KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
SYLLABUS

		Semester – I
17MMP103	**NUMERICAL ANALYSIS**	**L T P C**
		4 0 0 4

Course Objective: This course provides a deep knowledge to the learners to understand the basic concepts ofNumerical Methods which utilize computers to solve Engineering Problems that are not easily solved or even impossible to solve by analytical means.

Course Outcome: To familiarize with numerical solution of equations, ODE & PDE and get exposed withnumerical differentiation and integration.

UNIT I

Solutions of Non Linear Equations: Newton's method-Convergence of Newton's method-Bairstow's method for quadratic factors. Numerical Differentiation and Integration:
Derivatives from difference tables – Higher order derivatives – divided difference.
Trapezoidal rule – Romberg integration – Simpson's rules.

UNIT II

Solutions of system of Equations: The Elimination method: Gauss Elimination and Gauss Jordan Methods – LU decomposition method.
Methods of Iteration: Gauss Jacobi and Gauss Seidal iteration-Relaxation method.

UNIT III

Solutions of Ordinary Differential Equations: One step method: Euler and Modified Euler methods–Rungekutta methods. Multistep methods: Adams Moulton method – Milne's method

UNIT IV

Boundary Value Problem and Characteristic value problem: The shooting method: The linear shooting method – The shooting method for non-linear systems.
Characteristic value problems –Eigen values of a matrix by Iteration-The power method.

UNIT V

Numerical Solution of Partial Differential Equations: Classification of Partial Differential Equation of the second order – Elliptic Equations. Parabolic equations: Explicit method – The Crank Nicolson difference method. Hyperbolic equations – solving wave equation by Explicit Formula.

SUGGESTED READINGS

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.

Department of Mathematics

LECTURE PLAN

Subject: Numerical Analysis **Subject code: 17MMP103**

S.No	Lecture Hour	Topics to be Covered	Support Materials
UNIT 1			
1	1	Introduction and basic concepts of simultaneous equations	R1: Ch 2: Pg: 20-24
2	1	Problems on Newton Raphson method	R3: Ch 2: Pg: 33-34
3	1	Continuation of Problems on Newton Raphson method	R3: Ch 2: Pg: 34-35
4	1	Convergency on Newton Raphson method	R1: Ch 2: Pg: 35-36
5	1	Bairstows method for quadratic factors	T1: Ch 1: Pg: 66-68
6	1	Continuation of Bairstows method for quadratic factors	T1: Ch 1: Pg: 68-70
7	1	Derivative from difference table and higher order derivatives	T1: Ch 5: Pg: 357-367
8	1	Divided difference	R2: Ch 5: Pg: 124-129
9	1	Trapezoidal rule and Simpson's rule	R3: Ch 5: Pg: 198-201
10	1	Problems on Trapezoidal rule and Simpson's rule	R3: Ch 5: Pg: 209
11	1	Romberg's Integration	R3: Ch 5: Pg: 202-204
12	1	Problems on Romberg's Integration	R3: Ch 5: Pg: 208
13	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 13 hours			

T1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

R1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

R2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

R3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT II			
S.No	**Lecture Hour**	**Topics to be Covered**	**Support Materials**
1	1	Introduction: Gauss Elimination method problems	R3: Ch 6: Pg: 257-258
2	1	Continuation of problems on Gauss Elimination method	R3: Ch 6: Pg: 259-260
3	1	Problems on Gauss Jordan method	R3: Ch 6: Pg: 260-261
4	1	Continuation of Problems on Gauss Jordan method	R3: Ch 6: Pg: 261-262
5	1	Problems on LU decomposition method	R3: Ch 6: Pg: 265-266
6	1	Continuation of Problems on LU decomposition method	R3: Ch 6: Pg: 266-269
7	1	Problems on Gauss Jacobi method	R2: Ch 7: Pg: 450-452
8	1	Continuation of Problems on Gauss Jacobi method	R2: Ch 7: Pg: 452-454
9	1	Problems on Gauss Seidal method	R2: Ch 7: Pg: 454-455
10	1	Continuation of Problems on Gauss Seidal method	R2: Ch 7: Pg: 456-457
11	1	Problems on Relaxation method	R2: Ch 7: Pg: 462-463
12	1	Continuation of Problems on Relaxation method	R2: Ch 7: Pg: 464-466
13	1	Recapitulation and discussion of possible questions	
Total No. of Lecture hours planned – 13 hours			

R2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.
R3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT III			
S.No	**Lecture Hour**	**Topics to be Covered**	**Support Materials**
1	1	Introduction on solution of ODE	R3: Ch 7: Pg: 295-296
2	1	Problems on Euler method	R3: Ch 7: Pg: 300-301
3	1	Continuation of problems on Euler method	R3: Ch 7: Pg: 301-303
4	1	Problems on Modified Euler method	R3: Ch 7: Pg: 303-304
5	1	Continuation of problems on Modified Euler method	R3: Ch 7: Pg: 304-305
6	1	Problems on RungeKutta method	R3: Ch 7: Pg: 305-306
7	1	Continuation of problems on RungeKutta method	R3: Ch 7: Pg: 307-308
8	1	Problems on Multistep methods: Adams Moulton method	R3: Ch 7: Pg: 309-310

9	1	Continuation of problems on Adams Moulton method	R3: Ch 7: Pg: 310-311
10	1	Problems on Milne's method	R3: Ch 7: Pg: 311-312
11	1	Continuation of problems on Milne's method	R3: Ch 7: Pg: 313-314
12	1	Recapitulation and discussion of possible questions	

Total No. of Lecture hours planned – 12 hours

R3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT IV

S.No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Boundary value problems	R3: Ch 7: Pg: 318-323
2	1	Problems on linear shooting method	R2: Ch 11: Pg: 672-674
3	1	Continuation of problems on linear shooting method	R2: Ch 11: Pg: 674-676
4	1	Problems on shooting method for nonlinear systems	R2: Ch 11: Pg: 678-680
5	1	Continuation of problems on shooting method for nonlinear systems	R2: Ch 11: Pg: 680-683
6	1	Characteristics value problems	T1: Ch 7: Pg: 541-542
7	1	Problems on eigen values of a matrix by iteration	T1: Ch 7: Pg: 542-543
8	1	Continuation of problems on eigen values of a matrix by iteration	T1: Ch 7: Pg: 544-545
9	1	Problems on power method	R2: Ch 9: Pg: 580-581
10	1	Recapitulation and discussion of possible questions	

Total No. of Lecture hours planned – 10 hours

T1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

R2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

R3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

UNIT V

S. No	Lecture Hour	Topics to be Covered	Support Materials
1	1	Classification of PDE	R3: Ch8: Pg: 333-335

2	1	Problems on Elliptic equation	R3: Ch8: Pg: 338-341
3	1	Continuation of Problems on Elliptic equation	R3: Ch8: Pg: 341-345
4	1	Problems on Parabolic equation- Explicit method	R3: Ch8: Pg: 349-354
5	1	Problems on parabolic equation- Crank Nicolson difference method	R3: Ch8: Pg: 355-356
6	1	Continuation of problems on Crank Nicolson difference method	R3: Ch8: Pg: 356-357
7	1	Problems on Hyperbolic equations	R3: Ch8: Pg: 358-362
8	1	Problems on solving wave equation by explicit formula	T1: Ch8: Pg: 603-605
9	1	Recapitulation and discussion of possible questions	
10	1	Discussion of previous ESE question papers.	
11	1	Discussion of previous ESE question papers.	
12	1	Discussion of previous ESE question papers.	
Total No. of Lecture hours planned – 12 hours			
T1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi. **R3.** Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.			

SUGGESTED READINGS

TEXT BOOK

1.Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .
2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.
3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
Department of Mathematics

Subject: Numerical Analysis	**Subject Code: 17MMP103**	**L T P C**
Class : I – M.Sc. Mathematics	**Semester : I**	**4 0 0 4**

UNIT-I

Solutions of Non Linear Equations: Newton's method-Convergence of Newton's method-Bairstow's method for quadratic factors. Numerical Differentiation and Integration: Derivatives from difference tables – Higher order derivatives – divided difference. Trapezoidal rule – Romberg integration – Simpson's rules.

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

Introduction

Equations that can be cast in the form of a polynomial are referred to as ***algebraic equations***. Equations involving more complicated terms, such as trigonometric, hyperbolic, exponential, or logarithmic functions are referred to as **transcendental equations**. The methods presented in this section are numerical methods that can be applied to the solution of such equations, to which we will refer, in general, as ***non-linear equations***. In general, we will we searching for one, or more, solutions to the equation,

$$f(x) = 0.$$

We will present the ***Newton-Raphson*** algorithm, and the ***secant*** method. In the secant method we need to provide two initial values of x to get the algorithm started. In the Newton-Raphson methods only one initial value is required.

Newton-Raphson method (or Newton's method)

Let us suppose we have an equation of the form $f(x) = 0$ in which solution is lies between in the range (a,b). Also $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are opposite signs, then there exist atleast one real root between a and b.

Let $f(a)$ be positive and $f(b)$ negative. Which implies atleast one root exits between a and b. We assume that root to be either a or b, in which the value of f(a) or f(b) is very close to zero. That number is assumed to be initial root. Then we iterate the process by using the following formula until the value is converges.

$$X_{n+1} = Xn - \frac{f(X_n)}{f'(Xn)}$$

Steps:

1. Find a and b in which $f(a)$ and $f(b)$ are opposite signs for the given equation using trial and error method.
2. Assume initial root as $X_o = a$ i.e., if $f(a)$ is very close to zero or $Xo = b$ if $f(a)$ is very close to zero
3. Find X1 by using the formula

$$X_1 = Xo - \frac{f(X_o)}{f'(Xo)}$$

4. *Find X_2 by using the following formula*

$$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$$

5. *Find $X_3, X_4, \ldots X_n$ until any two successive values are equal.*

Example:

Find the positive root of f(x) = 2x3 - 3x-6 =0 by Newton – Raphson method correct to five decimal places.

Solution:

Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(1)$ = 2-3-6 = -7 = -ve

$f(2)$ = 16 – 6-6 = 4 = +ve

So, a root between 1 and 2 . In which 4 is closer to 0 Hence we assume initial root as 2.

Consider $x_0 = 2$

So $X_1 = X_0 - f(X_0)/f'(X_0)$

$= X_0 - ((2X_03 - 3X_0 - 6) / 6\alpha_0 - 3) = (4X_03 + 6)/(6X_02 - 3)$

$X_{i+1} = (4X_i3 + 6)/(6X_i2 - 3)$

$X_1 = (4(2)^2+6)/(6(2)^2 - 3) = 38/21 = 1.809524$

$X_2 = (4(1.809524)^3+6)/(6(1.809524)^2 - 3) = 29.700256/16.646263 = 1.784200$

$X_3 = (4(1.784200)^3+6)/(6(1.784200)^2 - 3) = 28.719072/16.100218 = 1.783769$

$X_4 = (4(1.783769)^3+6)/(6(1.783769)^2 - 3) = 28.702612/16.090991 = 1.783769$

Example:

Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$ correct to 5 decimal places.

Solution :

Let $f(x) = x^3 - 6x + 4$; $f(0) = 4 = +ve$; $f(1) = -1 = -ve$

So a root lies between 0 and 1

f(1) is nearer to 0. Therefore we take initial root as $X_0 = 1$

$f'(x) = 3x^2 - 6$

$$= x - \frac{f(x)}{f'(x)}$$

$= x - (3x^3 - 6x + 4)/(3x^2 - 6)$

$= (2x^3 - 4)/(3x^2 - 6)$

$X_1 = (2X_0 3 - 4)/(3X_0 2 - 6) = (2-4)/(3-6) = 2/3 = 0.66666$

$X_2 = (2(2/3)^3 - 4)/(3(2/3)^2 - 6) = 0.73016$

$X_3 = (2(0.73015873)^3 - 4)/(3(0.73015873)^2 - 6)$

$= (3.22145837 / 4.40060469)$

$= 0.73205$

$X_4 = (2(0.73204903)^3 - 4)/(3(0.73204903)^2 - 6)$

$= (3.21539602 / 4.439231265)$

$= 0.73205$

The root is 0.73205 correct to 5 decimal places.

Bairstow Method

Bairstow Method is an iterative method used to find both the real and complex roots of a polynomial. It is based on the idea of synthetic division of the given polynomial by a quadratic function and can be used to find all the roots of a polynomial. Given a polynomial say,

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \tag{B.1}$$

Bairstow's method divides the polynomial by a quadratic function.

$$x^2 - rx - s. \tag{B.2}$$

Now the quotient will be a polynomial $f_{n-2}(x)$, *i.e.*

$$f_{n-2}(x) = b_2 + b_3 x + b_4 x^2 + \dots + b_{n-1} x^{n-3} + b_n x^{n-2} \tag{B.3}$$

and the remainder is a linear function $R(x)$, i.e.

$$R(x) = b_1(x-r) + b_0 \tag{B.4}$$

Since the quotient $f_{n-2}(x)$ and the remainder $R(x)$ are obtained by standard synthetic division the co-efficients $b_i \quad (i = 0..n)$ can be obtained by the following recurrence relation.

$$b_n = a_n \tag{B.5a}$$

$$b_{n-1} = a_{n-1} + rb_n \tag{B.5b}$$

$$b_i = a_i + rb_{i+1} + sb_{i+2} \quad \text{for} \quad i = n-2 \quad to \quad 0 \tag{B.5c}$$

If $x^2 - rx - s$ is an exact factor of $f_n(x)$ then the remainder $R(x)$ is zero and the real/complex roots of $x^2 - rx - s$ are the roots of $f_n(x)$. It may be noted that $x^2 - rx - s$ is considered based on some guess values for r, s. So Bairstow's method reduces to determining the values of r and s such that $R(x)$ is zero. For finding such values Bairstow's method uses a strategy similar to Newton Raphson's method.

Since both b_0 and b_1 are functions of r and s we can have Taylor series expansion of b_0, b_1 as:

$$b_1(r+\Delta r, \quad s+\Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s + O(\Delta r^2, \Delta s^2) \tag{B.6a}$$

$$b_0(r+\Delta r, \quad s+\Delta s) = b_0 + \frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s + O(\Delta r^2, \Delta s^2) \tag{B.6b}$$

For $\Delta s, \Delta r << 1$, $O(\Delta r^2, \Delta s^2)$ terms ≈ 0 i.e. second and higher order terms may be neglected, so that $(\Delta r, \Delta s)$ the improvement over guess value (r, s) may be obtained by equating (B.6a),(B.6b) to zero i.e.

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1 \tag{B.7a}$$

$$\frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s = -b_0 \quad \text{(B.7b)}$$

To solve the system of equations $(B.7a)-(B.7b)$, we need the partial derivatives of b_0, b_1 w.r.t. r and s. Bairstow has shown that these partial derivatives can be obtained by synthetic division of $f_{n-2}(x)$, which amounts to using the recurrence relation $(B.5a)-(B.5c)$ replacing $a_i's$ with $b_i's$ and $b_i's$ with $c_i's$ i.e.

$$c_n = b_n \quad \text{(B.8a)}$$

$$c_{n-1} = b_{n-1} + rc_n \quad \text{(B.8b)}$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \quad \text{(B.8c)}$$

for $i = 1,2,\dots,n-2$

where

$$\frac{\partial b_0}{\partial r} = c_1, \quad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \quad and \quad \frac{\partial b_1}{\partial s} = c_3 \quad \text{(B.9)}$$

∴ The system of equations (B.7a)-(B.7b) may be written as.

$$c_2\Delta r + c_3\Delta s = -b_1 \quad \text{(B.10a)}$$

$$c_1\Delta r + c_2\Delta s = -b_o \quad \text{(B.10b)}$$

These equations can be solved for $(\Delta r, \Delta s)$ and turn be used to improve guess value (r, s) to $(r+\Delta r, s+\Delta s)$.

Now we can calculate the percentage of approximate errors in (r,s) by

$$|\varepsilon_{a,r}| = |\frac{\Delta r}{r}| \times 100; \quad \varepsilon_{a,s} = |\frac{\Delta s}{s}| \times 100 \quad \text{(B.11)}$$

If $|\varepsilon_{a,r}| > \varepsilon_s$ or $|\varepsilon_{a,s}| > \varepsilon_s$, where ε_s is the iteration stopping error, then we repeat the process with the new guess i.e. $(r + \Delta r, s + \Delta s)$. Otherwise the roots of $f_n(x)$ can be determined by

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2} \quad \text{(B.12)}$$

If we want to find all the roots of $f_n(x)$ then at this point we have the following three possibilities:

1. If the quotient polynomial $f_{n-2}(x)$ is a third (or higher) order polynomial then we can again apply the Bairstow's method to the quotient polynomial. The previous values of (r, s) can serve as the starting guesses for this application.
2. If the quotient polynomial $f_{n-2}(x)$ is a quadratic function then use (B.12) to obtain the remaining two roots of $f_n(x)$.
3. If the quotient polynomial $f_{n-2}(x)$ is a linear function say $ax + b = 0$ then the remaining single root is given by $x = -\frac{b}{a}$

Example:
Find all the roots of the polynomial

$$f_4(x) = x^4 - 5x^3 + 10x^2 - 10x + 4$$

by Bairstow method . With the initial values $r = 0.5, \quad s = -0.5 \quad and \quad \varepsilon_s = 0.01.$

Solution:
Set *iteration=1*

$$a_0 = 4, \quad a_1 = -10, \quad a_2 = 10, \quad a_3 = -5 \quad a_4 = 1$$

Using the recurrence relations (B.5a)-(B.5c) and (B.8a)-(B.8c) we get

$$b_4 = 1, \quad b_3 = -4.5, \quad b_2 = 7.25, \quad b_1 = -4.125, \quad b_0 = -1.6875$$

$$c_4 = 1, \quad c_3 = -4 \quad c_2 = 4.75, \quad c_1 = 0.25$$

$\therefore$ the simultaneous equations for Δr and Δs are:

$$4.75\Delta r - 4\Delta s = 4.125$$

$$0.25\Delta r + 4.75\,\Delta s = 1.6875$$

on solving we get $\Delta r = 1.1180371, \quad \Delta s = 0.296419084$

$$\therefore r = 0.5 + \Delta r = 1.6180371$$

$$s = -0.5 + \Delta s = -0.203580916$$

and

$$|\varepsilon_{a,r}| = \left|\frac{1.1180371}{1.6180371}\right| \times 100 = 69.0983582$$

$$|\varepsilon_{a}, s| = \left|\frac{0.296419084}{-0.203580916}\right| \times 100 = 145.602585$$

Set *iteration*=2

$$b_4 = 1.0, \quad b_3 = -3.38196278, \quad b_2 = 4.32427788, \quad b_1 = -2.31465483, \quad b_0 = -0.625537872$$

$$c_4 = 1.0, \quad c_3 = -1.76392567, \quad c_2 = 1.26659977, \quad c_1 = 0.0938522071$$

$\therefore$ now we have to solve

$$1.26659977\Delta r - 1.76392567\Delta s = 2.31465483$$
$$0.0938522071\Delta r + 1.26659977\Delta s = 0.625537872$$

On solving we get $\Delta r = 2.27996969, \quad \Delta s = 0.324931115$

$$\therefore r = 1.6180371 + \Delta r = 3.89800692$$
$$s = -0.203580916 + \Delta s = 0.121350199$$

$$|\varepsilon_{a,r}| = \left|\frac{2.27996969}{3.89800692}\right| \times 100 = 58.490654$$

$$|\varepsilon_{a,s}| = \left|\frac{0.32493115}{0.121350199}\right| \times 100 = 267.763153$$

Now proceeding in the above manner in about ten iteration we get $r = 3, \quad s = -2$ with

$$|\varepsilon_{a,r}| \sim 7.95 \times 10^{-6} < \varepsilon_s = 0.01$$

$$|\varepsilon_{a,s}| \sim 5.96 \times 10^{-6} < \varepsilon_s = 0.01$$

Now on using $x = \dfrac{r \pm \sqrt{r^2 + 4s}}{2}$ $(i.e. \quad eqn. \quad B.12)$ we get $x = \dfrac{3 \pm \sqrt{9-8}}{2} = 2, 1$

So at this point Quotient is a quadratic equation

$$f_2(x) = x^2 + 2x + 2$$

Roots of $f_2(x)$ are: $x = 1 - i, \quad 1 + i$

$\therefore$ Roots $f_4(x)$ are $= 1 - i, \quad 1 + i, \quad 1, \quad 2$

i.e $f_4(x) = (x - (1-i))(x - (1+i))(x-1)(x-2)$.

Numerical differentiation

The problem of Interpolation is finding the value of y for the given value of x among (x_i , y_j) for i= 1 to n. Now we find the derivatives of the corresponding arguments . If the required value of y lies in the first half of the interval then we call it as Forward interpolation .If the required value of y (derivative value) lies in the second half of the interval we call it as Backward interpolation also if the derivative of y lies in the middle of of class interval then we solve by central difference.

Newton's forward formula for Interpolation :

$Y= y_0 +u \Delta y_0+u(u-1)/2! \Delta^2 Y_0 +u (u-1)(u-2) / 3! \Delta^3 Y_0+\ldots\ldots$

Where $u = (x-x_0)/h$

Differentiating with respect to x ,

$dy/dx = (dy/du). (du/dx) = (1/h) (dy / du)$

$(dy / dx)\ x \neq x_0 = (1 / h) [\Delta y_0 +(2u-1)/2\ \Delta^2 y_0 +(3u^2 -6u+2)/ 6\ \Delta^3 y_0 +\ldots\ldots\ldots]$

$(dy / dx)\ x = x_0 = (1 / h) [\Delta y_0 -(1/2)\ \Delta^2 y_0 +(1/ 3)\ \Delta^3 y_0 +\ldots\ldots\ldots]$

$(d^2y / dx^2)\ x \neq x_0= d/dx (dy / dx) = d/dx(dy / du. du / dx)$

$= (1/h^2) [\Delta^2 y_0 + 6(u-1) / 6\ \Delta^3 y_0 + (12u^2 - 36 u +22) / 2\ \Delta^4 y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^2) [\Delta^2 y_0 - \Delta^3 y_0 +(11/12) \Delta^4 y_0 +\ldots\ldots.]$

Similarly,

$(d^3y / dx^3)\ x \neq x_0= (1/h^3) [\Delta^3 y_0 + (2u - 3) / 2\ \Delta^4 y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^3) [\Delta^3 y_0 -(3/2)\Delta^4 y_0 +\ldots\ldots.]$.

In a similar manner the derivatives using backward interpolation an also be found out.

Using backward interpolation .

$(dy / dx)\ x \neq x_n = (1 / h) [\nabla y_n +(2u+1)/2\ \nabla^2 y_n +(3u^2 +6u+2)/ 6\ \nabla^3 y_n +\ldots\ldots\ldots]$

$(dy / dx)\ x = x_n = (1 / h) [\nabla y_n -(1/2)\ \nabla^2 y_n +(1/ 3)\ \nabla^3 y_n +\ldots\ldots\ldots]$

$(d^2y / dx^2)\ x \neq x_0= (1/h^2) [\nabla^2 y_0 + 6(u-1) / 6\ \nabla^3 y_0 + (12u^2 - 36 u +22) / 2\ \nabla^4 y_0 +\ldots\ldots.]$

$(d^2y / dx^2)\ x= x_0=(1/h^2) [\nabla^2 y_0 - \nabla^3 y_0 +(11/12) \nabla^4 y_0 +.]$

Example

Find the first two derivatives of $x^{(1/3)}$ at x= 50 and x= 56, given the table below.

X:	50	51	52	53	54	55	56
Y:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	3.6840				
51	3.7084	0.0244			
52	3.7325	0.0241	-0.0003	0	
53	3.7563	0.0238	-0.0003	0	0
54	3.7798	0.0235	-0.0003	0	0
55	3.8030	0.0232	-0.0003	0	0
56	3.8259	0.0229	-0.0003		

At x= 50,

$$(dy/dx)_{x=x0} = (1/h)[\Delta y_0 - (1/2)\Delta^2 y_0 + (1/3)\Delta^3 y_0 + \ldots\ldots\ldots]$$

$$= (1/1)[0.024-(1/2)(-0.0003)+0] = 0.02455$$

$$(d^2y/dx^2)_{x=x0} = (1/h^2)[\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\Delta^4 y_0 + \ldots\ldots.]$$

$$= (1/1)[-0.003-0] = -.0003$$

At x=56,

$$(dy/dx)_{x=xn} = (1/h)[\nabla y_n + (1/2)\nabla^2 y_n + (1/3)\nabla^3 y_n + \ldots\ldots\ldots]$$

$$= (1/1)[0.0229+(1/2)(-0.0003)+0] = 0.02275.$$

$$(d^2y/dx^2)_{x=xn} = (1/h^2)[\nabla^2 y_n + \nabla^3 y_n + (11/12)\nabla^4 y_n + \ldots]$$

$$= (1/1)[-0.003-0] = -0.0003.$$

For the above ptroblem let us find the first two derivatives of x when x= 52 and x= 55.

When x=52, From Newton's forward formula

$$(dy/dx)\ x \neq x_0 = (1/h)[\Delta y_0 + (2u-1)/2\ \Delta^2 y_0 + (3u^2 - 6u+2)/6\ \Delta^3 y_0 + \ldots\ldots\ldots],$$

$$= (1/1)[0.0244+(3/2)(-0.0003)+0] = 0.02395,$$

Since here $u = (x-x_0)/h = (52-50)/1 = 2$.

$$(d^2y/dx^2)\ x \neq x_0 = (1/h^2)[\Delta^2 y_0 + 6(u-1)/6\ \Delta^3 y_0 + (12u^2 - 36u + 22)/2\ \Delta^4 y_0 + \ldots\ldots.]$$

= (1/)m [-0.0003+0] = -0.0003.

When x= 55,from backward interpolation

$(dy / dx)\ x \neq x_n = (1 / h)\ [\nabla y_n + (2v+1)/2\ \ \nabla^2 y_n + (3v^2 + 6v+2)/ 6\ \ \nabla^3 y_n + \ldots\ldots\ldots]$

= (1/1) [0.0229+(-1/2)(-0.0003)+0] = 0.02305,

Since here $v= (x-x_n) / h = (55-56)/1 = -1$.

$(d^2y / dx^2)\ x \neq x_n = (1/h^2)\ [\nabla^2 y_n + 6(v+1) / 6\ \nabla^3 y_n + (12v^2 + 36 v + 22) / 2\ \nabla^4 y_n + \ldots\ldots]$

= (1/1) [0.0229+(-1/2)(-0.0003)+0] = 0.02305.

Numerical Integration:

We know that $\int_a^b f(x)dx$ represents the area between y = f(x), x – axis and the ordinates x = a and x = b. This integration is possible only if the f(x) is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows: Given as set of (n+1) paired values (x_i, y_i), i = 0,1,2,…,n of the function y=f(x), where f(x) is not known explicitly, it is required to compute $\int_{x_0}^{x_n} y\, dx$.

A general quadrature formula for equidistant ordinates (or Newton – cote's formula)

For equally spaced intervals, we have Newton's forward difference formula as

$$y(x)=y(x_0+uh)=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\ldots \qquad \ldots\ldots(1)$$

Now, instead of $f(x)$, we will replace it by this interpolating formula of Newton.

Here, $u = \frac{x-x_0}{h}$ where h is interval of differencing.

Since $x_n = x_0 + nh$, and $u = \frac{x-x_0}{h}$ we have $\frac{x-x_0}{h} = n = u$.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$$= \int_{x_0}^{x_n+nh} P_n(x)\, dx \text{ where } P_n(x) \text{ is interpolating polynomial}$$

$$=\int_0^n \left(y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right)(hdu)$$

Since $dx = hdu$, and when $x = x_0$, $u = 0$ and when $x = x_0+nh$, $u = n$.

$$= h\left[y_0(u) + \frac{u^2}{2}\Delta y_0 + \frac{\left(\frac{u^3}{3} - \frac{u^2}{2}\right)}{2}\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$\int_{x_0}^{x_n} f(x)dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right] \ldots(2)$$

The equation (2), called Newton-cote's quadrature formula is a general quadrature formula. Giving various values for n, we get a number of special formula.

Trapezoidal rule:

By putting n = 1, in the quadrature formula (i.e there are only two paired values and interpolating polynomial is linear).

$\int_{x_0}^{x_n+nh} f(x)dx = h\left[1.y_0 + \frac{1}{2}\Delta y_0\right]$ since other differences do not exist if n = 1.

$$= \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_n+nh} f(x)dx$$

$$= \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_n+2h} f(x)dx + \ldots\ldots\ldots +$$

$$\int_{x_0+(n-1)h}^{x_n+nh} f(x)dx$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots . + y_{n-1})\right]$$

$= \frac{h}{2}$ [(sum of the first and the last ordinates) + 2(sum of the remaining ordinates)]

This is known as Trapezoidal Rule and the error in the trapezoidal rule is of the order h^2.

Romberg's method

For an interval of size h, let the error in the trapezoidal rule be kh^2 where k is a constant. Suppose we evaluate $I = \int_{x_0}^{x_n} y\, dx$, taking two different values of h, say h_1 and h_2, then

$$I = I_1 + E_1 = I_1 + kh_1^2 \quad I = I_2 + E_2 = I_2 + kh_2^2$$

Where I_1, I_2 are the values of I got by two different values of h, by trapezoidal rule and E_1, E_2 are the corresponding errors.

$$I_1 + kh_1^2 = I_2 + kh_2^2$$

$$k = \frac{I_1 - I_2}{h_2^2 - h_1^2}$$

substituting in (1), $I = I_1 + \frac{I_1 - I_2}{h_2^2 - h_1^2} h_1^2$ & $I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$

This I is a better result than either I_1, I_2.

If $h_1 = h$ and $h_2 = \frac{1}{2}h$, then we get

$$I = \frac{I_1\left(\frac{1}{4}h^2\right) - I_2 h^2}{\frac{1}{4}h^2 - h^2} \qquad = \frac{4I_2 - I_1}{3} = I_2 + \frac{1}{2}(I_2 - I_1), \quad I = I_2 + \frac{1}{2}(I_2 - I_1)$$

We got this result by applying trapezoidal rule twice. By applying the trapezoidal rule many times, every time halving h, we get a sequence of results $A_1, A_2, A_3, \ldots\ldots$ we apply the formula given by (3), to each of adjacent pairs and get the resultants $B_1, B_2, B_3 \ldots..$ (which are improved values). Again applying the formula given by (3), to each of pairs $B_1, B_2, B_3 \ldots..$we get another sequence of better results $C_1, C_2, C_3 \ldots.$continuing in this way, we proceed until we get two successive values which are very close to each other. This systematic improvement of Richardson's method is called Romberg method or Romberg integration.

Simpson's one-third rule:

Setting n = 2 in Newton- cote's quadrature formula, we have $\int_{x_0}^{x_n} f(x)dx = h \left[2y_0 + \frac{4}{2}\Delta y_0 + \left(\frac{1}{2}\right)\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0 \right]$ (since other terms vanish)

$$= \frac{h}{3}(y_2 + y_1 + y_0)$$

Similarly, $\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$

If n is an even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all the integrals, if n is an even positive integer, that is, the number of ordinates y_0, y_1, y_2....y_n is odd, we have

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots\ldots.+ \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[(y_0+y_n) + 2(y_2+ y_4+\ldots.) + \ldots\ldots. + 4(y_1+ y_3+\ldots..)\right]$$

$= \frac{h}{3}$ [(sum of the first and the last ordinates) + 2(sum of remaining odd ordinates) + 2(sum of even ordinates)]

Simpson's three-eighths rule:

Putting n = 3 in Newton – cotes formula

$$= \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+ y_2+y_4+y_5+\ldots.+y_{n-1})+2(y_3+ y_6+y_9+\ldots.+y_n)\right] \ldots.(2)$$

Equation (2) is called *Simpson's three – eighths rule* which is applicable only when n is a multiple of 3.Truncation error in simpson's rule is of the order h

Example

Evaluate $\int_{-3}^{3} x^4 dx$ by using (1) trapezoidal rule (2)simpson's rule. Verify your results by actual integration.

Solution.

Here $y(x) = x^4$. Interval length$(b – a)$ = 6. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

We form below the table

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

(i) By trapezoidal rule:

$\int_{-3}^{3} y\,dx = \frac{h}{2}$ [(sum of the first and the last ordinates) +

2(sum of the remaining ordinates)]

$=\frac{1}{2}$ [(81+81)+2(16+1+0+1+16)] =115

(ii) **By simpson's one - third rule** (since number of ordinates is odd):

$$\int_{-3}^{3} y\,dx = \frac{1}{3}[(81+81) + 2(1+1) + 4(16+0+16)] \qquad = 98.$$

(iii) Since $n = 6$, (multiple of three), we can also use **simpson's three - eighths rule.** By this rule,

$$\int_{-3}^{3} y\,dx = \frac{1}{3}[(81+81) + 3(16+1+1+16) + 2(0)] \qquad = 99$$

(iv) **By actual integration,**

$$\int_{-3}^{3} x^4 dx = 2*\left[\frac{x^5}{5}\right]_0^3 = \frac{2*243}{5} = 97.2$$

From the results obtained by various methods, we see that simpson's rule gives better result than trapezoidal rule

Part B (5x6=30 Marks)

Possible Questions

1. Find the positive root of f(x) = x^3- x -1= 0 by Newton –Raphson method correct to 5 decimal places.
2. By dividing the range into 10 equal parts evaluate $\int_0^{\pi} sinxdx$ by Trapezoidal &Simpson's rule. Verify your answer with integration.
3. Find a first two derivative of $x^{1/3}$ at x =50 &x =56 given the table below.

X	50	51	52	53	54	55	56
Y =$x^{1/3}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

4. Using Lin Bairstow's method, obtain the quadratic factors of the polynomial given by f(x)=x^3-2x^2+x-2.
5. Write Down the Derivative ofNewton's forward difference.
6. Find the positive root of f(x) =2x^3-3x-6 =0 by Newton –Raphson method correct to five decimal places.
7. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule.

Part C (1x10=10 Marks)

Possible Questions

1. Derive Newton –Raphson method.
2. Use Romberg's method to compute I = $\int_0^1 \frac{dx}{1+x}$ correct to 3 decimal places.
3. Find the values of y at x =21 and x =28 from the following data

X	20	23	26	29
Y	0.3420	0.3907	0.4384	0.4848

Evaluate $\int_{-3}^3 x^4\, dx$ using Simpson's rule.

4. Derive the formula for trapezoidal rule.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Numerical Analysis **Subject Code: 17MMP103**

Class : I - M.Sc. Mathematics **Semester : I**

Unit II

Solutions of system of Equations

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
What are the types of solving linear system of equations-----------?	Direct and Iterative	differentiation	integration	interpolation	Direct and Iterative
Gauss elimination method is a -----------------.	Direct method	InDirect method	Iterative method	convergent	Direct method
The rate of convergence in Gauss – Seidel method is roughly --------- times than that of Gauss Jacobi method.	2	3	4	0	2
Example for iterative method -------------.	Gauss elimination	Gauss Siedal	Gauss Jordan	none	Gauss Siedal
In the absence of any better estimates, the initial approximations are taken as---	x = 0, y = 0, z = 0	x = 1, y = 1, z = 1	x = 2, y = 2, z = 2	x = 3, y = 3, z = 3	x = 0, y = 0, z = 0
When Gauss Jordan method is used to solve AX = B, A is transformed into ---	Scalar matrix	diagonal matrix	Upper triangular matrix	lower triangularmatrix	diagonal matrix
The modification of Gauss – Elimination method is called --------------------	Gauss Jordan	Gauss Siedal	Gauss Jacobbi	Gauss Elimination	Gauss Jordan
------------ Method produces the exact solution after a finite number of steps.	Gauss Siedal	Gauss Jacobbi	Iterative method	Direct	Direct
In the upper triangular coefficient matrix, all the elements above the diagonal are --------------.	Zero	non-zero	unity	negative	non-zero

In the upper triangular coefficient matrix, all the elements below the diagonal are --------------.	Positive	nonzero	zero	negative	zero
Gauss Seidal method always --------- for a special type of systems.	Converges	diverges	oscillates	equal	Converges
Condition for convergence of Gauss Seidal method is ---------------.	Coefficient matrix is diagonally dominant	pivot element is Zero	Coefficient matrix is not diagonally dominant	pivot element is non Zero	Coefficient matrix is diagonally dominant
Modified form of Gauss Jacobi method is -------------------- method.	Gauss Jordan	Gauss Siedal	Gauss Jacobbi	Gauss Elimination	Gauss Siedal
In Gauss elimination method by means of elementary row operations, from which the unknowns are found by ----------------- method	Forward substitution	Backward substitution	random	Gauss Elimination	Backward substitution
In iterative methods, the solution to a system of linear equations will exist if the absolute value of the largest coefficient is ------------- the sum of the absolute values of all remaining coefficients in each equation.	less than	greater than or equal to	equal to	not equal	greater than or equal to
In ------------- iterative method, the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration.	Gauss Siedal	Gauss Jacobi	Gauss Jordan	Gauss Elimination	Gauss Siedal
The direct method fails if any one of the pivot elements become ----.	Zero	one	two	negative	Zero
In Gauss elimination method the given matrix is transformed into --------------.	Unit matrix	diagonal matrix	Upper triangular matrix	lower triangular matrix	Upper triangular matrix
If the coefficient matrix is not diagonally dominant, then by ----------- that diagonally dominant coefficient matrix is formed.	Interchanging rows	Interchanging Columns	adding zeros	Interchangingrow and Columns	Interchangingrow and Columns
Gauss Jordan method is a -----------------.	Direct method	InDirect method	iterative method	convergent	Direct method

Gauss Jacobi method is a -----------------.	Direct method	InDirect method	iterative method	convergent	InDirect method
The modification of Gauss – Jordan method is called ---------------------.	Gauss Jordan	Gauss Siedal	Gauss Jacobbi	gauss elemination	Gauss Siedal
Gauss Seidal method always converges for ----------- of systems	Only the special type	all types	quadratic types	first type	Only the special type
In solving the system of linear equations, the system can be written as ---	BX = B	AX = A	AX = B	AB = X	AX = B
In solving the system of linear equations, the augment matrix is --------	(A, A)	(B, B)	(A, X)	(A, B)	(A, B)
In the direct methods of solving a system of linear equations, at first the given system is written as ------------- form.	An augment matrix	a triangular matrix	constant matrix	Coefficient matrix	An augment matrix
All the row operations in the direct methods can be carried out on the basis of --	all elements	pivot element	negative element	positiveelement	pivot element
The direct method fails if -----------.	1st row elements 0	1st column elements 0	Either 1st or 2nd	2 nd row is dominant	Either 1st or 2nd
The elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal is called ------------	Gauss elimination	Gauss jordan	Gauss jacobi	Gauss siedal	Gauss jordan
In Gauss Jordan method, we get the solution ---------------------	without using back substitution method	By using back substitution method	by using forward substitution method	Without using forward substitution method	By using back substitution method
If the coefficient matrix is diagonally dominant, then ---------- method converges quickly.	Gauss elimination	Gauss jordan	Direct	Gauss siedal	Gauss siedal
Which is the condition to apply Jocobi's method to solve a system of equations	1st row is dominant	1st column is dominant	diagonally dominant	2 nd row is dominant	diagonally dominant
Iterative method is a ------------ method	Direct method	InDirect method	Interpolation	extrapolation	InDirect method

As soon as a new value for a variable is found by iteration it is used immediately in the equations is called --------------.	Iteration method	Direct method	Interpolation	extrapolation	Iteration method
-------------- is also a self-correction method.	Iteration method	Direct method	Interpolation	extrapolation	Iteration method
The condition for convergence of Gauss Seidal method is that the ------ should be diagonally dominant	Constant matrix	unknown matrix	Coefficient matrix	extrapolation	Coefficient matrix
In ----------- method, the coefficient matrix is transformed into diagonal matrix	Gauss elimination	Gauss jordan	Gauss jacobi	Gauss seidal	Gauss jordan
We get the approximate solution from the ---------------.	Direct method	InDirect method	fast method	Bisection	InDirect method
-------------- Method takes less time to solve a system of equations	Direct method	InDirect method	fast method	Bisection	Direct method
The iterative process continues till ------------- is secured.	convergency	divergency	oscillation	point	convergency
In Gauss elimination method, the solution is getting by means of ----------- from which the unknowns are found by back substitution.	Elementary operations	Elementary column operations	Elementary diagonal operations	Elementary row operations	Elementary row operations
The method of iteration is applicable only if all equation must contain one coefficient of different unknowns as ---------- than other coefficients.	smaller	larger	equal	non zero	larger
The -------------- is reduced to an upper triangular matrix or a diagonal matrix in direct methods.	Coefficient matrix	Constant matrix	unknown matrix	Augment matrix	Augment matrix
The augment matrix is the combination of -----------------.	Coefficient matrix and constant matrix	Unknown matrix and constant matrix	Coefficient matrix and Unknown matrix	Coefficient matrix, constant matrix and Unknown matrix	Coefficient matrix and constant matrix
The given system of equations can be taken as in the form of -----------	A = B	BX= A	AX= B	AB = X	AX= B

The sufficient condition of iterative methods will be satisfied if the large coefficients are along the ---------------- of the coefficient matrix.	Rows	Coloumns	Leading Diagonal	elements	Leading Diagonal
Which is the condition to apply Gauss Seidal method to solve a system of equations.	1st row is dominant	1st column is dominant	diagonally dominant	Leading Diagonal	diagonally dominant
In the absence of any better estimates, the ------------- of the function are taken as x = 0, y = 0, z = 0.	initialapproximations	roots	points	final value	initialapproximations
The solution of simultaneous linear algebraic equations are found by using-	Direct method	InDirect method	fast method	Bisection	InDirect method

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
Department of Mathematics

Subject: Numerical Analysis	**Subject Code: 17MMP103**	**L T P C**
Class : I – M.Sc. Mathematics	**Semester : I**	**4 0 0 4**

UNIT-II

Solutions of system of Equations: The Elimination method: Gauss Elimination and Gauss Jordan Methods – LU decomposition method.

Methods of Iteration: Gauss Jacobi and Gauss Seidal iteration-Relaxation method.

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

SOLUTIONS OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

INTRODUCTION

We will study here a few methods below deals with the solution of simultaneous Linear Algebraic Equations

GAUSS ELIMINATION METHOD (DIRECT METHOD).

This is a direct method based on the elimination of the unknowns by combining equations such that the *n* unknowns are reduced to an equation upper triangular system which could be solved by back substitution.

Consider the *n* linear equations in *n* unknowns, viz.

$$a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=b_n \quad \dots(1)$$

Where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to $AX=B$ .....(2)

Where $A = \begin{pmatrix} a_{11} & a_{12} \dots\dots\dots a_{1n} \\ a_{21} & a_{22} \dots\dots\dots a_{2n} \\ \dots\dots\dots & \dots\dots\dots \dots\dots\dots \\ a_{n1} & a_{n2} \dots\dots a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix (A,B) to upper triangular matrix.

$$(A,B) = \left(\begin{array}{cc|c} a_{11} & a_{12} \dots\dots\dots a_{1n} & b_1 \\ a_{21} & a_{22} \dots\dots\dots a_{2n} & b_2 \\ \dots\dots\dots & \dots\dots\dots \dots\dots\dots & \vdots \\ a_{n1} & a_{n2} \dots\dots\dots a_{nn} & b_n \end{array}\right) \quad (3)$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\frac{a_{i1}}{a_{11}}$ and add to the ith row of (A,B), where i=2,3,…,n. By thia, all elements in the first column of (A,B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \cdots & \cdots & \cdots & & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right) \quad \ldots\ldots(4)$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\frac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row (i=3,4,…,n). Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} & c_2 \\ 0 & 0 & c_{23} & \cdots & c_{3n} & d_3 \\ \cdots & \cdots & \cdots & & \cdots & \vdots \\ 0 & 0 & c_{n3} & \cdots & c_{nn} & d_n \end{array}\right) \ldots (5)$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A,B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \cdots & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdots & & b_{2n} & c_2 \\ 0 & 0 & c_{23} & c_{34} & \cdots & c_{3n} & d_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{nn} & d_n \end{array}\right) \ldots(6)$$

From, (6) the given system of linear equations is equivalent to

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots.+a_{1n}x_n&=b_1\\ b_{22}x_2+b_{23}x_3+\ldots.+b_{2n}x_n&=c_2\\ c_{33}x_3+\ldots.+c_{3n}x_n&=d_3\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{nn}x_n&=k_n \end{aligned}\right\}$$

Going from the bottom of these equation, we solve for $x_n=\frac{k_n}{a_{nn}}$. Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method for we solve x_n, x_{n-1}, x_{n-2}, ……x_2, x_1.

GAUSS – JORDAN ELIMINATION METHOD (DIRECT METHOD)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX=B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making the matrix A not above the leading diagonal of A also as zeros. By this way, the system $AX=B$ will reduce to the form.

$$\left(\begin{array}{ccccccc|c} a_{11} & 0 & 0 & 0 & \ldots\ldots\ldots\ldots & a_{1n} & & b_1\\ 0 & b_{22} & 0 & 0 & \ldots\ldots\ldots\ldots & b_{2n} & & c_2\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & & d_3\\ 0 & 0 & 0 & 0 & \ldots\ldots\ldots\ldots & a_{nn} & & k_n \end{array}\right) \quad \ldots(7)$$

From (7)

$$x_n=\frac{k_n}{a_{nn}},\ \ldots\ldots,\ x_2=\frac{c_2}{b_{22}},\ x_n=\frac{b_1}{a_{11}}$$

Note: By this method, the values of $x_1, x_2, \ldots.x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss – Jordan method.*

$x+2y+z=3,\ \ 2x+3y+3z=10,\ \ 3x-y+2z=13.$

Solution. (By Gauss method)

This given system is equivalent to

$$\begin{pmatrix}1 & 2 & 1\\2 & 3 & 3\\3 & -1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\10\\13\end{pmatrix}$$

$$AX = B$$

$$(A,B) = \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\2 & 3 & 3 & 10\\3 & -1 & 2 & 13\end{array}\right] \ldots\ldots\ldots\ldots\ldots (1)$$

Now, we will make the matrix A upper triangluar.

$$(A,B) = \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\2 & 3 & 3 & 10\\3 & -1 & 2 & 13\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\0 & -1 & 1 & 4\\0 & -7 & -1 & 4\end{array}\right] \quad R_2+(-2)R_1\,,\ R_3+(-3)R_1$$

Now, take b_{22}=-1 as the pivot and make b_{32} as zero.

$$(A,B) \sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\0 & -1 & 1 & 4\\0 & 0 & -8 & -24\end{array}\right] R_{32}(-7) \ldots\ldots\ldots\ldots (2)$$

From this, we get

$$x+2y+z = 3\,, \qquad -y+z=4 \qquad , \ -8z=-24$$

$\therefore$ $z=3,\ y=-1,\ x=2$ by back substitution.

$x=2,\ y=-1,\ z=3$

Solution. (Gauss – Jordan method)

In stage 2, make the element, in the position (1,2), also zero.

$$(A,B) \sim \left[\begin{array}{ccc|c}1 & 2 & 1 & 3\\0 & -1 & 1 & 4\\0 & 0 & -8 & -24\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1 & 0 & 3 & 11\\0 & -1 & 1 & 4\\0 & 0 & -8 & -24\end{array}\right] R_{12}(2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_{13}(3),\ R_{23}(1)$$

i.e., $x = 2,\ y = -1,\ \ z = 3$

METHOD OF TRIANGULARIZATION (OR METHOD OF FACTORIZATION) (*DIRECT METHOD*)

This method is also called as *decomposition* method. In this method, the coefficient matrix A of the system $AX = B$, decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. we will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system is equivalent to $AX = B$

Where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

And an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \text{ so that}$$

$LUX = B$ Let $UX = Y$ And hence $LY = B$

That is, $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\therefore$ $y_1 = b,\ l_{21}y_1+y_2 = b_2,\ l_{31}y_1+l_{32}y_2+y_3 = b_3$

By forward substitution, y_1, y_2, y_3 can be found out if L is known.

From (4), $\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$, $u_{22}x_2 + u_{23}x_3 = y_2$ and $u_{33}x_3 = y_3$

From these, x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.Now L and U can be found from $LU = A$

i.e., $\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

i.e.,

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3 l's and 6 u's.

That is, L and U re known. Hence X is found out. Going into details, we get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first rows of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$ $\quad l_{21}u_{12}+u_{22} = a_{22}$ $\quad l_{21}u_{13}+u_{23} = a_{23}$

$$l_{21} = \frac{a_{21}}{a_{11}},\quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.\, a_{12} \quad \text{and} \quad u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.\, a_{13}$$

again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12}+l_{32}u_{22} = a_{32}$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = a_{33}$

solving, $l_{31} = \dfrac{a_{31}}{a_{11}}$, $l_{32} = \dfrac{a_{32} - \frac{a_{21}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}}$

$$u_{33} = \left[a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{13} - \right] \left[\frac{a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}}\right] a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{13}$$

Therefore L and U are known.

Example 2 *By the method of triangularization, solve the following system.*

$5x - 2y + z = 4,\quad 7x + y - 5z = 8,\quad 3x + 7y + 4z = 10.$

Solution. The system is equivalent to

$$\begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$$A \quad X = B$$

Now, let $\quad LU = A$

That is, $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{pmatrix}$$

Multiplying and equating coefficients,

$$u_{11} = 5, \quad u_{12} = -2, \quad u_{13} = 1$$

$$l_{21}u_{11} = 7 \quad l_{21}u_{12} + u_{22} = 1 \quad l_{21}u_{13} + u_{23} = -5$$

$$l_{21} = \frac{7}{5}, \; u_{22} = 1 - \frac{7}{5}.(-2) = \frac{19}{5} \text{ and}$$

$$u_{23} = -5 - \frac{7}{5}.(1) = -\frac{32}{5}$$

Again equating elements in the third row,

$l_{31}u_{11} = 3, \; l_{31}u_{12} + l_{32}u_{22} = 7$ and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$

$$\therefore \qquad l_{31} = \frac{3}{5}, \quad l_{32} = \frac{7 - \frac{3}{5}\cdot(-2)}{\frac{19}{5}} = \frac{41}{19}$$

$$u_{33} = 4 - \frac{3}{5}\cdot(1) - \frac{41}{19}\left(-\frac{32}{5}\right) = 4 - \frac{3}{5} + \frac{1312}{95}$$

$$= \frac{1635}{95} = \frac{327}{19}$$

Now L and U are known.Since $LUX = B,$ $LY = B$ where $UX = Y.$

From $LY = B,$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix}$$

$y_1 = 4,\ \frac{7}{5}y_1 + y_2 = 8,\ \frac{3}{5}y_1 + \frac{41}{19}y_2 + y_3 = 10$

$$y_2 = 8 - \frac{28}{5} = \frac{12}{5}$$

$$y_3 = 10 - \frac{12}{5} - \frac{41}{19} \times \frac{12}{5} = 10 - \frac{12}{5} - \frac{492}{95} = \frac{46}{19}$$

$UX = Y$ gives
$$\begin{pmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{pmatrix}$$

$$5x - 2y + z = 4$$

$$\frac{19}{5}y - \frac{32}{5}z = \frac{12}{5}$$

$$\frac{327}{19} z = \frac{46}{19}$$

$$z = \frac{46}{327}$$

$$\frac{19}{5} y = \frac{12}{5} + \frac{32}{5}\left(\frac{46}{327}\right)$$

$$y = \frac{284}{327}$$

$$5x = 4+2y - z = 4 + 2\left(\frac{568}{327}\right) - \frac{46}{327}$$

$$\therefore \quad x = \frac{366}{327}$$

$$\therefore \quad x = \frac{366}{327}, \; y = \frac{284}{327}, \; z = \frac{46}{327}$$

ITERATIVE METHODS

This iterative methods is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. When this condition is satisfied, the system will be solvable by the iterative method. The system,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

will be solvable by this method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX=B$ are greater than the sum of absolute values of the other coefficients of that row. The condition is *sufficient* but not *necessary*.

JACOBI METHOD OF ITERATION OR GAUSS – JACOBI METHOD

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$a_1x+b_1y+c_1z = d_1$$

$$a_2x+b_2y+c_2z = d_2$$

$$a_3x+b_3y+c_3z = d_3 \quad \ldots\ldots\ldots\ldots (1)$$

Let us assume $|a_1| > |b_1|+|c_1|$

$$|b_2| > |a_2|+|c_2|$$

$$|c_3| > |a_3|+|b_3|$$

Then, iterative method can be used for the system (1). Solve for x, y, z (whose coefficients are the larger values) in terms of the other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \quad \ldots\ldots\ldots\ldots (2)$$

If x^0, y^0, z^0 are the initial values of x, y, z respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \quad \ldots\ldots\ldots (3)$$

Again using these values $x^{(2)}, y^{(2)}, z^{(2)}$ in (2), we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)}) \quad \ldots.(4)$$

Proceeding in the same way, if the rth iterates are $x^{(r)}$, $y^{(r)}$, $z^{(r)}$, the iteration scheme reduces to

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r)} - b_3 y^{(r)}) \quad .....(5)$$

The procedure is continued till the convergence is assured (correct to required decimals).

GAUSS – SEIDEL METHOD OF ITERATION:

This is only a refinement of Guass – Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values y^0, z^0 for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of x^0 as in Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ *and* $y^{(1)}$, use $x^{(1)}$ for x *and* $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss – seidel method is very fast when compared to Gauss – Jacobi method. The rate of convergence in Gauss – Seidel method is roughly two times than that of Gauss – Jacobi method. As we saw the sufficient condition already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest* coefficient

is greater than the sum of the absolute values of all the remaining coefficients. (The largest coefficients must be the coefficients for different unknowns).

Example 3 Solve the following system by Gauss – Jacobi and Gauss – Seidel methods:

10x-5y-2z = 3 ; 4x-10y+3z = -3 ; x+6y+10z = -3.

Solution: Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$ is diagonally dominant, since $|10| > |-5| + |-2|$.

$$|-10| > |4| + |3|.$$

$$|10| > |1| + |6|$$

Gauss – Jacobi method, solving for x, y, z we have

$$x = \frac{1}{10}(3+5y+2z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$y = \frac{1}{10}(3+4x+3z)$ ………………. (2)

$z = \frac{1}{10}(-3-x-6y)$ ………………. (3)

First iteration: Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}(3+5(0)+2(0)) = 0.3$$

$$y^{(1)} = \frac{1}{10}(3+4(0)+3(0)) = 0.3$$

$$z^{(1)} = \frac{1}{10}(-3-(0)-6(0)) = -0.3$$

Second iteration: using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}(3+5(0.3)+2(-0.3)) = 0.39$$

$$y^{(2)} = \frac{1}{10}(3+4(0.3)+3(-0.3)) = 0.33$$

$$z^{(2)} = \frac{1}{10}(-3-(0.3)-6(0.3)) = -0.51$$

Third iteration: using these values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1), (2), (3), we get,

$$x^{(3)} = \frac{1}{10}(3+5(0.33)+2(-0.51)) = 0.363$$

$$y^{(3)} = \frac{1}{10}(3+4(0.39)+3(-0.51)) = 0.303$$

$$z^{(3)} = \frac{1}{10}(-3-(0.39)-6(0.33)) = -0.537$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10}(3+5(0.303)+2(-0.537)) = 0.3441$$

$$y^{(4)} = \frac{1}{10}(3+4(0.363)+3(-0.537)) = 0.2841$$

$$z^{(4)} = \frac{1}{10}(-3-(0.363)-6(0.303)) = -0.5181$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10}(3 + 5(0.2841) + 2(-0.5181)) = 0.33843$$

$$y^{(5)} = \frac{1}{10}(3 + 4(0.3441) + 3(-0.5181)) = 0.2822$$

$$z^{(5)} = \frac{1}{10}(-3 - (0.3441) - 6(0.2841)) = -0.50487$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10}(3 + 5(0.2822) + 2(-0.50487)) = 0.340126$$

$$y^{(6)} = \frac{1}{10}(3 + 4(0.33843) + 3(-0.50487)) = 0.283911$$

$$z^{(6)} = \frac{1}{10}(-3 - (0.33843) - 6(0.2822)) = -0.503163$$

Seventh iteration:

Iteration	Gauss - jacobi method			Gauss – seidel method		
	x	y	z	x	y	z
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-0.51	0.3936	0.2828	-0.5090
3	0.363	0.303	-0.537	0.3396	0.2831	-0.5038
4	0.3441	0.2841	-0.5181	0.3407	0.2851	-0.5051
5	0.3384	0.2822	-0.5048	0.3415	0.2850	-0.5051
6	0.3401	0.2839	-0.5031	0.3414	0.2850	-0.5051
7	0.3413	0.2851	-0.5043	0.3414	0.2850	-0.5051

8	0.3416	0.2852	-0.5051			
9	0.3411	0.2851	-0.5053			

$$x^{(7)} = \frac{1}{10}(3 + 5(0.285039017) + 2(-0.5051728))$$

$$= 0.3414849$$

$$y^{(7)} = \frac{1}{10}(3 + 4(0.3414849) + 3(-0.5051728))$$

$$= 0.28504212$$

$$z^{(7)} = \frac{1}{10}(-3 - (0.3414849) - 6(0.28504212))$$

$$= -0.5051737$$

The values at each iteration by both methods are tabulated below:

The values correct to 3 decimal places are

$x = 0.342,\ y = 0.285,\ z = -0.505$

Relaxation Method

Consider the system of equations,

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} (1)$$

we define the residuals r_1, r_2, r_3 by the relations

$$\left.\begin{aligned} r_1 &= a_1x + b_1y + c_1z - d_1 \\ r_2 &= a_2x + b_2y + c_2z - d_2 \\ r_3 &= a_3x + b_3y + c_3z - d_3 \end{aligned}\right\} (2)$$

if we can find the values of *x, y, z* so that $r_1 = 0 = r_2 = r_3$ then those values of *x, y, z* are the exact values of the system. If it is not possible to make $r_1 = 0 = r_2 = r_3$, then we make simultaneously the values to r_1, r_2, r_3 to as close to zero as possible. In other words we "liquidate" the residuals r_1, r_2, r_3 by taking better approximate values of *x, y, z* what will be the slight change is made in the values of *x, y, z* what will be the corresponding changes in the residuals, r_1, r_2, r_3? We give below an 'operation table' from which we can easily know the corresponding changes in r_1, r_2, r_3 for a change of 1 unit in *x*, while there is no change in re is no change in *y* and *z*, for a change of 1 unit in *y* while there in no change in *x* and *z* for a change of 1 unit in *z* while there is no change in *y* and *x*.

Operation Table

Operation	*Change in (or increment in)*					
	X	*y*	*z*	r_1	r_2	r_3
R_1	1	0	0	a_1	a_2	a_3
R_2	0	1	0	b_1	b_2	b_3
R_3	0	0	1	c_1	b_3	c_3

What is the meaning of the above table ?

The operator R_1 increase the value of *x* by 1, *y* by zero, *z* by zero

(no change in *y* and *z*) and this operation increases the residuals r_1 by a_1, r_2 by a_2, and r_3 by a_3 (the increase in r_1, r_2, r_3 are the nothing but the coefficients of *x* in the equations given). Similarly R_3 increases the value of *z* by 1 (while *x, y* are kept constant) and the effect of this operation increases the values of r_1, r_2, r_3 by c_1, c_2, c_3 respectively.

One can easily see that the operation table consists of the unit matrix *I* and the transpose of the matrix *A* and *A'*, where *A* is the coefficient matrix of the system of equations.

Convergence of the relaxation method:

If the method should converge, the diagonal elements of the coefficient matrix A should be dominant; that is, A is diagonally dominant. Referring to the system of equations given above; the system can be solved by this method successfully only if

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Where at least once the strict inequality holds.

Example 1. *Solve the following equations using relaxation method*

$$10x - 2y - 2z = 6$$

$$-x + 10y - 2z = 7$$

$$-x - y + 10z = 8$$

Solution: Since the diagonal elements are dominant, we will do by relaxation method.

The residuals r_1, r_2, r_3 are given by

$$r_1 = 10x - 2y - 2z - 6$$

$$r_2 = -x + 10y - 2z - 7$$

$$r_3 = -x - y + 10z - 8$$

Operation Table (write 1,A')

Changes in

	x	y	z	r_1	r_2	r_3
R_1	1	0	0	10	-1	-1
R_2	0	1	0	-2	10	-1
R_3	0	0	1	-2	-2	10

We will take the initial values of x, y, z as 0, 0, 0.

Setting $x=0=y=z$, we get $r_1 = -6$, $r_2 = -7$, $r_3 = -8$

We write these residuals below and *relax* these values making changes in x, y, z as shown below:

	x	y	z	r_1	r_2	r_3
	0	0	0	-6	-7	-8
$R_1 \longrightarrow$	1	0	0	-8	-9	2
$R_2 \longrightarrow$	0	1	0	-10	1	1
$R_3 \longrightarrow$	0	0	1	0	0	0
	1	1	1	0	0	0

Analysis: In line (1), for $x=0, y= 0, z = 0$ the residuals are -6,-7,-8. The numerically largest residual is -8 which is encircled.

First, we liquidate the numerically largest residual r_3 = -8 by a proper multiple of R_3. Since R_3 operation increases r_3 by 10, by operation 1.R_3, we get (i.e., put $x=0, y=0, z=1$) r_1 = -6+(-2) = -8; r_2 = -7+(-2) = -9; r_3 = -8+10 = 2 giving line (2). Now, in line (2), numerically greatest residual is -9 which is encircled. We will liquidate this r_2 by proper multiple of R_2. An increase of 1 in y will increase r_2 by 10, r_1 by -2 and r_3 by -1. Hence doing the operation 1.R_2 new r_1=-8-2=-10,r_2=-9+10=1, r_3=2+(-1)=1 and we get the line (3). Now in line (3), r_1=-10 is the numerically greatest value. Now, we will liquidate this r_1=-10 by a proper multiple of R_1. Doing the operations R_1 (1, 0, 0), r_1=-10+10=0, r_2=1+(-1)=0, r_3=1+(-1)=0. Fortunately all the residuals have become zero after the 3 operations. Adding the values of x, y, z we get $x=1, y= 1, z = 1$ as the exact solution for the system.

Part B (5x6=30 Marks)

Possible Questions

1. Applying Gauss Jordan method to find the solution of the following system
   $10x+y+z = 12; 2x+10y+z = 13; x+y+5z = 7$
2. Solve the system of equation by Gauss Jacobi method.
   $5x-2y+z= -4$; $x+6y-2z= -1$; $3x+y+5z=13$
3. Solve the system of equation by Gauss Seidel method
   $10x-5y-2z =3$; $4x-10y+3z =-3$; $x+6y+10z =-3$
4. Solve the following system by Relaxation method.
   $10x-2y-2z =6$; $-x+10y+-2z =7$; $-x-y+10z =8$
5. By the Method of Triangularization solve the following system
   $5x-2y+z =4$; $7x+y-5z = 8$; $3x+7y+4z =10$
6. Solve the system of equations by Gauss Seidel method correct to 3 decimal places.
   $8x-3y+2z=20$; $4x+11y-z=33$; $6x+3y-12z=35$
7. Solve the system of equations by Gauss elimination method.
   $x+2y+z = 3$; $2x+3y+3z = 10; 3x-y+2z = 13$

Part C (1x10=10 Marks)

Possible Questions

1. By the Gauss Jordan method solve the following equations.
   $5x-2y+z=4$; $7x+y-5z=8$; $3x+7y+4z=10$
2. Explain the algorithm of LU decomposition method
3. Solve the following system of equations using Gauss Elimination method.
   $2x+y+z=10$; $3x+2y+3z=18$; $x+4y+9z=16$
4. Applying Gauss Jacobi method to find the solution of the following system
   $10x+2y+z=9$; $2x+20y-2z= - 44$; $-2x+3y+10z=22$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Numerical Analysis **Subject Code: 17MMP103**

Class : I - M.Sc. Mathematics **Semester : I**

Unit III

Solutions of Ordinary Differential Equations

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The Euler Method of second category are called ------------	Diagram	graph	line graph	Continuous line graph	Continuous line graph
In Euler's Method solution of the differential equation denoted by -----------	$X_{n+1} = X_n + h f(x_n, y_n)$ n= 0,1,2,3…	$Y_{n+1} = y_n - h f(x_n, y_n)$ n= 0,1,2,3…	$Y_{n+1} = y_n + f(x_n, y_n)$ n= 0,1,2,3…	$Y_{n+1} = y_n + h f(x_n, y_n)$ n= 0,1,2,3…	$Y_{n+1} = y_n + h f(x_n, y_n)$ n= 0,1,2,3…
Euler,s algorithm formula is ---------------------	points	slopes	slopes and points	chords	slopes
The error in Euler method is ----------------------------.	o(h2)	o(h4)	o(h3)	o(hn)	o(h2)
In Euler's Method averaged the ------------	points	slopes	slopes and points	chords	points
In Modified Euler's Method averaged the ------------	$Y_{n+1} = y_n + y_n'$	$Y_{n+1} = y_n + h y_n'$	$x_{n+1} = X_n + h X_n'$	$Y_{n+1} = y_n - h y_n'$	$Y_{n+1} = y_n + h y_n'$
The Euler Method Predicator is -------------	$Y_{n+1} = y_n + (y_n' + Y_{n+1}')$	$Y_{n+1} = y_n + y_n' + Y_{n+1})$	$Y_{n+1} = y_n + y_n' + Y_{n+1}')$	$Y_{n+1} = y_n - (y_n' + Y_{n+1}')$	$Y_{n+1} = y_n + (y_n' + Y_{n+1}')$
The Euler Method Corrector is -------------	convergent	slow convergent	divergent	fast convergent	slow convergent
The Euler Method and Modified Euler's Method are ------------	required	not required	may be required	must required	not required

In R - k method derivatives of higher order are ------------	constant	zero	variable	non-zero	constant
The n -divided difference of a polynomialof the n -th degree are	Trapezoidal rule	simpson s rule	3/8 simpson's rule	Booles rule	simpson s rule
Relation between Δ, ∇ and E	$K_1 = h f (x_n)$	$K_1 = h f (x_n , y_n)$	$K_1 = f (y_n)$	$K_1 = h f (y_n)$	$K_1 = h f (x_n , y_n)$
Given Initial value problem $y' = dy/dx\ f (x ,y)$ where $y (x_0) = y_0$, In Runge kutta	Trapezoidal rule	simpson s rule	3/8 simpson's rule	Booles rule	Booles rule
In Newton cote formula if f(x) is interpolate at equally spaced nodes by a polynomial of degree four then it represents	constant	variable	zero	negative	zero
n th difference of a polynomial of n th degree are constant and all higher order difference are	Independent	dependent	Inverse	not Independent	Independent
In divided difference the value of any difference is ----- of the order of their argument	Trapezoidal rule	simpson s rule	3/8 simpson's rule	Booles rule	3/8 simpson's rule
In Newton cote formula if f(x) is interpolate at equally spaced nodes by a polynomial of degree three then it represents	g has fixed point in [a, b]	g has not fixed point in [a, b]	g has fixed point in (a, b)	g has fixed point in (a, c)	g has fixed point in [a, b]
Which of the following relation is true ?	Trapezoidal rule	simpson s rule	3/8 simpson's rule	Booles rule	Trapezoidal rule
In Newton cote formula if f(x) is interpolate at equally spaced nodes by a polynomial of degree one then it represents	$E^{½} + E^{-½}$	$E^{½} - E^{-½}$	$E^{½} . E^{-½}$	$E^{½} / E^{-½}$	$E^{½} . E^{-½}$
central difference equivalent to shift operator is	$K_3 = h f (x_n + h , y_n + h)$	$K_3 = h f (x_n , y_n)$	$K_3 = h f (x_n + ½ h , y_n + ½ h)$	$K_3 = f (x_n , y_n)$	$K_3 = h f (x_n + ½ h , y_n + ½ h)$
In R-k methods, the derivatives of ----- are not require only the given function values at different	higher order	lower order	middle order	zero	higher order
A predictor formula is used to --------------------the values of y at xi+1.	correct	predict	increase	decrease	predict
If all the non zero terms involve only the dependent variable u and u' then differential equation is called	homogeneous	non homogeneous	linear	non linear	homogeneous
Sum of the eigen values of a matrix is equal to the----------- of the diagonal element of the matrix.	sum	product	divide	square	sum

Euler's method uses straight line segments to approximate the solution method is refered to as a------.	4th order	3rd order	2nd order	1st order	1st order
Adams Moulton method is --------------method	single step	multi step	direct	indirect	multi step
The Runge-Kutta method do not require prior calculation of the -----order derivatives.	middle	lower	higher	zero	higher
Taylor series and Euler methods are -----------------------then Runge- Kutta method.	fastly divergent	slowly divergent	fastly convergent	slowly convergent	slowly convergent
Which of these are multistep methods?	Milne's method	Runge-Kutta method	Euler	Modified euler	Milne's method
A predictor formula is used to ---------------------the values of y at xi+1.	correct	predict	increase	decrease	predict
If dy/dx is a function x alone, then fourth order Runge-Kutta method reduces to ------.	Trapezoidal rule	Taylor series	Euler method	Simpson method	Simpson method

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
Department of Mathematics

Subject: Numerical Analysis	**Subject Code: 17MMP103**	**L T P C**
Class : I – M.Sc. Mathematics	**Semester : I**	**4 0 0 4**

UNIT-III

Solutions of Ordinary Differential Equations: One step method: Euler and Modified Euler methods–Rungekutta methods. Multistep methods: Adams Moulton method – Milne's method

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11.32}$$

where $y(x_0) = y_0$

Suppose that we wish to find successively $y_1, y_2, \ldots, y_m$, where y_m is the value of y corresponding to $x = x_m$, where $x_m = x_0 + mh$, $m = 1, 2, \ldots$, h being small. Here, we use the property that in a small interval, a curve is nearly a straight line.

Thus, in the interval x_0 to x_1 of x, we approximate the curve by the tangent at the point (x_0, y_0)

Therefore, the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= f(x_0, y_0)(x - x_0) \quad \text{[from Eqn (11.32)]}$$

$$\text{or } y = y_0 + (x - x_0) f(x_0, y_0)$$

Hence, the value of y corresponding to $x = x_1$ is

$$y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$$

or

$$y_1 = y_0 + h f(x_0, y_c) \qquad (11.33)$$

Since the curve is approximated by the tangent in $[x_0, x_1]$, Eqn (11.33) gives the approximated value of y_1.

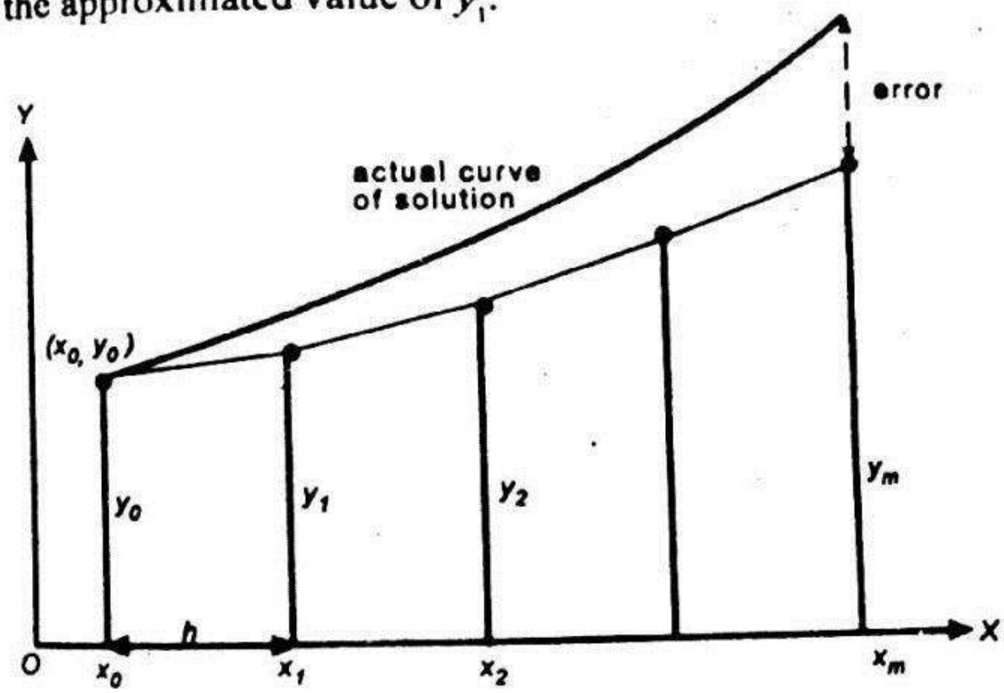

Fig 11.2

Similarly, approximating the curve in the next interval $[x_1, x_2]$ by a line through (x_1, y_1) with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1) \qquad (11.34)$$

Proceeding on, in general it can be shown that

$$y_{m+1} = y_m + hf(x_m, y_m) \qquad (11.35)$$

Remarks In Euler's method, the actual curve of a solution is approximated by a sequence of short lines as shown in Fig 11.2. It is possible that the sequence of lines may deviate from the curve of solution significantly. The process is very slow and to obtain it with reasonable accuracy using Euler's method, we have to take h very small. An improvement over this method is discussed in the following section.

11.11 IMPROVED EULER'S METHOD

Here, we consider a line passing through $A(x_0, y_0)$ whose slope is the average of the slopes at $A(x_0, y_0)$ and $P(x_1, y_1^{(1)})$ such that $y^1{}_1 = y_0 + hf(x_0, y_0)$.

In Fig. 11.3, let AL_1 be the tangent to the curve at $A(x_0, y_0)$ and PL_2 be the line through $P(x_1, y_1^{(1)})$ having the slope $f(x_1, y_1^{(1)})$. Now PM is the line having slope

$$\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

that is, average of two slopes $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$.

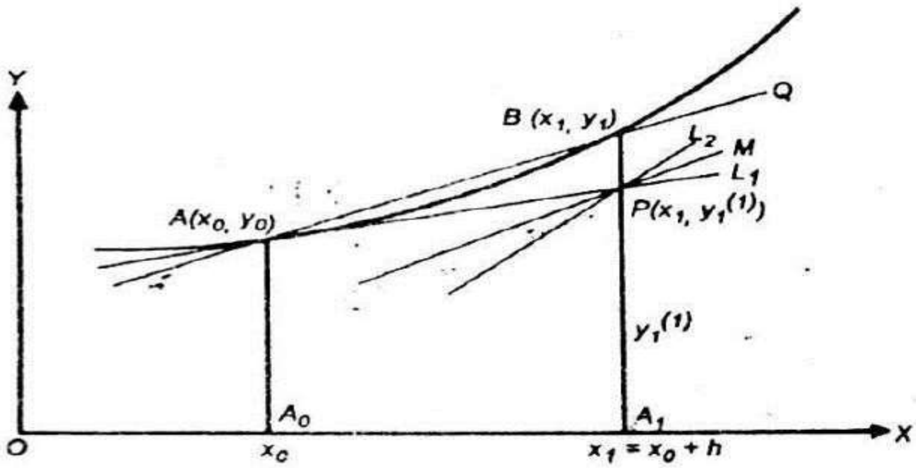

Fig 11.3

Line AQ through (x_0, y_0) and parallel to PM is used to approximate the curve. Then, ordinate of point E will give the value of y_1.

Therefore, equation to ABQ is

$$y - y_0 = (x - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\} \tag{11.36}$$

As we are assuming that $A_1B = y_1$, coordinates of B will be (x_1, y_1). This point will lie on AQ.

$$\therefore\ y_1 - y_0 = (x_1 - x_0)\frac{1}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

or $$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(1)})\}$$

$$= y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))\} \tag{11.37}$$

In general, we have the formula

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))\} \quad (11.38)$$

where $x_m - x_{m-1} = h$.

11.12 MODIFIED EULER'S METHOD

In this method the curve in the interval (x_0, x_1), where $x_1 = x_0 + h$, is approximated by the line through (x_0, y_0) with slope

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right) \quad (1)$$

that is, the slope at the middle point whose abscissa is the average of x_0 and x_1, i.e. $x_0 + \frac{h}{2}$.

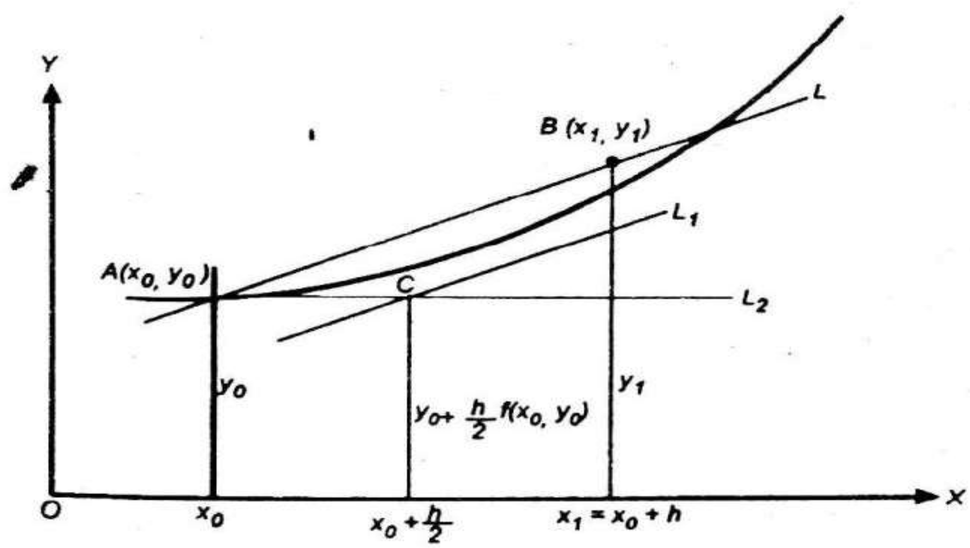

Fig 11.4

Geometrically, line L through (x_0, y_0) which is parallel to L_1, a line through $(x_0 + \frac{h}{2}, \frac{h}{2} f(x_0, y_0))$, with the slope (1) approximates the curve in the interval $[x_0, x_1]$. The ordinate at $x = x_1$, meeting the line L at B, will give the value of y_1.

The equation for line L is

$$y - y_0 = (x - x_0)\{f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))\}$$

Putting $x = x_1$, we get

$$y_1 = y_0 + (x_1 - x_0)\{f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))\}$$

$$= y_0 + h f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)) \qquad (11.39)$$

Proceeding in the same way, it can be shown that,

$$y_{m+1} = y_m + h f(x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)) \qquad (11.40)$$

Example 11.9 Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$ in the range $0 \le x \le 0.3$ using (i) Euler's method (ii) improved Euler's method, and (iii) modified Euler's method by choosing $h = 0.1$. Compare the answers with exact solution.

Solution Given $\frac{dy}{dx} = 1 - y$ and $y(0) = 0$ and $h = 0.1$. (i)

Now we have to find out the solutions at $x = 0.1, 0.2$ and 0.3.

(i) Euler's method : The algorithm is,

if $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

then, $y_{m+1} = y_m + h f(x_m, y_m)$

Here, $f(x, y) = 1 - y$, $h = 0.1$; $x_0 = 0, y_0 = 0$

$\therefore$ From Eqn (ii), $y_{m+1} = y_m + (0.1)(1 - y_m)$

or $y_{m+1} = 0.1 + 0.9 y_m$

Putting $m = 0, 1, 2$ successively, we get

$y_1 = 0.1 + 0.9 y_0 = 0.1 + (0.9)(0) = 0.1$

$y_2 = 0.1 + 0.9 y_1 = 0.1 + (0.9)(0.1) = 0.19$

$y_3 = 0.1 + 0.9 y_2 = 0.1 + (0.9)(0.19) = 0.271$

$\therefore y(0.1) = 0.1$; $y(0.2) = 0.19$; $y(0.3) = 0.271$

(ii) Improved Euler's method : Here, the formula is

$$y_{m+1} = y_m + \frac{h}{2}\{f(x_m, y_m) + f(x_{m+1}, y^{(1)}_{m+1})\}$$

where, $f(x_{m+1}, y^{(1)}_{m+1}) = f\{x_m + h, y_m + hf(x_m, y_m)\}$

Here, $f(x, y) = 1 - y$, $\therefore f(x_m, y_m) = 1 - y_m$

$$\therefore f(x_{m+1}, y^{(1)}_{m+1}) = 1 - \{y_m + hf(x_m, y_m)\}$$
$$= 1 - y_m - h(1 - y_m)$$
$$= (1 - h)(1 - y_m)$$

Substituting it in Eqn (iv), we get

$$y_{m+1} = y_m + \frac{h}{2}\{(1-y_m) + (1-h)(1-y_m)\}$$

$$= y_m + \frac{1}{2}h(2-h)(1-y_m)$$

$$= y_m + 0.095(1-y_m) \quad [\because h = 0.1]$$

$$y_{m+1} = 0.095 + 0.095\, y_m \qquad \text{(v)}$$

Putting $m = 0, 1, 2$ successively in Eqn (v), we get

$y_1 = 0.095 + 0.905\, y_0 = 0.095 + (0.905)(0) = 0.095$

$y_2 = 0.095 + 0.905\, y_1 = 0.095 + (0.905)(0.095) = 0.180975$

$y_3 = 0.095 + 0.905\, y_2 = 0.095 + (0.905)(0.180905) = 0.2587823$

(iv) *Modified Euler's method :* The formula is,

$$y_{m+1} = y_m + hf\{x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\}$$

$$= y_m + h\{1 - [y_m + \frac{h}{2} f(x_m, y_m)]\}$$

$$= y_m + h\{1 - y_m - \frac{h}{2}(1-y_m)\}$$

$$= y_m + h\{1 - \frac{h}{2})(1-y_m)\}$$

$$= y_m + 0.095(1-y_m) \quad [\because h = 0.1]$$

$$= 0.095 + 0.905\, y_m$$

which is identical to Eqn (vi). Putting $m = 0, 1, 2$ successively in Eqn (vi), we get

$$y_1 = 0.095, y_2 = 0.180975, y_3 = 0.2587823$$

Exact solution : We have

$$\frac{dy}{dx} = 1 - y \text{ or } \frac{dy}{1-y} = dx$$

On integrating, we get

$$-\log(1-y) + \log C = x \text{ or } \frac{C}{1-y} = e^x$$

or

$$e^x(1-y) = C \qquad \text{(vii)}$$

But $y = 0$ at $x = 0$. Therefore, from Eqn (vii), we get $C = 1$ and hence

$$e^x(1-y) = 1 \text{ or } y = 1 - e^{-x} \qquad \text{(viii)}$$

$$\therefore \quad y_1 = y(0.1) = 1 - e^{0.1} = 0.0951625$$
$$y_2 = y(0.2) = 1 - e^{0.2} = 0.1812692$$
and $$y_3 = y(0.3) = 1 - e^{0.3} = 0.2591817$$

- Now compare the results in the following table.

x	Euler's method	Improved Euler's method	Modified Euler's method	Exact solution
0	0	0	0	0
0.1	0.1	0.095	0.095	0.0951625
0.2	0.19	0.180975	0.180975	0.1812692
0.3	0.271	0.2587823	0.2587823	0.2591817

RUNGE KUTTA METHODS

In this section, we will study the formula for third and fourth order Runge–Kutta methods. Their derivations being tedious and unrequired, the formulae have not been derived here.

Runge–Kutta method of third order: It is defined by the following equations:

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2)$$
$$k_4 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$
$$k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

and $$y_1 = y_0 + K$$

We can see that this is identical to Runge's method.

Runge–Kutta method of fourth order: It is most commonly known as Runge–Kutta method and the working procedure is as follows. Consider the following equations.

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

To compute y_1, calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, \; y_0 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $$y_1 = y_0 + k \text{ and } x_1 = x_0 + h$$

The increment in y in second interval is computed in a similar manner by means of the formulae

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, \; y_1 + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Then $$y_2 = y_1 + k \text{ and } x_2 = x_1 + h$$

and so on for succeeding intervals.

You can notice that the only change in the formulae for the different intervals is in the values of x and y. Thus, to find k in ith interval, we should have substituted x_{i-1} and y_{i-1} in the expressions for k_1, k_2, k_3 and k_4.

Example 11.12 Given $y' = x^2 - y$, $y(0) = 1$, find $y(0.1)$, $y(0.2)$ using Runge–Kutta methods of (i) second order, (ii) third order and (iii) fourth order.

Solution Given

$$y' = f(x, y) = x^2 - y, \; x_0 = 0, \; y_0 = 1$$

$$\therefore \quad f(x_0, y_0) = -1.$$

Let $h = 0.1$

Runge–Kutta method of 2nd order : Here,

$$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = hf(0.1, 0.9)$$
$$= (0.1)\,[(0.1)^2 - 0.9] = -0.089$$

$$\therefore \quad k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.1) + (0.089)] = -0.0945$$

$$\therefore \quad y_1 = y(0.1) = y_0 + k = 1 - 0.0945 = 0.9055$$

Again, taking $x_1 = 0.1,\ y_1 = 0.9055$ in place of (x_0, y_0) and repeating the process,

$$k_1 = hf(x_1, y_1) = h(x_1^2 - y_1)$$
$$= (0.1)\,[(0.1)^2 - 0.9055] = -0.08955$$
$$k_2 = hf[x_1 + h,\ y_1 + k_1] = hf(0.2, 0.81595)$$
$$= (0.1)\,[(0.2)^2 - 0.81595] = -0.077595$$

$$\therefore k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}[(-0.08955 - 0.077595] = -0.0835725$$

$$\therefore y_2 = y(0.2) = y_1 + k = 0.9055 - 0.0835725$$
$$= 0.8219275$$

Runge–Kutta method of 3rd order : Here,

$$k_1 = hf(x_0, y_0) = -0.1$$
$$k_2 = hf(x_0 + h,\ y_0 + k_1) = -0.089$$
$$k_3 = hf(x_0 + h,\ y_0 + k_2) = hf(0.1, 0.911)$$
$$= (0.1)\,[(0.1)^2 - 0.911] = -0.0901$$
$$k_4 = hf(x_0 + h/2,\ y_0 + k_1/2) = hf(0.05, 0.95)$$
$$= (0.1)\,[(0.05)^2 - 0.95] = -0.09475$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$= \frac{1}{6}[(-0.1) + 4(-0.09475) + (-0.0901\ ] = -0.09485$$

$$\therefore y_1 = y(0.1) = 0.90515$$

Taking $x_1 = 0.1,\ y_1 = 0.90515,\ h = 0.1$ in place of (x_0, y_0) and repeating the process, we get

$$k_1 = hf(x_1, y_1) = (0.1)\,[(0.1)^2 - 0.90515] = -0.089515$$
$$k_2 = hf(x_1 + h,\ y_1 + k_1) = hf(0.2, 0.815635)$$
$$= (0.1)\,[(0.2)^2 - 0.815635] = -0.0775635$$
$$k_3 = hf(x_1 + h,\ y_1 + k_2) = hf(0.2, 0.8275865)$$
$$= (0.1)\,[(0.2^{\ 2} - 0.8275865] = -0.0787586$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.15, 0.8603925)$$
$$= (0.1)[(0.15)^2 - 0.8603925] = -0.0837892$$

$$\therefore k = \frac{1}{6}(k_1 + 4k_4 + k_3)$$
$$= \frac{1}{6}[(-0.089515) + 4(-0.0837892) - 0.0787586] = -0.0839051$$
$$\therefore y_2 = y(0.2) = y_1 + k = 0.90515 - 0.0839051 = 0.8212449$$

Runge-Kutta method of 4th order : Here,

$$k_1 = hf(x_0, y_0) = -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f[0.05, 0.95]$$
$$= (0.1)[(0.05)^2 - 0.95] = 0.09475$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.05, 0.952625)$$
$$= (0.1)[(0.05)^2 - 0.952625] = -0.0950125$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = hf[0.1, 0.9049875]$$
$$= (0.1)[(0.1)^2 - 0.9049875] = -0.0894987$$

$$\text{Now, } k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125) - 0.0894987]$$
$$= -0.0948372$$
$$\therefore y_1 = y(0.1) = y_0 + k = 1 - 0.0948372 = 0.9051627$$

Taking $x_1 = 0.1$, $y_1 = 0.9051627$ in place of x_0, y_0 and repeating the process, we get

$$k_1 = hf(x_1, y_1) = hf(0.1, 0.9051627)$$
$$= (0.1)[(0.1)^2 - 0.9051627] = -0.0895162$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf(0.15, 0.8604046)$$
$$= (0.1)[(0.15)^2 - 0.8604046] = -0.0837904$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf(0.15, 0.8632674)$$
$$= (0.1)[(0.15)^2 - 0.8632674] = -0.0840767$$

$$k_4 = hf(x_1+h,\ y_1+k_3) = hf(0.2,\ 0.8210859)$$
$$= (0.1)\,[(0.2)^2 - 0.8210859] = -0.0781085$$

$$k = \frac{1}{6}(k_1+2k_2+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}[-0.0895162+2(-0.0837904)+2(-0.0840767)-0.0781085]$$
$$= -0.0838931$$

$$\therefore\quad y_2 = y(0.2) = y_1 + k = 0.9051627 - 0.0838931$$
$$= 0.08212695$$

11.19 MILNE'S METHOD

Here, the equation to be solved numerically is

$$\frac{dy}{dx} = f(x,y)\ ;\ y(x_0) = y_0$$

The value $y_0 = y(x_0)$ being given, we calculate $y_1 = y(x_0+h) = y(x_1)$; $y_2 = y(x_0+2h) = y(x_2)$; $y_3 = y(x_0+3h) = y(x_3)\ \ldots$ where h is a suitably chosen spacing.

where $x = x_0 + uh$.

For $y = y'$ the above gives

$$y' = y'_0 + u\Delta y_0' + \frac{u(u-1)}{2!}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0' + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y'_0 + \cdots \qquad (11.55)$$

Now using $\Delta^n = (1-E)^n$, $n = 1, 2, 3$ and simplifying the above, we get

$$y_4 - y_0 = h\left[4y_0' + 8(y_1'-y_0') + \frac{20}{3}(y_2'-2y_1'+y_0') + \frac{8}{3}(y_3'-3y_2'+3y_1'-y_0') + \frac{14}{45}\Delta^4 y_0' + \cdots\right] \qquad (11.56)$$

$$\text{or } y^4 = y^0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad (11.57)$$

[by considering only differences upto third order]

Hence, the error in Eqn (11.57) is $= \frac{14}{45}\Delta^4 y_0' + \cdots$ and this can be proved to be $= \frac{14}{45}\, y^5(\xi)$, where ξ lies in between x_0 and x_4. Hence, Eqn (11.57) can be written as

$$y_4 = y_0 + \frac{4h}{3}\,(2y_1' - y_2' + 2y_3') + \frac{14h^5}{45}\, y^5(\xi) \qquad (11.58)$$

$\therefore x_0, x_1, x_2, x_3, x_4$ are any five consecutive values of x, the above, in general, can be written as

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\,(2y_{n-2}' - y_{n-1}' + 2y_n') + \frac{14h^5}{45} y^5(\xi_1) \qquad (11.59)$$

where ξ_1 lies between x_{n-3} and x_{n+1}. Eqn (11.59) is known as *Milne's predictor formula*.

To get *Milne's corrector formula*, integrate Eqn (11.55) with respect to x over the interval x_0 to $x_0 + 2h$. Then we have

$$\int_{x_0}^{x_0+2h}[y']dx = h\int_0^2\Big\{y_0' + u\Delta y_0' + \frac{1}{2}(u^2 - u)\Delta^2 y_0'$$

$$+\frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y_0'$$

$$+\frac{1}{24}(u^4 - 6u^3 + 11u^2 - 6u)\Delta^4 y_0' + \cdots\Big\}du$$

Or $$y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' - \frac{1}{3}\Delta^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta y_0' + \cdots\right]$$

$$= h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0') - \frac{h}{90}\Delta^4 y_0' + \cdots\right] \quad [\text{using } \Delta^r = (E-1)^r, n = 1, 2, 3]$$

or
$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \quad (11.60)$$

Considering only differences upto third order, Eqn (11.60) gives

$$y_2 = y_0 + \frac{h}{3}(y_0' - 4y_1' + y_2') \quad (11.61)$$

$\therefore$ Error $= -\frac{h}{90}\Delta^4 y_0' + \cdots$ and this can be proved to be $= -\frac{h^5}{90}y^5(\xi)$, where $x_0 < \xi < x_2$.

$\therefore$ Eqn (11.61) can be written as

$$y_2 = y_0 + \frac{h}{3}(y'_0 - 4y'_1 + y'_2) - \frac{h^5}{90}y^5(\xi) \quad (11.62)$$

Since x_0, x_1, x_2 are any three consecutive values of x, the above can be written in general as

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y_{n-1}' - 4y_n' + y_{n+1}') - \frac{h^5}{90}y^5(\xi_2) \quad (11.63)$$

where ξ_2 lies in between x_{n-1} and x_{n+1}. Eqn (11.63) is called as *Milne's corrector formula*.

Note : This method requires at least four values prior to the required value. If the initial four values are not given, we can obtain them by using Picard's method or Taylor's series method or Euler's method or Runge–Kutta method.

Example 11.16 Given $\frac{dy}{dx} = 1/x + y$, $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$, find $y(0.8)$ using Milne's method.

Solution In the usual notation, Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \quad (i)$$

where $y_{n+1,p}$ denotes the predicted value at y_{n+1}.

In the given problem, $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$ $h = 0.2, y_0 = 2,$ $y_1 = 2.0933, y_2 = 2.1755, y_3 = 2.2493$

and $y' = \frac{1}{x+y}$

Putting $n = 3$ in Eqn (i), the predictor is

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \quad \text{(ii)}$$

Now $y_1' = \frac{1}{x_1 + y_1} = \frac{1}{0.2 + 2.0933} = 0.4360528$

$$y_2' = \frac{1}{x_2 + y_2} = \frac{1}{0.4 + 2.1755} = 0.3882741$$

$$y_3' = \frac{1}{x_3 + y_3} = \frac{1}{0.6 + 2.2493} = 0.3509633$$

Substituting in Eqn (ii), we get

$y_{4,p} = 2 + 4(0.2)/3\,[2(0.4360528) - 0.3882741 + 2(0.3509633)]$

$= 2.3162022$ (iii)

Now, Milne's corrector formula in general form is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \quad \text{(iv)}$$

here $y_{n+1,c}$ denotes the corrected value of y_{n+1}.

Putting $n = 3$ in above, we get

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \quad \text{(v)}$$

From Eqn (iii), $y_{4,p} = 2.3162022$ and $x_4 = 0.8$.

$$y_4' = \frac{1}{x_4 + y_{4,p}} = \frac{1}{0.8 + 2.3162022} = 0.3209034$$

Hence, from Eqn (v),

$$y_{4,c} = 2.1755 + \frac{0.2}{3}[0.3882741 + 4(0.3509633) + 0.3209034]$$

$= 2.3163687$

$\therefore y(0.8) = y_4 = 2.3164$ corrected to four decimals.

11.20 ADAMS–BASHFORTH METHOD

Given

$$\frac{dy}{dx} = f(x, y) \tag{11.64}$$

and $y(x_0) = y_0$
we compute $y_{-1} = y(x_0 - h)$, $y_{-2} = y(x_0 - 2h)$, $y_{-3} = y(x_0 - 3h), \cdots$

Neglecting the fourth order and higher order differences and using $\nabla f_0 = f_0 - f_{-1}$, $\nabla^2 f_0 = f_0 - 2f_{-1} + f_{-2}$, $\nabla^3 f_0 = f_0 - 3f_{-1} + 3f_{-2} - f_{-3}$, in Eqn (11.66), we get, after simplification,

$$y_1 = y_0 + \frac{h}{24}\{55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\}$$

which is known *Adams–Bashforth predictor formula* and is denoted generally as

$$y_{n+1,p} = y_n + \frac{h}{24}\{55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\}$$

or $$y_{n+1,p} = y_n + \frac{h}{24}\{55 y_n' - 59 y_{n-1}' + 37 y_{n-2}' - 9 y_{n-3}'\} \tag{11.67}$$

Having found y_1, we find $f_1 = f(x_0 + h, y_1)$

Then to find a better value of y_1, we derive a corrector formula by substituting *Newton's backward interpolation formula* at f_1 in place of $f(x, y)$ in Eqn (11.65), i.e.

$$y_1 = y_0 + \int_{x_0}^{x_0+h} \left\{ f_1 + u\nabla f_1 + \frac{1}{2!}u(u+1)\nabla^2 f_1 + \frac{1}{3!}u(u+1)(u+2)\nabla^3 f_1 + \cdots \right\} dx$$

Example 11.18 Using Adams–Bashforth method, find $y(1.4)$ given $y' = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$.

Solution Given

$y' = x^2(1+y)$, $x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, y_0 = 1,$

$y_1 = 1.233, y_2 = 1.548, y_3 = 1.979, h = 0.1.$

Adams–Bashforth predictor formula is

$$y_{4,p} = y_3 + \frac{h}{24}(55\,y_3' - 59\,y_2' - 37\,y_1' - 9\,y_0') \qquad \text{(i)}$$

$$y_0' = x_0^2(1+y_0) = (1)^2[1+1] = 2$$
$$y_1' = x_1^2(1+y_1) = (1.1)^2[1+1.233] = 2.70193$$
$$y_2' = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.66912$$
$$y_3' = x_3^2(1+y_3) = (1.3)^2[1+1.979] = 5.03451$$

$$\therefore\ y_{4,p} = 1.979 + 0.1/24\,\{55(5.03451) - 59(3.66912) + 37(2.70193) - 9(2)\}$$
$$= 2.5722974$$

Now $y_{4,p}' = x_4^2(1+y_{4,p}) = (1.4)^2\{1+2.5722974\}$
$= 7.0017029$

The corrector formula is

$$y_{4,c} = y_3 + \frac{h}{24}\{9\,y_4' + 19\,y_3' - 5\,y_2' + y_1'\}$$

$$= 1.979 + \frac{0.1}{24}\{9(7.0017029) + 19(5.03451) - 5(3.66912) + 2.70193\}$$

$$= 2.5749473$$

$\therefore y(0.4) = 2.575$, correct to three decimal places.

Part B (5x6=30 Marks)

Possible Questions

1. i) Derive the Newton's Forward difference formula
   ii) Derive the Newton's Backward difference formula.
2. Use Euler's method to solve the equation y' = -y with the condition y(0) = 1.
3. Given $\frac{dy}{dx}$= y-x where y(0)=2, find y(0.1),y(0.2) using RungeKutta method.
4. Write the Derivative of Adams Moulton's Method
5. Solve y'= -y &y(0)=1 determine the values of y at x=(0.01)(0.01)(0.04) by Euler method.
6. Solve the equation $\frac{dy}{dx}$ = 1-y given y(0)=0 using Modified Euler method and tabulate the solutions at x=0.1,0.2.
7. Determine the value of y (0.4) using Milne's Method given y' = $xy+y^2$, y(0)=1 and get the values of y(0.1),y(0.2) andy(0.3)

Part C (1x10=10 Marks)

Possible Questions

1. Compute y at x=0.25 by Modified Euler method given y'=2xy, y(0)=1.
2. Apply the fourth order Runge Kutta method to find y(0.1), y(0.2) given that y'=x+y, y(0)=1.
3. Derivative of Milne's Predicator and Corrector Method.
4. Given $\frac{dy}{dx}$=$1+y^2$, where y=0 when x=0, find y(0.4) using Adams Moultan method.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Numerical Analysis **Subject Code: 17MMP103**

Class : I - M.Sc. Mathematics **Semester : I**

Unit IV

Boundary Value Problem and Characteristic value problem

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
In numerical methods, the boundary problems,are solved by using -----------------method.	finite difference	Euler	Milne's	Runge- kutta	finite difference
The boundary value problem by using the method for solving the initial value problem is called-----------	shooting method	difference method	finite element	iterative method	shooting method
--------------method is initial value problem methods.	Milne's	Euler	Shooting	Runge- kutta	Shooting
The -------- method is used to determine numerically largest eigen value and the corresponding eigen vector of matrix A.	Gauss Jordan	power	choleskey	Gauss seidel	power
The iterative process continues till --------- is secured.	convergency	divergency	oscillation	infinite	convergency
------------- is also a self-correction method.	Iteration method	Direct method	Interpolation	extrapolation	Iteration method
The condition for convergence of Gauss Seidal method is that the ------ should be diagonally dominant	Constant matrix	unknown matrix	Coefficient matrix	extrapolation	Coefficient matrix

In iterative methods, the solution to a system of linear equations will exist if the absolute value of the largest coefficient is ------------- the sum of the absolute values of all remaining coefficients in each equation.	less than	greater than or equal to	equal to	not equal	greater than or equal to
In ------------- iterative method, the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration.	Gauss Siedal	Gauss Jacobi	Gauss Jordan	Gauss Elimination	Gauss Siedal
The direct method fails if any one of the pivot elements become ----.	Zero	one	two	negative	Zero
In ---------- method, the coefficient matrix is transformed into diagonal matrix	Gauss elimination	Gauss jordan	Gauss jacobi	Gauss seidal	Gauss jordan
We get the approximate solution from the --------------.	Direct method	InDirect method	fast method	Bisection	InDirect method
------------- Method takes less time to solve a system of equations	Direct method	InDirect method	fast method	Bisection	Direct method
The ----------- problem by using the method for solving the initial value problem is called shooting method.	initial value	boundary value	secondary value	primary value	boundary value
Which involves the transformation of the boundary-value problem into an initial-value prolem?	Milne's	Euler	Shooting	Runge- kutta	Shooting
Which method involves the solution of the initial-value problem by any of the known methods?	Milne's	Euler	Shooting	Runge- kutta	Shooting
Which method involves the solution of the given boundary-value problem.	Milne's	Euler	Shooting	Runge- kutta	Shooting
The eienvalues of a 4x4 matrix [A] are given as 2, -3, 13 and 7. the |det(A)| then is---------	546	19	25	3	546
The --------------- process continues till convergency is secured.	first	second	iterative	non iterative	iterative
The -------------------method is used to find the eigen values of a real symmetric matrix.	Jordan	Seidal	choleskey	Jacobi	Jacobi

If the eigen values of A are 1,3,4 then the dominant eigen value of A is --------------.	0	4	1	3	4
The shooting method is --------- problem.	initial value	boundary value	secondary value	primary value	initial value
If all the non zero terms involve only the dependent variable u and u' then the differential equation is called ---------	homogeneous	non homogeneous	linear	non linear	homogeneous

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
Department of Mathematics

Subject: Numerical Analysis	**Subject Code: 17MMP103**	**L T P C**
Class : I – M.Sc. Mathematics	**Semester : I**	**4 0 0 4**

UNIT-IV

Boundary Value Problem and Characteristic value problem: The shooting method: The linear shooting method – The shooting method for non-linear systems.
Characteristic value problems –Eigen values of a matrix by Iteration-The power method.

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

Introduction

Consider the two point boundary value problem

$u'' = f(x, u, u'), \; x \in (a, b)$ (4.1)

Where a prime denotes differentiation with respect to x, with one of the following three boundary conditions.

Boundary condition of the first kind:

$u(a) = \gamma_1 \quad , u(b) = \gamma_2 .$ (4.2)

Boundary condition of the second kind:

$u'(a) = \gamma_1 \quad , u'(b) = \gamma_2 .$ (4.3)

Boundary condition of the third kind(or mixed kind):

$a_0 u(a) - a_1 u'(a) = \gamma_1$ (4.4i)

$b_0 u(b) + b_1 u'(b) = \gamma_2$ (4.4ii)

Where $a_0, b_0, a_1, b_1, \gamma_1, \gamma_2$ are constant such that

$a_0 a_1 \geq 0, \; |a_0| + |a_1| \neq 0$

$b_0 b_1 \geq 0, \; |b_0| + |b_1| \neq 0$ and , $|a_0| + |b_0| \neq 0.$

In (4.1), if all the non zero terms involve only the dependent variable u and u′, then the differential equation is called homogeneous, otherwise, it is inhomogeneous. Similarly, the boundary conditions are homogeneous when γ_1 and γ_2 are zero; otherwise, they are inhomogeneous. A homogeneous boundary value problem , that is a homogeneous differential equation along with homogeneous boundary condition , possesses only a trivial solution u(x)=0. we, therefore , consider those boundary value problems in which a parameter λ occurs either in the differential equation or in the boundary condition , and we determine value of λ , called **eigenvalues** for which the boundary value problem has a nontrivial solution. Such a solution is called **eigenfunction** and the entire problem is called an **eigenvalue** or a **characteristic value problem.**

The solution of the boundary (4.1) exists and is unique if the following conditions are satisfied:

Let $u' = z$ and $-\infty < u, z < \infty$

(i) $f(x, u, z)$ is continuous,
(ii) $\partial f/\partial u$ and $\partial f/\partial z$ exist and are continuous.
(iii) $\partial f/\partial u>0$ and $|\partial f/\partial z| \leq w$.

In what follows, we shall assume that the boundary value problems a unique solution and we shall attempt to determine it. The numerical methods for solving the boundary value problems may broadly be classified in to the following three types:

(i). *Shooting Methods*These are initial value problem methods. Here, we add sufficient number of conditions at one end point and adjust these conditions until the required conditions are satisfied at the other end.

(ii)*Difference methods*The differential equation is replaced by a set of difference
Equations which are solved by direct or iterative methods.

(iii) *Finite element methods*The differential equation is discretized by using approximate methods with a piece wise polynomial solution.
We shall now discuss in detail the shooting methods and for solving numerically both the linear and non linear second order boundary value problems.

Initial Value Problem Method (Shooting Method)

Consider the boundary value problem (4.1) (BVP) subject to the given boundary conditions

Since the differential equation is of second order, we require two linear independent conditions to solve the boundary value problem. one of the ways of solving the boundary value problem is the following.

(i) *Boundary conditions of the first kind*Here , we are given $u(a) = \gamma_1$.in order that an initial value method can be used, we guess the value of the slope at x=a as $u'(a)=s$.

(ii) *Boundary conditions of the second kind* Here, we are given $u'(a)= \gamma_1$.in order that an initial value method can be used, we guess the value of u(x) at x=a as $u(a)=s$.

(iii) *Boundary conditions of the third kind* Here, we guess the value u(a) or u′(a). if we assume that $u'(a)=s$, then from(4.4i), we get $u(a)=(a_1 s+ \gamma_1)/ a_0$.

The related initial value problem is solved upto x=b, by using single step or a multi-step method. If the problem is solved directly, then we use the methods for second order initial value problems. If the differential equation is reduced to a system of two first order equations, then we use the Runge-Kutta methods or the multi-step methods for a system of first order equations.

If the solution at x=b does not satisfy the given boundary condition at the other end x=b , then we take another guess value of u(a) or u′(a) and solve the initial value problem again upto x=b. these two solutions at x=b , of the initial value problems are used to obtain a better estimate of u(a) or u′(a) .A Sequence of such problems are solved, if necessary, to obtain the solution of the

given boundary value problem. For a *linear, non-homogenous boundary value problem, it is sufficient to solve two initial value problems with two linearly independent guess initial conditions.*

This technique of solving the boundary value problem by using the methods for solving the initial value problems is called the **shooting method**.

Linear SecondOrder Differential Equations

Consider the linear differential equations

$-u''+p(x)\,u'+q(x)\,u=r(x)\ ,\ a< x <b$ (4.5)

Subject to the given boundary conditions. We assume that the functions p(x), q(x)>0, and r(x) are continuous on [a, b], so that the boundary value problem(4.5) has a unique solution.

The general solution of (4.5) can written as

$u(x)=u_0(x)+\mu_1\,u_1(x)+\mu_2\,u_2(x)$ (4.6)

Where (i).$u_0(x)$ is a particular solution of the non homogeneous equation (4.5), that is

$-u_0''+p(x)\,u_0'+q(x)\,u_0=r(x)$ (4.7)

(ii) $u_1(x)$ and $u_2(x)$ are any two linearly independent , complementary solutions of the corresponding homogeneous equation of (4.5) , that is

$-u_1''+p(x)\,u_1'+q(x)\,u_1=0$ (4.8)

$-u_2''+p(x)\,u_2'+q(x)\,u_2=0$ (4.9)

We choose the initial conditions as follows:

Boundary conditions of the first kind Since $u(a) = \gamma_1$ is given, we take a guess value for $u'(a)$. We have the following two case.

Case 1:$\gamma_1 \neq 0$. We choose

$u_0(a) = u_1(a) = u_2(a) = \gamma_1$

$u_0'(a) = \eta_0^*$. $u_1'(a) = \eta_1^*$, $u_2'(a) = \eta_2^*$ (4.10i)

Where η_0^* , η_1^*, η_2^* are arbitrary. Since $u_1(x)$ and $u_2(x)$ are linearly independent solutions, a suitable choice of the initial conditions is

$$\eta_0^* = 0, \eta_1^* = 1, \eta_2^* = 0. \qquad (4.10ii)$$

Other choices of linearly independent values can also be considered.

We now solve the differential equation (4.7)-(4.9) along with the corresponding initial conditions, using value methods with the same lengths, and obtain $u_0(b)$, $u_1(b)$ and $u_2(b)$.Now since the solution (4.6) satisfies the boundary conditions at x=a and x=b , we obtain, at x=a: $u_0(a)+\mu_1 u_1(a)+ \mu_2 u_2(a)= \gamma_1$

Or $\gamma_1+\mu_1 \gamma_1+ \mu_2 \gamma_1= \gamma_1$ Or $\mu_1 + \mu_2 = 0$ x=b:

$u_0(b)+\mu_1 u_1(b)+ \mu_2 u_2(b)= \gamma_2$ (4.11i)

or $\mu_2 = \dfrac{\gamma_2 - u_0(b)}{u_2(b) - u_1(b)}$, $u_1(b) \neq u_2(b)$. (4.11ii)

Case 2:$\gamma_1 = 0$. In this case, we cannot (4.10i), since $[u_1(a), u'_1(a)]^T = [0,1]^T$ and $[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly dependent . We choose the conditions as

: $u_0(a) = \eta_0$, $u_1(a) = \eta_1$, $u_2(a) = \eta_2$

$u_0'(a) = \eta_0^*$. $u_1'(a) = \eta_1^*$, $u_2'(a) = \eta_2^*$

A suitable set of values is

$\eta_0 = \gamma_1 = 0$, $\eta_0^* = 0$; $\eta_1 = 1$, $\eta_1^* = 0$; $\eta_2 = 0$, $\eta_2^* = 1$. (4.12)

We note that the conditions $[u_1(a), u'_1(a)]^T = [0,1]^T$ and$[u_2(a), u'_2(a)]^T = [0,0]^T$ are linearly independent . Any other linearly independent set of values can be used.

We now solve the corresponding initial values problems upto x=b.

Now, since the solution (4.6) satisfies the boundary conditions at x=a and x=b, we obtain, at

x=a: $u_0(a)+\mu_1 u_1(a)+\mu_2 u_2(a)=\gamma_1=0$.

Or $\eta_0+\mu_1\eta_1+\mu_2\eta_2=0$

Or $\mu_1=0$(using (4.12))

X=b: $u_0(b)+\mu_1 u_1(b)+\mu_2 u_2(b)=\gamma_2$ (4.13i)

Or $\mu_2=\dfrac{\gamma_2-u_0(b)}{u_2(b)}$, $u_2(b)\neq 0$

(4.13ii)

We determine μ_1, μ_2 from (4.11) or (4.13) and obtain the solution of the given boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

Boundary conditions of the second kind Since $u'(a)=\gamma_1$ is given ,we guess the value of u(a). Again, we consider the following two cases.

Case 1:$\gamma_1\neq 0$. We choose

$u_0(a)=\eta_0$, $u_1(a)=\eta_1$, $u_2(a)=\eta_2$

$u_0'(a)=u_1'(a)=u_2'(a)=\gamma_2$ (4.14i)

A suitable set is values is

$\eta_0=0$, $\eta_1=1$, $\eta_2=0$. (4.14ii)

Since the initial conditions $[u_1(a), u'_1(a)]^T=[0,1]^T$ and$[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly independent , we obtain linearly independent solutions $u_1(x)$ and $u_2(x)$. Using these initial conditions, we solve the corresponding initial value problems, with the same step lengths, upto x=b.

Now, from (4.6) , we get

$u'(x)=u'_0(x)+\mu_1 u'_1(x)+\mu_2 u'_2(x)$

Using the given condition (4.3), we get , at x=a: $u'_0(a)+\mu_1 u'_1(a)+\mu_2 u'_2(a)=\gamma_1$

Or $\gamma_1+\mu_1\gamma_1+\mu_2\gamma_1=\gamma_1$ ………(4.16i)

Or $\mu_1+\mu_2=0$

x =b: $u'_0(b)+\mu_1 u'_1(b)+\mu_2 u'_2(b)=\gamma_2$

Or $\mu_2=\dfrac{\gamma_2-u_0(b)}{u_2(b)-u_1(b)}$, $u'_1(b)\neq u'_2(b)$. ……(4.16ii)

Case 2:$\gamma_1=0$. we cannot use the conditions as in case 1, since $[u_1(a), u'_1(a)]^T=[1,0]^T$ and $[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly dependent . In this case, we choose

: $u_0(a)=\eta_0$, $u_1(a)=\eta_1$, $u_2(a)=\eta_2$

$u'_0(a)=\eta_0^*$. $u'_1(a)=\eta_1^*$, $u'_2(a)=\eta_2^*$

A suitable set of values is

$\eta_0=0$, $\eta_0^*=\gamma_1=0$; $\eta_1=1$, $\eta_1^*=0$; $\eta_2=0$, $\eta_2^*=1$.(4.17)

We note that the conditions $[u_1(a), u'_1(a)]^T=[1,0]^T$ and$[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly independent . Any other linearly independent set of values can be used.

Using (4.6), (4.15) and the boundary conditions (4.3), we get, at

x=a: $u'_0(a)+\mu_1 u'_1(a)+\mu_2 u'_2(a)=\gamma_1=0.$

Or $\eta_0^*+\mu_1\eta_1^*+\mu_2\eta_2^*=0$ ……..(4.18i)

Or $\mu_2=0$

X=b: $u'_0(b)+\mu_1 u'_1(b)+\mu_2 u'_2(b)=\gamma_2$

Or $\mu_2=\dfrac{\gamma_2-u'_0(b)}{u'_1(b)}$, $u'_1(b)\neq 0$ ………(4.18ii)

We determine μ_1 , μ_2 from (4.16) or (4.18) and obtain the solution of the boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

*Boundary conditions of the third kind*In the case, we assume the arbitrary initial conditions as

$u_0(a)=\eta_0$, $u_1(a)=\eta_1$, $u_2(a)=\eta_2$

$u'_0(a)=\eta^*_0$. $u'_1(a)=\eta^*_1$, $u'_2(a)=\eta^*_2$ (4.19i)

A suitable set of values is

$\eta_0=0$, $\eta^*_0=0$; $\eta_1=1$, $\eta^*_1=0$; $\eta_2=0$, $\eta^*_2=1$. (4.19ii)

Again , We note that the conditions $[u_1(a), u'_1(a)]^T=[1,0]^T$ and$[u_2(a), u'_2(a)]^T=[0,0]^T$ are linearly independent . Using these initial conditions, we solve the corresponding initial value problems, using the same step lengths, upto x=b.

Using (4.6) (4.19) and the boundary conditions (4.4), we get, at x=a:

$a_0\,[u_0(a)+\mu_1\, u_1(a)+\mu_2\, u_2(a)] - a_1\,[u'_0(a)+\mu_1\, u'_1(a)+$

$\mu_2\, u'_2(a)]=\gamma_1$

Or $a_0[\eta_0+\mu_1\eta_1+\mu_2\,\eta_2]- a_1[\eta^*_0+\mu_1\,\eta^*_1+\mu_2\,\eta^*_2]=\gamma_1$

Or $a_0\mu_1 - a_1\mu_2=\gamma_1$ ……….(4.20i)

x=b: $b_0\,[u_0(b)+\mu_1\, u_1(b)+\mu_2\, u_2(b)] + b_1\,[u'_0(b)+\mu_1\, u'_1(b)+\mu_2\, u'_2(b)]=\gamma_2$

Or $\mu_1\,[b_0 u_1(b)+ b_1\, u'_1(b)]+\mu_2\,[b_0 u_2(b)+ b_1\, u'_2(b)]$

$=\gamma_2 -[b_0\, u_0(b)+ b_1\, u'_0(b)]$ …….. (4.20ii)

We determine μ_1 , μ_2 from (4.20) and obtain the solution of the boundary value problem, using (4.6), at mesh point used in integrated the initial value problems.

Boundary value problem of the first kind we solve the initial value problems (4.21i)(4.21ii) using the initial conditions

$u_1(a)=\gamma_1$, $u'_1(a)=0$

$u_2(a)=\gamma_1$, $u'_2(a)=1$ ……….(4.23i)

upto x =b. Any other value for $u'_2(a)$ can also be used. Since the general solution (4.22) satisfies the boundary condition at x =b, we get

$u(b) = \gamma_2 = \lambda u_1(b)+(1-\lambda) u_2(b)$

or $\lambda = \dfrac{\gamma_2 - u_2(b)}{u_1(b) - u_2(b)}$, $u_1(b) \neq u_2(b)$. ……...(4.23ii)

Boundary value problem of the second kind we solve the initial value problem (4.21i) , (4.21ii) using the initial conditions

$u_1(a) = 0$, $u'_1(a) = \gamma_1$ $u_2(a) = 1$, $u'_2(a) = \gamma_1$ ………(4.23iii)

upto x=b. since the general solution (4.22) satisfies the boundary condition at x=b, we have $u'(b) = \gamma_2 = \lambda u'_1(b)+(1+\lambda) u'_2(b)$

or $\lambda = \dfrac{\gamma_2 - u'_2(b)}{u'_1(b) - u'_2(b)}$, $u'_1(b) \neq u'_2(b)$. ……(4.23iv)

Boundary value problem of the third kind we solve the initial value problem (4.21i) , (4.21ii) using the initial conditions $u_1(a) = 0$, $u'_1(a) = -\gamma_1 / a_1$

$u_2(a) = 1$, $u'_2(a) = (a_0 - \gamma_1)/a_1$ ……………(4.23v)

upto x=b. the general solution (4.22) satisfies the boundary condition at x=b, we get

$\gamma_2 = b_0 u(b) + b_1 u'(b) = b_0 [\lambda u_1(b)+(1-\lambda) u_2(b)] + b_1[\lambda u'_1(b)+(1-\lambda) u'_2(b)]$

$= \lambda [b_0 u_1(b)+ b_1 u'_1(b)] + (1-\lambda) [b_0 u_2(b)+ b_1 u'_2(b)]$

Or $\lambda = \dfrac{\gamma_2 - b_0 u_2(b) + b_1 u_3(b)}{[b_0 u_1(b) + b_1 u'_1(b)] - [b_0 u_2(b) + b_1 u'_2(b)]}$ ………..(4.23vi)

The results obtained are identical in both the approaches.

Example 1

Using the shooting method, solve the first boundary value problem

$u'' = u+1$, $0 < x < 1$

$u(0) = 0$, $u(1) = e-1$.

Use the Euler-Cauchy method with h=0.25 to solve the resulting system of first order initial problems. Compare the solution with the exact solution $u(x) = e^x - 1$.

Since boundary value problem in linear and non-homogeneous, we assume the solution in the form

$u(x)=u_0(x)+\mu_1\ u_1(x)+\mu_2\ u_2(x)$ ………(4.24i)

Where $u_0(x)$ satisfies the non-homogeneous differential equation and $u_1(x)$, $u_2(x)$ satisfy the homogeneous differential equation. Therefore, we have

$u''_0-u_0(x)=1$, $u''_1-u_1(x)=0$ and $u''_2-u_2(x)=1$

We assume the initial conditions as given in (4.12), that is

$u_0(0)=0$, $u'_0(0)=0$; $u_1(0)=1$, $u'_1(0)=0$; $u_2(0)=0$, $u'_2(0)=1$.

For the sake of illustration, we shall follow the steps in the method and obtain the analytical solution also.

Solving the differential equations and using the initial conditions, we obtain

$u_0(x)=(1/2)(e^{x}+e^{-x})-1$, $u_1(x)=(1/2)(e^{x}+e^{-x})$,

$u_2(x)=(1/2)(e^{x}-e^{-x})$ ……….(4.24ii)

Now from (4.24i) we obtain

$u(0)=u_0(0)+\mu_1\ u_1(0)+\mu_2\ u_2(0)$

$u(1)=u_0(1)+\mu_1\ u_1(1)+\mu_2\ u_2(1)$

$=u_0(1)+\mu_2\ u_2(1)=e-1.$ ……….(4.24iii)

Now from (4.24ii) we obtain

$u_0(1)=(1/2)(e-e^{-1})-1$ and $u_2(1)=(1/2)(e-e^{-1})$

Hence, from (4.24iii), we get

$$\mu_2=\frac{(e-1)-u_0(1)}{u_2(1)}=\frac{2(e-1)-(e+e^{-1}-2)}{(e-e^{-1})}$$

$$=\frac{e-e^{-1}}{e-e^{-1}}=1$$

Therefore, the analytical of the problem is

$u(x)=u_0(x)+\mu_1\ u_1(x)+\mu_2\ u_2(x)$

$=(1/2)(e^{x}+e^{-x})-1+(1/2)(e^{x}-e^{-x})=e^{x}-1.$

The illustrates the general of implementation of the method.

We now determine the solution of the initial value problems, using the Euler –Cauchy method with h=0.25.

We need to solve the following three, second order initial problems in 0< x <1.

$u''_0 - u_0(x) = 1$, $u_0(0)=0$, $u'_0(0)=0$.

$u''_1 - u_1(x) = 0$, $u_1(0)=1$, $u'_1(0)=0$.

$u''_2 - u_2(x) = 1$, $u_2(0)=0$, $u'_2(0)=1$. ……(4.24iv)

We write these problems as equivalent first order systems.

Denote $u_0(x)=Y_0(x)$, $u'_0(x)=Y'_0(x) = Z_0(x)$,

$u_1(x)=Y_1(x)$, $u'_1(x)=Y'_1(x) = Z_1(x)$,

$u_2(x)=Y_2(x)$, $u'_2(x)=Y'_2(x) = Z_2(x)$.

then , we can write (4.24iv) as the following systems

$$\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}' = \begin{pmatrix} Z_0 \\ 1+Y_0 \end{pmatrix}, \begin{pmatrix} Y_0(0) \\ Z_0(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix}' = \begin{pmatrix} Z_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} Y_1(0) \\ Z_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} Y_2 \\ Z_2 \end{pmatrix}' = \begin{pmatrix} Z_2 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Y_2(0) \\ Z_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Applying the Euler-Cauchy method

$$u_{j+1} = u_j + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h\, f(t_j, u_j) \;,\; k_2 = h\, f(t_j + h, u_j + k_1)$$

We obtain the following systems:

System 1 we have $f_1 = Z_0$ and $f_2 = 1 + Y_0$

$$\begin{pmatrix} Y_{0,j+1} \\ Z_{0,j+1} \end{pmatrix} = \begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \frac{h}{2}\begin{pmatrix} Z_{0,j} \\ 1+Y_{0,j} \end{pmatrix} + \frac{h}{2}\begin{pmatrix} Z_{0,j}+h(1+Y_{0,j}) \\ 1+Y_{0,j}+hZ_{0,j} \end{pmatrix}$$

$$= \begin{pmatrix} 1+(h^2/2) & h \\ h & 1+(h^2/2) \end{pmatrix}\begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

$$= \mathbf{B}(\mathrm{h}) \begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} h^2/2 \\ h \end{pmatrix}$$

Where $\mathbf{B}(\mathrm{h}) = \begin{pmatrix} 1+(h^2/2) & h \\ h & 1+(h^2/2) \end{pmatrix}$.

The initial conditions are $Y_{0,0}=0$, $Z_{0,0}=0$.

The system 2 and 3 can be immediately written as

$$\begin{pmatrix} Y_{1,j+1} \\ Z_{1,j+1} \end{pmatrix} = \mathbf{B}(\mathrm{h})\begin{pmatrix} Y_{1,j} \\ Z_{1,j} \end{pmatrix} \quad , Y_{1,j}=1 \quad , \quad Z_{1,0}=0 .$$

And $\begin{pmatrix} Y_{2,j+1} \\ Z_{2.j+1} \end{pmatrix} = \mathbf{B}(\mathrm{h})\begin{pmatrix} Y_{2,j} \\ Z_{2,j} \end{pmatrix}$, $Y_{2,0}=0$, $Z_{2,0}=1$.

Where **B**(h) is same as above.

Using h=0.25 . We obtain

$$\begin{pmatrix} Y_{0,j+1} \\ Z_{0,j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix}\begin{pmatrix} Y_{0,j} \\ Z_{0,j} \end{pmatrix} + \begin{pmatrix} 0.03125 \\ 0.25 \end{pmatrix}$$

With $Y_{0,0}=0$, $Z_{0,0}=0$ for j=0,1,2,3, we get

$u_0(0.25) \approx Y_{0,1} = 0.03125$ $\quad u'_0(0.25) \approx Z_{0,1} = 0.025$

$u_0(0.50) \approx Y_{0,2} = 0.12598$ $\quad u'_0(0.50) \approx Z_{0,2} = 0.51563$

$u_0(0.75) \approx Y_{0,3} = 0.29007$ $\quad u'_0(0.75) \approx Z_{0,3} = 0.81324$

$u_0(1.00) \approx Y \quad = 0.53369$ $\quad u'_0(1.00) \approx Z_{0,4} = 1.16117$

we have $\begin{pmatrix} Y_{1,j+1} \\ Z_{1,j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix}\begin{pmatrix} Y_{1,j} \\ Z_{1,j} \end{pmatrix}$, $Y_{1,j}=1$, $Z_{1,0}=0$.

$u_1(0.25) \approx Y_{1,1} = 1.03125$ $u'_1(0.25) \approx Z_{1,1} = 0.025$

$u_1(0.50) \approx Y_{1,2} = 1.12598$ $u'_1(0.50) \approx Z_{1,2} = 0.51563$

$u_1(0.75) \approx Y_{1,3} = 1.29007$ $u'_1(0.75) \approx Z_{1,3} = 0.81324$

$u_1(1.00) \approx Y_{1,4} = 1.53369$ $u'_1(1.00) \approx Z_{1,4} = 1.16117$

$$\begin{pmatrix} Y_{2,j+1} \\ Z_{2.j+1} \end{pmatrix} = \begin{pmatrix} 1.03125 & 0.25 \\ 0.25 & 1.03125 \end{pmatrix} \begin{pmatrix} Y_{2,j} \\ Z_{2,j} \end{pmatrix}, \quad Y_{2,0} = 0, \quad Z_{2,0} = 1.$$

$u_2(0.25) \approx Y_{2,1} = 0.025$ $u'_2(0.25) \approx Z_{2,1} = 1.03125$

$u_2(0.50) \approx Y_{2,2} = 0.51563$ $u'_2(0.50) \approx Z_{2,2} = 1.12598$

$u_2(0.75) \approx Y_{2,3} = 0.81324$ $u'_2(0.75) \approx Z_{2,3} = 1.129007$

$u_2(1.00) \approx Y_{2,4} = 1.16117$ $u'_2(1.00) \approx Z_{2,4} = 1.53369$

From (4.13) , we get

$$\mu_1 = 0, \ \mu_2 = \frac{\gamma_2 - u_0(1)}{u_2(1)} = \frac{e - 1 - 0.53369}{1.16117} = 1.02017$$

we obtain the solution of the boundary value problem from

$u(x) = u_0(x) + 1.02017\, u_2(x)$.

the solution at the model points are given in table 4.1 . The maximum absolute error which

Occurs at x=0.50 is given by

max.abs.error=0.00329

TABLE 1 SOLUTIONOFEXAMPLE 1

x_j	Exact: $u(x_j)$	u_j
0.25	0.28403	0.28629
0.50	0.64872	0.65201
0.75	1.11700	1.11971
1.00	1.71828	1.71828

More accurate results can be obtained by using smaller step length h.

Iterative Method For Eigen Values

Power method

Power method is used to determine numerically largest eigen value and corresponding eigen vector of a matrix A.

Let A be a n×n square matrix and let $\lambda_1, \lambda_2 .... \lambda_n$ be distinct eigen value of so that

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > ...|\lambda_n| \qquad (1)$$

Let $v_1, v_2, ... v_n$ be their corresponding eigen vectors

$$\therefore \quad Av_i = \lambda v_i, i = 1,2,3,......n \qquad (2)$$

This method is applicable only if the vectors $v_1, v_2, ... v_n$ are linearly independent. This may be true even if the eigen value $\lambda_1, \lambda_2 .... \lambda_n$ are not distinct.

These n vectors constitute a vector space of which these vectors from a basis.

Let Y_0 be any vector of this space.

Then $Y_0 = C_1 v_1 + C_2 v_2 + C_3 v_3 + ... + C_n v_n$

Where C_i's are constants (scalars).

Pre-multiplying by A, we get

$$Y_1 = AY_0 = C_1 Av_1 + C_2 Av_2 + C_3 Av_3 + ... + C_n Av_n$$

$$= C_1 \lambda_1 v_1 + C_2 \lambda_2 v_2 + C_3 \lambda_3 v_3 + ... + C_n \lambda_n v_n$$

Similarly $Y_2 = C_1 \lambda^2{}_1 v_1 + C_2 \lambda^2{}_2 v_2 + C_3 \lambda^2{}_3 v_3 + ... + C_n \lambda^2{}_n v_n$

Continuing this process

$$Y_r = AY_{r-1} = C_1 \lambda^r{}_1 v_1 + C_2 \lambda^r{}_2 v_2 + C_3 \lambda^r{}_3 v_3 + ... + C_n \lambda^r{}_n v_n$$

$$= \lambda^r{}_1 \left[C_1 v_1 + C_2 \left(\frac{\lambda_2}{\lambda_1}\right)^r v_2 + C_3 \left(\frac{\lambda_3}{\lambda_1}\right)^r v_3 + ... + C_n \left(\frac{\lambda_n}{\lambda_1}\right)^r v_n \right]$$

Similarly,

$$Y_{r+1} = A^{r+1}Y_0 \quad = \lambda^{r+1}{}_1\left[C_1 v_1 + C_2\left(\frac{\lambda_2}{\lambda_1}\right)^{r+1} v_2 + C_3\left(\frac{\lambda_3}{\lambda_1}\right)^{r+1} v_3 + \ldots. + C_n\left(\frac{\lambda_n}{\lambda_1}\right)^{r+1} v_n\right]$$

As $r \to \infty, \left(\frac{\lambda_i}{\lambda_1}\right)^r \to 0, i = 2,3,\ldots n$

In the limit as $r \to \infty$

$Y_r \to \lambda_1{}^r C_1 v_1$

$Y_{r+1} \to \lambda_1{}^{r+1} C_1 v_1$

$\therefore \lambda_i = \lim_{r \to \infty} \frac{\left(A^{r+1}Y_0\right)}{\left(A^r Y_0\right)}$, I =1, 2….n.

Where the suffix i denotes ith component of the vector.

To get the convergence quicker, we normalize the vector before multiplication by A.

Method: Let v_0 be an arbitrary vector and find

Example 1: Find the dominant eigen value of A=$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by power method and hence find the other eigen value also. Verify your results by any other matrix theory.

Solution

Let an initial arbitrary vector be $X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A X_1=$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = 4X_2$.

$$A\,X_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 7.5 \end{pmatrix} = 4\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = 4X_3$$

$$A\,X_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ 5 \end{pmatrix} = 5\begin{pmatrix} \frac{7}{15} \\ 1 \end{pmatrix} = bX_4$$

$$A\,X_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{7}{15} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{37}{15} \\ \frac{81}{15} \end{pmatrix} = \frac{81}{15}\begin{pmatrix} \frac{37}{81} \\ 1 \end{pmatrix} = \frac{81}{15}X_5$$

$$A\,X_5 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{37}{81} \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4568 \\ 5.3704 \end{pmatrix} = 5.3704\begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = 5.3704X_6$$

$$A\,X_6 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4575 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4575 \\ 5.3724 \end{pmatrix} = 5.3704\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 54.3724X_7$$

$$A\,X_7 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix} = 5.3723\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = 5.3723X_8$$

$$A\,X_8 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.4574 \\ 5.3723 \end{pmatrix} = 5.3723\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$$

Hence $\lambda_1 = 5.3723$ and eigenvector $X_1 = \begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$.

Since $\lambda_1 + \lambda_2 =$ Trace of A=1+4=5

Second eigen value= $\lambda_2 = -0.3723$

Characterstic equation is $\lambda^2 - (1+4)\lambda + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0$

I.e., $\lambda^2 - 5\lambda - 2 = 0 \therefore \lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2} = 5.3723, -0.3723.$

The values got by power method exactly coinside with the solution from analytical method.

Part B (5x6=30 Marks)

Possible Questions

1. Using boundary value problem to solve y''+y+1=0, $0\leq x \leq 1$, where y(0)=0, y(1)=0 with h=0.5 use finite difference method to determine the value of y(0.5).
2. Use power method to find the eigen values of A= $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$
3. Find the dominant eigen value and the corresponding eigen vector of A= $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
4. Solve the boundary value problem y''=y(x),y(0)=0,y(1)=1.
5. Write the derivative of Power method.
6. Write the Derivative of Characteristic value Problems
7. Solve the boundary value problem $\frac{d^2y}{dx^2}$ –y = 0 with y(0) = 0 and y(2) = 3.62686

Part C (1x10=10 Marks)

Possible Questions

1. Solve the boundary value problem y''(x)=y(x), y(0)=0, y(1)=1.752 by the shooting methodtake m_0=0.7, m_1=0.8
2. Derive the Shooting method.
3. Find the eigen values and eigen vectors of the matrix A = $\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$
4. Find the dominant eigen value of A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method and hence find the other eigen values.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Numerical Analysis **Subject Code: 17MMP103**

Class : I - M.Sc. Mathematics **Semester : I**

Unit V

Numerical Solution of Partial Differential Equations

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
------------------ method is used to solve the Laplace's equation.	Crank-Nicholson	Liebmann's iteration	Laplace	Bender -schmidt	Liebmann's iteration
------------method is used to solve the parabolic equation.	Crank-Nicholson	Liebmann's iteration	Laplace	Bender -schmidt	Crank-Nicholson
Explicit method is used to solve the ------- --------------------- equation.	one dimensional	wave	laplace	poisson	wave
The error in the diagonal formula is ------- -------times the error in the standard formula.	3	2	5	4	4
In ---------- method, the coefficient matrix is transformed into diagonal matrix	Gauss elimination	Gauss jordan	Gauss jacobi	Gauss seidal	Gauss jordan
We get the approximate solution from the ---------------.	Direct method	InDirect method	fast method	Bisection	InDirect method
------------- Method takes less time to solve a system of equations	Direct method	InDirect method	fast method	Bisection	Direct method
The iterative process continues till ------ ------- is secured.	convergency	divergency	oscillation	point	convergency

We get the approximate solution from the ---------------.	Direct method	InDirect method	fast method	Bisection	InDirect method
-------------- Method takes less time to solve a system of equations	Direct method	InDirect method	fast method	Bisection	Direct method
The iterative process continues till ------------- is secured.	convergency	divergency	oscillation	point	convergency
Liebmann's iteration process is used to solve laplace equation in ---------------------dimension.	one	two	three	zero	two
solving numerically the hyperbolic equation utt = c^2uxx, solution is provided by the boundary condition	u(0,t) = 0	u(I,t) = 0	u(x,0)=0	u(x,0) =f(x)_	u(0,t) = 0
The simplest form of the explicit formula to solve utt =a2uxx, can be got if we select I as----------	h/a	h/k	I/a	I/ca	I/a
An important and frequently occuring elliptic equation is --------------- equation.	laplace	parabolic	hyperbolic	poisson	laplace
In numerical methods, the boundary problems,are solved by using ----------------method.	finite difference	Euler	Milne's	Runge-	finite difference

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
Department of Mathematics

Subject: Numerical Analysis	**Subject Code: 17MMP103**	**L T P C**
Class : I – M.Sc. Mathematics	**Semester : I**	**4 0 0 4**

UNIT-V

Numerical Solution of Partial Differential Equations: Classification of Partial Differential Equation of the second order – Elliptic Equations. Parabolic equations: Explicit method – The Crank Nicolson difference method. Hyperbolic equations – solving wave equation by Explicit Formula.

TEXT BOOK

1. Gerald, C. F., and Wheatley. P. O., (2006). Applied Numerical Analysis, sixth edition, Dorling Kindersley (India) Pvt. Ltd. New Delhi.

REFERENCES

1. Jain. M. K., Iyengar. S. R. K. and R. K. Jain., (2012). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi .

2. Burden R. L., and Douglas Faires.J,(2007). Numerical Analysis, Seventh edition, P. W. S. Kent Publishing Company, Boston.

3. Sastry S.S., (2008). Introductory methods of Numerical Analysis, Fourth edition, Prentice Hall of India, New Delhi.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section, we will classify partial differential equations of the second order. The general linear partial differential equation of second order in two independent variables is of the form,

$$A\frac{\partial^2 u}{dx^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$$\text{or } Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \tag{12.22}$$

where A, B, C, D, E and F are in general, functions of x and y.

Eqn (12.22) is said to be

(i) **Elliptic** at a point (x, y) in the plane if $B^2 - 4\,AC < 0$

(ii) **Parabolic** if $B^2 - 4\,AC = 0$

(iii) **Hyperbolic** if $B^2 - 4\,AC > 0$.

Note: It is possible for an equation to be of more than one type depending on the values of the coefficients.

For example, the equation $yu_{xx} + u_{yy} = 0$ is *elliptic* if $y > 0$, *parabolic* if $y < 0$.

But here, we are concerned with constant coefficients only.

Example 12.1 Classify the following equations

(i) $3u_{xx} + u_{xy} - 4u_{yy} + 3u_y = 0$

(ii) $x^2 u_{xx} + (a^2 - y^2) u_{yy} = 0, -\infty < x < \infty, -a < y < a$.

(iii) $u_{xx} - 6u_{xy} + 9u_{yy} - 17u_y = 0$

Solution (i) Here, $A = 3$, $B = 1$, $C = -4$

$\therefore B^2 - 4AC = (1)^2 - 4(3)(-4) > 0$

$\therefore$ The given equation is hyperbolic.

(ii) Here, $A = x^2$, $B = 0$, $C = a^2 - y^2$

$\therefore B^2 - 4AC = -4x^2(a^2 - y^2) = 4x^2(a^2 - y^2)$

Now x^2 is always (+)ve for all $-\infty < x < \infty$

and $a^2 - y^2$ is (–)ve for all $-a < y < a$.

$\therefore B^2 - 4AC = 4(+\text{ve})(-\text{ve}) = (-)$ ve i.e. < 0

So, the given equation is elliptic.

(iii) Here, $A = 1$, $B = -6$, $C = 9$

$\therefore B^2 - 4AC = (-6)^2 - 4(1)(9) = 0$

So, the given equation is parabolic.

12.5 ELLIPTIC EQUATIONS

An important equation of the elliptic type is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ i.e. } u_{xx} + u_{yy} = 0 \qquad (12.23)$$

This equation is called *Laplace's equation*.

Replacing the derivatives by the corresponding difference expressions n Eqn (12.23), we get

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

Taking a square mesh and putting $h = k$, we get from above

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] \qquad (12.24)$$

that is, the value of u at any interior mesh point is the arithmetic mean of its values at the four neighbouring mesh points to the left, right, below and above. This is called the *standard five point formula* (*SFPF*).

Instead of Eqn (12.24), we may also use the formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\right] \tag{12.25}$$

which shows that the value of $u_{i,j}$ is the arithmetic mean of its values at the four neighbouring diagonal mesh points. This is called the *diagonal five point formula* [*DFPF*].

The SFPF and DFPF are represented in Figs. 12.3 and 12.4 below.

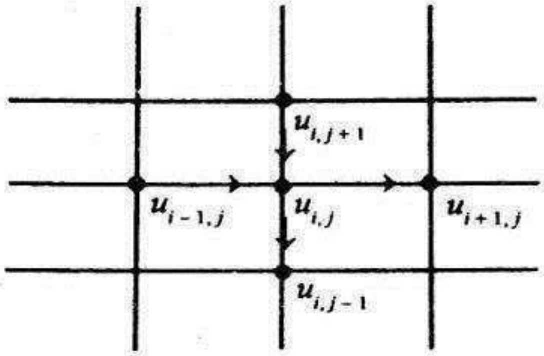

Fig. 12.3 *SFPF*

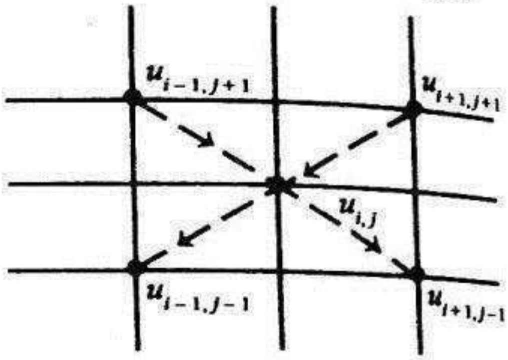

Fig. 12.4 *DFPF*

Note: The DFPF is valid since, we know that Laplace equation remains invariant when the coordinate axes are rotated through 45°. But the error in DFPF is four times the error in SFPF. Therefore, we prefer SFPF.

12.6 SOLUTION TO LAPLACE'S EQUATION BY LIEBMANN'S ITERATION PROCESS

Consider the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the given boundary conditions. For simplicity, we assume that function $u(x, y)$ is required over a rectangular region R with boundary C. Let R be divided into a network of small squares of side h. Let the values of $u(x, y)$ on boundary C be given by C_i and the interior mesh points and boundary points be as shown in Fig. 12.5

We know that the value of $u(x, y)$ satisfying the Laplacian equation can be replaced by either SFPF or DFPF. To start the iteration process, initially we find rough values at interior points and then improve it by iterative process, mostly using SFPF.

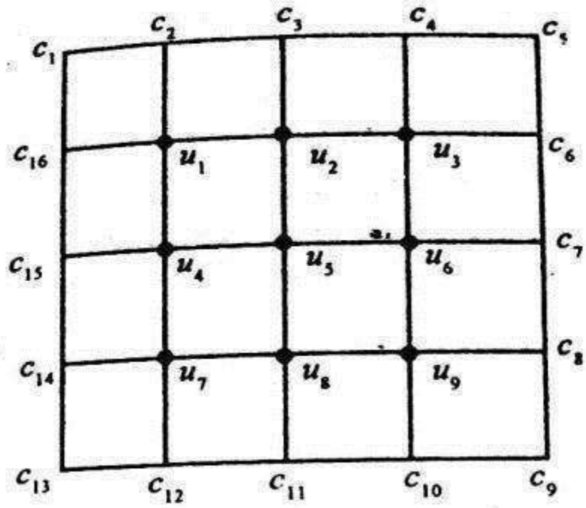

Fig. 12.5

We first find u_5, at the centre of the square by taking the average of four boundary values (SFPF).

$$\therefore u_5 = \frac{1}{4}(c_{15} + c_7 + c_3 + c_{11})$$

Next, we find the initial values at the centres of the four large inner squares using DFPF.

Thus, $$u_1 = \frac{1}{4}(u_5 + c_1 + c_3 + c_{15})$$

$$u_3 = \frac{1}{4}(u_5 + c_5 + c_3 + c_7)$$

$$u_7 = \frac{1}{4}(u_5 + c_{13} + c_{11} + c_{15})$$

$$u_9 = \frac{1}{4}(u_5 + c_9 + c_7 + c_{11})$$

The values at the remaining interior points are obtained by SFPF.

Thus, $$u_2 = \frac{1}{4}(u_1 + u_3 + c_3 + u_5)$$

$$u_4 = \frac{1}{4}(c_{15} + u_5 + u_1 + u_7)$$

$$u_6 = \frac{1}{4}\,(u_5 + c_7 + u_3 + u_9)$$

$$u_8 = \frac{1}{4}\,(u_7 + u_9 + u_5 + c_{11})$$

Now that we have got all the boundary values of u and rough values at every mesh (grid) point in the interior of the region R, we proceed with an iteration process to improve their accuracy. We start with u_1 and iterate it using the latest available values of the four adjacent points. Thus, we iterate all the mesh points systematically from left to right along successive rows. The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

Here, the superscript denotes iteration number. This is known as *Liebmann's iteration process.*

Example 12.2 Solve $u_{xx} + u_{yy} = 0$ in $0 \le x \le 4, 0 \le y \le 4$, given that $u(0, y) = 0$; $u(4, y) = 81.2y$;

$u(x, 0) = \dfrac{x^2}{2}$ and $u(x, 4) = x^2$. Take $h = k = 1$ and obtain the result correct to one decimal.

Solution Let us divide the given region R, i.e. $0 \le x \le 4, 0 \le y \le 4$ into 16 square meshes. The numerical values of the boundary, using the given analytical expression are calculated and exhibited in Fig. 12.6.

Let $u_1, u_2, u_3, \ldots, u_9$ be the values of u at the interior mesh points. Now the initial values of u's are calculated either by SFPF or DFPF as given below:

$$u_5^{(0)} = \frac{0 + 12 + 4 + 2}{4} = 4.5 \text{ (SFPF)}$$

$$u_1^{(0)} = \frac{0 + 4 + 0 + 4.5}{4} = 2.125 \text{ (DFPF)}$$

$$u_3^{(0)} = \frac{4.5 + 16 + 4 + 12}{4} = 9.125 \text{ (DFPF)}$$

$$u_7^{(0)} = \frac{0 + 2 + 0 + 4.5}{4} = 1.625 \text{ (DFPF)}$$

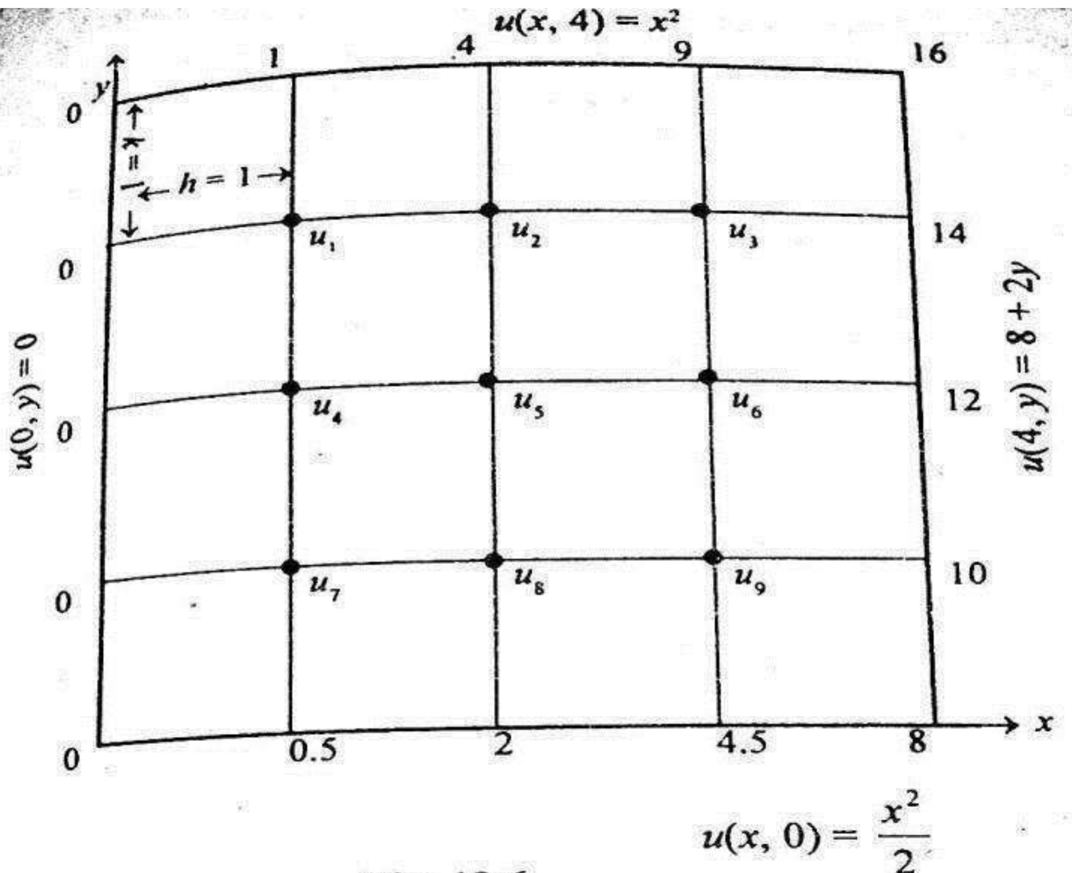

Fig. 12.6

$$u_9^{(0)} = \frac{4.5 + 8 + 2 + 12}{4} = 6.625 \text{ (DFPF)}$$

$$u_2^{(0)} = \frac{4 + 4.5 + 2.125 + 9.125}{4} = 4.9375 \text{ (SFPF)}$$

$$u_4^{(0)} = \frac{0 + 4.5 + 2.125 + 1.625}{4} = 2.0625 \text{ (SFPF)}$$

$$u_6^{(0)} = \frac{4.5 + 12 + 9.125 + 6.625}{4} = 8.0625 \text{ (SFPF)}$$

$$u_8^{(0)} = \frac{1.625 + 6.625 + 4.5 + 2}{4} = 3.6875 \text{ (SFPF)}$$

ow we use Liebmann's iteration formula, i.e.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n+1)}\right]$$

to improve above results. When we use this formula at points $u_1, u_2, \ldots, u_9$ we get the following equations for iteration:

$$u_1^{(n+1)} = \frac{1}{4}\left[0 + u_2^{(n)} + u_4^{(n)} + 1\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_3^{(n)} + u_5^{(n)} + 4\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[u_2^{(n+1)} + 14 + u_6^{(n)} + 9\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[0 + u_5^{(n)} + u_7^{(n)} + u_1^{(n+1)}\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_6^{(n)} + u_8^{(n)} + u_2^{(n+1)}\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[u_5^{(n+1)} + 12 + u_9^{(n)} + u_3^{(n+1)}\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + u_8^{(n)} + 0.5 + u_4^{(n+1)}\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[u_7^{(n+1)} + u_9^{(n)} + 2 + u_5^{(n+1)}\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[u_8^{(n+1)} + 10 + 4.5 + u_6^{(n+1)}\right]$$

First iteration $(n = 0)$

$$u_1^{(1)} = \frac{1}{4}\left[0 + u_2^{(0)} + u_4^{(0)} + 1\right]$$

$$= \frac{1}{4}\left[0 + 4.9375 + 2.0625 + 1\right] = 2$$

$$u_2^{(1)} = \frac{1}{4}\left[u_1^{(1)} + u_3^{(0)} + u_5^{(0)} + 4\right]$$

$$= \frac{1}{4}\left[2 + 9.125 + 4.5 + 4\right] = 4.90625$$

$$u_3^{(1)} = \frac{1}{4}\left[u_2^{(1)} + 14 + u_6^{(0)} + 9\right]$$

$$= \frac{1}{4}[4.90625 + 14 + 8.0625 + 9] = 8.9921875$$

$$u_4^{(1)} = \frac{1}{4}[0 + u_5^{(0)} + u_7^{(0)} + u_1^{(1)}]$$

$$= \frac{1}{4}[0 + 4.5 + 1.625 + 2] = 2.03125$$

$$u_5^{(1)} = \frac{1}{4}[u_4^{(1)} + u_6^{(0)} + u_8^{(0)} + u_2^{(1)}]$$

$$= \frac{1}{4}[2.03125 + 8.0625 + 3.6875 + 4.90625] = 4.671875$$

$$u_6^{(1)} = \frac{1}{4}[u_5^{(1)} + 12 + u_9^{(0)} + u_3^{(1)}]$$

$$= \frac{1}{4}[4.671875 + 12 + 6.625 + 8.9921875] = 8.0722656$$

$$u_7^{(1)} = \frac{1}{4}[0 + u_8^{(0)} + 0.5 + u_4^{(1)}]$$

$$= \frac{1}{4}[0 + 3.6875 + 0.5 + 2.03125] = 1.5546875$$

$$u_8^{(1)} = \frac{1}{4}[u_7^{(1)} + u_9^{(0)} + 2 + u_5^{(1)}]$$

$$= \frac{1}{4}[1.5546875 + 6.625 + 2 + 4.671875] = 3.7128906$$

$$u_9^{(1)} = \frac{1}{4}[u_8^{(1)} + 10 + 4.5 + u_6^{(1)}]$$

$$= \frac{1}{4}[3.7128906 + 10 + 4.5 + 8.0722656] = 6.5712891$$

Second iteration $(n = 1)$

$$u_1^{(2)} = \frac{1}{4}[0 + u_2^{(1)} + u_4^{(1)} + 1]$$

$$= \frac{1}{4}[0 + 4.90625 + 2.03125 + 1] = 1.984375$$

$$u_2^{(2)} = \frac{1}{4}[u_1^{(2)} + u_3^{(1)} + u_5^{(1)} + 4]$$

$$= \frac{1}{4}[1.984375 + 8.9921875 + 4.671875 + 4] = 4.9121094$$

$$u_3^{(2)} = \frac{1}{4}[u_2^{(2)} + 14 + u_6^{(1)} + 9]$$

$$= \frac{1}{4}[23 + 4.9121094 + 8.0722656] = 8.9960937$$

$$u_4^{(2)} = \frac{1}{4}[0 + u_5^{(1)} + u_7^{(1)} + u_1^{(2)}]$$

$$= \frac{1}{4}[0 + 4.671875 + 1.5546875 + 1.984375] = 2.0527344$$

$$u_5^{(2)} = \frac{1}{4}[u_4^{(2)} + u_6^{(1)} + u_8^{(1)} + u_2^{(2)}]$$

$$= \frac{1}{4}[2.0527344 + 8.0722656 + 3.7128906 + 4.9121094]$$

$$= 4.6875$$

$$u_6^{(2)} = \frac{1}{4}[u_5^{(2)} + 12 + u_9^{(1)} + u_3^{(2)}]$$

$$= \frac{1}{4}[4.6875 + 12 + 6.5712891 + 8.9960937] = 8.063720$$

$$u_7^{(2)} = \frac{1}{4}[0 + u_8^{(1)} + 0.5 + u_4^{(2)}]$$

$$= \frac{1}{4}[0.5 + 3.7128906 + 2.0527344] = 1.5664063$$

$$u_8^{(2)} = \frac{1}{4}[u_7^{(2)} + u_9^{(1)} + 2 + u_5^{(2)}]$$

$$= \frac{1}{4}[1.5664063 + 6.5712891 + 2 + 4.6875] = 3.7062988$$

$$u_9^{(2)} = \frac{1}{4}\,[u_8^{(2)} + 10 + 4.5 + u_6^{(2)}]$$

$$= \frac{1}{4}\,[3.7062988 + 14.5 + 8.0637207] = 6.5675049$$

Third iteration $(n = 2)$

$$u_1^{(3)} = \frac{1}{4}\,[0 + u_2^{(2)} + u_4^{(2)} + 1]$$

$$= \frac{1}{4}\,[0 + 4.9121094 + 2.0527344 + 1] = 1.991211$$

$$u_2^{(3)} = \frac{1}{4}\,[u_1^{(3)} + u_3^{(2)} + u_5^{(2)} + 4]$$

$$= \frac{1}{4}\,[1.991211 + 8.9960931 + 4.6815 + 4] = 4.9181012$$

$$u_3^{(3)} = \frac{1}{4}\,[u_2^{(3)} + 14 + u_6^{(2)} + 9]$$

$$= \frac{1}{4}\,[4.9187012 + 14 + 8.0637207 + 9] = 8.9956055$$

$$u_4^{(3)} = \frac{1}{4}\,[0 + u_5^{(2)} + u_7^{(2)} + u_1^{(3)}]$$

$$= \frac{1}{4}\,[0 + 4.6875 + 1.5664063 + 1.991211] = 2.0612793$$

$$u_5^{(3)} = \frac{1}{4}\,[u_4^{(3)} + u_6^{(2)} + u_8^{(2)} + u_2^{(3)}]$$

$$= \frac{1}{4}\,[2.0612793 + 8.0637207 + 3.7062988 + 4.9187012]$$

$$= 4.6875$$

$$u_6^{(3)} = \frac{1}{4}\,[u_5^{(3)} + 12 + u_9^{(2)} + u_3^{(3)}]$$

$$= \frac{1}{4}\,[4.6875 + 12 + 6.5675049 + 8.9956055] = 8.0626526$$

$$u_7^{(3)} = \frac{1}{4}\,[0 + u_8^{(2)} + 0.5 + u_4^{(3)}]$$

$$= \frac{1}{4}\,[0 + 3.7062988 + 0.5 + 2.0612793] = 1.5668945$$

$$u_8^{(3)} = \frac{1}{4}\,[u_7^{(3)} + u_9^{(2)} + 2 + u_5^{(3)}]$$

$$= \frac{1}{4}\,[1.5668945 + 6.5675049 + 2 + 4.6875] = 3.7054749$$

$$u_9^{(3)} = \frac{1}{4}\,[u_8^{(3)} + 10 + 4.5 + u_6^{(3)}]$$

$$= \frac{1}{4}\,[3.7054749 + 10 + 4.5 + 8.0626526] = 6.56707319$$

$\therefore$ $u_1 \approx 1.99$; $u_2 \approx 4.91$; $u_3 \approx 8.99$; $u_4 \approx 2.06$
$u_5 \approx 4.69$; $u_6 \approx 8.06$; $u_7 \approx 1.57$; $u_8 \approx 3.71$;

and $u_9 \approx 6.57$

12.7 POISSON'S EQUATION – ITS SOLUTION

The partial differential equation

$$\nabla^2 u = f(x, y) \text{ or } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ or } u_{xx} + u_{yy} = f(x, y) \qquad (12.26)$$

where $f(x, y)$ is a given function of x and y is called the Poisson's equation. It is of elliptic type.

To solve the Poisson equation numerically, the derivatives in Eqn (12.26) are replaced by difference expressions at the points $x = ih$, $y = jk$ (here, $h = k$). Then we get

$$\frac{1}{h^2}\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] + \frac{1}{h^2}\,[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = f(ih, jh)$$

or $$u_{i-1,j} + u_{i+1,j} + u_{i,j-1}, u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \qquad (12.27)$$

Applying the above formula at each mesh point, we get similar equations in the pivotal values i, j. These equations can be solved by iteration techniques.

Example 12.5 Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0, y = 0, x = 3, y = 3$ with $u = 0$ on the boundary and mesh length = 1.

Solution

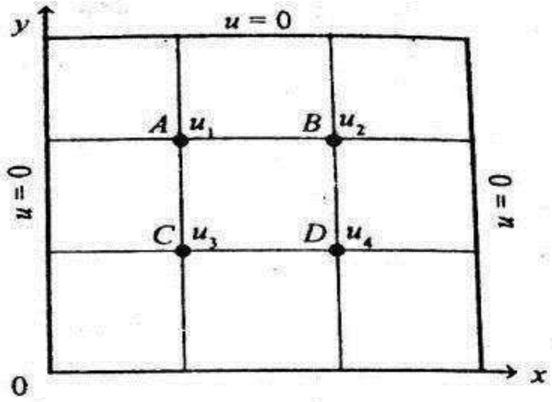

Fig. 12.9

The given differential equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10) \qquad \text{(i)}$$

Let u_1, u_2, u_3, u_4 be the values of u at the four mesh points A, B, C and D as shown in Fig 12.9. Replacing LHS of Eqn (1) by finite difference expressions and putting $x = ih, y = jh$ on the RHS of it, we get

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \qquad \text{(ii)}$$

At A, i.e. for u_1, by putting $i = 1, j = 2$ in Eqn (ii), we get

$$0 + u_2 + u_3 + 0 + - 4u_1 = -10\ (1 + 4 + 10)$$

$$\text{or } u_1 = \frac{1}{4}(u_2 + u_3 + 150) \qquad \text{(iii)}$$

At B, i.e. for u_2, by putting $i = 2, j = 2$ in Eqn (ii), we get

$$u_2 = \frac{1}{4}\ (u_1 + u_4 + 180) \qquad \text{(iv)}$$

At C, i.e. for u_3, by putting $i = 1, j = 1$ in Eqn (ii), we get

$$u_3 = \frac{1}{4}\ (u_1 + u_4 + 120) \qquad \text{(v)}$$

At D, i.e. for u_4, by putting $i = 2, j = 1$ in Eqn (ii), we get

$$u_2 = \frac{1}{4}\ (u_2 + u_3 + 150) \qquad \text{(iv)}$$

From Eqns (iii) and (vi) we can see that $u_4 = u_1$. So it is enough if we find u_1, u_2, and u_3.

Moreover, with $u_4 = u_1$, Eqns (iii)–(v) reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$$

$$u_2 = \frac{1}{2}(u_1 + 90)$$

$$u_3 = \frac{1}{2}(u_1 + 60)$$

Now let us solve these equations by Gauss–Seidal iteration method.

First iteration: We start the iteration by putting $u_2 = 0$, $u_3 = 0$

$$u_1^{(1)} = \frac{150}{4} = 37.5$$

$$u_2^{(1)} = \frac{1}{2}(37.5 + 90) = 63.75$$

$$u_3^{(1)} = \frac{1}{2}(37.5 + 60) = 48.75$$

Second iteration:

$$u_1^{(2)} = \frac{1}{4}(63.75 + 48.75 + 150) = 65.625$$

$$u_2^{(2)} = \frac{1}{2}(65.625 + 90) = 77.8125$$

$$u_3^{(2)} = \frac{1}{2}(65.625 + 60) = 62.8125$$

Third iteration:

$$u_1^{(3)} = \frac{1}{4}(77.8125 + 62.8125 + 150) = 72.65625$$

$$u_2^{(3)} = \frac{1}{2}(72.65625 + 90) = 81.328125$$

$$u_3^{(3)} = \frac{1}{2}(72.65625 + 60) = 66.32815$$

Fourth iteration:

$$u_1^{(4)} = \frac{1}{4}(81.328125 + 66.328125 + 150) = 74.414063$$

$$u_2^{(4)} = \frac{1}{2}(74.414063 + 90) = 82.207031$$

$$u_3^{(4)} = \frac{1}{2}(74.414063 + 60) = 67.207031$$

Fifth iteration:

$$u_1^{(5)} = \frac{1}{4}(82.207031 + 67.207031 + 150) = 74.853516$$

$$u_2^{(5)} = \frac{1}{2}(74.853516 + 90) = 82.426758$$

$$u_3^{(5)} = \frac{1}{2}(74.853516 + 60) = 67.426758$$

Sixth iteration:

$$u_1^{(6)} = \frac{1}{4}(82.426758 + 67.426758 + 150) = 74.963379$$

$$u_2^{(6)} = \frac{1}{2}(74.963379 + 90) = 82.481689$$

$$u_3^{(6)} = \frac{1}{2}(74.963379 + 60) = 67.481689$$

Since these values are the same as those of fifth iteration, we have, $u_1 = 75$, $u_2 = 82.5$ and $u_3 = 67.5$, $\therefore\ u_4 = 75$.

12.8 PARABOLIC EQUATIONS

The one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \text{ where } \alpha^2 = \frac{k}{c\rho} \qquad (12.28)$$

(c is the specific heat of the material, ρ is the density and k is the thermal conductivity) is a well-known example of a parabolic equation. The solution to it is a function of x and t i.e. $u(x, t)$. It is defined for values of x from $x = 0$ to $x = l$, and for values of time t from $t = 0$ to $t = \infty$. The solution is not defined in a closed form (as in the case of elliptic equations) but propagates in an open-ended region from initial values satisfying the prescribed boundary conditions (see Fig. 12.18).

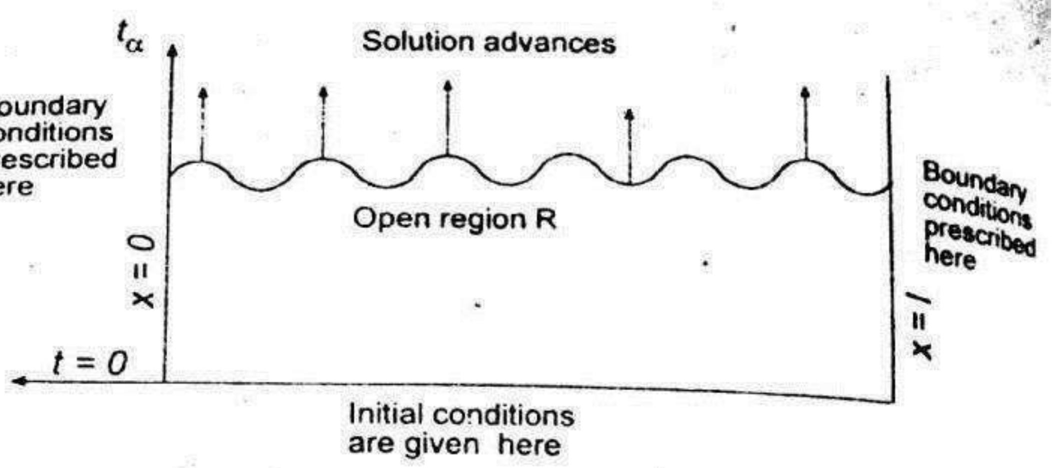

Fig. 12.18

12.8.1 Bender–Schmidt method

Solution to one dimensional heat equations can be obtained using Bender–Schmidt method. Here, consider the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \alpha^2 = \frac{k}{c\rho} \tag{12.29}$$

This can be wirtten as

$$u_{xx} = a u_t, \text{ where } a = \frac{1}{\alpha^2} \tag{12.30}$$

Now our aim is to solve Eqn (12.30) subject to the boundary conditions

$$u(0, t) = T_0 \tag{12.31}$$

$$u(l, t) = T_l \tag{12.32}$$

and the initial conditions

$$u(x, 0) = f(x) \tag{12.33}$$

by finite differences method. We select a spacing h for the variable x and a spacing k for the time variable t. We know that

$$u_{xx} = \frac{1}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

and $\quad u_t = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right]$

Substituting the above in Eqn (12.30) it becomes

$$\frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] = a\frac{1}{k}[u_{i,j+1} - u_{i,j}]$$

or $$u_{i,j+1} - u_{i,j} = \lambda [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

where $$\lambda = \frac{k}{h^2}a.$$

or $$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}) \quad (12.34)$$

The boundary conditions (12.31) and (12.32) can be put in difference form as

$$u_{0,j} = T_0 \quad (12.35a)$$

and $$u_{n,j} = T_l \quad (12.35b)$$

where $j = 1, 2, \ldots$ [here $nh = l$] and the initial condition (12.33) as

$$u_{i,0} = f(ih), \; i = 1, 2, \ldots \quad (12.36)$$

Eqn (12.34) gives the value of u at $x = ih$ at time t_{j+k} in terms of values of u at $x = (i-1)h$, ih and $(i+1)h$ at a time t_j.

$\because u(x, 0) = f(x)$, u is known at $t = 0$

Therefore, the recurrence relation (12.34) allows the evaluation of u at each pivotal point x_i at any t_j.

If h, k are chosen such that the coefficient of $u_{i,j}$ vanishes,

i.e. $1 - 2\lambda = 0$ or $\lambda = \frac{1}{2}$, then

Eqn (12.34) becomes

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}] \quad (12.37)$$

and $$k = \frac{a}{2}h^2 \quad (12.38)$$

Eqn (12.37) implies that the value of u at $x = x_i$ at time t_{j+1} is equal to the average of the values of u at the surrounding points x_{i-1} and x_{i+1} at the previous time t_j. Eqn (12.37) is called *Bender–Schmidt recurrence equation*.

Example 12.6 Solve

$$\frac{\partial u}{\partial t} = \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2}$$

with the conditions $u(0, t) = 0$, $u(4, t) = 0$, $u(x, 0) = x(4-x)$ taking $h = 1$ and employing Bender–Schmidt recurrence equation. Continue the s[illegible] through ten time steps.

Solution The general equation is $u_{xx} = au_t$

Here, $a = 2, h = 1$ $\quad\therefore\quad$ $\lambda = \dfrac{k}{h^2}$ $\qquad$ $a = \dfrac{k}{2}$

Therefore, $\lambda = \dfrac{1}{2}$ and k should be equal to 1. Now using Bender–Schmidt recurrence relation, the values of u_{ij} are tabulated below (Fig. 12.19)

direction of x →

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0
6	0	0.375	0.5	0.375	0
7	0	0.25	0.375	0.25	0
8	0	0.1875	0.25	0.1875	0
9	0	0.125	0.1875	0.125	0
10	0	0.094	0.125	0.094	0

direction of t ↓

Fig. 12.19

Explanation: Range for x is 0 [illegible]

$x = ih \quad\Rightarrow\quad x_i = ih = i$ (h = 1)

$t = jk \quad\Rightarrow\quad t_j = jk = j$ (k = 1)

Given $u(x, 0) = x(4 - x)$ or $u(i, 0) = i(4 - i)$

Now for $0 \le x \le 4$, i.e. $0 \le i \le 4$, we have $u(i, 0) = 0, 3, 4, 3, 0$. These are filled in the first row.

Given $u(0, t) = 0 \;\forall\; t$, i.e. $u(0, j) = 0 \;\forall\; j$. Hence the entries in the first column are zero.

Also, $u(4, t) = 0 \;\forall\; t$, i.e. $u(4, j) = 0 \;\forall\; j$. $\therefore$ The entries in the last column are zero.

The Bender–Schmidt's recurrence relation is

$$u_{i,j+1} = \frac{1}{2}\,[u_{i+1,j} + u_{i-1,j}] \qquad \text{(i)}$$

Putting $j = 0$ in Eqn (i), we get

$$u_{i,1} = \frac{1}{2}\,[u_{i+1,0} + u_{i-1,0}] \qquad \text{(ii)}$$

Putting $i = 1$ in Eqn (ii), we get

$$u_{1,1} = \frac{1}{2}\,[u_{2,0} + u_{0,0}] = \frac{1}{2}\,[4 + 0] = 2$$

Putting $i = 2$ in Eqn (ii), we get

$$u_{2,1} = \frac{1}{2}\,[u_{3,0} + u_{1,0}] = \frac{1}{2}\,[3 + 3] = 3$$

Putting $i = 3$ in Eqn (ii), we get

$$u_{3,1} = \frac{1}{2}\,[u_{4,0} + u_{2,0}] = \frac{1}{2}\,[0 + 4] = 2$$

Thus the second row is filled. Similarly, putting $j = 1, 2, 3, 4, 5, 6, 7, 8, 9,$ the other rows are filled.

12.8.2 Crank–Nicholson method

In this section, we will derive *Crank–Nicholson difference method* to solve parabolic equations.

$$\text{Let } u_{xx} = a u_t \qquad (12.39)$$

be the partial differential equation to be solved subject to the conditions

$$u(0, t) = T_0 \qquad (12.40)$$

$$u(l, t) = T_l \qquad (12.41)$$

and

$$u(x, 0) = f(x) \qquad (12.42)$$

We know that at point $u_{i,j}$, the finite difference approximation for u_{xx} is

$$u_{xx} = \frac{1}{h^2}\,\{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} \qquad (12.43)$$

At point $u_{i,j+1}$ the finite difference approximation for u_{xx} is

$$u_{xx} = \frac{1}{h^2}\,\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\} \qquad (12.44)$$

Average of Eqns (12.43) and (12.44) is

$$u_{xx} = \frac{1}{2h^2}\,\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} \qquad (12.45)$$

For u_t, the forward difference approximation is

$$u_t = \frac{1}{k}\{u_{i,j+1} - u_{i,j}\} \qquad (12.46)$$

Substituting Eqns (12.45) and (12.46) in Eqn (12.39), we get (after simplification)

$$\frac{\lambda}{2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\} = \{u_{i,j+1} - u_{i,j}\}$$

(where $\lambda = \frac{k}{h^2}a$)

or
$$\frac{\lambda}{2}u_{i+1,j+1} - (\lambda+1)u_{i,j+1} + \frac{\lambda}{2}u_{i-1,j+1}$$
$$= -\frac{\lambda}{2}u_{i+1,j} + (\lambda-1)u_{i,j} - \frac{\lambda}{2}u_{i-1,j}$$

or
$$\lambda\{u_{i+1,j+1} + u_{i-1,j+1}\} - 2(\lambda+1)u_{i,j+1}$$
$$= 2(\lambda-1)u_{i,j} - \lambda\{u_{i+1,j} + u_{i-1,j}\} \qquad (12.47)$$

Eqn (12.47) is called *Crank–Nicholson difference scheme or method.*

Note:(i) Choosing λ is very important. Very often a good choice of λ is $\lambda = 1$. In such a case, the Crank–Nicholson scheme becomes

$$u_{i,j+1} = \frac{1}{4}\{u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\} \qquad (12.48)$$

Subject to $k = ah^2$

(ii) The Crank–Nicholson's formula is convergent for all values of λ.

Example 12.8 Using Crank–Nicholson's method, solve $u_{xx} = 16u_t$, $0 < x < 1$, $t > 0$, given $u(x, 0) = 0$, $u(0, t) = 0$, $u(1, t) = 50t$.

Compute u for two steps in t direction taking $h = \frac{1}{4}$

Solution Here, $a = 16$, $h = \frac{1}{4}$.

$$\therefore \qquad k = ah^2 = 16\left(\frac{1}{16}\right) = 1.$$

The Crank–Nicholson scheme is given by

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right] \quad \text{(i)}$$

x increasing →

$j \backslash i$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	50
2	0	u_3	u_4	u_5	100

Fig. 12.21

Applying Eqn (i) at the mesh points u_1, u_2, u_3, we get

$$u_1 = \frac{1}{4}(0+0+0+u_2) = \frac{1}{4}u_2 \quad \text{(ii)}$$

$$u_2 = \frac{1}{4}(0+0+u_1+u_3) = \frac{1}{4}(u_1+u_3) \quad \text{(iii)}$$

$$u_3 = \frac{1}{4}(0+0+u_2+50) = \frac{1}{4}(u_2+50) \quad \text{(iv)}$$

Substituting (iv) and (ii) in (iii), we get

$$u_2 = \frac{1}{4}\left[\frac{1}{4}u_2 + \frac{1}{4}(u_2+50)\right]$$

or $$16u_2 = 2u_2 + 50 \therefore u_2 = 3.5714$$

$u_1 = 0.89285$; $u_3 = 13.39285$

Applying Eqn (i) again at the mesh points u_4, u_5, u_6, we get

$$u_4 = \frac{1}{4}u_5 \quad \text{(v)}$$

$$u_5 = \frac{1}{4}(u_4 + u_6) \quad \text{(vi)}$$

$$u_6 = \frac{1}{4}(u_3 + 100) \quad \text{(vii)}$$

On Solving, we get $u_4 = 1.7857$, $u_5 = 7.1429$ and $u_6 = 26.7857$

12.9 HYPERBOLIC EQUATIONS

Given below is a wave equation of one dimension,

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \text{ or } a^2 u_{xx} - u_{tt} = 0 \tag{12.49}$$

This is a hyperbolic equation.

We know that $B^2 - 4AC = 0 - 4(a^2)(-1) = 4a^2 > 0$

Solution by Method of Finite Differences

Eqn (12.49) is subject to the conditions

$$u(0, t) = 0 \tag{12.50}$$

$$u(l, t) = 0 \tag{12.51}$$

$$\text{and } u(x, 0) = f(x) \tag{12.52}$$

$$u_t(x, 0) = 0 \tag{12.53}$$

Substituting

$$u_{xx} = \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$u_{tt} = \frac{1}{k^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]$$

in Eqn (12.49) where h and k are selected spacings for the variables x and t, we get

$$\frac{a}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} - u_{i,j-1}) = 0$$

or $$\lambda^2 a^2 (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - (u_{i,j+1} - 2u_{i,j} - u_{i,j-1}) = 0$$

where $$\lambda = \frac{k}{h}$$

or $$u_{i,j+1} = 2(1 - \lambda^2 a^2)u_{i,j} + \lambda^2 a^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} \tag{12.54}$$

The boundary conditions (12.50) and (12.51) can be put in the difference form as

$$u_{0,j} = 0 = u_{n,j}; j = 1, 2, 3 \ldots \tag{12.55}$$

(Here it means $nh = l$)

The initial condition (12.52) as

$$u_{i,0} = f(ih),\ i = 1, 2, \ldots \tag{12.56}$$

and (12.53) as

$\frac{1}{k}\ [u_{i,j+1} - u_{i,0}] = 0$ when $t = 0$, i.e. $j = 0$

$$\therefore u_{i,1} - u_{i,0} = 0 \qquad (12.57)$$

$$\text{i.e. } u_{i,1} = u_{i,0} = f(i, h) \qquad (12.58)$$

using Eqn (12.56).

Now Eqns (12.56) and (12.57) give the values of u on the first two rows $j = 0$ and $j = 1$. Putting $j = 1$ in Eqn (12.54), we get

$$u_{i,2} = 2(1 - \lambda^2 a^2)u_{i,1} + \lambda^2 a^2(u_{i+1,1} + u_{i-1,1}) - u_{i,0} \qquad (12.59)$$

R.H.S of Eqn (12.59) involves the values of u on the first two rows $j=0$ and $j = 1$. These are known from the initial conditions (12.56) and (12.58). Hence $u_{i,2}$ is found explicitly.

Knowing $u_{i,2}$, we can calculate $u_{i,3}$ by putting $j = 2$ in (12.54) and so on. Thus, Eqn (12.54) is an explicit scheme for the solution to the given equation.

Note:

(i) If $k < h$ solution (12.54) is convergent.

(ii) The coefficient of $u_{i,j}$ in Eqn(12.50) will be zero if $\lambda^2 = \frac{1}{a^2}$, i.e.

$$k = \frac{h}{a}.$$

Then Eqn (12.54) takes the form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

That is, the value of u at $x = x_i$ at a time $t = t_j + k$

= Value of u at $x = x_{i-1}$ at previous time $t = t_j$
+ Value of u at $x = x_{i+1}$ at previous time $t = t_j$
– Value of u at $x = x_i$ at time $t = t_j - k$.

Example 12.9 Solve $u_{tt} = 4u_{xx}$ with the boundary conditions $u(0, t) = 0 = u(4, t)$, $u_t(x, 0)$ and $u(x, 0) = x(4 - x)$.

Solution Given equation is $u_{tt} = 4u_{xx}$

Here, $a^2 = 4$, i.e. $a = 2$. Taking $h = 1$, we get $k = \frac{h}{a} = \frac{1}{2} = 0.5$.

From the initial conditions,

$$u(0, t) = 0 \Rightarrow u = 0 \text{ along entire line } x = 0$$
$$u(4, t) = 0 \Rightarrow u = 0 \text{ along entire line } x = 4$$

In difference form, these are

$u_{0,j} = 0$ and $u_{4,j} = 0$ for all j

Now $u(x, 0) = x(4 - x) \Rightarrow u(0, 0) = 0, u(1, 0) = 3, u(2, 0) = 4,$

$u(3, 0) = 3, \ u(4, 0) = 0$

In difference notation,

$u_{i,0} = u(i, 0) = i(4 - i)$ for different i

Putting $i = 0, 1, 2, 3, 4$ we get

$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3, u_{4,0} = 0$

Now the condition $u_t(x, 0) = 0$

$$\Rightarrow \frac{1}{k}[u_{i,j+1} - u_{i,j}] = 0 \text{ when } j = 0$$

$$\Rightarrow u_{i,1} = u_{i,0} \ \forall\, i$$

i.e. u on the first two rows are equal.

Now consider the recurrence relation

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

If we put $j = 1$, we get

$$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$$

Putting $i = 1, 2, 3 \ldots$ successively, we get

$$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 4 + 0 - 3 = 1$$
$$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 3 + 3 - 4 = 2$$
$$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0 + 4 - 3 = 1$$

that is, the third row is filled in. In a similar way we can fill in the remaining rows as shown in the following table

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
1	0	3	4	3	0
2	0	1	2	1	0
3	0	−1	−2	−1	0
4	0	−3	−4	−3	0

Part B (5x6=30 Marks)

Possible Questions

1. Solve the Laplace equation.

	100	100	100	50	
0					0
0					0
0					0
0					0
	0	0	0		

2. Write the Derivative for Crank Nicholson method.
3. Using Crank –Nicholson method Solve $u_{xx} = 16u_t$,0< x < 1,t >0,given u (x ,0) =0 u (0 ,t) = 0 , u (1 ,t)= 50t Compute u for two steps in t taking direction h = 1/ 4
4. Write the derivative of explicit formula to solve the wave equation.
5. Write the derivative of Elliptic equations.
6. Solve the Poisson equation $u_{xx} + u_{yy} = -10(x^2+y^2+10)$.
7. Write the derivative of Bender Schmidt method to solve parabolic equations

Part C (1x10=10 Marks)

Possible Questions

1. Explain the classification of Partial differential Equations.
2. Use Bender Schmidt recurrence relation to solve the equation $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ with the conditions $u(x, 0)=4x-x^2$, u(0, t)=u(4, t)=0. Assume h=0.1. find the values of u upto t=5.
3. Solve by Crank Nicholson method the equation $u_{xx}= u_t$ subject to u(x, 0)=0, u(0, t)=0 & u(1, t)=t for two time steps.
4. Solve numerically $4u_{xx} = u_{tt}$ with the boundary condition, u(0, t)=u(4, t)=0 and the initial conditions $u_t(x, 0)=0$ & $u(x, 0)=4x-x^2$, taking h=1(for 4 time steps).

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Numerical Analysis **Subject Code: 17MMP103**

Class : I - M.Sc. Mathematics **Semester : I**

Unit I

Solutions of Non Linear Equations

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The formula for Newton Raphson method is ------------------.	x_{n+1} = f (x_n) /f ’(xn)	x_{n+1} =xn+ f (x_n) /f ’(xn)	x_{n+1} =xn- f (x_n) /f ’(xn)	x_{n+1} =xn- f '(x_n) /f (xn)	x_{n+1} = f (x_n) /f ’(xn)
The order of convergence of Newton Raphson method is ----------	4	2	1	0	2
The approximate value of the root of f(x) given by the bisection method is ----	x_0 = a + b	x_0 = f(a) + f(b)	x_0 = (a + b)/ 2	x_0 = (f(a) + f(b))/2	x_0 = (a + b)/ 2
In Newton Raphson method, the error at any stage is proportional to the ------- of the error in the previous stage.	cubic	square	square root	0	square root
----------------- method is also called method of tangents.	Gauss Seidal	secant	bisection	Newton raphson	Newton raphson
If f (x) contains some functions like exponential, trigonometric, logarithmic etc., then f (x) is called -------------- equation.	Algebraic	transcendental	numerical	polynomial	transcendental
A polynomial in x of degree n is called an algebraic equation of degree n if -----	f (x) = 0	f (x) = 1	f (x) <1	f (x) >1	f (x) = 0
The Newton Rapson method fails if ---------------.	f' (x) = 0	f (x) = 0	f (x) =1	f (x) =1	f' (x) = 0

The initial approximation root is not given, choose two values of x 'a' and 'b', such that f(a) and f(b) are of opposite signs. If $\|f(a)\| < \|f(b)\|$ take ------	"a"	"b"	0	1	"a"
Graeffe's root squaring method has a great advantage over other methods in that it does not require prior information about the --------------------.	initial value	approximate value	final value	small	approximate value
If we choose the initial approximation x_0 ------------ to the root then we get the root of the equation very quickly	Close	far	average	small	Close
In Newton Rapson method when f '(x) is very large and the interval h will be --- then the root can be calculated in even less time.	Small	large	Average of the roots	normal root	Small
The order of convergence in --------------- method is two.	Bisection	Regula falsi	False position	Newton raphson	Newton raphson
If f (x_1) and f (a) are of opposite signs, then the actual roots of the equation f(x)=0 in False position method lie between ------------------.	'a' and 'b'	'b' and 'x_1'	'a' and 'x_1'	'x_1' and 'x_2'	'a' and 'x_1'
The iterative procedure is repeated till the --------- is found to the desired degree of accuracy.	initial value	approximate value	Root	linear	Root
If we equate a function f(x) to zero, then f(x) = 0 will reprasent an -----equation	Polynomial	transcendental	algebraic	linear	Polynomial
The equation $3x - \cos x - 1 = 0$ is known as ------------- equation.	Polynomial	transcendental	algebraic	linear	linear
$x^4 + 2x - 1 = 0$ is --------- equation.	Polynomial	transcendental	algebraic	linear	algebraic
$x\, e^x - 3x + 1.2 = 0$ is known as --------- equation	Polynomial	transcendental	algebraic	linear	transcendental
If f(a) and f(b) have opposite signs then the root of f(x) = 0 lies between ----.	0&a	a & b	b & 0	1 & -1	a & b

The error at any stage is proportional to the square of the ------------------.	error in the previous stage	error in the next stage	error in the last stage	error in the first stage	error in the previous stage
The convergence of iteration method is ----------------.	Zero	Polynomial	Quadratic	linear	Zero
The sufficient condition for convergence of iterations is ----------------.	$\lvert f'(x) \rvert = 1$	$\lvert f'(x) \rvert > 1$	$\lvert f'(x) \rvert < 1$	$\lvert f'(x) \rvert = 0$	$\lvert f'(x) \rvert < 1$
Solution of an equation f(x) = 0 means we have to find its -----------	Roots or Zeros	initial value	final value	approximate value	approximate value
In Newton Rapson method if -----------, then 'a' is taken as the initial approximation to the root.	$\lvert f(a) \rvert + \lvert f(b) \rvert$	$\lvert f(a) \rvert = \lvert f(b) \rvert$	$\lvert f(a) \rvert > \lvert f(b) \rvert$	$\lvert f(a) \rvert < \lvert f(b) \rvert$	$\lvert f(a) \rvert < \lvert f(b) \rvert$
In iteration method the given equation is taken in the form of -------------.	y = f(x)	x = f(x)	x = f(y)	y = f(y)	y = f(x)
The convergence of the sequence is not guaranteed always unless the choice of --------------- is properly chosen.	x_0	y_0	x_2	y_2	x_0
The sequence will converge rapidly in Iteration method, if $\lvert f'(x) \rvert$ is -------------.	Zero	Very large	Very small	one	Very small
If -----------, $\lvert x_n - a \rvert$ will become very great and the sequence will not converge.	$\lvert f'(x) \rvert = 1$	$\lvert f'(x) \rvert > 1$	$\lvert f'(x) \rvert < 1$	$\lvert f'(x) \rvert < 0$	$\lvert f'(x) \rvert > 1$
If p >=1 can be found out such that |e i+1| intersecion |e i | p. k where k is a positive constant for every i, then p is called the ----------------.	order of convergence	order of divergent	order of oscillation	none	order of convergence
If p = 1, then the convergence is -------------.	cubic	Quadratic	Linear	zero	Linear
If p = 2, then the convergence is -------------.	cubic	Quadratic	Linear	zero	Linear
In Iteration method if the convergence is linear then the convergence is of order ---	four	three	two	one	one
If the function f(x) is $e^x - 3x = 0$, then for Iteration method the variable x can be taken as ------------.	$e^x / 3$	$3 / e^x$	$e^x / 3x$	$e^{3x} / 3$	$e^x / 3$

The values of x which makes f(x) as ----------- are known as roots or zeros of the function f(x).	Zero	one	f'(x)	f''(x)	Zero
In Iteration method if the convergence is ---------- then the convergence is of order one.	cubic	Quadratic	Linear	zero	Linear
In Newton Raphson method the choice of --------------- is very important for the convergence	initial value	final value	intermediate value	approximate value	approximate value
If f(a) and f(b) are of opposite signs, a root of ----------- lies between 'a' and 'b'.	f(y) = 0	f(x) =1	f(x) = 0	f(y) = 1	f(x) = 0
If f(a) and f(b) are of opposite signs, a root of f(x) = 0 lies between 'a' and 'b'. This idea can be used to fix an ---------------------.	approximate root	actual root	intermediate root	acurate root	approximate root
The polynomial equations is given in he form of $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots + an = 0$, where ai's are ----------.	Zero	one	complex	real	2^m th power
Newton Rapson method is also called method of ------------------.	Gauss Seidal	Regula Falsi	Bisection	tangents	real
If f (–1) and f (–2) are of opposite signs, then the negative roots of the equation f(x)=0 in False position method lie between ------------------.	–1 and –2	–1 and 1	1 and –2	1 and 2	tangents
The ------------------- method fails if f '(x) = 0.	Bisection	False Position	Newton Rapson	Graffe's root squaring	–1 and –2
-------------- Formula can be used for unequal intervals.	Newton's forward	Newton's backward	Lagrange	stirling	Lagrange
By putting n = 3 in Newton cote's formula we get ----------- rule.	Simpson's 1/3 rule	Simpson's 3/8 rule	Trapezoidal rule	Simpson's rule	Simpson's 3/8 rule
The process of computing the value of a function outside the range is called -----	interpolation	extrapolation	triangularisation	integration	extrapolation
The process of computing the value of a function inside the range is called ------	interpolation	extrapolation	triangularisation	integration	interpolation

The difference value $y_2 - y_1$ in a Newton's forward difference table is denoted by	D y_0	Ñy_1	Dy_2	$Ñ^2$ y_0	Ñy_1
The order of error in Trapezoidal rule is -------------.	h	h^2	h^3	h^4	h^2
The order of error in Simpson's rule is -------------	h	h^2	h^3	h^4	h^4
__________ Formula can be used for interpolating the value of f(x) near the end of the tabular values.	Newton's forward	Newton's backward	Lagrange	stirling	Newton's backward
The technique of estimating the value of a function for any intermediate value is	interpolation	extrapolation	forward method	backward method	interpolation
The $(n+1)^{th}$ and higher differences of a polynomial of the nth degree are -----	zero	one	two	three	zero

Reg No ……………….

(17MMP103)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Karpagam University

Coimbatore - 21

DEPARTMENT OF MATHEMATICS

First Semester

I Internal Test – August'2017

Numerical Analysis

Date : .08.2017( ) **Time: 2 Hours**

Class: I M.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 x 1 = 20 Marks)

1.The order of convergence of Newton Raphson method is ---.

a)4 b)2 c)1 d)0

2.Bairstow's is used to find the --------------------roots of polynomial without using complex arithmetic.

a)real b)complex valued c)square root d)cubic

3.Numerical differentiation can be used only when the difference of some order are -----

a)zero b)one c)constant d)two

4.Relation between D and E is D = -------------------

a)E-1 b)E+1 c)E*1 d)1-E

5.The n^{th} divided difference of a polynomial of degree n are---.

a)zero b)constant c)linear d)non-linear

6.The order of error in Trapezoidal rule is ----------------------.

a)h b)h^3 c)h^2 d)h^4

7.The------------------ method is also called method of tangents.

a)Gauss seidal b)Secant c)Bisection d)Newton Raphson

8. If f (x) contains some functions like exponential, trigonometric, logarithmic etc.,then f (x) is called ------------- equation.

a)algebraic b)transcendental c)numerical d)polynomial

9. The iterative procedure is repeated till the --------- is found to the desired degree of accuracy.

a)initial value b)approximate value c)root d)linear

10.Example for iterative method --------------------.

a)Gauss elimination b)Gauss Jordan

c)Gauss seidal d)Newton's forward

11.The modification of Gauss-Elimination method is called ---- ---------------------.

a)Gauss Jordan b)Gauss seidal c)Gauss Jacobi d)crout

12.In the upper triangular coefficient matrix, all the elements below the diagonal are ------

a)positive b)non zero c)zero d)Negative

13.Gauss Jordan method is a------------------------.

a)direct method b)indirect method

c)iterative method d)convergent

14.In solving the system of linear equations the system can be written as --------------------

a)BX =A b)AX =A c)AX =B d)AB =X

15.-------------------------is also a self-correction method.

a)direct method b)iteration method

c)interpolation d)extrapolation

16.Euler method is used for solving -----------------differential equation.

a)firstorder b)fourth order c)third order d)second order

17.The error in Euler method is ----------------------------.

a)o(h^2) b)o(h^4) c)o(h^3) d)o(h^n)

18.The Euler Method of second category are called ------------

a)diagram b)graph c)line graph d)continuous line graph

19.Which is the condition to apply Gauss Seidal method to solve a system of equations.

a)1st row is dominant b)1st column is dominant

c)diagonally dominant d)leading diagonal

20.The method of iteration is applicable only if all equation must contain onecoefficient of different unknowns as ---------- than other coefficients.

a)small b)larger c)equal d)non zero

PART – B (3 x 2 = 6 Marks)

21. Write the formula for Simpson's rule.
22. Explain Gauss Jordan method.
23. What is the formula for Newton forward difference?

PART – C (3 x 8 = 24 Marks)

24. a) Find the positive root of $f(x) = x^3 - x - 1 = 0$ by Newton Raphson method correct to 5decimal places.

(OR)

b) Use Romberg's method to compute $I = \int_0^1 \frac{dx}{1+x}$ correct to 3 decimal places.

25. a) By the Method of Triangularization solve the following

system

$5x-2y+z = 4$; $7x+y-5z = 8$; $3x+7y+4z = 10$

(OR)

b) Solve the system of equations by Gauss Seidel method correct to 3 decimal places.

$8x-3y+2z=20$; $4x+11y-z=33$; $6x+3y-12z=35$

26. a) Solve the system of equations by Gauss elimination method.

$x+2y+z = 3$; $2x+3y+3z = 10$;$3x-y+2z = 13$

(OR)

b) Use Euler's method to solve the equation $y' = -y$ with the condition $y(0) = 1$.

Reg No ……………….

(17MMP103)

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
Coimbatore - 21
DEPARTMENT OF MATHEMATICS
First Semester
II Internal Test – September'2017
Numerical Analysis

Date : .09.2017( ) **Time: 2 Hours**
Class: I M.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 x 1 = 20 Marks)

Answer all the questions:

1. The Runge-Kutta method do not require prior calculation of the -----order derivatives.
a) middle b) lower c) higher d) zero

2. Which of these are multistep methods?
a) Milne's method b) Runge-Kutta method
c) Euler d) Modified euler

3. Taylor series and Euler methods are -----------------------then Runge- Kutta method.
a) fastly divergent b) slowly divergent
c) fastly convergent d) slowly convergent

4. In R - k method derivatives of higher order are ------------
a) constant b) zero c) variable d) non zero

5. The Euler Method and Modified Euler's Method are ---------
a) required b) not required
c) may be required d) must required

6. In numerical methods, the boundary problems, are solved by using -----------------method.
a) finite difference b) Euler c) Milne's d) Runge-Kutta

7. ---------------method is initial value problem methods.
a) Euler b) Shooting c) Milne's d) Runge-kutta

8. The -------- method is used to determine numerically largest eigen value and the corresponding eigen vector of matrix A.
a) Gauss Jordan b) choleskey c) power d) Gauss Seidel

9. ---------------------method is used to find the eigen values of a real symmetric matrix.
a) Jordan b) Seidel c) choleskey d) Jacobi

10. .A predictor formula is used to ---------------------the values of y at xi+1.
a) correct b) predict c) increase d) decrease

11. If the eigen values of A are 1,3,4 then the dominant eigen value of A is --------------.
a) 0 b) 4 c) 1 d) 3

12. If dy/dx is a function x alone, then fourth order Runge-Kutta method reduces to ------.
a) Trapezoidal rule b) Taylor series
c) Euler method d) Simpson method

13. If all the non zero terms involve only the dependent variable u and u' then the differential equation is called ---------
a) homogeneous b) non homogeneous
c) linear d) non linear

14. The error in Euler method is -----------------------------.
a) $o(h^2)$ b) $o(h^4)$ c) $o(h^3)$ d) $o(h^n)$

15. In solving equation $u_{tt} = a^2 u_{xx}$ by the Crank-Nicholson method, to simplify method we take $(dx)^2/a^2k$ as -----------------

a) 3 b) 1 c) 2 d) 0

16. ----------------------method is used to solve the Laplace's equation.

a) Crank-Nicholson difference b) Liebmann's iteration

c) Bender-Schmidt d) Laplace

17. Classify the equation $u_{xx}+2u_{xy}+4u_{yy}=0$ is ------------------.

a) hyperbolic b) parabolic c) poisson d) elliptic

18. The ---------------------scheme converges for all values of I.

a) Crank-Nicholson difference b) Liebmann's iteration

c) Bender-Schmidt d) Explicit scheme

19. The differential equation $xu_{xx}+u_{yy} = 0$ is said to elliptic if --------------------.

a) $x<0$ b) $x=0$ c) $x>0$ d) $x\neq 0$

20. The wave equation in one dimension is ----------------.

a) hyperbolic b) parabolic c) Poisson d) elliptic

PART – B (3 x 2 = 6 Marks)

Answer all the questions:

21. Write the formula of Modified and Improved Euler method.
22. Explain briefly about the Boundary value problem.
23. Write the classification of partial differential equation of the second order.

PART – C (3 x 8 = 24 Marks)

Answer all the questions:

24. a) Apply the fourth order Runge Kutta method to find y(0.1), y(0.2) given that $y'=x+y$, $y(0)=1$.

(OR)

b) Find y(2) if y(x) is the solution of $y'=\frac{1}{2}(x+y)$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$.

25. a) Derive the Shooting method.

(OR)

b) Use power method to find the eigen values of

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

26. a) Solve the Laplace equation.

	11.1	17.0	19.7	18.6	
0					21.9
0					21.0
0					17.0
0					9.0
	8.7	12.1	12.8		

(OR)

b) Solve numerically $4u_{xx} = u_{tt}$ with the boundary condition, $u(0, t)=u(4, t)=0$ and the initial conditions $u_t(x, 0)=0$ & $u(x, 0)=4x-x^2$, taking h=1(for 4 time steps).

Reg. No.................................

[15MMP103]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2015

First Semester

MATHEMATICS

NUMERICAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a) Find the real root of $x^3-2x-5=0$ using Newton's method and correct to four decimal places.

Or

b) Using trapezoidal rule, evaluate $\int_{-1}^{1}\frac{dx}{1+x^2}$ taking 8 intervals.

22. a) Solve
$$x+3y+3z=16$$
$$x+4y+3z=18$$
$$x+3y+4z=19$$
by Gauss elimination method.

Or

b) Solve the following equations by Gauss-Sidel method
$$4x+2y+z=14$$
$$x+5y-z=10$$
$$x+y+8z=20$$

23. a) Evaluate y (1.2) correct to 3 decimal places by modified Euler method given that $\frac{dy}{dx}=(y-x^2)^3$ y(1)=0 taking h=0.2

Or

b) Apply the fourth order Runge - Kutta method, to find an approximate value of y when x=0.2 given that $y^1=x+y$, y(0) = 1with h=0.2

24. a) Explain briefly boundary value problems with an example.

Or

b) Find the Eigen values of matrix A, $A=\begin{pmatrix}3 & -1 & 0\\ -2 & 4 & -3\\ 0 & -1 & 1\end{pmatrix}$

25. a) Explain types of partial differential equations.

Or

b) Explain the text : PARABOLIC EQUATIONS

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ in $0<x<t, t\geq 0$ given that u(x , 0) =20 , u (5 , t) = 100
Compute u for the time step with h=1 by Crank – Nicholson method.
