KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act1956)

Coimbatore – 641 021.

DEPARTMENT OF MATHEMATICS

SEMESTER-II

18CSU202/18ITU202	**DISCRETE STRUCTURE**	**Semester – II** **4H – 4C**

Instruction Hours / week: L: 4 T: 0 P: 0 **Marks:** Int : **40** Ext : **60** Total: **100**

COURSE OBJECTIVES

- To provides a deep knowledge to the learners to develop and analyze algorithms as well as enable them to think about and solve problems in new ways.
- To express ideas using mathematical notation and solve problems using the tools of mathematical analysis.

COURSE OUTCOME

On successful completion of the course,students will be able to

1.Familiar with elementary algebraic set theory

2.Acquire a fundamental understanding of the core concepts in growth of functions.

3.Describe the method of recurrence relations

4.get wide knowledge about graphs and trees

5.initiate to knowledge from inference theory

UNIT I

Sets: Introduction, Sets , finite and infinite sets, uncountably infinite sets, functions, relations, properties of binary relations, closure, partial ordering relations, counting , Pigeonhole principle, Permutation and Combination, Mathematical Induction, Principle of inclusion and Exclusion.

UNIT II

Growth of Functions: Asymptotic Notations, Summation formulas and properties, Bounding Summations, approximation by Integrals

UNIT III

Recurrences: Recurrence relations, generating functions, linear recurrence relations with constant coefficients and their solution, Substitution Method, recurrence trees, Master theorem.

UNIT IV

Graph Theory : Basic terminology, models and types, multigraphs and weighted graphs, graph representation, graph isomorphism, connectivity, Euler and Hamiltonian Paths and circuits, Planar graphs, graph coloring, trees, basic terminology and properties of trees, introduction to Spanning trees

UNIT V

Prepositional Logic: Logical Connectives, Well-formed Formulas, Tautologies, Equivalences, Inference Theory.

SUGGESTED READINGS

1. Kenneth Rosen. (2006). Discrete Mathematics and Its Applications (6th ed.). New Delhi: McGraw Hill.
2. Tremblay , J .P. , & Manohar, R. (1997). Discrete Mathematical Structures with Applications to Computer Science. New Delhi: McGraw-Hill Book Company.
3. Coremen, T.H., Leiserson, C.E. , & R. L. Rivest. (2009). Introduction to algorithms, (3rd ed.). New Delhi: Prentice Hall on India.
4. Albertson, M. O.,& Hutchinson, J. P. (1988). Discrete Mathematics with Algorithms . New Delhi: John wiley Publication.
5. Hein, J. L. (2009). Discrete Structures, Logic, and Computability(3rd ed.). New Delhi: Jones and Bartlett Publishers.
6. Hunter, D.J. (2008). Essentials of Discrete Mathematics. New Delhi: Jones and Bartlett Publishers.

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: M.SANGEETHA
SUBJECT NAME: DISCRETE STRUCTURES SUB.CODE:18CSU202
SEMESTER: II CLASS: I B.Sc CS B

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
UNIT-I			
1	1	Introduction, Sets , finite and infinite sets, uncountably infinite sets.	S2:chapter-2,Pg.No:104-114
2	1	functions, relations, properties of binary relations	S2:chapter-2,Pg.No:148-155
3	1	closure, partial ordering relations, counting.	S2:chapter-2,Pg.No:183-191
4	1	Pigeonhole principle	S2:chapter-2,Pg.No:192-197
5	1	Permutation and Combination	S1: chapter -4 Pg.No:313-318
6	1	Mathematical Induction	S1: chapter -4 Pg.No:320-326
7	1	Principle of inclusion and Exclusion.	S6: chapter -5 Pg.No:-172-181
8	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit 1=8		
UNIT-II			
1	1	Introduction to growth of functions	S3: chapter -3 Pg.No:44-46
2	1	Asymptotic Notations	S3: chapter -3 Pg.No:47-50
3	1	Continuation on Asymptotic notations	S3: chapter -3 Pg.No:51-53

4	1	Summation- Definition and basic concepts	W_1:Staff.ust.edu.cn/ch3
5	1	Properties of Summation-problems	W_1:Staff.ust.edu.cn/ch3
6	1	Bounding Summation with Examples	W_1:Staff.ust.edu.cn/ch3
7	1	Approximation by integrals-Problems	W_1:Staff.ust.edu.cn/ch3
8	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit II=8		
		UNIT-III	
1	1	Recurrence relations-Definition and basic concepts	S1: chapter -7 Pg.No:449-455
2	1	Problems on Generation functions	S1: chapter -7 Pg.No:484 - 490
3	1	Linear recurrence relation with constant coefficient	S1: chapter -7 Pg.No:460-470
4	1	Problems on Substitution method	S3: chapter -4 Pg.No: 83-88
5	1	Problems on Recurrence tree	S3: chapter -4 Pg.No: 88-92
6	1	Master theorem	S3: chapter -4 Pg.No: 96-99
7	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit III= 7		
		UNIT-IV	
1	1	Introduction to Graph theory Basic terminology, models and types, multigraphs and weighted graphs,	S1: chapter -9 Pg.No:589-595
2	1	Graph Representation and isomorphism of graphs	S1: chapter -8 Pg.No:611-620
3	1	Connectivity- Definition and theorems	S1: chapter -8 Pg.No:621-630
4	1	Euler’s and Hamiltonian paths	S1: chapter -8 Pg.No:633-645
5	1	Planner graph-theorem	S1: chapter -8 Pg.No:657-665
6	1	Graph coloring-Definition and theorems	S1: chapter -8 Pg.No:666-674

7	1	Tree and its Properties, Spanning trees	S1: chapter -9 Pg.No:724-735
8	1	Recapitulation and Discussion of possible questions	
	Total No of Hours Planned For Unit IV=8		
		UNIT-V	
1	1	Introduction to Statement and Notation Logical Connectives	S6: chapter -1 Pg.No:2-6 S4: chapter-1 Pg. No: 11 -12
2	1	Well formed formulae	S5: chapter -7 Pg.No:356-358
3	1	Tautologies-Problems	S2: chapter -1 Pg.No:24-25
4	1	Equivalence of formulae-Problems	S5: chapter -7 Pg.No:368-373
5	1	Theory of Inference	S2: chapter -1 Pg.No:65-67
6	1	Recapitulation and Discussion of possible questions	
7	1	Discuss on Previous ESE Question Papers	
8	1	Discuss on Previous ESE Question Papers	
9	1	Discuss on Previous ESE Question Papers	
	Total No of Hours Planned for unit V=9		
Total Planned Hours	**40**		

SUGGESTED READINGS

1. Kenneth Rosen. (2012). Discrete Mathematics and Its Applications (6th ed.). New Delhi: McGraw Hill.
2. Tremblay , J .P. , & Manohar, R. (1997). Discrete Mathematical Structures with Applications to Computer Science. New Delhi: McGraw-Hill Book Company.
3. Coremen, T.H., Leiserson, C.E. , & R. L. Rivest. (2009). Introduction to algorithms, (3rd ed.). New Delhi: Prentice Hall on India.
4. Albertson, M. O.,& Hutchinson, J. P. (1988). Discrete Mathematics with Algorithms . New Delhi: John wiley Publication.

5. Hein, J. L. (2009). Discrete Structures, Logic, and Computability(3rd ed.). New Delhi: Jones and Bartlett Publishers.

6. Hunter, D.J. (2017). Essentials of Discrete Mathematics. New Delhi: Jones and Bartlett Publishers.

UNIT – I

Sets: Introduction, Sets , finite and infinite sets, uncountably infinite sets, functions, relations, properties of binary relations, closure, partial ordering relations, counting , Pigeonhole principle, Permutation and Combination, Mathematical Induction, Principle of inclusion and Exclusion.

KAHE

1. Introduction

Set: any collection of objects (individuals)

Naming sets: A, B, C,….

Members of a set: the objects in the set

Naming objects: a, b, c, ….

Notation: Let A be a set of 3 letters a, b, c.
We write A = {a, b, c}
a is a member of **A**, **a** is in **A**, we write **a ∈ A**
d is not a member of **A,** we write **d ∉ A**

Important:
1. {a}≠ a
   {a} - a set consisting of one element a.
   a - the element itself

2. A set can be a member of another set:
   B = { 1, 2, {1}, {2}, {1,2}}

Finite sets: finite number of elements
Infinite sets: infinite number of elements
Cardinality of a finite set **A**: the number of elements in **A**: #A, or |A|

Describing sets:

a. by enumerating the elements of A:
   for finite sets: {red, blue, yellow}, {1,2,3,4,5,6,7,8,9,0}
   for infinite sets we write: {1,2,3,4,5,….}

b. by property, using predicate logic notation
   Let P(x) is a property, D - universe of discourse

The set of all objects in D, for which P(x) is true, is :

$$A = \{x \mid P(x)\}$$

we read: A consists of all objects x in D such that P(x) is true

c. by recursive definition, e.g. sequences

KAHE

Examples:

1. The set of the days of the week:

{Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

2. The set of all even numbers :

$\{ x \mid \text{even}(x) \}$
$\{ 2,4,6,8,\ldots\}$

3. The set of all even numbers, greater than 100:

$\{ x \mid \text{even}(x) \wedge x > 100\}$
$\{ 102, 104, 106, 108,\ldots\}$

4. The set of integers defined as follows:
$a_1 = 1$, $a_{n+1} = a_n + 2$ (the odd natural numbers)

Universal set: **U** - the set of all objects under consideration

Empty set: Ø set without elements.

2. Relations between sets

2.1. Equality

Let **A** and **B** be two sets.
We say that **A** is equal to **B**, **A = B** if and only if they **have the same members**.

Example:

$A = \{2,4,6\}$, $B = \{2,4,6\}$ $A = B$

$A = \{a, b, c\}$, $B = \{c, a, b\}$ $A = B$

$A = \{1,2,3\}$, $B = \{1,3,5\}$, $A \neq B$

Written in predicate notation:

$$A = B \text{ if and only if } \forall x, x \in A \leftrightarrow x \in B$$

2.2. Subsets

The set of all numbers contains the set of all positive numbers. We say that the set of all positive numbers is a **subset** of the set of all numbers.

Definition: A is a subset of **B** if all elements of **A** are in **B.** However **B** may contain elements that are not in **A**

Notation: $A \subseteq B$
Formal definition:

$$A \subseteq B \text{ if and only if } \forall x, x \in A \rightarrow x \in B$$

Example: $A = \{2,4,6\}$, $B = \{1,2,3,4,5,6\}$, $A \subseteq B$

Definition: if **A** is a subset of **B, B** is called a **superset** of **A.**

Other definitions and properties:

a. If $A \subseteq B$ and $B \subseteq A$ then $A = B$

If **A** is a subset of **B**, and B is a subset of **A**, **A** and B are equal.

b. Proper subsets: **A** is a proper subset of **B**, $A \subset B$, if and only if **A** is a subset of **B** and there is at least one element in **B** that is not in **A**.

$$A \subset B \text{ iff } \forall x, x \in A \rightarrow x \in B \wedge \exists x, x \in B \wedge x \notin A$$

The above expression reads:

A is a proper subset of B if and only if all x in A are also in B and there is an element x in B that is not in A.

iff means if and only if

The empty set Ø is a subset of all sets.
All sets are subsets of the universal set **U**.

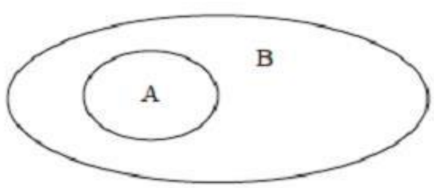
2.3. Disjoint sets

Definition: Two sets A and B are disjoint if and only if they have no common elements

A and B are disjoint if and only if $\sim\exists\, x,\ (x \in A) \wedge (x \in B)$
i.e. $\forall\, x,\quad x \notin A \vee x \notin B$

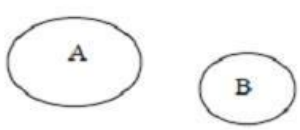
If two sets are not disjoint they have common elements.

Picturing sets: **Venn diagrams** - used to represent relations between sets

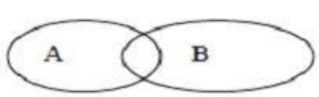


A is a (proper) subset of B

All elements in the set A are also elements in the set B



Disjoint sets



Not disjoint sets

3. Operations on sets

3.1. Intersections

The set of all students at Simpson and the set of all majors in CS have some elements in common - the set of all students in Simpson that are majoring in CS. This set is formed as the intersection of all students in CS and all students at Simpson.

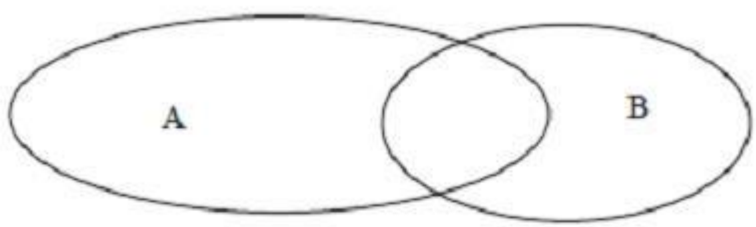
Definition: Let **A** and **B** are two sets. The set of all elements common to **A** and **B** is called the intersection of **A** and **B**

Notation: $\mathbf{A \cap B}$

Formal definition:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Venn diagram:



Example: $\mathbf{A = \{2,4,6\}, B = \{1,2,5,6\}, \quad A \cap B = \{2,6\}}$

Other properties:

$A \cap B \subseteq A, \quad A \cap B \subseteq B$

The intersection of two sets A and B is a subset of A, and a subset of B

$A \cap \emptyset = \emptyset$ The intersection of any set A with the empty set is the empty set
$A \cap U = A$ The intersection of any set A with the universal set is the set A itself.

Intersection corresponds to conjunction in logic.

Let $A = \{x \mid P(x)\}$, $B = \{x \mid Q(x)\}$

$A \cap B = \{x \mid P(x) \wedge Q(x)\}$

3. 2. Unions

The set of all rational numbers and the set of all irrational numbers taken together form the set of all real numbers - as a **union** of the rational and irrational numbers.

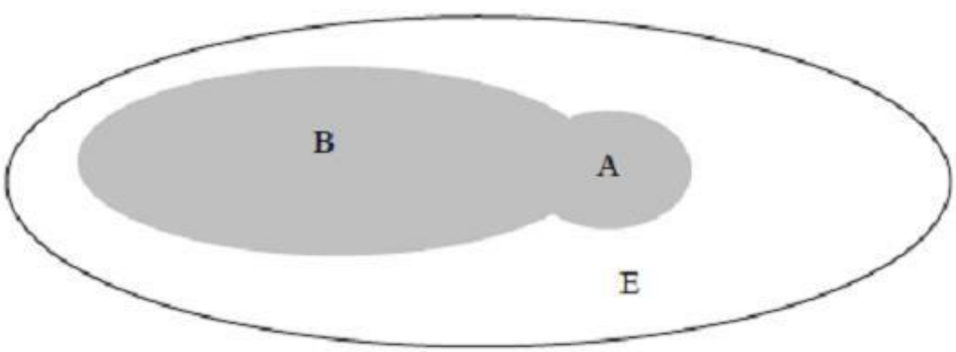
All classes at Simpson consist of students. If we take the elements of all classes, we will get all students - as the union of all classes.

Definition: The union of two sets A and B consists of all elements that are in A combined with all elements that are in B.
(note that an element may belong both to A and B)

Notation: $A \cup B$
Formal definition:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

Venn diagram:



Example: $A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5,6\}$, $A \cup B = \{1,2,3,4,5,6,8,10\}$
$A \cup B$ contains all elements in A and B without repetitions.

Other properties of unions:

$A \subseteq A \cup B$ $B \subseteq A \cup B$

A is a subset of the union of A and B,
B is a subset of the union of A and B

$A \cup \varnothing = A$ The union of any set A with the empty set is A
$A \cup U = U$ The union of any set A with the universal set E is the universal set.

Union corresponds to disjunction in logic.
Let $A = \{x \mid P(x)\}$, $B = \{x \mid Q(x)\}$
$A \cup B = \{x \mid P(x) \vee Q(x)\}$

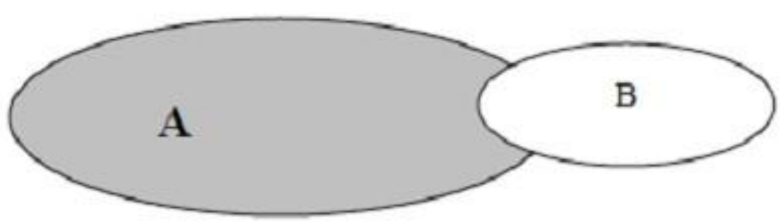
3.3. Differences

Definition: Let A and B be two sets. The set **A - B**, called the difference between A and B, is the set of all elements that are in A and are not in B.

Notation: **A – B or A \ B**
Formal definition:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

Venn diagram:



Example: **A = {2,4,6}, B = {1, 5,6}, A - B = {2,4}**

A – Ø = A The difference between A and the empty set is A
A – U = Ø The difference between A and the universal set is the empty set.

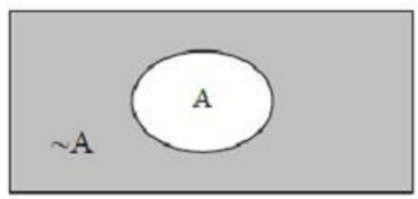
3.4. Complements

Definition: Let **A** be a set. The set of all objects within the universal set that are not in **A**, is called the complement of **A**.

Notation: ~**A**
Formal definition:

$$\sim A = \{x \mid x \notin A\}$$

Venn diagram:



SETS IDENTITIES

Using the operation unions, intersection and complement we can build expressions ov sets.

Example:
A - set of all black objects
B - set of all cats
$A \cap B$ - set of all black cats

The set identities are used to manipulate set expressions

$A \cup \sim A = U$	Complementation Law
$A \cap \sim A = \emptyset$	Exclusion Law
$A \cap U = A$ $A \cup \emptyset = A$	Identity Laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\sim(\sim A) = A$	Double Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$\sim(A \cap B) = \sim A \cup \sim B$ $\sim(A \cup B) = \sim A \cap \sim B$	De Morgan's Laws
$A - B = A \cap \sim B$	Alternate representation for set difference

Proof problems for sets

A. Element Proofs

Definitions used in the proofs

Def 1: $A \cup B = \{ x \mid x \in A \ \vee \ x \in B\}$

Def 2: $A \cap B = \{ x \mid x \in A \ \wedge \ x \in B\}$

Def 3: $A - B = \{ x \mid x \in A \ \wedge \ x \notin B\}$

Def 4: $\sim A = \{ x \mid x \notin A \}$

Inference rules often used:

$P \wedge Q \models P, Q$
$P, Q \models P \wedge Q$
$P \models P \vee Q$

How to prove that two sets are equal:
$A = B$
1) show that $A \subseteq B$, i.e. choose an arbitrary element in A and show that it is in B
2) show that $B \subseteq A$, i.e. choose an arbitrary element in B and show that it is in A

The element was chosen arbitrary, hence any element that is a member of the left se also a member of the right set, and vice versa.

Example:
Prove that $A - B = A \cap \sim B$

1. Show that $A - B \subseteq A \cap \sim B$

Let $x \in A - B$
By Def 3:
$x \in A \wedge x \notin B$ (1)
By (1) $x \in A$ (2)
By (1) $x \notin B$ (3)
By (3) and Def 4: $x \in \sim B$ (4)
By (2), (4)
$x \in A \wedge x \in \sim B$ (5)
By (5) and Def 2:
$x \in A \cap \sim B$

x was an arbitrary element in $A - B$, therefore $A - B \subseteq A \cap \sim B$ (6)

2. Show that $A \cap \sim B \subseteq A - B$

Let $x \in A \cap \sim B$

By Def 2:

$x \in A \wedge x \in \sim B$ (7)

By (7) $x \in A$ (8)

By (7) $x \in \sim B$ (9)

By (9) and Def 4: $x \notin B$ (10)

By (8), (10)

$x \in A \wedge x \notin B$ (11)

By (11) and Def 3:

$x \in A - B$

x was an arbitrary element in $A \cap \sim B$, therefore $A \cap \sim B \subseteq A - B$ (12)

by (6) and (12):

$$A - B \subseteq A \cap \sim B$$

Q.E.D.

B. Using set identities

Prove that $A \cap (\sim A \cup B) = A \cap B$

Method: Apply the set identities to the expression on the left, until the expression on th right is obtained.

By Distribution Laws: $A \cap (\sim A \cup B) = (A \cap \sim A) \cup (A \cap B)$

By the Exclusion Law $A \cap \sim A = \emptyset$

Hence $A \cap (\sim A \cup B) = \emptyset \cup (A \cap B)$

By the Identity Law: $\emptyset \cup (A \cap B) = A \cap B$

Hence $A \cap (\sim A \cup B) = A \cap B$

1. Set partitions

Two sets are disjoint if they have no elements in common, i.e. their intersection is the empty set.

A and B are disjoint sets iff $A \cap B = \varnothing$

Definition: Consider a set A, and sets $A_1, A_2, \ldots A_n$, such that:

a. $A_1 \cup A_2 \cup \ldots \cup A_n = A$

b. $A_1, A_2, \ldots A_n$, are mutually disjoint, i.e. for all i and j, $A_i \cap A_j = \varnothing$

The set $\{A_1, A_2, \ldots A_n\}$ is called a **partition** of A

Example:

1. $A = \{a, b, c, d, e, f, g\}$

$A_1 = \{a, c, d\}$
$A_2 = \{b, f\}$
$A_3 = \{e, g\}$

The set $\{\{a, c, d\}, \{b, f\}, \{e, g\}\}$ is a partition of A.

2. Cartesian product

Consider the identification numbers on license plates: $x_1x_2x_3$ $Y_1Y_2Y_3$
where $x_1x_2x_3$ is a 3-digit number and $Y_1Y_2Y_3$ is a combination of 3 letters

How do we make sure that each license plate would have a different identification number?

The program that assigns numbers uses Cartesian product of sets.

Definition: Let A and B be two sets. The Cartesian product of A and B is defined as t set

$$A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$$

Example 1:

$A = \{0, 1, 2, 3\}$
$B = \{a, b\}$

$A \times B = \{(0,a), (0,b), (1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Example 2:

A = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
A x A = {(0,0),(0,1), (0,2), …. (0,9),
(1,0),(1,1), (1,2), …….,(1,9),
……
(9,0),(9,1), (9,2), …. (9,9)}

We can consider the result to be the set of all 2-digit numbers.

3. Power sets

KAHE

Definition: The set of all subsets of a given set A is called power set of A.

Notation 2^A , or $\mathcal{P}(A)$

Example:

A - { a,b,c,d}

$\mathcal{P}(A)$ = {∅, {a},{b},{c},{d}
{a,b},{a,c},{a,d},{b,c},{b,d},{c,d}
{a,b,c},{a,b,d},{a,c,d}, {b, c, d}
{a,b,c,d}}

Number of elements in $\mathcal{P}(A)$ is 2^N, where N = number of elements in A
Why 2^N?

<u>Bit notation</u>: For a set A with **n** elements, each subset of A can be represented by a string of length *n* over {0,1}, i.e. a string consisting of 0s and 1s.

For example:

{a,b} = 1 1 0 0
{a,c} = 1 0 1 0
{b,c,d} = 0 1 1 1

The i-th element in the string is 1 if the element a_i is in the subset, otherwise it is 0.
Thus the subset {a,b,d} of the set {a,b,c,d} can be represented by the string '1101'

There are 2^n different strings with length **n** over {0,1}(why?), hence the number of the subsets is 2^n .

Set Relations

2. Definition

Let A and B be two sets. A relation **R** from **A** to **B** is any set of pairs (x,y), x ϵ A, y ϵ B, i.e. any subset of A x B.

If **x** and **y** are in relation **R**, we write xRy, or $(x,y) \in R$..

R is a set defined as
$R = \{(x,y) \mid x \in A, y \in B, xRb.\}$

3. Relations and Cartesian products

Relations between two sets **A** and **B** are sets of pairs of elements of **A** and **B**.
The Cartesian product **A x B** consists of all pairs of elements of **A** and **B.**

Thus relations between two sets are subsets of the Cartesian product of the sets.

Example:

Let $A = \{1, 3, 4, 5\}$
$B = \{2, 7, 8\}$

The relation R1 :"less than" from set A to set B is defined by the following set:

$R1 = \{(1, 2), (1, 7), (1, 8), (3, 7), (3, 8), (4, 7), (4, 8), (5, 7), (5, 8)\}$

This set is a subset of the Cartesian product of A and B:

A x B = **{(1,2),(1,7),(1,8),**
(3,2), **(3,7), (3,8),**
(4,2), **(4,7), (4,8)**,
(5,2),**(5,7),(5,8)}**
(the members of R1 are in boldface)

The relation R2: "greater than" from set A to set B is defined by the set:

$R2 = \{(3, 2), (4, 2), (5, 2)\}$

It is also a subset of A x B.

The relation R3 "equal to" from A to B is the empty set, since no element in A is equal to an element in B.

7. Domains and ranges

Let R be a relation from X to Y,
the **domain of R** is the set of all elements in X that occur in at least one pair of the relation,
the **range of R** is the set of all elements in Y that occur in at least one pair of the relation.

In the above example, the domain of **R: choose(x,y)** is the set of students {Ann, Tom , Paul}, and the range is the set of food items: {spaghetti, fish, pie, cake}.

The domain and the range are easily found using the matrix or the graph representations of the relation.

1. **Definition**

Let A and B be two sets. A relation **R** from **A** to **B** is any set of pairs (x,y), x ϵ A, y ϵ B, i.e. any subset of A x B.
The empty set is a subset of the Cartesian product – the empty relation

2. **How to write relations**

a. as set of pairs

$A = \{1,2,3\}, \{B = 4,5,6\}$
$R = \{\{1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$

b. using predicates

$A = \{1,2,3\}, \{B = 4,5,6\}$
$R = \{(x,y) \mid x \in A, y \in B, y \text{ is a multiple of } x\}$

3. Graph and matrix representation

$A = \{1,2,3\}, \{B = 4,5,6\}$
$R = \{\{1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$

1. Set operations and relations

Relations are sets. All set operations are applicable to relations

Examples:

Let A= {3, 5, 6, 7}
B ={4, 5, 9}

Consider two relations R and S from A to B:

$R = \{(x,y) | x \in A, y \in B, x < y\}$

If $(x,y) \in R$ we write xRy

R is a finite set and we can write down explicitly its elements:
R= {(3,4),(3,5),(3,9),(5,9), (6,9),(7,9)}

$S = \{(x,y) | x \in A, y \in B, |x-y| = 2\}$

If $(x,y) \in S$ we write xSy

S is a finite set and we can write down explicitly its elements:
S = {(3,5), (6,4), (7,5), (7,9)}

For R and S the universal set is A x B:
{(3,4), (3, 5), (3, 9),
(5, 4), (5, 5), (5, 9),
(6, 4), (6, 5), (6, 9),
(7, 4), (7, 5), (7, 9)}

a) intersection of R and S:

$R \cap S = \{(x,y) \mid xRy \wedge xSy\}$ $\qquad$ $R \cap S = \{(3,5),(7,9)\}$

b) union of R and S:

$R \cup S = \{(x,y) \mid xRy \vee xSy\}$

$R \cup S = \{ (3,4),(3,5),(3,9),(5,9), (6,9),(7,9), (6,4), (7,5) \}$

c) complementation:

$\sim R = \{(x,y) \mid \sim(xRy)\}$

$\sim R = U - R$

The universal set for R is the Cartesian product A x B

$A = \{ 3,5,6,7\}$

$B = \{4,5,9\}$

$U = A \times B = \{(3,4), (3,5), (3,9), (5,4), (5,5), (5,9),$
$(6,4), (6,5), (6,9), (7,4), (7,5), (7,9)\}$

$R = \{(3,4), (3,5), (3,9), (5,9), (6,9), (7,9)\}$

$U - R = \{(5,4), (5,5), (6,4), (7,4), (7,5)\}$

Note that for any two sets A and B, $A - B = A \cap \sim B$

d) difference R - S, S - R:

$R - S = \{(x,y) \mid xRy \wedge \sim(xSy)\}$

$R - S = \{(3,4),(3,9),(5,9), (6,9)\}$

2. Inverse relation

Let R: $A \to B$ be a relation from A to B. The inverse relation $R^{-1} : B \to A$ is defined as in the following way:

$$R^{-1} : B \to A \ \{(y,x) | \ (x,y) \in R\}$$

Thus $xRy \equiv yR^{-1} x$

Examples:

a. Let A = {1,2,3}, B = {1,4,9}

Let R: $B \to A$ be the set {(1,1), (1,4), (2,2), (2,4), (3,3)}
$R^{-1} : B \to A$ is the relation {(1,1), (4,1), (2,2), (4,2), (3,3)}

b. Let A = {1,2,3}, R: $A \to A$ be the relation {(1,2), (1,3), (2,3)}

R^{-1} is the relation: {(2,1), (3,1), (3,2)}

3. Composition of relations

Let X, Y and Z be three sets, R be a relation from X to Y, S be a relation from Y to Z.

A composition of R and S is a relation from X to Z :

$S \circ R = \{(x,z) | \ x \in X, z \in Z, \ \exists \ y \in Y$, such that xRy, and ySz$\}$

Note that the operation is right-associative, i.e. we first apply R and then S

Example 1:

Let X, Y, and Z be the sets:

X: {1,3,5}
Y: {2,4,8}
Z :{2,3,6}

Let $R : X \to Y$, and $S : Y \to Z$, be the relation "less than":

$R = \{(1,2),(1,4),(1,8),(3,4),(3,8),(5,8)\}$
$S = \{(2,3),(2,6),(4,6)\}$

$S \circ R : \{(1,3), (1,6), (3,6)\}$

The element (1,3) is formed by combining (1,2) from R and (2,3) from S
The element (1,6) is formed by combining (1,2) from R and (2,6) from S
Note, that (1,6) can also be obtained by combining (1,4) from R and (4,6) from S.
The element (3,6) is formed by combining (3,4) from R and (4,6) from S

4. Identity relation

Identity relation on a set A is defined in the following way:

$I = \{(x,x) | x \in A\}$

Example:

Let $A = \{a, b, c\}$, $I = \{(a,a), (b,b), (c,c)\}$

5. Problems:

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, $C = \{x, y, z\}$

a. Let $R = \{(1,a), (2,b), (3,a)\}$ and $S = \{(a,y),(a,z),(b,x),(b,z)\}$

Find $S \circ R$

b. Let $R = \{(1,a), (2,b), (3,a)\}$ and $S = \{(a,y),(a,z)\}$

Find $S \circ R$

a. Let R = {(1,a), (2,b)} and S = {(a,y), (b, y), (b,z)}
Find $S \circ R$

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}
Find R^{-1} , S^{-1} and $R^{-1} \circ S^{-1}$

Solutions

Let A = {1, 2, 3}, B = {a, b}, C = {x, y, z}

a. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}

Find $S \circ R$

Solution: {(1,y), (1, z), (2,x),(2,z), (3,y), (3, z)}

b. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z)}

Find $S \circ R$ Solution: {(1,y), (1, z), (3,y), (3, z)}

c. Let R = {(1,a), (2,b)} and S = {(a,y), (b, y), (b,z)}
Find $S \circ R$
Solution: {(1,y), (2,y), (2, z)}

d. Let R = {(1,a), (2,b), (3,a)} and S = {(a,y),(a,z),(b,x),(b,z)}
Find R^{-1} , S^{-1} and $R^{-1} \circ S^{-1}$

Solution:
R^{-1} = {(a,1), (b,2), (a,3)}
S^{-1} = {(y,a),(z,a),(x,b),(z,b)}

$R^{-1} \circ S^{-1}$ = {(y,1), (y,3), (x,2), (z,1), (z,3), (z,2)}

Definitions:

Let R be a binary relation on a set A.

1. R is **reflexive,** iff for all $x \in A$, $(x,x) \in R$, i.e. **xRx** is true.

2. R is **symmetric**, iff for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$

   i.e **xRy → yRx** is true

3. R is **transitive** iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y,z) \in R$, then $(x, z) \in R$

   i.e. **(xRy Λ yRz) → xRz** is true

A. Reflexive relations

Let R be a binary relation on a set A.

R is **reflexive,** iff for all $x \in A$, $(x,x) \in R$, i.e. **xRx** is true.

1. Examples:

1. Equality is a reflexive relation
   for any object x: $x = x$ is true.
2. "less then" (defined on the set of real numbers) is **not a reflexive relation.**
   for any number x: $x < x$ is **not** true
3. " less then or equal to" (defined on the set of real numbers) **is a reflexive relation**
   for any number x $x \le x$ is true

4. Reflexive and irreflexive relations

Compare the three examples below:

1. A = {1,2,3,4}, R1 = {(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4)}
2. A = {1,2,3,4}, R2 = {(1,2), (2,3), (3,4), (4,1)}
3. A = {1,2,3,4}, R3= {(1,1), (1,2), (3,4), (4,4)}

R1 is a reflexive relation. R2 ? R3 ?

Definition: Let R be a binary relation on a set A.
R is **irreflexive** iff for all $x \in A$, $(x,x) \notin R$

Definition: Let R be a binary relation on a set A.
R is **neither reflexive, nor irreflexive** iff
there is $x \in A$, such that $(x, x) \in R$. and there is $y \in A$ such that $(y, y) \notin R$

Thus R2 is irreflexive, R3 is neither reflexive nor irreflexive.

reflexive:	for all x: xRx
irreflexive:	for no x: xRx
neither:	for some x: xRx is true, for some y: yRy is false

B. Symmetric relations

R is **symmetric**, iff for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$

i.e **xRy** $\rightarrow$ **yRx** is true

This means: if two elements **x** and **y** are in relation R, then **y** and **x** are also in R, i.e. if xRy is true, yRx is also true.

1. Examples:

1. equality is a symmetric relation: if a = b then b = a
2. "less than" is <u>not</u> a symmetric relation : if a < b is true then b < a is false
3. "sister" on the set of females is symmetric
4. "sister" on the set of all human beings is not symmetric

4. Symmetric and anti-symmetric relations

Compare the relations:

1. A = {1,2,3,4}, R1 = {(1,1), (1,2), (2,1), (2,3), (3,2), (4,4)}
2. A = {1,2,3,4}, R2 = {(1,1), (1,2), (2,3), (4,4)}
3. A = {1,2,3,4}, R3 = {(1,1), (1,2), (2,1) , (2,3), (4,4)}

Definition: Let R be a binary relation on a set A.
R is **anti-symmetric** if for all x, y $\in$ A, x $\neq$ y, **if (x, y) $\in$ R, then (y, x) $\notin$ R.**

Definition: R is **neither symmetric nor anti-symmetric** iff it is not symmetric and not anti-symmetric.

symmetric: xRy → yRx for all x and y
anti-symmetric: xRy and yRx → x = y
neither: for some x and y: xRy, and yRx
for others xRy is true, yRx is not true

C. Transitive relations

Let R be a binary relation on a set A.
R is **transitive** iff for all x, y, z $\in$ A, if (x, y) $\in$ R and (y,z) $\in$ R , then (x, z) $\in$ R

i.e. **(xRy Λ yRz) → xRz** is true

1. Examples:

1. Equality is a transitive relation a = b, b = c, hence a = c
2. "less than" is a transitive relation a < b, b < c, hence a < c
3. mother_of(x,y) is <u>not</u> a transitive relation
4. sister(x,y) is a transitive relation
5. brother (x,y) is a transitive relation.
6. A = {1,2,3,4} R = {(1,1), (1,2), (1,3), (2,3), (4,3)} - transitive
7. A = {1,2,3,4} R = {(1,1), (1,2), (1,3), (2,3), (3,4)} - not transitive

Equivalence Relations, Partial Orders

1. Equivalence relations

Definition: A relation R is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

Examples:

Let **m** and **n** be integers and let **d** be a positive integer. The notation

$$m \equiv n \pmod{d}$$

is read "m is congruent to n modulo d".

The meaning is: the integer division of **d** into **m** gives the same remainder as the integer division of **d** into **n**.

Consider the relation

$R = \{(x,y) \mid x \bmod 3 = y \bmod 3\}$

4 mod 3 = 1, 7 mod 3 = 1, hence $(4,7) \in R$

The relation is reflexive: $x \bmod 3 = x \bmod 3$
symmetric: if $x \bmod 3 = y \bmod 3$, then $y \bmod 3 = x \bmod 3$
transitive: if $x \bmod 3 = y \bmod 3$, and $y \bmod 3 = z \bmod 3$, then $x \bmod 3 = z \bmod 3$

Consider the sets $[x] = \{y \mid yRx\}$

$[0] = \{0,3,6,9,12,\ldots.\}$
$[1] = \{1,4,7,10,13,\ldots.\}$
$[2] = \{2,5,8,11,14,\ldots\}$
……

From the definition of [x] it follows that

$[0] = [3] = [6] \ldots$
$[1] = [4] = \ldots$
$[2] = [5] = \ldots$

Thus the relation R produces three different sets [0], [1] and [2].
Each number is exactly in one of these sets. Thus {[0], [1], [2]} is a **partition** of the set of non-negative integers.

2. Partial Orders

Definition: Let R be a binary relation defined on a set A. R is a partial order relation iff R is transitive and anti-symmetric

Examples:

1. Let A be a set, and P(A) be the power set of A. The relation 'subset of' on P (A) is a partial order relation
   It is reflexive, anti-symmetric, and transitive

2. Let N be the set of positive integers, and R be a relation defined as follows:
   $(x, y) \in R$ iff y is a multiple of x
   e.g. $(3,12) \in R$, while $(3, 4) \notin R$
   R is a partial order relation. It is reflexive, anti-symmetric, and transitive

Functions

1. Definition: A function **f** from a set **X** to a set **Y** is a subset of the Cartesian pr
X x Y, $f \subseteq X \times Y$, such that

$\forall x \in X \quad \exists y \in Y$, such that $(x,y) \in f$, and

$(x,y1) \in f \wedge (x,y2) \in f \rightarrow y1 = y2$

i.e. if $(x,y1) \in f$ and $(x,y2) \in f$, then $y1 = y2$

Thus all elements in X can be found in exactly one pair of **f.**

Notation: Let **f** be a function from **A** to **B.** We write

$$f : A \to B$$
$$a \in A,\ f(a) = b,\ \ b \in B$$

Examples:

$A = \{1,2,3\}$, $B = \{a,b\}$

$R = \{(1,a),(2,a),(3,b)\}$ is a function

Other definitions:

Let **f** be a function from **A** to **B.**

1. **Domain** of **f:** the set **A**
2. **Range** of **f**: $\{b: b \in B$ and there is an $a \in A,\ f(a) = b\}$
3. **Image** of a under **f: f(a)**

Example:

$A = \{1,2,3\}$, $B = \{a,b\}$

$f = \{(1,a),(2,a),(3,b)\}$

domain: $\{1,2,3\}$,

range: $\{a,b\}$

a is image of 1 under f: $f(1) = a$, $f(2) = \ldots..$ $f(3) = \ldots\ldots$

2. Functions with more arguments

Let A = A1 x A2 , and **f** be a function from **A** to **B**

We write: f(a1,a2) = b

If A = A1 x A2 x … x An, we write f(a1,a2,…,an) = b

a1, a2, ..,an: arguments of **f**
b : value of **f**

3. Functions of special interest

a. one-to-one

distinct elements have distinct images

if a1 $\neq$ a2, then f(a1) $\neq$ f(a2)

Example:
A = {1,2,3}, B = {a,b,c,d}

one-to-one function **f** = {(1,a), (2,c), (3,b)}

b. onto

Every element in **B** is an image of some element in **A**

Example:

A = {1,2,3}, B = {a,b }

onto function f = {(1,a), (2,b), (3,b)}

c. bijection

f is **bijection** iff **f** is a **one-to-one** function and **f** is a **onto** function

Example:

A = {1,2,3}, B = {a,b,c }

bijection f = {(1,a), (2,c), (3,b)}

4. Inverse function

If f is a bijection, f^{-1} is a function, also a bijection.

$f^{-1} = \{(y,x) \mid (x,y) \in f\}$

Example:

A = {1,2,3}, B = {a,b,c }

f = {(1,a), (2,c), (3,b)}

$f^{-1} = \{(a,1), (b,3), (c,2)\}$

KAHE

5. Composition of functions

Let **f : A → B, g : B → C** be two functions.
The composition **h = g ° f** is a function from **A** to **C** such that **h(a) = g(f(a)**

Example: Let f (x) = x +1, g(x) = x^2.

The composition h(x) = f (x) ° g(x) = f(g(x)) = (x^2)+ 1

The composition p(x) = g(x) ° f(x) = g(f(x))= $(x+1)^2$

When **f** is a bijection and $\mathbf{f^{-1}}$ exists, we have:
$f^{-1}(f(a)) = a$, $f(f^{-1}(b)) = b$, a ∈ A, b ∈ B,

Counting Principles

The Multiplication Principle

The Multiplication Principle

Let $m \in \mathbb{N}$. For a procedure of m successive distinct and independent steps with n_1 outcomes possible for the first step, n_2 outcomes possible for the second step, . . . , and n_m outcomes possible for the mth step, the total number of possible outcomes is

$$n_1 \cdot n_2 \cdots n_m$$

Addition Principle

The Addition Principle

For a collection of m disjoint sets with n_1 elements in the first, n_2 elements in the second, . . . , and n_m elements in the mth, the number of ways to choose one element from the collection is

$$n_1 + n_2 + \cdots + n_m$$

Using the Pigeon-Hole Principle

The **Pigeon-Hole Principle** (see Section 4.6) states that if m objects are to be put in n locations, where $m > n > 0$, then at least one location must receive at least two objects. Thus, to prove that a set of objects has at least two elements with the same property, first count the number of distinct properties of objects in the set, and then count the number of distinct elements. If the total number of elements is larger than the number of distinct properties of objects, then the Pigeon-Hole Principle implies that at least two of the elements have the same property. The next example is an illustration of this type of argument.

Example 9. A local bank requires customers to choose a four-digit code to use with an ATM card. The code must consist of two letters in the first two positions and two digits in the other two positions. The bank has 75,000 customers. Show that at least two customers choose the same four-digit code.

Solution. First, use the Multiplication Principle to calculate the number of distinct codes possible:

$$\begin{aligned}(\#\text{ Four-symbol codes}) &= (\#\text{ Choices of letter 1}) \cdot (\#\text{ Choices for letter 2}) \\ &\quad \cdot (\#\text{ Choices for digit 1}) \cdot (\#\text{ Choices for digit 2}) \\ &= 26 \cdot 26 \cdot 10 \cdot 10 \\ &= 67{,}600\end{aligned}$$

Now, apply the Pigeon-Hole Principle. Since there are 75,000 customers and only 67,600 codes, the Pigeon-Hole Principle implies that at least two of the customers choose the same code. ■

Example 10. Suppose a group of vacationers is split into 159 teams. How many leagues must be formed if a league should contain at most 8 teams? 10 teams? 12 teams?

Solution. The Generalized Pigeon-Hole Principle tells us that the answers are

$$\left\lceil \frac{159}{8} \right\rceil = 20 \quad \left\lceil \frac{159}{10} \right\rceil = 16 \quad \left\lceil \frac{159}{12} \right\rceil = 14$$

It remains for the organizers to determine which size of a league is most manageable. ■

Permutations and Combinations

(c) The answer is the product of the number of ways to put four books on one shelf and the number of ways to put the remaining books on the second shelf. The number of ways to arrange four books on the first shelf is $P(8,4)$, and the four remaining books

(c) The answer is the product of the number of ways to put four books on one shelf and the number of ways to put the remaining books on the second shelf. The number of ways to arrange four books on the first shelf is $P(8,4)$, and the four remaining books

(c) The answer is the product of the number of ways to put four books on one shelf and the number of ways to put the remaining books on the second shelf. The number of ways to arrange four books on the first shelf is $P(8,4)$, and the four remaining books

Definition 1. Let $n, r \in \mathbb{N}$. A **permutation** of an n-element set is a linear ordering o the n elements of the set. For $n \geq r \geq 0$ an **r-permutation** of an n-element set is a line ordering of r elements of the set.

Example 1. List all permutation of the elements a, b, and c.

Solution. The permutations are abc, acb, bac, bca, cab, and cba.

Let $\boldsymbol{P(n, r)}$ denote the number of r-permutations of an n-element set. We defir $P(n, 0) = 1$ for all $n \in \mathbb{N}$.

Example 2.

(a) How many ways can eight different books be arranged on a shelf?
(b) How many ways can four of eight different books be arranged on a shelf?
(c) How many ways can eight different books be arranged on two shelves so that each shelf contains four books?

Solution.

(a) The answer is the number of ordered ways of arranging the books on the shelf. That is,

$$P(8,8) = 8! = 40{,}320$$

(b) The number of ways to arrange four of the eight books is

$$P(8,4) = 1680$$

(c) The answer is the product of the number of ways to put four books on one shelf and the number of ways to put the remaining books on the second shelf. The number of ways to arrange four books on the first shelf is $P(8,4)$, and the four remaining books

can be arranged in $P(4,4)$ ways on the second shelf. Therefore, the total number of arrangements will be

$$\begin{aligned}(\text{\# Arrangements of books on two shelves}) &= (\text{\# Arrangements on first shelf})\\ &\quad \cdot (\text{\# Arrangements on second shelf})\\ &= P(8,4)\cdot P(4,4)\\ &= (8!/4!)\cdot(4!/0!)\\ &= 8!\\ &= 40{,}320\end{aligned}$$

Combinations

Definition 2. Let $n, r \in \mathbb{N}$ such that $n \geq r \geq 0$. An unordered selection of r elements from an n element set is called a **combination.**

Example 4. List all the combinations of the set $\{a, b, c\}$.

Solution. The combinations will be of sizes 0, 1, 2, and 3. All combinations are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$. ■

Example 5. How many different poker hands are there?

Solution. This answer is just the number of ways of choosing five cards from the 52-card deck:

$$C(52,5) = \frac{52!}{47!\,5!} = 2{,}598{,}960$$ ■

Example 8. An examination consists of 20 questions, of which the student must answer any 12.

(a) How many different ways can a student choose questions to answer?
(b) The 20-question exam is split into three parts. There are 6 questions in the first part, 10 in the second part, and 4 in the third part. A student must choose three from the first part, eight from the second part, and one from the third part. How many ways can a student choose questions to answer?

Solution.

(a) The answer is just the number of different 12-element subsets of a 20-element set, or $C(20, 12) = 125{,}970$.

(b) By the Multiplication Principle, the answer will be the product of the number of ways to make choices in each category:

$$\begin{aligned}(\#\text{ Possible choices}) &= (\#\text{ Choices for part 1}) \cdot (\#\text{ Choices for part 2}) \\ &\quad \cdot (\#\text{ Choices for part 3}) \\ &= C(6, 3) \cdot C(10, 8) \cdot C(4, 1) \\ &= 3600\end{aligned}$$

■

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k + 1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k + 1)$ is true.

Sum of the First *n* Integers

Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution To construct a proof by induction, you must first identify the property $P(n)$. In this case, $P(n)$ is the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \qquad \leftarrow \text{the property } (P(n))$$

[To see that $P(n)$ is a sentence, note that its subject is "the sum of the integers from 1 to n" and its verb is "equals."]

In the basis step of the proof, you must show that the property is true for $n = 1$, or, in other words that $P(1)$ is true. Now $P(1)$ is obtained by substituting 1 in place of n in $P(n)$. The left-hand side of $P(1)$ is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus $P(1)$ is

$$\boxed{1 = \frac{1(1+1)}{2}.}$$

← basis ($P(1)$)

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

In the inductive step, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. *[This assumption is the inductive hypothesis.]* You must then show that $P(k+1)$ is true. What are $P(k)$ and $P(k+1)$? $P(k)$ is obtained by substituting k for every n in $P(n)$. Thus $P(k)$ is

$$\boxed{1 + 2 + \cdots + k = \frac{k(k+1)}{2}.}$$

← inductive hypothesis ($P(k)$)

Similarly, $P(k+1)$ is obtained by substituting the quantity $(k+1)$ for every n that appears in $P(n)$. Thus $P(k+1)$ is

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2},$$

or, equivalently,

$$\boxed{1+2+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.}\qquad \leftarrow \text{to show } (P(k+1))$$

Theorem 5.2.2 Sum of the First n Integers

For all integers $n \geq 1$,

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Proof (by mathematical induction):

Let the property $P(n)$ be the equation

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}. \qquad \leftarrow P(n)$$

Show that P(1) is true:

To establish $P(1)$, we must show that

$$1=\frac{1(1+1)}{2} \qquad \leftarrow \quad P(1)$$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2}=\frac{2}{2}=1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Suppose that k is any integer with $k \geq 1$ such that

$$1+2+3+\cdots+k=\frac{k(k+1)}{2} \qquad \begin{array}{l}\leftarrow P(k)\\ \text{inductive hypothesis}\end{array}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \quad \leftarrow P(k+1)$$

[We will show that the left-hand side and the right-hand side of $P(k+1)$ are equal to the same quantity and thus are equal to each other.]

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1 + 2 + 3 + \cdots + (k+1) \\
&\quad = 1 + 2 + 3 + \cdots + k + (k+1) && \text{by making the next-to-last term explicit} \\
&\quad = \frac{k(k+1)}{2} + (k+1) && \text{by substitution from the inductive hypothesis} \\
&\quad = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&\quad = \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&\quad = \frac{k^2+3k+1}{2} && \text{by algebra.}
\end{aligned}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+1}{2}.$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true *[as was to be shown]*.

POSSIBLE QUESTIONS

TWO MARKS

1. State Principle of Mathematical Induction.
2. Find the value of n if $np_3=5np_2$
3. State Pigeonhole Principle.
4. Define Permutation.
5. In how many words can letters of the word "INDIA" be arranged?

SIX MARKS

1. Explain about types of relation with examples.
2. From the 7 men and 4 women a committee of 6 to be formed can this be done when the committee contains i) Exactly 2 women ii) At least 2 women
3. Let A={1,2,3} and f,g,h and s be functions from A to A given by
   f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) };
   h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find $f_o g$, $g\,_o f$, $f\,_o\, h\,_o g$, $g_o s$, $s\,_o\, s$, $f\,_o\, s$.
4. State and prove Pigeonhole Principle.
5. Explain the types of sets.
6. Using Mathematical Induction prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. Find the number of integers between I to 250 that are not divisible by any of the integers 2,3,5 and 7.
8. From a committee consisting of 6 men and 7 women ,in how many ways can be select a committee of
   a) 3 men and 4 women.
   b) 4 members which has atleast one women.
   c) 4 persons that has atmost one man.
   d) 4 persons of both sexes.
   e) 4 persons in which Mr. and Mrs.Kannan is not included.
9. Find the number of distinct permutations that can be formed from all the letters of each word (I)RADAR (II) UNUSUAL
10. $\frac{1}{1.2} + \frac{1}{2.3} + \dots\dots\dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Questions	opt1	opt2	opt3	opt4	Answer
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then RoS = ------	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(1,5),(3,2),(2,5)}
If f(x) = x+2 and g(x) = x^2-1 then(gof)(x) = ------	x^2+4x+4	x^2+4x-3	x^2-4x+4	x^2+4x+3	x^2+4x+3
A relation R in a set X is ---------- if for every $x\in X,(x,x)\notin R$	transitive	symmetric	irreflexive	reflexive	irreflexive
Suppose in RxR, the ordered pairs (x-2, 2y+1) and (y-1, x+2) are equal. The values of x and y are	2,3	3,2	2,-3	3,-2	3,2
A relation R on a set is said to be an equivalence relation if it	Reflexive	Symmetric	Reflexive,Symmetric, Transitive	Transitive	Reflexive,Symmetric, Transitive
Let f : R→R where R is a set of real numbers.Then f(x) = -2x	One-to-one	Onto	into	bijection	bijection
A mapping f : x→y is called ------ if distinct elements of x are	one-to-one	Onto	into	many to one	one-to-one
If the relation R and S are both reflexive then R ∨ S is ---------	symmetric	reflexive	transitive	not reflexive	reflexive
A One – to –one function is also known as ---------	injective	surjective	bijective	objective	injective
A On to function is also known as ---------	injective	surjective	bijective	objective	surjective
A One – to –one and onto function is also known as ---------	injective	surjective	bijective	objective	bijective
Let f : x→y , g : y→x be the functions then g is equal to f^{-1} only if --------	fog = Iy	gof = I_x	gof=I_y	fog=I_x	gof = I_x

In N, define aRb if a+b = 7. This is symmetric when --------	b+a =7	a+a =7	b+c =7	a + c = 7	b+a =7
If the relation is ---------- relation if aRb,bRa →a = b -------	symmetric	reflexive	Antisymmetric	not reflexive	Antisymmetric
f : R→R, g : R→R defined by f(x) = 4x-1 and g(x) = cos x..The	4cosx –1	4cosx	4cosx +1	1/4cosx	4cosx –1
Let f : N→N be a function such that f(x) = 5 ,x∈N then the f(x) is	identity	inverse	equal	constant	constant
A binary relation R in a set X is said to be symmetric if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb⇒bRa
A binary relation R in a set X is said to be reflexive if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRa
A binary relation R in a set X is said to be antisymmetric if ------	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb,bRa⇒a=b
A binary relation R in a set X is said to be transitive if ----	aRa	aRb⇒bRa	aRb,bRc⇒aRc	aRb,bRa⇒a=b	aRb,bRc⇒aRc
If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)}	{(4,2),(3,2),(1,4)}	{(1,5),(3,2),(2,5)}	{(1,2),(2,2)}	{(4,5),(3,3),(1,1)}	{(4,5),(3,3),(1,1)}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the domain of R = ---------	{1,3}	{2,3}	{2,4}	{1,4}	{2,4}
Let x = {1,2,3,4}, R = {(2,3),(4,1)} then the range of R = ---------	{1,3}	{3,1}	{2,4}	{1,4}	{3,1}
In a relation matrix all the diagonal elements are one then it satisfies --------	symmetric	antisymmetric	transitive	reflexive	reflexive
In a relation matrix A=(aij) a_{ij} =a_{ji} then it satisfies ------- relation	symmetric	reflexive	transitive	antisymmetric	symmetric
An ordered arrangement of r - element of a set containning n - distinct	r permutation of n	r - combination of n	n permutation of r	n combination of r elements	r permutation of n elements
The r - permutation of n elements is denoted by ------------	P (r, n)	P(n,r)	c(r, n)	c(n, r)	P(n,r)

The r - permutation of n elements is denoted by P (n, r) where -------	r ≤ n	r = n	r ≥ n	r > n	r ≤ n
An unordered pair of r elements of a set containing n distinct	r permutati on of n	r - combinati on of n	n permutation of r	n combination of r elements	r - combination of n elements
The number of different permutations of the word BANANA is -------	720	60	120	360	60
The number of way a person roundtrip by bus from A to C by way of B will be ------	12	48	144	264	144
How many 10 digits numbers can be written by using the digits 1 and 2 ?	C (10, 9) + C (9, 2)	1024	C(10, 2)	10!	1024
The number of ways to arrange th a letters of the word CHEESE are ------	120	240	720	6	120
r - combination of n elements is denoted by ------	P (r, n)	P(n,r)	C(r, n)	C(n, r)	C(n, r)
The value of C(n,n) is ----------	0	1	n	n-1	1
C (n, n-r) = ------	C(n, r)	C(n-1, r)	C(n-1, r-1)	C(n, r-1)	C(n, r)
C (n, r) + C (n, r-1) = ------	C(n, r)	C (n+1, r-1)	C (n+1, r)	C(n, r+1)	C (n+1, r)
The number of arranging different crcular arrangement of n objects = ----------	n!	(n+1)!	(n -1)!	0!	(n -1)!
The number of ways of arranging n beads in the form of a necklace = ----------	(n-1)!	(n-1)!/2	n!	n!/2	(n-1)!/2
The value of C(10, 6) + C(9, 5) + C(8, 4) + C(8, 3) is ----------	C(10, 7)	C(9,7)	C(8, 5)	C(11, 5)	C(11, 5)

The value of C(10, 8) + C(10,7) is -------------	990	165	45	120	165
The number of different words can be formed out of the letters of the word	64	120	40320	720	720
The number of ways can a party of 7 persons arrange themselves	6!	7!	5!	7	6!
The sum of entries in the fourth row of Pascal's triangle is ----------	8	4	10	16	8
The number of wors can be formed out of the letters of the word	100	120	720	150	720
The value of P(n,n) = -------------	1	0	n	n-1	n
The value of P(10, 3) is -----------	120	720	60	45	720
If P (10, r) is 720, then the value of r is --------	2	3	4	5	3
In how many ways 5 children out of a class of 20 line for a picture?	P (20, 4)	P(20, 5)	P (5, 20)	P(5, 5)	P(20, 5)
The value of C(n, r) is ----------	an integer	a fraction	an integer or a fraction	a rational number less than 1	an integer
The value of P(n, r) / r! is ---------	r	C(n, r)	n /r	nr	c(n,r)

UNIT – II

Growth of Functions: Asymptotic Notations, Summation formulas and properties, Bounding Summations, approximation by Integrals

KAHE

Growth of Functions

- We will use something called *big-O notation* (and some siblings described later) to describe how a function grows.
  - What we're trying to capture here is how the function grows.
  - … without capturing so many details that our analysis would depend on processor speed, etc.
  - … without worrying about what happens for small inputs: they should always be fast.
- For functions $f(x)$ and $g(x)$, we will say that "$f(x)$ is $O(g(x))$" [pronounced "$f(x)$ is big-oh of $g(x)$"] if there are positive constants C and k such that

$$|f(x)| \le C|g(x)| \text{ for all } x>k.$$

  - The big-O notation will give us a order-of-magnitude kind of way to describe a function's growth (as we will see in the next examples).
  - Roughly speaking, the k lets us only worry about big values (or input sizes when we apply to algorithms), and C lets us ignore a factor difference (one, two, or ten steps in a loop).
  - I might also say "$f(x)$ is in $O(g(x))$", then thinking of $O(g(x))$ as the set of all functions with that property.
- *Example:* The function $f(x)=2x^3+10x$ is $O(x^3)$.

  Proof: To satisfy the definition of big-O, we just have to find values for C and k that meet the condition.

  Let $C=12$ and $k=2$. Then for $x>k$,

$$|2x^3+10x|=2x^3+10x<2x^3+10x^3=|12x^3|. \blacksquare$$

- Note: there's nothing that says we have to find the *best* C and k. Any will do.
  - Also notice that the absolute value doesn't usually do much: since we're worried about running times, negative values don't usually come up. We can just demand that x is big enough that the function is definitely positive and then remove the $|\cdots|$.
- Now it sounds too easy to put a function in a big-O class. But…
- *Example:* The function $f(x)=2x^3+10x$ is **not in** $O(x^2)$.

Proof: Now we must show that no C and k exist to meet the condition from the definition.

For any candidate C and k, we can take $x>k$ and $x>0$ and we would have to satisfy

$$|2x^3+10x| 2x^3+10x 2x^3 x < C|x^2| < Cx^2 < Cx^2 < C/2$$

So no such C and k can exist to let the inequality hold for large x. ■

- *Example:* The function $f(x)=2x^3+10x$ is $O(x^4)$.

  Proof idea: For large x, we know that $x^4>x^3$. We could easily repeat the $O(x^3)$ proof above, applying that inequality in a final step.

- *Example:* The function $f(x)=5x^2-10000x+7$ is $O(x^2)$.

  Proof: We have to be a little more careful about negative values here because of the "$-10000x$" term, but as long as we take $k\geq 2000$, we won't have any negative values since the $5x^2$ term is larger there.

  Let $C=12$ and $k=2000$. Then for $x>k$,

  $$|5x^2-10000x+7|=5x^2-10000x+7<5x^2+7x^2=|12x^2|. ∎$$

- It probably wouldn't take many more proofs to convince you that x^n is always in $O(x^n)$ but never in $O(x^{n-1})$.
  - We can actually do better than that…
- The big-O operates kind of like a $\leq$ for growth rates.

Big-O Results

- *Theorem:* Any degree-n polynomial, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is in $O(x^n)$.

  Proof: As before, we can assume that $x>1$ and then,

  $$|f(x)|=|a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0|\le|a_n|x^n+|a_{n-1}|x^{n-1}+\cdots+|a_1|x+|a_0|=x^n(|a_n|+|a_{n-1}|/x+\cdots+|a_1|/x^{n-1}+|a_0|/x^n)\le x^n(|a_n|+|a_{n-1}|+\cdots+|a_1|+|a_0|).$$

  Now, if we let $C=\sum|a_i|$ and $k=1$, we have satisfied the definition for $O(x^n)$. ■

- *Theorem:* If we have two functions $f_1(x)$ and $f_2(x)$ both $O(g(x))$, then $f_1(x)+f_2(x)$ is also $O(g(x))$.

  Proof: From the definition of big-O, we know that there are C_1 and k_1 that make $|f_1(x)|\le C|h(x)|$ for $x>k_1$, and similar C_2 and k_2 for $f_2(x)$.

  Let $C=C_1+C_2$ and $k=\max(k_1,k_2)$. Then for $x>k$,

  $$|f_1(x)+f_2(x)|\le|f_1(x)|+|f_2(x)|\le C_1|g(x)|+C_2|g(x)|=C|g(x)|.$$

  Thus, $f_1(x)+f_2(x)$ is $O(g(x))$. ■

- The combination of functions under big-O is generally pretty sensible…
  - *Theorem:* If for large enough x, we have $f(x)\le g(x)$, then $f(x)$ is $O(g(x))$.
    - Sometimes the big-O proof is even easier.
  - *Theorem:* If we have two functions $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$, then $f(x)+g(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.
    - When adding, the bigger one wins.

KAHE

- *Theorem:* If we have three functions f,g,h where $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, then $f(x)$ is $O(h(x))$.
  - Approximately: if h is bigger than g and g is bigger than f, then h is bigger than f.
- *Corollary:* Given $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$ **and** $g_1(x)$ is $O(g_2(x))$ then $f_1(x)+f_2(x)$ is $O(g_2(x))$.
  - That is, if we have two functions we know a big-O bound for, and we add them together, we can ignore the smaller one in the big-O.
- *Theorem:* If we have two functions $f_1(x)$ which is $O(g_1(x))$ and $f_2(x)$ which is $O(g_2(x))$, then $f(x)g(x)$ is $O(g_1(x)g_2(x))$.
  - Multiplication happens in the obvious way.
- *Theorem:* Any constant value is is $O(1)$.
  - Aside: You will often hear a constant running time algorithm described as $O(1)$.
- *Corollary:* Given $f(x)$ which is $O(g(x))$ and a constant a, we know that $af(x)$ is $O(g(x))$.
  - That is, if we have a function multiplied by a constant, we can ignore the constant in the big-O.

- All of that means that it's usually pretty easy to guess a good big-O category for a function.
  - $f(x)=2^x+x^2$ is in $O(\max(|2^x|,|x^2|))=O(2^x)$, since 2^x is larger than x^2 for large x.
  - $f(x)=1100x^{12}+100x^{11}-87$ is in $O(x^{12})$.
    - Directly from the theorem about polynomials.
    - For small x, the $100x^{11}$ is the largest, but as x grows, the $1100x^{12}$ term takes over.
  - $f(x)=14x2^x+x$ is in $O(x2^x)$.

- What is a good big-O bound for $17x^4-12x^2+\log_2 x$?
  - We can start with the obvious:

$$17x^4-12x^2+\log_2 x \text{ is in } O(17x^4-12x^2+\log_2 x).$$

  - From the above, we know we can ignore smaller-order terms:

$$17x^4-12x^2+\log_2 x \text{ is in } O(17x^4).$$

KAHE

- And we can ignore leading constants:

$$17x^4-12x^2+\log_2 x \text{ is in } O(x^4).$$

- The "ignore smaller-order terms and leading constants" trick is very useful and comes up a lot.

Big-Ω

- As mentioned earlier, big-O feels like $\leq$ for growth rates.
  - … then there must be $\geq$ and $=$ versions.
- We will say that a function $f(x)$ is $\Omega(g(x))$ ("big-omega of $g(x)$") if there are positive constants C and k such that when $x>k$,

$$|f(x)|\geq C|g(x)|.$$

  - This is the same as the big-O definition, but with a $\geq$ instead of a $\leq$.
- *Example:* The function $3x^2+19x$ is $\Omega(x^2)$.

  Proof: If we let $C=3$ and $k=1$ then for $x>k$,

$$|3x^2+19x|\geq 3x^2+19x\geq 3|x^2|.$$

  From the definition, we have that $3x^2+19x$ is $\Omega(x^2)$. ■

- As you can guess, the proofs of big-Ω are going to look just about like the big-O ones.
  - We have to be more careful with negative values: in the big-O proofs, we could just say that the absolute value was bigger and ignore it. Now we need smaller values, so can't be so quick.
  - But the basic ideas are all the same.
- *Theorem:* $f(x)$ is $O(g(x))$ iff $g(x)$ is $\Omega(f(x))$.

Proof: First assume we have $f(x)$ in $O(g(x))$. Then there are positive C and k so that when $x>k$, we know $|f(x)|\leq C|g(x)|$. Then for $x>k$, we have $|g(x)|\geq \frac{1}{C}|f(x)|$ and we can use k and $\frac{1}{C}$ as constants for the definition of big-Ω.

Similarly, if we assume that $g(x)$ is $\Omega(f(x))$, we have positive C and k so that when $x>k$, we have $|g(x)|\geq C|f(x)|$. As above we then have for $x>k$, $|f(x)|\leq \frac{1}{C}|g(x)|$. ■

Big-Θ

- We will say that a function $f(x)$ is $\Theta(g(x))$ ("big-theta of $g(x)$") if $f(x)$ is **both** $O(g(x))$ and $\Omega(g(x))$.
  - For a function that is $\Theta(g(x))$, we will say that that function "is order $g(x)$."
- *Example:* The function 2^x+x^2 is order 2^x.

Proof: To show that 2^x+x^2 is $O(2^x)$, we can take $C=2$ and $k=4$. Then for $x>k$,

$$|2^x+x^2|=2^x+x^2\leq 2\cdot 2^x.$$

To show that 2^x+x^2 is $\Omega(2^x)$, we can use $C=1$ and $k=1$. For $x>k$,

$$|2^x+x^2|=2^x+x^2\geq 2^x.$$

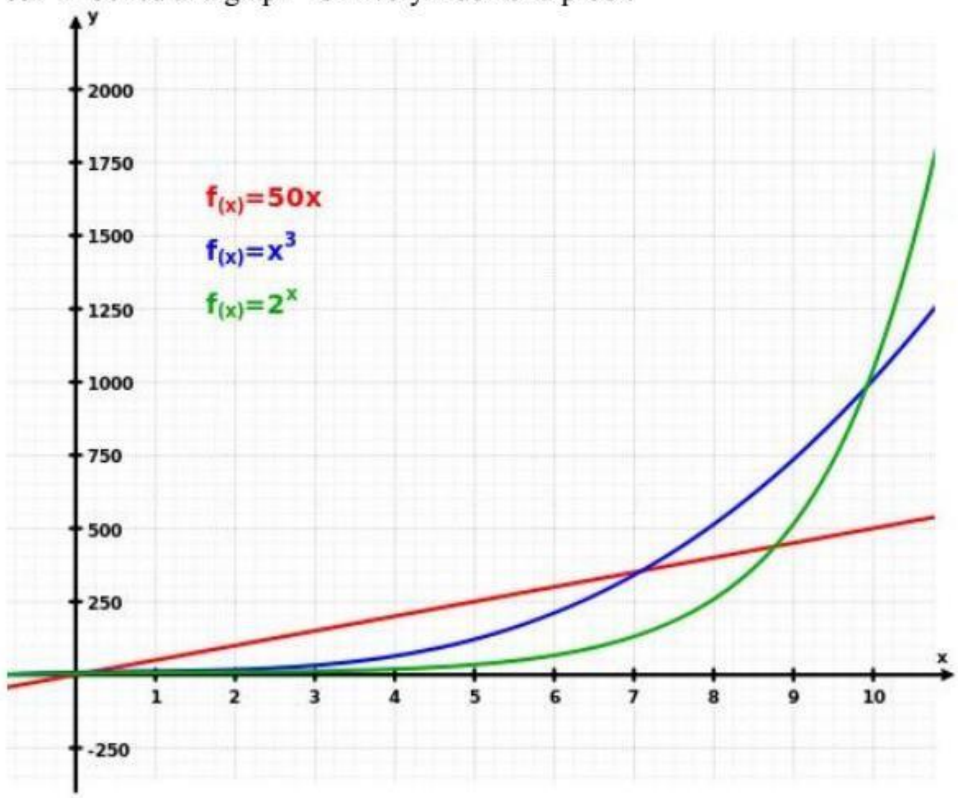
Thus, 2^x+x^2 is $\Theta(2^x)$. ■

- The above theorem gives another way to show big-Θ: if we can show that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, then $f(x)$ is $\Theta(g(x))$.
- *Theorem:* Any degree-n polynomial with $a_n\neq 0$, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with $a_n>0$ is in $\Theta(x^n)$.
- A few results on big-Θ…
  - *Theorem:* If we have two functions $f_1(x)$ which is $\Theta(g_1(x))$ and $f_2(x)$ which is $\Theta(g_2(x))$, and $g_2(x)$ is $O(g_1(x))$, then $f_1(x)+f_2(x)$ is $\Theta(g_1(x)))$.
    - That is, when adding two functions together, the bigger one "wins".

- *Theorem:* If we have two functions $f_1(x)$ which is $\Theta(g(x))$ and $f_2(x)$ which is $O(g(x))$, then $f(x)+g(x)$ is $\Theta(g(x)))$.
- *Theorem:* for a positive constant a, a function $af(x)$ is $\Theta(g(x))$ iff $f(x)$ is $\Theta(g(x))$.
  - That is, leading constants don't matter.

- *Corollary:* Any degree-n polynomial, $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with $a_n>0$ is in $\Theta(x^n)$.

- What functions have a "higher" big-Θ than others is usually fairly obvious from a graph, but "I looked at a graph" isn't very much of a proof.



Source: [Wikipedia Exponential.svg]

- The big-O notation sets up a hierarchy of function growth rates. Here are some of the important "categories":

$$n! \quad 2^n \quad n^3 \quad n^2 \quad n\log n \quad n \quad \sqrt{n}=n^{1/2} \quad \log n \quad 1$$

  - Each function here is big-O of ones above it, but not below.
  - e.g. $n\log n$ is $O(n^2)$, but n^2 is **not** $O(n\log n)$.
  - So in some important way, n^2 grows faster than $n\log n$.

- Where we are headed: we will be able to look at an algorithm and say that one that takes $O(n\log n)$ steps is faster than one that takes $O(n2)$ steps (for large input).

Asymptotic Notation

9.7.1 Little Oh

Definition 9.7.1. For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is *asymptotically smaller* than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

For example, $1000x^{1.9} = o(x^2)$, because $1000x^{1.9}/x^2 = 1000/x^{0.1}$ and since $x^{0.1}$ goes to infinity with x and 1000 is constant, we have $\lim_{x\to\infty} 1000x^{1.9}/x^2 = 0$. This argument generalizes directly to yield

9.7.2 Big Oh

Big Oh is the most frequently used asymptotic notation. It is used to give an upper bound on the growth of a function, such as the running time of an algorithm.

Definition 9.7.5. Given nonnegative functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = O(g)$$

iff

$$\limsup_{x\to\infty} f(x)/g(x) < \infty.$$

This definition[12] makes it clear that

Definition 9.7.12. Given functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = \Omega(g)$$

iff there exists a constant $c > 0$ and an x_0 such that for all $x \geq x_0$, we have $f(x) \geq c|g(x)|$.

In other words, $f(x) = \Omega(g(x))$ means that $f(x)$ is greater than or equal to $g(x)$, except that we are willing to ignore a constant factor and to allow exceptions for small x.

If all this sounds a lot like big-Oh, only in reverse, that's because big-Omega is the opposite of big-Oh. More precisely,

Little Omega

There is also a symbol called little-omega, analogous to little-oh, to denote that one function grows strictly faster than another function.

Definition 9.7.14. For functions $f, g : \mathbb{R} \to \mathbb{R}$ with f nonnegative, we say that

$$f(x) = \omega(g(x))$$

iff

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$

In other words,

$$f(x) = \omega(g(x))$$

iff

$$g(x) = o(f(x)).$$

Definition 9.7.15.

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } g = O(f).$$

The statement $f = \Theta(g)$ can be paraphrased intuitively as "f and g are equal to within a constant factor." Indeed, by Theorem 9.7.13, we know that

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } f = \Omega(g).$$

Example: $n^2 + n = O(n^3)$

Proof:

- Here, we have $f(n) = n^2 + n$, and $g(n) = n^3$
- Notice that if $n \geq 1$, $n \leq n^3$ is clear.
- Also, notice that if $n \geq 1$, $n^2 \leq n^3$ is clear.
- **Side Note:** In general, if $a \leq b$, then $n^a \leq n^b$ whenever $n \geq 1$. This fact is used often in these types of proofs.
- Therefore,

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- We have just shown that

$$n^2 + n \leq 2n^3 \text{ for all } n \geq 1$$

- Thus, we have shown that $n^2 + n = O(n^3)$ (by definition of Big-O, with $n_0 = 1$, and $c = 2$.)

Example: $n^3 + 4n^2 = \Omega(n^2)$

Proof:

- Here, we have $f(n) = n^3 + 4n^2$, and $g(n) = n^2$
- It is not too hard to see that if $n \geq 0$,

$$n^3 \leq n^3 + 4n^2$$

- We have already seen that if $n \geq 1$,

$$n^2 \leq n^3$$

- Thus when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 4n^2$$

- Therefore,

$$1n^2 \leq n^3 + 4n^2 \text{ for all } n \geq 1$$

- Thus, we have shown that $n^3 + 4n^2 = \Omega(n^2)$ (by definition of Big-Ω, with $n_0 = 1$, and $c = 1$.)

Example: $n^2 + 5n + 7 = \Theta(n^2)$

Proof:

- When $n \geq 1$,

$$n^2 + 5n + 7 \leq n^2 + 5n^2 + 7n^2 \leq 13n^2$$

- When $n \geq 0$,

$$n^2 \leq n^2 + 5n + 7$$

- Thus, when $n \geq 1$

$$1n^2 \leq n^2 + 5n + 7 \leq 13n^2$$

Thus, we have shown that $n^2 + 5n + 7 = \Theta(n^2)$ (by definition of Big-Θ, with $n_0 = 1$, $c_1 = 1$, and $c_2 = 13$.)

Show that $\frac{1}{2}n^2 + 3n = \Theta(n^2)$

Proof:

- Notice that if $n \geq 1$,

$$\frac{1}{2}n^2 + 3n \leq \frac{1}{2}n^2 + 3n^2 = \frac{7}{2}n^2$$

- Thus,

$$\frac{1}{2}n^2 + 3n = O(n^2)$$

- Also, when $n \geq 0$,

$$\frac{1}{2}n^2 \le \frac{1}{2}n^2 + 3n$$

- So

$$\frac{1}{2}n^2 + 3n = \Omega(n^2)$$

- Since $\frac{1}{2}n^2 + 3n = O(n^2)$ and $\frac{1}{2}n^2 + 3n = \Omega(n^2)$,

$$\frac{1}{2}n^2 + 3n = \Theta(n^2)$$

KAHE

Show that $(n \log n - 2n + 13) = \Omega(n \log n)$

Proof: We need to show that there exist positive constants c and n_0 such that

$$0 \leq c\,n \log n \leq n \log n - 2\,n + 13 \text{ for all } n \geq n_0.$$

Since $\quad n \log n - 2\,n \leq n \log n - 2\,n + 13,$

we will instead show that

$$c\,n \log n \leq n \log n - 2\,n,$$

which is equivalent to

$$c \leq 1 - \frac{2}{\log n}, \text{ when } n > 1.$$

If $n \geq 8$, then $2/(\log n) \leq 2/3$, and picking $c = 1/3$ suffices. Thus if $c = 1/3$ and $n_0 = 8$, then for all $n \geq n_0$, we have

$$0 \le c\,n \log n \le n \log n - 2\,n \le n \log n - 2\,n + 13.$$

Thus $(n \log n - 2\,n + 13) = \Omega(n \log n)$.

Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Proof:

- We need to find positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 n^2 \le \frac{1}{2}n^2 - 3\,n \le c_2 n^2 \text{ for all } n \ge n_0$$

- Dividing by n^2, we get

$$0 \le c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$$

- $c_1 \le \frac{1}{2} - \frac{3}{n}$ holds for $n \ge 10$ and $c_1 = 1/5$
- $\frac{1}{2} - \frac{3}{n} \le c_2$ holds for $n \ge 10$ and $c_2 = 1$.
- Thus, if $c_1 = 1/5$, $c_2 = 1$, and $n_0 = 10$, then for all $n \ge n_0$,

$$0 \le c_1 n^2 \le \frac{1}{2}n^2 - 3\,n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Thus we have shown that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.

Summary of the Notation

- $f(n) = O(g(n)) \Rightarrow f \preceq g$
- $f(n) = \Omega(g(n)) \Rightarrow f \succeq g$
- $f(n) = \Theta(g(n)) \Rightarrow f \approx g$
- It is important to remember that a Big-O bound is only an *upper bound*. So an algorithm that is $O(n^2)$ might not ever take that much time. It may actually run in $O(n)$ time.
- Conversely, an Ω bound is only a *lower bound*. So an algorithm that is $\Omega(n \log n)$ might actually be $\Theta(2^n)$.
- Unlike the other bounds, a Θ-bound is precise. So, if an algorithm is $\Theta(n^2)$, it runs in quadratic time.

POSSIBLE QUESTION

TWO MARKS

1. Prove that the function $f(x) = 2x^3+10x$ is $O(x^3)$.
2. Prove that the function $3x^2+19x$ is $\Omega(x^2)$.
3. Prove that $n^2 + 5n + 7 = \Theta(n^2)$.
4. Evaluate $\sum_{k=1}^{9}(5k+8)$.
5. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{(2n+1)^2}{5n^2+2n+1}$

SIX MARKS

1. Show that $(n \log n - 2n + 13) = \Omega(n \log n)$.
2. Evaluate the sum $\sum_{k=1}^{8}(5k^2+8k+1)$
3. Evaluate the sum $\sum_{k=7}^{12}(k+1)$
4. Evaluate the sum -5-4-3-2-1+0+1+2+3+4+.....+30
5. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(k\frac{7}{n}\right)^2$
6. The function $f(x)=2x^3+10x$ is $o(x^4)$.
7. Evaluate the limit n tends to infinity $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k+9}{n}$.
8. Show that if we have two functions $f_1(x)$ and $f_2(x)$ both $O(g(x))$, then $f_1(x)+f_2(x)$ is also $O(g(x))$.
9. If $\sum_{k=1}^{n}k^4 = \frac{4n(n+1)(2n+1)(3n^2+3n-1)}{A}$ then find A?
10. Show that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$.
11. The integral $\int_0^5 x^2dx$ is computed as the limit of the sum $\sum_{k=1}^{n}\frac{A}{n}\left(k\frac{A}{n}\right)^2$. What value of A must appear in the sum ?

KAHE

Questions	opt1	opt2	opt3	opt4	Answer
The growth of …………………….. is directly related to the complexity of algorithms.	functions	relations	parameters	polynomials	functions
Growth of ……………………. is simple characterization of algorithm efficiency	polynomials	parameters	functions	relations	functions
Growth of functions allows to compare relative performance of ……………….. algorithms	relations	alternative	same	parameters	alternative
If there are multiple input parameters, we will try to reduce them to a ………………………..parameter, expressing some parameters in terms of the selected parameter.	triple	more than three	single	double	single
Effect of input size ………………………..without bound on running time of algorithm	constant	varies	increase	decrease	increase
If $m \leq n \rightarrow f(m) \leq f(n)$ then it is monotonically……………..	increasing	decreasing	strictly increasing	strictly decreasing	increasing
If greatest integer $\leq$ x then it is……………………………. …….	constant(x)	floor(x)	ceiling(x)	function(x)	floor(x)
If smallest integer $\geq$ x then it is ……………………………… ….	ceiling(x)	constant(x)	function(x)	floor(x)	ceiling(x)
Floor and ceiling functions are monotonically …………………………	strictly increasing	strictly decreasing	increasing	decreasing	increasing
Polynomial is asymptotically……………… ………….. iff ad > 0	zero	positive	non zero	negative	positive
Any positive exponential function grows ………………. than any polynomial	slower	constantly	faster	varies	faster

If lg n = $\log_2 n$ then it is ……………………..	binary logarithm	composition	exponentiation	natural logarithm	binary logarithm
If ln n = $\log_e$ n then it is ……………………..	composition	exponentiation	natural logarithm	binary logarithm	natural logarithm
If lg k n = (lg n)k then it is ……………………..	natural logarithm	binary logarithm	composition	exponentiation	exponentiation
If lg lg n = lg(lg n) then it is ……………………..	binary logarithm	composition	natural logarithm	exponentiation	composition
A function f(n) is ………………………….. bounded if f(n) = $\lg^{O(1)} n$	exponentially	logarithmically	polylogarithmically	polyexponentially	polylogarithmically
Any positive polynomial function grows …………………….. than any polylogarithmic function	varies	faster	slower	constantly	faster
…………………… can be sandwiched between $c_1g(n)$ and $c_2g(n)$, for sufficiently large n	f(n)+g(n)	f(n)	g(n)	f(n)-g(n)	f(n)
O-notation denotes asymptotic …………………………	strictly lower bound	upper bound	lower bound	strictly upper bound	upper bound
Asymptotic notation gives us a method for classifying ……………... according to their rate of growth	functions	series	relations	variables	functions
If f(x) = (x3 – 1) / (3x + 1) then f(x) is	$O(x^2)$	$O(x)$	$O(x^2 / 3)$	$O(1)$	$O(x^2)$
If $f(x) = 3x^2 + x^3\log x$, then f(x) is	$O(x^2)$	$O(x^3)$	$O(x)$	$O(1)$	$O(x^3)$
The big-O notation for $f(n) = (n\log n + n^2)(n^3 + 2)$ is	$O(n^2)$	$O(3^n)$	$O(n^4)$	$O(n^5)$	$O(n^5)$
The big-theta notation for function $f(n) = 2n^3 + n – 1$ is	n	n^2	n^3	n^4	n^3
The big-theta notation for $f(n) = n\log(n^2 + 1) + n^2\log n$ is	$n^2\log n$	n^2	logn	$n\log(n^2)$	$n^2\log n$

The big-omega notation for $f(x, y) = x^5y^3 + x^4y^4 + x^3y^5$ is	x^5y^3	x^5y^5	x^3y^3	x^4y^4	x^3y^3
If $f_1(x)$ is O(g(x)) and $f_2(x)$ is o(g(x)), then $f_1(x) + f_2(x)$ is	O(g(x))	o(g(x))	O(g(x)) + o(g(x))	O(g(x)) - o(g(x))	O(g(x))
The little-o notation for f(x) = xlogx is	x	x^3	x^2	xlogx	x^2
The big-O notation for f(n) = $2\log(n!) + (n^2 + 1)\log n$ is	n	n^2	nlogn	n^2logn	n^2logn
The big-O notation for f(x) = 5logx is	1	x	x^2	x^3	x
The big-Omega notation for $f(x) = 2x^4 + x^2 – 4$ is	x^2	x^3	x	x^4	x^4
f is ………………………….... of g if f is both O(g) and Ω(g)	small-O	big-Θ	big-O	big-Ω	big-Θ
f (n) is asymptotically smaller than g(n) if…………………………..	f (n) = ω(g(n))	f (n) = o(g(n))	f (n) = Ω(g(n))	f (n) = Θ(g(n))	f (n) = o(g(n))
f (n) is asymptotically larger than g(n) if …………………………..	f (n) = Ω(g(n))	f (n) = Θ(g(n))	f (n) = ω(g(n))	f (n) = o(g(n))	f (n) = ω(g(n))

UNIT – III

Recurrences: Recurrence relations, generating functions, linear recurrence relations with constant coefficients and their solution, Substitution Method, recurrence trees, Master theorem.

KAHE

Solving the Recurrence

Claim 10.1.1. *$T_n = 2^n - 1$ satisfies the recurrence:*

$$T_1 = 1$$
$$T_n = 2T_{n-1} + 1 \qquad (\text{for } n \geq 2).$$

Proof. The proof is by induction on n. The induction hypothesis is that $T_n = 2^n - 1$. This is true for $n = 1$ because $T_1 = 1 = 2^1 - 1$. Now assume that $T_{n-1} = 2^{n-1} - 1$ in order to prove that $T_n = 2^n - 1$, where $n \geq 2$:

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 1. \end{aligned}$$

Linear Recurrences

In general, a *homogeneous linear recurrence* has the form

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)$$

where $a_1, a_2, \ldots, a_d$ and d are constants. The *order* of the recurrence is d. Commonly, the value of the function f is also specified at a few points; these are called *boundary conditions*. For example, the Fibonacci recurrence has order $d = 2$ with coefficients $a_1 = a_2 = 1$ and $g(n) = 0$. The boundary conditions are $f(0) = 1$ and $f(1) = 1$. The word "homogeneous" sounds scary, but effectively means "the simpler kind". We'll consider linear recurrences with a more complicated form later.

Theorem 10.3.1. *If $f(n)$ and $g(n)$ are both solutions to a homogeneous linear recurrence, then $h(n) = sf(n) + tg(n)$ is also a solution for all $s, t \in \mathbb{R}$.*

Proof.

$$\begin{aligned} h(n) &= sf(n) + tg(n) \\ &= s\,(a_1 f(n-1) + \ldots + a_d f(n-d)) + t\,(a_1 g(n-1) + \ldots + a_d g(n-d)) \\ &= a_1(sf(n-1) + tg(n-1)) + \ldots + a_d(sf(n-d) + tg(n-d)) \\ &= a_1 h(n-1) + \ldots + a_d h(n-d) \end{aligned}$$

Solving First-Order Recurrences Using Back Substitution

Theorem 2. (Solution of First-Order Recurrence Relations) The solution of

$$T(n) = \begin{cases} cT(n-1) + f(n) & \text{for } n \geq k \\ f(k) & \text{for } n = k \end{cases}$$

where c is a constant and f is a nonzero function of n for $n \geq k$ is

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Motivation for the Proof. First, use back substitution to decide what the general form of the solution might be, and then prove by induction that this is the solution:

$$\begin{aligned} T(n) &= cT(n-1) + f(n) \\ &= c(cT(n-2) + f(n-1)) + f(n) \\ &= c^2T(n-2) + cf(n-1) + f(n) \\ &= c^2(cT(n-3) + f(n-2)) + cf(n-1) + f(n) \\ &= c^3T(n-3) + c^2 f(n-2) + cf(n-1) + f(n) \end{aligned}$$

Using back substitution one more time gives

$$\begin{aligned} T(n) &= c^3\left[cT(n-4) + f(n-3)\right] + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + c^3 f(n-3) + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + \sum_{l=n-3}^{n} c^{n-l} f(l) \end{aligned}$$

If back substitution is continued until the argument of T is k—that is, for $n-k$ steps—then the expression for $T(n)$ becomes

$$\begin{aligned} T(n) &= c^{n-k}T(n-(n-k)) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \\ &= c^{n-k}T(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \end{aligned}$$

Since $T(k) = f(k)$, replace the reference to T on the right-hand side of the equation, getting

$$T(n) = c^{n-k} f(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l)$$
$$= \sum_{l=n-k}^{n} c^{n-l} f(l)$$

Proof. By induction, show that

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Let $n_0 = k$. Let $\mathcal{T} = \{n \in \mathbb{N} : n \geq k \text{ and } T(n) \text{ is a solution}\}$.

(Base step) First, show that

$$\sum_{l=k}^{n} c^{n-l} f(l)$$

is a solution for $n = k$ so that $k \in \mathcal{T}$.

$$\sum_{l=k}^{k} c^{k-l} f(l) = c^{k-k} f(k) = f(k) = T(k)$$

(Inductive step) Now, assume that $T(n)$ is given by this expression for $n \geq n_0$, that is, $T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$. Now prove that $T(n+1)$ is also given by this expression: In this case, prove that $T(n+1) = \sum_{l=k}^{n+1} c^{n-l} f(l)$.

$$T(n+1) = cT(n) + f(n+1) \quad \text{(Definition of recurrence relation)}$$
$$= c\sum_{l=k}^{n} c^{n-l} f(l) + f(n+1) \quad \text{(Inductive hypothesis)}$$
$$= \sum_{l=k}^{n} c^{n-l+1} f(l) + f(n+1)$$
$$= \sum_{l=k}^{n+1} c^{n+1-l} f(l)$$

This proves $n + 1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \{n \in \mathbb{N} : n \geq k\}$. ■

Example 1. Solve

$$T(n) = \begin{cases} T(n-1) + n^2 & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = n^2$ for $n \geq 0$, $c = 1$, and $k = 0$. Since $T(0) = f(0)$, by Corollary 1 the solution is

$$T(n) = \sum_{l=1}^{n} l^2 = \frac{1}{6} \cdot (2n+1) \cdot n \cdot (n+1)$$

See Theorem 9(b) in Section 7.10 for a derivation of this formula. ■

Example 2. Solve

$$T(n) = \begin{cases} 3T(n-1) + 4 & \text{for } n \geq 1 \\ 4 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = 4$ for $n \geq 0$, $c = 3$, and $k = 0$. By Corollary 2, the solution is

$$T(n) = 4 \cdot \frac{3^{n+1} - 1}{3 - 1} = 2 \cdot (3^{n+1} - 1)$$ ■

Rules for Solving Second-Order Recurrence Relations

Solving Second-Order Homogeneous Recurrence Relations with Constant Coefficients Using the Complementary Equation with Distinct Real Roots

$$H(n) + AH(n-1) + BH(n-2) = 0,$$
$$H(n_1) = D, \text{ and } H(n_2) = E.$$

STEP 1: Assume $f(n) = c^n$ is a solution, and substitute for $H(n)$, yielding the characteristic equation

$$c^2 + Ac + B = 0$$

STEP 2: Find the roots of the characteristic equation: c_1 and c_2. Use the quadratic formula if the equation does not factor. If $c_1 \neq c_2$, then the general solution is

$$S(n) = Ac_1^n + Bc_2^n$$

STEP 3: Use the initial conditions to form the system of equations

$$H(n_1) = D = Ac_1^{n_1} + Bc_2^{n_2}$$
$$H(n_2) = E = Ac_1^{n_2} + Bc_2^{n_2}$$

STEP 4: Solve the system of equations found in step 3, getting A_0 and B_0 as the two solutions. Form the particular solution

$$H(n) = A_0c_1{}^n + B_0c_2{}^n$$

Example 1. Solve the recurrence relation $a_n - 6a_{n-1} - 7a_{n-2} = 0$ for $n \geq 5$ where $a_3 = 344$ and $a_4 = 2400$.

Solution. Form the characteristic equation and then factor it:

$$c^2 - 6c - 7 = 0$$
$$c = 7, -1$$

Form the general solution of the recurrence relation $a_n = A7^n + B(-1)^n$, and solve the system of equations determined by the boundary values $a_3 = 344$ and $a_4 = 2400$ to get the particular solution:

$$a_3 = A7^3 + B(-1)^3$$
$$a_4 = A7^4 + B(-1)^4$$

Now, substituting 344 and 2400 for a_3 and a_4 gives

$$344 = 343A - B$$
$$2400 = 2401A + B$$

Adding the two equations gives

$$2744 = 2744A$$
$$1 = A$$

It follows that $B = -1$. Therefore, $a_n = 7^n + (-1)^{n+1}$ for $n \geq 3$ is the particular solution. ■

Substitution Method

- Guess the form of solution and use induction to find constants
- Determine upper bound on the recurrence

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

Guess the solution as: $T_n = O(n \lg n)$
Now, prove that $T_n \leq cn \lg n$ for some $c > 0$
Assume that the bound holds for $\lfloor \frac{n}{2} \rfloor$
Substituting into the recurrence

$$\begin{aligned} T_n &\leq 2(c \left\lfloor \frac{n}{2} \right\rfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor)) + n \\ &\leq cn \lg\left(\frac{n}{2}\right) + n \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\leq cn \lg n \quad \forall c \geq 1 \end{aligned}$$

Boundary condition: Let the only bound be $T_1 = 1$

$$\nexists c \mid T_1 \leq c1 \lg 1 = 0$$

Problem overcome by the fact that asymptotic notation requires us to prove

$$T_n \leq cn \lg n \text{ for } n \geq n_0$$

Include T_2 and T_3 as boundary conditions for the proof

$$T_2 = 4 \quad T_3 = 5$$

Choose c such that $T_2 \leq c2 \lg 2$ and $T_3 \leq c3 \lg 3$
True for any $c \geq 2$

– If a recurrence is similar to a known recurrence, it is reasonable to guess a similar solution

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

If n is large, difference between $T_{\lfloor \frac{n}{2} \rfloor}$ and $T_{\lfloor \frac{n}{2} \rfloor + 17}$ is relatively small

– Prove upper and lower bounds on a recurrence and reduce the range of uncertainty.
Start with a lower bound of $T_n = \Omega(n)$ and an initial upper bound of $T_n = O(n^2)$. Gradually lower the upp
bound and raise the lower bound to get asymptotically tight solution of $T_n = \Theta(n \lg n)$

- Recursion trees
  - Recurrence

$$T_n = 2T_{\frac{n}{2}} + n^2$$

Assume n to be an exact power of 2.

$$\begin{aligned}
T_n &= n^2 + 2T_{\frac{n}{2}} \\
&= n^2 + 2\left(\left(\frac{n}{2}\right)^2 + 2T_{\frac{n}{4}}\right) \\
&= n^2 + \frac{n^2}{2} + 4\left(\left(\frac{n}{4}\right)^2 + 2T_{\frac{n}{8}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + 8\left(\left(\frac{n}{8}\right)^2 + 2T_{\frac{n}{16}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{8} + \cdots \\
&= n^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) \\
&= \Theta(n^2)
\end{aligned}$$

The values above decrease geometrically by a constant factor.

– Recurrence

$$T_n = T_{\frac{n}{3}} + T_{\frac{2n}{3}} + n$$

Longest path from root to a leaf

$$n \rightarrow \left(\frac{2}{3}\right) n \rightarrow \left(\frac{2}{3}\right)^2 n \rightarrow \cdots 1$$

KAHE

$\left(\frac{2}{3}\right)^k n = 1$ when $k = \log_{\frac{3}{2}} n$, k being the height of the tree
Upper bound to the solution to the recurrence – $n \log_{\frac{3}{2}} n$, or $O(n \log n)$

The Master Method

- Suitable for recurrences of the form

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where $a \geq 1$ and $b > 1$ are constants, and
$f(n)$ is an asymptotically positive function

- For mergesort, $a = 2$, $b = 2$, and $f(n) = \Theta(n)$
- Master Theorem

Theorem 2 *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let T_n be defined on the nonnegative integers by the recurrence*

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where we interpret $\frac{n}{b}$ to mean either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then T_n can be bounded asymptotically as follows

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T_n = \Theta(n^{\log_b a})$*
2. *If $f(n) = \Theta(n^{\log_b a})$, then $T_n = \Theta(n^{\log_b a} \lg n)$*
3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T_n = \Theta(f(n))$*

- In all three cases, compare $f(n)$ with $n^{\log_b a}$
- Solution determined by the larger of the two
  - * Case 1: $n^{\log_b a} > f(n)$
    Solution $T_n = \Theta(n^{\log_b a})$
  - * Case 2: $n^{\log_b a} \approx f(n)$
    Multiply by a logarithmic factor
    Solution $T_n = \Theta(n^{\log_b a} \lg n) = \Theta(f(n) \lg n)$
  - * Case 3: $f(n) > n^{\log_b a}$
    Solution $T_n = \Theta(f(n))$
- In case 1, $f(n)$ must be asymptotically smaller than $n^{\log_b a}$ by a factor of n^{ϵ} for some constant $\epsilon > 0$
- In case 3, f_n must be polynomially larger than $n^{\log_b a}$ and satisfy the "regularity" condition that $af(\frac{n}{b}) \leq cf(n)$

- Using the master method
  - Recurrence

$$T_n = 9T_{\frac{n}{3}} + n$$

$a = 9,\ b = 3,\ f(n) = n$
$n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
$f(n) = O(n^{\log_3 9-\epsilon})$, where $\epsilon = 1$
Apply case 1 of master theorem and conclude $T_n = \Theta(n^2)$

  - Recurrence

$$T_n = T_{\frac{2n}{3}} + 1$$

$a = 1,\ b = \frac{3}{2},\ f(n) = 1$
$n^{\log_b a} = n^{\log_{\frac{3}{2}} 1} = n^0 = 1$
$f(n) = \Theta(n^{\log_b a}) = \Theta(1)$
Apply case 2 of master theorem and conclude $T_n = \Theta(\lg n)$

  - Recurrence

$$T_n = 3T_{\frac{n}{4}} + n\lg n$$

$a = 3,\ b = 4,\ f(n) = n\lg n$
$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$
$f(n) = \Omega(n^{\log_4 3+\epsilon})$, where $\epsilon \approx 0.2$
Apply case 3, if regularity condition holds for $f(n)$
For large n, $af(\frac{n}{b}) = 3\frac{n}{4}\lg(\frac{n}{4}) \leq \frac{3}{4}n\lg n = cf(n)$ for $c = \frac{3}{4}$
Therefore, $T_n = \Theta(n\lg n)$

(13)

Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Ex:

If the seq $a_n = 3.2^n$, $n \geq 1$ then find the corresponding recurrence relation.

Soln:

For $n \geq 1$

$a_n = 3.2^n$.

$a_{n-1} = 3.2^{n-1}$

$= 3 \cdot \frac{2^n}{2}$

$a_{n-1} = \frac{a_n}{2}$

$a_n = 2(a_{n-1})$, for $n \geq 1$ with $a_0 = 3$

Ex:

Solve the recurrence relation defined by $S_0 = 100$ and $S_k = (1.08) S_{k-1}$ for $k \geq 1$

Soln:

Gn $S_0 = 100$

$S_k = (1.08) S_{k-1}$, $k \geq 1$

$$16(\alpha_1+2\alpha_2) = 16$$

$$\alpha_1+2\alpha_2 = 1 \longrightarrow (1)$$

$$S_3 = 80.$$

$$\Rightarrow (\alpha_1+3\alpha_2)\,4^3 = 80$$

$$\Rightarrow 64(\alpha_1+3\alpha_2) = 80.$$

$$\Rightarrow \alpha_1+3\alpha_2 = \frac{80}{64} = \frac{5}{4}$$

$$\alpha_1+3\alpha_2 = \frac{5}{4} \longrightarrow (2)$$

$$(2)-(1) \Rightarrow \boxed{\alpha_2 = \frac{1}{4}}$$

$$\Rightarrow \alpha_1 + 2\left(\tfrac{1}{4}\right) = 1$$

$$\alpha_1 + \tfrac{1}{2} = 1$$

$$\boxed{\alpha_1 = \tfrac{1}{2}}$$

$$S(k) = (\alpha_1+\alpha_2 k)\,4^k$$

$$= (2+k)\,4^{k-1}$$

which is the reqd soln

POSSIBLE QUESTIONS

TWO MARKS

1. Define characteristics equation.

2. Solve $a_n - 4a_{n-1} = 0$ for n ≥ 2 with $a_0 = 1, a_1 = 1$.

3. State Master theorem.

4. Write the methods for solving recurrence.

5. If the sequence $a_n = 3.2^n, n \geq 1$ then find the corresponding recurrence relation.

SIX MARKS

1.Solve the recurrence relation $a_n = a_{n-1}+2a_{n-2}$ with a_0=2 and a_1=7.

2.Solve $T(n) = 2\,T\,(\frac{n}{2}) + n$ using subtitution method

3.Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10$, $a_1 = 41$ using generating function.

4.Find the sequence whose generating function is $\frac{32-22x}{2-3x+x^2}$ using partial functions.

5.Solve the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$ with the initial condition $a_0 = a_1 = 1$.

6.Solve $T(n) = 7\,T\,(\frac{n}{2}) + \Theta\,(n^2)$ by iterative method.

7.Solve the recurrence relation a_n+2-6a_{n+1}+9a_n=0 with a_0=1 & a_1 =4.

8.Using the generating function, solve the recurrence relation a_n =3a_{n-1} for n≥1 with a_0=2.

9.Solve $T(n) = 2\,T(\sqrt{n}) + \log n$

10.Using generating function, solve the recurrence relation a_n =3a_{n-1} +1for n≥ 1 with a_0=1.

Questions	opt1	opt2	opt3	opt4	Answer
The procedure for finding the terms of a sequence in a recursive manner is called ………………………..	reflexive relation	recurrence relation	transitive relation	linear relation	recurrence relation
An equation or inequality that describes a function in terms of its value on smaller inputs known as ……………………..	non linear relation	linear relation	symmetric relation	recurrence relation	recurrence relation
Recurrence can be solved to derive the …………….. time	generating	running	starting	terminating	running
In recurrence tree, T_n = …………………………	2Tn/2 - n2	2Tn/2 * n2	2Tn/2 + n2	3Tn/2 + n2	2Tn/2 + n2
A ………………………... recurrence has the form f (n)= a1f (n – 1)+ a2f (n – 2)+…….+adf(n –d)	non homogeneous linear	non linear	linear	homogeneous linear	homogeneous linear
If f (n) and g(n) are both solutions to a …………………... recurrence, then h(n)= sf(n) + tg(n) is also a solution for all s, t ϵ R.	non linear	linear	homogeneous linear	non homogeneous linear	homogeneous linear
Generating functions can be used to find a solution to any ………………………..	homogeneous recurrence	non homogeneous linear recurrence	linear recurrence	non linearrecurrence	linear recurrence
The generating function for choosing elements from a union of disjoint sets is the …………………... of the generating functions for choosing from each set.	product	difference	equal	sum	product
Recurrence relation is a formula that relates two or more successive terms in a ………………………………	values	series	sequence	variables	sequence

Any recurrence relation is accompanied by ……………………………. which specifies the first term	zero condition	initial condition	boundary condition	final condition	initial condition
The purpose of solving a recurrence relation is to find a formula for the general term of the sequence given by that ………………………………	symmetric relation	transitive relation	recurrence relation	reflexive relation	recurrence relation
Solving ……………………... is used in computer science to assess the running time of recursive algorithms	transitive relation	recurrence relations	reflexive relation	symmetric relation	recurrence relations
Linear homogeneous recurrence relations with…………..coefficients and their sequences.	non zero	constant	varied	zero	constant
A recurrence relation is homogeneous if …………………………….	h(n) = 1	h(n) = 0	h(n) = x	h(n) = x+y	h(n) = 0
Methods for solving recurrences is/are …………………….	both Substitution method and Iteration method	direct method	Iteration method	Substitution method	both Substitution method and Iteration method
Recursion-tree method and Master method are ……………………………. method	constant	Substitution	Iteration	direct	Iteration
Sometimes recurrences can be reduced to simpler ones by changing ……………………….	variables	values	series	constants	variables
A……………………………… ………….. can be used to visualize the iteration procedure	fibonacci series	Generating functions	power series	recurrence tree	recurrence tree
The classical Tower of Hanoi problem gives us the recurrence T(n) = 2T(n − 1) + 1 with base case ………………….	T(1) = 0	T(0) = 1	T(0) = 0	T(1) = 1	T(0) = 0

A common class of recurrences arises in the context of recursive backtracking algorithms and counting problems is called	non homogeneous recurrence	linear recurrences	non linear recurrences	homogeneous recurrence	linear recurrences
A recurrence T(n) = f(n)T(n − 1) + g(n) is called a ……………………….. linear recurrence.	higher order	third order	first order	second order	first order
A recurrence in which T(n) is expressed in terms of a sum of constant multiples of T(k) for certain values k < n is called a ………………………………	varied	constant	different	zero	constant
The idea of a Recursion Tree is to expand T (n) to a tree with the ……………………………...total cost.	zero	same	unit	different	same
Recurrences can be used to represent the runtime of ………………………………………	Generating functions	recursive functions	homogeneous functions	linear functions	recursive functions.
The pattern in recurrence tree method is typically ……………………………..	constant series	fibonacci series	a arithmetic or geometric series.	taylor series	a arithmetic or geometric series
In linear recurrence each term of a sequence is a…………………………... of earlier terms in the sequence.	linear function	non linear functions	recursive functions	Generating functions	linear function
Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal …………………	fibonacci series	power series	taylor series	constant series	power series
…………………….. can be used for solving a variety of counting problems.	Generating functions	non linear functions	linear functions	homogeneous functions	Generating functions

Generating functions can be used for solving ……………………	homogeneous functions	recurrence relations	non linear function	linear functions	recurrence relations
………………………….. can be used for proving some of the combinatorial identities	linear functions	homogeneous functions	Generating functions	non linear functions	Generating functions
Generating functions can be used for finding asymptotic formulae for terms of ………………	relations	sequences	functions	seires	sequences
If the recurrence equations is Fn = Fn-1 + Fn-2 with initial values a1 = a2 = 1 then it is …………………	Pell number	Fibonacci number	Padovan sequence	Lucas number	Fibonacci number
If the recurrence equations is Fn = Fn-1 + Fn-2 with initial values a1 = 1, a2 = 3 then it is …………………	Fibonacci number	Padovan sequence	Lucas number	Pell number	Lucas number
If the recurrence equations is Fn = Fn-2 + Fn-3 with initial values a1 = a2 = a3 = 1 then it is …………………	Padovan sequence	Pell number	Fibonacci number	Lucas number	Padovan sequence
If the recurrence equations is Fn = 2Fn-1 + Fn-2 with initial values a1 = 0, a2 = 1 then it is …………………	Lucas number	Padovan sequence	Fibonacci number	Pell number	Pell number
The recurrence for Fibonacci numbers Fn = Fn-1 + Fn-2 is a linear homogeneous recurrence relation of degree …………………………..	three	two	four	one	two

UNIT – III

Recurrences: Recurrence relations, generating functions, linear recurrence relations with constant coefficients and their solution, Substitution Method, recurrence trees, Master theorem.

KAHE

Solving the Recurrence

Claim 10.1.1. $T_n = 2^n - 1$ *satisfies the recurrence:*

$$T_1 = 1$$
$$T_n = 2T_{n-1} + 1 \qquad \textit{(for } n \geq 2\textit{)}.$$

Proof. The proof is by induction on n. The induction hypothesis is that $T_n = 2^n - 1$. This is true for $n = 1$ because $T_1 = 1 = 2^1 - 1$. Now assume that $T_{n-1} = 2^{n-1} - 1$ in order to prove that $T_n = 2^n - 1$, where $n \geq 2$:

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 1. \end{aligned}$$

Linear Recurrences

In general, a *homogeneous linear recurrence* has the form

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)$$

where $a_1, a_2, \ldots, a_d$ and d are constants. The *order* of the recurrence is d. Commonly, the value of the function f is also specified at a few points; these are called *boundary conditions*. For example, the Fibonacci recurrence has order $d = 2$ with coefficients $a_1 = a_2 = 1$ and $g(n) = 0$. The boundary conditions are $f(0) = 1$ and $f(1) = 1$. The word "homogeneous" sounds scary, but effectively means "the simpler kind". We'll consider linear recurrences with a more complicated form later.

Theorem 10.3.1. *If $f(n)$ and $g(n)$ are both solutions to a homogeneous linear recurrence, then $h(n) = sf(n) + tg(n)$ is also a solution for all $s, t \in \mathbb{R}$.*

Proof.

$$\begin{aligned} h(n) &= sf(n) + tg(n) \\ &= s\,(a_1 f(n-1) + \ldots + a_d f(n-d)) + t\,(a_1 g(n-1) + \ldots + a_d g(n-d)) \\ &= a_1(sf(n-1) + tg(n-1)) + \ldots + a_d(sf(n-d) + tg(n-d)) \\ &= a_1 h(n-1) + \ldots + a_d h(n-d) \end{aligned}$$

Solving First-Order Recurrences Using Back Substitution

Theorem 2. (Solution of First-Order Recurrence Relations) The solution of

$$T(n) = \begin{cases} cT(n-1) + f(n) & \text{for } n \geq k \\ f(k) & \text{for } n = k \end{cases}$$

where c is a constant and f is a nonzero function of n for $n \geq k$ is

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Motivation for the Proof. First, use back substitution to decide what the general form of the solution might be, and then prove by induction that this is the solution:

$$\begin{aligned} T(n) &= cT(n-1) + f(n) \\ &= c(cT(n-2) + f(n-1)) + f(n) \\ &= c^2T(n-2) + cf(n-1) + f(n) \\ &= c^2(cT(n-3) + f(n-2)) + cf(n-1) + f(n) \\ &= c^3T(n-3) + c^2f(n-2) + cf(n-1) + f(n) \end{aligned}$$

Using back substitution one more time gives

$$\begin{aligned} T(n) &= c^3[cT(n-4) + f(n-3)] + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + c^3f(n-3) + \sum_{l=n-2}^{n} c^{n-l} f(l) \\ &= c^4T(n-4) + \sum_{l=n-3}^{n} c^{n-l} f(l) \end{aligned}$$

If back substitution is continued until the argument of T is k—that is, for $n-k$ steps—then the expression for $T(n)$ becomes

$$\begin{aligned} T(n) &= c^{n-k}T(n-(n-k)) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \\ &= c^{n-k}T(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l) \end{aligned}$$

Since $T(k) = f(k)$, replace the reference to T on the right-hand side of the equation, getting

$$T(n) = c^{n-k} f(k) + \sum_{l=n-k+1}^{n} c^{n-l} f(l)$$
$$= \sum_{l=n-k}^{n} c^{n-l} f(l)$$

Proof. By induction, show that

$$T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$$

Let $n_0 = k$. Let $\mathcal{T} = \{n \in \mathbb{N} : n \geq k \text{ and } T(n) \text{ is a solution}\}$.

(Base step) First, show that

$$\sum_{l=k}^{n} c^{n-l} f(l)$$

is a solution for $n = k$ so that $k \in \mathcal{T}$.

$$\sum_{l=k}^{k} c^{k-l} f(l) = c^{k-k} f(k) = f(k) = T(k)$$

(Inductive step) Now, assume that $T(n)$ is given by this expression for $n \geq n_0$, that is, $T(n) = \sum_{l=k}^{n} c^{n-l} f(l)$. Now prove that $T(n+1)$ is also given by this expression: In this case, prove that $T(n+1) = \sum_{l=k}^{n+1} c^{n-l} f(l)$.

$$T(n+1) = cT(n) + f(n+1) \quad \text{(Definition of recurrence relation)}$$
$$= c\sum_{l=k}^{n} c^{n-l} f(l) + f(n+1) \quad \text{(Inductive hypothesis)}$$
$$= \sum_{l=k}^{n} c^{n-l+1} f(l) + f(n+1)$$
$$= \sum_{l=k}^{n+1} c^{n+1-l} f(l)$$

This proves $n + 1 \in \mathcal{T}$.

By the Principle of Mathematical Induction, $\mathcal{T} = \{n \in \mathbb{N} : n \geq k\}$. ■

Example 1. Solve

$$T(n) = \begin{cases} T(n-1) + n^2 & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = n^2$ for $n \geq 0$, $c = 1$, and $k = 0$. Since $T(0) = f(0)$, by Corollary 1 the solution is

$$T(n) = \sum_{l=1}^{n} l^2 = \frac{1}{6} \cdot (2n+1) \cdot n \cdot (n+1)$$

See Theorem 9(b) in Section 7.10 for a derivation of this formula. ■

Example 2. Solve

$$T(n) = \begin{cases} 3T(n-1) + 4 & \text{for } n \geq 1 \\ 4 & \text{for } n = 0 \end{cases}$$

Solution. In the general formula, $f(n) = 4$ for $n \geq 0$, $c = 3$, and $k = 0$. By Corollary 2, the solution is

$$T(n) = 4 \cdot \frac{3^{n+1} - 1}{3 - 1} = 2 \cdot (3^{n+1} - 1)$$ ■

Rules for Solving Second-Order Recurrence Relations

Solving Second-Order Homogeneous Recurrence Relations with Constant Coefficients Using the Complementary Equation with Distinct Real Roots

$$H(n) + AH(n-1) + BH(n-2) = 0,$$
$$H(n_1) = D, \text{ and } H(n_2) = E.$$

STEP 1: Assume $f(n) = c^n$ is a solution, and substitute for $H(n)$, yielding the characteristic equation

$$c^2 + Ac + B = 0$$

STEP 2: Find the roots of the characteristic equation: c_1 and c_2. Use the quadratic formula if the equation does not factor. If $c_1 \neq c_2$, then the general solution is

$$S(n) = Ac_1^n + Bc_2^n$$

STEP 3: Use the initial conditions to form the system of equations

$$H(n_1) = D = Ac_1^{n_1} + Bc_2^{n_2}$$
$$H(n_2) = E = Ac_1^{n_2} + Bc_2^{n_2}$$

STEP 4: Solve the system of equations found in step 3, getting A_0 and B_0 as the two solutions. Form the particular solution

$$H(n) = A_0 c_1{}^n + B_0 c_2{}^n$$

Example 1. Solve the recurrence relation $a_n - 6a_{n-1} - 7a_{n-2} = 0$ for $n \geq 5$ where $a_3 = 344$ and $a_4 = 2400$.

Solution. Form the characteristic equation and then factor it:

$$c^2 - 6c - 7 = 0$$
$$c = 7, -1$$

Form the general solution of the recurrence relation $a_n = A7^n + B(-1)^n$, and solve the system of equations determined by the boundary values $a_3 = 344$ and $a_4 = 2400$ to get the particular solution:

$$a_3 = A7^3 + B(-1)^3$$
$$a_4 = A7^4 + B(-1)^4$$

Now, substituting 344 and 2400 for a_3 and a_4 gives

$$344 = 343A - B$$
$$2400 = 2401A + B$$

Adding the two equations gives

$$2744 = 2744A$$
$$1 = A$$

It follows that $B = -1$. Therefore, $a_n = 7^n + (-1)^{n+1}$ for $n \geq 3$ is the particular solution. ■

Substitution Method

- Guess the form of solution and use induction to find constants
- Determine upper bound on the recurrence

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

Guess the solution as: $T_n = O(n \lg n)$
Now, prove that $T_n \leq cn \lg n$ for some $c > 0$
Assume that the bound holds for $\lfloor \frac{n}{2} \rfloor$
Substituting into the recurrence

$$\begin{aligned} T_n &\leq 2(c \left\lfloor \frac{n}{2} \right\rfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor)) + n \\ &\leq cn \lg\left(\frac{n}{2}\right) + n \\ &= cn \lg n - cn \lg 2 + n \\ &= cn \lg n - cn + n \\ &\leq cn \lg n \quad \forall c \geq 1 \end{aligned}$$

Boundary condition: Let the only bound be $T_1 = 1$

$$\nexists c \mid T_1 \leq c1 \lg 1 = 0$$

Problem overcome by the fact that asymptotic notation requires us to prove

$$T_n \leq cn \lg n \text{ for } n \geq n_0$$

Include T_2 and T_3 as boundary conditions for the proof

$$T_2 = 4 \quad T_3 = 5$$

Choose c such that $T_2 \leq c2 \lg 2$ and $T_3 \leq c3 \lg 3$
True for any $c \geq 2$

– If a recurrence is similar to a known recurrence, it is reasonable to guess a similar solution

$$T_n = 2T_{\lfloor \frac{n}{2} \rfloor} + n$$

If n is large, difference between $T_{\lfloor \frac{n}{2} \rfloor}$ and $T_{\lfloor \frac{n}{2} \rfloor + 17}$ is relatively small

– Prove upper and lower bounds on a recurrence and reduce the range of uncertainty.
Start with a lower bound of $T_n = \Omega(n)$ and an initial upper bound of $T_n = O(n^2)$. Gradually lower the upp
bound and raise the lower bound to get asymptotically tight solution of $T_n = \Theta(n \lg n)$

- Recursion trees
  - Recurrence

$$T_n = 2T_{\frac{n}{2}} + n^2$$

Assume n to be an exact power of 2.

$$\begin{aligned}
T_n &= n^2 + 2T_{\frac{n}{2}} \\
&= n^2 + 2\left(\left(\frac{n}{2}\right)^2 + 2T_{\frac{n}{4}}\right) \\
&= n^2 + \frac{n^2}{2} + 4\left(\left(\frac{n}{4}\right)^2 + 2T_{\frac{n}{8}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + 8\left(\left(\frac{n}{8}\right)^2 + 2T_{\frac{n}{16}}\right) \\
&= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{8} + \cdots \\
&= n^2(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) \\
&= \Theta(n^2)
\end{aligned}$$

The values above decrease geometrically by a constant factor.

– Recurrence

$$T_n = T_{\frac{n}{3}} + T_{\frac{2n}{3}} + n$$

Longest path from root to a leaf

$$n \rightarrow \left(\frac{2}{3}\right) n \rightarrow \left(\frac{2}{3}\right)^2 n \rightarrow \cdots 1$$

KAHE

$\left(\frac{2}{3}\right)^k n = 1$ when $k = \log_{\frac{3}{2}} n$, k being the height of the tree
Upper bound to the solution to the recurrence – $n \log_{\frac{3}{2}} n$, or $O(n \log n)$

The Master Method

- Suitable for recurrences of the form

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where $a \geq 1$ and $b > 1$ are constants, and
$f(n)$ is an asymptotically positive function

- For mergesort, $a = 2$, $b = 2$, and $f(n) = \Theta(n)$
- Master Theorem

Theorem 2 *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let T_n be defined on the nonnegative integers by the recurrence*

$$T_n = aT_{\frac{n}{b}} + f(n)$$

where we interpret $\frac{n}{b}$ to mean either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then T_n can be bounded asymptotically as follows

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T_n = \Theta(n^{\log_b a})$*
2. *If $f(n) = \Theta(n^{\log_b a})$, then $T_n = \Theta(n^{\log_b a} \lg n)$*
3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T_n = \Theta(f(n))$*

– In all three cases, compare $f(n)$ with $n^{\log_b a}$

– Solution determined by the larger of the two

* Case 1: $n^{\log_b a} > f(n)$
  Solution $T_n = \Theta(n^{\log_b a})$
* Case 2: $n^{\log_b a} \approx f(n)$
  Multiply by a logarithmic factor
  Solution $T_n = \Theta(n^{\log_b a} \lg n) = \Theta(f(n) \lg n)$
* Case 3: $f(n) > n^{\log_b a}$
  Solution $T_n = \Theta(f(n))$

– In case 1, $f(n)$ must be asymptotically smaller than $n^{\log_b a}$ by a factor of n^{ϵ} for some constant $\epsilon > 0$

– In case 3, f_n must be polynomially larger than $n^{\log_b a}$ and satisfy the "regularity" condition that $af(\frac{n}{b}) \leq cf(n)$

- Using the master method
  - Recurrence

$$T_n = 9T_{\frac{n}{3}} + n$$

$a = 9,\ b = 3,\ f(n) = n$
$n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
$f(n) = O(n^{\log_3 9-\epsilon})$, where $\epsilon = 1$
Apply case 1 of master theorem and conclude $T_n = \Theta(n^2)$

  - Recurrence

$$T_n = T_{\frac{2n}{3}} + 1$$

$a = 1,\ b = \frac{3}{2},\ f(n) = 1$
$n^{\log_b a} = n^{\log_{\frac{3}{2}} 1} = n^0 = 1$
$f(n) = \Theta(n^{\log_b a}) = \Theta(1)$
Apply case 2 of master theorem and conclude $T_n = \Theta(\lg n)$

  - Recurrence

$$T_n = 3T_{\frac{n}{4}} + n\lg n$$

$a = 3,\ b = 4,\ f(n) = n\lg n$
$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$
$f(n) = \Omega(n^{\log_4 3+\epsilon})$, where $\epsilon \approx 0.2$
Apply case 3, if regularity condition holds for $f(n)$
For large n, $af(\frac{n}{b}) = 3\frac{n}{4}\lg(\frac{n}{4}) \le \frac{3}{4}n\lg n = cf(n)$ for $c = \frac{3}{4}$
Therefore, $T_n = \Theta(n\lg n)$

(13)

Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Ex:

If the seq $a_n = 3.2^n$, $n \geq 1$ then find the corresponding recurrence relation.

Soln:

For $n \geq 1$

$$a_n = 3.2^n.$$

$$a_{n-1} = 3.2^{n-1}$$

$$= 3 \cdot \frac{2^n}{2}$$

$$a_{n-1} = \frac{a_n}{2}$$

$$a_n = 2(a_{n-1}), \text{ for } n \geq 1 \text{ with } a_0 = 3$$

Ex:

Solve the recurrence relation defined by $S_0 = 100$ and $S_k = (1.08) S_{k-1}$ for $k \geq 1$

Soln:

Given $S_0 = 100$

$S_k = (1.08) S_{k-1}$, $k \geq 1$

$$16(\alpha_1+2\alpha_2) = 16$$

$$\alpha_1+2\alpha_2 = 1 \longrightarrow (1)$$

$$S_3 = 80.$$

$$\Rightarrow (\alpha_1+3\alpha_2)4^3 = 80$$

$$\Rightarrow 64(\alpha_1+3\alpha_2) = 80.$$

$$\Rightarrow \alpha_1+3\alpha_2 = \frac{80}{64} = \frac{5}{4}$$

$$\alpha_1+3\alpha_2 = \frac{5}{4} \longrightarrow (2)$$

$$(2)-(1) \Rightarrow \boxed{\alpha_2 = \frac{1}{4}}$$

$$\Rightarrow \alpha_1+2\left(\tfrac{1}{4}\right) = 1$$

$$\alpha_1 + \tfrac{1}{2} = 1$$

$$\boxed{\alpha_1 = \tfrac{1}{2}}$$

$$S(k) = (\alpha_1+\alpha_2 k)4^k$$

$$= (2+k)4^{k-1}$$

which is the reqd soln

POSSIBLE QUESTIONS

TWO MARKS

1. Define characteristics equation.

2. Solve $a_n - 4a_{n-1} = 0$ for n ≥ 2 with $a_0 = 1$, $a_1 = 1$.

3. State Master theorem.

4. Write the methods for solving recurrence.

5. If the sequence $a_n = 3.2^n$, $n \geq 1$ then find the corresponding recurrence relation.

SIX MARKS

1.Solve the recurrence relation $a_n = a_{n-1}+2a_{n-2}$ with $a_0=2$ and $a_1=7$.

2.Solve $T(n) = 2\,T\,(\frac{n}{2}) + n$ using subtitution method

3.Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10$, $a_1 = 41$ using generating function.

4.Find the sequence whose generating function is $\frac{32-22x}{2-3x+x^2}$ using partial functions.

5.Solve the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$ with the initial condition $a_0 = a_1 = 1$.

6.Solve $T(n) = 7\,T\,(\frac{n}{2}) + \Theta\,(\mathrm{n}^2)$ by iterative method.

7.Solve the recurrence relation $a_n+2-6a_{n+1}+9a_n=0$ with $a_0=1$ & $a_1=4$.

8.Using the generating function, solve the recurrence relation $a_n=3a_{n-1}$ for $n\geq1$ with $a_0=2$.

9.Solve $T(n) = 2\,T(\sqrt{n}) + \log n$

10.Using generating function, solve the recurrence relation $a_n=3a_{n-1}+1$for $n\geq 1$ with $a_0=1$.

Questions	opt1	opt2	opt3	opt4	Answer
If X and Y be the sets. Then the set (X - Y) union (Y- X) union (X intersection Y) is equal to?	X U Y	X^c U Y^c	X ∩Y	$X^c \cap Y^c$	X U Y
If G is an undirected planar graph on n vertices with e edges then ?	e ≤ n	e ≤ 2n	e ≤ 3n	e > n	e ≤ 2n
The number of circuits that can be created by adding an edge between any two vertices in a tree is -----	Two	Exactly one	More than one	At least two	Exactly one
In a tree between every pair of vertices there is ----	Exactly one path	A self loop	Two circuits	n number of paths	Exactly one path
A graph is a collection of....	Row and columns	Vertices and edges	Equations	lines	Vertices and edges
The degree of any vertex of graph is	The number of edges incident with	Number of vertex in a graph	Number of vertices adjacent to that	Number of edges in a graph	The number of edges incident with vertex
If for some positive integer k, degree of vertex d(v)=k for every vertex v of the graph G, then G is called...	K graph	K-regular graph	Empty graph	Trivial graph	K-regular graph
A graph with no edges is known as empty graph. Empty graph is also known as...	Trivial graph	Regular graph	Bipartite graph	cycle graph	Trivial graph

Length of the walk of a graph is	The number of vertices in walk W	The number of edges in walk W	Total number of edges in a graph	Total number of vertices in a graph	The number of edges in walk W
If the origin and terminus of a walk are same, the walk is known as....?	open	closed	path	neither open nor closed	closed
A graph G is called a ..... if it is a connected acyclic graph	Cyclic graph	Regular graph	Tree	path	Tree
The complete graph K, has... different spanning trees	n^{n-2}	n*n	n^2	n	n^{n-2}
A continuous non - intersecting curve in the plane whose origin and terminus coincide ?	Planar	Jordan	Hamiltanion	unique	Jordan
A path in graph G, which contains every vertex of G once and only once ?	Eular tour	Hamiltanion Path	Eular trail	Hamiltanion Tour	Hamiltanion Path
A tree having a main node, which has no predecessor is....	Spanning Tree	Rooted Tree	Weighted Tree	forest	Rooted Tree
Diameter of a graph is denoted by diam(G) is defined by....	max (e(v) : v belongs to V)	max(d(u,v))	both max (e(v) : v belongs to V) and max(	min (d(u,v))	both max (e(v) : v belongs to V) and max(d(u,v)
A vertex of a graph is called even or odd depending upon ----	number of edges	number of vertices	degree	eccentricity	degree

An edge having the same vertex as both its end vertices is called --------	graph	tree	self-loop	node	self-loop
The maximum number of edges in a simple graph with n vertices is------	n	(n-2)/2	(n-1)/2	n+1	(n-1)/2
A vertex of degree zero is called an -------------------------	null vertex	isolated vertex	null graph	pendant vertex	null vertex
Vertices with which a walk begins or ends are called its------	isolated vertex	null vertex	pendant vertex	terminal vertices	terminal vertices
A graph with no vertices is a ------	null graph	trivial	Empty graph	parallel	null graph
A ------ is connected graph without circuit	graph	directed graph	undirected graph	tree	tree
The sum of the degrees of all vertices of a graph is equal to ----- the number of edges.	twice	thrice	same	any	twice
A node with no children is called ------	siblings	node	leaf	tree	leaf
A graph is ------ if it has no parallel edges or self-loops	simple	directed	adjacent	self-loop	simple

A graph in which some edges are directed and some are undirected is called -----	mixed graph	regular graph	complete graph	simple graph	mixed graph
Every graph is its own ------	mixed graph	sub graph	simple graph	complete graph	sub graph
------ is also called cycle.	circuit	walk	path	closed walk	circuit
If no vertex appears more than once in an open walk then it is called a ------	closed walk	circuit	walk	path	path
The number of edges in a path is called the------- of the path.	length	walk	same	circuit	length
A simple graph G with n vertices is said to be a----- if the degree of every vertex is n-1.	regular graph	complete graph	simple graph	null graph	complete graph
A walk is also called -------	chain	edge	vertex	graph	chain
. A------- - is a closed , non intersecting walk.	closed walk	circuit	walk	path	circuit
The total number of degrees of an isolated node is -------	0	2	1	3	0
A tree is an ------- graph.	cyclic	directed	acyclic	disconnected	acyclic

A-------- is a graph whose components are all trees.	tree	graph	forest	walk	forest
. A---------- consists of set of vertices and edges such that each edge is incident with vertices.	graph	path	forest	walk	graph
A vertex having no edge incident on it is calleD -------	end vertex	pendant vertex	isolated vertex	null graph	isolated vertex
A graph is said to be -------- if there exists at least one path between every pair of vertices in G.	connected	disconnected	null graph	hamiltanion	connected
A tree with n vertices has ------ edges	n	n-1	n-2	n+1	n-1
A graph in which all nodes are of equal degrees is known as…………………	regular graph	complete graph	simple graph	null graph	regular graph

UNIT – IV

Graph Theory : Basic terminology, models and types, multigraphs and weighted graphs, graph representation, graph isomorphism, connectivity, Euler and Hamiltonian Paths and circuits, Planar graphs, graph coloring, trees, basic terminology and properties of trees, introduction to Spanning trees

INTRODUCTION : GRAPH THEORY

Graph theory is used to analyses problems of combinatorial nature that arise in computer science, operations research , physical science and economics . The term graph is familiar to you because it has been used in the context of straight lines and linear in equalities .In this chapter , first we will combine the concepts of graph theory with digraph of a relation to define a more general type of graph that has more than one edge between a pair of vertices. Second , we will identify basic components of a graph ,its features any many applications of graphs.

Definitions and Examples

Definition: A **graph** G = (V,E) is a mathematical structure consisting of two finite sets V and E. The elements of V are called **Vertices (or nodes)** and the elements of E are called Edges. Each edge
is associated with a set consisting of **either one** or **two vertices** called its **endpoints**.

The correspondence from edges to endpoints is called **edge-endpoint function**. This function is generally denoted by γ. Due to this function, some author denote graph by $G = (V, E, \gamma)$.

Definition: A graph consisting of one vertex and no edges is called a **trivial graph**.

Definition: A graph whose vertex and edge sets are empty is called a **null graph**.

Definition: An edge with just one end point is called a **loop** or a **self loop**.
Thus, a loop is an edge that joins a single endpoint to itself.

Definition: An edge that is not a self-loop is called a **proper edge**.

Definition: If two or more edges of a graph G have the same vertices, then these edges are said to be **parallel** or **multi-edges**.

Definition: Two vertices that are connected by an edge are called **adjacent**.

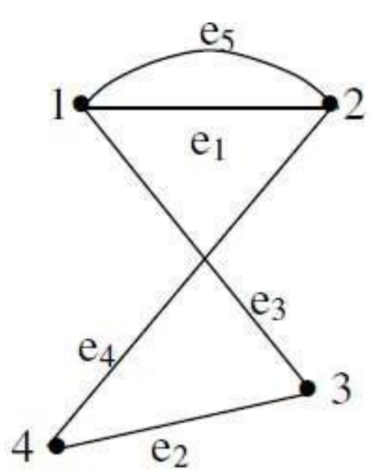
Definition: An endpoint of a loop is said to be **adjacent to itself**.

Definition: An edge is said to be **incident** on each of its endpoints.

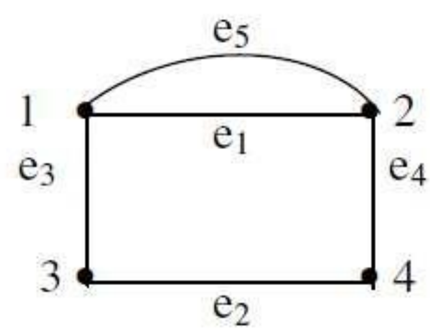
Definition: Two edges incident on the same endpoint are called **adjacent edges**.

Definition: The number of edges in a graph G which are incident on a vertex is called the degree of that **vertex**.

Definition: A vertex of degree zero is called an **isolated vertex**.
Thus, a vertex on which no edges are incident is called isolated.
Definition: A graph without multiple edges **(parallel edges)** and loops is called **Simple graph**.
Notation: In pictorial representations of a graph, the vertices will be denoted by dots and edges by line segments.

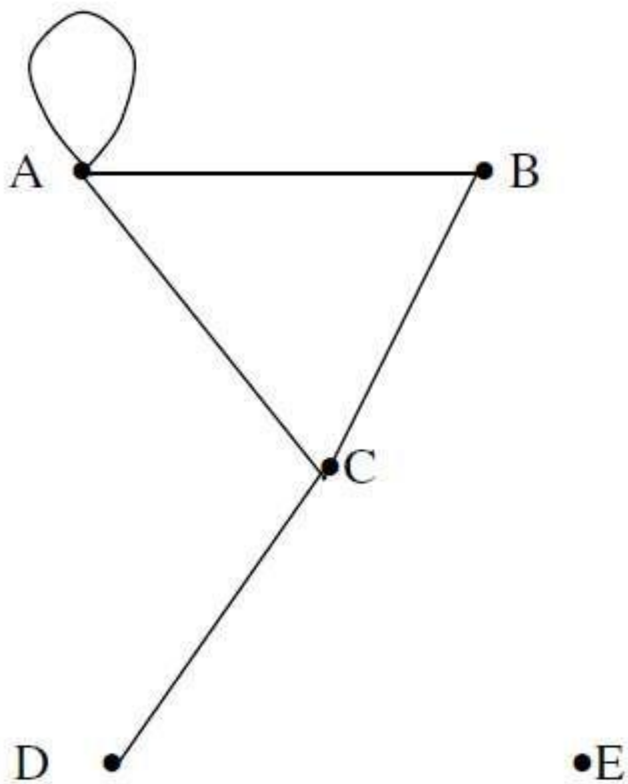


or



The edges e_2 and e_3 are adjacent edges because they are incident on the same vertex B.

2. Consider the graph with the vertices A, B , C, D and E pictured in the figure below.



In this graph, we note that

No. of edges = 5

Degree of vertex A = 4

Degree of vertex B = 2

Degree of vertex C = 3

Degree of vertex D = 1

Degree of vertex E = 0

Sum of the degree of vertices = 4 + 2 + 3 + 1 + 0 =10
Thus, we observe that

$$\sum_{i=1}^{5} \deg(v_i) = 2e \ ,$$

where $\deg(v_i)$ denotes the degree of vertex v_i and e denotes the number of edges.

Euler's Theorem: (The First Theorem of Graph Theory): The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.

(Thus, total degree of a graph is even)

Proof: Each edge in a graph contributes a count of 1 to the degree of two vertices (end points of
the edge), That is, each edge contributes 2 to the degree sum. Therefore the sum of degrees of the
vertices is equal to twice the number of edges.

Corollary: There must be an even number of vertices of odd degree in a given graph G.
Proof: We know, by the Fundamental Theorem, that

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{no. of edges}$$

Thus the right hand side is an even number. Hence to make the left-hand side an even number there
can be only even number of vertices of odd degree.

Theorem: A non-trivial simple graph G must have at least one pair of vertices whose degrees are
equal.

Proof: Let the graph G has n vertices. Then there appear to be n possible degree values, namely 0, 1, ….,n – 1. But there cannot be both a vertex of degree 0 and a vertex of degree n – 1 because if there is a vertex of degree 0 then each of the remaining n – 1 vertices is adjacent to atmost n–2 other vertices. Hence the n vertices of G can realize atmost $n-1$ possible values for their degrees. Hence the pigeonhole principle implies that at least two of the vertices have equal degree.

Definition: A graph G is said to **simple** if it has no parallel edges or loops. In a simple graph, an edge with endpoints v and w is denoted by $\{v, w\}$.

Definition: For each integer $n \geq 1$, let D_n denote the graph with **n vertices** and **no edges**. Then D_n is called the **discrete graph on n vertices**.

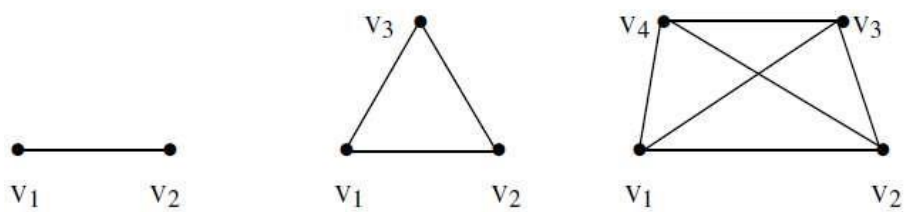
For example, we have

• • • and • • • • •

D_3 D_5

Definition: Let $n \geq 1$ be an integer. Then a simple graph with n vertices in which there is an edge between each pair of distinct vertices is called the **complete Graph** on n vertices. It is denoted by K_n.

For example, the complete graphs K_2, K_3 and K_4 are shown in the figures below:



K_2 K_3 K_4

Definition: If each vertex of a graph G has the same degree as every other vertex, then G is called a **regular graph**.

A **k-regular graph** is a regular graph whose common degree is k.

But this graph is not complete because v_2 and v_4 have not been connected through an edge. Similarly, v_1 and v_3 are not connected by any edge.

Thus

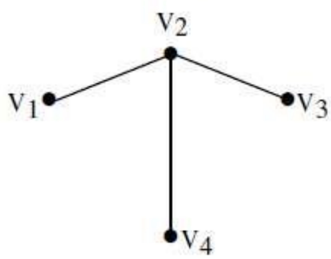
A Complete graph is always regular but **a regular graph need not be complete**.

Definition: If G is a simple graph, the **complement of G**, (**Edge complement**), denoted by G′ or G^c is a graph such that

(i) The vertex set of G′ is identical to the vertex set of G, that is $V_{G'} = V_G$

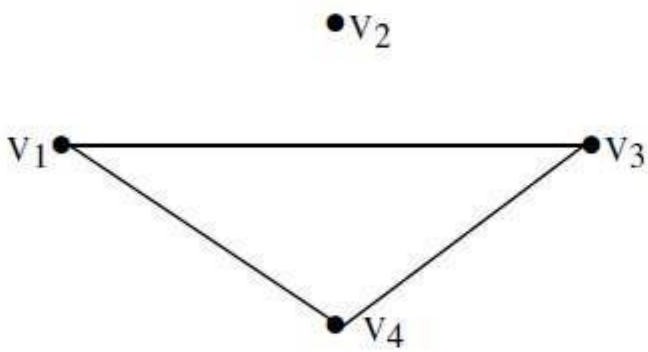
(ii) Two distinct vertices v and w of G′ **are connected by** an edge if and only if v and w **are not connected** by an edge in G.

For example, consider the graph G



G

Then complement G′ of G is the graph



G′

Definition: The property of mapping endpoints to endpoints is called **preserving incidence** or **the continuity rule** for graph mappings.

As a consequence of this property, a self-loop must map to a self-loop.

Thus, two isomorphic graphs are same except for the labeling of their vertices and edges.

Walks, Paths and Circuits

Definition: In a graph G, **a walk from vertex v_0 to vertex v_n** is a finite alternating sequence:

$$\{v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n\}$$

of vertices and edges such that v_{i-1} and v_i are the endpoints of e_i.

The **trivial walk** from a vertex v to v consists of the single vertex v.

Definition: In a graph G, a **path** from the vertex v_0 to the vertex v_n is a walk from v_0 to v_n that does not contain a repeated edge.

Thus a **path** from v_0 to v_n is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n\},$$

where all the edges e_I are distinct.

Definition: In a graph, a simple path from v_0 to v_n is a path that does not contain a repeated vertex.

Thus a simple path is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{i-1}, e_n, v_n\},$$

where all the e_i are distinct and all the v_i are distinct.

Definition: A walk in a graph G that starts and ends at the same vertex is called a **closed walk**.

Definition: A closed walk that does not contain a repeated edge is called a **circuit**.

Thus, closed a closed path is called a circuit (or a cycle) and so a circuit is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n\},$$

where $v_0 = v_n$ and all the e_i are distinct.

Definition: In a graph the number of edges in the path $\{v_0, e_1, v_1, e_2, \ldots\ldots, e_n, v_n\}$ from v_0 to v_n is called the **length of the path**.

Theorem: If there is a path from vertex v_1 to v_2 in a graph with n vertices, then there does not exist a path of more than n-1 edges from vertex v_1 to v_2.

Proof: Suppose there is a path from v_1 to v_2. Let

$$v_1, \ldots\ldots, v_i, \ldots\ldots\ldots, v_2$$

be the sequence of vertices which the path meets between the vertices v_1 and v_2. Let there be m edges in the path. Then there will be m + 1 vertices in the sequence. Therefore if $m > n-1$, then there will be more than n vertices in the sequence. But the graph is with n vertices. Therefore some vertex, say v_k, appears more than once in the sequence. So the sequence of vertices shall be

$$v_1, \ldots\ldots, v_i, \ldots\ldots, v_k, \ldots\ldots, v_k, \ldots\ldots, v_2.$$

Deleting the edges in the path that lead v_k back to v_k we have a path from v_1 to v_2 that has less edges than the original one. This argument is repeated untill we get a path that has n-1 or less edges.

CONNECTED AND DISCONNECTED GRAPHS :

Definition: Two vertices v_1 and v_2 of a graph G are said to be **connected** if and only if there is a walk from v_1 to v_2.

Definition: A graph G is said to be **connected** if and only if given any two vertices v_1 and v_2 in G, there is a walk from v_1 to v_2.

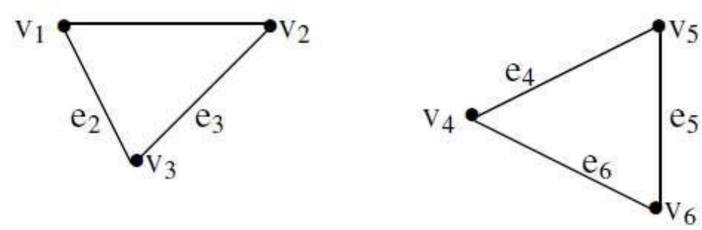
Thus, a graph G is connected if there exists a walk between every two vertices in the graph.

Definition: A graph which is not connected is called **Disconnected Graph.**

Example: Which of the graph below are connected?

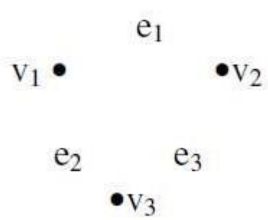
Definition: If a graph G is disconnected, then the various connected pieces of G are called the **connected components of the graph**.

Example: Consider the graph given below:



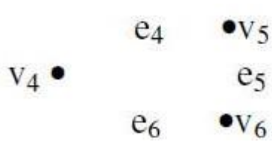
This graph is disconnected and have two connected components:

H_1 :



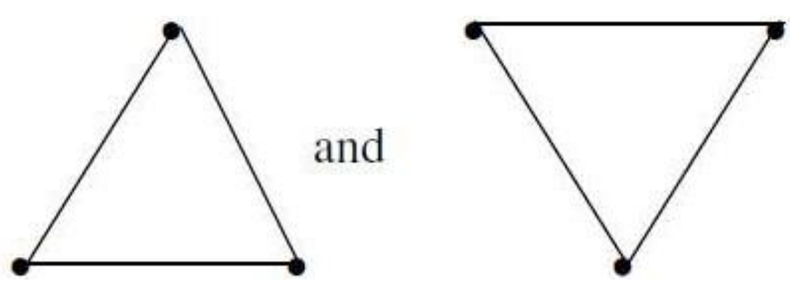
with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{e_1, e_2, e_3\}$

H_2 :

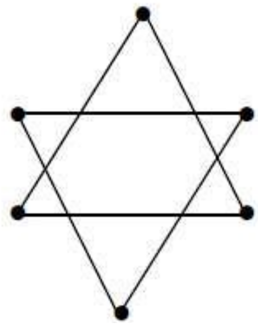


with vertex set $\{v_4, v_5, v_6\}$ and edge set $\{e_4, e_5, e_6\}$.

Solution: The connected components are :



Example: Find the number of connected components in the graph



Eulerian Paths And Circuits

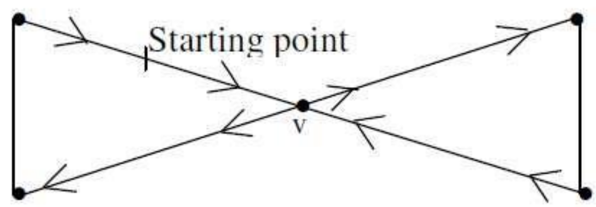
Definition: A path in a graph G is called an **Euler Path** if it includes **every edge exactly once**.

Definition: A graph is called **Eulerian graph** if there exists a Euler circuit for that graph.

Definition: A circuit in a graph G is called an **Euler Circuit** if it includes every edge exactly once. Thus, an Euler circuit (Eulerian trail) for a graph G is a sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses **every edge of G exactly once**.

Theorem 1. If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof: Let G be a graph which has an Euler circuit. Let v be a vertex of G. We shall show that degree of v is even. By definition, Euler circuit contains every edge of graph G. Therefore the Euler circuit contains all edges incident on v. We start a journey beginning in the middle of one of the edges adjacent to the start of Euler circuit and continue around the Euler circuit to end in the middle of the starting edge. Since Euler circuit uses every edge exactly once, the edges incident on v occur



in entry / exist pair and hence the degree of v is a multiple of 2. Therefore the degree of v is even. This completes the proof of the theorem.

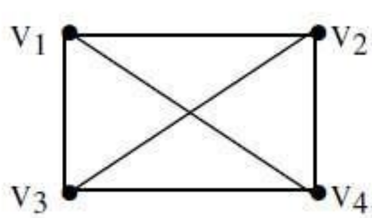
We know that contrapositive of a conditional statement is logically equivalent to statement. Thus the above theorem is equivalent to:

Theorem:2. If a vertex of a graph is not of even degree, then it does not have an Euler circuit.

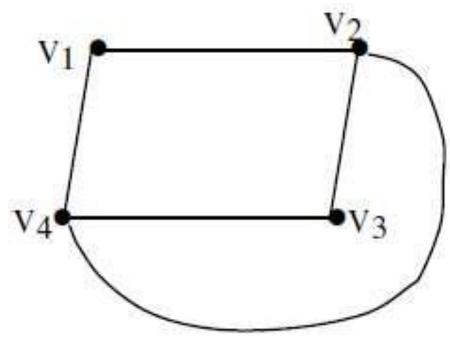
or

"If some vertex of a graph has odd degree, then that graph does not have an Euler circuit".

Example: Show that the graphs below do not have Euler circuits.
(a)



(b)



Solution: In graph (a), degree of each vertex is 3. Hence this **does not** have a Euler circuit.

In graph (b), we have

$$\deg(v_2) = 3$$
$$\deg(v_4) = 3$$

Since there are vertices of odd degree in the given graph, therefore it **does not** have an Euler circuit.

are graphs in which each vertex has degree 2 but these graphs do not have Euler circuits since there is no path which uses each vertex at least once.

Theorem 3. If G is a connected graph and every vertex of G has even degree, then G has an Euler circuit.

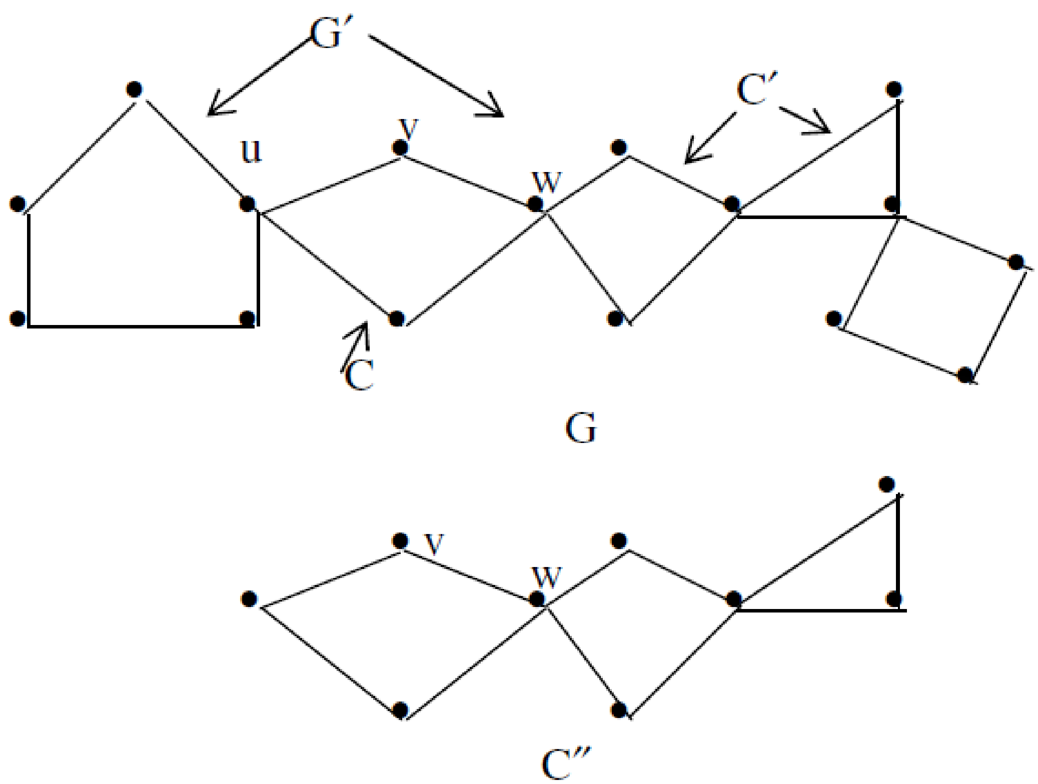
Proof: Let every vertex of a connected graph G has even degree. If G consists of a single vertex, the trivial walk from v to v is an Euler circuit. So suppose G consists of more than one vertices. We start from any verted v of G. Since the degree of each vertex of G is even, if we reach each vertex other than v by travelling on one edge, the same vertex can be reached by travelling on another previously unused edge. Thus a sequence of distinct adjacent edges can be produced indefinitely as long as v is not reached. Since number of edges of the graph is finite (by definition of graph), the sequence of distinct edges will terminate. Thus the sequence must return to the starting vertex. We thus obtain a sequence of adjacent vertices and edges starting and ending at v without repeating any edge. Thus we get a circuit C.

If C contains every edge and vertex of G, then C is an Eular circuit.

If C does not contain every edge and vertex of G, remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Let the resulting subgraph be G′. We note that when we removed edges of C, an even number of edges from each vertex have been removed. Thus degree of each remaining vertex remains even.

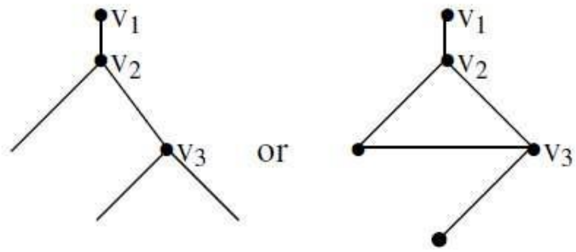
Further since G is connected, there must be at least one vertex common to both C and G′. Let it be w(in fact there are two such vertices). Pick any sequence of adjacent vertices and edges of G′ starting and ending at w without repeating an edge. Let the resulting circuit be C′.

Join C and C′ together to create a new circuit C″. Now, we observe that if we start from v and follow C all the way to reach w and then follow C′ all the way to reach back to w. Then continuing travelling along the untravelled edges of C, we reach v.



Theorem 5. If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.

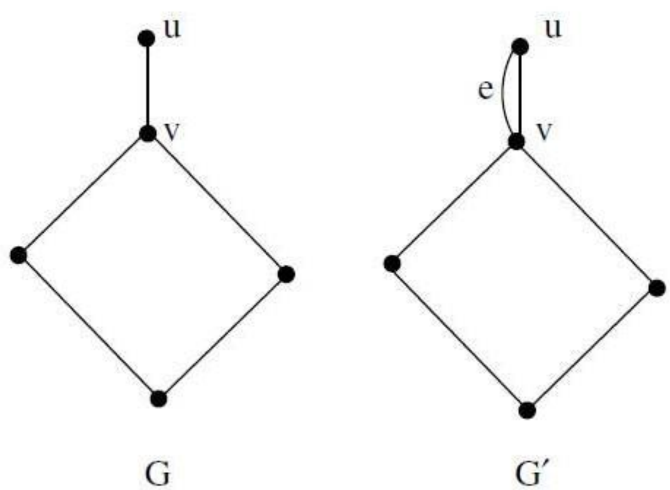
Proof : Let v_1, v_2 and v_3 be vertices of odd degree. Since each of these vertices had odd degree, any possible Euler path must leave (arrive at) each of v_1, v_2, v_3 with no way to return (or leave). One vertex of these three vertices may be the beginning of Euler path and another the end but this leaves the third vertex at one end of an untravelled edge. Thus there is no Euler path.



(Graphs having more than two vertices of odd degree).

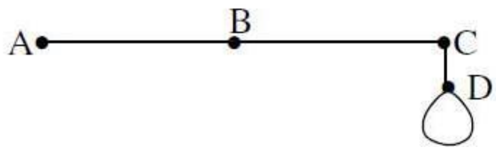
Theorem 6. If G is a connected graph and has exactly two vertices of odd degree, then there is an Euler path in G. Further, any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof: Let u and v be two vertices of odd degree in the given connected graph G.



If we add the edge e to G, we get a connected graph G′ all of whose vertices have even degree. Hence there will be an Euler circuit in G′. If we omit e from Euler circuit, we get an Euler path beginning at u(or v) and edning at v(or u).

Examples. Has the graph given below an Eulerian path?



Solution: In the given graph,

$$\deg(A) = 1$$

$$\deg(B) = 2$$

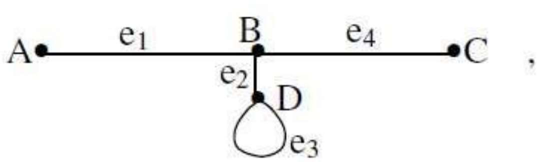
$$\deg(C) = 2$$

$$\deg(D) = 3$$

Thus the given connected graph has exactly two vertices of odd degree. Hence, it has an Eulerian path.

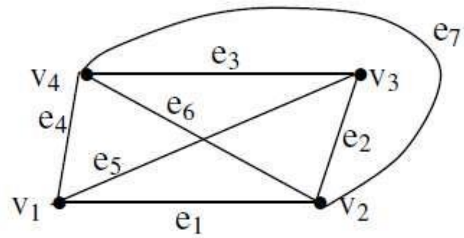
If it starts from A(vertex of odd degree), then it ends at D(vertex of odd degree). If it starts from D(vertex of odd degree), then it ends at A(vertex of odd degree).

But on the other hand if we have the graph as given below :



then deg(A) = 1, deg(B) = 3 deg(C) = 1, degree of D = 3 and so we have four vertices of odd degree. Hence it does not have Euler path.

Example: Does the graph given below possess an Euler circuit?



Solution: The given graph is connected. Further

$$\deg(v_1) = 3$$

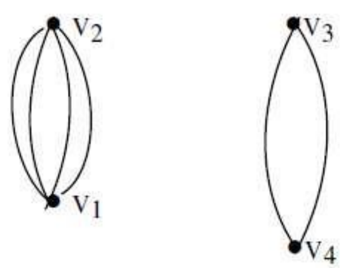
$$\deg(v_2) = 4$$

$$\deg(v_3) = 3$$

$$\deg(v_4) = 4$$

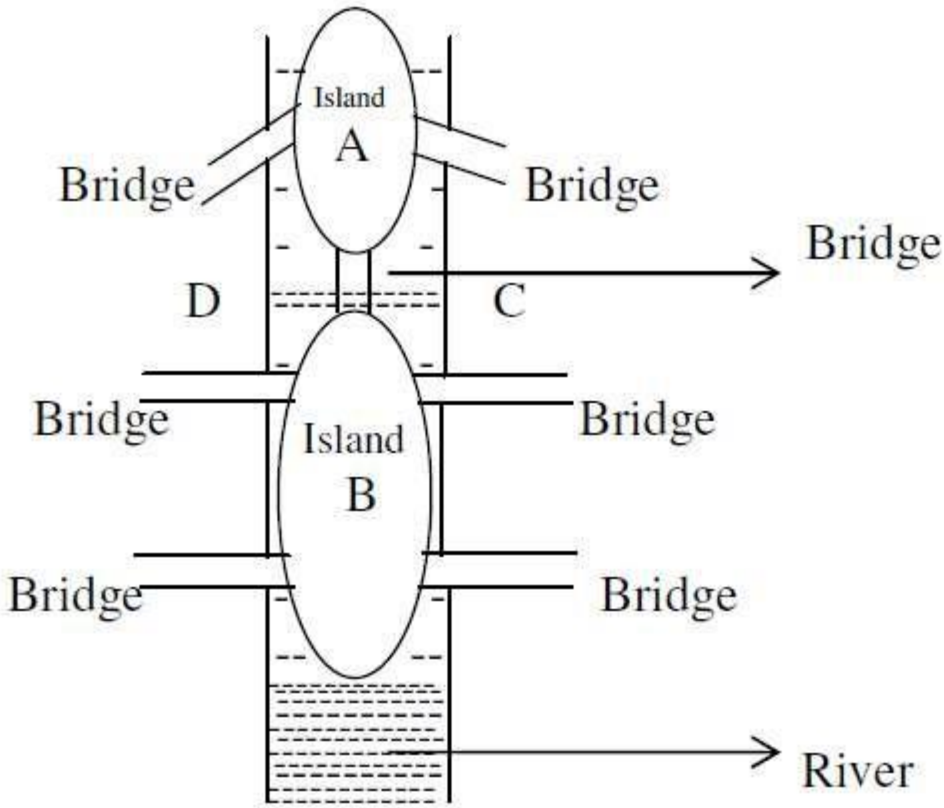
Since this connected graph has vertices with odd degree, it cannot have Euler circuit. But this graph has Euler path, since it has exactly two vertices of odd degree. For example, $v_3\, e_2\, v_2\, e_7\, v_4\, e_6\, v_2\, e_1\, v_1\, e_4\, v_4\, e_3\, v_3\, e_5\, v_1$

Example: Consider the graph



Here, $\deg(v_1) = 4$, $\deg(v_2) = 4$, $\deg(v_3) = 2$, $\deg(v_4) = 2$. Thus degree of each vertex is even. But the graph is not Eulerian since it is **not connected**.

Example 4:. The bridges of Konigsberg: The graph Theory began in 1736 when Leonhard Euler solved the problem of seven bridges on Pregel river in the town of Konigsberg in Prussia (now Kaliningrad in Russia). The two islands and seven bridges are shown below:

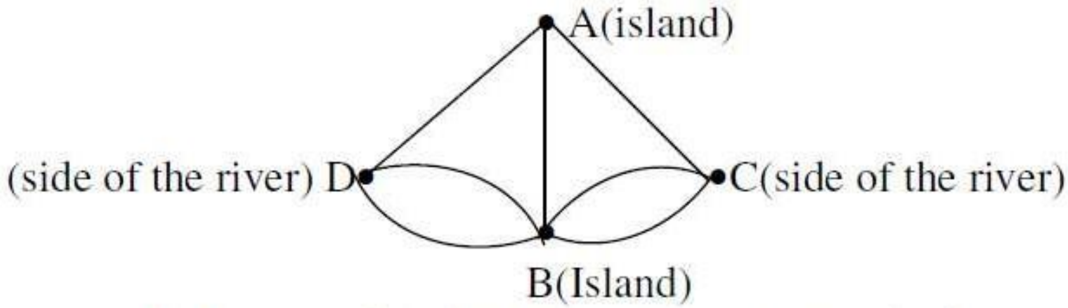


The people of Konigsgerg posed the following question to famous Swiss Mathematician Leonhard Euler:
"Beginning anywhere and ending any where, can a person walk through the town of Konigsberg crossing all the seven bridges exactly once?

Euler showed that such a walk is impossible. He replaced the islands A, B and the two sides (banks) C and D of the river by vertices and the bridges as edges of a graph. We note then that

$$\deg(A) = 3$$
$$\deg(B) = 5$$
$$\deg(C) = 3$$
$$\deg(D) = 3$$

Thus the graph of the problem is



(Euler's graphical representation of seven bridge problem)

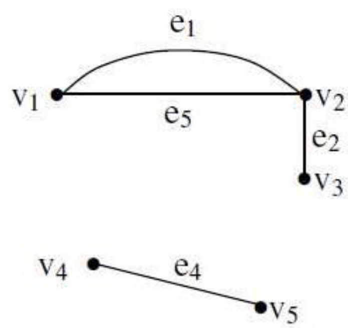
The problem then reduces to

"Is there any Euler's path in the above diagram?".

To find the answer, we note that there are more than two vertices having odd degree. Hence there exist no Euler path for this graph.

Definition: An edge in a connected graph is called a **Bridge** or a **Cut Edge** if deleting that edge creates a disconnected graph.

In this graph, if we remove the edge e_3, then the graph breaks into two Connected Component given below:

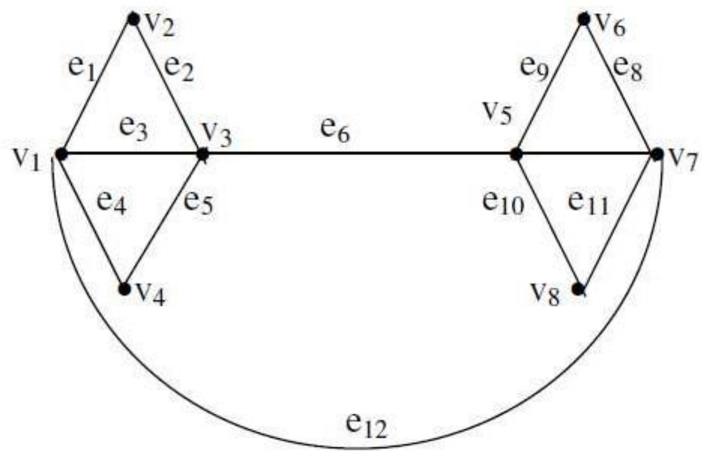


Hence the edge e_3 is a bridge in the given graph.

METHOD FOR FINDING EULER CIRCUIT

We know that if every vertex of a non empty connected graph has even degree, then the graph has an Euler circuit. We shall make use of this result to find an Euler path in a given graph.

Consider the graph



We note that

$$\deg(v_2) = \deg(v_4) = \deg(v_6) = \deg(v_8) = 2$$

$$\deg(v_1) = \deg(v_3) = \deg(v_5) = \deg(v_7) = 4$$

Hence all vertices have even degree. Also the given graph is connected. Hence the given has an Euler circuit. We start from the vertex v_1 and let C be

$$C : v_1\, v_2\, v_3\, v_1$$

Then C is not an Euler circuit for the given graph but C intersect the rest of the graph at v_1 and v_3. Let C' be

$$C' : v_1 v_4\, v_3\, v_5\, v_7\, v_6\, v_5\, v_8\, v_7\, v_1$$

(In case we start from v_3, then C' will be $v_3\, v_4\, v_1\, v_7\, v_6\, v_5\, v_7\, v_8\, v_5$)

Path C' into C and obtain

$$C'' : v_1 v_2\, v_3\, v_1\, v_4\, v_3\, v_5\, v_7\, v_6\, v_5\, v_8\, v_7\, v_1$$

Or we can write

$$C'' : e_1 e_2\, e_3\, e_4\, e_5\, e_6\, e_7\, e_8\, e_9\, e_{10}\, e_{11}\, e_{12}$$

(If we had started from v_2, then C'' : $v_1 v_2\ v_3\ v_4\ v_1\ v_7\ v_6\ v_5\ v_7\ v_8\ v_5\ v_3\ v_1$ **or**

$e_1 e_2\ e_5\ e_4\ e_{12}\ e_8\ e_9\ e_7\ e_{11}\ e_{10} e_6\ e_3$)

In C'' all edges are covered exactly once. Also every vertex has been covered at least once. Hence C'' is a Euler circuit.

Hamiltonian Circuits

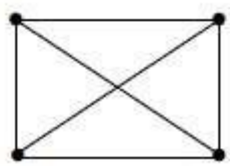
Definition: A **Hamiltonian Path** for a graph G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

Definition: A **Hamiltonian Circuit** for a graph G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last which are the same.

Definition: A graph is called **Hamiltonian** if it admits a Hamiltonian circuit.

Example 1 : A complete graph K_n has a Hamiltonian Circuit. In particular the graphs

K_3 and



K_4

are Hamiltonian.

Theorem: Let G be a connected graph with n vertices and let u and v be two vertices of G that are not adjacent. If

$$\deg(u) + \deg(v) \geq n,$$

then G has a Hamiltonian circuit.

Matrix Representation of Graphs

A graph can be represented inside a computer by using the adjacency matrix or the incidence matrix of the graph.

Definition: Let G be a graph with n ordered vertices $v_1, v_2, \ldots\ldots, v_n$. Then the **adjacency matrix of G** is the $n \times n$ matrix $A(G) = (a_{ij})$ over the set of non-negative integers such that

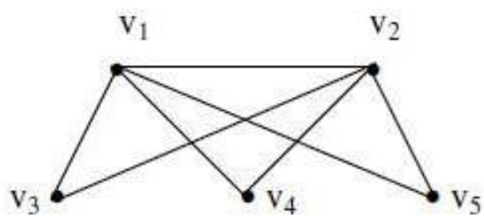
a_{ij} = the number of edges connecting v_i and v_j for all i, j = 1, 2,…,n.

We note that if G has no loop, then there is no edge joining v_i to v_i, i = 1, 2,…,n. Therefore, in this case, all the entries on the main diagonal will be 0.

Further, if G has no parallel edge, then the entries of A(G) are either 0 or 1.
It may be noted that adjacent matrix of a graph is symmetric.

Conversely, given a $n \times n$ symmetric matrix $A(G) = (a_{ij})$ over the set of non-negative integers, we can associate with it a graph G, whose adjacency matrix is A(G), by letting G have n vertices and joining v_i to vertex v_j by a_{ij} edges.

Example 1: Find the adjacency matrix of the graph shown below:



Solution: The adjacency matrix $A(G) = (a_{ij})$ is the matrix such that

a_{ij} = No. of edges connecting v_i and v_j.

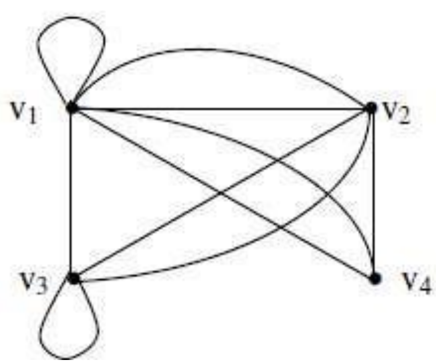
So we have for the given graph

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Example 2 : Find the graph that have the following adjacency matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Solution: We note that there is a loop at v_1 and a loop at v_3. There are parallel edges between v_1, v_2; v_1, v_4; v_2, v_1; v_2, v_3, v_3, v_2 ; v_4, v_1. Thus the graph is



Trees

Definition: A graph is said to be a **Tree** if it is a connected acyclic graph.

THEOREM:

A graph G with e = v – 1, that has no circuit is a tree.

Proof: It is sufficient to show that G is connected. Suppose G is not connected and let G', G''….. be connected component of G. Since each of G', G'',…. is connected and has no cycle, they all are tree. Therefore, by Lemma 3,

$$e' = v' - 1$$
$$e'' = v'' - 1$$

-------------- ,

where e', e'', … are the number of edges and v', v'',… are the number of vertices in G', G'', …respectively. We have, on adding

$$e' + e'' + \ldots\ldots = (v' - 1) + (v'' - 1) + \ldots\ldots$$

Since

$$e = e' + e'' + \ldots..$$
$$v = v' + v'' + \ldots. \quad ,$$

we have

$$e < v - 1 \quad ,$$

which contradicts our hypotheses. Hence G is connected. So G is connected and acyclic and is therefore a tree.

Definition: A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is 0 and the incoming degrees of all other vertices are 1.

Definition: In a rooted tree, a vertex, whose outgoing degree is 0 is called a **leaf** or **terminal node**, whereas a vertex whose outgoing degree is non - zero is called a **branch node** or an **internal node**.

Definition: Let u be a branch node in a rooted tree. Then a vertex v is said to be **child** (**son** or **offspring**) of u if there is an edge from u to v. In this case **u** is called **parent** (**father**) of v.

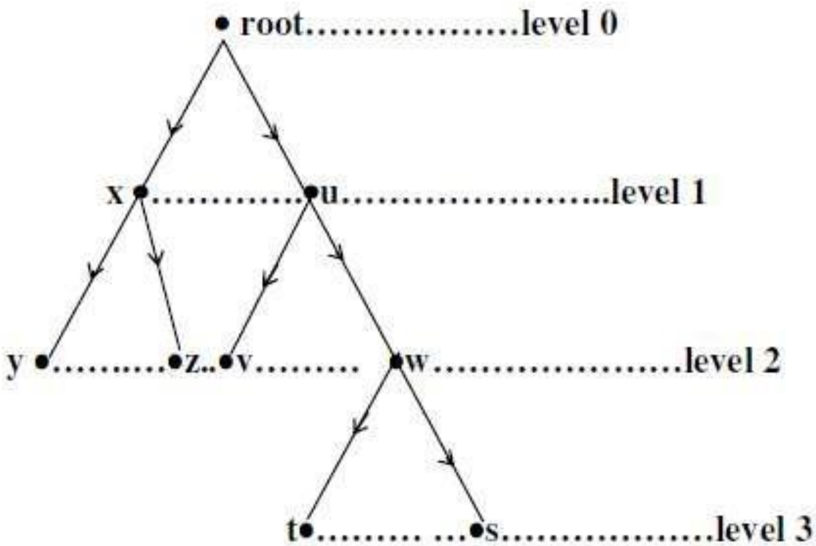
Definition: Two vertices in a rooted tree are said to be **siblings** (**brothers**) if they are both children of same parent.

Definition: A vertex v is said to be a **descendent** of a vertex u if there is a unique directed path from u to v.
In this case u is called the **ancestor** of v.

Definition: The **level** (or **path length**) of a vertex u in a rooted tree is the number of edges along the unique path between u and the root.

Definition: The **height** of a rooted tree is the maximum level to any vertex of the tree.

As an example of these terms consider the rooted tree shown below:



POSSIBLE QUESTIONS

PART-B (TWO MARKS)

1.Define directed graph.

2.How many vertices does a regular graph of degree 4 with 10 edges have

3.Define Hamiltonian path

4.Define isomorphic graph.

5.Define chromatic number

PART – C(SIX MARKS)

1. State and prove handshaking lemma

2.Define (i) Proper coloring graph (ii) Chromatic Number (iii) Independent set.

3.Give an example of a graph which is

(i).Eulerian but not Hamiltonian

(ii).Hamiltonian but not Eulerian

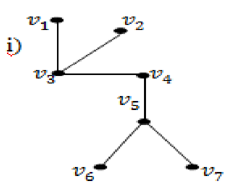
(iii).Both eulerian and Hamiltonian

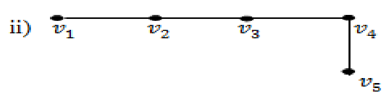
(iv).Non Eulerian and non Hamiltonian

4.Show that if a fully binary tree has i internal vertices then it has (i+1) terminal vertices and (2i+1) total vertices.

5.Describe about konigsberg bridge problem.

6.Find the eccentricity of all vertices, center, radius and diameter of the following graph.





7.Prove that the number of vertices of odd degree in a graph is always even.

8.Prove that the number of pendent vertices of a tree is equal to $\frac{n+1}{2}$

9. Define graph. Explain the various types of graph with an example.

10.In a undirected graph,the number of odd degree vertices are even.

Questions	opt1	opt2	opt3	opt4	Answer
If X and Y be the sets. Then the set (X - Y) union (Y- X) union (X intersection Y) is equal to?	X U Y	X^c U Y^c	X ∩Y	$X^c \cap Y^c$	X U Y
If G is an undirected planar graph on n vertices with e edges then ?	e ≤ n	e ≤ 2n	e ≤ 3n	e > n	e ≤ 2n
The number of circuits that can be created by adding an edge between any two vertices in a tree is -----	Two	Exactly one	More than one	At least two	Exactly one
In a tree between every pair of vertices there is ----	Exactly one path	A self loop	Two circuits	n number of paths	Exactly one path
A graph is a collection of....	Row and columns	Vertices and edges	Equations	lines	Vertices and edges
The degree of any vertex of graph is	The number of edges incident with	Number of vertex in a graph	Number of vertices adjacent to that	Number of edges in a graph	The number of edges incident with vertex
If for some positive integer k, degree of vertex d(v)=k for every vertex v of the graph G, then G is called...	K graph	K-regular graph	Empty graph	Trivial graph	K-regular graph
A graph with no edges is known as empty graph. Empty graph is also known as...	Trivial graph	Regular graph	Bipartite graph	cycle graph	Trivial graph

Length of the walk of a graph is	The number of vertices in walk W	The number of edges in walk W	Total number of edges in a graph	Total number of vertices in a graph	The number of edges in walk W
If the origin and terminus of a walk are same, the walk is known as ....?	open	closed	path	neither open nor closed	closed
A graph G is called a ..... if it is a connected acyclic graph	Cyclic graph	Regular graph	Tree	path	Tree
The complete graph K, has... different spanning trees	n^{n-2}	n*n	n^2	n	n^{n-2}
A continuous non - intersecting curve in the plane whose origin and terminus coincide ?	Planar	Jordan	Hamiltanion	unique	Jordan
A path in graph G, which contains every vertex of G once and only once ?	Eular tour	Hamiltanion Path	Eular trail	Hamiltanion Tour	Hamiltanion Path
A tree having a main node, which has no predecessor is....	Spanning Tree	Rooted Tree	Weighted Tree	forest	Rooted Tree
Diameter of a graph is denoted by diam(G) is defined by....	max (e(v) : v belongs to V)	max(d(u,v))	both max (e(v) : v belongs to V) and max(	min (d(u,v))	both max (e(v) : v belongs to V) and max(d(u,v)
A vertex of a graph is called even or odd depending upon ----	number of edges	number of vertices	degree	eccentricity	degree

An edge having the same vertex as both its end vertices is called --------	graph	tree	self-loop	node	self-loop
The maximum number of edges in a simple graph with n vertices is------	n	(n-2)/2	(n-1)/2	n+1	(n-1)/2
A vertex of degree zero is called an -------------------------	null vertex	isolated vertex	null graph	pendant vertex	null vertex
Vertices with which a walk begins or ends are called its------	isolated vertex	null vertex	pendant vertex	terminal vertices	terminal vertices
A graph with no vertices is a ------	null graph	trivial	Empty graph	parallel	null graph
A ------ is connected graph without circuit	graph	directed graph	undirected graph	tree	tree
The sum of the degrees of all vertices of a graph is equal to ----- the number of edges.	twice	thrice	same	any	twice
A node with no children is called ------	siblings	node	leaf	tree	leaf
A graph is ------ if it has no parallel edges or self-loops	simple	directed	adjacent	self-loop	simple

A graph in which some edges are directed and some are undirected is called -----	mixed graph	regular graph	complete graph	simple graph	mixed graph
Every graph is its own ------	mixed graph	sub graph	simple graph	complete graph	sub graph
------ is also called cycle.	circuit	walk	path	closed walk	circuit
If no vertex appears more than once in an open walk then it is called a ------	closed walk	circuit	walk	path	path
The number of edges in a path is called the------- of the path.	length	walk	same	circuit	length
A simple graph G with n vertices is said to be a----- if the degree of every vertex is n-1.	regular graph	complete graph	simple graph	null graph	complete graph
A walk is also called -------	chain	edge	vertex	graph	chain
. A------- - is a closed , non intersecting walk.	closed walk	circuit	walk	path	circuit
The total number of degrees of an isolated node is -------	0	2	1	3	0
A tree is an ------- graph.	cyclic	directed	acyclic	disconnected	acyclic

A-------- is a graph whose components are all trees.	tree	graph	forest	walk	forest
. A---------- consists of set of vertices and edges such that each edge is incident with vertices.	graph	path	forest	walk	graph
A vertex having no edge incident on it is calleD -------	end vertex	pendant vertex	isolated vertex	null graph	isolated vertex
A graph is said to be -------- if there exists at least one path between every pair of vertices in G.	connected	disconnected	null graph	hamiltanion	connected
A tree with n vertices has ------ edges	n	n-1	n-2	n+1	n-1
A graph in which all nodes are of equal degrees is known as…………………	regular graph	complete graph	simple graph	null graph	regular graph

UNIT – V

Prepositional Logic: Logical Connectives, Well-formed Formulas, Tautologies, Equivalences, Inference Theory.

Propositions. Compound Statements. Truth Tables

Statements (Propositions): Sentences that claim certain things, either true or false

Notation: A, B, …P, Q, R, …., p, q, r, etc.

Examples of statements: Today is Monday. This book is expensive
If a number is smaller than 0 then it is positive.

Examples of sentences that **are not statements**: Close the door! What is the time?

Propositional variables: A, B, C, …, P., Q, R, … Stand for statements. May have true or false value.

Propositional constants:

T – true
F - false

Basic logical connectives: NOT, AND, OR
Other logical connectives can be represented by means of the basic connectives

Logical connectives	pronounced	Symbol in Logic
Negation	NOT	¬, ~, '
Conjunction	AND	Λ
Disjunction	OR	V
Conditional	if then	→
Biconditional	if and only if	↔
Exclusive or	Exclusive or	⊕

Truth tables - Define formally the meaning of the logical operators.
The abbreviation **iff** means **if and only if**

a. Negation (NOT, ~, ¬, ')

P **~P** ------------------ **T** F F **T**	~P is true if and only if P is false

b. Conjunction (AND, Λ, &&)

P	Q	P Λ Q	
T	**T**	**T**	P Λ Q is true iff both P and Q are true. In all other cases P Λ Q is false
T	F	F	
F	**T**	F	
F	F	F	

c. Disjunction / Inclusive OR (OR, V, ||)

P	Q	P V Q	
T	**T**	**T**	P V Q is true iff P is true or Q is true or both are true.
T	F	**T**	P V Q is false iff both P and Q are false
F	**T**	**T**	
F	F	F	

d. Conditional , known also as implication (→)

P	Q	P→ Q	
T	**T**	**T**	The implication P→ Q is false iff P is true however Q is false.
T	F	F	In all other cases the implication is true
F	**T**	**T**	
F	F	**T**	

e. Biconditional (↔)

P	Q	P ↔ Q	
T	**T**	**T**	P↔ Q is true iff P and Q have same values - both are true or both are false.
T	F	F	If P and Q have different values, the biconditional is false.
F	**T**	F	
F	F	**T**	

f. Exclusive OR (⊕)

P	Q	P ⊕ Q	
T	**T**	F	P⊕ Q is true iff P and Q have different values
T	F	**T**	We say: "P or Q but not both"
F	**T**	**T**	
F	F	F	

Precedence of the logical connectives:

Connectives within parentheses, innermost parentheses first

¬	**negation**
Λ	**conjunction**
V	**disjunction**
→	**conditional**
↔, ⊕	**biconditional, exclusive OR**

Compound Statements: Logical expressions that consist of propositional variables and logical connectives. They may contain also propositional constants.

Evaluating compound statements : by building their truth tables

Example: ¬ P V Q

P	Q	¬ P	¬ P V Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(P V Q) Λ ¬ (P Λ Q)

P	Q	P V Q A	P Λ Q B	¬ (P Λ Q) ¬ B	(P V Q) Λ ¬ (P Λ Q) A Λ ¬B
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

(the letters A and B are used as shortcuts)

1. Tautologies and Contradictions

A propositional expression is a **tautology** if and only if for all possible assignments of truth values to its variables its truth value is **T**

Example: P V ¬ P is a tautology

P	¬ P	P V ¬ P
T	F	T
F	T	T

A propositional expression is a **contradiction** if and only if for all possible assignments of truth values to its variables its truth value is **F**

Example: P Λ ¬ P is a contradiction

P	¬ P	P Λ ¬ P
T	F	F
F	T	F

Usage of tautologies and contradictions - in proving the validity of arguments; for rewriting expressions using only the basic connectives.

Definition: Two propositional expressions P and Q are logically equivalent, if and only if P ↔ Q is a tautology. We write P ≡ Q or P ⇔ Q.

Note that the symbols ≡ and ⇔ are **not logical connectives**

Exercise:

a) Show that P → Q ↔ ¬ P V Q is a tautology, i.e. P → Q ≡ ¬ P V Q

P	Q	¬ P	¬ P V Q	P → Q	P → Q ↔ ¬ P V Q
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

2. Logical equivalences

Similarly to standard algebra, there are **laws** to manipulate logical expressions, given as logical equivalences.

1. Commutative laws	$P \vee Q \equiv Q \vee P$	
	$P \wedge Q \equiv Q \wedge P$	
2. Associative laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	
	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
3. Distributive laws:	$(P \vee Q) \wedge (P \vee R) \equiv P \vee (Q \wedge R)$	
	$(P \wedge Q) \vee (P \wedge R) \equiv P \wedge (Q \vee R)$	
4. Identity	$P \vee F \equiv P$	
	$P \wedge T \equiv P$	
5. Complement properties	$P \vee \neg P \equiv T$	(excluded middle)
	$P \wedge \neg P \equiv F$	(contradiction)
6. Double negation	$\neg(\neg P) \equiv P$	
7. Idempotency (consumption)	$P \vee P \equiv P$	
	$P \wedge P \equiv P$	
8. De Morgan's Laws	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	
	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	
9. Universal bound laws (Domination)	$P \vee T \equiv T$	
	$P \wedge F \equiv F$	
10. Absorption Laws	$P \vee (P \wedge Q) \equiv P$	
	$P \wedge (P \vee Q) \equiv P$	
11. Negation of T and F:	$\neg T \equiv F$	
	$\neg F \equiv T$	

1. Truth table of the conditional statement

P	Q	$P \to Q$
T	T	T
T	**F**	**F**
F	T	T
F	F	T

P is called **antecedent**
Q is called **consequent**
Meaning of the conditional statement: The truth of P implies (leads to) the truth of Q

Note that when P is false the conditional statement is true no matter what the value of Q is. We say that in this case the conditional statement is **true by default or vacuously true.**

2. Representing the implication by means of disjunction

$$\mathbf{P \to Q \equiv \neg P \vee Q}$$

P	Q	$\neg P$	$\mathbf{P \to Q}$	$\mathbf{\neg P \vee Q}$
T	T	F	T	T
T	F	F	**F**	**F**
F	T	T	T	T
F	F	T	T	T

Same truth tables

Usage:

1. To rewrite "OR" statements as conditional statements and vice versa (for better understanding)
2. To find the negation of a conditional statement using De Morgan's Laws

3. Rephrasing "or" sentences as "if-then" sentences and vice versa

Consider the sentence:
(1) "The book can be found in the library or in the bookstore".

Let

A = The book can be found in the library
B = The book can be found in the bookstore

Logical form of (1): **A V B**

Rewrite A V B **as a conditional statement**

In order to do this we need to use the commutative laws, the equivalence $\neg(\neg P) \equiv P$, and the equivalence $P \rightarrow Q \equiv \neg P \vee Q$

Thus we have:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$

The last expression $\neg A \rightarrow B$ is translated into English as
"If the book cannot be found in the library, it can be found in the bookstore".

Here the statement "The book cannot be found in the library" is represented by $\neg A$

There is still one more conditional statement to consider.
$A \vee B \equiv B \vee A$ (commutative laws)

Then, following the same pattern we have:

$B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

The English sentence is: **"If the book cannot be found in the bookstore, it can be found in the library.**

We have shown that:

$A \vee B \equiv \neg(\neg A) \vee B \equiv \neg A \rightarrow B$
$A \vee B \equiv B \vee A \equiv \neg(\neg B) \vee A \equiv \neg B \rightarrow A$

Thus the sentence **"The book can be found in the library or in the bookstore"** can be rephrased as:
"If the book cannot be found in the library, it can be found in the bookstore".
"If the book cannot be found in the bookstore, it can be found in the library.

4. Negation of conditional statements

Positive: The sun shines
Negative: The sun does not shine

Positive: " If the temperature is 250°F then the compound is boiling "
Negative: ?
In order to find the negation, we use De Morgan's Laws.

Let
P = the temperature is 250°F
Q = the compound is boiling

Positive: $P \rightarrow Q \equiv \neg P \vee Q$
Negative: $\neg (P \rightarrow Q) \equiv \neg (\neg P \vee Q) \equiv \neg (\neg P) \wedge \neg Q \equiv P \wedge \neg Q$

Negative: The temperature is 250°F however the compound is not boiling

IMPORTANT TO KNOW:

The negation of a disjunction is a conjunction.
The negation of a conjunction is a disjunction

The **negation of a conditional statement is a conjunction**, not another if-then statement

Question: Which logical connective when negated will result in a conditional statement?

5. Necessary and sufficient conditions

Definition:

"P is a **sufficient condition** for Q" means : **if P then Q, $P \rightarrow Q$**
"P is a **necessary condition** for Q" means: **if not P then not Q, $\sim P \rightarrow \sim Q$**
The statement $\sim$**P → $\sim$Q is equivalent to Q → P**

Hence given the statement **P → Q,**
P is a sufficient condition for Q, and Q is a necessary condition for P.

Examples:

If n is divisible by 6 then n is divisible by 2.
The sufficient condition to be divisible by 2 is to be divisible by 6.
The necessary condition to be divisible by 6 is to be divisible by 2

If n is odd then n is an integer.
The sufficient condition to be an integer to be odd.
The necessary condition to be odd is to be an integer.

If and only if - the biconditional

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q$ is true whenever P and Q have same values. Otherwise it is false.

This means that **both P → Q and Q → P have to be true**

P	Q	**P → Q**	**Q → P**	P↔Q
T	**T**	**T**	**T**	**T**
T	F	F	T	F
F	T	T	F	F
F	**F**	**T**	**T**	**T**

Contrapositive

Definition: The expression ~Q → ~P is called **contrapositive** of P → Q

The conditional statement P → Q and its contrapositive ~Q → ~P **are equivalent**.
The proof is done by comparing the truth tables

The truth table for P → Q and ¬ Q → ¬ P is:

P	Q	¬ P	¬ Q	P→ Q	¬ Q → ¬ P
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

We can also prove the equivalence by using the disjunctive representation:

$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to \neg P$

Converse and inverse

Definition: The converse of P → Q is the expression Q → P

Definition: The inverse of P → Q is the expression ~P → ~Q

Neither the converse nor the inverse are equivalent to the original implication.
Compare the truth tables and you will see the difference.

P	Q	¬ P	¬ Q	P→ Q	Q→ P	**¬ P → ¬ Q**
T	T	F	F	T	T	T
T	F	F	T	**F**	**T**	**T**
F	T	T	F	**T**	**F**	**F**
F	F	T	T	T	T	T

Valid and Invalid Arguments.

Definition: An argument is a sequence of statements, ending in a conclusion. All the statements but the final one (the conclusion) are called premises(or assumptions, hypotheses)

Verbal form of an argument:

(1) If Socrates is a human being then Socrates is mortal.
(2) Socrates is a human being

Therefore (3) Socrates is mortal

Another way to write the above argument:

$$\begin{array}{l} P \rightarrow Q \\ P \\ \therefore\ Q \end{array}$$

2. Testing an argument for its validity

Three ways to test an argument for validity:

A. Critical rows

1. Identify the assumptions and the conclusion and assign variables to them.
2. Construct a truth table showing all possible truth values of the assumptions and the conclusion.
3. Find the **critical rows - rows in which all assumptions are true**
4. For each critical row determine whether the conclusion is also true.
   a. If the conclusion is **true in all critical rows**, then **the argument is valid**
   b. If there is at least one row **where the assumptions are true, but the conclusion is false**, then the argument is **invalid**

B. Using tautologies

The argument is true if the conclusion is true whenever the assumptions are true.
This means: If all assumptions are true, then the conclusion is true.
"All assumptions" means the conjunction of all the assumptions.

Thus, let A1, A2, … An be the assumptions, and B - the conclusion.

For the argument to be valid, the statement
If (A1 Λ A2 Λ… Λ An) then B must be **a tautology -** true for all assignments of values to its variables, i.e. its column in the truth table must contain only **T**

i.e.

$$(A1 \land A2 \land \ldots \land An) \rightarrow B \equiv \mathbf{T}$$

C. Using contradictions

If the argument is valid, then we have $(A1 \land A2 \land \ldots \land An) \rightarrow B \equiv \mathbf{T}$
This means that the negation of $(A1 \land A2 \land \ldots \land An) \rightarrow B$ should be a contradiction - containing only **F** in its truth table

In order to find the negation we have first to represent the conditional statement as a disjunction and then to apply the laws of De Morgan

$(A1 \land A2 \land \ldots \land An) \rightarrow B \equiv \sim(A1 \land A2 \land \ldots \land An) \lor B \equiv$

$\sim A1 \lor \sim A2 \lor \ldots \lor \sim An \lor B.$

The negation is:

$\sim((A1 \land A2 \land \ldots \land An) \rightarrow B) \equiv \sim(\sim A1 \lor \sim A2 \lor \ldots \lor \sim An \lor B)$

$\equiv A1 \land A2 \land \ldots \land An \land \sim B$

The argument is valid if $A1 \land A2 \land \ldots \land An \land \sim B \equiv F$

There are two ways to show that a logical form is a tautology or a contradiction:

a. by constructing the truth table
b. by logical transformations applying the logical equivalences (logical identities)

Examples:

1. Consider the argument:

$$\begin{array}{ll} & P \to Q \\ & P \\ \therefore & Q \end{array}$$

Testing its validity:

a. by examining the truth table:

P	Q	P→ Q
T	**T**	**T**
T	F	F
F	T	T
F	F	T

b. By showing that the statement 'If all premises then the conclusion" is a tautology: The premises are P and P→ Q. The statement to be considered is:

(P Λ (P→ Q)) → Q

We shall show that it is a tautology by using the following identity laws:

(1) P→ Q ≡ ~P V Q
(2) (P V Q) V R ≡ P V (Q V R) commutative laws
(P Λ Q) Λ R ≡ P Λ (Q Λ R)
(3) (P Λ Q) V R ≡ (P V R) Λ (Q V R) distributive law
(4) P Λ ~P ≡ F
(5) P V ~P ≡ T
(6) P V F ≡ P
(7) P V T ≡ T
(8) P Λ T ≡ P
(9) P Λ F ≡ F
(10) ~(P Λ Q) ≡ ~P V ~Q De Morgan's Laws

		(P Λ (P→ Q)) → Q
by (1)	≡	~(P Λ (P→ Q)) V Q
by (10)	≡	(~P V ~(P→ Q)) V Q
by (1)	≡	(~P V ~(~P V Q)) V Q
by (10)	≡	(~P V (P Λ ~Q)) V Q
by (3)	≡	((~P V P) Λ (~P V ~Q)) V Q
by (5)	≡	(**T** Λ (~P V ~Q)) V Q
by (8)	≡	(~P V ~Q) V Q
by (2)	≡	~P V (~Q V Q)
by (5)	≡	~P V **T**
by(7)	≡	**T**

2. Consider the argument

$$\begin{array}{ll} & P \to Q \\ & Q \\ \therefore & P \end{array}$$

We shall show that this argument is invalid by examining the truth tables of the assumptions and the conclusion. The critical rows are in boldface.

P	Q	P→ Q	
T	**T**	**T**	
T	F	F	
F	**T**	**T**	**here the assumptions are true, however the conclusion is false**
F	F	T	

Exercise:

Show the validity of the argument:

1. P V Q (premise)
2. ~Q (premise)

Therefore P (conclusion)

a. by using critical rows
b. by contradiction using logical identities

Solution:

a. by critical rows

conclusion		**Premises**		
P	**Q**	**P V Q**	**~Q**	
T	T	T	F	
T	F	**T**	**T**	**Critical row**
F	T	T	F	
F	F	F	T	

b. By contradiction using identities

$((P \vee Q) \wedge \sim Q) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee (Q \wedge \sim Q)) \wedge \sim P \equiv$

$((P \wedge \sim Q) \vee \mathbf{F}) \wedge \sim P \equiv$

$(P \wedge \sim Q) \wedge \sim P \equiv$

$P \wedge \sim P \wedge \sim Q \equiv \mathbf{F} \wedge \sim Q \equiv \mathbf{F}$

POSSIBLE QUESTIONS

2 MARKS

1. Construct the truth table for $\neg(P \wedge Q)$.

2. Define tautology .

3. Prove that without using truth table $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$ is a tautology.

4. Prove that $P \rightarrow (Q \vee R) \leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$.

5. Construct the truth table for $\neg(P) \vee \neg(Q)$.

6 MARKS

1.construct the truth table $\neg(P \vee (Q \wedge R)$

2.show that $(x)(H(x) \rightarrow M(x)) \wedge H(S) \rightarrow M(S)$

3.Define disjunctive normal form and conjunctive normal form. Also obtain disjunctive normal form of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

4.Prove that $(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R))$ is a tautology.

5.Verify that a proposition $P \vee \neg(P \wedge Q)$ is a tautology.

6.Obtain the PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

7.Lions are dangerous animals. There are lions. There are dangerous animals.

8.Construct the truth table for $(P \leftrightarrow R) \wedge (\neg Q \rightarrow S)$

9.Obtain PDNF of $(\neg((P \vee Q) \wedge R)) \wedge (P \vee R))$

10.Demonstrate that R is a valid inference from the premises $P \rightarrow Q, Q \rightarrow R, and\ P$.

Questions	opt1	opt2	opt3	opt4	Answer
Let p be “He is tall” and let q “He is handsome”. Then the statement “It is false that he is short or handsome” is:	p^q	~(~pv q)	~pv q	p v q	~(~pv q)
The proposition p^ (~ p v q) is………………………	A tautology	a contradiction	Logically equivalent to p ^q	an assumption	Logically equivalent to p ^ q
Which of the following is/are tautology?	a v b → b ^ c	a ^ b → b v c	a v b → (b → c)	a v b → b v c	a ^ b → b v c
Identify the valid conclusion from the premises Pv Q, Q → R, P → M, ˥M	P ^ (R v R)	P ^ (P ^ R)	R ^ (P v Q)	Q ^ (P v R)	Q ^ (P v R)
Let a, b, c, d be propositions. Assume that the equivalence a ↔ (b v ˥b) and b ↔ c hold. Then truth value of the formula (a ^ b) → ((a ^ c) v d) is always	TRUE	FALSE	Same as the truth value of a	Same as the truth value of b	TRUE
Which of the following is a declarative statement?	It's right	He says	Two may not be an	I love you	He says

P → (Q → R) is equivalent to-------------	(P ^ Q) → R	(P v Q) → R	(P v Q) → ⅂R	(P v Q) → P	(P ^ Q) → R
If F1, F2 and F3 are propositional formulae such that F1 ^ F2 → F3 and F1 ^ F2→F3 are both tautologies, then which of the following is TRUE?	Both F1 and F2 are tautologies	The conjuction F1 ^ F2 is not satisfiable	Neither is tautologies	F1v F2 is tautology	Both F1 and F2 are tautologies
Consider two well-formed formulas in propositional logic F1 : P →⅂P F2 : (P →⅂P) v (⅂P →), then	F1 is satisfiable, F2 is unsatisfiable	F1 is unsatisfiable, F2 is satisfiable	F1 is unsatisfiable, F2 is valid	F1 & F2 are both satisfiable	F1 is unsatisfiable, F2 is valid
What can we correctly say about proposition P1 : (p v ⅂q) ^ (q →r) v (r v p)	P1 is tautology	P1 is satisfiable	If p is true and q is false and r is false	If p as true and q is true and r	If p is true and q is false and r is false
(P v Q) ^ (P → R)^ (Q →S) is equivalent to	S ^ R	S → R	S v R	S U R	S v R
In propositional logic , which of the following is equivalent to p → q?	~p → q	~p v q	~p v~ q	p →q	~p v q
⅂(P → Q) is equivalent to	P ^ ⅂Q	P ^ Q	⅂P v Q	⅂P ^ Q	P ^ ⅂Q
(P v Q) ^ (P→R) ^ (Q → R) is equivalent to	P	Q	R	True= T	R

How many rows would be in the truth table for the following compound proposition: (p ∨ q) ∧ ¬(q ∧ t) ∨ (r → s)	32	34	27	25	32
Which of the following statement is the negation of the statement,"2 is even and –3 is negative"?	2 is even and –3 is not negativ	2 is odd and –3 is not negative.	2 is even or –3 is not negativ	2 is odd or –3 is not negati	2 is odd or –3 is not negativ
p→q is logically equivalent to	~ q→p	~ p→q	~ p ^ q	~ p v q	~ p v q
Which of the following is not a well formed formula?	"for all x [P(x)→ f (x)^	for all x1,x2,x3 { x1 = x2 ^ x2 = x3	~ (p →q)→ q	[T v P(a, b)]→z Q(z)	for all x1,x2,x3 { x1 = x2 ^
[~ q ^ (p→q)]→~ p is,	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
An and statement is true if, and only if, both components are	TRUE	FALSE	not true	neither true nor false	TRUE
If P : It is hot & Q : It is humid,then what does P ^ (~ Q):mean?	It is not hot and it is not humid	It is hot and it is humid	It is hot and it is not humid	It is not hot and it is not humid	It is hot and it is not humid
An or statement is false if, and only if, both components are ………………..	TRUE	FALSE	not true	neither true nor false	FALSE

Two statement forms are logically equivalent if, and only if they always have……………………. …….	not same truth values	the same truth values	the different truth values	the same false values	the same truth values
A tautology is a statement that is always ………………….	TRUE	FALSE	not true	neither true nor false	TRUE
A contradiction is a statement that is always ………………….	FALSE	TRUE	not true	neither true nor false	FALSE
The statement (p^q) Þ p is a…………………	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
In propositional logic which one of the following is equivalent to p→q ?	p→q	p→q	p v q	p v ~~q~~	~~p~~ v q
Which of the following proposition is a tautology?	(p v q)→p	p v (q→p)	p v(p→q)	(p v q)→q	p v(p→q)
Which one is the contrapositive of q → p ?	~p → ~ q	p → ~ q	~p → q	p → q	~p → ~ q
The statement form pv(~p) is a……………………..	Satisfiable	Unsatisfiable	Tautology	Invalid	Tautology
Let p and q be statements given by “p →q". Then q is called	hypothesis	conclusion	TRUE	#####	conclusion

The statement form p^(~p) is a……………………..	contradiction	Unsatisfiable	Tautology	Invalid	contradiction
If p and q are statement variables, the conditional of q by p is given by ……………..	~p → ~ q	p → ~ q	~p → q	p → q	p → q
Let p and q be statements given by "p →q". Then p is called………………	hypothesis	conclusion	TRUE	#####	hypothesis
The statement (p → r) ∧ (q → r) is equivalent to…………	p ∨ q →~ r	p ∨ q → r	p ∨~ q → r	~p ∨ q → r	p ∨ q → r
The Negation of a Conditional Statement p →q is given by ………………..	p ∧ ~q	~p ∧ ~q	p V ~q	p ∧ q	p ∧ ~q
Given statement variables p and q, the biconditional of p and q is given by	p«~q	p→q	~p«q	p«q	p«q
The inverse of "if p then q" is …………………………	if ~p then ~q	if ~p then ~q	if ~p then ~q	if ~p then ~q	if ~p then ~q

"R is a………………………. condition for S" means "if R then S ."	valid	inevitable	sufficient	necessary	sufficient
A conditional statement and its contrapositive are …………………	A tautulogy	a contradiction	Logically equivalent	an assumption	Logically equivalent
A rule of inference is a form of argument that is …………….	valid	a contradiction	an assumption	A tautulogy	valid

Reg no-------------

(18ITU202/18CSU202/18CTU202/18CAU202)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Coimbatore-21

DEPARTMENT OF MATHEMATICS

Second Semester

I Internal Test - Nov'2018

Discrete Structure

Date: -11-2018 **Time: 2 Hours**

Class: I-B.Sc IT,CT,CS,BCA **Maximum Marks:50**

PART-A(20X1=20 Marks)

Answer all the Questions:

1. If f(x) = x+2 and g(x) = x^2 –1 then $(g_o f)(x)$ = ------
   (a) x^2 +4x+4 (b)x^2 +4x-3 (c) x^2 - 4x+4 (d) x^2 +4x+3
2. A mapping f : x→y is called ------ if distinct elements of x are mapped into distinct elements
   (a) one-one (b) onto (c) into (d) many-one
3. Let f : N → N be a function such that f(x) = 5 ,x є N then the f(x) is called---------function.
   (a) identity (b) constant (c) inverse (d) equal
4. The number of different permutations of the word BANANA is -------
   (a) 720 (b) 360 (c) 120 (d) 60
5. Growth of functions allows to compare relative performance of ________ algorithms
   (a) relations (b) alternative (c)same (d) parameters
6. The r - permutation of n elements is denoted by ______
   (a) P (r, n) (b) P(n,r) (c) c(r, n) (d) c(n, r)
7. If R= {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)} are relations then RoS = ______
   (a) {(4,2),(3,2),(1,4)} (b) {(1,5),(3,2),(2,5)}
   (c) {(1,2),(2,2)} (d) {(4,5),(3,3),(1,1)}
8. If the relation R and S are both reflexive then R Ú S is _______
   (a) transitive (b) not reflexive
   (c) reflexive (d) symmetric
9. Let f : x→y , g : y→ x be the functions then g is equal to f^{-1} only if ______
   (a) fog = I_y (b)gof = I_y (c) fog = I_x (d) gof = I_x
10. The number of ways can a party of 7 persons arrange themselves around a circular table______
    (a) 6! (b) 7! (c) 0 (d) 3!
11. The value of C(n,n) is ________
    (a) 0 (b) -1 (c) 2 (d) 1
12. Let f : N→N be a function such that f(x) = 5 ,for every x in N then the f(x) is called________function.
    (a) constant (b) identity (c) unit (d) zero
13. In N, define aRb if a+b = 7. This is symmetric when _____
    (a) b+a=7 (b) a=b (c) ab=7 (d) a+a=7
14. An Onto function is also known as_________
    (a) injective (b) surjective (c) bijective (d) into
15. The value of C(10, 8) + C(10,7) is _____________
    (a) 990 (b) 165 (c) 45 (d) 120
16. The sum of entries in the fourth row of Pascal's triangle is ____.
    (a) 10 (b) 4 (c)10 (d) 16
17. The growth of ___________is directly related to the complexity of algorithms.
    (a) Functions (b) relations (c) parameters (d) polynomials
18. How many 10 digits numbers can be written by using the digits 1 and 2 ?
    (a) C (10, 9) + C (9, 2) (b) 1024 (c)C(10, 2) (d) 10!
19. A binary relation R in a set X is said to be antisymmetric if _________.
    (a) aRa (b) aRb→bRa
    (c) aRb,bRc→aRc (d) aRb,bRa→a=b
20. If log n = $\log_2$ n then it is ______________
    (a) Binary logarithm (b) composition
    (c)exponentiation (d) relation

PART-B (3X2=6 Marks)

Answer all the Questions:

21. If A = {a, b, c} and B = {1, 2} then find A $\times$ B and B $\times$ A

22. Give an example for one – one function.

23. Define Permutation

PART-C (3X8=24 Marks)

Answer all the Questions:

24. (a) Prove that the associative property under union.

(OR)

(b) Prove that $1^2+2^2+3^2+\ldots +n^2= n(n+1)(2n+1)/6$ by Principle of Mathematical induction.

25. (a) State and prove Pigeonhole Principle.

(OR)

(b) Explain about properties of relation.

26. (a) Let A={1,2,3} and f,g,h and s be functions from A to A given by f ={ (1,2), (2,3),(3,1) } ; g = { (1,2), (2,1), (3,3) } ;

h = { (1,1), (2,2), (3,1) } and s = { (1,1), (2,2), (3,3) }. Find $f \circ g,\ g \circ f,\ f \circ h \circ g,\ g \circ s, s \circ s,\ f \circ s.$

(OR)

(b) Write about the types of function with example.