KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021.
SYLLABUS

Semester – I

17MMP104	**ORDINARY DIFFERENTIAL EQUATIONS**	**L T P C**
		4 0 0 4

Course Objectives: On successful completion of this course the learner gains knowledge about Second order linear equation, Legendre equation and Bessel equations etc., which provides the essential motivation in applied mathematics.

Course Outcome: To be familiar with formulation and solutions of ordinary differential equations and get exposed to physical problems with applications.

UNIT I
Second order linear equations with ordinary points – Legendre equation and Legendre polynomial – Second order equations with regular singular points – Bessel equation.

UNIT II
System of first order equations – existence and uniqueness theorems – fundamental matrix.

UNIT III
Non homogeneous linear system – linear systems with constant coefficient – Linear systems with periodic coefficients.

UNIT IV
Successive approximation – Picard's theorem – Non uniqueness of solution – continuation and dependence on initial conditions – existence of solution in the large existence and uniqueness of solution in the system.

UNIT V
Fundamental results – Sturms comparison theorem – elementary linear oscillations – comparison theorem of Hille winter – Oscillations of $x'' + a(t)x = 0$ elementary non linear oscillations.

SUGGESTED READINGS

TEXT BOOK

1.Earl A. Coddington, (2002). An introduction to Ordinary differential Equations, Prentice Hall of India Private limited, New Delhi.

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

2. Rai. B, Choudhury, D. P. and Freedman, H. I. (2004). A course of Ordinary differential Equations, Narosa Publishing House, New Delhi.

3. George F. Simmons, (1991). Differential Equations with application and historical notes, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

4. Ordinary Differential Equations:An Introduction, Author(s): B. Rai, D. P. Choudhury ISBN: 978-81-7319-650-8, Publication Year: Reprint 2017

KARPAGAM ACADEMY OF HIGHER EDUCATION
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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
Department of Mathematics

Subject: Ordinary Differential Equation **Subject Code: 17MMP104**

Class : I-M.Sc Mathematics **Semester: I**

LESSON PLAN

UNIT -I

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
1.	1	Second order linear equation with ordinary	R1: Ch 3: Page no: 69-70
2.	1	Points-Definition and Example	R1: Ch 3: Page no:70-71
3.	1	Continuations of example on second order	R1: Ch 3: Page no:72-76
4.	1	Legendre equation	T1: Ch 3: P. no:130-134
5.	1	Legendre polynomial	T1: Ch 3: P. no:130-134
6.	1	Second order equation with regular points	R1: Ch 3: Page no:76-78
7.	1	Power series solution of order n	R1: Ch 3: Page no:76-78
8.	1	Bessel equation with example	R1: Ch 3: Page no:78-80
9.	1	Bessel Functions	R1: Ch 3: Page no:78-80
10.	1	Properties of Bessel equations	R1: Ch 3: Page no:78-80
11.	1	Derivation of bessels function	R1: Ch 3: Page no:80-84
12.	1	Recapitations and Discussion of possible questions	R1: Ch 3: Page no:84-88
Total	**12 hrs**		

TEXT BOOK

1. Earl A. Coddington, (2002). An introduction to Ordinary differential Equations, Prentice Hall of India Private limited, New Delhi.

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

UNIT-II

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
1.	1	System of first order equation- definitions	R1: Ch 4: Page no:92-94
2.	1	System of first order equation example	R1: Ch 4: Page no:92-94
3.	1	System of first order equation example	R1: Ch 4: Page no:92-94
4.	1	Existence and uniqueness theorem	R1: Ch 4: Page no:99-102
5.	1	Continuation of theorem	R1: Ch 4: Page no:99-102
6.	1	Example for existence theorem	R1: Ch 4: Page no:102-104
7.	1	Fundamental Matrix –definition and theorem	R1: Ch 7: Page no:254
8.	1	Theorem for Fundamental Matrix	R1: Ch 4: Page no:105-107
9.	1	Fundamental matrix Examples	R1: Ch 4: Page no:105-107
10.	1	Fundamental matrix Examples	R1: Ch 4: Page no:105-107
11.	1	Fundamental matrix Examples	R1: Ch 4: Page no:107-108
12.	1	Recapitulation and discussion of possible questions	
Total	**12 hrs**		

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

UNIT-III

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
1.	1	Non Homogenous linear system	R1: Ch 4: Page no:108-110
2.	1	Linear System with Constant coefficient	R1: Ch 4: Page no:110-112
3.	1	Linear System with Constant coefficient theorem	R1: Ch 4: Page no:110-112
4.	1	Example for linear system with constant coefficient	R1: Ch 4: Page no:112-116
5.	1	Example for linear system with constant co efficient	R1: Ch 4: Page no:119-120
6.	1	Example for linear system with constant co efficient	R1: Ch 4: Page no:119-120
7.	1	Linear system with periodic co efficient concept	R1: Ch 4: Page no:121-123
8.	1	Linear system with periodic co efficient concept and theorem	R1: Ch 4: Page no:121-123
9.	1	Linear system with periodic co efficient lemmas	R1: Ch 4: Page no:121-123
10.	1	Linear system with periodic co efficient concept ant theorem	R1: Ch 4: Page no:123-124
11.	1	Recapitulation and discussion of possible questions	
Total	**11 hrs**		

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

UNIT-IV

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
1.	1	Successive approximation introduction	T1: Ch 5: Page no:200-202
2.	1	Theorem for successive approximation	R1: Ch 5: Page no:134-135
3.	1	Picard’s theorem	R1: Ch 5: Page no:136-139
4.	1	Picard’s theorem Lemma	R1: Ch 5: Page no:136-139
5.	1	Example for Picard’s theorem	R1: Ch 5: Page no:140-142
6.	1	Non Uniqueness Solution	R1: Ch 5: Page no:143-146
7.	1	Continuous and dependence of initial conditions	R1: Ch 5: Page no:143-146
8.	1	Theorem Continuous and dependence of initial conditions	R1: Ch 5: Page no:147-149
9.	1	Existence and uniqueness of solution of system-definition and lemma	R1: Ch 5: Page no:147-149
10.	1	Existence and uniqueness of solution of system-Theorem	R1: Ch 5: Page no:147-149
11.	1	Existence of solution in large theorem	R1: Ch 5: Page no:149-151
12.	1	Recapitulation and discussion of possible questions	
Total	**12 hrs**		

TEXT BOOK

1.Earl A. Coddington, (2002). An introduction to Ordinary differential Equations, Prentice Hall of India Private limited, New Delhi.

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

UNIT-V

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
1.	1	Fundamental results-concept and theorem	R1: Ch 8: Page no:204-207
2.	1	Strum's comparison theorem	R3: Ch 8: Page no: 161-163
3.	1	Strum separation theorem	R1: Ch 8: Page no:208-209
4.	1	Strum separation theorem with example	R1: Ch 8: Page no:208-209
5.	1	Elementary linear oscillation theorem	R1: Ch 8: Page no:210-212
6.	1	Lemma for comparison theorem of Hille winder	R1: Ch 8: Page no:213-215
7.	1	Hille Theorem	R1: Ch 8: Page no:216-217
8.	1	Winder Theorem	R1: Ch 8: Page no:216-217
9.	1	Oscillations of x"+a(t)x=0 of elementary non linear oscillations	R1: Ch 8: Page no:218-219
10.	1	Recapitulation and discussion of important questions	
11.	1	Discuss on Previous ESE question papers	
12.	1	Discuss on Previous ESE question papers	
13.	1	Discuss on Previous ESE question papers	
Total	**13 hrs**		

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

2. George F. Simmons, (1991). Differential Equations with application and historical notes, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equation	**Semester: I**	**L T P C**
Subject Code: 17MMP104	**Class: I- M.Sc Mathematics**	**4 0 0 4**

UNIT -I

Second order linear equations with ordinary points – Legendre equation and Legendre polynomial – Second order equations with regular singular points – Bessel equation.

TEXT BOOK

1.Earl A. Coddington, (2002). An introduction to Ordinary differential Equations, Prentice Hall of India Private limited, New Delhi.

REFERENCES

1. Deo. S. G, Lakshmikantham, V. and Raghavendra, V. (2003). of Ordinary differential Equations, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

2. Rai. B, Choudhury, D. P. and Freedman, H. I. (2004). A course of Ordinary differential Equations, Narosa Publishing House, New Delhi.

3. George F. Simmons, (1991). Differential Equations with application and historical notes, Second edition, Tata Mc Graw Hill Publishing Company limited, New Delhi.

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UNIT – II

SOLUTION IN POWER SERIES

SECOND ORDER LINEAR EQUATIONS WITH ORDINARY POINTS

Consider the second order linear homogeneous equation of the form

$$x'' + a_1(t)x' + a_2(t)x = 0. \quad \text{---1}$$

Definition:

(Analytic functions) A function g, defined on an interval I is said to be analytic at $t = a$ where $a \in I$, if g can be expanded in a power series $\sum_{n=0}^{\infty} K_n(t-a)^n$ with a positive radius of convergence.

Example:

Trivially, any polynomial in t is analytic at $t = 0$. The elementary functions e^t, $\sin t$, $\cos t$ are analytic at all points of the real line.

Consider the differential equation

$$C_0(t)x'' + C_1(t)x' + C_2(t)x = 0, \quad t \in I. \quad \text{--2}$$

Let $$d_1(t) = \frac{C_1(t)}{C_0(t)} \quad \text{and} \quad d_2 = \frac{C_2(t)}{C_0(t)}.$$

Definition:

A point $a \in I$ is called an ordinary point for the Equation 2 if $d_1(t)$ and $d_2(t)$ are analytic at $t = a$.

Example:

The Hermite equation

$$x'' - 2tx' + 2x = 0 \quad \text{----3(3.9)}$$

has an ordinary point at $t = 0$ since $-2t$ and 2 are analytic functions at $t = 0$.

Example:

The point $t=2$ is not an ordinary point for the equation $(t-2)x''+x=0$ because the function $\frac{1}{(t-2)}$ does not admit a power series around 2 with a positive radius of convergence.

In all of what follows, we would be interested in studying the series solution around an ordinary point. Firstly, we illustrate the method by an example and later generalize it to a linear second order equation around an ordinary point.

Example:

Consider the Hermite equation (3.9). Assume that

$$z(t)=\sum_{k=0}^{\infty} a_k t^k$$

is a solution of (3.9). The aim now is to determine the constants a_k. First of all note that

$$z''(t)-2tz'(t)+2z(t)=0, \tag{3.10}$$

Here term by term differentiation for the series would be valid in the interior of the interval of convergence. Differentiating $z'(t)$ we get

$$z''(t)=\sum_{k=0}^{\infty}(k+1)(k+2)a_{k+2}t^k.$$

Substituting the values of z'', z' and z in (3.10), one obtains

$$2a_2+\sum_{k=1}^{\infty}[(k+2)(k+1)a_{k+2}-2a_k(k-1)]t^k+2a_0=0. \tag{3.11}$$

Since (3.11) holds for all t, the coefficients of powers of t vanish individually.

Hence

$$2a_2+2a_0=0, \tag{3.12}$$

$$(k+2)(k+1)a_{k+2}-2a_k(k-1)=0, \quad k\geq 1. \tag{3.13}$$

From (3.12), we get $a_2=-a_0$. It is easy to see from (3.13) that $a_3=0$ and hence successively it can be deduced that $a_{2k+1}=0$ for $k=1,2,\ldots$. From (3.13) we get

$$a_{2k+2}=\frac{2(2k-1)}{(2k+2)(2k+1)}a_{2k}.$$

Substituting for a_{2k} from (3.13) and repeating the process, we obtain

$$a_{2k+2} = \frac{2(2k-1)2(2k-3)\dots(2\cdot 3)(2\cdot 1)}{(2k+2)(2k+1)\dots 4\cdot 3}\, a_2. \tag{3.14}$$

But from (3.12), we have $a_2 = -a_0$ and so

$$a_{2k+2} = \frac{2^{k+1}(-1)(-1+2)(-1+4)\dots(-1+2k)}{(2k+2)!}\, a_0.$$

So the series solution $z(t)$ is

$$z(t) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{2^k(-1)(-1+2)\dots(-1+2k-2)}{(2k)!}\, t^{2k}\right] + a_1 t \tag{3.15}$$

where a_0 and a_1 are arbitrary constants. Let

$$z_1(t) = 1 + \sum_{k=1}^{\infty} \frac{2^k(-1)(-1+2)\dots(-1+2k-2)}{(2k)!}\, t^{2k} \quad \text{and} \quad z_2(t) = t$$

Since (3.15) is a solution of (3.9) whatever be a_0 and a_1, in particular we see that z_1 and z_2 are two solutions of (3.9). Also z_1 and z_2 are linearly independent on any interval of the real line. Thus we have established the existence of two linearly independent solutions of (3.9). This is an outcome of the power series method.

Relation (3.15) has many implications. It can be used to obtain approximate solutions of (3.9) in an interval around zero.

Example 3.7 illustrates that it is possible to obtain solutions of second order linear equations by the method of power series. For this purpose, we assumed that the coefficients which occur in the equation are analytic at t_0. But the question is, can we assume that any second order linear equation admits a power series solution around an ordinary point? The answer to this question is in the affirmative as can be seen from the following result.

Theorem:

Consider the second order linear Equation (3.7) where $a_1(t)$ and $a_2(t)$ are analytic at a point t_0. Then there exists a unique function $z(t)$, analytic at t_0, which is a solution of (3.7) in a certain neighbourhood of t_0 and in addition $z(t_0) = \alpha_1$ and $z'(t_0) = \alpha_2$ where α_1 and α_2 are given constants. Further, if the power series of $a_1(t)$ and $a_2(t)$ converge on the interval $|t - t_0| < r$, then so does the power series expansion for $z(t)$.

LEGENDRE EQUATION AND LEGENDRE POLYNOMIALS

The equation

$$(1-t^2)x'' - 2tx' + p(p+1)x = 0 \qquad (3.18)$$

where p is a real number, is called the Legendre equation of order p. Let us employ the power series method to solve (3.18) The standard form of (3.18) is given by

$$x'' - \frac{2t}{1-t^2}x' + \frac{p(p+1)}{1-t^2}x = 0, \quad t \neq \pm 1. \qquad (3.19)$$

Comparison of (3.19) with (3.7) yields

$$a_1(t) = -\frac{2t}{1-t^2} \quad \text{and} \quad a_2(t) = \frac{p(p+1)}{(1-t^2)}.$$

We know that the binomial expansions of $a_1(t)$ and $a_2(t)$ converge for $|t| < 1$. Hence from Theorem 3.1 the Equation (3.18) admits a power series solution valid for $|t| < 1$. Let us assume that

$$z(t) = \sum_{k=0}^{\infty} a_k t^k \qquad (3.20)$$

is a solution of (3.18).

Then we have $\quad (1-t^2)z'' - 2tz' + p(p+1)z = 0. \qquad (3.21)$

We obtain the following relations from (3.20)

$$\left.\begin{aligned} -2tz'(t) &= \sum_{k=0}^{\infty} -2ka_k t^k, \\ -t^2 z''(t) &= \sum_{k=0}^{\infty} -k(k-1)a_k t^k \end{aligned}\right\}. \qquad (3.22)$$

Using (3.20) and (3.22) in (3.21) we get, after simplification,

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + (p+k+1)(p-k)a_k]t^k = 0.$$

Since the above equation is valid for $|t| < 1$, the coefficients of t^k, for all k, vanish. This gives the recursion formula

$$a_{k+2} = -\frac{-(p+k+1)(p-k)}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, \ldots \tag{3.23}$$

The formula (3.23) shows that for even k, a_k is a multiple of a_0 while for odd k, a_k is a multiple of a_1. Let us list some of the values of a_k. From (3.23), we have

$$a_2 = -\frac{(p+1)p}{2 \cdot 1} a_0,$$

$$a_3 = -\frac{(p+2)(p-1)}{3 \cdot 2} a_1,$$

$$a_4 = -\frac{(p+3)(p-2)}{4 \cdot 3} a_2 = \frac{(p+3)(p+1)p(p-2)}{4!} a_0,$$

$$a_5 = -\frac{(p+4)(p-3)}{5 \cdot 4} a_3 = \frac{(p+4)(p+2)(p-1)(p-3)}{5!} a_1.$$

In general

$$a_{2m} = \frac{(-1)^m (p+2m-1)(p+2m-3)\ldots(p+1)p(p-2)\ldots(p-2m+2)}{(2m)!} a_0,$$

$$a_{2m+1} = \frac{(-1)^m (p+2m)(p+2m-2)\ldots(p+2)(p-1)(p-3)\ldots(p-2m+1)}{(2m+1)!} a_1$$

where $m = 1, 2, \ldots$. Thus we have evaluated coefficients a_{2m} and a_{2m+1} in terms of a_0 and a_1 respectively. Substituting these values in (3.20), we get the required power series solution for (3.18) as follows:

$$z(t) = a_0 \left[1 - \frac{(p+1)p}{2!} t^2 + \frac{(p+3)(p+1)p(p-2)}{4!} t^4 - \ldots \right]$$
$$+ a_1 \left[t - \frac{(p+2)(p-1)}{3!} t^3 + \frac{(p+4)(p+2)(p-1)(p-3)}{5!} t^5 - \ldots \right]. \tag{3.24}$$

Let us write

$$z(t) = a_0 z_1(t) + a_1 z_2(t), \quad |t| < 1, \tag{3.25}$$

where $z_1(t)$ and $z_2(t)$ represent the series

$$z_1(t) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (p+2m-1)(p+2m-3)\ldots(p+1)p(p-2)\ldots(p-2m+2)}{(2m)!} t^m,$$

(3.26)

$$z_2(t) = t + \sum_{m=1}^{\infty} \frac{(-1)^m (p+2m)(p+2m-2)\ldots(p+2)(p-1)(p-3)\ldots(p-2m+1)}{(2m+1)!} t^{2m+1}.$$

The constants a_0 and a_1 are arbitrary. If we choose $a_0 = 1$ and $a_1 = 0$, then $z(t) = z_1(t)$ and similarly if $a_0 = 0$ and $a_1 = 1$, then $z(t) = z_2(t)$. Indeed, $z_1(t)$ and $z_2(t)$ are two linearly independent solutions of (3.18) on $|t| < 1$. The general solution of (3.18) is thus given by (3.25).

In deriving $z_1(t)$ and $z_2(t)$ we have assumed that p is a real number. If p is a non-negative integer then $z_1(t)$ or $z_2(t)$ reduces to a polynomial in t of degree p if p is even, or of degree $p - 1$ if p is odd respectively. For example,

$$z_1(t) = 1\ (p = 0),\quad z_1(t) = 1 - 3t^2\ (p = 2),\quad z_1(t) = 1 - 10t^2 + \frac{35}{3}t^4\ (p = 4).$$

For these values of p, $z_2(t)$ is still an infinite power series. In case p is odd then $z_1(t)$ is an infinite power series and $z_2(t)$ reduces to a polynomial. For example,

$$z_2(t) = t\ (p = 1),\quad z_2(t) = t - \frac{5}{3}t^3\ (p = 3),\quad z_2(t) = t - \frac{14}{3}t^3 + \frac{21}{5}t^5\ (p = 5).$$

Legendre Polynomials

Let us now consider the Legendre equation when $p \geq 0$ is an integer n, namely,

$$(1 - t^2)x'' - 2tx' + n(n + 1)x = 0. \tag{3.28}$$

It is already seen that (3.28) admits a polynomial solution. Let us denote this solution by $P_n(t)$. We say $P_n(t)$ is a Legendre polynomial when $P_n(1) = 1$, $n = 0, 1, 2 \ldots$ These polynomials play an important role in mathematical physics. We obtain below some of their important properties.

Let V denote the polynomial $(t^2 - 1)^n$. Then we show that the nth derivative of V, denoted by D^nV, satisfies (3.28). By definition we have

$$V = (t^2 - 1)^n \tag{3.29}$$

and so $\frac{dV}{dt} = n(t^2 - 1)^{n-1} \cdot 2t$ which for $t \neq \pm 1$ can be rewritten as

$$(t^2 - 1)\frac{dV}{dt} - 2ntV = 0 \tag{3.30}$$

Differentiating (3.30), $(n + 1)$ times by using Leibnitz's theorem, we get

$$(1 - t^2)\frac{d^2}{dt^2}(D^nV) - 2t\frac{d}{dt}(D^nV) + n(n + 1)D^nV = 0$$

which proves that D^nV is a solution of (3.28). Hence the Legendre polynomial

$P_n(t)$ is a constant multiple of D^nV and so $P_n(t) = AD^nV$. To evaluate A, note that the Legendre polynomial satisfies $P_n(1) = 1$. Now,

$$P_n(t) = AD^n[(t-1)^n(t+1)^n]$$
$$= A(t+1)^n D^n(t-1)^n + \text{terms with } (t-1) \text{ as a factor}$$
$$= n!\, A(t+1)^n + \text{terms with } (t-1) \text{ as a factor.}$$

Hence $$P_n(1) = A\, n!\, 2^n = 1$$

which determines the value A, namely, $A = 1/(n!\, 2^n)$.

We have thus proved the following result.

Theorem:

The Legendre polynomial $P_n(t)$ is given by

$$P_n(t) = \frac{1}{2^n\, n!} \frac{d^n}{dt^n} (t^2 - 1)^n. \tag{3.31}$$

As a consequence of Theorem 3.2 we obtain the following result which exhibits one of the important properties of $P_n(t)$.

Theorem:

If P_n is a Legendre polynomial, then

$$\int_{-1}^{1} P_n^2(t)\, dt = \frac{2}{2n+1}. \tag{3.32}$$

Proof Let us denote, as before, $(t^2 - 1)^n$ by V.

Then $$\int_{-1}^{1} P_n^2(t)\, dt = \int_{-1}^{1} \left[\frac{1}{n!\, 2^n}\right]^2 \frac{d^n}{dt^n} V(t) \frac{d^n}{dt^n} V(t)\, dt.$$

Let us evaluate the integral given below

$$I = \int_{-1}^{1} \frac{d^n}{dt^n} V(t) \frac{d^n}{dt^n} V(t)\, dt.$$

Note that $$V^{(m)}(-1) = V^{(m)}(1) = 0, \quad \text{if} \quad 0 \le m < n. \tag{3.33}$$

We successively integrate by parts the integral I and get

$$I = \int_{-1}^{1} \left[\frac{d^{2n}}{dt^{2n}} V(t)\right] (-1)^n V(t)\, dt = (2n)! \int_{-1}^{1} (1 - t^2)^n\, dt.$$

With the help of the transformation $t = \cos\theta$ and using the formula for $\int_0^{\pi/2} \sin^m \theta\, d\theta$, we arrive at $\int_{-1}^{1} P_n^2(t)\, dt = 2/(2n+1)$.

Theorem:

If $P_n(t)$ and $P_m(t)$ are Legendre Polynomials, then

$$\int_{-1}^{1} P_n(t)\, P_m(t)\, dt = 0 \quad \text{if} \quad m \neq n. \tag{3.34}$$

Proof Equation (3.28) can also be written as

$$\frac{d}{dt}[(1-t^2)P_n'] = -n(n+1)P_n,$$

$$\frac{d}{dt}[(1-t^2)P_m'] = -m(m+1)P_m.$$

Multiply the first relation by P_m and the second relation by P_n and subtract the resulting expressions. Hence, we get

$$\frac{d}{dt}[(1-t^2)(P_m P_n' - P_n P_m')] = [m(m+1) - n(n+1)]P_m P_n.$$

Now integrate between the limits −1 and 1. The conclusion (3.34) follows.

Theorem 3.4 essentially says that the Legendre polynomials form an orthogonal set of functions with weight function unity on [−1, 1]. This property of $P_n(t)$ is crucially used in the expansion of a given function $g(t)$ defined and continuous on [−1, 1] in terms of $P_n(t)$.

Theorem:

If $g(t)$ is any continuous function of t defined on [−1, 1], then g admits an expansion of the form

$$g(t) = \sum_{n=0}^{\infty} C_n P_n(t), \quad t \in [-1, 1],$$

where C_n are constants given by

$$C_n = \frac{(2n+1)}{2} \int_{-1}^{1} g(t)\, P_n(t)\, dt, \quad n = 0, 1, 2, \ldots$$

The proof of this theorem is a consequence of Theorems 3.3 and 3.4 and hence is omitted.

SECOND ORDER EQUATION WITH REGULAR SINGULAR POINT

Consider the second order equation

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = 0. \qquad (3.35)$$

Suppose that $\frac{a_1(t)}{a_0(t)}, \frac{a_2(t)}{a_0(t)}$ are analytic functions at a point t_0 on an interval I. The point t_0 is then called an ordinary point of the given equation.

A point $t_0 \in I$ is defined to be a singular point for the given equation if it is not an ordinary point. Thus, at a singular point either $\frac{a_1(t)}{a_0(t)}$ or $\frac{a_2(t)}{a_0(t)}$ fails to be analytic at $t = t_0$. However, if the singularity is not of a predominant nature in the form of irregular one then the extension of the series method is possible for a class of such equations. We classify singular points as follows:

Example:

A point $t_0 \in I$ is called a regular singular point for the Equation (3.35) if t_0 is a singular point and, in addition, the functions $(t - t_0)\frac{a_1(t)}{a_0(t)}$ and $(t - t_0)^2\frac{a_2(t)}{a_0(t)}$ are analytic at $t = t_0$. If a singular point t_0 is not regular, it is called an irregular singular point.

Example:

The Bessel equation of order p

$$L(x)(t) = t^2x'' + tx' + (t^2 - p^2)x = 0, \quad \text{Re } p \geq 0, \qquad (3.36)$$

possesses a regular singular point at $t = 0$. Observe that the functions

$$t\left(\frac{t}{t^2}\right), \text{ i.e. } 1 \quad \text{and} \quad t^2\left(\frac{t^2 - p^2}{t^2}\right), \text{ i.e. } t^2 - p^2$$

are both analytic at $t = 0$.

Example:

In the case of equation

$$t(t-1)^2(t+3)x'' + t^2x' - (t^2+t-1)x = 0$$

observe that $t=0$, $t=1$ and $t=-3$ are singular points. It is easy to verify that the points 0 and -3 are regular singular points whereas since

$$\frac{(t-1)t^2}{t(t-1)^2(t+3)} \quad \text{i.e.} \quad \frac{t^2}{t(t-1)(t+3)}$$

is not analytic at $t=1$, we conclude that 1 is not a regular singular point of the given equation.

The aim of this section is to extend the series solution method to Equation (3.35) with regular singular points. To begin with we assume that series solutions for such equations exist. Suppose further that the singular point t_0 is at zero. There is no loss of generality in this assumption. We seek a solution $\phi(t)$ for (3.35) in the form

$$\phi(t) = t^m \sum_{k=0}^{\infty} c_k t^k \tag{3.37}$$

where the coefficients c_k are constants to be determined and m is a number so chosen that the power series $\phi(t)$ satisfies the Equation (3.35). After expanding $a_1(t)/a_0(t)$ and $a_2(t)/a_0(t)$ in power series at $t=0$ and substituting these in (3.35), we equate the coefficient of the first term to zero. This coefficient is of the form

$g(m)$, a polynomial of second degree in m. The equation $g(m)=0$ is called the 'indicial equation'. Assume that $c_0 \neq 0$. The indicial equation has two roots $m=m_1$ and $m=m_2$. We obtain two sets of constants c_k's which lead to two series solutions $\phi_1(t)$ and $\phi_2(t)$ respectively. There are several cases to be dealt with in detail depending on the nature of the roots m_1 and m_2.

To illustrate the method of power series in the case of second order equations with a regular singular point we propose to discuss the Bessel Equation (3.36) in this section. We need some properties of well known Gamma function defined by

$$\Gamma(\gamma) = \int_0^\infty e^{-t} t^{\gamma-1}\, dt, \quad \operatorname{Re}\gamma > 0.$$

They are listed below

(i) $\Gamma(\gamma+1) = \gamma\,\Gamma(\gamma)$

(ii) $\Gamma(1) = 1$

(iii) $\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots$

(iv) $\Gamma(1/2) = \sqrt{\pi}$.

Gamma function is not defined at $0, -1, -2, \ldots$. Limiting values of the Gamma function at these arguments are $\pm\infty$,

$$\Gamma(\gamma) = \int_0^\infty e^{-t} t^{\gamma-1}\, dt, \quad \text{Re}\,\gamma > 0$$

$$= \frac{\Gamma(\gamma+N)}{\gamma(\gamma+1)\ldots(\gamma+N-1)}, \quad \text{Re}\,\gamma < 0, \quad -N < \text{Re}\,\gamma \le -N+1, \quad \gamma \ne -N+1,$$

N being a positive integer.

We state below a theorem concerning existence and the nature of solutions of the Bessel Equation (3.36). The proof of this theorem is omitted.

Theorem:

Let m_1 and m_2 be the roots of the indicial equation $g(m) = 0$ of the Bessel Equation (3.36). Then

(1) There exists a solution ϕ_1 such that

$$\phi_1(t) = t^{m_1} \sum_{k=0}^{\infty} c_k t^k, \quad c_0 = 1, \quad t > 0;$$

if $m_1 - m_2 \ne 0$ or a positive integer, there exists a second solution ϕ_2 for $t > 0$ of the form

$$\phi_2(t) = t^{m_2} \sum_{k=0}^{\infty} \tilde{c}_k t^k, \quad \tilde{c}_0 = 1.$$

(ii) When $m_1 = m_2$, there are two linearly independent solutions ϕ_1 and ϕ_2 defined for $t > 0$ of the form

$$\phi_1(t) = t^{m_1} \sigma_1(t)$$

$$\phi_2(t) = t^{m_1+1} \sigma_2(t) + (\log t)\, \phi_1(t)$$

where σ_1 and σ_2 have power series representations and are convergent for all finite values of $t > 0$ and $\sigma_1(0) \neq 0$.

(iii) When $m_1 - m_2$ is a positive integer there are two linearly independent solutions ϕ_1 and ϕ_2 for $t > 0$ of the form

$$\phi_1(t) = t^{m_1} \sigma_1(t)$$

$$\phi_2(t) = t^{m_2} \sigma_2(t) + c(\log t)\, \phi_1(t)$$

where σ_1 and σ_2 have power series representations and are convergent for $t > 0$, $\sigma_1(0) \neq 0$, $\sigma_2(0) \neq 0$ and c is a constant.

Before proceeding to the study of Bessel Equation (3.36), we give below an example which illustrates the need for assuming a solution of the form (3.37) when a second order equation possesses a regular singular point.

Example:

Consider the equation $t^2x'' - (1 + t)x = 0$ having a regular singular point at $t = 0$.

Let $$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{k=0}^{\infty} a_k t^k.$$

Then $$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}, \quad x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}.$$

We then have

$$t^2x'' - (1 + t)x = -a_0 - (a_1 + a_0)t + \sum_{k=2}^{\infty} [k(k-1)a_k - (a_k + a_{k-1})]t^k$$

$$= 0$$

Hence, $a_0 = 0$, $a_1 = 0$ and in turn $a_n = 0$, $n = 2, 3, \ldots$. This proves that $x(t) = 0$ is a solution of the given equation. But the situation is different. For, let the series solution be of the form

$$x(t) = t^m \sum_{k=0}^{\infty} a_k t^k, \quad m \neq 0, \quad a_0 \neq 0.$$

$$t^2 x'' - (1+t)x = [m(m-1) - 1]a_0 + \sum_{k=1}^{\infty} [(k+m)(k+m-1)a_k - (a_k + a_{k-1})]t^k$$

$$= 0$$

We conclude that $g(m) = m(m-1) - 1 = m^2 - m - 1 = 0$ is the indicial equation having roots $m_1 = \frac{1}{2}(1+\sqrt{5})$ and $m_2 = \frac{1}{2}(1-\sqrt{5})$. Further,

$$(k+m)(k+m-1)a_k - (a_k + a_{k-1}) = 0, \quad k = 1, 2, \ldots.$$

Choose $a_0 = 1$. We get a recurrence relation

$$a_k = \frac{a_{k-1}}{(k+m)(k+m-1)-1}, \quad k = 1, 2, \ldots$$

yielding a solution $x(t)$ given by

$$x(t) = t^m \left[1 + \frac{t}{(m+1)m - 1} + \frac{t^2}{[(m+2)(m+1) - 1][(m+1)m - 1]} + \ldots \right]$$

where $m = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$. Observe that this solution is different from $x(t) \equiv 0$.

Bessel Function

We are now in a position to study the Bessel Equation (3.36). Assume a solution in the form

$$\phi(t) = t^m \sum_{k=0}^{\infty} c_k t^k, \quad c_0 \neq 0, \quad t > 0.$$

Clearly $$t^2 \phi''(t) + t\phi'(t) + (t^2 - p^2)\phi(t) = 0. \tag{3.38}$$

We have $$\phi'(t) = \sum_{k=0}^{\infty} (m+k)c_k t^{m+k-1}$$

and $$\phi''(t) = \sum_{k=0}^{\infty} (m+k)(m+k-1)c_k t^{m+k-2}.$$

From (3.38) , we have, for $t > 0$,

$$c_0(m^2 - p^2)t^m + c_1[(m+1)^2 - p^2]t^{m+1} + \sum_{k=2}^{\infty} [\{(m+k)^2 - p^2\}c_k + c_{k-2}]t^{m+k} = 0.$$

Hence, the indicial equation $g(m) = m^2 - p^2 = 0$ has roots $m_1 = p$ and $m_2 = -p$. Assume that $m_1 - m_2$ is not an integer. Further, note that $c_1 = 0$ and

$$\{(m+k)^2 - p^2\}c_k + c_{k-2} = 0, \quad k = 2, 3, \ldots .$$

Case (i) We determine a solution corresponding to the root $m_1 = p$.

Hence, $$\{(p+k)^2 - p^2\}c_k + c_{k-2} = 0, \quad k = 2, 3, \ldots,$$

which yields $$c_k = \frac{-c_{k-2}}{k(2p+k)}, \quad k = 2, 3, 4, \ldots .$$

Since $c_1 = 0$, it follows that all coefficients

$$c_{2k+1} = 0, \quad k = 1, 2, \ldots .$$

Further $$c_2 = \frac{-c_0}{4(p+1)},$$

$$c_4 = \frac{-c_2}{4(2p+4)} = \frac{c_0}{2 \cdot 4^2(p+1)(p+2)}.$$

In general

$$c_{2k} = (-1)^k \frac{c_0}{k!\, 4^k(p+k)(p+k-1)\ldots(p+1)}.$$

Hence, one solution $\phi_1(t)$ of the Bessel Equation (3.36) is given by

$$\phi_1(t) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^{p+2k}}{k!\, 4^k(p+k)(p+k-1)\ldots(p+1)}.$$

Employing Gamma function, we have

$$\phi_1(t) = c_0 2^p\, \Gamma(p+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(p+k+1)} \left(\frac{t}{2}\right)^{p+2k}$$

Note that $c_0 \neq 0$ is an arbitrary constant. For convenience, we choose

$$c_0 = \frac{1}{2^p \Gamma(p+1)}.$$

Then $$\phi_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} \left(\frac{t}{2}\right)^{p+2k}, \quad t > 0.$$

The solution $\phi_1(t)$ is called the Bessel function of order p and is denoted by $J_p(t)$.

Thus $$J_p(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(p+k+1)} \left(\frac{t}{2}\right)^{p+2k}, \quad t > 0.$$

Case (ii) We now consider the second root of the indicial equation, namely $m_2 = -p$. It can be observed that there is a minor change in the above discussions. We need to replace p by $-p$ everywhere. Hence, we get another solution $J_{-p}(t)$ given by

$$J_{-p}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-p+k+1)} \left(\frac{t}{2}\right)^{-p+2k}, \quad t > 0.$$

In arriving at this solution we have assumed that

$$c_0 = \frac{1}{2^{-p}\,\Gamma(-p+1)}.$$

We assume that the solutions $J_p(t)$ and $J_{-p}(t)$ exists. We can prove that the series representing them are convergent for $t > 0$ and that these two solutions are linearly independent when p is not a positive integer or zero. Hence, the general solution of the Bessel Equation (3.36) of order which is neither a positive integer nor zero and Re $p \neq 0$ is given by

$$x(t) = A\,J_p(t) + B\,J_{-p}(t), \quad t > 0$$

where A and B are arbitrary constants. The solution $J_p(t)$ of the Bessel Equation (3.36) is called Bessel function of order p of the first kind.

In case $p = 0$, the Bessel equation takes the form

$$t^2x'' + tx' + t^2x = 0. \tag{3.39}$$

Assume that solution $\phi(t)$ of this equation has the form

$$\phi(t) = \sum_{k=0}^{\infty} c_k t^k.$$

We can now proceed as in the previous case and arrive at a solution $J_0(t)$ given by

$$J_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{t}{2}\right)^{2k}, \quad t > 0. \tag{3.40}$$

It is easy to show that the solution series $J_0(t)$ converges for all finite values of $t > 0$. The solution $J_0(t)$ is called Bessel function of order zero of the first kind.

In fact, one can obtain $J_0(t)$ from $J_p(t)$ by substituting $p = 0$ and noting that $\Gamma(k+1) = k!$.

The Bessel Equation is of the second order and hence it possesses two linearly independent solutions. It has been possible to determine two such solutions when constant p in the Equation (3.36) is such that $p \neq 0$, Re $p \neq 0$ and p is not a positive integer. It remains to determine solutions when two roots m_1 and m_2 of the indicial equation are such that $m_1 \neq m_2$ and $m_1 - m_2$ is an integer.

Assume that the two roots differ by an integer. Let $m_1 - m_2 = 2n$. Employing the Theorem 3.6 we find that the Equation (3.36) has two solutions

$$\phi_1(t) = J_n(t)$$

and

$$\phi_2(t) = t^{-n} \sum_{k=0}^{\infty} c_k t^k + c(\log t)\, J_n(t). \tag{3.41}$$

We already have the function $J_n(t)$ satisfying

$$L(J_n)(t) = 0.$$

To determine solution ϕ_2 of the Equation (3.36) we need to determine the coefficients c_k for $k = 0, 1, 2, \ldots$. For this purpose, let us substitute ϕ_2 in (3.36). We first find ϕ_2' and ϕ_2''. It is seen that

$$\phi_2'(t) = \sum_{k=0}^{\infty} c_k(k-n)t^{k-n-1} + c(\log t)\, J_n'(t) + \frac{c}{t} J_n(t)$$

and

$$\phi_2''(t) = \sum^{\infty} c_k(k-n)(k-n-1)t^{k-n-2} + c(\log t)\, J_n''(t)$$

$$+ \frac{c}{t} J_n'(t) - \frac{c}{t^2} J_n(t) + \frac{c}{t} J_n'(t).$$

Hence, we get

$$\begin{aligned} L(\phi_2(t)) &= t^2 \phi_2''(t) + t\,\phi_2'(t) + (t^2 - n^2)\,\phi_2(t) \\ &= 0 \cdot c_0 t^{-n} + c_1[(1-n)^2 - n^2]t^{1-n} \\ &\quad + t^{-n} \sum_{k=2}^{\infty} \{[(k-n)^2 - n^2]c_k + c_{k-2}\}t^k \\ &\quad + 2ct\, J_n'(t) + c(\log t) L(J_n)(t) = 0. \end{aligned} \tag{3.42}$$

The last term on the right side is zero. Further

$$\begin{aligned} J_n(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \frac{t^{n+2k}}{2^{n+2k}} \\ &= \sum_{k=0}^{\infty} b_{2k} t^{n+2k} \end{aligned}$$

where

$$b_{2k} = \frac{(-1)^k}{2^{n+2k}\, k!\,(n+k)!}.$$

From (3.42), it follows that

$$(1-2n)c_1 t + \sum_{k=2}^{\infty} [k(k-2n)c_k + c_{k-2}]t^k = -2c \sum_{k=0}^{\infty} (2k+n)b_{2k}t^{2k+2n}.$$

The first term on the left side is a multiple of t and the first term on the right side is a multiple of t^{2n}. Hence, $c_1 = 0$. For $n > 1$

$$k(k-2n)c_k + c_{k-2} = 0, \quad k = 2, 3, \ldots, 2n-1$$

yielding

$$c_1 = c_3 = c_5 = \ldots = c_{2n-1} = 0.$$

Also

$$c_{2k} = \frac{c_0}{2^{2k}\, k!\, (n-1)\ldots(n-k)}, \quad k = 1, 2, \ldots, n-1.$$

Comparing the coefficients of t^{2n} on both sides of (3.42) we get

$$c = \frac{-c_0}{2^{n-1}(n-1)!}.$$

Also

$$c_{2n+1} = c_{2n+3} = \ldots = 0.$$

Thus, all coefficients c_{2i+1}, $i = 0, 1, 2, \ldots$ are zero. The coefficients c_{2k} for $k = 1, 2, \ldots, n-1$ are known. Now from (3.42) we have

$$2k(2n+2k)c_{2n+2k} + c_{2n+2k-2} = -2c(n+2k)b_{2k}$$

for $k = 1, 2, \ldots$.

For $k = 1$, we get

$$c_{2n+2} = -\frac{cb_2}{2}\left(1 + \frac{1}{n+1}\right) - \frac{c_{2n}}{4(n+1)}.$$

Observe that c_{2n} is still not determined. We choose

$$c_{2n} = -\frac{cb_0}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$

This choice is made for convenience. We then have

$$c_{2n+2} = -\frac{cb_2}{2}\left(1 + 1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right)$$

(Note that $4(n+1)b_2 = -b_0$)

and recursively

$$c_{2n+2k} = -\frac{cb_{2k}}{2}\left[\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{n+k}\right)\right]; \quad k = 1, 2, \ldots.$$

Observe that we have now determined all the coefficients. In view of the relation (3.41) we have

$$\phi_2(t) = c_0 t^{-n} + c_0 t^{-n} \sum_{k=1}^{n-1} \frac{t^{2k}}{2^{2k} \cdot k!\,(n-1)\ldots(n-k)} - \frac{cb_0}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)t^n$$

The series representing K_n is convergent for $t > 0$. (Apply ratio test.)

PROPERTIES OF BESSEL FUNCTIONS

Several interesting properties of Bessel functions are known. We prove below some of them.

(i) Show that

$$\frac{d}{dt}[t^p J_p(t)] = t^p J_{p-1}(t) \tag{3.43}$$

and

$$\frac{d}{dt}[t^{-p} J_p(t)] = -t^{-p} J_{p+1}(t). \tag{3.44}$$

Proof We have

$$\frac{d}{dt}[t^p J_p(t)] = \frac{d}{dt}\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+2p}}{2^{2k+p} k!\, \Gamma(k+p+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+2p-1}}{2^{2k+p-1} k!\, \Gamma(k+p)}$$

$$= t^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(k+p)} \left(\frac{t}{2}\right)^{2k+p-1}$$

$$= t^p t J_{p-1}(t).$$

The other relation follows similarly. Expanding the relations (3.43) and (3.44), we get

$$J_p' + \frac{p}{t} J_p = J_{p-1};$$

$$J_p' - \frac{p}{t} J_p = -J_{p+1}.$$

Addition and subtraction yield

$$J_p' = \frac{1}{2}[J_{p-1} - J_{p+1}],$$

$$p J_p = \frac{t}{2}[J_{p-1} + J_{p+1}].$$

(ii) Let $a_1, a_2, \ldots$ be the positive zeros of the Bessel function $J_p(t)$.

Then $$\int_0^1 t J_p(a_m t) J_p(a_n t)\, dt = \begin{cases} 0 & , \quad m \neq n, \\ \frac{1}{2} J_{p+1}^2(a_n) & , \quad m = n. \end{cases}$$

Part -B (5x6=30 Marks)

Possible Questions:

1. If P_n is a legendre polynomial, then prove that $\int_{-1}^{1} P_n^2(t)dt = \frac{2}{2n+1}$.

2. Show that $d/dt[t^pJ_p(t)]=t^pJ_{p-1}(t)$.

3. Find the power series solution for the Bessel equation of order p.

4. Solve: x”-2tx’+2x=0

5. Show that $d/dt[t^{-p}J_p(t)]=-t^{-p}J_{p+1}(t)$

6. Show that (i) $J_p'(t) =\frac{1}{2}[J_{p-1}(t) - J_{p+1}(t)]$ (ii) $p J_p(t) =\frac{1}{2}[J_{p-1}(t) + J_{p+1}(t)]$

7. Solve: x”-2tx’+2nx=0

8. Show that the legendre polynomial Pn(t) can be expressed as $Pn(t)=1/2^n n! \; d^n/dt^n(t^2-1)^n$

9. Consider the equation $t(t-1)^2(t+3)x''+t^2x'-(t^2+t-1)x=0$. Check whether the point t=0, 1,-3 are regular singular points (or) not.

10. If $P_n(t)$ and $P_m(t)$ are legendre polynomial the $\int_{-1}^{1} P_n(t).P_m(t)dt = 0$ if m≠n.

Part -C (1x10=10 Marks)

Possible Questions:

1.Find the power series solution for the Bessel equation of order p.

2. If $P_n(t)$ and $P_m(t)$ are legendre polynomial the $\int_{-1}^{1} P_n(t).P_m(t)dt = 0$ if m≠n.

3. If a1,a2,…., be the positive zeros of the Bessel function Jp(t),then prove that $\int_0^1 tJ_p[a_n(t)].J_p[a_m(t)]dt = \begin{cases} 0 & if\ m \neq n \\ \frac{1}{2} & J_{p+1}^2(t) if\ m = n \end{cases}$

4. Solve: i. x”-2tx’+2x=0

ii. x”-2tx’+2nx=0

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Subject: Ordinary Differential Equations **Subject Code: 17MMP104**
Class : I-M.Sc Mathematics **Semester : I**

UNIT I

SOLUTION IN POWER SERIES

Part A (20x1=20 Marks)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Consider the equation $c_o(t)x''+c_{\ }(t)x=0$ then a point a is an ordinary point if d (t) and d (t) are analytic at ________	t=0	t=a	t=1	t=a²	t=a
An hermite equation has an ordinary point at______	t=0	t=a	t=1	t=a²	t=0
An analytic function for an hermite equation at t=0 is _______	–t and 1	t and 2	-2t and 2	2t and 1	-2t and 2
The legendre equation of order p is_________	(1-t) x″-2tx′+p(p+1)x=0	(1-t) x″-2tx′+(p+1)x=0	t2 x″-2tx′+p(p+1)x=0	(1-t2) x″-2tx′+p(p+1)x=0	(1-t2) x″-2tx′+p(p+1)x=0
When $p_n(t)$ is called an legendre polynomial?	Pn(1)=0	Pn(0)=1	Pn(1)=1	Pn(t)=1	Pn(1)=1
If $p_n(t)$ is a legendre polynomial then $\int_{-}^{1}$ pn(t)dt=_________________	1/(2n+1)!	2/(2n+1)!	2/(n+1)!	1/(n+2)!	2/(2n+1)!
If $p_m(t)$ and $p_n(t)$ are legendre polynomials then $\int_{-}^{1}$pn(t) pm(t)dt=______ if m≠n	1	-1	2	0	0
If $p_n(t)$ is a legendre polynomial then $p_n(-1)$= 1 if n is______	Negative	odd	Even	positive	odd
The Bessel equation of order p is______	t^2x''+tx+tx′+(t^2-p^2)x=0	tx″+(1-t)x″+px=0	2x″+(1-t)x′+(1-p2)x=0	tx″+(1-t)x+p^2x=0	t^2x''+tx+tx′+(t^2-p^2)x=0
The Bessel function of the first kind d/dt (tpJp(t))=_________	$t^{-p}J_p(t)$	$t^pJ_{p-1}(t)$	$t^{-p}J_{p+1}(t)$	$t^pJ_p(t)$	$t^pJ_{p-1}(t)$
If $p_n(t)$ is the generating function then $p_n(-1)$=______	-1	0	$(1)^n$	$(-1)^n$	$(-1)^n$
The hermite equation is________	t2x″-2tx′+x=0	x″+tx′-2x=0	x″-2tx′+2x=0	tx″-tx′+x=0	x″-2tx′+2x=0
The legendre polynomial $p_n(t)$ can be express as________	$1/2^nn!\ D^n(t^2-1)^n$	$1/2^nn!\ D^n(t^2-1)^{n-1}$	$1/n!\ D^n(t^2-1)^n$	$1/2^nn!\ D^n(t^2-1)$	$1/2^nn!\ D^n(t^2-1)^n$
The order of equation is (D^2+2D-8)y=0 is______	1	2	0	8	2
The solution of ordinary differential equation of n order contains __________ arbitrary constants	More than n	no	n	Atleast n	n
The n^{th} order ordinary linear homogeneous differential equation have_______	(n-1) singular solution	one singular solution	n-singular solution	no singular solution	no singular solution
The linearity principle for ordinary differential equation holds for_______	Non-homogeneous equation	linear differential equation	Homogeneous equation	non-linear equation	linear differential equation
A singular point which in______ is called an irregular singular point	Regular	ordinary point	analytic point	analytic function	Regular
If $p_m(t)$ and $p_n(t)$ are legendre polynomials then $\int_{-}^{1}$pn(t) pm(t)dt=______ if m=n	0	1/n+1	2/(2n+1)	1	2/(2n+1)
On Bessel's function, where n is any integer then J-n(x)=________	$(-1)^nJ_{-n}(x)$	$(-1)^nJ_n(x)$	$(-1)^nJ_{n+1}(x)$	$(-1)^nJ_{n-1}(x)$	$(-1)^nJ_n(x)$
When the hermite equation has an ordinary point?	t=0	t=-2	t=0	t=0	t=0
The second order linear homogeneous equation is of the form________	x″+a1(t)x′+a2(t)x	x″+a1(t)x′+a2(t)x=constant	x″+a1(x)x=0	x″+a1(x)x′=constant	x″+a1(t)x′+a2(t)x
The regular singular point of the equation tx″+(1-t)x′+nx=0 is_______	t=1	t=-1	t=0	t=n	t=0
The equation tx″+(1-t)x′+nx=0 where n is a constant, is called the____________	aagrange equation	legendre equation	Bessel equation	hermite equation	lagrange equation
The singular point of the equation t(t-1)^2 (t+3)x″+t ^{2} 1)x=0 is__________	t=0 and =1	t=0, t=1 and t=-3	t=1 qnd t=-3	t=0 and t=-3	t=0, t=1 and t=-3
The equation t^2x''-(1+t)x=0 having a regular singular point at_______	t=-1	t=1	t=√-1	t=0	t=0
If Jp(t) is a Bessel function then d/dx[t-pJp(t)]=_________	$-t^pJ_{p-1}(t)$	$t\text{-}^pJ_{p+1}(t)$	$-t\text{-}^pJ_{p+1}(t)$	$t^pJ_{p-1}(t)$	$-t\text{-}^pJ_{p+1}(t)$
The regular singular point of the equation t^2 n(n+1)x=0 is________	0	infinity	1	2	infinity
The Bessel equation is of the second order then it possesses two _______	linearly dependent solution	independent solutions	dependent solutions	linearly independent solutions	linearly independent solutions
A point to is defined to be a singular point for the equations a0(t)x″+a1(t)x′+a2(t)x=0 if it is______	not an ordinary point	ordinary point	not an irregular point	irregular point	not an ordinary point

The regular singular points of the equations $(t-t^2)x''+[\gamma-(\alpha+\beta+1)]tx-\beta\alpha x=0$ is______________________	0and 1	0 and ∞	0,1 and ∞	1 and ∞	0,1 and ∞
The Bessel function of	$(1/\pi)Jn(t)$	$\pi J_n(t)$	$\pi/Jn(t)$	$Jn(t)$	$\pi J_n(t)$
The consider non-linear differential equation $x'= t^2-x^2$, $x=1/2$ when t=0 then the value of $x'(0)$=_____________	1/2	- 1/2	1/4	-1/4	-1/4
The equation $(1-t^2)x''-2tx'+p(p+1)x=0$ where p is a real number is called the ______of order p	legendre equation	laguerse equation	Bessel equation	Hermite equation	legendre equation
The Bessel equation possesses a ________ at t=0	ordinary point	analytic function	regular singular point	singular point	regular singular point
The equation $t(t-1)^2(t+3)x''+t^2x'-(t^2+t-1)=0$ is not analytic at ________	t=0	t=-1	t=-3	t=1	t=1
The Bessel function when n is _______	even or odd	odd	costant	even	odd
A regular singular point of the equation $2t^2x''+(2t+1)x'-x=0$ is ___________	t=0	t=2	t=1	t=-1	t=0
An _________ equation has an ordinary point at t = 0.	Legendre	Bessel	Hermite	Lagrange	Hermite
The _________ order linear homogeneous equation is of the form x" + a1(t)x'+a2(t)x= 0	first	second	third	fourth	second

KARPAGAM ACADEMY OF HIGHER EDUCATION
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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equation **Semester :I** **L T P C**

Subject Code: 17MMP104 **Class : I- M.Sc Mathematics** **4 0 0 4**

UNIT -II

System of first order equations – existence and uniqueness theorems – fundamental matrix

TEXT BOOK

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UNIT – II

System of Linear Differential Equations

SYSTEMS OF FIRST ORDER EQUATIONS

In Chapter 1 we observed that a general non-linear differential equation of order one is denoted in the form

$$x' = f(t, x) \qquad (4.1)$$

where x is a real valued scalar function defined on an interval I contained in the real line. The first order non-homogeneous linear equation

$$x' + a(t)x = b(t), \quad t \in I, \qquad (4.2)$$

is a special case of (4.1). We show that the Equations (4.1) and (4.2) are special cases in a more general set-up.

Suppose that n is a positive integer. Let $f_1, f_2, \ldots, f_n$ be given n real valued functions defined on some open connected set D contained in $(n+1)$-dimensional space. Consider a system of n equations

linear systems on which we concentrate in this chapter. Consider a system of equations

$$\left.\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + b_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + b_n(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + b_n(t) \end{aligned} \quad (t \in I) \right\} \qquad (4.6)$$

where all the functions $a_{ij}, b_i, i, j = 1, 2, \ldots, n$ are given. Let

$$f_i(t.x_1, x_2, \ldots, x_n) = a_{i1}(t)x_1 + a_{i2}(t)x_2 + \ldots + a_{in}(t)x_n + b_i(t)$$

for $i = 1, 2, \ldots, n$. It is then clear that the system (4.6) is a special case of the system (4.3). Define the matrix $A(t)$ by the relation

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}$$

and the vectors $b(t)$ and $x(t)$ by

$$b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix} \quad \text{and} \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

respectively. With these notations (4.6) reduces to

$$x' = A(t)x + b(t), \quad t \in I. \tag{4.7}$$

It is easy to observe that the system (4.6) is linear in $x_1, x_2, \ldots, x_n$. Equation (4.7) is a vector matrix representation of a linear non-homogeneous system (4.6). If $b(t) \equiv 0$ on I, then the system (4.7) reduces to the homogeneous system

$$x' = A(t)x, \quad t \in I. \tag{4.8}$$

Example:

Consider the system of equations

$$x_1' = 5x_1 - 2x_2$$
$$x_2' = 2x_1 + x_2.$$

This system of two equations can be represented in the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It can be verified that a solution of this system is given by

$$x_1(t) = (c_1 + c_2 t)e^{3t}, \quad x_2(t) = \left(c_1 - \frac{1}{2}c_2 + c_2 t\right)e^{3t}.$$

In Chapter 1, it has been shown that a general nth order IVP in normal form is

$$x^{(n)} = g(t, x, x', \ldots, x^{(n-1)}), \quad t \in I \tag{4.9}$$

$$x(t_0) = \alpha_0, \quad x'(t_0) = \alpha_1, \quad \ldots, \quad x^{(n-1)}(t_0) = \alpha_{n-1}, \quad t_0 \in I \tag{4.10}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are given constants. The theory concerning nth order equations is deducible from the theory of a system of n equations. For this purpose let us define $x_1, x_2, \ldots, x_n$ by

$$x_1 = x, \quad x' = x_2, \quad \ldots, \quad x^{(n-1)} = x_n.$$

Then

$$x_1' = x_2$$

$$x_2' = x_3$$
$$\vdots \qquad (4.11)$$
$$x_{n-1}' = x_n$$
$$x_n' = g(t, x_1, x_2, \ldots, x_n).$$

Let $\varphi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a solution of (4.11). Then

$$\phi_2 = \phi_1', \quad \phi_3 = \phi_2' = \phi_1'', \quad \ldots, \quad \phi_n = \phi_1^{(n-1)},$$

$$g[t, \phi_1(t), \phi_2(t), \ldots, \phi_n(t)] = g[t, \phi_1(t), \phi_1'(t), \ldots, \phi_1^{(n-1)}(t)]$$
$$= \phi_1^{(n)}(t).$$

Clearly the component ϕ_1 is a solution of (4.9). Conversely, if ϕ_1 is a solution of (4.9) on I then the vector $\varphi = (\phi_1, \phi_2, \ldots, \phi_n)$ is a solution of (4.11). Thus the system (4.11) is equivalent to (4.9). Further, if

$$\phi_1(t_0) = \alpha_0, \quad \phi_1'(t_0) = \alpha_1, \quad \ldots, \quad \phi_1^{(n-1)}(t_0) = \alpha_{n-1},$$

then the vector $\varphi(t)$ is so defined that $\varphi(t_0) = \alpha$ where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$. It is not difficult to observe that the system (4.11) is a special case of the vector equation $x' = f(t, x)$.

In particular, consider a special case of (4.9), namely, a linear equation of nth order of the form

$$a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = h(t), \quad t \in I$$

where $a_0(t) \neq 0$ for $t \in I$. This is equivalent to

$$x^{(n)} + \frac{a_1(t)}{a_0(t)} x^{(n-1)} + \ldots + \frac{a_n(t)}{a_0(t)} x = \frac{h(t)}{a_0(t)}. \qquad (4.12)$$

Equation (4.12) can be represented in the form of a system by defining

$$\left.\begin{aligned} x(t) &= x_1(t) \\ x_1'(t) &= x_2(t) \\ &\vdots \\ x_{(n-1)}'(t) &= x_n(t) \end{aligned}\right\} \quad t \in I. \qquad (4.13)$$

$$x_n'(t) = -\frac{a_n(t)}{a_0(t)} x_1 - \frac{a_{n-1}(t)}{a_0(t)} x_2 - \ldots - \frac{a_1(t)}{a_0(t)} x_n + \frac{h(t)}{a_0(t)}.$$

Let $$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dfrac{h(t)}{a_0(t)} \end{bmatrix}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n/a_0 & -a_{n-1}/a_0 & -a_{n-2}/a_0 & \dots & -a_1/a_0 \end{bmatrix}.$$

With these notations the system (4.13) is

$$x' = A(t)x + b(t), \quad t \in I. \tag{4.14}$$

Thus it has been established that (4.12) and (4.14) are equivalent. The representations (4.7) and (4.14) provide us considerable simplicity in studying certain aspects of systems of n equations and an nth order equation respectively.

Example:

For illustration consider a linear equation

$$x''' - 6x'' + 11x' - 6x = 0.$$

Denote

$$x_1 = x, \quad x_1' = x_2 = x', \quad x_2' = x'' = x_3$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

The equation takes the form $\bar{x}' = A(t)\bar{x}$.

Notice that the first component x_1 of the system is a solution of the given equation. It is easy to check, in the present case, that $x_1(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$ where c_1, c_2 and c_3 are arbitrary constants.

MODEL FOR ARMS COMPETITION BETWEEN TWO NATIONS

Let $x(t)$ denote the war potential of the nation A and $y(t)$ the war potential of the nation B at a given time t. The war potential can be evaluated on the basis of the budgetary provisions made for defence by a nation as also the type of weapons possessed by a nation and the involvement of man-power for this purpose.

In a simplestic way it is natural to expect that the nation A will keep its rate of change of $x(t)$ keeping in view the change in the war potential $y(t)$ of the nation B.

Hence, $$\frac{dx}{dt} = \alpha y.$$

However, quite some investment in armament race is required to keep the available arms in order and keep them fit for subsequent use. This and such other factors retard the rate of growth of $x(t)$. Naturally the retarding factor is proportional to the existing accumulated strength $x(t)$. Thus the above equation gets modified and we have

$$\frac{dx}{dt} = \alpha y - \beta x.$$

The war-like situation prevails in a nation when there are occasional disputes between the two nations. Rise in disputes immediately results into the rate of change of war potential. These considerations lead to the following mathematical model

$$\frac{dx}{dt} = \alpha y - \beta x + \lambda$$

$$\frac{dy}{dt} = \gamma x - \delta y + \mu.$$

Here λ and μ are assumed to be constants and represent the level of occasional disputes between two quarreling nations A and B.

This is a system of two linear equations. The model will faithfully represent the real situation provided the constants α, β, λ, γ, δ, μ are calculated properly. The effectiveness of the model will increase if these constants can be replaced by variable coefficients. But then the model will become complicated.

In case $\lambda = 0, \mu = 0$, i.e. the current disputes between two nations are at zero level, we have

$$\frac{dx}{dt} = \alpha y - \beta x \quad \text{and} \quad \frac{dy}{dt} = \gamma x - \delta y.$$

Suppose that $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$, i.e. there is no increase in war potential between two nations. This state indicates that the war-like atmosphere is absent in both the nations, i.e. peace prevails between them. The two nations attain an equilibrium position when

$$\alpha y - \beta x = 0,$$
$$\gamma x - \delta y = 0.$$

This is possible when $x = y = 0$; $(\alpha\delta \neq \beta\gamma)$.

Obviously even when equilibrium position $x = 0$, $y = 0$ is attended by two nations, local grievances between two nations may initiate increase in war potentials of the two nations, i.e.

$$\frac{dx}{dt} = \lambda \quad \text{and} \quad \frac{dy}{dt} = \mu; \quad \lambda > 0, \quad \mu > 0.$$

In case the constants α and γ known as 'defence term' are very large in comparison to β, δ, λ and μ the war potential increases rapidly since the equations representing the war potential are

$$\frac{dx}{dt} = \alpha y \quad \text{and} \quad \frac{dy}{dt} = \gamma x;$$

i.e. $\frac{d^2x}{dt^2} = \alpha\gamma x$ having a solution

$$x(t) = Ae^{\sqrt{\alpha\gamma}\, t} + Be^{-\sqrt{\alpha\gamma}\, t}.$$

Clearly when $A > 0$, $x(t) \to \infty$. This situation is an indication of actual war between two nations.

In order to create sympathetic atmosphere between two warring nations, one of the nations may resolve to adopt unilateral disarmament. Let us say that the nation B adopts this policy at a time t making $y = 0$. In this case the equations take the form

$$\frac{dx}{dt} = -\beta x + \lambda,$$

and

$$\frac{dy}{dt} = \gamma x + \mu.$$

In case γ is positive or x is positive, y will not remain zero in future. Hence unilateral disarmament decision cannot acquire a permanent status.

The model given above has been tested for some realistic situations prevailing in the first and second world war between conflicting nations. It has been experienced that it yields fairly correct conclusions.

The above model represents armament race between two nations. It is possible to extend the same model further to represent the armament race among three or more nations. Suitable modifications are then necessary.

It is our experience that while there exists an atmosphere of war between two

It is our experience that while there exists an atmosphere of war between two
nations, there are other factors such as trade and cooperation between two nations which reduce the fear of war in the minds of people. These factors also play their role in the armament race. It is possible to incorporate these factors while modelling.

EXISTENCE AND UNIQUENESS THEOREM

Theorem:2.1

Let $A(t)$ be an $n \times n$ matrix that is continuous in t on a closed and bounded interval I. Then there exists a solution to the IVP (4.15) on I and, in addition, this solution is unique.

Proof Assume that $t \geq t_0$ without loss of generality. We write the IVP (4.15) in the following equivalent integral form

$$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds.$$

Define the successive approximations by the relations, $x_0(t) \equiv x_0$,

$$x_{n+1}(t) = x_0 + \int_{t_{0_0}}^{t} A(s)\, x_n(s)\, ds, \quad t \in I,$$

for $n = 0, 1, 2, \ldots$. Note that the sequence of the functions $\{x_n\}$ exists, since x_0 is a given vector. First of all it is proved that $\{x_n(t)\}$ is uniformly convergent on I. Consider the series

$$x_0(t) + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t)).$$

$$x_0(t) + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t)).$$

The convergence of this series implies the convergence of the sequence $\{x_n(t)\}$. It is clear that

$$x_{n+1}(t) - x_n(t) = \int_{t_0}^{t} A(s)(x_n(s) - x_{n-1}(s))\, ds,$$

and so it follows that

$$\|x_{n+1}(t) - x_n(t)\| \le \int_{t_0}^{t} \|A(s)\|\, \|x_n(s) - x_{n-1}(s)\|\, ds.$$

Since I is a closed and bounded interval and $A(t)$ is continuous, there exists a constant $k_1 > 0$ such that $k_1 = \max_{t \in I} \|A(t)\|$. Thus

$$\|x_{n+1}(t) - x_n(t)\| \le k_1 \int_{t_0}^{t} \|x_n(s) - x_{n-1}(s)\|\, ds.$$

Further it is seen that

$$\|x_1(t) - x_0(t)\| \le k_1 \|x_0\| (t - t_0),$$

assuming that $t \ge t_0$. Using this inequality and the method of induction it is easy to obtain the estimate

$$\|x_{n+1}(t) - x_n(t)\| \le \frac{k_1^{n+1} \|x_0\| (t - t_0)^{n+1}}{(n+1)!}.$$

Since the right-hand side in the above estimate can be made arbitrarily small by choosing n sufficiently large. $\left(\text{Note here that } \frac{k_1^{n+1}(t - t_0)^{n+1}}{(n+1)!} \text{ is the } (n+2)\text{th term}\right.$

in the expansion of $e^{k_1(t-t_0)}$ and t is an element of the closed and bounded interval I). We claim that $\{x_n(t)\}$ is a uniform Cauchy sequence on I. This implies that the sequence $\{x_n(t)\}$ converges uniformly to a continuous function $x(t)$ on I. Thus, it is seen that

$$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds, \quad t \in I,$$

which follows by taking the limit as $n \to \infty$ on both sides of

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} A(s)\, x_n(s)\, ds, \quad t \in I.$$

This clearly proves that $x(t)$ is a solution of the integral equation equivalent to the system (4.15) existing on I.

To establish the uniqueness, assume that $y(t)$, $t \in I$, is another solution of (4.15). Then observe that

$$x(t) = x_0 + \int_{t_0}^{t} A(s)\, x(s)\, ds$$

and

$$y(t) = x_0 + \int_{t_0}^{t} A(s)\, y(s)\, ds.$$

Thus we obtain

$$x(t) - y(t) = \int_{t_0}^{t} A(s)(x(s) - y(s))\, ds, \quad t \in I,$$

from which it follows that

$$\|x(t) - y(t)\| \le \int_{t_0}^{t} \|A(s)\|\, \|x(s) - y(s)\|\, ds$$

$$\le k_1 \int_{t_0}^{t} \|x(s) - y(s)\|\, ds.$$

So, for any $\varepsilon > 0$, it is seen that

$$\|x(t) - y(t)\| < \varepsilon + k_1 \int_{t_0}^{t} \|x(s) - y(s)\|\, ds, \quad t \in I.$$

Let $z(t) = \|x(t) - y(t)\|$. Then,

$$z(t) < \varepsilon + k_1 \int_{t_0}^{t} z(s)\, ds, \quad t \in I.$$

Let $r(t)$ denote the right side of this inequality. Clearly, $r(t_0) = \varepsilon$ and $z(t) < r(t)$. Now $r'(t) = k_1 z(t) < k_1 r(t)$. So

$$r'(t) - k_1 r(t) < 0.$$

Multiplying by $\exp\{-k_1(t - t_0)\}$ on either side it is seen that

$$[r(t)\exp\{-k_1(t - t_0)\}]' < 0.$$

After integration, between t_0 and t, the following inequality

$$z(t) < r(t) < \varepsilon \exp[k_1(t - t_0)]$$

is obtained.

Since this is true for each $\varepsilon > 0$, it is seen that $z(t) \le 0$. This proves that $x(t) = y(t)$ on I.

Theorem 2.2:

The set of all solutions of the system (4.15 (a)) on I forms an n-dimensional vector space over the field of complex numbers.

Proof Let y_1 and y_2 be any two solutions of (4.15 (a)) on I and let c_1 and c_2 be any scalars. Then it is easy to show that $c_1y_1 + c_2y_2$ is a solution of (4.15 (a)) on I. This establishes that the set of solutions of the system forms a vector space.

It is now shown that this vector space is of dimension n.

Let $e_i \in R^n$ $(i = 1, 2, \ldots, n)$ be an n-tuple such that the ith component is 1 and all other components are zeros. It is clear that the vectors $e_1, e_2, \ldots, e_n$ are linearly independent. The system (4.15 (a)) has n solutions $y_1, y_2, \ldots, y_n$ such that

$$y_1(t_0) = e_1, \quad y_2(t_0) = e_2, \quad \ldots, \quad y_n(t_0) = e_n$$

where t_0 is some point of I. It is now shown that $\{y_1, y_2, \ldots, y_n\}$ is a linearly independent set of n vectors. Consider n scalars c_i, $i = 1, 2, \ldots, n$ such that

$$c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t) = 0, \quad t \in I.$$

In particular for $t = t_0 \in I$ it is seen that

$$c_1y_1(t_0) + c_2y_2(t_0) + \ldots + c_ny_n(t_0) = 0.$$

But $y_i(t_0) = e_i$, $i = 1, 2, \ldots, n$ are linearly independent and so the above equation clearly implies that $c_1 = c_2 = \ldots = c_n = 0$. Thus the vectors $y_i(t)$, $i = 1, 2, \ldots, n$ are linearly independent of I.

The proof is concluded by showing that any solution φ of (4.15 (a)) is a linear combination of $y_1, y_2, \ldots, y_n$. Let $\varphi(t_0) = B_1e_1 + B_2e_2 + \ldots + B_ne_n$, and the vector $B = (B_1, B_2, \ldots, B_n)$. So the vector $\sum_{i=1}^{n} B_iy_i(t)$, $t \in I$, is a solution of (4.15 (a)) and, in addition, this solution passes through the point (t_0, B). Hence from the uniqueness property proved in Theorem 4.1, $\sum_{i=1}^{n} B_i y_i(t)$ has to coincide with $\varphi(t)$ since $\varphi(t_0) = B$ and $\varphi(t)$ is a solution of (4.15 (a)). This completes the proof.

To sum up, the set of n linearly independent solutions thus obtained forms a fundamental set of solutions of the system (4.15 (a)).

FUNDAMENTAL MATRIX

Theorem 2.3:

Let $A(t)$ be an $n \times n$ matrix which is continuous on I. Suppose a matrix Φ satisfies (4.17). Then det Φ satisfies the first order equation

$$(\det \Phi)' = (\operatorname{tr} A)(\det \Phi). \tag{4.18}$$

Or, in other words, for $\tau \in I$,

$$\det \Phi(t) = \det \Phi(\tau) \exp \int_{\tau}^{t} \operatorname{tr} A(s)\, ds. \tag{4.19}$$

Proof By definition the n columns of Φ are n solutions $\varphi_1, \varphi_2, \ldots, \varphi_n$ of (4.15 (a)). Denote

$$\varphi_i = \{\phi_{1i}, \phi_{2i}, \ldots, \phi_{ni}\}, \quad i = 1, 2, \ldots, n.$$

Let $a_{ij}(t)$ be the (i, j)th element of $A(t)$. Then

$$\phi'_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)\, \phi_{kj}(t); \quad i, j = 1, 2, \ldots, n. \tag{4.20}$$

Now

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

and so it is seen that

$$(\det \Phi)' = \begin{vmatrix} \phi'_{11} & \phi'_{12} & \dots & \phi'_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi'_{21} & \phi'_{22} & \dots & \phi'_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{vmatrix} + \dots + \begin{vmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi'_{n1} & \phi'_{n2} & \dots & \phi'_{nn} \end{vmatrix}.$$

Substituting the values of $\phi'_{11}, \phi'_{12}, \dots, \phi'_{1n}$ from (4.20), the first term on the right side of the above equation reduces to

$$\begin{vmatrix} \sum_{k=1}^{n} a_{1k}\phi_{k1} & \sum_{k=1}^{n} a_{1k}\phi_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}\phi_{kn} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{vmatrix}$$

which is $a_{11} \det \Phi$. Carrying this out for the remaining terms it is seen that

$$(\det \Phi)' = (a_{11} + a_{22} + \dots + a_{nn}) \det \Phi = (\operatorname{tr} A) \det \Phi.$$

The equation thus obtained is a linear differential equation. The proof of the theorem is complete since it is known that the solution of this equation is given by (4.19).

Theorem 2.4:

A solution matrix Φ of (4.17) on I is a fundamental matrix of (4.15 (a)) on I if and only if $\det \Phi(t) \neq 0$, $t \in I$.

Proof Let $\Phi(t)$ be a solution matrix such that $\det \Phi(t) \neq 0$, $t \in I$. Then the columns of Φ are linearly independent on I. Hence Φ is a fundamental matrix.

Conversely, let $\Phi(t)$ be a fundamental matrix and let φ_j, $j = 1, 2, \dots, n$ be the columns of Φ. Let φ be any solution of (4.15 (a)). Then there exist constants

$$\varphi = \sum_{i=1}^{n} c_i \varphi_i = \Phi \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \Phi c, \text{ where } c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This is a system of linear equations for the unknowns $c_1, c_2, \dots, c_n$. For a fixed $\tau \in I$ the above system has a solution and hence $\det \Phi(\tau) \neq 0$. Now from Theorem 4.3 it is clear that $\det \Phi(t) \neq 0$, $t \in I$, which completes the proof.

Some of the useful properties of the fundamental matrix are established in the following results.

Theorem 2.6:

Let $\Phi(t)$, $t \in I$, denote a fundamental matrix of the system

$$x' = Ax \tag{4.21}$$

such that $\Phi(0) = E$, where A is a constant matrix. Here E denotes the identity matrix. Then Φ satisfies

$$\Phi(t+s) = \Phi(t)\Phi(s) \tag{4.22}$$

for all values of t and $s \in I$.

Proof By the uniqueness theorem there exists a unique fundamental matrix $\Phi(t)$ for the given system such that $\Phi(0) = E$. It is to be noted here that $\Phi(t)$ satisfies the matrix equation

$$X' = AX. \tag{4.23}$$

Define for any real number s

$$Y(t) = \Phi(t+s).$$

Then $$Y'(t) = \Phi'(t+s) = A\Phi(t+s) = AY(t).$$

Hence $Y(t)$ is a solution of the matrix Equation (4.23) such that $Y(0) = \Phi(s)$. Now, suppose $Z(t) = \Phi(t)\,\Phi(s)$, for all t and s. Observe that $Z(t)$ is a solution of (4.23). Clearly $Z(0) = \Phi(0)\,\Phi(s) = E\Phi(s) = \Phi(s)$. So there are two solutions $Y(t)$ and $Z(t)$ of (4.23) such that $Y(0) = Z(0) = \Phi(s)$. By uniqueness property therefore it must be seen that $Y(t) = Z(t)$, whence the relation (4.22). The proof of the theorem is complete.

Example:

Consider the linear system $x' = A(t)x$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

We show that the matrix

$$\Phi(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & e^{-3t}\, t^2/2! \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

is fundamental. For this, we need to show that the three columns

$$e^{-3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e^{-3t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}, \quad e^{-3t}\begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}$$

are linearly independent. We can show that

$$e^{-3t}\left[c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix} \right]$$

implies that $c_1 = c_2 = c_3 = 0$.

Further we show that $\Phi(t)$ satisfies the given linear equation. Clearly

$$\Phi'(t) = e^{-3t}\begin{bmatrix} -3 & 1-3t & t-(3t^2/2!) \\ 0 & -3 & 1-3t \\ 0 & 0 & -3 \end{bmatrix}$$

$$= e^{-3t}\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= A\Phi(t).$$

Hence $\Phi(t)$ is a fundamental matrix.

In a subsequent section, we provide a method of finding fundamental matrix when the matrix A is constant.

Part -B (5x6=30 Marks)

Possible Questions:

1. Prove that the solution matrix φ of X'=A(t)X (t$\in I$) on I is a fundamental matrix of x'=A(t)x on I iff det φ (t)$\neq 0$ for t$\in I$.

2. Solve $x_1' = 5x_1 - 2x_2$; $x_2' = 2x_1 + x_2$.

3. Let $\varphi($ t), t$\in I$ denote a fundamental matrix of the system x'+Ax such that $\varphi($ 0)=E, denotes identity matrix , then P.T φ satisfies, $\varphi($ t+s)=$\varphi($ t)$\varphi($ s), for all values of t,s$\in I$

4. Find the first three successive approximation for the system $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $\begin{bmatrix} x_1\,(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5. State and prove the existence and uniqueness theorem on IVP.

6. Let A(t) be an n x n matrix which is continuous on I. Suppose a matrix φ satisfies the matrix X'=A(t)X, tϵI .Then Prove that det φ satisfies the first order equation (det φ)'=(trA)(det φ).

7. Find the four approximations of a solution to x”-2x’+x=0,x(0)=0,x’(0)=1.

8. Prove that the set of all solutions of the system x’=A(t)x on I form an n dimensional vector space over the field of complex numbers.

9. Solve 3 $x_1'+3x_1+4x_2=0$; $3x_2'+2x_1+3x_2=0$.

10. Find the fundamental matrix of the system x'(t) = A(t).x(t) where $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

Part -C (1x10=10 Marks)

Possible Questions

1.State and prove the existence and uniqueness theorem on IVP.

2. Find the fundamental matrix of the system x'(t) = A(t).x(t) where $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

3. Let φ be a fundamental matrix, for the system x'=A(t)x(t∈ I) ---(1) and let C be a constant non-singular matrix. Then prove that φc is also a fundamental matrix of x'=A(t)x .In addition prove that every fundamental matrix of (1) is of this type for some non-singular matrix C.

4. Solve : i) $3x_1'+3x_1+4x_2=0$; $3x_2'+2x_1+3x_2=0$.

ii) $x_1' = 5x_1 - 2x_2$; $x_2' = 2x_1 + x_2$.

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equations — **Subject Code: 17MMP104**
Class : I-M.Sc Mathematics — **Semester : I**

UNIT- II

System of Linear Differential Equations

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The IVP problems has __________ solution.	unique	infinte	finite	uncountable	unique
The general non-linear differential equation of order one is denoted in the form__________.	x′=f(x,t)	x′=f(t,x)	x=f(t,x)	x=f(x,t)	x′=f(t,x)
The first order non-homoeneous linear equation _______________ is a special case of x′=f(t,x).	x'-a(t)x=b(t), t є I	x'+x=b(t), t є I	x'+a(t)x=b(t), t є I	x'-x=b(t), t є I	x'+a(t)x=b(t), t є I
If the columns are linearly independent in the matrix φ then the matrix is called_________	a fundamental matrix	fundamental matrix of period w	non singular matrix	singular matrix	a fundamental matrix
The set of all solutions of the system x'=A(t)x on I forms an n-dimensional vector space over the field of _______ numbers.	complex	real	whole	integers	complex
The general non-linear differential equation of order _______ is denoted in the form x′=f(t,x).	1	2	3	4	1
In the inequality | f(t, x)-f(t,x)|≤K|x -x |, K is________________________	Variable in t	constant	Variable in x	Variable in x	constant
In lipschitz conditions, the value of K is ____________________________	≤ 0	≥0	<0	>0	<0
f(t, x)-f(t,x)=_____________________	∂f(t,x)	∂f(t, x)	∂f(t,x)	∂f(t,x)	∂f(t,x)
In the inequality | f(t, x)-f(t,x)|/|x -x |is _______________theorem	Intermediate value	average value	mean value	bounded value	mean value
The variable x(t,t ,x) is a function of ________________	t	x	t	x	t
The second approximation of x’=-x, x(0)=1, t≥0 is_________________________	1+ t	1-t	t	1	1-t
The solution for x’=x², x(0)=1 is ____________________	x(t)=1/t	x(t)=1/(t-1)	x(t)=1/(1-t)	x(t)=1/(1+t)	x(t)=1/(1-t)
The value of e^{-t} at t= ∞ is______________________	x(t)=0	x(t)=-1	x(t)= - ∞	x(t)=∞	x(t)=0
Let x(t)=1/(1-t) is the solution in________ interval	-∞ < t ≤1	-∞≤ t ≤1	-∞ < t <1	-∞ ≤ t < 1	-∞ < t <1
The solution for x’=-x, x(0)=1, t≥ 0 is ____________________	$x(t)=2e^{-t}$	$x(t)=-e^{-t}$	$x(t)=e^{-t}$	$x(t)=e^{t}$	$x(t)=e^{-t}$
The value of $1/e^{t}$ at t= ∞ is______________________	x(t)=0	x(t)=-1	x(t)= - ∞	x(t)=∞	x(t)=0
The solution for x’=x, x(0)=2, t≥ 0 is ____________________	$x(t)=2e^{-t}$	$x(t)=2e^{t}$	$x(t)=e^{t}$	$x(t)=e^{-t}$	$x(t)=2e^{t}$
The solution for x’= 2x/t, x(0)=0, t> 0 is ____________________	$x(t)=e^{t}$	$x(t)=t^{-2}$	$x(t)=t^{2}$	x(t)=t	$x(t)=t^{2}$
The solution for x’=x, x(0)=-2, t≥ 0 is ____________________	$x(t)=-2e^{-t}$	$x(t)=-2e^{t}$	$x(t)=e^{-2t}$	$x(t)=e^{2t}$	$x(t)=-2e^{t}$
The solution for x’=-x, x(0)=3, t≥ 0 is ____________________	$x(t)=3e^{-t}$	$x(t)=3e^{t}$	$x(t)=e^{-3t}$	$x(t)=e^{3t}$	$x(t)=3e^{t}$
The solution for x’=-x, x(0)=a (a is constant), t≥ 0 is ____________________	$x(t)=ae^{-t}$	$x(t)=ae^{t}$	$x(t)=e^{-at}$	$x(t)=e^{at}$	$x(t)=ae^{t}$
The existence of the solution x(t) in -∞<t<∞ is called __________________existence	local	non local	neighbourhood	solution	non local
The solution for x’=-x, x(0)=0, t≥ 0 is ____________________	$x(t)=e^{t}$	x(t)=0	$x(t)=e^{-t}$	x(t)=1	x(t)=0
The solution for x’=-x, x(0)=13, t≥ 0 is ____________________	$x(t)=13e^{-t}$	$x(t)=13e^{t}$	$x(t)=e^{-13t}$	$x(t)=e^{13t}$	$x(t)=13e^{-t}$
The solution for x’=-x, x(0)=c (c is constant), t≥ 0 is ____________________	$x(t)=ce^{-t}$	$x(t)=ce^{t}$	$x(t)=e^{-ct}$	$x(t)=e^{ct}$	$x(t)=ce^{-t}$
The solution for x’=x, x(0)=3p, t≥ 0 is ____________________	$x(t)=3pe^{-t}$	$x(t)=3pe^{t}$	$x(t)=e^{-3pt}$	$x(t)=e^{3pt}$	$x(t)=3pe^{t}$
The solution for x’=-x, x(0)=31, t≥ 0 is ____________________	$x(t)=31e^{-t}$	$x(t)=31e^{t}$	$cx(t)=e^{-81t}$	$x(t)=e^{81t}$	$x(t)=31e^{-t}$
The solution for x’=-x, x(0)=4.9, t≥ 0 is ____________________	$x(t)=4.9e^{-t}$	$x(t)=4.9e^{t}$	$x(t)=e^{-4.9t}$	$x(t)=e^{4.9t}$	$x(t)=4.9e^{-t}$
The solution for x’=-x, x(0)=9, t≥ 0 is ____________________	$x(t)=9e^{-t}$	$x(t)=9e^{t}$	$x(t)=e^{-9t}$	$x(t)=e^{9t}$	$x(t)=9e^{-t}$
The __________has unique solution.	boundary value problem	local exixtence problem	cinitial value problem	none of the above	initial value problem

The __________ equation possesses a reular singular point at t=0.	Bessel equation	legendre equation	lagrange equation	hermite equation	Bessel equation
The second order linear _____________ equation is of the form x'' + a1(t)x'+a2(t)x= 0.	non-homogeneous	homogeneous	lagrange equation	hermite equation	homogeneous
If the columns are linearly _______________ in the matrix φ then the matrix is called fundamental matrix.	linear	non-linear	independent	dependent	independent
The variable _____________ is a function of t.	$x(t,t_0)$	$x(t,t,x_0)$	$x(t,t_0,x)$	$x(t,t_0,x_0)$	x(t,t ,x)
In ___________ conditions, the value of K is <0.	lipschitz	non-linear	independent	dependent	lipschitz
The linearity principle for ______________________________ holds for linear differential equation.	Non-homogeneous equation	ordinary differential equation	Homogeneous equation	dnon-linear equation	ordinary differential equation
If A=2I,then the ___________is 4.	tr(A')	tr(A.A)	tr(A)	tr(0)	tr(A)
A real or complex-valued function φ defined on a non-empty subset is said to be a ________ if it possesses the first order derivative.	function	solution	order	degree	solution
A differential equation of ___________order of the form x'=g(t)h(x) is called an equation with variables separable.	first	second	third	fourth	first

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equation	**Semester :I**	**L T P C**
Subject Code: 17MMP104	**Class : I- M.Sc Mathematics**	**4 0 0 4**

UNIT -III

Non homogeneous linear system – linear systems with constant coefficient – Linear systems with periodic coefficients.

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UNIT – III

System of Linear Differential Equations

NON-HOMOGENEOUS LINEAR SYSTEMS

Theorem 3.1:

Let $\Phi(t)$ be a fundamental matrix for the system (4.15 (a)) for $t \in I$. Then ϕ, defined by (4.28), is a solution of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = 0. \tag{4.29}$$

Now let us assume that $x_h(t)$ is a solution of the IVP

$$x' = A(t)x, \quad x(t_0) = x_0, \quad t, t_0 \in I. \tag{4.30}$$

Then $F(t) = x_h(t) + \phi(t)$ is also a solution of the Equation (4.25). For

$$\begin{aligned} F'(t) &= x_h'(t) + \phi'(t) \\ &= A(t)\, x_h(t) + A(t)\, \phi(t) + b(t) \\ &= A(t)[x_h(t) + \phi(t)] + b(t) \\ &= A(t)\, F(t) + b(t). \end{aligned}$$

Further $\quad F(t_0) = x_h(t_0) + \phi(t_0) = x_0$

Hence $\quad F(t) = x_h(t) + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, b(s)\, ds \quad (4.31)$

is the solution of $x' = A(t)x + b(t), x(t_0) = x_0$.

Since $\Phi(t)$ is a fundamental matrix, the solution $x_h(t)$ may be written as

$$x_h(t) = \Phi(t)c$$

where c is a constant vector. Further, since $x_h(t_0) = x_0$, we have

$$x_h(t_0) = \Phi(t_0)c = x_0$$

LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

systems in an explicit form several difficulties are encountered. In fact there are very few situations when the solution can be found explicitly. The aim of this section is to develop a method to find the solution of (4.15) with the assumption that $A(t)$ is a constant matrix. The method involves first finding the characteristic values of the matrix A. If the characteristic values of a matrix A are known then, in general, a solution can be obtained in an explicit form. Note that when the matrix $A(t)$ is variable, it is usually difficult to find solutions.

Before proceeding further, recall the definition of the exponential of a given matrix A. It is defined as follows

$$\exp A = E + \sum_{p=1}^{\infty} \frac{A^p}{p!}.$$

Also, if A and B are two matrices which commute then

$$\exp (A + B) = \exp A \exp B.$$

For the present assume the proofs of the convergence of the sum through which $\exp A$ is defined and the result stated above. So by definition

$$\exp (tA) = E + \sum_{p=1}^{\infty} \frac{t^p A^p}{p!}.$$

Here it is noted that the infinite series for $\exp (tA)$ converges uniformly on every compact interval.

Now consider a linear homogeneous system with a constant matrix, namely

$$x' = Ax, \quad t \in I \tag{4.32}$$

where I is an interval in R. From Chapter 1, recall that the solution of (4.32), when A and x are scalers, is $x(t) = ce^{tA}$ for an arbitrary constant c. A similar situation prevails when we deal with (4.32). We prove the following theorem.

Theorem 3.2

The general solution of the system (4.32) is $x(t) = e^{tA}c$ where c is an arbitrary constant vector. Further, the solution of (4.32) with the initial condition $x(t_0) = x_0$, $t_0 \in I$ is given by

$$x(t) = e^{(t-t_0)A}x_0, \quad t \in I. \tag{4.33}$$

Proof Let $x(t)$ be any solution of (4.32). Define a vector $u(t)$ by $u(t) = e^{-tA}x(t)$, $t \in I$. Then it follows that

$$u'(t) = e^{-tA}(-Ax(t) + x'(t)), \quad t \in I.$$

Since $x(t)$ is a solution of (4.32), $u'(t) \equiv 0$. It means that $u(t) = c$, $t \in I$; where c is some constant vector. Substituting the value c for $u(t)$, it is seen that $x(t) = e^{tA}c$. Employing the given initial condition $x(t_0) = x_0$, it follows that $c = e^{-t_0A}x_0$. Hence, we get $x(t) = e^{tA} \cdot e^{-t_0A}x_0$. Since A commutes with itself, it is seen that $x(t) = e^{(t-t_0)A}x_0$ which is (4.33).

In particular, choose $t_0 = 0$ and n linearly independent vectors e_j, $j = 1, 2, \ldots, n$, the vector e_j being the vector with 1 at the jth component and zero elsewhere. In this case we get n linearly independent solutions corresponding to the set of n vectors $(e_1, e_2, \ldots, e_n)$. Thus a fundamental matrix for (4.32) is

$$\Phi(t) = e^{tA}E = e^{tA}, \quad t \in I; \tag{4.34}$$

since the matrix with columns represented by $e_1, e_2, \ldots, e_n$ is the identity matrix E. Thus e^{tA} solves the matrix differential equation

$$X' = AX, \; x(0) = E; \; t \in I. \tag{4.35}$$

Example:

Consider a similar example to determine a fundamental matrix for

$x' = Ax$, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$. Notice that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.$$

By the remark given before Theorem 4.8, it is known that the fundamental matrix in this case is given by

$$\exp(tA) = \exp\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t \exp\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t,$$

since $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ commute. But

$$\exp\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t = \exp\begin{bmatrix} 3t & 0 \\ 0 & 3t \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}.$$

Observe that

$$\exp\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t = E + \sum_{p=1}^{\infty} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^p \frac{t^p}{p!}$$

and

$$\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{2p} = \begin{bmatrix} 2^{2p} & 0 \\ 0 & 2^{2p} \end{bmatrix}, \quad p = 1, 2, 3, \ldots.$$

$$\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{2p+1} = \begin{bmatrix} 0 & -2^{2p+1} \\ -2^{2p+1} & 0 \end{bmatrix}, \quad p = 0, 1, 2, \ldots.$$

Hence

$$\exp\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{p=1}^{\infty} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{2p} \frac{t^{2p}}{(2p)!}$$

$$+ \sum_{p=0}^{\infty} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{2p+1} \frac{t^{2p+1}}{(2p+1)!}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{p=1}^{\infty} \begin{bmatrix} 2^{2p} & 0 \\ 0 & 2^{2p} \end{bmatrix} \frac{t^{2p}}{(2p)!}$$

$$+ \sum_{p=0}^{\infty} \begin{bmatrix} 0 & -2^{2p+1} \\ -2^{2p+1} & 0 \end{bmatrix} \frac{t^{2p+1}}{(2p+1)!}$$

$$= \begin{bmatrix} 1 + \sum_{p=1}^{\infty} \frac{(2t)^{2p}}{(2p)!} & - \sum_{p=0}^{\infty} \frac{(2p)^{2p+1}}{(2p+1)!} \\ - \sum_{p=0}^{\infty} \frac{(2t)^{2p+1}}{(2p+1)!} & 1 + \sum_{p=1}^{\infty} \frac{(2t)^{2p}}{(2p)!} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{2t} + e^{-2t} & e^{-2t} - e^{2t} \\ e^{-2t} - e^{2t} & e^{2t} + e^{-2t} \end{bmatrix}.$$

Hence, if follows that

$$e^{tA} = \frac{1}{2} \begin{bmatrix} e^{5t} + e^{t} & e^{t} - e^{5t} \\ e^{t} - e^{5t} & e^{5t} + e^{t} \end{bmatrix}.$$

From Theorem 4.8 it is learnt that the general solution of the system (4.32) is $e^{tA}c$ but the nature of e^{tA} is yet to be known. Once e^{tA} is determined the solution of (4.32) is completely obtained.

In order to be able to do this the procedure given below is followed. Choose a solution of (4.32) in the form

$$x(t) = e^{\lambda t}c \tag{4.36}$$

where c is a constant vector and λ is a scalar. $x(t)$ is determined if λ and c are known. Substituting (4.36) in (4.32), we get

$$(\lambda E - A)c = 0. \tag{4.37}$$

Observe that c is a constant vector $(c_1, \ldots, c_n)$. Hence (4.37) is equivalent to

$$\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{1n} \\ -a_{21} & \lambda - a_{22} & -a_{2n} \\ \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \lambda - a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0.$$

Hence

$$\begin{aligned} (\lambda - a_{11})c_1 - a_{12}c_2 - \ldots - a_{1n}c_n &= 0 \\ -a_{21}c_1 + (\lambda - a_{22})c_2 - \ldots - a_{2n}c_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad & \ \ \vdots \\ -a_{n1}c_1 - a_{n2}c_2 - \ldots + (\lambda - a_{nn})c_n &= 0. \end{aligned} \tag{4.37 (a)}$$

This is a system of n-algebraic homogeneous linear equations in unknowns $c_1, c_2, \ldots, c_n$. This system of equations has a nontrivial solution (i.e. different from $c_1 = c_2 = \ldots = c_n = 0$) if and only if the determinant of the coefficients, namely

$$\det(\lambda E - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \ldots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & & \lambda - a_{nn} \end{vmatrix} = 0.$$

This determinant is a polynomial of degree n in λ. Let us denote it by $p(\lambda)$, i.e.

$$p(\lambda) = \det(\lambda E - A) = 0 \qquad (4.38)$$

is called the characteristic equation for the matrix A. This being an nth order polynomial equation in λ, it admits n solutions which may be distinct, repeated, real or complex.

The roots of (4.38) are called the "eigenvalues" or the "characteristic values" of A. Let λ_1 be an eigenvalue of A and corresponding to this eigenvalue, let c_1 be the non-trivial solution of (4.37). The vector c_1 is called an "eigenvector" of A corresponding to the eigenvalue λ_1. Note that any constant multiple of c_1 is also an eigenvector. Then

$$x_1(t) = e^{\lambda_1 t} c_1$$

is a solution of the system (4.32). Now suppose that all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct and that $c_1, c_2, \ldots, c_n$ are the distinct eigenvectors respectively. Then it is clear that $x_k(t) = e^{\lambda_k t} c_k$ $(k = 1, 2, \ldots, n)$ are n solutions of the system (4.32). Note that the eigenvectors corresponding to distinct eigenvalues are linearly independent. Thus $\{x_k(t)\}, k = 1, 2, \ldots, n$ is a set of n linearly independent vector functions, which are solutions of (4.32). So by the principle of superposition the general solution of the linear system is

$$k = 1$$

Now consider the vectors

$$e^{\lambda_1 t} c_1, e^{\lambda_2 t} c_2, \ldots, e^{\lambda_n t} c_n.$$

Let these vectors be columns of an $n \times n$ matrix $\Phi(t)$. So by construction, Φ has n linearly independent columns which are solutions of (4.32) and hence Φ is a fundamental matrix. Since e^{tA} is also a fundamental matrix, from Theorem 4.5, it is therefore seen that $e^{tA} = \Phi(t)D$ where D is some non-singular constant matrix. A word of caution is warranted here. Note that the above discussion is based on the assumption that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.

LINEAR SYSTEMS WITH PERIODIC COEFFICIENTS

Theorem 3.3:

The necessary and sufficient condition for the system (4.43) to admit a non-zero periodic solution of period ω is that $E - e^{A\omega}$ is singular. (E is the identity matrix.)

Proof The general non-zero solution of (4.43) is $x(t) = e^{At}c$ where c is an arbitrary non-zero constant vector. So by definition $x(t)$ is periodic, of period $\omega \neq 0$, if and only if $x(t) = x(t+\omega) = e^{At}e^{A\omega}c = e^{At}c$.

From the above equation it follows that (4.43) has a non-zero periodic solution if and only if $(E - e^{A\omega})c = 0$. But it is known that c is a non-zero vector and so system (4.43) has a non-zero periodic solution of period ω if and only if $E - e^{A\omega}$ is singular. The proof is complete.

It is to be observed that Theorem 4.9 is also interesting in itself. It throws light on the non-singularity of the matrix $E - e^{A\omega}$. In fact, it states a criterion for the non-singularity of $E - e^{A\omega}$.

Consider the forced system

$$x' = Ax + f(t), \quad t \in (-\infty, \infty) \tag{4.44}$$

where f is a continuous vector function on $(-\infty, \infty)$. Firstly a characterisation for a periodic solution of period ω for (4.44) is dealt with under the assumption that $f(t)$ is periodic with period ω. Then we try to connect the criterion for periodic solutions for (4.44) in the light of the corresponding unforced system (4.43).

Theorem 3.4:

Let $f(t)$ be periodic with period ω. Then a solution $x(t)$ of (4.44) is periodic of period ω if and only if $x(0) = x(\omega)$.

Proof Let $x(t)$ be a periodic solution with period ω. Then $x(0) = x(\omega)$. The condition is necessary. For sufficiency, assume that $x(t)$ is a solution of (4.44) such that $x(0) = x(\omega)$. Let $u(t) = x(t+\omega)$. Then $u'(t) = x'(t+\omega) = Ax(t+\omega) + f(t+\omega)$ $= Au(t) + f(t)$. This shows that $u(t)$ is a solution of (4.44) and in addition $u(0) = x(\omega) = x(0)$. The uniqueness of solutions therefore shows that $x(t) \equiv u(t) \equiv x(t+\omega)$ which shows that $x(t)$ is periodic with period ω.

Many times it is interesting to study properties of solutions of (4.44) in the light of the associated system (4.43). A study of this type indicates the many-sided implications of the forcing term $f(t)$. The following is one such implication.

Theorem 3.5

Let $f(t)$ be continuous on $(-\infty, \infty)$ and periodic with period ω. A necessary and sufficient condition for the system (4.44) to have a unique periodic solution with period ω is that the system (4.43) has no non-zero periodic solution of period ω.

Proof The general solution of (4.44) is given by

$$x(t) = e^{At}c + \int_0^t e^{A(t-s)} f(s)\, ds.$$

Note here that $x(0) = c$. Now

$$x(\omega) = e^{A\omega}c + \int_0^\omega e^{A(\omega-s)} f(s)\, ds.$$

But from Theorem (4.10) there is a periodic solution of period ω for (4.44) if and only if

$$x(0) = c = x(\omega) = e^{A\omega}c + \int_0^\omega e^{A(\omega-s)} f(s)\, ds,$$

that is, if and only if, for some c

$$(E - e^{A\omega})c = \int_0^\omega e^{A(\omega-s)} f(s)\, ds. \qquad (4.45)$$

Hence there exists a unique periodic solution for (4.44) if and only if the Equation (4.45) has a unique solution c for any periodic function f. But it has a unique solution c if and only if $E - e^{A\omega}$ is non-singular. Thus the conclusion of the theorem follows by an application of Theorem 4.9.

Let us now consider a linear system

$$x' = A(t)x \qquad (4.46)$$

where $A(t)$ is a continuous $n \times n$ matrix such that

$$A(t+\omega) = A(t),\ \ \omega \neq 0,\ -\infty < t < \infty \qquad (4.47)$$

and that ω is the minimal period. Let $\Phi(t)$ denote a fundamental matrix for (4.46). We prove the following basic result.

Theorem 3.6

Let Φ denote a fundamental matrix for (4.46). Then $\Phi(t+\omega)$, $(-\infty < t < \infty)$, is also a fundamental matrix for (4.46).

Proof The fundamental matrix Φ satisfies the relation

$$\Phi'(t) = A(t)\,\Phi(t), \quad (-\infty < t < \infty).$$

Clearly,

$$\Phi'(t+\omega) = A(t+\omega)\,\Phi(t+\omega)$$
$$= A(t)\,\Phi(t+\omega), \quad (-\infty < t < \infty).$$

Further, note that $\det \Phi(t+\omega) \not\equiv 0$. Hence, in view of Theorem 4.4 we conclude that $\Phi(t+\omega)$ is also a fundamental matrix. The proof is complete.

Since $\Phi(t)$ and $\Phi(t+\omega)$ are fundamental solution matrices for (4.46), there exists a non-singular constant matrix C such that

$$\Phi(t+\omega) = \Phi(t)C. \tag{4.48}$$

It is known that corresponding to a non-singular constant matrix C there exists a matrix R such that

$$C = e^{\omega R} \tag{4.49}$$

We make use of this fact in the following well-known result due to Floquet.

Theorem 3.7

Let $\Phi(t)$ be a fundamental matrix for (4.46) where the matrix $A(t)$ satisfies the condition (4.47). Then there exists a periodic non-singular matrix P such that $P(t+\omega) = P(t)$, $-\infty < t < \infty$, and a constant matrix R such that

$$\Phi(t) = P(t)\,e^{tR}, \quad (-\infty < t < \infty). \tag{4.50}$$

Proof In view of the relations (4.48) and (4.49), we get

$$\Phi(t+\omega) = \Phi(t)\,e^{\omega R}, \quad (-\infty < t < \infty).$$

Let $P(t)$ denote the matrix $\Phi(t)e^{-tR}$. Then

$$P(t+\omega) = \Phi(t+\omega)\,e^{-(t+\omega)R}$$
$$= \Phi(t)\,e^{\omega R}e^{-(t+\omega)R}$$
$$= \Phi(t)\,e^{-tR}$$
$$= P(t).$$

Part -B (5x6=30 Marks)

Possible Questions:

1 Find a fundamental matrix for X'=AX, where A = $\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

2. Let P(t) and R be the matrices obtained in Floquet theorem. Prove that the transformation x =P(t) reduce the linear system x'=A(t)x to the system z' = Rz.

3. Let f(t) be periodic with period ω. Prove that there is a solution x(t) of x' = Ax + f(t), t ϵ (-∞, ∞) is periodic of period ω iff x(0) = x(ω).

4. Determine the variation of parameter formula.

5. Find a fundamental matrix e^{tA} where A = $\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

6. Show that $\varphi(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & e^{-3t}t\frac{2}{2!} \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$ is a fundamental matrix of the linear system x(t)' = A(t).x(t) where x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, A = $\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

7. Determine e^{tA} for the system x^{-1} = Ax where A=$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

8. State and prove Floquet theorem.

9. The general solution of system x'=Ax, t∈ I is $x(t)$= $e^{t(A)}$c where c is an arbitrary constant vector. Then prove that the solution of x'=Ax with initial condition $x(t_0)=x_0$, $t_0 \in I$,is given by $x(t)=e^{(t-t_0)A}x_0$, t∈ I.

10. . Determine e^{tA} for the system X'=AX where A=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Part –C (1x10=10 Marks)

Possible Questions

1.Show that $\varphi(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & e^{-3t}t^{\frac{2}{2!}} \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$ is a fundamental matrix of the linear system

x(t)' = A(t).x(t) where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

2. Determine e^{tA} for the system X'=AX where $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

3. Determine e^{tA} for the system $x^{-1} = Ax$ where $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. Determine the variation of parameter formula and Find a fundamental matrix e^{tA}

where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

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DEPARTMENT OF MATHEMATICS
Subject: Ordinary Differential Equations **Subject Code: 17MMP104**
Class : I-M.Sc Mathematics **Semester : I**

UNIT- III

System of Linear Differential Equation

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Let $f_1, f_2, \ldots f_n$ be given n real valued functions defined on some open connected set be contained in ______ dimensional space	n-1	n	n+1	n+1/2	n+1
Let A=(t_0 , α , α ………..αn) is a point in D . then the dimension of a is _____	n-1	n	n+1	n+1/2	n+1
The system of the equation x′=A(t)x where t ЄI is called_______	Homogeneous	non homogenous	linear	non linear	Homogeneous
The n^{th} order differential equation can be reduced from ______ system of equation	n	n+1	n-1	1	n
The set of all solution of the system is in the field of ______	real	complex	rational	exponential	complex
The solution of the x''-2x'+x=0 x(0)=0 , x'(0)=1 where tЄ[0,a] is	(te^t, e^t)	$(te^t+(1+t)e^t)$	(e^t, te^t)	$(0, e^t)$	$(te^t+(1+t)e^t$
The solution of x'=A(a) is _______________	x(a) =x $_o$+ $_{ao}\int^a$ A(s) X(s) dx, tЄI	x(a) = $_{ao}\int^a$ A(s) X(s) dx, tЄI	x(a) =x $_o$.$_{ao}\int^a$ A(s) X(s) dx, tЄI	x(a) =x $_o$+ $_{ao}\int^a$ A(s) X(s) ds, tЄI	
If A=2I,then the tr(A) is ____________	6	4	14	7	4
A solution matrix of x'=A(t)x on is a fundamental matrix on iff ______	det φ(t)=0	det φ(t)≠0	det φ'(t)≠0	A(t)≠0	det φ(t)≠0
If φ is a fundamental matrix of X'=A(t)X on I. If C be any constant,then _______ is also a fundamental matrix.	φ+C	φ-C	Cφ	φ	Cφ
The solution of x='A(t)x is___________________________	x(a) =x $_o$+ $_{ao}\int^a$ A(s)x(s))ds	x(a) =x + $_{ao}\int^a$ A(s)x(s))ds	x(a) =x $_o$+ $_{ao}\int^a$ A(a)x(s))ds	x(a) =x + $_{ao}\int^a$ A(a)x(s))ds	x(a) =x $_o$+ $_{ao}\int^a$ A(s)x(s))ds
The set of all solution of the system are ______	linearly independent	linearly dependent	linear	unique	linearly independent
exp(A+B)=___________________________	exp(A)+exp(B)	exp(A)exp(B)	exp(A)-exp(B)	exp(A)/exp(B)	exp(A)+exp(B)
exp(AB)=___________________________	exp(A)+exp(B)	exp(A)exp(B)	exp(A)-exp(B)	exp(A)/exp(B)	exp(A)exp(B)
log(ab)=____________________________	log(a)+log(b)	log(a)-log(b)	log(a)log(b)	log(a)/log(b)	log(a)/log(b)
log(a+b)	log(a)+log(b)	log(a)-log(b)	log(a)log(b)	log(a)/log(b)	log(a)-log(b)
If φ(t) is a fundamental matrix, then φ'(t)=________________	A(t)φ'(t)	A(t)φ(t)	A'(t)φ'(t)	A(t+w)φ(t)	A(t)φ(t)
The system x'=-A^t(t)(x) is __________ to x'=A(t)x	adjacent	adjoint	opposite	equal	adjoint
If x(t)=de^{at}, then x(t+w)=_________________________________w is period	e^{at}	de^{at}	e^{dat}	Ae^{dat}	de^{at}
If φ(t)and φ(t+w) are a fundamental matrix for x'=A(t)x, then(1/ φ(t))φ(t+w)=________________	I	singular	scalar matrix	constant matrix	constant matrix
exp(r +r +……………….r_a)w=_________________where r_e are characteristic roots	tr φ(w)	det(1/ φ(w))	det φ(w)	det(r_e)	det φ(w)
A solution matrix of x'=a(t)x t ЄI with the initial condition x(t)=x , t ЄI is ______	$e^{at-ato}x_0$	e^{ato} x	e^{at} x	$e^{at\ \ ato}$ x	$e^{at\ \ ato}$ x
For any two differential matrix X and Y , d/dt(XY)_________________	d/dt(X)Y+Xd/dt(Y)	d/dt(X)+Xd/dt(Y)	d/dt(X)+d/dt(Y)	d/dt(XY)+d/dt(Y)	d/dt(X)Y+Xd/dt(Y)
For any two differential matrix A, d/dt(1/A)=_____________	-(1/A)d/dt(1/A) A	-(1/A)(d/dt(A))(1/A)	d/dt(1/A)A	(1/A)d/dt(1/A)A	-(1/A)(d/dt(A))(1/A)
If the columns are linearly independent in the matrix φ then the matrix is called_________	column matrix	fundamental matrix	soltion matrix	Identity matrix	fundamental matrix
If φ(t) is a fundamental matrix for x'=A(t)x, then φ(t+w)=________________	a fundamental matrix	fundamental matrix of period w	non singular matrix	singular matrix	a fundamental matrix
which of the following equation is periodic but solution is not periodic	x'=cos²t	x'=x	x'=cos²x	x'=cos t	x'=cos²t
If A(t) is nXn matrix continuous in t on_______	closed	bounded	closed and bounded	open	closed and bounded
If φ(t) is a fundamental matrix , then φ(t+s)=________________	φ(t)φ(s)	φ(t)+φ(s)	φ(t)-φ(s)	φ(t)/φ(s)	φ(t)φ(s)
If φ(t) is a fundamental matrix , then φ()=________________	column matrix	fundamental matrix	soltion matrix	Identity matrix	Identity matrix

If $(1/\varphi(t))^t$ is a fundamental matrix for the equation________________	d/dt $(1/\varphi(t))^t$=-A^t $(1/\varphi(t))^t$	d/dt $(1/\varphi(t))^t$=A^t $(1/\varphi(t))^t$	d/dt $(1/\varphi(t))^t$=-A^t $(1/\varphi(t))^t$	d/dt $(1/\varphi(t))^t$=-A $(1/\varphi(t))^t$	d/dt $(1/\varphi(t))^t$=-A^t $(1/\varphi(t))^t$
The system x'=-A^t(t)x, t ЄI has the fundamental matrix of the form_____________	(1/φ(t))	$(\varphi(t))^t$	$(1/\varphi(t))^t$	(1/φ(t))	$(1/\varphi(t))^t$
The solution of x'=Ax+f(t),t Є(-∞,∞) is period w iff x(0)=___	x(∞)	x(-∞)	x(w)	x(t)	x(w)
The solution of x"+x=cost is _______________	x(t)=acos(t+b)+(1/2)t sint	x(t)=acos(t+b)	x(t)=cos(t)+(1/2)t sint	x(t)=acos(t-b)+(1/2)t sint	x(t)=acos(t+b)+(1/2)t sint
If x'=ax has non zero periodic solution of period b then $\|E-e^{ab}\|$is___________	zero	not zero	1	not define	zero
A differential equation of first order of the form__________is called an equation with variables separable.	x'=g'(t)h'(x)	x'=g'(t)h(x)	x'=g(t)h'(x)	x'=g(t)h(x)	x'=g(t)h(x)
The class of nth order equations is divided mainly into ______ sub-classes.	one	two	three	fourth	two
A real or complex-valued function φ defined on a _________ subset is said to be a solution if it possesses the first order derivative.	Linearly independent	Independent	non-empty	empty	non-empty
The order of equation is $(D^2+2D-8)y=0$ is______	1	2	0	8	2
A function is said to be homogeneous function of degree _______ if for any t, f(tx, ty) = t^n f(x, y)	n-1	n+1		n	n

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equation **Semester : I** **L T P C**

Subject Code: 17MMP104 **Class : I- M.Sc Mathematics** **4 0 0 4**

UNIT -IV

Successive approximation – Picard's theorem – Non uniqueness of solution – continuation and dependence on initial conditions – existence of solution in the large existence and uniqueness of solution in the system.

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UNIT – IV

EXISTENCE AND UNIQUENCE OF SOLUTION

Definition: A function $f(t, x)$ defined in a region $D \subset R^2$ is said to satisfy Lipschitz condition in the variable x with a Lipschitz constant K, if the inequality

$$|f(t, x_1) - f(t, x_2)| \le K|x_1 - x_2| \quad (5.1)$$

holds whenever $(t, x_1), (t, x_2)$ are in D. In such a case we denote f to be a member of the class Lip (D, K).

As a consequence of the definition, a function $f(t, x)$ satisfies Lipschitz condition if and only if there exists a constant $K > 0$ such that

$$\frac{|f(t, x_1) - f(t, x_2)|}{|x_1 - x_2|} \le K, \quad x_1 \ne x_2$$

whenever $(t, x_1), (t, x_2)$ belong to D.

The question which may arise is to find a general criterion which would ensure the Lipschitz condition. The following theorem shows the existence of a typical class of such functions. For simplicity, we assume the region D to be a closed rectangle.

Theorem 4.1:

Let $f(t, x)$ be a continuous function defined over a rectangle $R = \{(t, x) : |t - t_0| \le p, |x - x_0| \le q\}$. Here p, q are some positive real numbers. Let $\frac{\partial f}{\partial x}$ be defined and continuous on R. Then $f(t, x)$ satisfies the Lipschitz condition in R.

Proof Since $\frac{\partial f}{\partial x}$ is continuous on R there exists a positive constant A such that

$$\left|\frac{\partial f}{\partial x}(t, x)\right| \le A \quad (5.2)$$

for all (t, x) in R. Let $(t, x_1), (t, x_2)$ be any two points in R. Then by the mean value theorem of differential calculus, there exists a number s which lies between x_1 and x_2 such that

$$f(t, x_1) - f(t, x_2) = \frac{\partial f}{\partial x}(t, s)(x_1 - x_2),$$

Since the point (t, s) lies in R and the inequality (5.2) holds, it is seen that

$$\left|\frac{\partial f}{\partial x}(t, s)\right| \le A.$$

Hence we have

$$|f(t, x_1) - f(t, x_2)| \le A|x_1 - x_2|$$

whenever (t, x_1), (t, x_2) are in R. The proof is complete.

The following example illustrates that the existence of partial derivative of f is not necessary for f to be a Lipschitz function.

Example:

Let $f(t, x) = |x|$ on the unit square R around the origin, namely,

$$R = \{(t, x) : |t| \le 1, \quad |x| \le 1\}.$$

The partial derivative of f at $(t, 0)$ fails to exist but f satisfies Lipschitz condition in x on R with Lipschitz constant $K = 1$.

The example below shows that there exist functions which do not satisfy the Lipschitz condition.

Example:

Let $f(t, x) = x^{1/2}$ be defined on the rectangle

$$R = \{(t, x) : |t| \le 2, \quad |x| \le 2\}.$$

Then f does not satisfy the inequality (5.1) in R. This is because

$$\frac{f(t, x) - f(t, 0)}{x - 0} = x^{-1/2}, \quad x \ne 0,$$

is unbounded in R, since it can be made as large as possible by choosing x close to zero.

Gronwall Inequality

The integral inequality, due to Gronwall, plays a useful part in the study of several

Theorem 4.2: Assume that $f(t)$ and $g(t)$ are non-negative continuous functions for $t \geq t_0$. Let $k > 0$ be a constant. Then the inequality

$$f(t) \leq k + \int_{t_0}^{t} g(s) f(s)\, ds, \quad t \geq t_0$$

implies the inequality

$$f(t) \leq k \exp\left(\int_{t_0}^{t} g(s)\, ds\right), \quad t \geq t_0.$$

Proof By hypothesis we have

$$\frac{f(t)\, g(t)}{k + \int_{t_0}^{t} g(s) f(s)\, ds} \leq g(t), \quad t \geq t_0.$$

Noting that $f(t)\, g(t)$ is the derivative of $k + \int_{t_0}^{t} g(s) f(s)\, ds$, integration of this inequality between the limits t_0 to t, leads to

$$\log\left(k + \int_{t_0}^{t} g(s) f(s)\, ds\right) - \log k \leq \int_{t_0}^{t} g(s)\, ds$$

or, in other words,

$$k + \int_{t_0}^{t} g(s) f(s)\, ds \leq k \exp\left(\int_{t_0}^{t} g(s)\, ds\right).$$

This inequality together with the hypothesis leads to the desired conclusion.

Corollary : If, for $t \geq t_0$,

$$f(t) \leq k \int_{t_0}^{t} f(s)\, ds$$

where f and k are as given in Theorem 5.2 then, $f(t) = 0$ for $t \geq t_0$.

Proof From the hypothesis it is clear that for $t \geq t_0$ and any $\varepsilon > 0$

$$f(t) < \varepsilon + k \int_{t_0}^{t} f(s)\, ds.$$

The application of the above theorem yields

$$f(t) < \varepsilon \exp k(t - t_0), \quad t \geq t_0.$$

Let $\varepsilon \to 0$. This leads to the fact that $f(t) = 0$ for $t \geq t_0$.

SUCCESSIVE APPROXIMATIONS

Lemma:

$x(t)$ is a solution of (5.3) on some interval I if and only if $x(t)$ is a solution of (5.4).

Proof If $x(t)$ is a solution of (5.3) then it is easy to show that $x(t)$ satisfies (5.4). Let $x(t)$ be a solution of (5.4). Obviously $x(t_0) = x_0$. Differentiating both sides of (5.4), and noting that $f(t, x)$ is continuous in (t, x), it is seen that $x'(t) = f(t, x(t))$ which completes the proof.

Now we are set to define certain approximations to a solution of (5.3). First of all we start with an approximation to a solution and improve it by iteration. It is expected that these iterations converge to a solution of (5.3) in the limit. The importance of Equation (5.4) now springs up. In this connection, we mention that the estimates can be handled easily with integrals rather than derivatives.

A rough approximation to a solution of (5.3) is just the constant function $x_0(t) = x_0$. We may get a better approximation by substituting $x_0(t)$ in the right hand sides of (5.4), thus obtaining a new approximation $x_1(t)$ given by

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds.$$

To get a still better approximation we repeat the process thereby defining

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\, ds.$$

In general,

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))\, ds, \quad n = 1, 2, \ldots \tag{5.5}$$

This procedure is known in the literature as "Picards' method of successive approximations". We conclude this section with a few examples. In the next section we show that the sequence $\{x_n(t)\}$ does converge to a unique solution of (5.3) provided $f(t, x)$ satisfies the desired condition.

Example:

Consider the IVP $x' = -x$, $x(0) = 1$, $t \geq 0$. It is equivalent to the integral equation

$$x(t) = 1 - \int_{t_0}^{t} x(s)\, ds.$$

The first approximation is given by $x_0(t) \equiv 1$. The second approximation is

$$x_1(t) = 1 - \int_0^t x_0(s)\, ds = 1 - t.$$

By the definition of successive approximations, it follows that

$$x_2(t) = 1 - \int_{t_0}^{t} (1 - s)\, ds = 1 - \left(t - \frac{t^2}{2}\right).$$

In general, the $(n + 1)$th approximation is

$$x_n(t) = 1 - \left[t - \frac{t^2}{2} + \ldots + (-1)^n \frac{t^n}{n!}\right].$$

We recognize here that $x_n(t)$ is the $(n + 1)$th partial sum of the series for e^{-t}. It is easy to note that e^{-t} is the solution of the IVP under consideration.

Example:

Consider the IVP $x' = x^2$, $x(0) = 1$. The equation is equivalent to the integral equation

$$x(t) = 1 + \int_{t_0}^{t} x^2(s)\, ds.$$

The first approximation is $x_0(t) = 1$. Now

$$x_1(t) = 1 + \int_{t_0}^{t} 1\, ds = 1 + t$$

$$x_2(t) = 1 + \int_0^t (1 + s)^2\, ds = 1 + t + t^2 + \frac{t^3}{3},$$

$$x_3(t) = 1 + \int_0^t \left(1 + s + s^2 + \frac{s^3}{3}\right)^2 ds = 1 + t + t^2 + t^3 + \frac{2t^4}{3} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{t^7}{63}.$$

All $x_n(t)$, $n = 0, 1, 2, \ldots$ are polynomials.

Observe that the IVP can be solved explicitly by the method of separation of variables. Here

$$x(t) = \frac{1}{1 - t}$$

is a solution existing on $-\infty < t < 1$.

PICARD'S THEOREM

Theorem 4.3 :

Let $h = \min\left(a, \frac{b}{L}\right)$. Then the successive approximations given by

(5.5) are valid on $I = |t - t_0| \leq h$. Further

$$|x_j(t) - x_0| \leq L|t - t_0| \leq b, \quad j = 1, 2, \ldots, t \in I. \tag{5.6}$$

Proof The method of induction is used to prove the lemma. Since we start with any point (t_0, x_0) in R^2, it is clear that $x_0(t) = x_0$ satisfies (5.6). Now assume that, by induction hypothesis, for any $j = n > 0$, x_n is defined on I and satisfies (5.6). Hence $(s, x_n(s))$ is in R^2 for all s in I. Therefore x_{n+1} is defined on I. Because of the definition, we have

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds, \quad t \in I.$$

Using the induction hypothesis, it is seen that

$$|x_{n+1}(t) - x_0| = \left|\int_{t_0}^{t} f(s, x_n(s))\, ds\right| \leq \int_{t_0}^{t} |f(s, x_n(s))|\, ds \leq L|t - t_0| \leq Lh \leq b.$$

Thus x_{n+1} satisfies (5.6). This completes the proof.

We now state and prove the Picard's Theorem, a fundamental result dealing with the problem of existence of a unique solution for a class of nonlinear initial value problems.

Theorem 4.4:

(Picard's Theorem) Let $f(t, x)$ be continuous and be bounded by L and satisfy Lipschitz condition with Lipschitz constant K on the closed rectangle R. Then the successive approximations x_n, $n = 1, 2, \ldots$, given by (5.5) converge uniformly on an interval $I = \|t - t_0\| \leq h$, $h = \min(a, b/L)$, to a solution x of the IVP (5.3). In addition, this solution is unique.

Proof We know that the IVP (5.3) is equivalent to the integral Equation (5.4). Our aim is to show that the successive approximations x_n converge to the unique solution of (5.4) and hence to the unique solution of the IVP (5.3). First, note that

$$x_n(t) = x_0(t) + \sum_{i=1}^{n} [x_i(t) - x_{i-1}(t)]$$

is the nth partial sum of the series

$$x_0(t) + \sum_{i=1}^{\infty} [x_i(t) - x_{i-1}(t)]. \tag{5.7}$$

Hence the convergence of the sequence $\{x_n\}$ is equivalent to the convergence of the series (5.7). We complete the proof by showing that

(a) the series (5.7) converges uniformly to a continuous function $x(t)$;

(b) x satisfies the integral Equation (5.4);

(c) x is the unique solution of (5.3).

To start with we fix a positive number $h = \min(a, b/L)$. Because of Lemma 5.1 the successive approximations $x_n(t), n = 1, 2, \ldots$ in (5.5) are well defined on $I = |t - t_0| \le h$. Henceforth, we stick to the interval $t_0 \le t \le t_0 + h$. The proof on the interval $[t_0 - h, t_0]$ is similar except for minor changes.

We estimate $x_{j+1}(t) - x_j(t)$ on the interval $[t_0, t_0 + h]$. Let us denote $m_j(t) = |x_{j+1}(t) - x_j(t)|; j = 0, 1, 2, \ldots$. Since $f(t, x)$ satisfies Lipschitz condition, by the definition of successive approximations, we obtain

$$m_j(t) = \left| \int_{t_0}^{t} [f((s, x_j(s)) - f(s, x_{j-1}(s))]\, ds \right|$$

$$\le K \int_{t_0}^{t} |x_j(s) - x_{j-1}(s)|\, ds$$

or, in other words,

$$m_j(t) \le K \int_{t_0}^{t} m_{j-1}(s)\, ds. \tag{5.8}$$

By direct computation,

$$m_0(t) = |x_1(t) - x_0(t)| = \left| \int_{t_0}^{t} f(s, x_0(s))\, ds \right|$$

$$\le \int_{t_0}^{t} |f(s, x_0(s))|\, ds$$

$$\le L(t - t_0). \tag{5.9}$$

We assert that

$$m_j(t) \le LK^j \frac{(t - t_0)^{j+1}}{(j+1)!}, \tag{5.10}$$

for $j = 0, 1, 2, \ldots$ and $t_0 \le t \le t_0 + h$. The proof of the assertion follows by induction. When $j = 0$, (5.10) is, in fact, (5.9). Assume that for an integer $j = p \ge 1$ the assertion (5.10) is valid. Therefore,

$$m_{p+1}(t) \le K\int_{t_0}^{t} m_p(s)\,ds \le K\int_{t_0}^{t} LK^p \frac{(s-t_0)^{p+1}}{(p+1)!}\,ds$$

$$\le LK^{p+1}\frac{(t-t_0)^{p+2}}{(p+2)!}, \quad t_0 \le t \le t_0+h,$$

which shows that (5.10) holds when $j = p+1$. Thus (5.10) holds for all $k \ge 0$. Hence the series $\sum_{j=0}^{\infty} m_j(t)$ is dominated by the series $\frac{L}{K}\sum_{j=0}^{\infty}\frac{K^{j+1}h^{j+1}}{(j+1)!}$ which converges $L(e^{kh}-1)/K$. Hence the series (5.7) converges uniformly and absolutely on the interval $t_0 \le t \le t_0+h$. Let

$$x(t) = x_0(t) + \sum_{n=1}^{\infty}[(x_n(t)-x_{n-1}(t))]; \quad t_0 \le t \le t_0+h. \tag{5.11}$$

Since the convergence is uniform, the limit function $x(t)$ in (5.11) is continuous on $[t_0, t_0+h]$. It is easy to show that the points $(t, x(t))$ are in the rectangle R for all $t \in I$. This completes the proof of (a).

We now show that the limit function $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds. \tag{5.12}$$

By the definition of successive approximations

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))\,ds. \tag{5.13}$$

In view of (5.13), we have

$$\left|x(t) - x_0 - \int_{t_0}^{t} f(s, x(s))\,ds\right|$$

$$= \left|x(t) - x_n(t) + \int_{t_0}^{t} f(s, x_{n-1}(s))\,ds - \int_{t_0}^{t} f(s, x(s))\,ds\right|$$

$$\leq |x(t) - x_n(t)| + \int_{t_0}^{t} |f(s, x_{n-1}(s)) - f(s, x(s))| \, ds. \tag{5.14}$$

Since $x_n(t) \to x(t)$ uniformly, and $|x_n(t) - x_0| \leq b$ for all n and for t in $[t_0, t_0 + h]$, it follows that $|x(t) - x_0| \leq b$ on $[t_0, t_0 + h]$. Using the Lipschitz condition in (5.14), it is seen that

$$\left| x(t) - x_0 - \int_{t_0}^{t} f(s, x(s)) \, ds \right| \leq |x(t) - x_n(t)| + K \int_{t_0}^{t} |x(s) - x_{n-1}(s)| \, ds$$

$$\leq |x(t) - x_n(t)| + Kh \max_{t_0 \leq s \leq t_0 + h} |x(s) - x_{n-1}(s)|. \tag{5.15}$$

The uniform convergence of $x_n(t)$ to $x(t)$ now implies that the right hand side of

(5.15) tends to zero as $n \to \infty$. But the left side of (5.15) is independent of n. Thus $x(t)$ satisfies the Integral Equation (5.4). This proves (b).

Let us now prove that if $\bar{x}(t)$ and $x(t)$ are any two solutions of the IVP (5.3), then they coincide on $[t_0, t_0 + h]$. $\bar{x}(t)$ and $x(t)$ satisfy (5.4). Therefore

$$|\bar{x}(t) - x(t)| \leq \int_{t_0}^{t} |f(s, \bar{x}(s)) - f(s, x(s))| \, ds. \tag{5.16}$$

$$|\bar{x}(t) - x(t)| \leq \int_{t_0}^{t} |f(s, \bar{x}(s)) - f(s, x(s))| \, ds. \tag{5.16}$$

Both $\bar{x}(s)$ and $x(s)$ lie in R for all s in $[t_0, t_0 + h]$ and hence it follows from (5.16) that

$$|\bar{x}(t) - x(t)| \leq K \int_{t_0}^{t} |\bar{x}(s) - x(s)| \, ds.$$

By the application of the Gronwall inequality, we arrive at

$$|\bar{x}(t) - x(t)| = 0 \quad \text{on} \quad [t_0, t_0 + h]$$

which means $\bar{x}(t) = x(t)$ on $[t_0, t_0 + h]$. This proves (c), completing the proof of the theorem.

Another important feature of Picard's theorem is that a bound for the error in the case of truncated computation at the n-th iteration can also be obtained. The theorem that follows is a result dealing with such a bound on the error.

Theorem 4.5: The error $x(t) - x_n(t)$ satisfies the estimate

$$|x(t) - x_n(t)| \leq \frac{L(Kh)^{n+1}}{K(n+1)!} e^{Kh}; \quad t \in [t_0, t_0 + h]. \qquad (5.17)$$

Proof Since $x(t) = x_0(t) + \sum_{j=0}^{\infty} [x_{j+1}(t) - x_j(t)]$, we have

$$x(t) - x_n(t) = \sum_{j=n}^{\infty} [x_{j+1}(t) - x_j(t)].$$

The above relation implies, in view of (5.10), that

$$|x(t) - x_n(t)| \leq \sum_{i=n}^{\infty} |x_{j+1}(t) - x_j(t)| \leq \sum_{j=n}^{\infty} m_j(t)$$

$$\leq \sum_{j=n}^{\infty} \frac{L(Kh)^{j+1}}{K(j+1)!} = \frac{L(Kh)^{n+1}}{K(n+1)!}\left[1 + \sum_{j=1}^{\infty} \frac{(Kh)^j}{(n+2)\dots(n+j+1)}\right]$$

$$\leq \frac{L(Kh)^{n+1}}{K(n+1)!} e^{Kh} \qquad t \in [t_0, t_0 + h]$$

which is (5.17). The proof is complete.

Example:

Consider the IVP $x' = x$, $x(0) = 1$; $t \geq 0$. Observe that all the conditions of the Picard's theorem are satisfied. To find a bound on the error $x(t) - x_n(t)$ we have to determine K and L. It is quite clear that $K = 1$. Let R be the

closed rectangle around (0, 1) i.e. $R = \{(t, x) : |t| \leq 1 \text{ and } |x - 1| \leq 1\}$. Then $L = 1$ and $h = 1$. Suppose the error is not to exceed ε. The question is to find a number n such that $|x - x_n| \leq \varepsilon$. To achieve this, a sufficient condition is that

$$\frac{L(Kh)^{n+1}}{K(n+1)!} e^{Kh} < \varepsilon.$$

We have to find an n such that $\frac{1}{(n+1)!} < \varepsilon e^{-1}$ or, in other words, $(n+1)! > \varepsilon^{-1}e$. This inequality can be achieved since $\varepsilon^{-1}e$ is finite and $(n+1)! \to \infty$. Thus, if $\varepsilon = 1$, we can choose $n \geq 2$, so that the error is less than 1.

CONTINUATION AND DEPENDENCE ON INITIAL CONDITIONS

Theorem 4.6: Let

(i) $f(t, x)$ be defined and continuous on an open connected set $D \subset R^{n+1}$ and satisfy the Lipschitz condition on D;

(ii) $f(t, x)$ is bounded on D;

(iii) $x(t)$ be the unique solution of the IVP (5.3) existing on $h_1 < t < h_2$. Then $\lim_{t \to h_2 - 0} x(t)$ exists. If the point $[h_2, x(h_2 - 0)]$ is in D, then $x(t)$ can be continued to the right of h_2.

The remaining part of this section deals with the continuous dependence of solutions on initial conditions. We start with the IVP

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{5.22}$$

Let $x(t; t_0, x_0)$ be a solution of (5.22). Then $x(t; t_0, x_0)$ is a function of the time variable t, the initial time t_0 and the initial state x_0. The problem of dependence of initial conditions is to know how $x(t; t_0, x_0)$ behaves as a function of t_0 and x_0. We show, under certain conditions, that $x(t; t_0, x_0)$ is a continuous function of t_0 and x_0. This amounts to saying that the solution $x(t; t_0, x_0)$ of a physical problem (5.22) stays in a neighbourhood of solutions $x^*(t; t_0^*, x_0^*)$ of

$$x' = f(t, x), \quad x(t_0^*) = x_0^* \tag{5.23}$$

provided that $|t_0 - t_0^*|$ and $|x_0 - x_0^*|$ are sufficiently small.

Theorem 4.6:

Let $x(t) = x(t; t_0, x_0)$ and $x^*(t) = x(t; t_0^*, x_0^*)$ be solutions of the IVPs (5.22) and (5.23) respectively on an interval $a \leq t \leq b$. Let $(t, x(t))$, $(t, x^*(t))$ lie in a domain D for $a \leq t \leq b$. Further, let $f \in$ Lip (D, K) be bounded by L in D. Then for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|x(t) - x^*(t)| < \varepsilon, \quad a \leq t \leq b \tag{5.24}$$

whenever $|t_0 - t_0^*| < \delta$ and $|x_0 - x_0^*| < \delta$.

Proof It is first of all clear that for $a \le t_0, t_0^* \le b$ the solutions $x(t)$ and $x^*(t)$ with $x(t_0) = x_0$ and $x^*(t_0^*) = x_0^*$ exists uniquely. Let $t_0^* \ge t_0$. From Lemma 5.2, we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \qquad (5.25)$$

$$x^*(t) = x_0^* + \int_{t_0^*}^{t} f(s, x^*(s))\, ds. \qquad (5.26)$$

From (5.25) and (5.26), we obtain

$$x(t) - x^*(t) = x_0 - x_0^* + \int_{t_0^*}^{t} [f(s, x(s)) - f(s, x^*(s))]\, ds$$

$$+ \int_{t_0}^{t_0^*} f(s, x(s))\, ds. \qquad (5.27)$$

Taking absolute values on both sides of (5.27) and using the hypothesis it is seen that

$$|x(t) - x^*(t)| \le |x_0 - x_0^*| + \int_{t_0^*}^{t} |f(s, x(s)) - f(s, x^*(s))|\, ds$$

$$+ \int_{t_0}^{t^*} |f(s, x(s))|\, ds$$

$$\le |x_0 - x_0^*| + \int_{t_0^*}^{t} K|x(s) - x^*(s)|\, ds + L\,|t_0 - t_0^*|.$$

Hence, by the Gronwall inequality, it follows that

for all $t : a \le t \le b$. Given any $\varepsilon > 0$, now choose

$$\delta(\varepsilon) = \frac{\varepsilon}{2 \exp [K(b-a)]} \min \left[1, \frac{1}{L}\right].$$

From (5.28) it is easy to see that

$$|x(t) - x^*(t)| \le \left[\frac{\varepsilon}{2 \exp \{K(b-a)\}} + \frac{L\varepsilon}{2L \exp \{K(b-a)\}}\right] \exp K[(b-a)] = \varepsilon$$

provided $|t_0 - t_0^*| \le \delta(\varepsilon)$ as well as $|x_0 - x_0^*| \le \delta(\varepsilon)$.

This completes the proof of the theorem.

EXISTENCE OF SOLUTIONS IN THE LARGE

Theorem 5.7 :

Assume that $f(t, x)$ is continuous on the strip S defined by

$$S : |t - t_0| \le T \quad \text{and} \quad |x| < \infty$$

where T is some finite positive real number. Let $f \in \text{Lip}\,(S,K)$. Then the successive approximations defined by (5.5) for the IVP (5.29) exist on $|t - t_0| \le T$ and converge to a solution x of (5.29).

Proof Recall that the definition of successive approximations (5.5) is

$$\left.\begin{aligned} &x_0(t) = x_0 \\ &x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))\, ds, \quad |t - t_0| \le T. \end{aligned}\right\} \tag{5.30}$$

We prove the theorem for the interval $[t_0, t_0 + T]$. The proof for the interval $[t_0 - T, t_0]$ is similar. First note that (5.30) defines the successive approximations on $t_0 \le t \le t_0 + T$. Further

$$|x_1(t) - x_0(t)| = \left| \int_{t_0}^{t} f(s, x_0)\, ds \right| . \tag{5.31}$$

Since $f(t, x)$ is assumed to be continuous, $f(t, x_0)$ is continuous on $[t_0, t_0 + T]$ which implies that there exists a positive constant L such that

$$|f(t, x_0)| \le L \quad \text{for all} \quad t \in [t_0, t_0 + T].$$

Using this bound on $f(t, x_0)$ in (5.31), we get

$$|x_1(t) - x_0(t)| \le L(t - t_0) \le LT, \quad t \in [t_0, t_0 + T]. \tag{5.32}$$

Once we arrive at the estimate (5.32), then

$$|x_n(t) - x_{n-1}(t)| \le \frac{LK^{n-1}T^n}{n!}, \quad t \in [t_0, t_0 + T] \tag{5.33}$$

follows by induction. From (5.33), as in the proof of Theorem 5.3, the uniform convergence of the series

$$x_0(t) + \sum_{n=0}^{\infty} [x_{n+1}(t) - x_n(t)]$$

and hence, the uniform convergence of the sequence $\{x_n\}$ on $[t_0, t_0 + T]$ can be easily established. Let $x(t)$ denote the limit function, namely,

$$x(t) = x_0(t) + \sum_{n=0}^{\infty} [x_{n+1}(t) - x_n(t)], \quad t \in [t_0, t_0 + T]. \tag{5.34}$$

In view of (5.33), it follows that

$$\begin{aligned}
|x_n(t) - x_0| &= \left| \sum_{p=1}^{n} [x_p(t) - x_{p-1}(t)] \right| \\
&\le \sum_{p=1}^{n} |x_p(t) - x_{p-1}(t)| \\
&\le \frac{L}{K} \sum_{p=1}^{n} \frac{K^p T^p}{p!} \\
&\le \frac{L}{K} \sum_{p=1}^{\infty} \frac{K^p T^p}{p!} = \frac{L}{K}(e^{KT} - 1).
\end{aligned}$$

Since $x_n(t)$ converges to $x(t)$ on $t_0 \le t \le t_0 + T$, it is seen that

$$|x(t) - x_0| \le \frac{1}{K}(e^{KT} - 1).$$

Note that the function $f(t, x)$ is continuous on the rectangle

$$R : |t - t_0| \le T, \quad |x - x_0| \le \frac{L}{K}(e^{KT} - 1).$$

Hence, there exists a real number L_1 such that

$$|f(t, x)| \le L_1, \quad (t, x) \in R.$$

The convergence of the sequence $\{x_n(t)\}$ is uniform. Hence the limit function $x(t)$ is continuous. From (5.17), it follows that

$$|x(t) - x_n(t)| \le \frac{L_1 (KT)^{n+1}}{K(n+1)!} e^{KT}.$$

Now our aim is to prove that the function $x(t)$ is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \quad t_0 \le t \le t_0 + T. \tag{5.35}$$

The continuity of x is a consequence of the uniform convergence of $\{x_n\}$ on $[t_0, t_0 + T]$. Now

$$\left| x(t) - x_0 - \int_{t_0}^{t} f(s, x(s))\, ds \right|$$

$$= \left| x(t) - x_n(t) + \int_{t_0}^{t} [f(s, x_n(t)) - f(s, x(s))]\, ds \right|$$

$$\le |x(t) - x_n(t)| + \int_{t_0}^{t} |f(s, x(t)) - f(s, x_n(s))|\, ds. \tag{5.36}$$

Since $x_n \to x$ uniformly on $[t_0, t_0 + T]$, the right side of (5.36) tends to zero as $n \to \infty$. So by letting $n \to \infty$, it follows from (5.36) that

$$|x(t) - x_0 - \int_{t_0}^{t} f(s, x(s))\, ds| \le 0, \quad t \in [t_0, t_0 + T]$$

EXISTENCE AND UNIQUENCE OF SOLUTION OF SYSTEM

Definition 5.2 A vector function $f(t, x)$ defined on D is said to satisfy the Lipschitz condition in the variable x, with Lipschitz constant K on D, if

$$\|f(t, x_1) - f(t, x_2)\| \le K\|x_1 - x_2\|. \tag{5.41}$$

uniformly in t for all (t, x_1), (t, x_2) in D.

It is easy to show the continuity of $f(t, x)$ in x for each fixed t in case $f(t, x)$ is Lipschitzian in x. If $f(t, x)$ is Lipschitzian on D then there exists a non-negative, real-valued function $L(t)$ such that $|f(t, x)| \le L(t)$, for all (t, x) in D.

It might possibly happen that $L(t)$ is continuous on $|t - t_0| \le a$. We know then that there exists a constant $L > 0$ such that $L(t) \le L$.

Lemma :

Let $f(t, x)$ be a continuous function in (t, x) on D. $x(t, t_0, x_0)$ denoted by $x(t)$ is a solution of (5.40) on some interval I contained in $|t - t_0| \leq a$ if and only if $x(t)$ is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \quad t \in I. \tag{5.42}$$

Proof We indicate the line of the proof. First of all we prove that the component $x_i(t)$ of $x(t)$ satisfies

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} f_i(s, x(s))\, ds, \quad t \in I, \quad i = 1, 2, \ldots, n$$

if and only if $x_i'(t) = f_i(t)),\ i = 1, 2, \ldots, n$ hold. The proof of the above assertion is exactly the same as that of Lemma 5.1. Once this step is established it is obvious that Lemma 5.3 follows.

As expected, the integral Equation (5.42) is now exploited to define successive approximations $\{y_j(t), j = 0, 1, 2, \ldots\}$ where each y_j is an n-vector. We define them by the relations

$$\begin{cases} y_0(t) = x_0 \\ y_n(t) = x_0 + \int_{t_0}^{t} f(s, y_{j-1}(s))\, ds \end{cases}, \quad t \in I. \tag{5.43}$$

The following lemma establishes that, under certain conditions, the successive approximations are indeed well defined.

Lemma :

Let $f(t, x)$ be defined and continuous in $(t, x) \in D$ and let $f(t, x)$ be bounded by $L > 0$ on D. Define $h = \min(a, b/L)$. Then the successive approximations are well defined by (5.43) on the interval $I = |t - t_0| \leq h$. Further

$$\|y_j(t) - x_0\| \leq L|t - t_0| \leq b, \quad j = 1, 2, \ldots$$

The proof is very similar to the proof of Lemma 5.2.

Theorem 4.9

(Picard's theorem for system of equations). Let all the conditions of Lemma 5.4 hold and let $f(t, x)$ satisfy the Lipschitz condition with Lipschitz constant K on D. Then the successive approximations defined by (5.43) converge uniformly on $I : |t - t_0| \leq h$ to a unique solution of the IVP (5.40).

Corollary :

The error left over by truncation at the nth approximation for $x(t)$ has a bound given by

$$|x(t) - y_n(t)| \le \frac{L(Kh)^{n+1}}{K(n+1)!} e^{Kh}, \quad t \in [t_0, t_0 + h]. \tag{5.44}$$

As seen earlier the Lipschitz property of $f(t, x)$ in Theorem 5.9 cannot be altogether dropped. We show this by the following example.

Example :

Consider the nonlinear IVP given by the system of equations

$$x_1' = 2x_2^{1/3}, \quad x_1(0) = 0,$$

$$x_2' = 3x_1, \quad x_2(0) = 0.$$

This IVP can be written in the vector form as follows:

$$x' = f(t, x), \quad x(0) = 0,$$

where $x = (x_1, x_2), f(t, x) = (2x_2^{1/3}, 3x_1)$ and 0 is the zero vector (0, 0). Note that $x(t) \equiv 0$ is a solution. It can be verified that $x(t) = (t^2, t^3)$ is yet another solution of the IVP. Thus the uniqueness of solutions of IVP is violated. However, it is clear

that the IVP has solutions. It is not difficult to verify, in this case, that $f(t, x)$ is continuous in (t, x) in the neighbourhood of $(_0, 0)$.

Part -B (5x6=30 Marks)

Possible Questions:

1. The error x(t)-x_n(t)satisfies the estimates $|x(t) - x_n(t)| \leq \frac{L(kh)^{n+1}}{K(n+1)!} e^{kh}; t \in [t_0, t_0 + h]$.

2. State and prove the Gronwall inequality.

3. Prove that x(t) is a solution of x'=f(t,x), x(t_0)=x_0 on some interval I if x(t) is a solution of x(t)=x_0+$\int_{t0}^{t} f(s, x(s))\, ds$.

4. Prove that Picard's theorem

5. Assume that f(t,x) is continuous on the strip S defined by S: $|t - t_0| \leq$T and $|x| < \infty$ where T is some finite positive real number. Let $f \in$ Lip(S,K). Then prove that the successive approximations defined by x_n(t) = x_0 + $\int_{t_0}^{t} f(s, x_{n-1}(s))ds$,n=1,2…for the x'=f(t,x), x(t_0)=x_0 exist on $|t - t_0| \leq$T and converge to a solution x of x'=f(t,x).

6. Consider the IVP x' = x^2 + $\cos^2$t, x(0) = 0. Determine the largest interval of existence of its solution.

7. Let h=min(a,$\frac{b}{L}$)then P.T.the successive approximations given by

x_n(t) = x_0 + $\int_{t_0}^{t} f(s, x_{n-1}(s))ds$,n=1,2……..are valid on I = $|t - t_0| \leq h$ further $|x_j(t) - x_0| \leq$ L$|t - t_0| \leq$ b, j = 1,2,…, t$\in I$.

8. State and prove the theorem on non-local existence of solution of IVP x'= f(t,x),x(t_0)=x_0.

9. Assume that f (t, x) is a continuous function on $|t| \leq \infty, |x| < \infty$. Further , let f satisfy Lipschitz condition on the strip S_a for all $a > 0$ where $S_a = \{(t, x): |t| \leq a, |x| < \infty\}$. Then prove that the initial value problem x' = f (t, x), x(t_0) = x_0.

10. Prove that x(t) is a solution of x(t)=x_0+$\int_{t0}^{t} f(s, x(s))\, ds$ if x(t) is a solution of

x'=f(t,x), x(t_0)=x_0 on some interval I

Part -C (1x10=10 Marks)

Possible Questions:

1.State and prove Picard's theorem

2.Consider the IVP x' = $x^2 + \cos^2 t$, x(0) = 0. Determine the largest interval of existence of its solution.

3. Assume that f (t, x) is a continuous function on $|t| \leq \infty, |x| < \infty$. Further , let f satisfy Lipschitz condition on the strip S_a for all $a > 0$ where $S_a = \{(t, x): |t| \leq a, |x| < \infty\}$. Then prove that the initial value problem x' = f (t, x), $x(t_0) = x_0$ has a unique solution existing for all

4. State and prove the theorem on non-local existence of solution of IVP x'= f(t,x),$x(t_0)=x_0$.

Let h=min(a,$\frac{b}{L}$)then P.T.the successive approximations given by

$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))ds$,n=1,2……..are valid on I = $|t - t_0| \leq h$ further $|x_j(t) - x_0| \leq L|t - t_0| \leq b$, j = 1,2,…, t$\in I$.

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equations **Subject Code: 17MMP104**
Class : I-M.Sc Mathematics **Semester : I**

UNIT-IV
Existence And Uniqueness of Solution

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The derivative of $x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ with respect to a	x'(a)= f(a , s)	x'(a)= f(a, x)	x'(a)= f(a)	x'(a)= f(x)	x'(a)= f(a, x)
The Picard's theorem deal with the problem of existence of a ___________ solution for a class of non-linear initial value problem.	finite	unique	infinite	none of the above	unique
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+1	β-1	β-2	β	β-2
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+1	β-3	β-2	β-1	β-3
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+8	β-8	β-7	β+8	β-8
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+53	β-52	β-51	β+52	β-52
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+87	β-86	β-85	β+86	β-86
$x(a) = x_o + {}_{ao}\int^{a} f(s,x_{\beta}(s))ds$ is the ______________________th approximation	β+34	β-33	β-32	β+32	β-33
Successive approximations is ________________________ process	finite	infinite	n	n-1	infinite
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	53	52	51	54	52
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	4	5	3	1	4
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	3	2	1	4	2
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	55	54	52	56	54
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	33	32	31	34	32
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	53	52	51	54	52
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	32	33	31	34	33
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	56	55	54	52	55
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	78	77	76	79	77
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	89	88	87	90	88
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	93	92	91	94	92
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	83	82	81	84	82
$x(a) = x_o + {}_{ao}\int^{a} f(s,x(s))ds$ is the ______________________th approximation	2	1	0	3	1
The initial value problem furnishing a solution around (t ,x) is called the________________ for an initial value problem.	boundary value problem	local exixtence problem	initial value problem	none of the above	local exixtence problem
The __________ deals with the problem of existence of a unique solution for a class of nonlinear initial value problems.	existence theorem	uniquenes theorem	hermite equation	Picard's theorem	Picard's theorem
Existence of solutions in the large is also known as _________________.	existence theorem	non-local existence	local existence	uniqueness theorem	non-local existence
The __________________ is an infinite process.	existence theorem	non-local existence	local existence	successive approximations	successive approximations
The_______________ in the large is alo known as non-local existence.	existence theorem	non-local existence	existence of solutions	uniqueness theorem	existence of solutions
The _____________ furnishing a solution around (t ,x) is called the local existene problem for an initial value problem.	boundary value problem	local exixtence problem	initial value problem	none of the above	initial value problem

The Picard's theorem deal with the problem of existence of a unique solution for a class of ___________ initial value problem.	linear	non-linear	independent	dependent	non-linear
The solution of ___________,t Є(-∞,∞) is period w iff x(0)=x(w).	x'=Ax+f(t)	x(t)=1/(t-1)	x(t)=1/(1-t)	x(t)=1/(1+t)	x'=Ax+f(t)
A real or complex-valued function φ defined on a non-empty subset is said to be a solution if it possesses the _____order derivative.	first	second	third	fourth	first
A differential equation of first order of the form x'=g(t)h(x) is called an equation with variables ____________.	differentiable	separable	not separable	not differential	separable
The function f(x)=4x +2 is of degree	2	1	3	0	1
___________ of a Differential equation is the degree of the highest ordered derivative.	derivative	Homogeneous	order	degree	degree
The equation F(x)G(y)dx + f(x)g(y) = 0 is called	constant equation	first order equation	second order equation	separable equation	separable equation
∂M/∂y = ∂N/∂x is a_________________	Separable	Non separable	Exact	Non exact	Exact
The exact equation is _________	∂M/∂y = ∂N/∂x	∂M/∂y + ∂N/∂x =c	∂M/∂y - ∂N/∂x =2	∂M/∂y -∂N/∂x =1	∂M/∂y = ∂N/∂x
The solution of ordinary differential equation of n order contains __________ arbitrary constants	More than n	no	n	Atleast n	n
___________________ of differential equation is the graph of general particular solution	differential curve	curve	integral curve	differential line	integral curve
If ________________,then the tr(A) is 4.	A=4I	A=I	A=3I	A=2I	A=2I

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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equation	**Semester :I**	**L T P C**
Subject Code: 17MMP104	**Class : I-M.Sc Mathematics**	**4 0 0 4**

UNIT V

Fundamental results – Sturms comparison theorem – elementary linear oscillations – comparison theorem of Hille winter – Oscillations of $x'' + a(t)x = 0$ elementary non linear oscillations.

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UNIT V

Oscillations of Second Order Equations

Fundamental Result

Definition:

The equation $(py')' + qy = 0$ is said to be oscillatory on an interval $\mathbb{I}$ if there exists a non-trivial solution of the equation with infinitely many zeros on $\mathbb{I}$.

Sturm's Comparison Theorem

The phrase "comparison theorem" for a pair of differential equations is used in the sense stated below:

' If a solution of the first differential equation has a certain known property P then the solution of a second differential equation has the same or some related property P under certain hypothesis.'

Sturm's comparison theorem is a result in this direction concerning zeros of solutions of a pair of linear homogeneous differential equations. Sturm's theorem has varied interesting implications in the theory of oscillations. We remind that a solution means a nonzero solution.

Theorem 5.1:

(Sturm's Comparison Theorem)

Let r_1, r_2 and p be continuous functions on (a,b) and $p>0$. Assume that x and y are real solutions of

$$(px')' + r_1 x = 0, \tag{4.4}$$

$$(py')' + r_2 y = 0 \tag{4.5}$$

respectively on (a,b). If $r_2(t) \geq r_1(t)$ for $t \in (a,b)$ then between any two consecutive zeros t_1, t_2 of x in (a,b) there exists at least *one zero of y (unless $r_1 \equiv r_2$) in $[t_1,t_2]$. Moreover, when $r_1 \equiv r_2$ in $[t_1,t_2]$ the conclusion still holds if x and y are linearly independent .*

Proof. The proof is by the method of contradiction. Suppose y does not vanish in $(0,1)$. Then either y is positive in $(0,1)$ or y is negative in $(0,1)$. Without loss of generality, let us assume that $x(t)>0$ on (t_1,t_2) . Multiplying (4.4) and (4.5) by y and x respectively and subtraction leads to

$$(px')'y - (py')'x - (r_2 - r_1)xy = 0,$$

which, on integration gives us

$$\int_{t_1}^{t_2} [(px')'y - (py')'x]\; dt = \int_{t_1}^{t_2} (r_2 - r_1)xy\; dt.$$

If $r_2 \neq r_1$ on (t_1,t_2), then, $r_2(t) > r_1(t)$ in a small interval of (t_1,t_2). Consequently

$$\int_{t_1}^{t_2} [(px')'y - (py')'x] > 0. \tag{4.6}$$

Using the identity

$$\frac{d}{dt}[p(x'y - xy')] = (px')'y - (py')'x,$$

now the inequality (4.6) implies

$$p(t_2)x'(t_2)y(t_2) - p(t_1)x'(t_1)y(t_1) > 0, \tag{4.7}$$

since $x(t_1) = x(t_2) = 0$. However, $x'(t_1) > 0$ and $x'(t_2) < 0$ as x is a non-trivial solution which is positive in (t_1,t_2). As py is positive at t_1 as well as at t_2, (4.7) leads to a contradiction.

Again, if $r_1 \equiv r_2$ on $[t_1,t_2]$, then in place of (4.7), we have

$$p(t_2)y(t_2)x'(t_2) - p(t_1)y(t_1)x'(t_1) \geq 0.$$

which again leads to a contradiction as above unless y is a multiple of x. This completes the proof. □

Remark : What Sturm's comparison theorem asserts is that the solution y has *at least* one zero between two successive zeros t_1 and t_2 of x. Many times y may vanish more than once between t_1 and t_2. As a special case of Theorem 4.2.1,we have

Theorem 5.2:

Let r_1 and r_2 be two continuous functions such that $r_2 \geq r_1$ on (a, b). Let x and y be solutions of equations

$$x'' + r_1(t)x = 0 \tag{4.8}$$

and

$$y'' + r_2(t)y = 0 \tag{4.9}$$

on the interval (a, b). Then y has at least a zero between any two successive zeros t_1 and t_2 of x in (a, b) unless $r_1 \equiv r_2$ on $[t_1, t_2]$. Moreover, in this case the conclusion remains valid if the solutions y and x are linearly independent.

Proof. the proof is immediate if we let $p \equiv 1$ in Theorem 4.2.1. Notice that the hypotheses of Theorem 4.2.1 are satisfied. □

The celebrated Sturm's separation theorem is an easy consequence of Sturm's comparison theorem as shown below.

Theorem 5.3:

(Sturm's Separation Theorem) Let x and y be two linearly independent real solutions of

$$x'' + a(t)x' + b(t)x = 0, \ t \geq 0 \tag{4.10}$$

where a, b are real valued continuous functions on $(0, \infty)$. Then, the zeros of x and y separate each other, i.e. between any two consecutive zeros of x there is one and only one zero of y. (Note that the roles of x and y are interchangeable.)

Proof. First we note that all the hypotheses of Theorem 4.2.1 are satisfied by letting

$$r_1(t) \equiv r_2(t) = b(t) \exp\Big(\int_0^t a(s)ds\Big)$$
$$p(t) = \exp\Big(\int_0^t a(s)ds\Big)$$

So between any two consecutive zeros of x, there is at least one zero of y. By repeating the argument with x in place of y, it is clear that between any two consecutive zeros of y there is a zero of x which completes the proof. □

Corollary :

Let r be a continuous function on $(0, \infty)$ and let x and y be two linearly independent solutions of

$$x'' + r(t)x = 0.$$

Then, the zeros of x and y separate each other.

A few comments are warranted on the hypotheses of Theorem 4.2.1. Example (given below) shows that Theorem 4.2.1 fails if the condition $r_2 \geq r_1$ is dropped.

Example:

Consider the equations

(i) $x'' + x = 0, r_1(t) \equiv +1, t \geq 0,$

(ii) $x'' - x = 0, r_2(t) \equiv -1, t \geq 0.$

All the conditions of Theorem 4.2.1 are satisfied except that r_2 is not greater than r_1. We note that between any consecutive zeros of a solution x (of (i), any solution y of (ii) does not admit a zero. Thus, Theorem 4.2.1 may not hold true if the condition $r_2 \geq r_1$ is dropped.

Example:

Consider

$$x'' + x = 0, r_1(t) \equiv 1$$
$$y'' + 4y = 0, r_2(t) \equiv 4.$$

Note that $r_2 \geq r_1$ and also that the remaining conditions of Theorem 4.2.1 are satisfied. $x(t) = \sin t$ is a solution of the first equation and $y(t) = \sin(2t)$ is a solution of the second equation which has zero at $t_1 = 0$ and $t_2 = \pi/2$. It is obvious that $x(t) = \sin t$ does not vanish at any point in $(0, \pi/2)$. This clearly shows that, under the hypotheses of Theorem 4.2.1, between two successive zeros of y there need not exist a zero of x.

Elementary Linear Oscillations

Presently we restrict our discussion to a class of second order equations of the type

$$x'' + a(t)x = 0, t \geq 0, \tag{4.11}$$

where a is a real valued continuous function defined for $t \geq 0$. A very interesting implication of Sturm's separation theorem is

Theorem 5.3:

(a) The equation (4.11) *is oscillatory if and only if, it has no solution with finite number of zeros in* $[0, \infty)$.

(b) Equation (4.11) *is either oscillatory or non-oscillatory but cannot be both.*

Proof. (a) *Necessity* It has an immediate consequence of the definition.
Sufficiency Let z be the given solution which does not vanish on (t^*, ∞) where $t^* \geq 0$. Then any non-trivial solution $x(t)$ of (4.11) can vanish atmost once in (t^*, ∞), i.e, there exists $t_0 (> t^*)$ such that $x(t)$ does not have a zero in $[t_0, \infty)$.
The proof of (b) is obvious. □

We conclude this section with two elementary results.

Theorem 5.4:

Let x be a solution of (4.11) existing on $(0, \infty)$. If $a < 0$ on $(0, \infty)$, then x has utmost one zero.

Proof. Let t_0 be a zero of x. It is clear that $x'(t_0) \neq 0$ for $x(t) \not\equiv 0$. Without loss of generality let us assume that $x'(t_0) > 0$ so that x is positive in some interval to the right of t_0. Now $a < 0$ implies that x'' is positive on the same interval which in turn implies that x' is an increasing function, and so, x does not vanish to the right of t_0. A similar argument shows that x has no zero to the left of t_0. Thus, x has utmost one zero. □

Remark Theorem is also a corollary of Sturm's comparison theorem. For the equation

$$y'' = 0$$

any non-zero constant function $y \equiv k$ is a solution. Thus, if this equation is compared with the equation (4.11) (observe that all the hypotheses of Theorem are satisfied) then, x vanishes utmost once, for otherwise if x vanishes twice then y necessarily vanishes at least once by Theorem 4.2.1,which is not true. So x cannot have more than one zero.

From Theorem the question arises: If a is continuous and $a(t) > 0$ on $(0, \infty)$, is the equation (4.11) oscillatory ? A partial answer is given in the following theorem.

Theorem 5.5:

Let a be continuous and positive on $(0, \infty)$ with

$$\int_1^\infty a(s)ds = \infty. \tag{4.12}$$

Also assume that x is any (non-zero) solution of (4.11) existing for $t \geq 0$. Then, x has infinite zeros in $(0, \infty)$.

Proof. Assume, on the contrary, that x has only a finite number of zeros in $(0, \infty)$. Then, there exists a point $t_0 > 1$ such that x does not vanish on $[t_0, \infty)$. Without loss of generality we assume that $x(t) > 0$ for all $t \geq t_0$. Thus

$$v(t) = \frac{x'(t)}{x(t)}, \ t \geq t_0$$

is well defined. It now follows that

$$v'(t) = -a(t) - v^2(t).$$

Integration on the above leads to

$$v(t) - v(t_0) = -\int_{t_0}^{t} a(s)ds - \int_{t_0}^{t} v^2(s)ds.$$

The condition (4.12) now implies that there exist two constants A and T such that $v(t) < A(< 0)$ if $t \geq T$ since $v^2(t)$ is always non-negative and

$$v(t) \leq v(t_0) - \int_{t_0}^{t} a(s)ds.$$

This means that x' is negative for large t. Let $T(\geq t_0)$ be so large that $x'(T) < 0$. Then, on $[T, \infty)$ notice that $x > 0, x' < 0$ and $x'' < 0$. But

$$\int_{T}^{t} x''(s)ds = x'(t) - x'(T) \leq 0$$

Now integrating once again we have

$$x(t) - x(T) \leq x'(T)(t - T), t \geq T \geq t_0. \tag{4.13}$$

Since $x'(T)$ is negative, the right hand side of (4.13) tends to $-\infty$ as $t \to \infty$ while the left hand side of (4.13) either tends to a finite limit (because $x(T)$ is finite) or tends to $+\infty$ (in case $x(t) \to \infty$ as $t \to \infty$). Thus, in either case we have a contradiction . So the assumption that x has a finite number of zeros in $(0, \infty)$ is false. Hence, x has infinite number of zeros in $(0, \infty)$, which completes the proof. □

It is not possible to do away with the condition (4.12) as shown by the following example.

COMPARISON THEOREM OF HILLE-WINTNER

Lemma 1. *The function $P(u,v)$ defined in* (10) *satisfies the following inequalities*

$$P(u,v) \geq \frac{1}{2}|u|^{2-p}\big(v-\Phi(u)\big)^2 \quad for \quad p \leq 2\,,$$
$$P(u,v) \leq \frac{1}{2}|u|^{2-p}\big(v-\Phi(u)\big)^2 \quad for \quad p \geq 2\,,\ u \neq 0\,.$$

Futhermore, let $T>0$ be arbitrary. There exists a constant $K = K(T) > 0$ such that

$$P(u,v) \geq K|u|^{2-p}\big(v-\Phi(u)\big)^2 \quad for \quad p \geq 2$$
$$P(u,v) \leq K|u|^{2-p}\big(v-\Phi(u)\big)^2 \quad for \quad p \leq 2\,,$$

and every $u, v \in \mathbb{R}$ satisfying $\left|\frac{v}{\Phi(u)}\right| \leq T$.

Now we derive the so-called modified Riccati equation which plays the crucial role in the proof of our main result. Let $x \in C^1$ be any function and w be a solution of the Riccati equation (8). Then from Picone's identity (9) we have

$$(11) \qquad (w|x|^p)' = r|x'|^p - c|x|^p - pr^{1-q}|x|^p P(\Phi^{-1}(w_x), w),$$

where $w_x = r\Phi(x'/x)$ and Φ^{-1} is the inverse function of Φ. At the same time, let h be a (positive) solution of (6) and $w_h = r\Phi(h'/h)$ be the solution of the Riccati equation associated with (6), then

$$(12) \qquad (w_h|x|^p)' = r|x'|^p - \tilde{c}|x|^p - pr^{1-q}|x|^p P\big(\Phi^{-1}(w_x, w_h)\big)\,.$$

Substituting $x = h$ into (11), (12) and subtracting these equalities we get the equation (in view of the identity $P(\Phi^{-1}(w_h), w_h) = 0$)

$$(13) \qquad ((w-w_h)h^p)' + (c-\tilde{c})h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0\,.$$

Observe that if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (13) reduces to (8) and this is also the reason why we call this equation the *modified Riccati equation.*

Finally, let us recall the concept of the principal solution of nonoscillatory equation (1) is introduced by Mirzov in [12] and later independently by Elbert and Kusano in [7]. If (1) is nonoscillatory, as mentioned at the beginning of this section, there exists a solution w of Riccati equation (8) which is defined on some interval $[T, \infty)$. It can be shown that among all solutions of (8) there exists the

minimal one $\tilde{w}$ (sometimes called the *distinguished* solution), minimal in the sense that any other solution of (8) satisfies the inequality $w(t) > \tilde{w}(t)$ for large t. Then the principal solution of (1) is given by the formula

$$\tilde{x} = K \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}\big(\tilde{w}(s)\big)\,ds\right\}\,,$$

i.e., the principal solution $\tilde{x}$ of (1) is a solution which "produces" the minimal solution $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$ of (8).

Theorem 5.6:

Let $\int^\infty r^{1-q}(t)\,dt = \infty$. Suppose that equation (6) is nonoscillatory and possesses a positive principal solution h such that there exist a finite limit

(14) $$\lim_{t\to\infty} r(t)h(t)\Phi\big(h'(t)\big) =: L > 0$$

and

(15) $$\int^\infty \frac{dt}{r(t)h^2(t)\big(h'(t)\big)^{p-2}} = \infty\,.$$

Further suppose that $0 \le \int_t^\infty C(s)\,ds < \infty$ and

(16) $$0 \le \int_t^\infty \big(c(s) - \tilde{c}(s)\big)h^p(s)\,ds \le \int_t^\infty \big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds < \infty\,,$$

all for large t. If equation (3) is nonoscillatory, then (1) is also nonoscillatory.

Proof. As we have already mentioned before, to prove that (1) is nonoscillatory, it is sufficient to find a solution of associated Riccati equation (8) which is defined on some interval $[T,\infty)$. This solution we will find (using the Schauder-Tychonov theorem) as a fixed point of a suitably constructed integral operator.

By our assumption, equation (3) is nonoscillatory, i.e., there exists an eventually positive principal solution x of this equation. Denote by $w := r\Phi(x'/x)$ the solution of the associated Riccati equation

$$w' + C(t) + (p-1)r^{1-q}(t)|w|^q = 0\,.$$

From the previous section, with (1) replaced by (3), i.e., with c replaced by C, we know that the modified Riccati equation

$$((w - w_h)h^p)' + (C - \tilde{c})h^p + pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = 0$$

holds, where h is the principal solution of (6) and $w_h = r\Phi(h'/h)$ is the minimal solution of the Riccati equation corresponding to equation (6). By integrating we get

(17)
$$h^p(w_h - w)\big|_T^t = \int_T^t \big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds + p\int_T^t r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,.$$

Since $\int^{\infty} r^{1-q}(t)\,dt = \infty$ and $0 \le \int_t^{\infty} C(s)\,ds < \infty$, w solves also the integral Riccati equation (see [3, p. 207])

$$w(t) = \int_t^{\infty} C(s)\,ds + (p-1)\int_t^{\infty} r^{1-q}(s)|w(s)|^q\,ds,$$

and therefore $w(t) \ge 0$ for large t. Hence

$$h^p(w_h - w)|_T^t \le h^p w_h(t) + h^p\big(w(T) - w_h(T)\big)$$

and letting $t \to \infty$ in (17) we have (with L given by (14))

$$\begin{aligned} L + h^p\big(w(T) - w_h(T)\big) \ge \int_T^{\infty} &\big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds \\ &+ p\int_T^{\infty} r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,. \end{aligned}$$

Since $P(u, v) \ge 0$ and (16) holds, this means that

(18) $$\int^{\infty} r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi\big(h(t)\big)\big)\,dt < \infty\,.$$

Now, since (14), (16), (18) hold, from (17) it follows that there exists a finite limit

$$\lim_{t\to\infty} h^p(t)\big(w(t) - w_h(t)\big) =: \beta$$

and also the limit

(19) $$\lim_{t\to\infty}\frac{w(t)}{w_h(t)} = \lim_{t\to\infty}\frac{h^p(t)w(t)}{h^p(t)w_h(t)} = \frac{L+\beta}{L}\,.$$

Therefore, letting $t \to \infty$ in (17) and then replacing T by t, we get the equation

(20) $$\begin{aligned} h^p(t)\big(w(t) - w_h(t)\big) - \beta = \int_t^{\infty} &\big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds \\ &+ p\int_t^{\infty} r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,. \end{aligned}$$

Since (19) holds, according to Lemma 1 there exists a positive constant K such that

$$K|\Phi^{-1}(w_h)|^{2-p}(w - w_h)^2 \le P\big(\Phi^{-1}(w_h), w\big)\,,$$

and hence

(21) $$\begin{aligned} &\frac{K}{r(t)h^2(t)\big(h'(t)\big)^{p-2}}\Big[\big(w(t) - w_h(t)\big)h^p(t)\Big]^2 \\ &\qquad \le r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi(t)\big)\,. \end{aligned}$$

Denote $G(t) = r^{-1}(t)h^{-2}(t)\big(h'(t)\big)^{2-p}$, then the last inequality after integrating over $[T, \infty)$ reads

$$K\int_T^{\infty} G(t)\Big[\big(w(t) - w_h(t)\big)h^p(t)\Big]^2\,dt \le \int_T^{\infty} r^{1-q}(t)P\big(r^{q-1}(t)h'(t), w(t)\Phi(h(t))\big)\,dt\,.$$

By (15) we have $\int^t G(s)\,ds \to \infty$ as $t \to \infty$. This implies that $\beta = \lim_{t\to\infty} h^p(t)\big(w(t) - w_h(t)\big) = 0$ since if $\beta \neq 0$, we have

$$\int^\infty G(t)\big[\big(w(t) - w_h(t)\big)h^p(t)\big]^2\,dt = \infty\,,$$

which, in view of (21), implies that $\int^\infty r^{1-q}P\big(r^{q-1}h', w\Phi(h)\big)\,dt = \infty$ and this contradicts (18). Consequently from (20), we get the integral equation

$$\text{(22)}\qquad h^p(t)\big(w(t) - w_h(t)\big) = \int_t^\infty \big(C(s) - \tilde{c}(s)\big)h^p(s)\,ds + p\int_t^\infty r^{1-q}(s)P\big(r^{q-1}h', w\Phi(h)\big)\,ds\,,$$

and this equation we use in constructing the integral operator whose fixed point is a solution of (8) which we are looking for.

Define the function set U and the mapping F by

$$U = \{u \in C[T,\infty) : w_h(t) \le u(t) \le w(t) \text{ for } t \in [T,\infty)\}\,,$$

where T is sufficiently large,

$$F(u)(t) = w_h(t) + h^{-p}(t)\left\{\int_t^\infty \big(c(s) - \tilde{c}(s)\big)h^p(s)\,ds + p\int_t^\infty r^{1-q}(s)h^p(s)P\big(\Phi^{-1}(w_h), u\big)\,ds\right\}$$

Observe that the set U is well defined since $w(t) \ge w_h(t)$ for large t by (16) and (22). Obviously, U is a convex and closed subset of the Frechet space $C[T,\infty)$ with the topology of the uniform convergence on compact subintervals of $[T,\infty)$. Denote $H(s) := \frac{|s|^q}{q} - \Phi^{-1}(w_h)s$. Then $H'(s) = \Phi^{-1}(s) - \Phi^{-1}(w_h) \ge 0$ for $s \ge w_h$. This means that $P(\Phi^{-1}(w_h), u)$ is nondecreasing in the second variable and hence if $w_h(t) \le u_1(t) \le u_2(t) \le w(t)$, $t \in [T,\infty)$, we have $F(u_1)(t) \le F(u_2)(t)$ for $t \in [T,\infty)$.

Next we show that F maps U into itself. To this end, it is sufficient to show that $w_h(t) \le F(w_h)(t) \le F(u)(t) \le F(w)(t) \le w(t)$ for large t. We have

$$F(w_h)(t) = w_h(t) + h^{-p}(t)\left\{\int_t^\infty \big(c(s) - \tilde{c}(s)\big)h^p(s)\,ds\right\} \ge w_h(t)$$

Part -B (5x6=30 Marks)

Possible Questions

1 State and prove Hille-Wintner comparison theorem.

2. Let a(t) be a continuous and positive on (0,-∞) with $\int_1^{\infty} a(s)\, ds = \infty$ also assume that x(t) is any solution of x"+a(t)x=0,existing for t≥ 0 then P.T. x(t) has infinite zero's in (0,∞).

3. Show that the equation x'' + x = 0 is oscillatory.

4. State and prove Strum's separation.

5. For large t, let a (t) be a continuous function for which f(t) exists and f(t) > pt^{-1} where p>1/4. Then prove that x'' (t) + a(t)x = 0 is oscillatory.

6. Let x(t) be a x''(t) + a(t)x = 0, t≥ 0 existing on (0, ∞).If a(t)<0 on (0, ∞) then prove that x(t) has atmost one zero.

7. Let a(t) in x''(t) + a(t)x = 0 be continuous on (0, ∞) and let $a^* = \limsup_{t\to\infty} tf(t) < 1/4$ then prove that x''(t) + a(t)x = 0 is non- oscillatory.

8. State and prove Strum's comparison.

9. Let a(t) be a continuous and positive on (0,-∞) with $\int_1^{\infty} a(s)\, ds = \infty$ also assume that x(t) is any solution of x"+a(t)x=0,existing for t≥ 0 then P.T. x(t) has infinite zero's in (0,∞).

10. Assume that f(t)= $\int_t^{\infty} a(s)ds$ exists on (0, ∞). Let v(t) be a continuous function such that v(T) – v(t)+ $\int_t^T v^2(s)ds = -\int_t^T a(s)ds$ for each T ≥ t and for each t in (0, ∞). Then prove that the integral $\int_t^{\infty} v^2(s)ds$ converges and v(T) →0 as T→ ∞.

Part –C (1x10=10 Marks)

Possible Questions

1. State and prove Hille-Winter comparison theorem.

2. State and prove Strum's comparison.

3. If x(t) is a solution of equation x"+a(t)x=0,there exists x(t) does not vanish for $t \geq t_0$ then prove that $V(t)=x'(t)/x(t), t \geq t_0$ is well defined and satisfy the Riccati equation $V'(t)+V^2(t)=-a(t)$.

4. State and Prove Strum's Separation Theorem

5. State and prove Hille theorem and winter theorem

KARPAGAM ACADEMY OF HIGHER EDUCATION
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DEPARTMENT OF MATHEMATICS

Subject: Ordinary Differential Equations **Subject Code: 17MMP104**
Class : I-M.Sc Mathematics **Semester : I**

Unit V

Oscillations of Second Order Equations

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If x(t*)=0 then a point t=t* ≥0 is a solution of x’’=f(t,x,x’) is called ________________	Oscillatory	Zero solution	Non oscillatory	Non zero solution	Zero solution
A zero of a solution x(t) of x’’=f(t,x,x’) if x(t*)=0 at a point t=_____________	0	t*>1	t*≥0	t*=0	t*≥0
The zero’s of solution of x’’+a(t)x’+b(t)x=0 are ____________	Isolated	Parallel	Oscillatory	Non oscillatory	Isolated
Elementary linear equation is of the form____________________	x’+a(t)x=0	x”+a(t)x≠0	x’’+a(t)x=0	x’+a(t)x≠0	x’’+a(t)x=0
If the equation x’’+a(t)x=0, t ≥ 0 is non oscillatoryiff it has a solution with only ___________________	Infinite	One	Two	Finite	Finite
When the euler equation is oscillatory?	K=1/4	K>1/4	K<1/4	K≥1/4	K>1/4
When the euler equation is non oscillatory?	K=1/4	K>1/4	K<1/4	K≤1/4	K≤1/4
The Euler equation of the form___________________	x’’+a(t)x=0	x’’+(k /t)x=0	x’’+(k/t²) x=0	x’’+kx=0	x’’+(k/t²) x=0
The Riceatin equation is_________________	v’(t)+ v² (t)+a(t)=0	v’(t)+v(t)=0	v’’(t)+v’(t)+a(t)= 0	v’’(t)+a(t)=0	v’(t)+ v² (t)+a(t)=0
If the solution of x’’+a(t)x=0, t ≥ 0 on (0,), a(t)<0 then x(t) has ________________	atleast one zero	atmost one zero	more then one zero	one zero	atmost one zero
The solution of x’’=f(t, x, x’) ,t≥0 existing in ____________________________	[0, ∞)	[0, ∞]	(0, ∞)	(0, ∞]	[0, ∞)
The solution of x’’=f(t, x, x’), t≥0 in [0,∞] is_______________ solution	Non trivial	trivial	isolated	non isolated	Non trivial
The solution of x’’=f(t, x, x’), t≥0 is non oscillatory if it does not have _________in [t ,∞]	Solution	Value	Zero	Non zero	Value
The solution of x’’=f(t, x, x’), t≥0 is non oscillatory if it does not have zero in ______	[t , ∞)	[t , ∞]	(t , ∞]	(t , ∞)	(t , ∞]
The solution of x’’=f(t, x, x’), t≥0 is ______________ if it does have zero in [t ,∞]	Isolate	Parallel	Oscillatory	Non oscillatory	Oscillatory
The solution of x’’=f(t, x, x’), t≥0 is Oscillatory if it does have ______ in [t ,∞]	Solution	Value	Zero	Non zero	Solution
The solution of x’’=f(t, x, x’), t≥0 is oscillatory if it does have zero in ______	[t , ∞)	[t , ∞]	(t , ∞]	(t , ∞)	(t , ∞)
The solution of x’’=f(t, x, x’), t≥0 is ______________ if it does have zero in [t ,∞]	Isolate	Parallel	Oscillatory	Non oscillatory	Isolate
Let x(t) & y(t) are two linearly independent solution then w(x(t), y(t))=______________________	Zero	Non zero	Infinity	Infinity	Non zero
Let x(t) & y(t) are two linearly dependent solution then w(x(t), y(t))=______________________	Zero	Non zero	Infinity	– Infinity	Zero
If w(x(t), y(t))=0 then x(t) & y(t) are ______________________ solution	Linearly independent	dependent	Same	Different	dependent
If w(x(t), y(t))≠ 0 then x(t) & y(t) are ______________________ solution	Linearly independent	Independent	Same	Different	Linearly independent
If w(x(a), y(a))≠0, here ‘a’ should be ______________________ point	Linearly independent	common	dependent	Different	common
If w(x(a), y(a))≠0, here ‘a’ is called ______________________ of the equation	Linearly independent	zero	dependent	point	zero
The zero of y=sin2t is t=_____________________	0,½π,π…………	0,π,2π…………	0,½π,…………	0	0,½π,π…………
The zero of y=sin t is t=_____________________	0,½π,π…………	0,π,2π…………	0,½π,…………	0	0,π,2π…………
The zero of y=cost is t=____________________	0,½π,π…………	0,½π,…………	½π	½π,…………	½π,…………
The zero of y=sint cost is t=_____________________	0,½π,π…………	0,π,2π…………	0,½π,…………	0	0,½π,π…………
The zero of y=sin(t/2) is t=_____________________	0,½π,π…………	0,2π…………	0,½π,…………	0	0,2π…………
The zero of y=cos(t/4) is t=_______________	0,½π,π…………	2π,6π,…………	½π	0	2π,6π,…………

The zero of y=cos(2t) is t=_________________	0,π/4,…………	π/2,3π/2,…………	π/4	π/4,3π/4,…………	π/4,3π/4,…………
The zero of y=sin4t is t=____________________	0,π/4,…………	0,2π,4π…………	0,π/4,…………		0 0,π/4,…………
The zero of y=2sint cost is t=____________________	0,½π,π……… …	0,π,2π…………	0,½π,…………		0 0,½π,π…………
The zero of y= (t-1)(t-2) is t=____________________	1, -2	1, 2	-1, -2	-1,2	1, 2
The zero of y= (t+1)(t+2) is t=____________________	1, -2	1, 2	-1, -2	-1,2	-1, -2
The zero of y= (t-1)(t+2) is t=____________________	1, -2	1, 2	-1, -2	-1, 2	1, -2
The zero of y= (t+1)(t-2) is t=____________________	1, -2	1, 2	-1, -2	-1, 2	-1, 2
The zero of y= (t-2)(t-2) is t=____________________	-2, -2	2, 2	-1, -2	-1, 2	2, 2
The zero of y= (t-1)(t-2)(t-3) is t=____________________		1, 2, 3	-1, -2, 3	-1, 2, -3	1, 2, 3
The zero of y=(t-1)(t-1)(t-2) is t=__________________________		1, 1, 2	-1, -1, -2	-1, 2, -2	1, 1, 2

Reg. No ----------------

(17MMP104)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Karpagam University

(Established Under Section 3 of UGC Act 1956)

COIMBATORE-21

M.Sc., DEGREE EXAMINATION- NOV 2017

First Semester

I-Internal

Mathematics

Ordinary Differential Equations

Time: 3 Hours **Maximum: 50 Marks**

PART - A (20 x 1 =20 Marks)

1. If A(t) is n x n matrix continuous in t on_______

a) closed b) bounded
c) closed and bounded d) open

2. The solution of x''+x=cost is _______________

a) x(t)=acos(t+b)+(1/2)t sint b) x(t)=acos(t+b)
c) x(t)=cos(t)+(1/2)t sint d) x(t)=acos(t-b)+(1/2)t sint

3. If φ(t) is a fundamental matrix, then φ(t+s)=________________

a) φ(t)φ(s) b) φ(t)+φ(s)
c) φ(t)-φ(s) d) φ(t)/φ(s)

4. The solution of x'=Ax + f(t),t Є(-∞,∞) is period w iff x(0)=___

a) x(∞) b) x(-∞)
c) x(w) d) x(t)

5. The system x'=-A^t(t)x, t ЄI has the fundamental matrix of the form_____________

a) (1/φ(t)) b) $(φ(t))^t$
c) $(1/φ(t))^t$ d) (1/φ(t))

6. The __________________ is an infinite process.

a) existence theorem b) non-local existence
c) local existence d) successive approximations

7. The Picard's theorem deal with the problem of existence of a unique solution for a class of ___________ initial value problem.

a) linear b) non-linear
c) independent d) dependent

8. The Picard's theorem deal with the problem of existence of a ___________ solution for a class of non-linear initial value problem.

a) finite b) unique
c) infinite d) none of the above

9. The initial value problem furnishing a solution around (t_0,x_0) is called the________________ for an initial value problem.

a) boundary value problem
b) local exixtence problem
c) initial value problem
d) none of the above

10. The_______________ in the large is also known as non-local existence.

a) existence theorem
b) non-local existence
c) existence of solutions
d) uniqueness theorem

11. The __________ deals with the problem of existence of a unique solution for a class of non-linear initial value problems.

a) existence theorem
b) uniquenes theorem
c) hermite equation
d) Picard's theorem

12. Successive approximations is _________________________ process

a) finite
b) infinite
c) n
d) n-1

13. Existence of solutions in the large is also known as _________________.

a) existence theorem
b) non-local existence
c) local existence
d) uniqueness theorem

14. A zero of a solution x(t) of x''=f(t,x,x') if x(t*)=0 at a point t=____________

a) 0
b) t*>1
c) t*≥0
d) t*=0

15. The Euler equation of the form_________________

a) x''+a(t)x=0
b) x''+(k /t)x=0
c) $x''+(k/t^2)\ x=0$
d) x''+kx=0

16. The solution of x''=f(t, x, x'), t≥0 is non oscillatory if it does not have zero in ______

a) $[t_0,\ \infty)$
b) $[t_0,\ \infty]$
c) $(t_0,\ \infty]$
d) $(t_0,\ \infty)$

17. The zero of y=cost is t=__________________

a) 0,½π,π………..
b) 0,½π,………..
c) ½π
d) ½π,………..

18. The zero's of solution of x'' + a(t)x' + b(t)x=0 are ___________

a) Isolated
b) Parallel
c) Oscillatory
d) Non oscillatory

19. Let x(t) & y(t) are two linearly dependent solution then w(x(t), y(t))=_________.

a) zero
b) non zero
c) infinity
d) –infinity

20. The zero of y = (t-1)(t-2) is t=____________________

a) 1, -2 b) 1, 2
c) -1, -2 d) -1,2

PART-B (3 x 2 = 6 Marks)

Answer all the questions

21. State the Floquet theorem.

22. State and Gronwall inequality.

23. Define Isolated.

PART-B (3 x 8 = 24 Marks)

Answer all the questions

24. a) Find a fundamental matrix for X'=AX, where A = $\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

(Or)

b) Determine e^{tA} for the system X'=AX where A=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

25. a) Prove that x(t) is a solution of x'=f(t,x), $x(t_0)=x_0$ on some interval I iff x(t) is a solution of $x(t)=x_0+\int_{t0}^{t} f(s, x(s))\, ds$.

(Or)

b) Consider the IVP $x^1 = x^2 + \cos^2 t$, x (0) = 0. Determine the largest interval of existence of its solution.

26. a) State and prove Strum's separation.

(Or)

b) Let a(t) be a continuous and positive on (0,-∞) with $\int_1^\infty a(s)\, ds = \infty$ also assume that x(t) is any solution of x''+a(t)x=0,existing for t≥ 0 then P.T. x(t) has infinite zero's in (0,∞).

Reg. No ----------------

(17MMP104)

KARPAGAM ACADEMY OF HIGHER EDUCATION

Karpagam University

COIMBATORE –21

DEPARTMENT OF MATHEMATICS

First SEMESTER

II INTERNAL TEST-Sep'17

Ordinary Differential Equation

Date : .10.2017 **Time: 2 Hours**

Class : I M.Sc Mathematics **Maximum: 50 Marks**

PART - A (20 x 1 =20 Marks)

1 $x(a) = x_0 + \int_{a_0}^{a} f(s, x_{\beta-87}(s))ds$ is the ________ approximation

a) β+87 b) β-86

c) β-85 d) β+86

2. The solution of x''+x=cost is ________________

a) x(t)=acos(t+b)+(1/2)t sint b) x(t)=acos(t+b)

c) x(t)=cos(t)+(1/2)t sint d) x(t)=acos(t-b)+(1/2)t sint

3. If φ(t) is a fundamental matrix, then φ(t+s)=________________

a) φ(t)φ(s) b) φ(t)+φ(s)

c) φ(t)-φ(s) d) φ(t)/φ(s)

4. The solution of x'=Ax + f(t),t Є(-∞,∞) is period w iff x(0)=___

a) x(∞) b) x(-∞)

c) x(w) d) x(t)

5. The system $x' = -A^t(t)x$, t ЄI has the fundamental matrix of the form_____________

a) (1/φ(t)) b) $(\varphi(t))^t$

c) $(1/\varphi(t))^t$ d) (1/φ(t))

6. The __________________ is an infinite process.

a) existence theorem b) non-local existence

c) local existence d) successive approximations

7. The Picard's theorem deal with the problem of existence of a unique solution for a class of ___________ initial value problem.

a) Linear b) non-linear

c) independent d) dependent

8. If x(t*)=0 then a point t=t* ≥0 is a solution of x''=f(t,x,x') is called ________________

a) Oscillatory b) Zero solution

c) Non oscillatory d) Non zero solution

9. The initial value problem furnishing a solution around (t_0, x_0) is called the_________________ for an initial value problem.

a) boundary value problem b) local exixtence problem

c) initial value problem d) none of the above

10. The________________ in the large is also known as non-local existence.

a) existence theorem b) non-local existence

c) existence of solutions d) uniqueness theorem

11. The __________ deals with the problem of existence of a unique solution for a class of non-linear initial value problems.

a) existence theorem b) uniquenes theorem

c) hermite equation d) Picard's theorem

12. Successive approximations is ________________process

a) finite b) infinite

c) n d) n-1

13. Elementary linear equation is of the form_____________

a) $x'+a(t)x=0$ b) $x''+a(t)x\neq 0$

c) $x''+a(t)x=0$ d) $x'+a(t)x\neq 0$

14. A zero of a solution x(t) of $x''=f(t,x,x')$ if $x(t^*)=0$ at a point t=____________

a) 0 b) $t^*>1$

c) $t^*\geq 0$ d) $t^*=0$

15. The Euler equation of the form_________________

a) $x''+a(t)x=0$ b) $x''+(k/t)x=0$

c) $x''+(k/t^2)x=0$ d) $x''+kx=0$

16. The solution of $x''=f(t, x, x')$, $t\geq 0$ is non oscillatory if it does not have zero in ______

a) $[t_0, \infty)$ b) $[t_0, \infty]$

c) $(t_0, \infty]$ d) (t_0, ∞)

17. The zero of y=cost is t=__________________

a) 0,½π,π………… b) 0,½π,…………

c) ½π d) ½π,…………

18. The zero's of solution of $x'' + a(t)x' + b(t)x=0$ are ___________

a) Isolated b) Parallel

c) Oscillatory d) Non oscillatory

19. Let x(t) & y(t) are two linearly dependent solution then w(x(t), y(t))=_________.

a) zero b) non zero

c) infinity d) –infinity

20. The zero of y = (t-1)(t-2) is t=___________________

a) 1, -2 b) 1, 2

c) -1, -2 d) -1,2

PART-B (3 x 2 = 6 Marks)

Answer all the questions

21. State the Floquet theorem.

22. State and Gronwall inequality.

23. Define Isolated.

PART-C (3 x 8 = 24 Marks)

Answer all the questions

24. a) Find a fundamental matrix for X'=AX, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

(Or)

b) Determine e^{tA} for the system X'=AX where $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

25. a Consider the IVP $x' = x^2 + \cos^2 t$, $x(0) = 0$. Determine the largest interval of existence of its solution.

(Or)

b) Consider the IVP $x^1 = x^2 + \cos^2 t$, $x(0) = 0$. Determine the largest interval of existence of its solution.

26. a) State and prove Strum's separation.

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b) Let a(t) be a continuous and (+ve) on (0,-∞) with $\int_1^\infty a(s)\,ds = \infty$ also assume that x(t) is any solution of $x''+a(t)x=0$,existing for $t\geq 0$ then Prove that x(t) has infinite zero's in (0,∞).

Reg. No................................

[16MMP104]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2016 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2016

First Semester

MATHEMATICS

ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. If $P_n(t)$ and $P_m(t)$ are legendre polynomials, then prove that $\int_{-1}^{1} P_n(t)\, P_m(t)\, dt = 0$, if $m \neq n$

(OR)

b. Show that $\frac{d}{dt}\left[t^{-p} J_p(t)\right] = \left[-t^{-p} J_{p+1}(t)\right]$

22. a. Let $\Phi(t), t\epsilon I$ denote a fundamental matrix of the system $x^1 = Ax$ such that $\Phi(0) = E$, where A is a constant matrix. Here E denotes the identity matrix. Then Φ satisfies $\Phi(t+s) = \Phi(t)\Phi(s)$ for all values of t and $s \epsilon I$.

(OR)

b. Prove that the set of all solutions of the system $x^1 = A(t)x$ on I forms an n – dimensional vector space over the field of complex numbers .

23. a. Find e^{At} when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(OR)

b. Let $\Phi(t), t\epsilon I$ denote a fundamental matrix for the system $x^1 = Ax$. Then $\Phi(t+\omega)$, $-\infty < t < \infty$, is also a fundamental matrix.

24. a. Let f(t,x) be a continuous function defined over a rectangle $R = \{\{(t,x) : |t - t_0| \le p ,: |x - x_0| \le q\}$.Here p, q are some positive real numbers. Let $\frac{\partial f}{\partial x}$ be defined and continuous on R. Then prove that f(t,x) satisfies the Lipschitz condition in R.

(OR)

b. The error $x(t) - x_n(t)$ satisfies the estimate

$$|x(t) - x_n(t)| \le \frac{L(Kh)^n}{k(n+1)!} e^{kh} \quad , t \in [t_0, t_0 + h]$$

25. a. Prove that the zeros of a solution of $x'' + a(t)x' + b(t)x = 0$, $t \ge 0$ are isolated.

(OR)

b. Let a(t) in $x'' + a(t)x = 0$ be continuous on $(0, \infty)$ and let $a^* = \lim_{t\to\infty} \sup tf(t) < 1/4$ where f(t) is defined in Hille – winter. Then the equation $x'' + a(t)x = 0$ is non oscillatory.

PART – C (1x10 = 10 marks)

26. **Compulsory:**

Show that $\emptyset(t) = \begin{pmatrix} e^{-3t} & te^{-3t} & \frac{t^2 e^{-3t}}{2!} \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{pmatrix}$ is fundamental, where $A = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$.
