56 -5 61	48 -33 81	27 ∈	u_1 = 27
82 68 14	35 49 -14	21 **45**	81 **2**	u_2 = 81
99 50 49	31 **35**	71 3 68	63 **58**	u_3 = 63
v_1 = -13	v_2 = -32	v_3 = 60	v_4 = 0	

Since d_{22}=-14<0, the solution under the test is not optimal.

Now let us from a new basic feasible solution by giving maximum allocation to the cell (2,2) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (2,2) and having its other corners at some occupied cells. Along this closed loop, indicate +θ and -θ alternatively at the corners.

14 **70**	56	48	27 ∈
82	35 +θ	21 **45**	81 **2** -θ
99	31 -θ **35**	71	63 +θ **58**

From the two cells (2,4),(3,2) having –θ we find that the minimum of the allocations 2,35 is 2. Add this 2 to the cells with +θ and subtract this 2 to the cells with +θ. Hence the new basic feasible solution is given by

14 **70**	56	48	27 ∈
82	35 **2**	21 **45**	81
99	31 **33**	71	63 **60**

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent position. We apply MODI method for optimality.

14 **70**	56 -5 61	48 -19 81	27 ∈	u_1 = -40
82 54 28	35 **2**	21 **45**	81 67 14	u_2 = 0
99 50 49	31 **33**	71 17 54	63 **60**	u_3 = -4
v_1 = 54	v_2 = 35	v_3 = 21	v_4 = 67	

Since $d_{ij}>0$, the solution under the test is optimal.

∴ The optimal allocation schedule is given by $x_{11} = 70$, $x_{14} = \in$, $x_{22} = 2$, $x_{23} = 45$, $x_{32} = 33$, $x_{34} = 60$ and the optimum (minimum) transportation cost

=Rs. 14 x 70 + 27 x ∈ + 35 x 2 + 21 x 45 + 31 x 33 + 63 x 60

=Rs.6798/- as ∈→0.

Unbalanced Transportation Problems

If the given transportation problems is unbalanced one, i.e., if $\sum a_i \neq \sum b_j$, then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vector (zero unit transportation costs) as the case may be and then solve by usual method.

When the total supply if greater than the total demand, a dummy destination is included in the matrix with zero cost vectors. The excess supply is entered as a rim requirement for the dummy destination.

When the total demand is greater than the total supply, a dummy source is included in the matrix with zero cost vectors. The excess demand is entered as rim requirements for the dummy source.

Example 1: Solve the transportation problem

Destination

	A	B	C	D	supply
1	11	20	7	8	50
Source 2	21	16	20	12	40
3	8	12	18	9	70
Demand	30	25	35	40	

Solution: Since the total supply ($\sum a_i$ =160) is greater than the total demand ($\sum b_j$ =130), the given problem is an unbalanced transportation problem. To convert this i9nto a balanced one, we introduce a dummy destination E with zero unit transportation costs and having demand equal to 160-130=30 units.

∴ The given problem becomes

Destination

	A	B	C	D	E	supply
1	11	20	7	8	0	50
Source 2	21	16	20	12	0	40
3	8	12	18	9	0	70
Demand	30	25	35	40	30	160

By using VAM the initial solution is as shown in the following table

11	20	7 **35**	8 **15**	0

21	16	20	12 **10**	0 **30**
8 **30**	12 **25**	18	9 **15**	0

∴ The initial transportation cost

=Rs. 7 x 35 + 8 x 15 + 12 x 10 + 0 x 30 + 8 x 30 + 12 x 25 + 9 x 15

=Rs. 1160/-

For Optimality: Since the number non-negative allocations at independent position is (m + n – 1), we apply the MODI method.

11 7 4	20 11 9	7 **35**	8 **15**	0 -4 4	u_1 = 8
21 11 10	16 15 1	20 11 9	12 **10**	0 **30**	u_2 = 12
8 **30**	12 **25**	18 8 10	9 **15**	0 -3 3	u_3 = 9
v_1 = -1	v_2 = 3	v_3 = -1	v_4 = 0	v_5 = -12	

Since all d_{ij}>0, the solution under the test is optimum and unique.

∴ The optimum allocation schedule is x_{13} = 35, x_{14} = 15, x_{24} = 10, x_{25} = 30, x_{31} = 30, x_{32} = 25, x_{34} = 15

It can be noted that x_{25}=30 means that 30 units are dispatched from source 2 to the dummy destination E. In other words, 30 units are left undespatched from source 2.

The optimum (minimum) transportation cost

= Rs. 7 x 35 + 8 x 15 + 12 x 10 + 0 x 30 + 8 x 30 + 12 x 25 + 9 x 15

=Rs. 1160/-

Example 2: Solve the transportation problem with unit transportation costs, demands and supplies as given below:

Destination

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
Source S_2	11	5	2	8	55
S_3	10	12	4	7	70
Demand	85	35	50	45	

Solution: Since the total demand ($\sum b_j$ =215) is greater than the total supply ($\sum a_i$ =195), the given problem is unbalanced transportation problem. To convert this into a balanced one, we introduce a dummy source S_4 with zero unit transportation costs and having supply equal to 215-195=20 units. ∴ The given problems becomes

Destination

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
Source S_2	11	5	2	8	55
S_3	10	12	4	7	70
S_4	0	0	0	0	20
Demand	85	35	50	45	215

As this problem is balanced, there exists a basic feasible solution to this problem. By using Vogel's approximation method, the initial solution is as shown in the following table.

6 **65**	1 **5**	9	3
11	5 **30**	2 **25**	8
10	12	4 **25**	7 **45**
0 **20**	0	0	0

∴ The initial transportation cost

= Rs. 6x65+1x5+5x30+2x25+4x25+7x45+0x20

=Rs.1010/-

For optimality: Since number of non-negative allocations at independent positions is (m + n – 1), we apply the MODI method.

6 **65**	1 **5**	9 -2 11	3 1 2	u_1 = 6
11 10 1	5 **30**	2 **25**	8 5 3	u_2 = 10
10 12 -2	12 7 5	4 **25**	7 **45**	u_3 = 12
0 **20**	0 -5 5	0 -8 8	0 -5 5	u_4 = 0
v_1 = 0	v_2 = -5	v_3 = -8	v_4 = -5	

Since d_{31}=-2<0, the solution under the test is not optimal.

Now let us form a new basic feasible solution by giving maximum empty. For this, we draw a closed path consisting of horizontal and vertical lines beginning and ending at this cell (3,1) and having its other corners at some occupied cells. Along this closed loop, indicate $+\theta$ and $-\theta$ alternatively at the corners.

We have,

6 **65** $-\theta$	1 **5** $+\theta$	9	3
11	5 **30** $-\theta$	2 **25** $+\theta$	8
10 $+\theta$	12	4 **25** $-\theta$	7 **45**

0 **20**	0	0	0

From the three cells (1,1), (2,2), (3,3) having $-\theta$, we find that the minimum of the allocations 65,30,25 is 25. Add this 25 to the cells with $+\theta$ and subtract this 25 to this cells with$-\theta$. Finally, the new feasible solution is displayed in the following table.

6 **40**	1 **30**	9	3
11	5 **5**	2 **50**	8
10 **25**	12	4	7 **45**
0 **20**	0	0	0

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent positions. Now we check for optimality.

6 **40**	1 **30**	9 -2 11	3 3 0
11 10 1	5 **5**	2 **50**	8 7 1
10 **25**	12 5 7	4 2 2	7 **45**
0 **20**	0 -5 5	0 -8 8	0 -3 3

Since all d_{ij}>0 with d_{14}=0, the solution under the test is optimal and an alternative optimal solution exists.

∴ The optimum allocation schedule is given by $x_{13} = 35$, $x_{14} = 15$, $x_{24} = 10$, $x_{25} = 30$, $x_{31} = 30$, $x_{32} = 25$, $x_{34} = 15$, $x_{41} = 20$.

It can be noted that x_{41}=20 means that 20 units are dispatched from the dummy source S_4 to the destination D_1. In other words, 20 units are not fulfilled for the destination D_1.

The optimum (minimum) transportation cost

= Rs. 6 x 40 + 1 x 30 + 5 x 5 + 2 x 50 + 10 x 25 + 7 x 45 + 0 x 20

=Rs.960/-

Example 3:

Solve the transportation problem with unit transportation costs in rupees, demand and supplies as given below:

Destination

	D_1	D_2	D_3	Supply(units)
A	5	6	9	100
Origin B	3	5	10	75
C	6	7	6	50
D	6	4	10	75
Demand (units)	70	80	120	

Solution: Since the total supply ($\sum a_i$ =270), the given transportation problem is unbalanced.

To convert this into a balanced one, we introduce a dummy source D_4 with zero unit transportation costs and having demand equal to 300-270=30 units. ∴ The given problem becomes

Destination

	D_1	D_2	D_3	D_4	Supply(units)
A	5	6	9	0	100
Origin B	3	5	10	0	75
C	6	7	6	0	50
D	6	4	10	0	75
Demand (units)	70	80	120	30	300

By using VAM the initial solution is given by

5	6	9 **100**	0
3 **70**	5 **5**	10	0
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Since the number of non-negative allocations is 6, which is less than (m + n – 1) = 4+4-1 =7, this basic feasible solution is degenerate.

To resolve this degeneracy, we allocate a very small quantity ∈ to the cell (2,4), so that the number of occupied cells becomes (m + n – 1). Hence the non-degenerate basic feasible solution is given in the following table.

5	6	9 **100**	0
3 **70**	5 **5**	10	0 ∈
6	7	6 **20**	0 **30**
6	4 **75**	10	0

Now the number of non-negative allocations at independent positions is (m + n – 1). We apply MODI method.

5 6 -1	6 8 -2	9 **100**	0 3 -3	u_1 = 3
3 **70**	5 **5**	10 6 4	0 ∈	u_2 = 0

6 3 3	7 5 2	6 **20**	0 **30**	u_3 = 0
6 2 4	4 **75**	10 5 5	0 -1 1	u_4 = -1
v_1 = 3	v_2 = 5	v_3 = 6	v_4 = 0	

Since there are some $d_{ij} < 0$, the current solution is not optimal.

Since d_{14}= -3 is the most negative, let is form a new basic feasible solution is giving maximum allocations to the corresponding cell(1,4) by making an occupied cell empty. We draw a closed loop consisting of horizontal and vertical lines beginning and ending at this cell (1,4) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ Alternatively at the corners.

5	6	9 **100** $-\theta$	0 $+\theta$
3 **70**	5 **5**	10	0 ∈
6	7	6 **20** $+\theta$	0 **30** $-\theta$
6	4 **75**	10	0

From the two cells (1, 3), (3, 4) having $-\theta$, we find that the minimum of the allocations 100, 30 is 30. Add this 30 the cells with $+\theta$ and subtract this 30 to the cells with $-\theta$. Hence the new basic feasible solution is given in the following table.

5	6	9 **70**	0 **30**
3 **70**	5 **5**	10	0 ∈
6	7	6 **50**	0
6	4 **75**	10	0

We see that the above table satisfies the rim conditions with (m + n – 1) non-negative allocations at independent positions. We apply MODI method.

5 3 2	6 5 1	9 **70**	0 **30**	$u_1 = 0$
3 **70**	5 **5**	10 9 1	0 ∈	$u_2 = 0$
6 0 6	7 2 5	6 **50**	0	$u_3 = -3$
6 2 4	4 **75**	10 8 2	0 -1 1	$u_4 = -1$
$v_1 = 3$	$v_2 = 5$	$v_3 = 9$	$v_4 = 0$	

Since all $d_{ij}>0$, the current solution is optimal and unique.

The optimum allocation schedule is given by $x_{13} = 70$, $x_{14} = 30$, $x_{21} = 70$, $x_{22} = 5$, $x_{24} = \in$, $x_{33} = 50$, $x_{42} = 75$ and the optimum (minimum) transportation cost

= Rs. 9 x 70 + 0 x 30 + 3 x 70 + 5 x 5 + 0 x ∈ + 6 x 50 + 4 x 75

=Rs. 1465/-

Maximization case in Transportation Problems

So far we have discussed the transportation problems in which the objectives has been to minimize the total transportation cost and algorithms have been designed accordingly.

If we have a transportation problems where the objective is to maximize the total profit, first we have to convert the maximization problem into a minimization problem by multiplying all the entries by -1 (or) by subtracting all the entries from the highest entry in the given transportation table. The modified minimization problem can be solved in the usual manner.

Assignment Problem:

Introduction

The assignment problem is a particular case of the transportation problem in which the objective is to assign a number of tasks (Jobs or origins or sources) to an equal number of facilities (machines or persons or destinations) at a minimum cost (or maximum profit).

Suppose that we have '*n*' jobs to be performed on '*m*' machines (one Job to one machine) and our objective is to assign the jobs to the machines at the minimum cost (or maximum profit) under the assumption that each machine can perform each job but with varying degree of efficiencies.

The assignment problem can be stated in the form of *m* x *n* matrix (c_{ij}) called a cost matrix (or) Effectiveness matrix where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

	1	2	3	……………	n
1	c_{11}	c_{12}	c_{13}	……………	c_{1n}
2	c_{21}	c_{22}	c_{23}	……………	c_{2n}
Machines 3	c_{31}	c_{32}	c_{33}	……………	c_{3n}
.	…..	…..	…..	……………	…..
.					
.	…..	…..	…..	……………	…..
.					
.	…..	…..	…..	……………	…..
.					
m	c_{m1}	c_{m2}	c_{m3}	……………	c_{mn}

Mathematical formulation of an assignment problem.

Consider an assignment problem of assigning *n* jobs to *n* machines (one job to one machine). Let c_{ij} be the unit cost of assigning i^{th} nachine to the j^{th} job and

$$\text{Let } x_{ij} = \begin{cases} 1, \text{ if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, \text{ if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is then given by the following LPP

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,2,\ldots,n$$

$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\ldots,n$

and $x_{ij} = 0$ (or) 1.

Difference between the transportation problem and the assignment problem.

	Transportation problem	*Assignment problem*
(a)	Supply at any source may be any positive quantity a_i	Supply at any source (machine) will be 1 i.e., $a_i = 1$.
(b)	Demand at any destination may be any positive b_j	Demand at any destination (job) will be 1 i.e., $b_j = 1$.
(c)	One or more source to any Number of destinations	One source (machine) to only one destination (job).

Assignment Algorithm (or) Hungarian Method.

First check whether the number of rows is equal to the number of columns. If it is so, the assignment problem is said to be **balanced**. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm.

Step 1: Subtract the smallest cost element of each row from all the elements in the row of the row of the given cost matrix. See that each row contains atleast one zero.

Step 2: Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step 3: (Assigning the zeros)

(a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this enriched zero, as these will not be considered for any future assignment. Continues in this way until all the rows have been examined.

(b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

Step 4: (Apply optimal Test)

(a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

(b) It atleast one row/column is without an assignment (i.e., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows.
(a) Mark (✓) the rows that do not have assignment.
(b) Mark (✓) the columns (not already marked) that have zeros in marked columns.
(c) Mark (✓) the rows (not already marked) that have assignments in marked columns.
(d) Repeat (b) and (c) until no more marking is required.
(e) Draw lines through all unmarked rows and columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

Step 6: Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7: Repeat steps (1) to (6). Until an optimum assignment is attained.

Note 1: In case some rows or columns contain more than one zero, encircle any unmarked zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zero is left unmarked or encircled.

Note 2: The above assignment algorithm is only for minimization problems.

Note 3: If the given assignment problem is of maximization type, convert it to a minimization assignment problem by max Z = - min (-Z) and multiply all the given cost elements by -1 in the cost matrix and then solve by assignment algorithm.

Note 4: Sometimes a final cost matrix contains more than required number of zeros at independent positions. This implies that there is more than one optimal solution (multiple optimal solutions) with the same optimum assignment cost.

Example 1:

Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

Job

	1	**2**	**3**	**4**	**5**
A	**8**	**4**	**2**	**6**	**1**
B	**0**	**9**	**5**	**5**	**4**
From C	**3**	**8**	**9**	**2**	**6**

D	**4**	**3**	**1**	**0**	**3**
E	**9**	**5**	**8**	**9**	**5**

Determine the optimum assignment schedule.

Solution: The cost matrix of the given assignment problem is

$$\begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix}$$

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: Select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix

$$\begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

Step 2: select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{pmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{pmatrix}$$

Since each row and each column at least one zero, we shall make assignments in the reduced matrix.

Step 3: Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Likewise examine the columns successively. The assignments in rows and columns in the reduced matrix is given by

$$\begin{pmatrix} 7 & 3 & 0 & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & 0 & 3 \\ 4 & (0) & 2 & 4 & 0 \end{pmatrix}$$

Step 4: Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

∴ The optimal assignment schedule is given by A $\longrightarrow$ 5, B $\longrightarrow$ 1, C $\longrightarrow$ 4, D $\longrightarrow$ 3, E $\longrightarrow$ 2.

The optimum (minimum) assignment cost = (1 + 0 + 2 + 1 + 5) cost units = 9 units of cost.

Example 2:

The processing time in hours for the when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

Machines

	M_1	M_2	M_3	M_4	M_5
J_1	9	22	58	11	19
J_2	43	78	72	50	63
Jobs J_3	41	28	91	37	45
J_4	74	42	27	49	39
J_5	36	11	57	22	25

Solution:

The cost matrix of the given problem is

9	22	58	11	19
43	78	72	50	63
41	28	91	37	45
74	42	27	49	39
36	11	57	22	25

Since the number of rows is equal to the number of columns in the cost matrix, the given assignment problem is balanced.

Step 1: select the smallest cost element in each row and subtract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{pmatrix}$$

Step 2: Select the smallest cost element in each column and subtract this from all the elements of the corresponding column, we get the following reduced matrix.

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

Step 3: Now we shall examine the rows successively. Second row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Third row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. Fourth row contains a single unmarked zero, encircle this zero and cross all other zero in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns. We get

$$\begin{pmatrix} 0 & 13 & 49 & (0) & 0 \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{pmatrix}$$

Step 4: Since the 5th column do not have any assignment, the current assignment is not optimal.

Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows:

(a) Mark (✓) the rows that do not have assignment. The row 5 is marked.
(b) Mark (✓) the columns (not already marked) that have zeros in marked rows. Thus column 2 is marked.
(c) Mark the rows (not already marked) that have assignment in, marked columns. Thus row 3 is marked.
(d) Repeat (b) and (c) until no more marking is required. In the present case this repetition is not necessary.
(e) Draw lines through all unmarked rows (rows 1, 2 and 4). And marked columns (column 2). We get

$$\begin{pmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & (4) \end{pmatrix}$$

Step 6: Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered element and add this 4 to all those elements which are lying in the intersections of these straight lines and do not change the remaining elements which lie on these straight lines. We get the following matrix.

$$\begin{pmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{pmatrix}$$

Since each row and each column contains at least one zero, we examine the rows and columns successively, i.e., repeat step 3 above, we get

$$\begin{pmatrix} 0 & 17 & 49 & (0) & 0 \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & 0 & 42 & 5 & (0) \end{pmatrix}$$

In the above matrix, each row and each column contains exactly one assignment (i.e., exactly one encircled zero), therefore the current assignment is optimal.

∴ The optimum assignment schedule is $J_1 \longrightarrow M_4,\ J_2 \longrightarrow M_1, J_3 \longrightarrow M_2, j_4 \longrightarrow M_3,$

$J_5 \longrightarrow M_5$ and the optimum (minimum) processing time

= 11+43+28+27+25 hours = 134 hours.

Unbalanced Assignment Models

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problems, then the given assignment problems is said to be unbalanced.

First convert the unbalanced assignment problems in to a balanced one by adding dummy rows or dummy columns with zero cost element in the cost matrix depending upon whether m<n or m>n and then solve by the usual method.

Example 1: A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.

Machines

	1	**2**	**3**	**4**
A	**18**	**24**	**28**	**32**
B	**8**	**13**	**17**	**19**
Jobs C	**10**	**15**	**19**	**22**

What are job assignments which will minimize the cost?

Solution:

The cost matrix of the given assignment problems is

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{pmatrix}$$

Since the number of rows is less than the number of columns in the cost matrix, the given assignment problems is unbalanced.

To make it a balanced one, add a dummy job D (row) with zero cost elements. The balanced cost matrix is given by

$$\begin{pmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now select the smallest cost element in each row (column) and subtract this from all the elements of the corresponding row (columns), we get the reduced matrix

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this reduced matrix, we shall make the assignment in rows and columns having single zero. We have

$$\begin{pmatrix} (0) & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & (0) & 0 & 0 \end{pmatrix}$$

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover the all zeros by drawing a minimum number of straight lines. Choose the smallest cost element not covered by these straight lines.

$$\begin{pmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & (5) & 9 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here 5 is the smallest cost element not covered by these straight lines. Subtract this 5 from all the uncovered element, add this 5 to those elements which lie in the intersections of these straight lines and do not change the remaining element which lie on the straight lines. We get

0	1	5	9
0	0	4	6
0	0	4	7
5	0	0	0

Since each row and each column contains atleast one zero, we shall make assignment in the rows and columns having single zero. We get

(0)	1	5	9
0	(0)	4	6
0	0	4	7
5	0	(0)	0

Since there are some rows and columns without assignment, the current assignment is not optimal.

Cover all the zeros by drawing a minimum number of straight lines.

0	1	5	9
0	0	4	6
0	0	(4)	7
5	0	0	0

Choose the smallest cost element not covered by these straight line, subtract this from all the uncovered elements, add this to those elements which are in the intersection of the lines and do not change the remaining elements which lie on these straight lines. Thus we get

0	1	1	5
0	0	0	2
0	0	0	3
9	4	0	0

Since each row and each column contains atleast one zero, we shall make the assignment in the rows and columns having single zero. We get

(0)	1	1	5
0	(0)	0	2
0	0	(0)	3
9	4	0	(0)

Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

∴ The optimum assignment schedule is given by A $\longrightarrow$ 1, B $\longrightarrow$ 2, C $\longrightarrow$ 3, D $\longrightarrow$ 4 and the optimum (minimum) assignment cost

= (18+13+19+0) cost unit = 50/- units of cost

Note 1: For this problem, the alternative optimum schedule is A $\longrightarrow$ 1, B $\longrightarrow$ 2, C $\longrightarrow$ 3, D $\longrightarrow$ 4, with the same optimum assignment cost= Rs. (18+17+15+0) = 50/- units of cost.

Note 2: Here the assignment D $\longrightarrow$ 4 means that the dummy Job D is assigned to the 4^{th} Machine. It means that machine 4 is left without any assignment.

Maximization case in Assignment Problems

In an assignment problem, we may have to deal with maximization of an objective function. For example, we may have to assign persons to jobs in such a way that the total profit is maximized. The maximization problems has to be converted into an equivalent minimization problem and then solved by the usual Hungarian Method.

The conversion of the maximization problem into an equivalent minimization problems can be done by any of the following methods:

(i) Since max Z = - min (-Z), multiply all the cost element c_{ij} of the cost matrix by -1.

(ii) Subtract all the cost elements c_{ij} of the cost matrix from the highest cost element in that cost matrix.

Example:

Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

	P	Q	R	S
A	51	53	54	50
B	47	50	48	50

Jobs C	49	50	60	61
D	63	64	60	60

Solution:

The profit matrix of the given assignment problem is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem, it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit from the highest profit element 64 of this profit matrix. Thus the cost matrix of the equivalent minimization problem is

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column. We get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Since each row and each column contains atleast one zero, we shall make the assignment in rows and columns having single zero. We get

2	1	(0)	4
2	(0)	2	0
11	11	1	(0)
(0)	0	4	4

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

∴ The optimum assignment schedule is given by A ⟶ R, B ⟶ Q, C ⟶ S, D ⟶ P and the optimum (maximum) profit = Rs. (54 + 50 + 61 + 63)

= Rs. 228/-

POSSIBLE QUESTIONS

PART-B(5× 2 =10)

1.What do you understand by transportation problem?

2. Define feasible solution of a transportation problem.
3. What is the optimality test used while solving an Assignment Problem using Hungarian method?

1. What is an assignment problem? Give two applications.
2. What is the optimality test used while solving an Assignment Problem using Hungarian method?
3. Define feasible solution of a transportation problem.
4. What do you understand by transportation problem?

PART-C (5× 4 =20)

1**.** a) Find the initial basic feasible solution for the following transportation problem by VAM.

Distribution centres

		D_1	D_1	D_1	D_1	Availability
	S_1	11	13	17	14	250
Origin	S_2	16	18	14	10	300
	S_3	21	24	13	10	400
Requirements		200	225	275	250	

2). Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using (i) North West Corner rule

(ii).Least Cost method

3) Explain the procedure of Hungarian method to solve Assignment Problem.

4.Solve the following transportation problems using vogel's method.

		A	B	C	D	E	F	Available
	1	9	12	9	6	9	10	5
Factory	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	

PART- D (1× 10 =10)

1) Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

	P	Q	R	S
A	51	53	54	50
B	47	50	48	50
Jobs C	49	50	60	61
D	63	64	60	60

2) Solve the transportation problem with unit transportation costs, demands and supplies as given below:

Destination

		D_1	D_2	D_3	D_4	Supply
	S_1	6	1	9	3	70
Source	S_2	11	5	2	8	55
	S_3	10	12	4	7	70
Demand		85	35	50	45	

3.Determine basic feasible solution to the following transportation problems using North West Corner Rule:

Sink

		A	B	C	D	E	Supply
Origin	P	2	11	10	3	7	4
	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)
Subject Name: REAL ANALYSIS **Subject Code: 17MMP102**

UNIT IV	OPTION1	OPTION2	OPTION3	OPTION4	ANSWERS
Sequence [1/n] is	unbounded & convergent	decreasing sequence	monotonic sequence	oscillating sequence	bounded
The sequence { 1,0,1,0,1,…… } is	increasing sequence	unbounded	bounded below	bounded above	divergent
Every convergent sequence is	bounded	convergent	unbounded	both converges and diverges	bounded
The series 1+3+5+7+…..	divergent	convergent	bounded	divergent	comparison test
Cauchy sequence is	unbounded & convergent	root test	ratio test	leibnitz test	
Which one of the following test does not give absolute convergence series	comparison test	$\lim \text{Sup}(x_n + y_n) > \lim \text{Sup}\, x_n + \lim \text{Sup}\, y_n$	$\lim \text{Sup}(x_n + y_n) = \lim \text{Sup}\, x_n + \lim \text{Sup}\, y_n$	$\lim \text{Sup}(x_n + y_n) < \lim \text{Sup}\, x_n + \lim \text{Sup}\, y_n$	$\lim \text{Sup}(x_n + y_n) \leq \lim \text{Sup}\, x_n + \lim \text{Sup}\, y_n$
If < x n > and < y n > sequence of real number	$\lim \text{Sup}(x_n + y_n) \leq \lim \text{Sup}\, x_n + \lim \text{Sup}\, y_n$	exactly two constant sub sequence	exactly three constant sub sequence	exactly four constant sub sequence	exactly two constant sub sequence
The sequence < 1+ (-1) n > has	exactly one constant sub sequence	bounded	having a subsequence converging to 3	convergent	bounded
Let < a n > = least power of 2 that divides n.then < a n > is	divergent to infinity	convergent but not absolutely convergent	absolutely divergent	divergent	convergent but not absolutely convergent
A conditionally converges series is a series which is	absolutely convergent	bounded	not necessarily bounded	neither bounded nor unbounded	bounded
The set of limit points of a bounded sequence is	unbounded	sequence of rational numbers	sequence of irrational numbers	bounded sequence of rational numbers	sequence of real numbers
Cauchy sequence is convergent if it is a	sequence of real numbers	convergent sequencee	bounded sequencee	unboundedt sequencee	divergent sequencee
If a sequence is not a cauchy sequence then it is	divergent sequencee	bounded	not necessarily bounded	neither bounded nor unbounded	bounded
The set of limit points of a bounded sequence is	unbounded	2	3	4	2
If { x n } and { x n+1 }=√(2+x n) then the sequence { x n } converges to	1	need not be convergent	may be convergent	divergent subsequence	convergent subsequence
Every cauchy sequence contains	convergent subsequence	divergent	convergent	unbounded	bounded
The series ∑_(n=1)^∞▒〖(-1)〗 n n	bounded	infinite limit	unique limit	no limit	unique limit
Every convergent sequence is bounded and it has	finite limit	f is not Riemann integrable on [a,b]	f is Riemann integrable on R	f is integrable on R	f is Riemann integrable on [a,b]
If f : [a,b] →R is continuous and monotonic functions then	f is Riemann integrable on [a,b]	Q	[S]	{S n }	{S n }
The notation of a sequence is	S	divergent	Uniformly Convergent	does not Uniformly Convergent	Uniformly Convergent
The series ∑_(n=1)^∞(-1)〗 n 1/n	convergent	need not be measurable	measurable	may be measurable	measurable
Union of two measurable sets is	not measurable	1	infinity	2	0
Cantor ternary set is measurable and its measure is	0	uncountable	bounded	un bounded	uncountable
A set without measure different from zero is	countable	only one subcover	finite subcover	no subcover	finite subcover
A set A is said to be Compact if it has a	many subcover	closed	open as well as closed	neither open nor closed	open as well as closed
The empty set φ and whole set X	open	one limit	many limit	no limit point	no limit point
Finite sets in a metric space have	more than one limit	it ia set	It is a limit point	it is empty	it is an interval
A subset A of R is connnected if and only if	it is an interval	closed	semi open	semi closed	closed
In a metric space every singleton set { p } is	open	B - A is semi open set	B - A is closed set	B - A is empty set	B - A is closed set
If A is open set and B is closed set then	B - A is open set	unbounded	totally bounded	bounded below	totally bounded
A sequentially compact metric space is	bounded	non negative finite number	extended real number	extended rational number	non negative finite number
The total Variation on [a,b] is	non positive finite number	it is not of bounded variation of [a,b]	it is not of bounded variation of R	it is of bounded variation of R	it is of bounded variation of [a,b]
If f is absolutely continuous on [a,b]	it is of bounded variation of [a,b]	it is always a bounded variation	its never a function of bounded variation	may be a function of bounded variation	may or may not be a function of bounded variation
A continuous function is	may or may not be a function of bounded variation	atmost one cluster point	atleast one cluster point	unique cluster point	atleast one cluster point
Every infinite sequence { x n } in X has	more cluster point	A is complete metric space	undefined	need not be a complete metric space	A is incomplete metric space
If A is closed subset of a complete metric space	A is incomplete metric space	any infinite subcollection of F has empty intersection	any finite subcollection of F has empty set	any finite subcollection of F has non-empty intersection	any finite subcollection of F has empty intersection
A collection F of sets have finite intersection property if	any finite subcollection of F has empty intersection	A is complete	completment of A is closed	completment of A is open	completment of A is closed
If A is ao open subset of complete metric space X then	A is incomplete	every sequence in X is divergent	every cauchy sequence in X is convergent	every cauchy sequence in X is divergent	every cauchy sequence in X is convergant
A metric space (X , p) is complete , if	every sequence in X is convergent	closed set	empty set	non empty set	closed set
The union of any finite collection of non empty closed set is	open set	closed	open and closed	does not exist	open and closed
The empty set ϕ of a metric space is	open	not countable	may be countable	need not be a countable	not countable
The set [0,1] is	countable	C is of measure zero	C is uncountable and of measure zero	C is uncountable and of positive measure	C is uncountable and of measure zero
Let C be Cantor's middle third set then	C is not countable	1	2	3	0
The set of rational numbers lebesque outer measure is	0	infinite subcovering of F	no finite subcovering of F	no infinite subcovering of F	finite subcovering of F
If F is a closed and bounded aet of real numbers then each open covering is	finite subcovering of F	it is uncountable	it ia dense	it is perfect set	it ia dense
what is not correct about cantor ternary set	it is closed	not measurable	neither measurable nor not measurable	need not be measurable	measurable
If f is a measurable function and f =g almost everywhere,then g is	measurable	sum of an end points of the interval	Product of an end points of the interval	division of an end points of the interval	difference of an end points of the interval
The length of an interval I is	difference of an end points of the interval	every interval is measurable	every open set in R is measurable	every closed set in R is measurable	every closed set in R is measurable

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Subject : REAL ANALYSIS	**SEMESTER: I**	**L T P C**
SUBJECT CODE: 17MMP102	**CLASS : I M.Sc**	**4 0 0 4**

UNIT V

Implicit functions and extremum problems:

Introduction – Functions with non zero Jacobian determinant – Inverse function theorem – Implicit function theorem – Extrema of real valued functions of one variable and several variables.

TEXT BOOK

1. Rudin. W., (1976) .Principles of Mathematical Analysis, Mcgraw Hill, New york .

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1. Tom .M. Apostol., (2002). Mathematical Analysis, Second edition, Narosa Publishing House, New Delhi.
2. Gupta.S.L. and Gupta.N.R.,(2003).Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd, Singapore.
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2. The definition of a local maximum or minimum is as follows:

Def. f has a local maximum (minimum) at a point $x^o \in X$ if $\exists$ $N(x^o)$ such that for all $x \in N(x^o)$, $f(x) - f(x^o) < 0$.($f(x) - f(x^o) > 0$ for a minimum.)

3. The generalization of the first order condition is as follows:

Proposition 1. If a differentiable function f has a maximum or a minimum at $x^o \in X$, then $f_i(x^o) = 0$, for all i.

The n equations generated by setting each partial derivative equal to zero represent the first order conditions. If a solution exists, then they may be solved for the n solution values x_i^o.

4. As in the case of n choice variables, there are second order conditions which determine whether a critical point is a maximum or a minimum. The complication is that there is no longer one second order derivative which can be checked for negativity or positivity. In fact, there are n^2 such derivatives, $f_{ij}(x^o)$, i, j = 1,...,n. The relevant second order condition for a maximum is that

$$(*) \qquad d^2f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j < 0 \text{ for all } (dx_{1,\dots,}dx_n) \neq 0.$$

This condition is that the quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j$ is *negative definite*.

In (*), the discriminate is the Hessian matrix of f (the objective function). As discussed above, the rather cumbersome (*) condition is equivalent to a fairly simple sign condition. This is as follows:

(SOC) (max) $|PM_i|$ of $H = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & f_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ f_{n1} & \cdot & \cdot & \cdot & f_{nn} \end{bmatrix}$, evaluated at x^o, have signs $(-1)^i$.

The analogous conditions for a minimum are that

$$(**) \quad d^2f(x^o) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^o)\, dx_i dx_j > 0 \text{ for all } (dx_1, \ldots, dx_n) \neq 0,$$

meaning that $d^2f(x^o)$ is positive definite, and this condition is equivalent to

(SOC) (min) $|PM_i|$ of $H = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & f_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ f_{n1} & \cdot & \cdot & \cdot & f_{nn} \end{bmatrix}$, evaluated at x^o, have positive signs

If f satisfies the SOC for a maximum globally, then f is strictly concave. If it satisfies the SOC for a minimum globally, then f is strictly convex. For an n variable function, the definition of strict concavity reads the same: $f(\alpha x + (1-\alpha)x') > \alpha f(x) + (1-\alpha)f(x')$, $x \neq x'$, $\alpha \in (0,1)$.

Proposition 2. If at a point x^o we have

(i) $f_i(x^o) = 0$, for all i, and

(ii) SOC for a maximum (minimum) is satisfied at x^o, Then x^o is a local maximum (minimum). If in addition the SOC is met for all $x \in X$ or if f is strictly concave (convex), then x^o is a unique global maximum (minimum).

Examples: #1 Maximizing a profit function over two strategy variables. Let profit be a function of the two variables x_i, $i = 1,2$. The profit function is $\pi(x_1,x_2) = R(x_1,x_2) - \Sigma r_i x_i$, where r_i is the unit cost of x_i and R is revenue. We wish to characterize a profit maximal choice of x_i. The problem is written as

$$\operatorname*{Max}_{\{x_1,x_2\}} \pi(x_1,x_2).$$

The FOC are

$$\pi_1(x_1,x_2) = 0$$

$$\pi_2(x_1,x_2) = 0.$$

The second order conditions are

$$\pi_{11} < 0,\ \pi_{11}\pi_{22} - \pi_{12}^2 > 0 \text{ (recall Young's Theorem } \pi_{ij} = \pi_{ji}).$$

The effect of a change in r_1 can be determined by differentiating the FOC with respect to r_1. We obtain

$$H\begin{bmatrix} \partial x_1 / \partial r_1 \\ \partial x_2 / \partial r_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$ where H is the relevant Hessian.

Using Cramer's rule,

$$\partial x_1/\partial r_1 = \frac{\begin{vmatrix} 1 & \pi_{12} \\ 0 & \pi_{22} \end{vmatrix}}{|H|} = \pi_{22}/|H| < 0.$$

Likewise

$$\partial x_2/\partial r_1 = \frac{\begin{vmatrix} \pi_{11} & 1 \\ \pi_{21} & 0 \end{vmatrix}}{|H|} = -\pi_{21}/|H| .$$

The sign of π_{21} is positive if 1 and 2 are complements in profit and it is negative if they are substitutes.

#2. $\underset{\{x,y\}}{\text{Min}}\ x^2 + xy + 2y^2$. The FOC are

$$2x + y = 0,$$

$$x + 4y = 0.$$

Solving for the critical values x = 0 and y = 0. $f_{11} = 2$, $f_{12} = 1$ and $f_{22} = 4$. The Hessian is

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \text{ with } f_{11} = 2 > 0 \text{ and } |H| = 8 - 1 = 7 > 0.$$

Thus, (0,0) is a minimum. Further, it is global, because the Hessian sign conditions are met for any x,y.

Existence

In the case of a function of many variables, we want to generalize our existence argument above. To do this, we must introduce a few concepts.

Def. 1. A set $X \subset R^N$ is said to be *open* if for all $x \in X\ \exists\ N(x)$ such that $N(x) \subset X$. The set X is said to be *closed* if its complement is open.

Def. 2. A set $X \subset R^N$ is said to be *bounded* if the distance between any two of its points is finite.

That is, $[\sum_{i=1}^{n}(x_i - x_i^{'})^2]^{1/2} < \infty$, for all x,x' $\in$ X.

Def. 3 A set $X \subset R^N$ is said to be *compact* if it is both closed and bounded.

We can now state a basic existence result.

Proposition. Let f: X→ R, where X is a subset of R^N. If X is compact and f is continuous, then f has a maximum and a minimum on X. If X is both compact and convex and f is strictly concave, then f has a unique maximum on X. If X is both compact and convex and f is strictly convex, then f has a unique minimum on X.

This proposition does not distinguish between boundary optima and interior optima. As in the case of a function of single variable, the results can be used to show the existence of interior optima by showing that boundary optima are dominated. The technique is as described above.

Suppose that there are $m < n$ constraints $g_j(x) = 0$, $j = 1, \ldots, m$. The Lagrangian is written as $L(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_n) = f(x) + \Sigma\lambda_j g_j(x)$. The FOC are that the derivatives of L in x_i and λ_j vanish:

$$f_i + \Sigma\lambda_j \partial g_j/\partial x_i = 0, \text{ for all } i,$$

$$g_j(x) = 0, \text{ for all } j.$$

The bordered Hessian becomes

$$|\overline{H}| = \begin{bmatrix} \underset{m\times m}{0} & J_g \\ J_g' & L_{ij} \end{bmatrix},$$

where J_g is the Jacobian of the constraint system in x, and $[L_{ij}]$ is the Hessian of the function L in x. The condition $m < n$ must be met, and the sign conditions for a maximum and a minimum are written it terms of the principle minors of the above bordered Hessian. For a maximum, the condition is

(SOC)(max) PM_i of $|\overline{H}|$ of order $i > 2m$ has sign $(-1)^r$, where r is the order of the largest order square $[L_{ij}]$ embedded in $|PM_i|$.

For a minimum, the condition is

(SOC)(min) $|PM_i|$ of $|\overline{H}|$ of order $i > 2m$ has sign $(-1)^m$.

<u>Examples</u>:

#1. Find the critical 4-tuple for the function $y = f(x_1, x_2, x_3)$, $f: R_+^3 \to R$

$$y = x_1x_2x_3$$

subject to

$x_1^2 + x_2^2 - 1 = 0$

$x_1 - x_3 = 0.$

Restrict your choice of x_i, i = 1, 2, 3, to the positive reals.

$L = x_1x_2x_3 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(x_1 - x_3)$

(1) $x_2x_3 + 2\lambda_1x_1 + \lambda_2 = 0$

(2) $x_1x_3 + 2\lambda_1x_2 = 0$

(3) $x_1x_2 - \lambda_2 = 0$

(4) $x_1^2 + x_2^2 - 1 = 0$

(5) $x_1 - x_3 = 0$

From (5)

$x_1 = x_3$

From (3)

$x_1x_2 = x_3x_2 = \lambda_2$

Solve for λ_1 from 2

$x_1x_3 + 2\lambda_1x_2 = 0$

$x_1^2 = -2\lambda_1x_2$

$\therefore \lambda_1 = \frac{-x_1^2}{2x_2}$

Go to (1)

$x_1x_2 + 2\left(\frac{-x_1^2}{2x_2}\right)x_1 + x_1x_2 = 0$

$2x_1x_2 - \frac{x_1^3}{x_2} = 0$

Multiply by x_2/x_1 (both sides)

$$(\alpha) \quad 2x_2^2 - x_1^2 = 0$$

Use (4)

$$(\beta) \quad x_1^2 + x_2^2 - 1 = 0$$

Solve x_1, x_2:

$$(\alpha) \quad x_1^2 = 2x_2^2$$

$$\therefore \quad 2x_2^2 + x_2^2 = 1$$

$$3x_2^2 = 1$$

$$x_2^2 = 1/3$$

$$x_2 = \sqrt{1/3}$$

but

$$x_1^2 = 2x_2^2 = 2(1/3) = 2/3$$

$$x_1 = \sqrt{2/3} \qquad x_3 = \sqrt{2/3}$$

and

$$y^0 = (1/3)^{1/2}(2/3)^{1/2}(2/3)^{1/2}$$

$$y^0 = \frac{2}{3\sqrt{3}}$$

$$\therefore \quad (\sqrt{2/3},\ \sqrt{1/3},\ \sqrt{2/3},\ \tfrac{2}{3}\sqrt{3}).$$

The Problem's Bordered Hessian.

$$\left[\overline{H}\right] = \begin{bmatrix} 0 & 0 & 2x_1 & 2x_2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 2x_1 & 1 & 2\lambda_1 & x_3 & x_2 \\ 2x_2 & 0 & x_3 & 2\lambda_1 & x_1 \\ 0 & -1 & x_2 & x_1 & 0 \end{bmatrix}$$

Remark 2: The FOC (1) and (2) are necessary conditions only if the Constraint Qualification holds. This rules out particular irregularities by imposing restrictions on the boundary of the feasible set. These irregularities would invalidate the FOC (1) and (2) should the solution occur there. Let x^o be the point at which (1) and (2) hold and let index set k = 1,…, K represent the set of g_j which are satisfied with equality at x^o. Then the matrix

$$J = \begin{vmatrix} \frac{\partial g_1(x^o)}{\partial x_1} & . \quad . \quad . & \frac{\partial g_1(x^o)}{\partial x_n} \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial g_K(x^o)}{\partial x_1} & & \frac{\partial g_K(x^o)}{\partial x_n} \end{vmatrix}$$

Has rank K ≤ n. That is the gradient vectors of the set of equality constraints are linearly independent.

Proof. 1. Choose $\delta_1 > 0$ and $\epsilon_1 > 0$ so that $F_y(x, y) > 0$ for $\|x - a\| < \delta_1$, $|y - b| < \epsilon_1$. Since $F(a, b) = 0$ and $F(a, y)$ is strictly increasing in y, $F(a, b + \epsilon_1/2) > 0$ and $F(a, b - \epsilon_1/2) < 0$. Let $\epsilon = \epsilon_1/2$ and choose $\delta < \delta_1$ so that $F(x, b + \epsilon) > 0$ and $F(x, b - \epsilon) < 0$ if $\|x - a\| < \delta$. These dimensions define B. For fixed x with $\|x - a\| < \delta$, since $F(x, b - \epsilon) < 0$, $F(x, b + \epsilon) > 0$, and $F(x, y)$ is strictly increasing in y, the intermediate value theorem implies that there is a unique y with $|y - b| < \epsilon$ such that $F(x, y) = 0$. The uniquely determined y defines a function$f(x)$. This proves the first statement.

2. We prove that f is continuous at a. Let $e > 0$ be given. Assume that $e < \epsilon$ Then by the proof of the first statement, there is a $d > 0$ (we may choose $d < \delta$) so that the uniquely defined $f(x)$ in $\{\|x - a\| < d\}$ satisfies $|f(x) - b| < d$. This proves continuity at a. We can repeat this argument at any point $(a_1, f(a_1)) \in B$, proving that f is continuous on $\{\|x - a\| < \epsilon\}$.

3. By differentiability

$$\begin{aligned} 0 = F(x, f(x)) &= F(a, b) + \sum_j P_j(x, f(x))(x - a_j) + Q(x, f(x))(f(x) - f(a)) \\ &= \sum_j P_j(x, f(x))(x - a_j) + Q(x, f(x))(f(x) - f(a)), \end{aligned}$$

where $P_j(x, f(x)), Q(x, f(x))$ are continuous at a. Rewrite this as

$$Q(x, f(x))(f(x) - f(a)) = -\sum_j P_j(x, f(x))(x - a_j).$$

Since $Q(x, f(x))$ is continuous at a and $Q(a, (f(a)) = f_y(a, b) > 0$, $Q(x, f(x)) > 0$ for x near a and we can divide by it to get

$$f(x) = f(a) + -\sum_j \frac{P_j(x, f(x))}{Q(x, f(x))}(x - a_j).$$

Each term $\dfrac{P_j(x, f(x))}{Q(x, f(x))}$ is continuous at a so f is differentiable at a. Moreover

$$f_{x_j}(a, b) = -\frac{F_j(a, b)}{F_y(a, b)}.$$

You might like this bad notation:

$$\frac{\partial y}{\partial x_j} = -\frac{\partial F_{x_j}}{\partial F_y}.$$

□

The geometrical significance of the Jacobian determinant is outlined here. Consider a transformation of a single rectangular Cartesian coordinate x to a new coordinate ξ. The line element dx is transformed to the new coordinate via

$$dx = \frac{dx}{d\xi} d\xi$$

In this case, the Jacobian determinant is simply the derivative $\frac{dx}{d\xi}$.

Now, consider an area element $dx\,dy$. For convenience in later generalization, we label the coordinates (x_1, x_2). Therefore, $x_1 = x$, and $x_2 = y$. Let us make a transformation to a new set of coordinates (ξ_1, ξ_2). The area element transforms as follows.

$$dx_1\, dx_2 = \left| \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2} \right| d\xi_1\, d\xi_2$$

How did we obtain the above result? First, consider a differential change in the new variable ξ_1 while keeping the variable ξ_2 fixed. The components of the infinitesimal vector resulting from this change are

$$\left(\frac{\partial x_1}{\partial \xi_1} d\xi_1, \frac{\partial x_2}{\partial \xi_1} d\xi_1 \right)$$

In a like manner, we can write the components of the vector obtained by making a differential change in the second variable ξ_2 while keeping the variable ξ_1 fixed.

$$\left(\frac{\partial x_1}{\partial \xi_2} d\xi_2, \frac{\partial x_2}{\partial \xi_2} d\xi_2 \right)$$

These two vectors need not be orthogonal in general. Therefore, we need a result for the area of a parallelogram whose sides are the differential vectors written above. We know that this is the magnitude of the vector product (cross product) of the two vectors. This is the result given above for the area element.

POSSIBLE QUESTIONS

UNIT-V

PART-B (5 × 6 = 30)

1. State and prove implicit function theorem.

2. Find implicit function defined by relation $x + y + z - xyz = 0$ near point (0,0,0) and find derivative with respect to x and y using implicit function theorem.

3. Prove that functions $u = x + y - z$, $v = x - y + z$ and $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another .Also find relation between them.

4.State and prove functions with non Zero Jacobian determinant.

5. Define saddle point with example and Define Jacobian determinant .

6. State and prove Inverse Function Theorem.

7. prove that if $f = u + iv$ is a complex valued function with derivative at a point z in $\boldsymbol{C}$,

then $J_f(z) = |f'(z)|^2$.

PART- C (1 × 10 =10)

1. Let A be an open subset of R^n and assume that f: A→ R^n has continous partial derivatives on A. If $J_f(x) \neq 0$ for every x in A, then prove f is an open mapping.

2. Examine the following functions extreme values

(i) $f(x, y) = y^2 + 4xy + 3x^2 + x^3$

(ii)) $f(x, y, z) = 2xy^2 + 3y^2 + 4z^2 - 3xy + 8z$

3.Prove: For some integer n ≥ 1, Let f have continuous n th derivative in the open interval (a,b). Suppose for some interior point c in (a,b) , we have $f'(c)$, $f''(c) \ldots\ldots f^{(n+1)}(c) = 0$ such that $f^n(c) \neq 0$.Then for n even,f has a local minimum at c if $f^n(c) > 0$ & a local maximum at c if $f^n(c) < 0$. If n is odd ,there is neither a local minimum nor a local maximum at c.

4. State and prove Inverse function Theorem

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Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: REAL ANALYSIS **Subject Code: 17MMP102**

UNIT-V	OPTION1	OPTION2	OPTION3	OPTION4	ANSWERS		
The outer measure of its interval is its	value	length	size of the interval	modulus of its value	length		
Every increasing function of its	bounded variation	variational value	bounded	unbounded variation	bounded variation		
v (x) is monotonic---------------- function	increasing	decreasing	either increasing or decreasing	neither increasing nor decreasing	increasing		
A function f of bounded variation is the expressible as a	difference of two monotonic increasing functions	sum of two monotonic increasing functions	division of two monotonic increasing functions	product of two monotonic increasing functions	difference of two monotonic increasing functions		
Outer Lebesgue measure is also known as	Lebesgue exterior measure	measure	Lebesgue measure	Lebesgue interior measure			
If set A is Lebesgue measurable and m *(A Δ B) =---------------	0	1	2	3	0		
The sets S 1,S 2,……S n are called the -------------------- of the partition P	components	parts	partition	combined	components		
The Refinement of P is denoted by	p1	p2	p '	P *			
Every singleton set is ----------------- set	disconnected set	connected	measurable	unmeasurable	connected		
Intersection of finite number of open set is	closed	open	neither open nor not closed	semiopen	open		
Union of finite number of closed set is	closed	open	neither open nor not closed	semi closed	closed		
Every closed interval is	closed	compact	open as well as closed	not compact	compact		
In a metric space (X , d) a non-empty X is	closed	compact	open as well as closed	not compact	compact		
Every infinite set A has a	no limit point	neibourhood	limit point	need not be a limit point	limit point		
If f is a continuous mapping of a compact metric space X into M.space y	f is uniformly continuous	not uniformly continuous	continuous	discotinuous	f is uniformly continuous		
If E1 and E 2 are lebesgue measurable set then	E1 union E2 is also lebesgue measurable sets	E1 = E 2	E1 >E 2	E1 <E 2	E1 union E2 is also lebesgue measurable sets		
Let f be an open covering of A , then	there exist countable collection of F which covers A	there exist Uncountable collection of F which covers A	there exist collection of F which covers A	there exist Uncountable collection of F which not covers A	there exist countable collection of F which covers A		
If f is continuous Real valued function of Compact metric space then	f is bounded	f is unbounded	f is constant	f is a function	f is bounded		
If f is an open covering of a closed and bounded set A then	there exist a finite subcollection of F which covers A	a finite subcollection of F which covers A	there exist a infinite subcollection of F which covers A	there exist a finite collection of F which covers A	there exist a finite subcollection of F which covers A		
Any Countable set of points on the real axis line	no measure	measure is 1	measure zero	not measurable	measure zero		
$\int(-f)d\mu$	$-\int(f)d\mu$	$\int(f)d\mu$ 0	1	2	$-\int(f)d\mu$	$\int(f)d\mu$	
A real valued function φ is --------------- function if it is lebesgue measurable	large	simple	minor	major	simple		
Every open and closed set is ------------------------- measurable	zero	lebesgue	not measurable	need not be measurable	lebesgue		
Every ------------------- set is lebesgue measurable	cantor set	empty set	heine borel	non -empty set	Heine borel		
Which one of the following is true ?	Family M of Lebesgue measurable sets is an algebra of sets	Family M of Lebesgue measurable sets is not algebra of sets	Family M of Lebesgue measurable sets	Lebesgue measurable sets	Family M of Lebesgue measurable sets is an algebra of sets		
If Lebesgue outer measure of a set E, m * E= 0 then	E is measurable	E is not measurable	E is need not be measurable	neither measurable nor not measurable	E ist measurable		
Which one of the following is true ?	Lebesgue exterior measure , m * is translation is invariant	Lebesgue exterior measure , m * is not translation is invariant	need not be measurable	Lebesgue interior measure , m * is not translation is invariant	Lebesgue exterior measure , m * is translation is invariant		
If A is countable , then	m * A =0	m * A =infinity	m * A =1	m * A ≠ 0	m * A =0		
If f : [a,b] →R monotonic then	f is of bounded variation	f is unbounded	the set of discontinuity of f are uncountable	the set of continuity of f are uncountable	f is of bounded variation		
Every sequence x n in a metric space X is convergent then every cauchys sequence	convergent	divergent	constant	convergent as well as divergent	convergent		
If function f is the difference of two monotonic real valued functions on [a,b] then	f is of bounded variation	f is of not bounded variation	f is of need not bounded variation	f is of may be bounded variation	f is of bounded variation		
If x is an accumulation point of S⊂ R	every open ball B (x) contains X	every open ball B (x) contains finitely many points	every open ball B (x) does not contains many of itspoints	every open ball B (x) contains few points	every open ball B (x) contains X		
If a set S ⊂ R^ n contains all its adherent points then S is	closed	open	neither open nor not closed	not closed	closed		
If R ^ n -S is open ,then S ⊂ R ^n is	open	closed	not open	not closed	closed		
Let X is metric Space If X is sequentially compact then X is	unbounded	not compact	compact	bounded	compact		
A metric space X has the BolZana weierstrass property , if X is	sequentially compact	unbounded	not compact	compact	sequentially compact		
If we take g(x)=x and h(x) =1 in general mean value theorem we obtain	Lagrange's mean value theorem	Cauchy's mean value theorem	Rolle's theorem	Taylor's theorem	Cauchy's mean value theorem		
Any closed interval with usual metric is	compact	not compact	need not be compact	sequentially compact	not compact		
The Euclidean Space R^n is	not separable	separable	connected	disconnected	seperable		
Every dense subset is ----------------------- in l_∞	countable	uncountable	bounded	unbounded	countable		
The usual metric space (R ,d) is	seperable	not separable	not compact	compact	compact		
The set of rational numbers lebesque outer measure is =	1	0	3	4	0		
Every measurable set is nearly a finite union of	set	open sat	closed	intervals	intervals		
Everuy convergent sequence of measurable functions is nearly	convergence	divergence	uniformly convergence	absolutely convergence	uniformly convergence		

Reg No : ------------

[17MBAP105]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education

COIMBATORE – 641021

First Semester

II Internal Test Sep'17

Quantitative Methods For Management

Class: I MBA **Time :2 Hours**

Date : 27.10.17 **Maximum: 50 Marks**

PART – A (15 x 1 = 15 marks)

ANSWER ALL THE QUESTIONS

1. Two cards are drawn from a pack of 52 cards. Find the probability that both are red cards---------
   a) $26C_2$ b) $52C_4$ c) $52C_2$ d) $26C_3$
2. The mean of Binomial distribution is measured by
   a) np b) npq c) pq d) nq.
3. Optimization means-----------------------
   a) Maximization of profit or minimization of cost
   b) minimization
   c) Maximization
   d) cost
4. An assignment problem is a particular case of ---------
   a) Transportation problem b) LPP
   c) Network problem problem d) Integer programming problem
5. Assignment problem is a ---------------form of a transportation problem.
   a) non-degenerate b) feasible
   c) degenerate d) infeasible
6. In Birth–death model, the probability distribution of queue length is given by --------.
   a) $\rho^n / (1-\rho)$ b) $\rho^2 / (1-\rho)$ c) $\rho / (1-\rho)$ d) $(1-\rho) / \rho^n$
7. The probability of an empty system is given by ---------------.
   a) $1 - (\lambda / \mu)$ b) $\lambda / (\mu - \lambda)$ c) $\lambda / \mu (\mu - \lambda)$ d) λ / μ
8. A_________ is a decision of the player to always select the same strategy.
   a) competitor b)Mixed Strategy
   c) pure Strategy d) strategy
9. The variance of a binomial distribution is measured by
   a) np b) np(1 – p) c) Pq d) Nq
10. A BFS for transportation problem must have exactly-------- non-negative allocation
    a) m+n b) m+n-2 c) m+n-1 d) m+n+1
11. The arriving people in a queueing system are called --------
    a) Input b) servers c) customers d) queue
12. A loss is considered as a ______ gain.
    a) Positive b) negative c) finite d) infinite
13. If binomial distribution is symmetrical if p=q=?
    a) 1 b) 0.4 c) 0 d) 0.5
14. Assignment technique is essentially a ---------- technique
    a) Maximization b) minimization
    c) minimization or maximization d) none
15. The Birth–death model is called -----------.
    a) M / M / 1 b) M / M / N
    c) M / M / ∞ d) M / M / 2

PART-B (3 x 8 =24 Marks)

ANSWER ALL THE QUESTIONS

16. a) Three coins are tossed simultaneously. Find the probability that
i) no head ii) one head iii) two heads

(OR)

b) Two persons A and B appeared for an interview for a job. The probability of selection of A is 1/3 and that of B is ½. Find the probability that
(i) both of them will be selected
(ii) only one of them will be selected

17.a) Solve the transportation problem with unit transportation costs, demands and supplies as given below:

Destination

	D_1	D_2	D_3	D_4	Supply
S_1	6	1	9	3	70
Source S_2	11	5	2	8	55
S_3	10	12	4	7	70
Demand	85	35	50	45	

(OR)

b) Solve the assignment problem for maximization given the profit matrix (profit in rupees).

Machines

	P	Q	R	S
A	51	53	54	50
B	47	50	48	50
Jobs C	49	50	60	61
D	63	64	60	60

18.a) Find the starting solution of the following transportation model

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Using (i) North West Corner rule
(ii)Least Cost method

(OR)

b) What is the application of the Theory of Game in the contemporary business environment? Explain.

PART- C (1 X 11 = 11 Marks)

COMPULSORY

19.) A box containing 100 transistors ,20 of which are Defective ,10 are selected for insepectin ,Indicate what Is the probability that (a) all10 are defective
(b) all 10 are good (c) atleast one is defective
(d) at most 3 are defective
(i) Write the properties of Binomial Distribution.
(ii) Comment on the following “The mean of a binomial distribution is 5 and its variance is 9.

Reg. No ----------------

(17MMP102)

KARPAGAM UNIVERSITY

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE-21

DEPARTMENT OF MATHEMATICS

First Semester

I Internal Test – Aug'17

REAL ANALYSIS

Class : I M.Sc (MATHEMATICS) **Time:2 hours**

Date : 29.08.17 (FN) **Max Marks: 50 Marks**

PART – A (20 x 1 = 20 Marks)

ANSWER ALL THE QUESTIONS:

1. If P is a partition of closed interval, then || P || is the length of ………………………
   (a) largest subinterval (b) smallest subinterval
   (c) last subinterval (d) first subinterval
2. The least upper bound is called -----------
   (a) bounded above (b) bounded below
   (c) Supremum (d) infimum
3. A continous mapping ***r*** of [a,b] is said to be closed if………………………
   (a) r(a)=r(b (b) r(a)<r(b) (c) r(a)>r(b) (d) r(a)=r(b)=0
4. A function which is Reimann Steiltjes integrable w.r.t S can be denoted as
   (a) f in R(S) (b) f not in R(S) (c) f in R (d) f not in R
5. A set S is closed if and only if S = ----------
   (a). int S (b) closure of S (c)limit (d) 0
6. The set N of natural numbers is ----------
   (a) bounded (b) not bounded (c)countable (d)uncountable
7. If f is Reimann Steiltjes integrable w.r.t S, then its ……………is also similar
   (a) $|f|$ (b) f' (c) $|f|/f$ (d) $f/|f|$
8. If f : [a,b] →R is continuous and monotonic functions then
   (a) f is Riemann integrable (b) Infinite limit
   (c) Unique limit (d) No limit
9. The series 1+3+5+7+…..-----------
   (a) divergent (b) Convergent
   (c) Bounded (d) conditionally convergent
10. The series $1+ r+ r^2 +r^3 +$………… is oscillatory if
   (a) r = 1 ` (b) r = 0 (c) r = -1 (d) r = 2
11. If S = [0, 1) then sup S = ----------------
   (a) (0, 1) (b) [0,1] (c) 0 (d) 1
12. If f *and* g are Reimann Steiltjes integrable w.r.t S, then…………..is also.
   (a) $f * g$ (b) $f + g/g$
   (c) $f - g/g$ (d) f/g
13. If a sequence $\{a_n\}$ converges to a real number then the given sequence is----------
   (a) unbounded sequence (b) convergent
   (c) divergent & bounded (d) bounded
14. …………is a range of a function from N x N to R.
   (a) Double sequence (b) Double series
   (c) finite sequence (d) finite series
15. The n[th] partial sum of series Σf_n is given by-----
   (a) $f_1 - f_2 + \cdots \dots\dots\dots\dots.. + f_n$
   (b) $f_1 - f_2 - \cdots \dots\dots\dots\dots.. - f_n$
   (c) $f_1 + f_2 - \cdots \dots\dots\dots\dots.. - f_n$
   (d) $f_1 + f_2 + \cdots \dots\dots\dots.. + f_n$

16. If an unbounded sequence --------
(a) has a limit point
(b)does not have a limit point
(c) has a unique limit point
(d) may or may not have a limit point
17. Double sequence of real numbers can be denoted by…………….
(a) $(x_{m,n})$ (b) $\Sigma x_{m,n}$ (c) Σx_m (d) (x_m)
18. The sequence {1/n } is ------
(a) convergent (b)increasing sequence
(c)monotonic sequence (d)oscillating sequence
19. The set of limit points of a bounded sequence is
(a) unbounded (b) not necessarily bounded
(c) bounded (d) neither bounded nor unbounded
20. Double series of real numbers depends on -------
(a) a single parameter (b) 2 parameters
(c) 3 parameters (d) 4 parameters

PART – B (3 x 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21.Show that Reimann Steiljes integral can be reduced to a finite sum.
22.State the Root test for convergence of series
23.State and prove linearity property in Reimann Steiltjes integrals.

PART – C (3x8 = 24 Marks)

ANSWER ALL THE QUESTIONS

24. a) If f ∈ R (α) on [a, b] then α ∈ R (f) on [a,b,] prove that

$$\int_a^b f(x)d\,\alpha(x) + \int_a^b \alpha(x)d\,f(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

(OR)

b) For any f ∈ R (α) on [a,b] and g∈ R (α) on [a,b] then c_1f+c_2g ∈R (α).
we have $\int_a^b (c_1f + c_2g)d\alpha = c_1\int_a^b (fd\alpha) + c_2\int_a^b (g\,d\alpha)$.

25. a) State and prove Riemann – Stieltjes condition.

(OR)

b) Assume that α is increasing on [a,b] then prove that the following are equivalent
(i) f ∈ R (α) on [a,b]
(ii) f satisfies Riemann condition w.r.to α on [a,b]
(iii) $\underline{I}(f,\alpha) = \overline{I}(f,\alpha)$

(OR)

26.a) Let $\sum a_n$ be an absolutely convergent series f having sum S. then every rearrangement of $\sum a_n$ also converges absolutely f has sum S.

(OR)

b) State and prove Ratio Test Theorem.

Reg. No...............................

[15MMP102]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2015

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours — Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a) If $f \in R(\alpha)$ on [a,b] and $f \in R(\beta)$ on [a,b] then $f \in R(c_1 \alpha + c_2 \beta)$ on [a,b] we have $\int_a^b f d(c_1 \alpha + c_2 \beta) = c_1 \int_a^b f d\alpha + c_2 \int_a^b f d\beta$.

Or

b) State and prove change of variable in Riemann – Stieltjes integral .

22 . a) State and prove Merten's Theorem

Or

b) State and prove Ratio Test Theorem.

23. a) State and prove Cauchy's condition for Uniform convergence .

Or

b) Assume that $\Sigma f_n(x) = f(x)$ (uniformly on S) if each f_n is continuous at a point x_0 of S then f is also continuous at x_0

24. a) State and prove Lesgue dominated convergence Theorem

Or

b) Assume f is Riemann integrable on [a,b] $\forall\, b \geq a$ and assume there is a positive constant M such that $\int_a^b |f(x)|\, dx \leq M \ \forall\, b \geq a$.

25. a) Let A be an open subset of R^n and assume that $f : A \rightarrow R^n$ has continuous partial Derivatives $D_j f_i$ on A.If f is 1- 1 on A and if $J_f(x) \neq 0 \ \forall x$ in A , then f (A) is an open.

Or

b) State and prove functions with non Zero Jacobian determinant.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. State and prove Rearrangement Theorem for double sequence.

Reg. No.................................

[14MMP102]

KARPAGAM UNIVERSITY
(Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2014 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2014

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours

Maximum : 60 marks

PART – A (10 x 2 = 20 Marks)
Answer any TEN Questions

1. What do you mean by Partition?
2. Write about refinement.
3. Define Riemann stieltjes sum.
4. Define rearrangement of a series.
5. Define double series.
6. What do you mean by Cesaro summability?
7. What do you say about limit function?
8. When a series is said to be converges uniformly?
9. Define mean convergence.
10. Define Lebesgue intergrable.
11. When we say that a function f is measurable?
12. State Lebesgue bounded convergence theorem.
13. Write a brief note on Jacobian determinant.
14. Define open mapping.
15. Define quadratic form.

PART B (5 X 8= 40 Marks)
Answer ALL the Questions

16. a) If $f \in R(\alpha)$ on [a,b], then $\alpha \in R(f)$ on [a,b] and prove that

$$\int_a^b f(x)\, d\alpha(x) + \int_a^b \alpha(x)\, d f(x)\, dx = f(b)\alpha(b) - f(a)\alpha(a)$$

Or

b) Let α be increasing on [a, b]. Then show the following

i) If P' is finer than P, prove that

$U(P, f, \alpha) \le L(P, f, \alpha)$ and $L(P', f, \alpha) \ge L(P, f, \alpha)$

ii) For any two partitions P_1 and P_2, prove that $L(P_1, f, \alpha) \le U(P_2, f, \alpha)$

17. a) State and prove Riemann's theorem on conditionally convergent series.

Or

b) If $\sum a_n$ converges absolutely, then show that the subseries $\sum b_n$ also converges absolutely. Also, $|\sum_{n=1}^{\infty} b_n| \le \sum_{n=1}^{\infty} |b_n| \le \sum_{n=1}^{\infty} a_n$

18. a) State and prove Dirichlet's test for uniform convergence

Or

b) Assume that $\sum f_n(x) = f(x)$ (uniformly on S) if each f_n is continuous at a point x_0 of S, then show that f is also continuous at x_0.

19. a) Assume $f \in L(I)$ and $g \in L(I)$. Then show that

i. $(af + bg) \in L(I)$ for ever real a and b, and $\int_I (af + bg) = a \int_I f + b \int_I g$

ii. $\int_I f \ge 0$ if $f(x) \ge 0$ a.e. on I.

iii. $\int_I f \ge \int_I g$ if $f(x) \ge g(x)$ a.e. on I.

iv. $\int_I f = \int_I g$ if $f(x) = g(x)$ a.e. on I.

Or

b) State and prove Levi theorem for sequences of Lebesgue-integrable functions.

20. **Compulsory : -**

State and prove implicit function theorem.

QP-488

Reg. No...............................

[12MCP102]

KARPAGAM UNIVERSITY

(Under Section 3 of UGC Act 1956)

COIMBATORE – 641 021

(For the candidates admitted from 2012 onwards)

M.Sc. DEGREE EXAMINATION, NOVEMBER 2012

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours Maximum : 100 marks

PART – A (15 x 2 = 30 Marks)
Answer ALL the Questions

1. Write about the family of all functions of bounded variation?
2. What do you mean by compact interval?
3. What is called Partition of [a,b]?
4. Describe about limit superior?
5. Define about Strictly finer?
6. Write about lower stieltjes integral?
7. Describe about Infinite series?
8. Explain about Supremum?
9. Describe about sum of the series?
10. Describe about Converge pointwise?
11. Explain about Converges uniformly on sequence of functions?
12. Mention about power series?
13. Define about Directional derivative?
14. Describe about Gradient vector?
15. Mention about Chain rule?

PART B (5 X 14= 70 Marks)
Answer ALL the Questions

16. a) i) Let $f:[a,b]\to R^n$ and $g:[c,d]\to R^n$ be two paths in R^n ,each of which is one to one on its domain .Then prove that f and g are equivalent iff they have same graph.
ii) If f is an increasing function on[a,b] and c ε(a,b), then both f(c-) and f(c+) exist and satisfy the inequalities $f(c-)\leq f(c)\leq f(c+)$.Then f(a+) and f(b-)exist and $f(a)\leq f(a+)$ and $f(b-)\leq f(b)$.
(Or)
b) Let f be of bounded variation on[a,b].If x ε[a,b] ,let $V(x)=V_f(a,x)$ and put V(a)=0. Then show that every point of continuity of f is also a point of continuity of V and Converse is also true.

17. a) i) Let α be of bounded variation on [a,b] and assume that f ε R(α) on[a,b] .then f εR(α) On every subinterval [c,d] of [a,b].
ii) Assume that α be an increasing sequence on [a,b].if f εR(α) on[a,b]. If gεR(α) on [a,b] then the product f.g ε R(α) on [a,b].
(Or)
b) State and prove Riemann conditions.

18. a) The infinite product $\prod U_n$ converges iff for all ∈>0 there exist an integer N, Such that for all
n> N ⇒ |Un+1Un+2..........Un+k |< |Un+1 | |Un+2 |.......... |Un+k |.
(Or)
b) State and prove Merten's theorem.

19. a) Let α be of bounded variation on[a,b]. Assume that each term of the sequence $\{f_n\}$ Is a real valued function such that f_n ε R(α) on[a,b] for each n = 1, 2......... Assume That $f_n\to f$ uniformly on[a,b] and define $g_n(x)=\int_a^x f_n(t)d\alpha(t)$.
i) f ε R(α) on [a,b].
ii) $g_n\to g$ uniformly on[a,b] where $g(x)=\int_a^x f(t)d\alpha(t)$.
(Or)
b) i) State and prove Cauchy condition for uniform convergence.
ii) Assume that $f_n\to f$ uniformly on S . If each f_n is continuous at a point c of S, then show that the limit function f is also continuous at c.

20. a) i) Let A be an open subset of R^n and assume that $f: A\to R^n$ is continuous and has finite partial derivatives Dj f_i on A .If f is one to one on A and if $J_f(x)\neq 0$ for each x in A ,then prove that f(A) is open .
ii) Assume that $f=(f_1,f_2,.........f_n)$ has continuous partial derivatives D_jf_i on an open set S in R^n , nad that the Jacobian determinant $J_f(a)\neq 0$ for some point a in S . Then there is an n-ball B(a) on which f is one- to-one.
(Or)
b) State and prove second derivative test for extrema.
