KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Post)
Coimbatore –641 021

Semester – I

17MMP105A	**ADVANCED DISCRETE MATHEMATICS**	**L T P C**
		4 0 0 4

Scope: On successful completion of this course the learner gains good knowledge about the concept of algebraic structures, lattices and its special categories, graph theory which play an important role in the field of computers.

Objectives: To be familiar with Algebraic Structures, Lattices, connected and disconnected graphs and be thorough with trees, spanning trees.

UNIT I

Algebraic Structures: Introduction- Algebraic Systems: Examples and General Properties: Definition and examples - Some Simple Algebraic Systems and General properties - Homomorphism and isomorphism - congruence relation - Semigroups and Monoids: Definitions and Examples - Homomorphism of Semigroups and Monoids.

UNIT II

Lattices: Lattices as Partially Ordered Sets: Definition and Examples - Principle of duality - Some Properties of Lattices - Lattices as Algebraic Systems – Sublattices - Direct product, and Homomorphism.

UNIT III

Some special Lattices - e.g. Complete, Complemented and Distributive Lattices - Boolean Algebra: Definition and Examples - Subalgebra - Direct product and Homomorphism - join irreducible - atoms and antiatoms.

UNIT IV

Graph Theory: Definition of a graph - applications, Incidence and degree - Isolated and pendant vertices - Null graph, Path and Circuits: Isomorphism - Subgraphs, Walks -Paths and circuits - Connected graphs, disconnected graphs – components - Euler graph.

UNIT V

Trees: Trees and its properties - minimally connected graph - Pendant vertices in a tree - distance and centers in a tree - rooted and binary tree. Levels in binary tree - height of a tree - Spanning trees - rank and nullity.

SUGGESTED READINGS

TEXT BOOKS

1. Tremblay J. P. and Manohar, R., (1997). Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III).

2. Deo N., (2000). Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V)

REFERENCES

1. Liu C.L., (2000). Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.

2. Wiitala S., (2003),Discrete Mathematics- A Unified Approach, McGraw-Hill Book Co, New Delhi.

3. Seymour Lepschutz, (2007) ,Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.

4..Advance Discrete Mathematics Paperback – 2011 by G.C.Sharma (Author), Madhu Jain (Author) Publisher: Laxmi Publications; Second edition (2011)

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SUBJECT: ADVANCED DISCRETE MATHEMATICS	SEMESTER: I	L T P C
SUBJECT CODE: 17MMP105A		4 0 0 4

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
	UNIT-I		
1	1	Algebraic structures :Introduction and basic concepts ;Definition, General properties and Examples.	T1: Chap: 3: Pg. No :270-271
2	1	Continuation of Algebraic structures General properties and Examples	T1:Chap :3:pg.No:272-274
3	1	Some Simple Algebraic Systems and General properties: Homomorphism and isomorphism	T1: Chap: 3: Pg. No:274-276
4	1	Continuation of Homomorphism and isomorphism	T1:Chap:3:pg.No:277-279
5	1	Congruence Relation	T1: Chap: 3: Pg. No: 279-282
6	1	Continuation of Congruence Relation	T1: Chap: 3: Pg. No: 279-282
7	1	Semigroups and Monoids : Definitions and Problems.	T1: Chap: 3: Pg. No : 282-286
8	1	Continuation of Problems on Semigroups and Monoids	T1: Chap: 3: Pg. No: 284-286
9	1	Homomorphism of Semigroups and Monoids – Problems.	T1: Chap: 3: Pg. No:287-292
10	1	Continuation of Problems on Homomorphism of Semigroups and Monoids	T1: Chap: 3: Pg. No:290-292
11	1	Recapitulation and discussion of possible questions on unit I	
Total		11 HOURS	
		T1. J .P.Tremblay & R. Manohar,	

		1997.Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III)	
		UNIT-II	
1	1	Introduction of Lattices Lattices as Partial Ordered Sets: Definition and Examples	T1: Chap: 4: Pg. No: 378-3782
2	1	Continuation of lattice of partial order sets	T1:Chap :4:pg.No:383-386
3	1	Principle of duality	R3: Chap: 15: Pg. No: 478-479
4	1	Continuation of Principle of duality	R3: Chap: 15: Pg. No: 480-484
5	1	Properties of Lattices	T1: Chap: 4: Pg. No: 382-385
6	1	Continuation of Properties of Lattices	R2: Chap: 6: Pg. No:413-415
7	1	Lattices as Algebraic Systems	T1: Chap: 4: Pg. No: 385-386
8	1	Continuation of Lattices as Algebraic Systems	R2: Chap: 6: Pg. No:416-419
9	1	Sublattices , Direct product, and Homomorphism- Problems	T1: Chap: 4: Pg. No: 387-389
10	1	Continuation of problems on Sublattices , Direct product, and Homomorphism	T1: Chap: 4: Pg. No: 390-391
11	1	Recapitulation and discussion of possible questions on unit-II	
Total		**11 HOURS**	
		T1. J .P.Tremblay & R. Manohar, 1997.Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III) R2. S. Wiitala, Discrete Mathematics- A Unified Approach, McGraw-Hill Book Co, New Delhi. R3. Seymour Lepschutz, Discrete	

		Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.	
UNIT-III			
1	1	Introduction of Some special Lattices	T1: Chap: 4: Pg. No: 392-394
2	1	Complete, Complemented and Distributive Lattices - Problems	T1: Chap: 4: Pg. No:395-399
3	1	Continuation of Complete, Complemented and Distributive Lattices - Problems	R3: Chap: 14: Pg. No: 454-458
4	1	Boolean Algebra: Definition and Problems	T1: Chap: 4: Pg. No: 398-400
5	1	Sub algebra , Direct product and Homomorphism	T1: Chap: 4: Pg. No: 401-406
6	1	Continuation of Sub algebra , Direct product and Homomorphism	T1: Chap: 4: Pg. No: 401-406
7	1	Join irreducible , atoms and antiatoms - Problems	T1: Chap: 4: Pg. No: 407-410
8	1	Continuation of Join irreducible , atoms and antiatoms - Problems	R3: Chap: 14: Pg. No: 411-415
9	1	Recapitulation and discussion of possible questions on unit III	
Total		**9 HOURS**	
		T1. J .P.Tremblay & R. Manohar, 1997.Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III) R3. Seymour Lepschutz, Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.	
UNIT-IV			
1	1	Introduction and basic definition of a graph and applications of graph theory	T2: Chap: 1: Pg. No: 1-3 T2:Chap:1:pg.No:3-6
2	1	Incidence and degree	T2: Chap: 1: Pg. No: 7-10

3	1	Continuation of Incidence and degree	R1: Chap: 4: Pg. No: 190-193
4	1	Isolated and pendant vertices , Null graph,	T2: Chap: 1: Pg. No: 11-13
5	1	Path and Circuits: Isomorphism- sub graphs	T2: Chap: 2: Pg. No: 14-16
6	1	Continuation of Path and Circuits: Isomorphism- sub graphs	R1: Chap: 4: Pg. No: 196-198
7	1	Walks, Paths and circuits - Problems	T2: Chap: 2: Pg. No: 17-21
8	1	Connected graphs , disconnected graphs, components - Problems	T2: Chap: 2: Pg. No: 21-23
9	1	Continuation of Connected graphs , disconnected graphs, components - Problems	T2: Chap: 2: Pg. No: 24-26
10	1	Euler graph – Introduction and examples	T2: Chap: 2: Pg. No: 28-32
11		Continuation of Euler graph – Introduction and examples	T2: Chap: 2: Pg. No: 33-37
12	1	Recapitulation and discussion of possible questions on unit IV	
Total		**12 HOURS**	
		T2. N. Deo, 2000. Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V) R1. C. L. Liu, 2000. Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.	
UNIT-V			
1	1	Introduction of Trees and its properties	T2: Chap: 3: Pg. No: 39-41 R1: Chap: 5: Pg. No: 255-257
2	1	Minimally connected graph	T2: Chap: 3: Pg. No:41-43
3	1	Continuation of minimally connected graph theorems	T2: Chap: 3: Pg. No:48-48
4	1	Pendant vertices in a tree – introduction and examples	T2: Chap: 3: Pg. No: 43-44
5	1	Pendant vertices in a tree – theorems	R2: chap : 7:pg: 156-158
6	1	Distance and centers in a tree	T2: Chap: 3: Pg. No: 45-47
7	1	Continuation of Distance and centers	R2: chap : 7:pg: 162-165

		in a tree	
8	1	Rooted and binary tree and Levels in binary tree, height of a tree-Problem.	T2: Chap: 3: Pg. No: 48-49,T2:Chap:3:pg.No:50-54
9	1	continuation of Rooted and binary tree and Levels in binary tree, height of a tree-Problem.	R1: Chap: 5: Pg. No: 262-264
10	1	Spanning trees- Problems	T2: Chap: 3: Pg. No: 55-56
11	1	Continuation of Spanning trees-Problems	R1: Chap: 5: Pg. No: 272-276
12	1	Rank and nullity-Introduction	T2: Chap: 3: Pg. No: 57-58
13	1	Rank and nullity-Problems	T2: Chap: 3: Pg. No: 59-60
14	1	Recapitulation and discussion of possible questions on unit V	
15	1	Discussion of Previous year ESE question paper	
16	1	Discussion of Previous year ESE question paper	
17	1	Discussion of Previous year ESE question paper	
Total		**17 HOURS**	
		T2. N. Deo, 2000. Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V) R1. C. L. Liu, 2000. Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.	

TEXT BOOKS

T1. J .P.Tremblay & R. Manohar, 1997.Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III)

T2. N. Deo, 2000. Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V)

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R1. C. L. Liu, 2000. Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.

R2. S.Wiitala, Discrete Mathematics- A Unified Approach, McGraw-Hill Book Co,New Delhi.

R3. Seymour Lepschutz, Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.

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DEPARTMENT OF MATHEMATICS

SUBJECT: ADVANCED DISCRETE MATHEMATICS	**SEMESTER: I**	**L T P C**
SUBJECT CODE: 17MMP105A	**CLASS:I PG (MATHEMATICS)**	**4 0 0 4**

UNIT I

Algebraic Structures: Introduction- Algebraic Systems: Examples and General Properties: Definition and examples - Some Simple Algebraic Systems and General properties - Homomorphism and isomorphism - congruence relation - Semigroups and Monoids: Definitions and Examples - Homomorphism of Semigroups and Monoids.

TEXT BOOKS

1. Tremblay J. P. and Manohar, R., (1997). Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III).

REFERENCES

2.Advance Discrete Mathematics Paperback – 2011 by G.C.Sharma (Author), Madhu Jain (Author) Publisher: Laxmi Publications; Second edition (2011)

ALGEBRAIC SYSTEMS

INTRODUCTION:

The algebraic systems contained two binary operations which were denoted by + and X in each case. The choice of these examples was dictated by our familiarity with the systems of integers and real numbers. These algebraic system are not simplest ones. In this section we give examples of algebraic systems consisting of a single unary or binary operation. It is possible to obtain such algebraic systems form those given earlier by simply considering one of the two binary operations; for example, (I,+) and (R,X) are perfectly.

Semigroups are the simplest algebraic structures which satisfy the properties of closure and associativity.They are very important in the theory of sequential machines, formal languages, and in certain applications relating to computer arithmetic such as multiplication.

A Monoid in addition to being a semigroup,also satisfies the identity property. Monoids are used in a number of applications but most particularly in the area of syntactic analysis and formal language.

For such algebraic systems , certain properties are taken as axioms of the system. Any result that is valid for an abstract systems holds for all those algebraic systems for which the axioms are true.

Definition:

A non-empty set together with a number of binary operations on it is called an algebraic system.

In what follows,

we shall define some algebraic systems :

Definition: A non-empty set S is said to be a **semigroup** if in S there is defined a binary operation $*$ satisfying the following property :

If a, b, c $\in$ S, then $a * (b * c) = (a * b) * c$ (Associative Law)

Thus

A non-empty set S together with an associative binary operation $*$ **defined on S is called a Semi-group.**

We denote the semi group by (S, $*$).

Definition. A semi group (S, $*$) is called **commutative** if the binary operation $*$ is a commutative operation, i.e., if $a * b = b * a$ for a, b $\in$ S.

Examples. 1. Let **Z** be the set of all integers. Then (**Z**, +) is a commutative semigroup. In fact, if a, b, c $\in$ **Z**, then

a.a $*$ b = a+b is an integer. Therefore, the operation + on **Z** is a binary operation.

b.a + (b+c) = (a+b) + c, because associative law holds in the set of integers.

c.a + b = b + a, because addition in **Z** is commutative.

2.The set **Z** of integers with the binary operation of subtraction is not a semi- group since subtraction is not associative in **Z**.

3.Let S be a finite set and let F(S) be the collection of all functions f : S $\rightarrow$ S under the operation of **composition of functions.** We know that composition of functions is associative, i.e fo(goh) = (fog)oh where f , g , h $\in$ F(S) .

Hence F(s) is a semigroup.

4.The set P(S), where S is a set, together with the operation of union is a commutative semigroup.

5.The integers modulo m, denoted by **Z**$_m$, refer to the set **Z**$_m$ = {0, 1, 2,…, m−1} .

6. The addition in **Z$_m$** is defined as a + b = r, where r is the remainder when a+b is divided by m.

7.The multiplication in **Z**$_m$ is defined by a**.**b = r, where r is the remainder when a+ b is divided by m .

For example, consider $\mathbf{Z}_4 = \{0, 1, 2,3\}$

The addition table is

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

We note

$(1+2)+3 = 3+3=2$ and $1+(2+3)=1+1=2$

Hence $(1+2)+3 =1+(2+3)$

In general , $(a+b)+c = a+(b+c)$, $a,b,c\in Z_4$

Hence Z_4 is a semigroup.

Definition. A non-empty set S is said to be a **monoid** if in S there is defined a binary operation $*$ satisfying the following properties :

1.If a, b, c $\in$ S, then $a * (b * c) = (a * b) * c$ (Associative Law)

2.There exists an element e $\in$ S such that $e * a = a * e = a$ for all $a \in S$ (Existence of identity element)

Thus

An algebraic system $(S, *)$ is said to be a **monoid** if

$*$ is a binary operation on non-empty set S

$*$ is an associative binary operation on S

There exists an identity element e in S.

It, therefore, follows that A monoid is a semi-group (S, $*$) that has an identity element.

Example.1. In example 3 above, identity function is an identity element for F(S).

Hence F(S) is a monoid.

Let **M** be the set of all n $\times$ n matrices and let the binary operation $*$ of **M** be taken as addition of matrices. Then (**M**, $*$) is a monoid. In fact,

(i)The sum of two n $\times$ n matrices is again a matrix of order n $\times$ n . Thus the operation of matrix addition is a binary operation.

(ii)If A, B, C $\in$ **M**, then A + (B+C) = (A+B) + C (Associative Law)

(iii)The zero matrix acts as additive identity of this monoid because

A + 0 = 0 + A = A for A $\in$ **M** .

Definition. Let A be a non-empty set. **A word** w on A is a finite sequence of its elements.

For example ,

$$w = ab\ ab\ bb = ab\ ab^3$$

is a word on A = {a, b} .

Definition. The number of elements in a word w is called **its length** and is denoted by l(w)**.**

For example, length of w in the above example is

$$l(w) = 6$$

Definition. Let u and v be two words on a set A. Then the word obtained by writing down the elements of u followed by the elements of v is called the **concatenation** of the words u and v on A.

For example, if A = {a, b, c} and

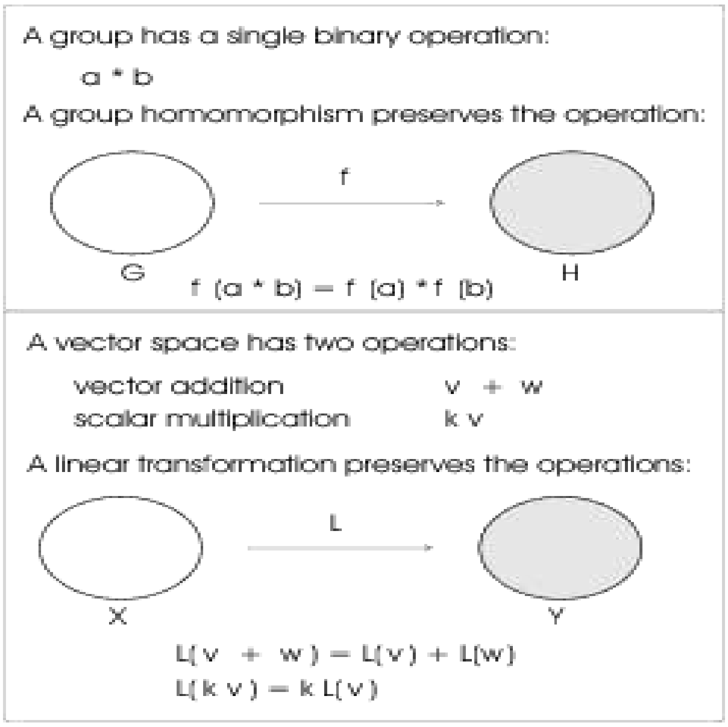
$$u = ab\ a\ bbb \text{ and } v = a\ c\ b\ a\ b$$

then $w = ab\ abbb\ ac\ bab = abab^3acbab$ is the concatenation of u and v.

HOMOMORPHISM AND ISOMORPHISM:

A homomorphism is a map between two algebraic structures of the same type (that is of the same name), that preserves the operations of the structures. This means a map f: $A \to B$ between two sets A, B equipped with the same structure such that, if $*$ is an operation of the structure (supposed here, for simplification, to be a binary operation), then f(x * y)= f(x) * f(y)

For example



A group has a single binary operation:

a * b

A group homomorphism preserves the operation:

f (a * b) = f (a) * f (b)

A vector space has two operations:

vector addition	v + w
scalar multiplication	k v

A linear transformation preserves the operations:

L(v + w) = L(v) + L(w)

L(k v) = k L(v)

:

Isomorphism, in modern algebra, a one-to-one correspondence (mapping) between two sets that preserves binary relationships between elements of the sets. For example, the set of natural numbers can be mapped onto the set of even natural numbers by multiplying each natural number by 2. The binary operation of adding two numbers is preserved—that is, adding two natural numbers and then multiplying the sum by 2 gives the same result as multiplying each natural number by 2 and then adding the products together—so the sets are isomorphic for addition.

Theorem:

The algebraic system (N,+) and (Z_4 ,+) where N is the set of natural numbers and + is the operation of addition on N, show that there exists a homomorphism from (N,+) to (Z_4 ,+)

Proof:

Define g:N → Z_4 given by g(a) = [a(mod 4)] for any a∈ N

For a, b ∈ N , let g(a)=[i] and g(b)=[j] ;then

g(a+b) =[(i+j)(mod 4)] = [i]$+_4$[j] = g(a) $+_4$ g(b)

observe that g(0) =[0] ; that is, the mapping g also preserves the identity element.

CONGRUENCE RELATION:

If two numbers b and c have the property that their difference b-c is integrally divisible by a number m (i.e., (b-c)/m is an integer), then b and c are said to be "congruent modulo m. The number m is called the modulus, and the statement b is congruent to c (modulo m,) is written mathematically as

$$b\equiv c(mod\ m)$$

If b-c is *not* integrally divisible by m, then it is said that b is *not* congruent to c (modulo m), which is written

$$b\not\equiv c(mod\ m)$$

The explicit "(mod m)" is sometimes omitted when the modulus m is understood by context, so in such cases, care must be taken not to confuse the symbol ≡ with the equivalence sign.

$$\rightarrow m\ [(a-b) + (b-c)]$$

$$\rightarrow m|(a-c)$$

$$\rightarrow a \equiv c \pmod m, \text{ which means that } a\ R\ c.$$ **Definition:**

An equivalence relation R on a semigroup (S, $*$) is called a **congruence relation** if a R a′ and b R b′ imply (a $*$ b) R (a′ $*$ b′).

Examples:

1.Let (**Z**, +) be the semigroup of integers. Consider the relation R defined on **Z** by A R b if and only if a ≡ b (mod m).

We know that a ≡ b (mod m) if m divides a–b. We note that

(i)For any integer a, we have a ≡ a (mod m), i.e., a R a

(ii)If a R b, then a ≡ b (mod m) → m | (a–b) → m|(b–a) and so b ≡ a (mod m) which means b R a.

(iii)If a R b and b R c, then

$$a \equiv b(mod\ m) \text{ and } b \equiv c(mod\ m)$$

$$\rightarrow m|(a-b) \text{ and } m|(b-c)$$

Thus R is reflexive, symmetric and transitive and so is an **equivalence relation**. Further, if

Then a ≡ c (mod m) and b ≡ d (mod m),

m| (a–c) and m | (b–d)

→m | [(a–c) + (b–d)]|

→m|[(a+b) – (c+d)]

→(a+b) ≡ (c+d) (mod m)

→(a+b) R (c+d)

Hence R is a congruence relation.

SEMIGROUPS AND MONOID

Binary Operation and its Properties

Definition. Let A be a non-empty set. Then a mapping $f : A \times A \to A$ is called a **binary operation.** Thus, a binary operation is a rule that assigns to each ordered pair $(a, b) \in A\times A$ an element of A.

Examples. 1. Let **Z** be the set of integers. Then $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ defined by $f(a,b) = a * b = a+b$, $a, b \in \mathbf{Z}$ is a binary operation on **Z** because the sum of two integers a and b is again an integer.

Thus, **addition of integers** is a binary operation.

2. Let **N** be the set of positive integers. Then $f : N \times N \to N$ defined by $f(a,b) = a * b = a-b$ is **not a binary operation** because difference of two positive integers need not be positive integer. For example 2-5 is not a positive integer.

3. For the set **N** of positive integers, let $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be defined by $f(a,b) = \frac{a}{b}$. Then f is not a binary operation. For example, if a = 2, b = 7, then $\frac{a}{b} = \frac{2}{7}$ is not a positive integer.

4. Let **Z** be the set of all integers. Then $f : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ defined by

$$f(a,b) = \max(a, b)$$

is a binary operation. For example,

$$f(2, 4) = 2 * 4 = \max(2,4) = 4 \in \mathbf{Z}.$$

5. Let A = {a, b, c}. Define * by

$$x * y = x, \ x, y \in A.$$

Then the table given below defines the operation $*$

*	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Further, if we define . by

$$x.y = y,\ x,\quad y \in A,$$

then the table given below defines the operation .

.	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

6. If A = {0, 1}. Then the binary operations $\wedge$ and $\vee$ are defined by the following tables :

$\wedge$	**0**	**1**
0	0	0
1	0	0

and

$\vee$	**0**	**1**
0	0	1
1	1	1

Properties of Binary Operation

1. Commutative Law :- A binary operation $*$ on a set A is said to be **commutative if**

$$a * b = b * a$$

for any elements a and b in A.

For example, consider the set **Z** of integers. Since

$$a+b = b+a \quad \text{and } a.b = b.a,$$

for a, b $\in$ **Z**, the addition and multiplication operations on **Z** are commutative.

But, on the other hand, subtraction in **Z** is not commutative since, for example,

$$2 - 3 \neq 3 - 2$$

Theorem. Let $*$ be a binary operation on a set A. Then any product $a_1 * a_2 * \dots * a_n$ requires no parenthesis, that is, all possible products are equal.

Proof. We shall prove this result by induction on n. Since $*$ is associative, the theorem holds for n = 1, 2 and 3. Suppose $[a_1 a_2 \dots a_n]$ denote any product and

$$(a_1 a_2 \dots a_n) = (\dots (a_1 a_2)a_3 \dots)a_n$$

It is sufficient then to show that

$$[a_1a_2 \dots a_n] = (a_1a_2 \dots a_n)$$

Since $[a_1 a_2 \dots a_n]$ denote arbitrary product, there is an m < n such that induction yields

$$\begin{aligned}
[a_1 a_2 \dots a_n] &= [a_1 a_2 \dots a_m] [a_{m+1} \dots a_n] \\
&= [a_1 a_2 \dots a_m] (a_{m+1} \dots a_n) \\
&= [a_1 a_2 \dots a_m] ((a_{m+1} \dots a_{n-1})a_n) \\
&= ([a_1 a_2 \dots a_m] (a_{m+1} .. a_{n-1}))a_n \\
&= [a_1 \dots a_{n-1}] a_n \\
&= (a_1 \dots a_{n-1})a_n \\
&= (a_1 a_2 \dots a_n) ,
\end{aligned}$$

which proves the result.

Definition. Let $*$ be a binary operation on a set A. An element e in A is called an **identity** element for $*$ if for any element $a \in A$,

$$a * e = e * a = a.$$

Further e is called right identity if $a * e = a$ and left identity if $e * a = a$ for any $a \in A$.

Let e_1 the left identity and e_2 be the right identity for a binary operation $*$. Then

$$e_1e_2 = e_2 \qquad \text{since } e_1 \text{ is left identity}$$

and

$$e_1 e_2 = e_1 \qquad \text{since } e_2 \text{ is right identity}$$

Hence $e_1 = e_2$ and so **identity element for a binary operation is unique**.

Definition. Let $*$ be a binary operation on a set A and let A has identity element e. Then inverse of an element a in A is an element b such that

$$a * b = b * a = e.$$

We shall see later on that if $*$ is associative, then the inverse of an element, if it exits, is unique.

Definition. A binary operation $*$ on a set A is said to satisfy the **left cancellation law** if

$$a * b = a * c \Rightarrow b = c$$

A binary operation $*$ on a set A is said to obey **right cancellation law** if

$$b * a = c * a \Rightarrow b = c$$

Let **Z** be the set of integers. Since

$$a + b = a + c \Rightarrow b = c$$

and

$$b + a = c + a \Rightarrow b = c \text{ for a, b, c} \in \mathbf{Z},$$

it follows that addition of integers in **Z** obeys both cancellation laws.

On the other hand, matrix multiplication does not obey cancellation laws. **To see it, let**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -3 \\ 1 & 5 \end{bmatrix}.$$

Then

$$AB = AC = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

but $B \neq C$.

Proposition 2. *Let $(M, \cdot, e)$ be a monoid. If an element x in M is invertible, then there is a unique inverse element, i.e., $xx' = x'x = e \wedge xx" = x"x = e \Rightarrow x' = x"$.*

Proof. Let x be invertible and x' and x" be its two inverses, i.e., $xx' = x'x = e$ and $xx" = x"x = e$. Then we have $x' = x'e = x'(xx") = (x'x)x" = ex" = x"$. □

In order to make all operations explicit in the flavor of universal algebra, the following equivalent alternative definition is sometimes preferred.

Definition 2. *A group is an algebra $(G, \cdot, (-)^{-1}, e)$ with a carrier set G and three operations: a binary operation $\cdot\colon G^2 \to G$, a unary operation $(-)^{-1}\colon G \to G$, and constant (nullary operation) $e \in G$ that satisfy the following identities*[1]

$$\begin{aligned} &\textit{[Associativity]} && x(yz) = (xy)z \\ &\textit{[Unit]} && ex = xe = x \\ &\textit{[Inverse element]} && xx^{-1} = x^{-1}x = e. \end{aligned}$$

As the notation suggests, the image of an element $x \in G$ under the unary operation $(-)^{-1}$ is denoted by x^{-1}. In this notation, common elsewhere a well, $(-)$ denotes a hole to be replaced by an argument. A group $(G, \cdot, (-)^{-1}, e)$ is commutative or abelian *if also $xy = yx$.*

Example 3. Examples of groups are $(\mathbb{Z}, +, -(-), 0)$, $(\mathbb{Q}, +, -(-), 0)$, $(\mathbb{R}, +, -(-), 0)$, $(\mathbb{Q} \setminus \{0\}, \cdot, 1/(-), 1)$, $(\mathbb{R} \setminus \{0\}, \cdot, 1/(-), 1)$. Convince yourselves that these are indeed groups! Note that the monoid $(\mathbb{N}, +, 0)$ is not a group, since there are no inverse elements with respect to addition. The additive inverse of an element x of a group, in e.g., $(\mathbb{Z}, +, -(-), 0)$, is denoted as usual by $-x$. The monoid $(\mathbb{Z}, \cdot, 1)$ is not a group since there are no inverse elements with respect to multiplication.

Let A be a set and let $P(A)$ denote the set of all permutations on A, i.e.,

$$P(A) = \{f\colon A \to A \mid f \text{ is bijective}\}.$$

Then $(P(A), \circ, (-)^{-1}, id_A)$ is a group, known as the group of permutations on A. Convince yourself in this as well. Here, as usual, $\circ$ denotes function composition, f^{-1} is the inverse function of a bijection f, and $id_A\colon A \to A$ is the identity function mapping every element to itself.

Let A be a set and let $+$ denote the operation of symmetric difference of sets, i.e, for two subsets B and C of A, we have

$$B + C = (B \setminus C) \cup (C \setminus B) = (B \cap C^c) \cup (C \cap B^c).$$

Then $(\mathcal{P}(A), +, id_{\mathcal{P}(A)}, \emptyset)$ is a group.

In the sequel we will use both ways to denote a group as convenient. The following simple property shows the relationship between the unary operation (inverse elements) and the binary operation of a group.

Proposition 3. *Let $G(\cdot)$ be a group. Then for any $x, y \in G$ it holds that*

$$(xy)^{-1} = y^{-1}x^{-1}.$$

Proof. Let $x, y \in G$. We have, applying associativity and unit law,

$$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xex^{-1} = xx^{-1} = e$$

and

$$(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}ey = y^{-1}y = e.$$

We next show that every group is cancellative.

Theorem 1. *Let $G_1(*, (-)^{-1}, e_1)$ and $G_2(\cdot, (-)^{-1}, e_2)$ be two groups and $h\colon G_1 \to G_2$ a (group) homomorphism. Then the following three statements hold*

(1) $\ker(h) = \{(x, y) \mid h(x) = h(y)\} \subseteq G_1 \times G_1$ *is a congruence of $G_1(*, (-)^{-1}, e_1)$,*

(2) $h(G_1)$ is a subgroup of G_2, and

(3) $G_1/\ker(h) \cong h(G_1)$.

where $G_1/\ker(h)$ denotes the quotient group of $G_1(*, (-)^{-1}, e_1)$ under the congruence $\ker(h)$. Since the operations of a quotient group and a subgroup are canonical, we do not write them in (3).

Theorem. Let (S, $*$) and (T, $*'$) be monoids with identities e and e′ respectively. Let F : S → T be a homomorphism from (S $*$) onto (T, $*'$). Then f(e) = e′.

Proof. Let b be any element of T. Since f is surjective, there is an element a $\in$ S such that f(a) = b. Since e is identity of S, we have

$$a * e = a = e * a \qquad \text{(i)}$$

and so

$$b = f(a) = f(a * e)\text{ , by (i)}$$
$$= f(a) *' f(e)\text{ , because f is homomorphism}$$

$$= b *' f(e)$$

Also,

$$\begin{aligned} b = f(a) &= f(e * a) \\ &= f(e) *' f(a) \\ &= f(e) *' b \end{aligned}$$

Hence

$$b *' f(e) = f(e) *' b = b$$

and so f(e) is identity for T. Thus, $f(e) = e'$.

Remark. The converse of the above theorem is not true.

Theorem. If f is a homomorphism from a commutative semigroup $(S, *)$ onto a semigroup $(T, *')$, then $(T, *')$ is also commutative, **that is, homomorphic image of an abelian (commutative) semigroup is abelian.**

Proof. Let $t_1, t_2 \in T$. Since f is onto, there exist $s_1, s_2 \in S$ such that

$$f(s_1) = t_1 \text{ and } f(s_2) = t_2$$

Then

$$\begin{aligned} t_1 *' t_2 &= f(s_1) *' f(s_2) \\ &= f(s_1 * s_2) \text{ , since f is homomorphism} \\ &= f(s_2 * s_1)\text{, since S is abelian} \\ &= f(s_2) *' f(s_1)\text{, since f is homomorphism} \\ &= t_2 *' t_1 \text{ .} \end{aligned}$$

Hence $(T, *')$ is abelian.

Remark. The converse of the above theorem is not true.

Theorem. Let f : (S, $*$) → (T, $*'$) be semigroup homomorphism. If S′ is a subsemigroup of (S, $*$), then the image of S′ under f is a subsemigroup of (T, $*'$).

Proof. Let f (S′) be the image of S′ under f and let t_1, t_2 be in f (S′). Then there are s_1 and s_2 in S′ such that

$$t_1 = f(s_1) \text{ and } t_2 = f(s_2)$$

We claim that f(S′) is closed under the binary operation $*'$. It is sufficient to show that $t_1 *' t_2 \in f(S')$. We have, in this direction,

$$t_1 *' t_2 = f(s_1) *' f(s_2)$$

$$= f(s_1 * s_2), \text{ because f is homomorphism.}$$

Now since S′ is a semigroup and $s_1, s_2 \in S'$, we have $s_1 * s_2 \in S'$(due to closeness of the peration $*$). Hence $f(s_1 * s_2) \in f(S')$. It follows, therefore, that $t_1 *' t_2 \in f(S')$.

Further, since the associativity hold in T, it also holds in f(S′). Hence f(S′) is a subsemigroup of (T, $*'$).

Theorem. The intersection of two subsemigroups of a semigroup (S, $*$) is subsemigroup of (S, $*$).

Proof. Let (S_1, $*$) and (S_2, $*$) be two subsemigroups of the semigroup (S, $*$). Let $a \in S_1 \cap S_2$ and $b \in S_1 \cap S_2$. Then

$$a \in S_1 \cap S_2 \Rightarrow a \in S_1 \text{ and } a \in S_2$$

$$b \in S_1 \cap S_2 \Rightarrow b \in S_1 \text{ and } b \in S_2$$

Since S_1 is a subsemigroup, therefore, a, b $\in S_1$ implies $a * b \in S_1$. Similarly, since S_2 is a subsemigroup, a, b $\in S_2$ implies $a * b \in S_2$. Hence

$$a * b \in S_1 \cap S_2$$

Hence $S_1 \cap S_2$ is closed under the operation $*$. Further associativity in S_1 and S_2 implies the associativity of $S_1 \cap S_2$ since $S_1 \cap S_2 \subseteq S_1$ and $S_1 \cap S_2 \subseteq S_2$. Hence $S_1 \cap S_2$ is a subsemigroup of (S, $*$).

Corollary. Intersection of two submonoids of a monoid (S, $*$) is a semimonoid of (S, $*$).

Proof follows the same line as that in the above Theorem.

Remark. Union of two subsemigroups of a semigroup $(S, *)$ need not be subsemigroup of $(S, *)$.

For example,

$$(S_1, *) = \{0, \pm 2, \pm 4, \pm 6, + \ldots.\}$$

and

$$(S_2, *) = \{0, \pm 3, \pm 6, \pm 9, \pm, \ldots\}$$

are subsemigroups of the semigroup $(\mathbf{Z}, +)$ of integers. But

$$S_1 \cup S_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm \ldots.\}$$

is not a subsemigroup of $(\mathbf{Z}, +)$, because

$$2 \in S_1 \cup S_2,\ 3 \in S_1 \cup S_2\ ,$$

but $2+3 = 5 \notin S_1 \cup S_2$ showing that $S_1 \cup S_2$ is not closed under addition.

Theorem. Let R be a congruence relation on the semigroup $(S, *)$. Then $\odot$: $S/R \times S/R \to S/R$ defined by

$$\odot\ ([a], [b]) = [a] \odot [b] = [a * b]\ , a, b \in S$$

is a binary operation on S/R and $(S/R, \odot)$ is a semigroup.

Proof. Suppose that $([a], [b]) = [a'], [b'])$. Then $a\ R\ a'$ and $b\ R\ b'$. Since R is congruence relation, this implies $a * b\ R\ a' * b'$. Thus $[a * b] = [a' * b']$, that is, $\odot$ is a well defined function. Hence $\odot$ is a **binary operation** S/R.

Further we note that

$$\begin{aligned}
[a] \odot ([b] \odot [c]) &= [a] \odot [b * c] \quad \text{(by definition of } \odot \text{)}\\
&= [a * (b * c)] \quad \text{(by definition of } \odot \text{)}\\
&= [(a * b) * c] \quad \text{(Associativity of } * \text{ in S)}\\
&= [a * b] \odot [c] \quad \text{(by definition of } \odot \text{)}\\
&= ([a] \odot [b]) \odot [c] \quad \text{(by definition of } \odot \text{)}
\end{aligned}$$

Hence $\odot$ is an associative operation. This implies that $(S/\mathbf{R}, \odot)$ is a **semigroup.**

The operation $\odot$ is called **quotient binary relation** on S/R constructed from the given binary relation $*$ on S by the congruence relation R.

The semigroup (S/R, $\odot$) is called Quotient Semigroup or **Factor Semigroup** or **the Quotient of S by R.**

Theorem. Let R be the congruence relation on the monoid (S, *), then (S/R, $\odot$) is a monoid.

Proof. We have shown above that (S/R, $\odot$) is a semigroup. Further if e is identity element in(S, $*$), then [e] is the identity in (S/R, $\odot$). Thus (S/R, $\odot$) is semigroup having identity element [e] and so is a monoid.

Theorem. Let R be a congruence relation on a semigroup (S,$*$) and let (S/R, $\odot$) be the corresponding quotient semigroup. Then the mapping $\phi : S \to S/R$ (called the **natural mapping**) defined by

$$\phi(a) = [a]$$

is an **onto homomorphism**, known as **Natural homomorphism.**

Proof. According to definition of ϕ, to each [a] in S/R, there is $a \in S$ such that $\phi[a] = [a]$. Hence ϕ is subjective. Now let $a, b \in S$. Then

$$\begin{aligned}\phi(a * b) &= [a * b] \\ &= [a] \odot [b] \\ &= \phi(a) \odot \phi(b)\end{aligned}$$

Hence ϕ is homomorphism onto.

Theorem (Fundamental Theorem of Semi-group Homomorphism). Let $f : S \to T$ be a homomorphism of the semigroup $(S, *)$ onto the semigroup $(T, *')$. Let R be the relation on S defined by

$$a \, R \, b \quad \text{if } f(a) = f(b) \text{ for } a, b \in S$$

Then

(i) R is a congruence relation on S

(ii) $(S/R, \odot)$ is isomorphic to $(T, *')$.

(If f is not onto, them (ii) shall be "S/R is isomorphic to f(S)".

Proof. First we show that R is an equivalence relation. We note that

(i) Since $f(a) = f(a)$, we have $a \, R \, a$.

(ii) If $a \, R \, b$, then $f(a) = f(b)$ or $f(b) = f(a)$ and hence $b \, R \, a$.

(iii) If a R b and b R c , then

$$f(a) = f(b) \text{ and } f(b) = f(c)$$

and hence

$$f(a) = f(c)$$

and so a R c.

Thus the relation R is reflexive, symmetric and transitive and so an equivalenc relation.

Suppose now that

$$a\,R\,a' \quad \text{and} \quad b\,R\,b'.$$

Then

$$f(a) = f(a') \text{ and } f(b) = f(b')$$

Since f is homomorphism,

$$\begin{aligned} f(a * b) &= f(a) *' f(b) \\ &= f(a') *' f(b') \\ &= f(a' * b') \end{aligned}$$

Hence

$$(a * b)\,R(a' * b')$$

and so R is a congruence relation.

Define

$$\psi : S/R \to T$$

by

$$\psi([a]) = f(a).$$

We claim that ψ is well defined. Suppose [a] = [b]. ψ will be well defined i f(a) = f(b). Now [a] = [b] implies a R b, that is, f(a) = f(b). Hence ψ is function (well defined).

Further, if [a], [b] ∈ S/R, then

$$\begin{aligned} \psi([a] \odot [b]) &= \psi([a * b]),\ a, b \in S \\ &= f(a * b) \\ &= f(a) *' f(b), \text{ because f is homomorphism} \end{aligned}$$

ψ ([a] $\odot$ [b]) = ψ ([a $*$ b]), a, b $\in$ S

= f(a $*$ b)

= f(a) $*'$ f(b), because f is homomorphism

= ψ [a] $*'$ ψ[b]

So ψ is semigroup homomorphism.

Also

ψ ([a] = ψ ([b]) $\Rightarrow$ f(a) = f(b)

$\Rightarrow$ a R b

$\Rightarrow$ [a] = [b],

and so ψ is one – to – one .

Thus ψ, as a map, is bijective and homomorphism. Hence ψ is an isomorphism and

$$S/R \cong \mathbf{T}$$

Remark. We have proved that the mapping $\phi : S \to S/R$ is natural homomorphism. Also, we proved that the mapping $\psi : S/R \to T$ is an isomorphism. Thus diagram of the situation becomes

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[dr, "\phi"'] & & T \\
& S/R \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Also, we note that

$$\begin{aligned}
(\psi \circ \phi)(a) &= \psi(\phi(a)) \\
&= \psi([a]) \\
&= f(a) \text{ for all } a \in S .
\end{aligned}$$

Hence

$$\psi \circ \phi = f$$

Direct product of semigroups :

Let $(S, *)$ and $(T, *')$ be two semigroups. Consider the cartesian product $S \times T$. Define a binary operation $*''$ on $S \times T$ by

$$(s_1, t_1) *'' (s_2, t_2) = (s_1 * s_2, t_1 *' t_2)$$

In what follows, we prove that $(S \times T, *'')$ is a semigroup.

Theorem. Let $(S, *)$ and $(T, *')$ be semigroups. Then $(S \times T, *'')$ is a semigroup under the binary operation $*''$ defined by

$$(s_1, t_1) *'' (s_2, t_2) = (s_1 * s_2,\ t_1 *' t_2)\,.$$

Proof. If (s_1, t_1) , (s_2, t_2) and $(s_3, t_3) \in S \times T$, then

$$\begin{aligned}[\,(s_1, t_1) *'' (s_2, t_2)\,] *'' (s_3, t_3) &= (s_1 * s_2, t_1 *' t_2) *'' (s_3, t_3)\\ &= ((s_1 * (s_2 * s_3), t_1 *' (t_2) *' t_3))\\ &= (s_1 * (s_2 * s_3), t_1 *' (t_2 *' t_3))\\ &= (s_1, t_1) *'' (s_2 * s_3, t_2 *' t_3)\\ &= (s_1, t_1) *'' [(s_2, t_2) *'' (s_3, t_3)\,]\end{aligned}$$

Hence $*''$ is associative and so $(S \times T, *'')$ is a semigroup.

Corollary. If $(S, *)$ and $(T, *')$ are monoids, then $(S \times T, *'')$ is also a monoid.

Proof. We have proved above that $(S \times T, *'')$ is a semigroup. We further note that if e_S is identity of $(S, *)$ and e_T is identity of $(T, *')$, then for $(s_1, t_1) \in S \times T$, we have

$$\begin{aligned}(e_S, e_T) *'' (s_1, t_1) &= (e_S * s_1, e_T *' t_1)\\ &= (s_1, t_1)\end{aligned}$$

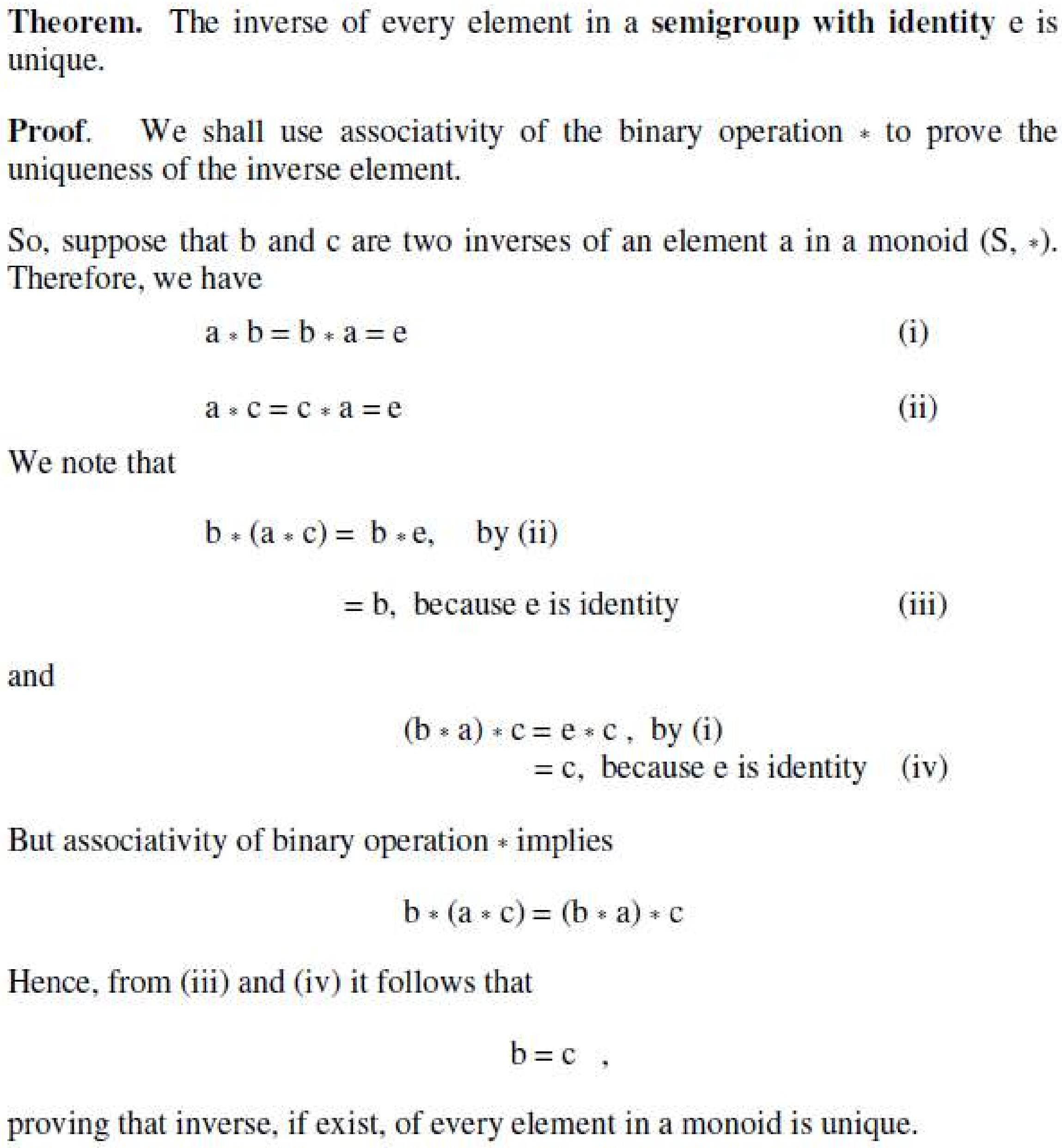
and

$$\begin{aligned}(s_1, t_1) *'' (e_S, e_T) &= (s_1 * e_S, t_1 *' e_T)\\ &= (s_1, t_1)\end{aligned}$$

Thus

$$(s_1, t_1) *'' (e_S, e_T) = (e_S, e_T) *''(s_1, t_1) = (s_1, t_1)$$

showing that (e_S, e_T) is identity element of $(S \times T, *'')$, that is, $(S \times T, *'')$ is a semigroup with identity (e_S, e_T) and hence is a monoid.



Theorem. The inverse of every element in a **semigroup with identity** e is unique.

Proof. We shall use associativity of the binary operation $*$ to prove the uniqueness of the inverse element.

So, suppose that b and c are two inverses of an element a in a monoid (S, $*$). Therefore, we have

$$a * b = b * a = e \qquad \text{(i)}$$

$$a * c = c * a = e \qquad \text{(ii)}$$

We note that

$$b * (a * c) = b * e, \quad \text{by (ii)}$$

$$= b, \ \text{because e is identity} \qquad \text{(iii)}$$

and

$$(b * a) * c = e * c\,, \ \text{by (i)}$$
$$= c, \ \text{because e is identity} \qquad \text{(iv)}$$

But associativity of binary operation $*$ implies

$$b * (a * c) = (b * a) * c$$

Hence, from (iii) and (iv) it follows that

$$b = c \quad ,$$

proving that inverse, if exist, of every element in a monoid is unique.

Theorem :

If (S,*) and (T,∘) are commutative semigroups then their product is also commutative semigroup.

Proof:

We have already shown that if (S,*) and (T,∘) are semigroups then their product is semigroup.

we now show that product SxT is commutative.

Let (a,b) ,(c,d) be any two elements in SxT .

Then

(a,b) + (c,d) = (a*c ,b∘d)

=(c*a , d∘b)

Because both * and ∘ are commutative

=(c,d) (a,b)

Thus + is a commutative operation on S*T.

Hence (SxT , +) is commutative semigroup.

Theorem:

Let f : s→ T be an onto mapping from a semigroup (S,*) to an algebraic structure (T,∘) where ∘ is a binary operation on T .If f is semigroup homomorphism then (T,∘) is a semigroup.

Proof:

In order to prove that (T,∘) is a semigroup.

we must show that ∘ is an associative operation on T.

Let x,y,z be any three elements in T.

Since f onto mapping the exists a,b,c is S such that x=f(a) , y=f(b) and z=f(c)

Now (x∘y)z=f(a)∘f(b)∘f(c)

=f(a*b)∘f(c) f is homomorphism

=f(a*b)*c) f is homomorphism

=f(a*(b*c)) * is associative

=f(a)∘f(b*c) f is homomorphism

=f(a) ∘(f(b)∘f(c)) f is homomorphism

$=x\circ(y\circ z)$

Hence ∘ is associative and S∘(T,∘) is a semigroup.

Theorem:

If (M,*) is a commutative monoid then the set of all idempotent elements of M forms a submonoid.

Proof:

Let S be the set of all idempotent element of M>.

That is $S=\{ x\epsilon M \ ; x^2 = x\}$

Since the identity element $e\epsilon M$ is idempotent , We have $e\epsilon S$.

We now show that S is closed with respect to * .

Let a,b be any two elements of S.

Then $a^2 = a$ and $b^2 = b$

Now

$(a*b)^2= (a*b)(a*b)$

$=a*(b*a)*b$ * is associative

$=a*(a*b)*b$ * is commutative

$= (a*a)*(b*b)$ * is associative

$=a^2 * b^2$

$=a*b$ $a^2 = a$ and $b^2 = b$

Thus a*b is idempotent element of M .

Hence $a*b\epsilon S$ and s∘(S,*) is a submonoid.

Theorem:

Let (M,*) and (T,∘) be two monoids with identity e and e' respectively. If f is an onto mapping from M onto T such that f(a*b) =f(a) ∘ f(b) ∀a,bϵ M then f(e) = e'

Proof:

Let y be any element of T.

Since f is onto,there exists an element xϵM such that f(x) = y.

Now , Y=f(x)=f(x*e) (e is the identity of (M,*))

=f(x) ∘ f(e)

=y∘f(e)

Similarly

Y=f(x) = f(e*x)

=f(e)∘f(x)

=f(e)∘y

Thus f(e) ∘y =y ∘f(e) =y

Which implies f(e) is the identity for T.

Since Identity element in a monoid is unique, we have e' =f(e).

PART - B

POSSIBLE QUESTIONS – SIX MARKS

1. Prove that under the semigroup homomorphism the properties associativity , idempotency and commutative are preserved.

2. Show that every monoid <M, *, e> is isomorphic to a submonoid of $<M^M, \circ\ ,\ \Delta>$ where Δ is the identity mapping of M.

3. Given the algebraic system $<N, +>$ and $<Z_4, +_4>$, where N is the set of natural numbers , show that there exists a homomorphism from $<N, +>$ to $<Z_4, +_4>$.

4.Show that the set of all the invertible elements of a monoid form a group under the same operation as that of the monoid.

5. Show that the intersection of any two congruence relations on a set is also a congruence relation.

6. Let <S, *> be a given semigroup. There exists a homomorphism g: S→ S^S, where $< S^S ,\circ >$ is a semigroup of functions from S to S under the operation of (left) composition.

7. Show that the set of all semigroup endomorphisms of a semigroup is a semigroup under the operation of left composition.

8. Define homomorphism with example.

9. Show that the composition of two homomorphisms is also a homomorphism.

10. Let <S, *> , <T, Δ> and <V, +> be semigroups and g: S→T and h: T →V be semigroup homomorphisms. Then $(h \circ g)$: S→ V is a semigroup homomorphism from <S, *> to <V, +>.

11. Let I be the set of integers and · denote the operation of multiplication so that <I, · , 1> is a monoid. Show that <{0}, ·> is a semigroup but not a submonoid.

PART – C

POSSIBLE QUESTIONS – TEN MARKS

1. State and prove the function theorem of semigroup homomorphism.

2.Let (M, *) be a monoid .Then there exists a subset $T \subseteq M^M$ such that (M, *) is isomorphic to the monoid (T, o).

3.Prove that every finite semigroup has an idempotent element.(That is an element a such that a^2=a).

4.Let f: S→ T be an onto mapping from a semigroup (S,*) to an algebraic structure (T,o) , where o is a binary operation on T. If f is semigroup homomorphism then (T,o) is a semigroup.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Advanced Discrete Mathematics **Subject Code: 17MMP105A**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
The mathematical structure (G, *) is said to be a ----------------- if binary operation * satisfies closure property only.	Semigroup	Quasigroup	Abelian Group	Monoid	Quasigroup
The algebraic structure (G, *) is said to be a ----------------- if binary operation * satisfies closure and associative property only.	Semigroup	Quasigroup	Abelian Group	Monoid	Semigroup
An equivalence relation R defined on the semigroup (S, *) is called a _________________________ if a R b and c R d then a*c R b * d.	equivalence classes	Monoid	congruence relation	relation	congruence relation
A semigroup with more than one idempotent element is ----------------	group	semigroup	not a group	subgroup	not a group
An operation * is said to be commutative law if -----------------	a*b= b*a	a+b=b+a	a*(b*c)= (a*b)*c	a*b=a*c	a*b= b*a
The semigroup S/~ is called the ___________________of S by ~.	relation	equivalence class	quotient	remainder	quotient
A function f: (S , *)→ (S',+) is called a ___________________ if f(a*b) = f(a) + f(b)	homomorphism	automorphism	isomorphism	epimorphism	homomorphism
Let g: <X,°> →<Y, *> is a homomorphism and if g is onto then g is called ----------------------	homomorphism	automorphism	isomorphism	epimorphism	epimorphism
Let g: <X,°> →<Y, *> is a homomorphism and if g is one to one then g is called ----------------------	homomorphism	monomorphism	isomorphism	epimorphism	monomorphism
Let g: <X,°> →<Y, *> is a homomorphism and if g is one to one and onto then g is called ----------------------	homomorphism	monomorphism	isomorphism	epimorphism	isomorphism
The intersection of two congruence relation is -----------------	Not a congruence relation	congruence relation	subalgebra	Directproduct	congruence relation
A function f: (S , *)→ (S',+) is called a homomorphism if	f(a*b) = f(a) + f(b)	f(a+b) = f(a) + f(b)	f(a+b) = f(a) * f(b)	f(a*b) = f(a) *f(b)	f(a*b) = f(a) + f(b)
Let g: <X,°> →<Y, *> is a homomorphism and if g is -----------------then g is called epimorphism	constant	one to one and onto	onto	one to one	onto
Let g: <X,°> →<Y, *> is a homomorphism and if g is --------------- then g is called monomorphism	constant	one to one and onto	onto	one to one	one to one
Let g: <X,°> →<Y, *> is a homomorphism and if g is ------------------ then g is called isomorphism	constant	one to one and onto	onto	one to one	one to one and onto
The algebraic structure (G, *) is said to be a semigroup if binary operation * satisfies -------------	closure and identity	identity and inverse	closure and associativity	associativity and identity	closure and associativity
The set of all semigroup endomorphisms of a semigroup is a -------------	group	Monoid	semigroup	Not a group	semigroup
Every semigroup homomorphism induces a --------------------	semigroup	congruence relation	Monoid	subalgebra	congruence relation
Every congruence relation induces a --------------------	semigroup homomorphism	congruence relation	semigroup	homomorphism	semigroup homomorphism
If <S, *> , <T, Δ> are both commutative groups then their direct product is ---------------	commutative	associative	Identity	closure	commutative
A set together with a number of operations on the set is called an ----------------	Monoid	semigroup	group	. Algebraic system	Algebraic system
The semigroup (S, *) which has also an identity element with respect to * is called ___________________	Semigroup	Quasigroup	Abelian Group	Monoid	Monoid

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DEPARTMENT OF MATHEMATICS

SUBJECT: ADVANCED DISCRETE MATHEMATICS	**SEMESTER: I**	**L T P C**
SUBJECT CODE:17MMP105A	**CLASS:I M.Sc (MATHEMATICS)**	**4 0 0 4**

UNIT II

Lattices: Lattices as Partially Ordered Sets: Definition and Examples - Principle of duality - Some Properties of Lattices - Lattices as Algebraic Systems – Sublattices - Direct product, and Homomorphism.

TEXT BOOKS

1.Tremblay J. P. and Manohar, R., (1997). Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III).

REFERENCES

1.Wiitala S., (2003),Discrete Mathematics- A Unified Approach, McGraw-Hill Book Co, New Delhi.

2.Seymour Lepschutz, (2007) ,Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.

LATTICES

Definitions and Examples

Definition: A **lattice** is a partially ordered set (L, ≤) in which every subset {a, b} consisting of **two element** has a **least upper bound** and a **greatest lower bound.**

We denote lub({a, b}) by a ∨ b and call it **join** or **sum of a and b.**

Similarly,

we denote GLB({a, b}) by a ∧ b and call it **meet** or **product of a and b.**
Other symbol used are:

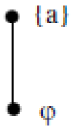
LUB : ⊕ , +, ∪
GLB : *, **.** , ∩

Thus **Lattice is** a mathematical structure with **two binary operations, join and meet.** Lattice structures often appear in computing and mathematical applications.

A totally ordered set is obviously a lattice but not all partially ordered sets are lattices.

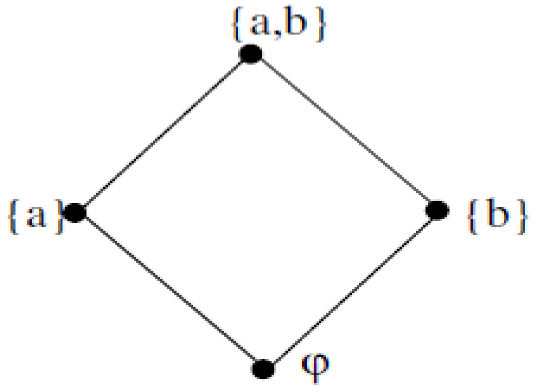
Example 1. Let A be any set and P(A) be its power set. The partially ordered set (P(A), ⊆) is a lattice in which the meet and join are the same as the operations ∩ and ∪ respectively. If A has single element, say a, then P(A) = {φ, {a}} and
LUB({ φ, {a}) = {a}
GLB({φ, {a}) = φ

The Hasse diagram of (P(A), ⊆) is a chain containing two elements φ and {a} as shown below:



If A has two elements, say a and b. Then P(A) = {φ, {a}, {b}, {a, b}}. The

Hasse diagram of {P(A), $\subseteq$) is then as shown below :



We note that

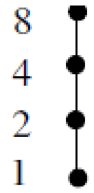
1. LUB exists for every two subsets and is L $\cup$ M

2. GLB exists for every two subsets and is in L $\cap$ M for L, M $\in$ P(A).

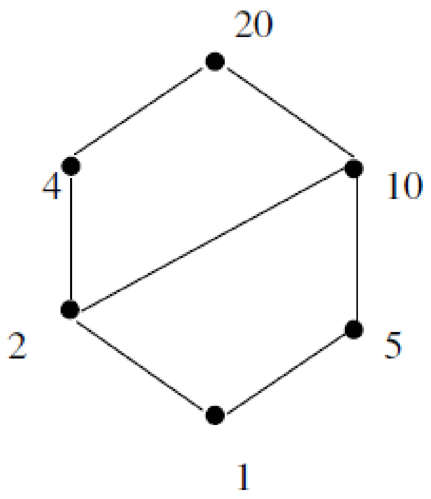
Hence P(A) in a lattice.

Example 2. Consider the poset (**N**, $\leq$), where $\leq$ is relation of divisibility. Then **N** is a lattice in which
join of a and b = a $\vee$ b = L C M(a, b)
meet of a and b = a $\wedge$ b = G C D (a, b) for a, b $\in$ **N**.

Example 3. Let n be a positive integer and let D_n be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility**.** The Hasse diagram of the lattices D_8, D_{20} and D_{30} are respectively.

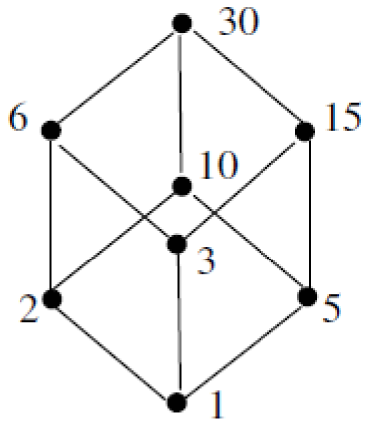


$D_8 = \{1, 2, 4, 8\}$



$D_{20} = \{1, 2, 4, 5, 10, 20\}$

and

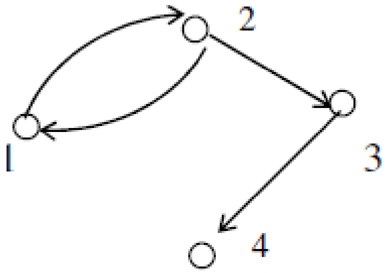


$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

The TransiDefinition: The **Transitive closure** of a relation R is the smallest transitive relation containing R. It is denoted by R_∞.

Example: Let A = {1, 2, 3, 4} and R = [(1, 2), (2, 3), (3, 4), (2, 1)] Find the transitive closure of R.

Solution: The digraph of R is



We note that from vertex 1, we have paths to the vertices 2, 3, 4 and 1. Note that path from 1 to 1proceeds from 1 to 2 to 1. Thus we see that the ordered pairs (1, 1), (1, 2), (1, 3) and (1, 4) are in R_∞. Starting from vertex 2, we have paths to vertices 2, 1, 3 and 4 so the ordered pairs (2, 1), (2, 2), (2, 3) and (2, 4)
are in R_∞. The only other path is from vertex 3 to 4, so we have
R_∞ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3,4)}

Example: Let R be the set of all equivalence relations on a set A. As such R consists of subsets of A × A and so R is a partially ordered set under the partial order of set inclusion. If R and S are equivalence relations on A, the same property may be expressed in relational notations as follows:
$R \subseteq S$ if and only if x R y _ x S y for all x y $\in$ A.

Then **(**R, $\subseteq$) is a poset. R is a lattice**,** where the meet of the equivalence relations R and S is their intersection $R \cap S$ and their join is $(R \cup S)_\infty$, the transitive closure of their union.

Definition: Let (L, $\leq$) be a poset and let (L, $\geq$) be the dual poset. If (L, $\leq$) is a lattice, we can show that (L, $\geq$) is also a lattice. In fact, for any a and b in L, the
L U B of a and b in (L, $\leq$) is equal to the GLB of a and b in (L, $\geq$). Similarly, the GLB of a and b in (L, $\leq$) is equal to L U B in (L, $\geq$).
The operation $\vee$ and $\wedge$ are called **dual of each other**.

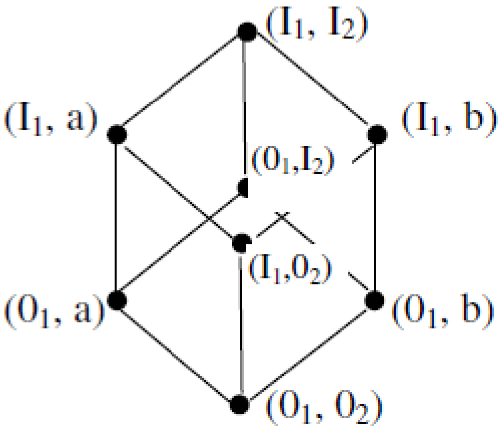
Example: Let S be a set and L = P(S). Then (L, $\subseteq$) is a lattice and its **dual lattice** is (L, $\supseteq$), where $\supseteq$ represents "contains". We note that in the poset (L, $\supseteq$), the join $A \vee B$ is the set $A \cap B$ and the meet $A \wedge B$ is the set $A \cup B$.

Cartesian Product of Lattices

Theorem: If (L_1, $\leq$) and (L_2, $\leq$) are lattices, then (L, $\leq$) is a lattice, where
$L = L_1 \times L_2$ and the partial order $\leq$ of L is the product partial order.

Proof: We denote the join and meet in L_1 by $\vee_1$, and $\wedge_1$ and the join and meet

in L_2 by $\vee_2$ and $\wedge_2$ respectively.
We know that Cartesian product of two posets is a poset.
Therefore $L = L_1 \times L_2$ is a poset. Thus all we need to show is that if
(a_1, b_1) and $(a_2, b_2) \in L$,
Then $(a_1, b_1) \vee (a_2, b_2)$ and $(a_1, b_1) \wedge (a_2, b_2)$ exist in L.
Further, we know that
$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2 , b_1 \vee b_2)$ and
and
$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2 , b_1 \wedge b_2)$
Since L_1 is lattice, $a_1 \vee_1 a_2$ and $a_1 \wedge_1 a_2$ exist. Similarly, since L_2 is a lattice, $b_1 \vee b_2$ and $b_1 \wedge b_2$ exist. Hence $(a_1, b_1) \vee (a_2, b_2)$ and $(a_1, b_1) \wedge (a_2, b_2)$ both exist and therefore $(L, \leq)$ is a lattice, called **the direct product of $(L_1, \leq)$ and $(L_2, \leq)$.**



$L = L_1 \times L_2$

Properties of Lattices:

Let $(L, \leq)$ be a lattice and let a, b , c $\in$ L. Then, from the definition of $\vee$ (join) and $\wedge$ (meet)

we have

(i) $a \leq a \vee b$ and $b \leq a \vee b$; $a \vee b$ is an upper bound of a and b.

(ii) if $a \leq c$ and $b \leq c$, then $a \vee b \leq c$; $a \vee b$ is the least bound of a and b.

(iii) $a \wedge b \leq a$ and $a \wedge b \leq b$; $a \wedge b$ is a lower bound of a and b.

(iv) if $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$; $a \wedge b$ is the greatest lower bound of a and b

Theorem:

Let L be a lattice. Then for every a and b in L,

(i) $a \vee b = b$ if and only if $a \leq b$

(ii) $a \wedge b = a$ if and only if $a \leq b$

(iii) $a \wedge b = a$ if and only if $a \vee b = b$

Proof:

(i) Let $a\vee b = b$. Since $a \leq a \vee b$, we have $a \leq b$.

Conversely, if $a \leq b$, then since $b \leq b$, it follows that b is an upper bound of a and b. Therefore, by the definition of least upper bound, $a \vee b \leq b$. Also $a \vee b$ being an upper bound, $b \leq a \vee b$. Hence $a \vee b = b$.

(ii) Let $a \wedge b = a$. Since $a \wedge b \leq b$, we have $a \leq b$. Conversely, if $a \leq b$ and since $a \leq a$, a is a lower bound of a and b and so, by the definition of greatest lower bound, we have

$$a \leq a \wedge b$$

Since $a \wedge b$ is lower bound,

$$a \wedge b \leq a$$

Hence

$$a \wedge b = a.$$

(iii) From (ii)

$$a \wedge b = a \Leftrightarrow a \leq b\text{…….(iv)}$$

From (i)

$$a \leq b \Leftrightarrow a \vee b = b\text{……….(v)}$$

Hence, combining (iv) and (v),

we have

$$a \wedge b = a \Leftrightarrow a \vee b = b.$$

Example: Let L be a linearly (total) ordered set. Therefore $a, b \in L$ imply either $a \leq b$ or $b \leq a$. Therefore, the above theorem implies that

$a \vee b = a$

$a \wedge b = a$

Thus for every pair of elements a, b in L, $a \vee b$ and $a \wedge b$ exist. Hence **a linearly ordered set is a lattice.**

Theorem :

Let $(L, \leq)$ be a lattice and let $a, b, c \in L$. Then we have

L_1 : **Idempotent property**

(i) $a \vee a = a$

(ii) $a \wedge a = a$

L_2 : **Commutative property**

(i) $a \vee b = b \vee a$

(ii) $a \wedge b = b \wedge a$

L_3 : **Associative property**

(i) $a \vee (b \vee c) = (a \vee b) \vee c$

(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

L_4 : **Absorption property**

(i) $a \vee (a \wedge b) = a$

(ii) $a \wedge (a \vee b) = a$

Proof: L_1 : The idempotent property follows from the definition of LUB and GLB.

L_2 : Commutativity follows from the symmetry of a and b in the definition of LUB and GLB.

L_3 : (i) From the definition of LUB, we have

$$a \leq a \vee (b \vee c) \text{ ....(1)}$$

$$b \vee c \leq a \vee (b \vee c) \text{ ....(2)}$$

Also $b \leq b \vee c$ and $c \leq b \vee c$ and so transitivity implies

$$b \leq a \vee (b \vee c)\text{................. (3)}$$

and

$$c \leq a \vee (b \vee c) \text{ ....(4)}$$

Now, (1) and (3) imply that $a \vee (b \vee c)$ is an upper bound of a and b and hence by the definition of least upper bound, we have

$$a \vee b \leq a \vee (b \vee c) \text{ ....(5)}$$

Also by (4) and (5), $a \vee (b \vee c)$ is an upper bound of c and $a \vee b$. Therefore

$$(a \vee b) \vee c \leq a \vee (b \vee c) \text{ ....(6)}$$

Similarly

$$a \vee (b \vee c) \leq (a \vee b) \vee c\text{.......... (7)}$$

Hence, by antisymmetry of the relation $\leq$, (6) and (7) yield

$a \vee (b \vee c) = (a \vee b) \vee c$

The proof of (ii) is analogous to the proof of part (i).

L_4 : (i) Since $a \wedge b \leq a$ and $a \leq a$, it follows that a is an upper bound of $a \wedge b$ and a. Therefore, by the definition of least upper bound

$$a \vee (a \wedge b) \leq a \text{ ....(8)}$$

On the other hand, by the definition of LUB, we have

$$a \leq a \vee (a \wedge b) \text{ ....(9)}$$

The expression (8) and (9) yields

$$a \vee (a \wedge b) = a.$$

(ii) Since $a \leq a \vee b$ and $a \leq a$, it follows that a is a lower bound of $a \vee b$ and a.

Therefore, by the definition of GLB,

$$a \leq a \wedge (a \vee b) \text{ ....(10)}$$

Also, by the definition of GLB, we have

$$a \wedge (a \vee b) \leq a \text{ ....(11)}$$

Then (10) and (11) imply

$$a \wedge (a \vee b) = a$$

and the proof is completed.

In view of L_3, we can write $a \vee (b \vee c)$ and $(a \vee b) \vee c$ as $a \vee b \vee c$. Thus, we can express

LUB ({$a_1, a_2, \ldots. a_n$) as $a_1 \vee a_2 \vee \ldots\ldots \vee a_n$

GLB ({$a_1, a_2, \ldots. a_n$) as $a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_n$

Remark:

Using commutativity and absorption property, part (ii) of previous Theorem can be proved as follows :

Let $a \wedge b = a$.

We note that

$$b \vee (a \wedge b) = b \vee a$$
$$= a \vee b \text{ (Commutativity)}$$

But

$$b \vee (a \wedge b) = b \text{ (Absorption property)}$$

Hence

$$a \vee b = b$$

and so by part (i), $a \leq b$. Hence $a \wedge b = a$ if and only if $a \leq b$.

Theorem: Let $(L, \leq)$ be a lattice. Then for any $a, b, c \in L$, the following properties hold :

1. (**Isotonicity) :** If $a \leq b$, then

(i) $a \vee c \leq b \vee c$

(ii) $a \wedge c \leq b \wedge c$

This property is called "Isotonicity".

2. $a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$

3. $c \leq a$ and $c \leq b$ if and only if $c \leq a \wedge b$

4. If $a \leq b$ and $c \leq d$, then

(i) $a \vee c \leq b \vee d$

(ii) $a \wedge c \leq b \wedge d$.

Proof : 1 (i). We know that

$a \vee b = b$ if and only if $a \leq b$.

Therefore, to show that $a \vee c \leq b \vee c$, we shall show that

$$(a \vee c) \vee (b \vee c) = b \vee c.$$

We note that

$$\begin{aligned}(a \vee c) \vee (b \vee c) &= [(a \vee c) \vee b] \vee c = a \vee (c \vee b) \vee c \\ &= a \vee (b \vee c) \vee c \\ &= (a \vee b) \vee (b \vee c) \\ &= b \vee c \ (\Theta a \vee b = b \text{ and } c \vee c = c)\end{aligned}$$

The part 1 (ii) can be proved similarly.

2. If $a \leq c$, then 1(i) implies

$$a \vee b \leq c \vee b$$

But

$$\begin{aligned}b \leq c &\Leftrightarrow b \vee c = c \\ &\Leftrightarrow c \vee b = c \text{ (commutativity)}\end{aligned}$$

Hence $a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$

3. If $c \leq a$, then 1(ii) implies $c \wedge b \leq a \wedge b$

But

$$c \leq b \Leftrightarrow c \wedge b = c$$

Hence

$$c \leq a \text{ and } c \leq b \text{ if and only if } c \leq a \wedge b.$$

4 (i) We note that 1(i) implies that if $a \leq b$, then $a \vee c \leq b \vee c = c \vee b$

if $c \leq d$, then $c \vee b \leq d \vee b = b \vee d$

Hence, by transitivity

$$a \vee c \leq b \vee d$$

(ii) We note that 1(ii) implies that

if $a \leq b$, then $a \wedge c \leq b \wedge c = c \wedge b$

$$\text{if } c \le d, \text{ then } c \wedge b \le d \wedge b = b \wedge d.$$

Therefore transitivity implies

$$a \wedge c \le b \wedge d.$$

Theorem:

Let (L, ≤) be a lattice. If $a, b, c \in L$, then

(1) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$

(2) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

These inequalities are called "Distributive Inequalities".

Proof: We have

$a \le a \vee b$ and $a \le a \vee c$ (i)

Also, by the above theorem, if $x \le y$ and $x \le z$ in a lattice, then $x \le y \wedge z$. Therefore (i) yields

$$a \le (a \vee b) \wedge (a \vee c) \text{................ (ii)}$$

Also

$$b \wedge c \le b \le a \vee b$$

and

$$b \wedge c \le c \le a \vee c ,$$

that is, $b \wedge c \le a \vee b$ and $b \wedge c \le a \vee c$ and so, by the above argument, we have

$$b \wedge c \le (a \vee b) \wedge (a \vee c) \text{ (iii)}$$

Also, again by the above theorem if $x \le z$ and $y \le z$ in a lattice, then

$$x \vee y \le z$$

Hence, (ii) and (iii) yield

$$a\ c\ (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$

This proves (1).

The second distributive inequality follows by using the **principle of duality.**

Theorem: (Modular Inequality) : Let (L, ≤) be a lattice. If $a, b, c \in L$, then

$a \le c$ if and only if $a \vee (b \wedge c) \le (a \vee b) \wedge c$

Proof: We know that $a \le c \Leftrightarrow a \vee c = c$ ....(1)

Also, by distributive inequality,

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Therefore using (1) $a \leq c$ if and only if

$a \vee (b \wedge c) \leq (a \vee c) \wedge c$,

which proves the result.

The modular inequalities can be expressed in the following way also:

$(a \wedge b) \vee (a \wedge c) \leq a \wedge [b \vee (a \wedge c)]$

$(a \vee b) \wedge (a \vee c) \geq a \vee [b \wedge (a \vee c)]$

Example: Let $(L, \leq)$ be a lattice and a, b, c $\in$ L. If $a \leq b \leq c$, then (i) $a \vee b = b \wedge c$, (ii) $(a \wedge b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Solution: (i) We know that

$a \leq b \Leftrightarrow a \vee b = b$

and

$b \leq c \Leftrightarrow b \wedge c = b$

Hence $a \leq b \leq c$ implies $a \vee b = b \wedge c$.

(ii) Since $a \leq b$ and $b \leq c$, we have

$a \wedge b = a$ and $b \wedge c = b$

Thus

$(a \wedge b) \vee (b \wedge c) = a \vee b$

$= b$,

since $a \leq b \Leftrightarrow a \vee b = b$.

Also, $a \leq b \leq c$ _ $a \leq c$ by transitivity. Then

$a \leq b$ and $a \leq c$ _ $a \vee b = b$, $a \vee c = c$

and so

$(a \vee b) \wedge (a \vee c) = b \wedge c$

$= b$ since $b \leq c \Leftrightarrow b \wedge c = b$.

Hence

$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$,

which proves (ii).

1.21. Lattices as Algebraic System

Definition. A **Lattice** is an algebraic system (L, $\vee$, $\wedge$) with two binary operations $\vee$ and $\wedge$, called **join** and **meet** respectively, on a non-empty set L
which satisfy the following axioms for a, b, c $\in$ L :

1. Commutative Law :
$a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.
2. Associative Law :

$(a \vee b) \vee c = a \vee (b \vee c)$
and
$(a \wedge b)\wedge c = a \wedge (b \wedge c)$

3. Absorption Law :
(i) $a \vee (a \wedge b) = a$
(ii) $a \wedge (a \vee b) = a$
We note that Idempotent Law follows from axiom 3 above. In fact,
$a \vee a = a \vee [a \wedge (a \vee b)]$ using ....3(ii)
= a using ....3(i)
The proof of $a \wedge a = a$ follows by principle of duality.

1.22 Partial Order Relations on a Lattice

A partial order relation on a lattice (L) follows as a consequence of the axioms for the binary operations $\vee$ and $\wedge$.
We define a relation $\leq$ on L such that for a, b $\in$ L ,

$$a \leq b \Leftrightarrow a \vee b = b$$

or analogously,

. $$a \leq b \Leftrightarrow a \wedge b = a .$$

We note that

(i) For any a $\in$ L
$a \vee a = a$ (idempotent law),
therefore $a \leq a$ showing that $\leq$ is **reflexive**.

(ii) Let $a \leq b$ and $b \leq a$. Therefore

$$a \vee b = b$$
$$b \vee a = a$$

But

$$a \vee b = b \vee a \text{ (Commutative Law in lattice)}$$

Hence

$$a = b ,$$

showing that $\leq$ is **antisymmetric.**

(iii) Suppose that $a \leq b$ and $b \leq c$. Therefore $a \vee b = b$ and $b \vee c = c$. Then

$$\begin{aligned} a \vee c &= a \vee (b \vee c) \\ &= (a \vee b) \vee c \text{ (Associativity in lattice)} \\ &= b \vee c \\ &= c , \end{aligned}$$

showing that $a \leq c$ and hence $\leq$ is transitive.

This shows that a **lattice is a partially ordered set**

1.23 Least Upper Bounds and Latest Lower Bounds in a Lattice

Let $(L, \vee , \wedge)$ be a lattice and let $a, b \in L$. We now show that LUB of $\{a, b\} \subseteq L$ with respect to the partial order introduced above is $a \vee b$ and GLB of $\{a, b\}$ is $a \wedge b$.

From absorption law

$a \wedge (a \vee b) = a$

$b \wedge (a \vee b) = b$

Therefore $a \leq a \vee b$ and $b \leq a \vee b$, showing that $a \vee b$ is upper bound for $\{a,b\}$. Suppose that there exists $c \in L$ such that $a \leq c$, $b \leq c$. Thus we have $a \vee c = c$ and $b \vee c = c$

and then

$$(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$$

implying that $a \vee b \leq c$.

Hence $a \vee b$ is the least upper bound of a and b.

Similarly, we can show that a $\wedge$ b is GLB of a and b.
The above discussion shows that the two definitions of lattice given so far are equivalent.

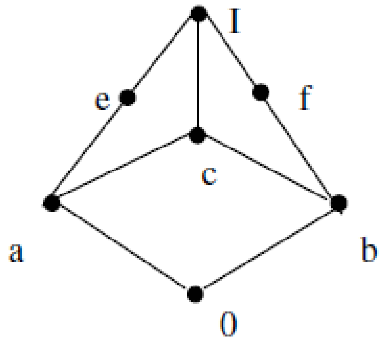
Sublattices

Definition: Let (L, $\leq$) be a lattice. A non-empty subset S of L is called a **sublattice** of L if a $\vee$ b $\in$ S and a $\wedge$ b $\in$ S whenever a $\in$ S, b $\in$ S.

(Or)

Let (L, $\vee$, $\wedge$) be a lattice and let S $\subseteq$ L be a subset of L. Then (S, $\vee$, $\wedge$) is called a sublattice of (L, $\vee$, $\wedge$) if and only if S is closed under both operations of join($\vee$) and meet($\wedge$).

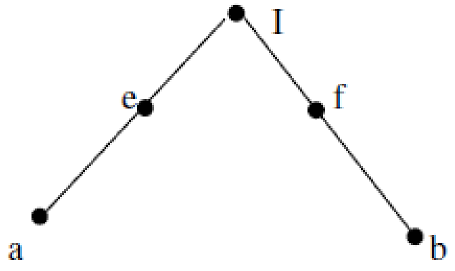
From the definition it is clear that **sublattice itself is a lattice.** However, **any subset of L which is a lattice need not be a sublattice.** For example, consider the lattice shown in the diagram:



L

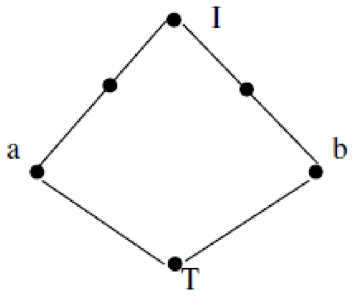
We note that

(i) the subset S shown by the diagram below is not a sublattice of L, since a $\wedge$ b $\notin$ S and a $\vee$ b $\notin$ S.



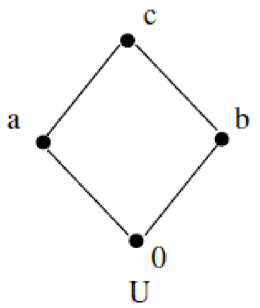
S

(ii) the set T shown below is not a sublattice of L since $a \vee b \notin T$.



However, T is a lattice when considered as a poset by itself.

(iii) the subset ∪ of L shown below is a sublattice of L:



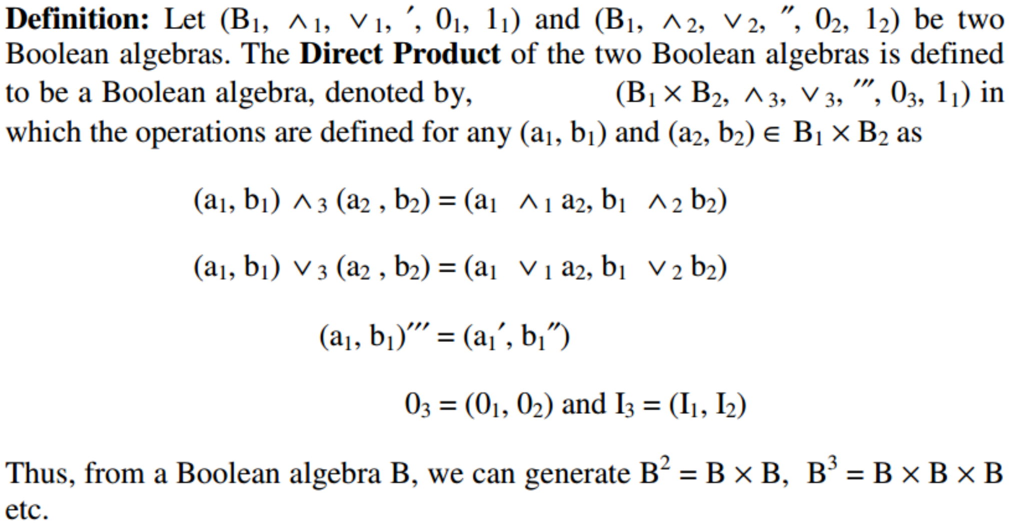
Example: Let A be any set and P(A) its power set. Then (P(A), ∨ , ∧) is a
lattice in which join and meet are union of sets and intersection of sets respectively.

A family _ of subsets of A such that S ∪ T and S ∩ T are in _ for S,

T ∈ _ is a sublattice of (P(A), ∨ , ∧). **Such a family _ is called a ring of**

subsets of A and is denoted by (R(A), ∨ **,** ∧ **)** (This is not a ring in the sense of algebra). **Some author call it lattice of subsets.**

Example: The lattice (D_n, ≤) is a sublattice of (**N**, ≤), where ≤ is the relation of divisibility.



Definition: Let (B_1, $\wedge_1$, $\vee_1$, ′, 0_1, 1_1) and (B_1, $\wedge_2$, $\vee_2$, ″, 0_2, 1_2) be two Boolean algebras. The **Direct Product** of the two Boolean algebras is defined to be a Boolean algebra, denoted by, ($B_1 \times B_2$, $\wedge_3$, $\vee_3$, ‴, 0_3, 1_1) in which the operations are defined for any (a_1, b_1) and (a_2, b_2) ∈ $B_1 \times B_2$ as

$$(a_1, b_1) \wedge_3 (a_2 , b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2)$$

$$(a_1, b_1) \vee_3 (a_2 , b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$$

$$(a_1, b_1)''' = (a_1', b_1'')$$

$$0_3 = (0_1, 0_2) \text{ and } I_3 = (I_1, I_2)$$

Thus, from a Boolean algebra B, we can generate $B^2 = B \times B$, $B^3 = B \times B \times B$ etc.

Lattice Isomorphism

Definition: Let (L_1, $\vee_1$, $\wedge_1$) and (L_2, $\vee_2$, $\wedge_2$) be two lattices. A mapping f :

$L_1 \rightarrow L_2$ is called a **lattice homomorphism** from the lattice the lattice (L_1, $\vee_1$,

$\wedge_1$) to (L_2, $\vee_2$, $\wedge_2$) if for any a, b ∈ L_1,

$f(a \vee_1 b) = f(a) \vee_2 f(b)$ and $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$

Thus, here both the binary operations of join and meet are preserved. **There may be mapping which preserve only one of the two operations. Such mapping are not lattice homomorphism**

Let $\leq_1$ and $\leq_2$ be partial order relations on (L_1, $\vee_1$, $\wedge_1$) and

(L_2, $\vee_2$, $\wedge_2$) respectively. Let f : $L_1 \rightarrow L_2$ be lattice homomorphism. If

a, b ∈ L_1, then

$a \leq_1 b \Leftrightarrow a \vee_1 b = b$

and so

$f(b) = f(a \vee_1 b)$

$= f(a) \vee_2 f(b)$

$\Leftrightarrow f(a) \leq_2 f(b)$

Thus

$a \leq_1 b \Leftrightarrow f(a) \leq_2 f(b)$

Thus order **relations are also preserved** under lattice homomorphism.

If a lattice homomorphism f: $L_1 \rightarrow L_2$ is one-to-one and onto, then it is called **lattice isomorphism.**

If there exists an isomorphism between two lattices, then the lattices are called **isomorphic**.

Since lattice isomorphism preserves order relation, therefore isomorphic lattices can be represented by the same diagram in which nodes are replaced by images .

Theorem: Let A = $\{a_1, a_2,\ldots,a_n\}$ and B = $\{b_1, b_2,\ldots\ldots b_n\}$ be any two finite sets with n elements. Then the lattices (P(A), $\subseteq$) and (P(B), $\subseteq$) are isomorphic
and so have identical Hasse-diagram.

Proof: Consider the mapping f : P(A) $\rightarrow$ P(B)
defined by

$$f(\{a_n\} = \{b_n\},\ f(\{a_1, a_2,\ldots,a_m\}) = \{b_1, b_2,\ldots\ldots b_n\} \text{ for } m \leq n .$$

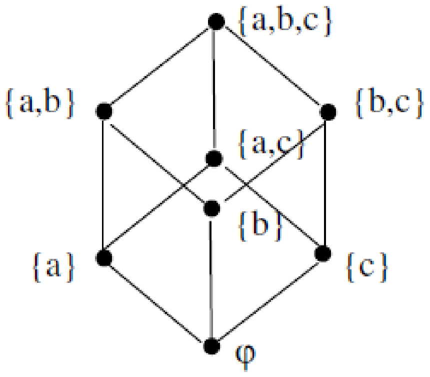
Then f is bijective mapping and L $\subseteq$ M $\Leftrightarrow$ f(L) $\subseteq$ f(M) for subsets L and M of P(A).

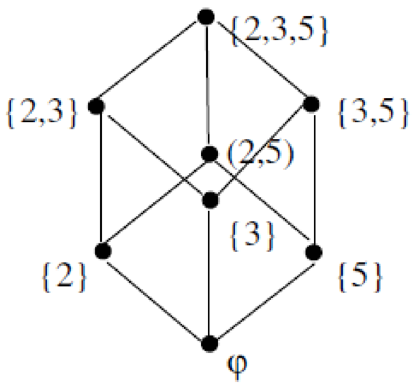
Hence P(A) and P(B) are isomorphic.

For example,

let A = {a, b, c}, B = {2, 3, 5}. The Hasse-diagram of

P(A) and P(B) are then given below:





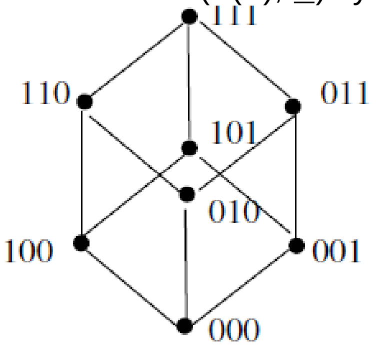
Define a mapping f : P(A) → P(B) by

$f(\phi) = \varphi$, $f(\{a\}) = \{2\}$, $f(\{b\}) = \{3\}$, $f(\{c\}) = \{5\}$
$f(\{a, b\}) = \{2, 3\}$, $f(\{b, c\}) = \{3, 5\}$, $f(\{a, c\}) = \{2, 5\}$

and
$f(\{a, b, c\}) = \{2, 3, 5\}$.

This is a bijective mapping satisfying the condition that if S and T are subsets of A, then $S \subseteq T$ if and only if $f(S) \subseteq f(T)$. Hence f is isomorphism and (P(A), ⊆) and (P(B), ⊆) are isomorphic.
Thus, for each n = 0, 1, 2,…., there is only one type of lattice and this lattice depends only on n, the number of elements in the set A, and not on A. It has 2^n elements. Also, we know that if A has n elements, then all subsets of A can be represented by sequences of 0's and 1's of length n. We can therefore label the

Hasse diagram of a lattice (P(A), $\subseteq$) by such sequence of 0's and 1's.



The lattice so obtained is named B_n. The properties of the partial order in B_n can be described directly as follows:

Let $x = a_1 a_2\ldots..a_n$ and $y = b_1 b_2\ldots..b_n$ be any two elements of B_n. Then

(1) $x \leq y$ if and only if $a_k < b_k$, $k = 1, 2,\ldots..,n$, where a_k and b_k are 0 or 1.

(2) $x \wedge y = c_1 c_2\ldots.c_n$, where $c_k = \min(a_k, b_k)$.
(3) $x \vee y = d_1 d_2 \ldots.d_n$, where $d_k = \max(a_k, h_k)$.

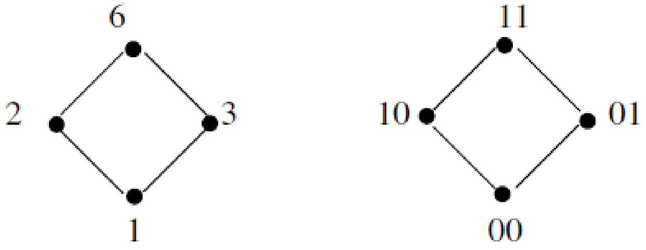
(4) x has a complement $x' = z_1 z_2\ldots\ldots z_n$ where $z_k = 1$ if $x_k = 0$ and $z_k = 0$ if $x_k = 1$.

Remark: $(B_n, \leq)$ under the partial order $\leq$ defined above is isomorphic to (P(A), $\subseteq$), when A has n elements. In such a case $x \leq y$ corresponds to $S \subseteq T$, $x \vee y$ corresponds to $S \cup T$ and x' corresponds to A^c.

Example : Let $D_6 = \{1, 2, 3, 6\}$, set of divisors of 6. Then D_6 is isomorphic to B_2. In fact $f : D_6 \rightarrow B_2$ defined by

$f(1) = 00, f(2) = 10, f(3) = 01, f(6) = 11$

is an isomorphism.



D_6 B_2

Bounded, Complemented and Distributive Lattices

Definition: A lattice L is said to be **bounded** if it has a greatest element I and a least element 0.

For the lattice (L, $\vee$, $\wedge$) with L = {a_1, a_2,….,a_n},

$a_1 \vee a_2 \vee\ldots.. \vee a_n = I$ and $a_1 \wedge a_2 \wedge\ldots\ldots \wedge a_n = 0$.

Example : The lattice $\mathbf{Z}_+$
of all positive integers under partial order of
divisibility **is not a bounded lattice** since it has a least element (the integer 1)
but no **greatest element.**

Example: The lattice **Z** of integers under partial order $\le$ (less than or equal to)
is **not bounded since it has neither a greatest element nor a least element.**

Example: Let A be a non-empty set. Then the lattice $(P(A), \subseteq)$ **is bounded.**
Its greatest element is A and the least element is empty set ϕ.

If $(L, \le)$ is a bounded Lattice, then for all $a \in L$

$0 \le a \le I$

$a \vee 0 = a$, $a \wedge 0 = 0$

$a \vee I = I$, $a \wedge I = a$

Thus 0 acts as identity of the operation $\vee$ and I acts as identity of the operation $\wedge$.

Definition: Let $(L \vee , \wedge , 0, I)$ be a bounded lattice with greatest element I and the least element 0. Let $a \in L$. Then an element $b \in L$ is called a **complement** of a if

$a \vee b = I$ and $a \wedge b = 0$

It follows from this definition that

0 and I are complement of each other.

Further, I is the only complement of 0. For suppose that $c \neq I$ is a complement of 0 and $c \in L$, then

$0 \vee c = I$ and $0 \wedge c = 0$

But $0 \vee c = c$. Therefore $c = I$ which contradicts $c \neq I$.

Similarly, 0 is the only complement of I.

Definition: A lattice $(L, \vee , \wedge , 1, 0)$ is called **complemented** if it is bounded and if every element of L has at least one complement.

Example:

The lattice $(P(A), \subseteq)$ of the power set of any set A is a bounded lattice, where meet and join operations on e(A) are $\cap$ and $\cup$ respectively. Its bounds are φ and A. The lattice $(P(A), \subseteq)$ is complemented in which the complement of any subset B of A is A – b

Definition:

A lattice $(L, \vee , \wedge)$ is called a **distributive lattice** if for any elements a, b and c in L,

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Properties (1) and (2) are called **distributive properties.**

Thus, in a distributive lattice, the operations $\wedge$ and $\vee$ are distributive over each other.

We further note that, by the principle of duality, the condition (1) holds if and only if (2) holds. Therefore it is sufficient to verify any one of these two equalities for all possible combinations of the elements of a lattice.

If a lattice L is not distributive, we say that L is **non-distributive.**

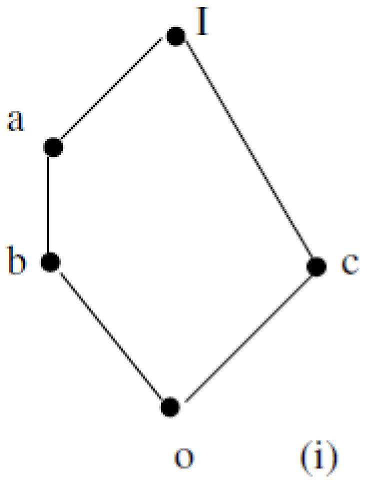
Example: For a set S, the lattice (P(S), $\subseteq$) is distributive. The meet and join operation in P(S) are $\cap$ and $\cup$ respectively. Also we know, by set

theory, that for A, B, C $\in$ P(S),

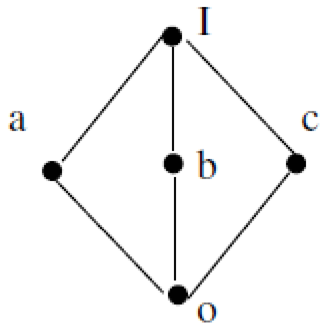
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example:
The **five elements** lattices given in the following diagrams are **non distributive.**



(i)



(ii)

In fact for the lattice (i), we note that $a \wedge (b \vee c) = a \wedge I = a$,
while

$(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

Hence

$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$,

showing that (i) is non-distributive.

For the lattice (ii) ,

we have

$a \wedge (b \vee c) = a \wedge I = a$,

while

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$.

Hence

$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$,

showing that (ii) is also non-distributive

POSSIBLE QUESTIONS (SIX MARKS)

1.Define sublattice, lattice homomorphism, order isomorphic.

2. Show that in a bounded distributive lattice, the elements which have complements form a sublattice.

3. Show that a lattice is distributive iff (a * b) + (b* c) + (c * a) = (a + b) * (b +c) * (c+ a).

4. Define complete, distributive lattice, Complemented lattice.

5.Every chain is a distributive lattice.

6. Show that every distributive lattice is modular but not conversely.

7. Show that a lattice is distributive iff (a * b) + (b* c) + (c * a) = (a + b) * (b +c) * (c+ a).

8. Show that a lattice homomorphism on a Boolean algebra which preserves 0 and 1 is Boolean homomorphism.

9. The direct product of any two distributive lattices is a distributive lattice.

10. Prove that two bounded lattices A and B are complemented iff A ×B is complemented.

11.Prove that two lattices A and B are relatively complemented iff A×B is relatively complemented.

POSSIBLE QUESTIONS (TEN MARKS)

1. If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. If the join operation is distributive over the meet operation, then the meet operation is also distributive over the join operation.

2. Let L be a finite distributive lattice. Then every a in L can be written uniquely (except for order) as the join of irredundant join irreducible elements.

3. In a distributive lattice, if an element has a complement then this complement is unique.

4.Every finite lattice is a complete .

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: ADVANCED DISCRETE MATHEMATICS Subject Code: 17MMP105A

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	ANSWER
The least member is denoted by -------	0	1	-1	2	0
The greatest member is denoted by -------	0	1	-1	2	1
The greatest lower bound of a,b$\in$ L is denoted by -------and is also called meet.	a*b	a+b	a-b	a/b	a*b
The least upper bound of a,b$\in$ L is denoted by -------- and is also called join.	a*b	a+b	a-b	a/b	a+b
The greatest lower bound of a,b$\in$ L is denoted by a*b and is also called -----	join	sum	meet	multiply	meet
The least upper bound of a,b$\in$ L is denoted by a + b and is also called ------	join	product	meet	multiply	join
Idempotent law is --------	$a \vee a = a$	$a \vee b = b \vee a$	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \vee (a \wedge b) = a$	$a \vee a = a$
Commutative law is --------	$a \vee a = a$	$a \vee b = b \vee a$	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \vee (a \wedge b) = a$	$a \vee b = b \vee a$
Associative law is -------	$a \vee a = a$	$a \vee b = b \vee a$	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \vee (a \wedge b) = a$	$a \vee (b \vee c) = (a \vee b) \vee c$
Absorption law is ------	$a \vee a = a$	$a \vee b = b \vee a$	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \vee (a \wedge b) = a$	$a \vee (a \wedge b) = a$
$a \vee a = a$ this law is called -------	Idempotent law	Commutative law	Associative law	Absorption law	Idempotent law
$a \vee b = b \vee a$, this law is called -------	Idempotent law	Commutative law	Associative law	Absorption law	Commutative law
$a \vee (b \vee c) = (a \vee b) \vee c$, this law is called -------	Idempotent law	Commutative law	Associative law	Absorption law	Associative law
$a \vee (a \wedge b) = a$, this law is called ------	Idempotent law	Commutative law	Associative law	Absorption law	Absorption law
$a \wedge a = a$, this law is called ------	Idempotent law	Commutative law	Associative law	Absorption law	Idempotent law
$a \wedge b = b \wedge a$, this law is called ------	Idempotent law	Commutative law	Associative law	Absorption law	Commutative law
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$, this law is called------	Idempotent law	Commutative law	Associative law	Absorption law	Associative law
$a \wedge (a \vee b) = a$, this law is called -------	Idempotent law	Commutative law	Associative law	Absorption law	Absorption law
A lattice which has both a least element and a greatest element is called ------	sub lattice	bounded lattice	complement lattice	lattice homomorphism	bounded lattice
A ----- is a poset in which every pair of element has a greatest lower bound and least upper bound.	lattice	sub lattice	bounded lattice	complement lattice	lattice
A partially ordered set { L, $\leq$} in which every pair of elements has a least upper bound and greatest lower bound is called ------------	Lattice	Boolean algebra	sub lattice	duals	Lattice
In a lattice, $\leq$ denotes --------------	addition of	multiple of	divisor of	subtraction of	divisor of
In a lattice, $\geq$ denotes --------------	addition of	multiple of	divisor of	subtraction of	multiple of
The lattices { L, $\leq$} and { L, $\geq$ } are called the ------------ each other	duals	one to one	unique	exist	duals
For a totally ordered set (p, $\leq$), the hasse diagram consists of -------- one below the other.	dot	cross	circles	arrow	circles
If B is a Boolean Algebra, then which of the following is true	B is a finite but not complemented lattice	B is a finite, complemented and distributive lattice	B is a finite, distributive but not complemented lattice	B is not distributive lattice.	B is a finite, complemented and distributive lattice
A partial ordered relation is transitive, reflexive and	antisymmetric	bisymmetric	antireflexive	asymmetric	antisymmetric
Which of the following pair is not congruent modulo 7...........	10, 24	25, 56	-31, 11	-64, -15	25, 56
A lattices which is commplemented and distributive is called a ---------------	sub algebra	Boolean algebra	sub lattice	duals	Boolean algebra

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DEPARTMENT OF MATHEMATICS

SUBJECT: ADVANCED DISCRETE MATHEMATICS **SEMESTER: I** **L T P C**
SUBJECT CODE: 17MMP105A **CLASS:I PG (MATHEMATICS)** **4 0 0 4**

UNIT III

Some special Lattices - e.g. Complete, Complemented and Distributive Lattices - Boolean Algebra: Definition and Examples - Subalgebra - Direct product and Homomorphism - join irreducible - atoms and antiatoms.

TEXT BOOKS

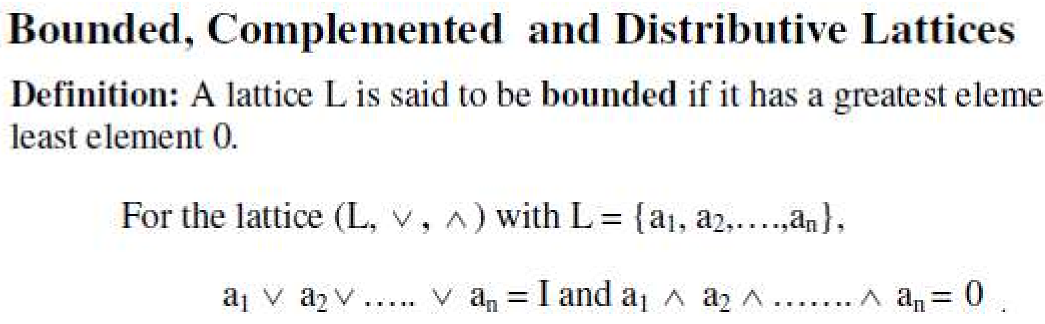
1. Tremblay J. P. and Manohar, R., (1997). Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III).

REFERENCES

1.Seymour Lepschutz, (2007) ,Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.
2. Advance Discrete Mathematics Paperback – 2011 by G.C.Sharma (Author), Madhu Jain (Author) Publisher: Laxmi Publications; Second edition (2011)

Introduction: SOME SPECIAL LATTICES

In this chapter we will consider mathematical objects known as Lattices. Lattices is a set of points in n dimensional space with a periodic structure.More recently,Lattices have become a topic of active research in computer science .They are used as an algorithmic tool to solve a wide variety of problems ; and they have have some unique properties from a computational complexity point of view.



Bounded, Complemented and Distributive Lattices

Definition: A lattice L is said to be **bounded** if it has a greatest element I and a least element 0.

For the lattice (L, $\vee$, $\wedge$) with L = {$a_1, a_2, \ldots, a_n$},

$$a_1 \vee a_2 \vee \ldots.. \vee a_n = I \text{ and } a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_n = 0 .$$

Definition: Let $(L \vee, \wedge, 0, I)$ be a bounded lattice with greatest element I and the least element 0. Let $a \in L$. Then an element $b \in L$ is called a **complement** of a if

$$a \vee b = I \text{ and } a \wedge b = 0$$

It follows from this definition that

0 and I are complement of each other.

Further, I is the only complement of 0. For suppose that $c \neq I$ is a complement of 0 and $c \in L$, then

$$0 \vee c = I \text{ and } 0 \wedge c = 0$$

But $0 \vee c = c$. Therefore $c = I$ which contradicts $c \neq I$.

Similarly, 0 is the only complement of I.

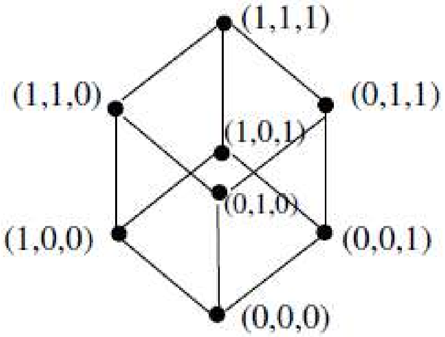
Definition: A lattice $(L, \vee, \wedge, 1, 0)$ is called **complemented** if it is bounded and if every element of L has at least one complement.

Example: The lattice $(P(A), \subseteq)$ of the power set of any set A is a bounded lattice, where meet and join operations on e(A) are $\cap$ and $\cup$ respectively. Its bounds are φ and A. The lattice $(P(A), \subseteq)$ is complemented in which the complement of any subset B of A is $A - b$.

Example: Let L^n be the lattice of n tuples of 0 and 1, where partial ordering is defined for $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots.., b_n) \in L^n$ by

$$a \leq_n b \Leftrightarrow a_i \leq b_i \qquad \text{for all } i = 1, 2, \ldots, n ,$$

where $\leq$ means less than or equal to. Then $(L^n, \leq_n)$ is lattice which is bounded. For example, the bounds are (0, 0, 0) and (1, 1, 1) for L^3.



The complement of an element of L^n can be obtained by interchanging 1 by 0 and 0 by 1 in the n-tuple representing the element. For example, complement of (1, 0, 1) in L^3 is (0, 1, 0).

Definition: A lattice $(L, \vee, \wedge)$ is called a **distributive lattice** if for any elements a, b and c in L,

$$(1)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$(2)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Properties (1) and (2) are called **distributive properties.**

Thus, in a distributive lattice, the operations $\wedge$ and $\vee$ are distributive over each other.

We further note that, by the principle of duality, the condition (1) holds if and only if (2) holds. Therefore it is sufficient to verify any one of these two equalities for all possible combinations of the elements of a lattice.

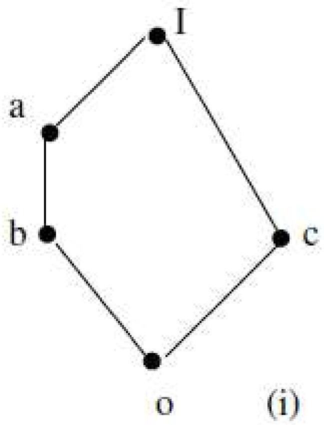
If a lattice L is not distributive, we say that L is **non-distributive.**

Example: For a set S, the lattice $(P(S), \subseteq)$ is distributive. The meet and join operation in P(S) are $\cap$ and $\cup$ respectively. Also we know, by set theory, that for A, B, C $\in$ P(S),

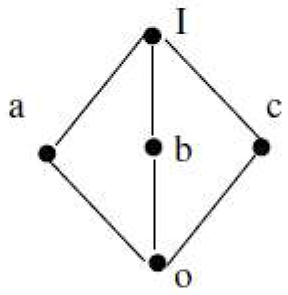
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example: The **five elements** lattices given in the following diagrams are **non distributive.**



(i)



(ii)

In fact for the lattice (i), we note that

$$a \wedge (b \vee c) = a \wedge I = a\,,$$

while

$$(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$$

Hence

$$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)\,,$$

showing that (i) is non-distributive.

For the lattice (ii) , we have

$$a \wedge (b \vee c) = a \wedge I = a\,,$$

while

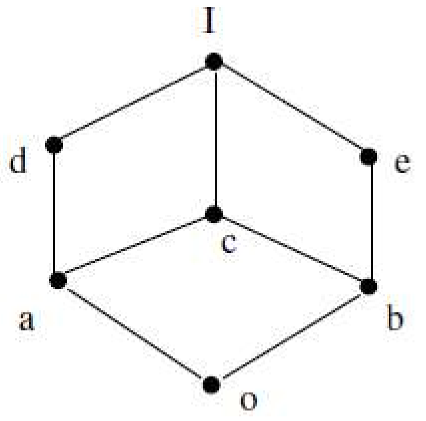
$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0\,.$$

Hence

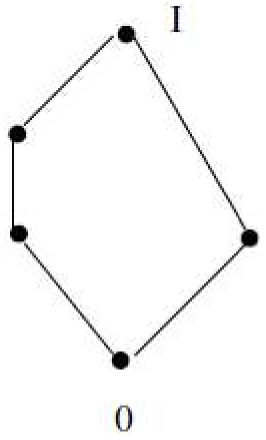
$$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)\,,$$

showing that (ii) is also non-distributive

Example: Is the following lattice a distributive lattice



Solution: The given lattice is **not distributive** since {0, a, d, e, I} is a sublattice which is isomorphic to the five-element lattice shown below :



Theorem: Every chain is a distributive lattice.

Proof: Let (L, ≤) be a chain and a, b, c ∈ L. We shall show that distributive law holds for any a, b, c ∈ L. Two cases arise :

Theorem: The direct product of any two distributive lattices is a distributive lattice.

Proof: Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be two lattices in which meet and join are $\wedge_1$, $\vee_1$ and $\wedge_2$, $\vee_2$ respectively. Then meet and join in $L_1 \times L_2$ are defined by

$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2) \qquad (1)$$

and

$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2) \qquad (2)$$

Since L_1 is distributive,

$$a_1 \wedge_1 (a_2 \vee_1 a_3) = (a_1 \wedge_1 a_2) \vee_1 (a_1 \wedge_1 a_3) \qquad (3)$$

Since L_2 is distributive,

$$b_1 \wedge_2 (b_2 \vee_2 b_3) = (b_1 \wedge_2 b_2) \vee_2 (b_1 \wedge_2 b_3) \qquad (4)$$

Therefore

$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)]$

$$= (a_1, b_1) \wedge [(a_2 \vee_1 a_3, b_2 \vee_2 b_3)]$$

$$= [(a_1 \wedge_1 (a_2 \vee_1 a_3), b_1 \wedge_2 (b_2 \vee_2 b_3)]$$

$$= [(a_1 \wedge_1 a_2) \vee_1 (a_1 \wedge_1 a_3), (b_1 \wedge_2 b_2) \vee_2 (b_1 \wedge_2 b_3)]$$

(using (3) and (4))

and using (1) and (2), we have

$$[(a_1, b_1) \wedge (a_2, b_2)] \vee [((a_1, b_1) \wedge (a_3, b_3)]$$

$$= (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2) \vee (a_1 \wedge_1 a_3, b_1 \wedge_2 b_3)$$

$$= [(a_1 \wedge_1 a_2) \vee_1 (a_1 \wedge_1 a_3), (b_1 \wedge_2 b_2) \vee_2 (b_1 \wedge_2 b_3)]$$

Hence

$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] = [(a_1, b_1) \wedge (a_2, b_2)] \vee [((a_1, b_1) \wedge (a_3, b_3)]$,

proving that $L_1 \times L_2$ is distributive.

Theorem: Let L be a bounded distributive lattice. If a complement of any element exists, it is unique.

Proof: Suppose on the contrary that b and c are complements of the element a $\in$ L. Then

$$a \vee b = I \qquad a \vee c = I$$

$$a \wedge b = 0 \qquad a \wedge c = 0$$

Using distributive law, we have

$$\begin{aligned} b &= b \vee 0 \\ &= b \vee (a \wedge c) \\ &= (b \vee a) \wedge (b \vee c) \\ &= (a \vee b) \wedge (b \vee c) \\ &= I \wedge (b \vee c) \\ &= b \vee c \end{aligned}$$

Similarly,

$$\begin{aligned} c &= c \vee 0 \\ &= c \vee (a \wedge b) \\ &= (c \vee a) \wedge (c \vee b) \\ &= (a \vee c) \wedge (c \vee b) \\ &= I \wedge (c \vee b) \\ &= I \wedge (b \vee c) \\ &= b \vee c \end{aligned}$$

Hence $b = c$.

BOOLEAN ALGEBRA

Definitions and Examples

Definition: A non-empty set B with two binary operations $\vee$ and $\wedge$, a unary operation $'$, and two distinct elements 0 and I is called a **Boolean Algebra** if the following axioms holds for any elements $a, b, c \in B$:

[B$_1$]: **Commutative Laws:**

$$a \vee b = b \vee a \quad \text{and} \quad a \wedge b = b \wedge a$$

[B$_2$]: **Distributive Law:**

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

[B$_3$]: **Identity Laws:**

$$a \vee 0 = a \quad \text{and} \quad a \wedge I = a$$

[B$_4$]: **Complement Laws:**

$$a \vee a' = I \quad \text{and} \quad a \wedge a' = 0$$

We shall call 0 as zero element, 1 as unit element and a' the complement of a.

We denote a Boolean Algebra by $(B, \vee, \wedge, \sim, 0, I)$.

Example 1. Let A be a non-empty set and P(A) be its power set. Then the set algebra $(P(A), \cup, \cap, -, \phi, A)$ is a Boolean algebra.

Example 2 : Let B = {0, 1} be the set of bits (binary digits) with the binary operations $\vee$ and $\wedge$ and the unary operation $'$ defined by the following tables:

$\vee$	1	0
1	1	1
0	1	0

,

$\wedge$	1	0
1	1	0
0	0	0

$'$	1	0
	0	1

Here the operations $\vee$ and $\wedge$ are logical operations and complement of 1 is 0 whereas complement of 0 is 1. Then $(B, \vee, \wedge, ', 0, 1)$ is a Boolean Algebra. It is the simplest example of a two-element algebra.

Further, a two element Boolean algebra is the only Boolean algebra whose diagram is a chain.

Example 3 : Let B_n be the set of n tuples whose members are either 0 or 1. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be any two members of B_n. Then we define

$$a \vee_1 b = (a_1 \vee b_1, a_2 \vee b_2, \ldots, a_n \vee b_n)$$

$$a \wedge_1 b = (a_1 \wedge b_1, a_2 \wedge b_2, \ldots, a_n \wedge b_n) \ ,$$

where $\vee$ and $\wedge$ are logical operations on {0, 1}, and

$$a' = (\sim a_1, \sim a_2, \ldots, \sim a_n) \ ,$$

where $\sim 0 = 1$ and $\sim 1 = 0$.

If 0_n represents (0, 0,…..,0) and $1_n = (1, 1, \ldots, 1)$, then $(B_n, \vee_1, \wedge_1, ', 0_n, 1_n)$ is a Boolean algebra.

Example 4. The poset D_{30} = {1, 2, 3, 5, 6, 10, 15, 30} has eight element. Define $\vee$, $\wedge$ and $'$ on D_{30} by

$$a \vee b = \text{lcm}(a, b) \ , \quad a \wedge b = \gcd(a, b) \qquad \text{and} \qquad a' = \frac{30}{a}.$$

Then D_{30} is a Boolean Algebra with 1 as the zero element and 30 as the unit element.

Example 5: Let S be the set of statement formulas involving n statement variables. The algebraic system (S, $\wedge$, $\vee$, ~, F, T) is a Boolean algebra in which $\wedge$, $\vee$, ~ denotes the operations of conjunction, disjunction and negation respectively. The element F and T denotes the formulas which are contradictions and Tautologies respectively. The partial ordering corresponding to $\wedge$, $\vee$ is implication $\Rightarrow$.

We have seen that B_n is a Boolean algebra. Using this fact, we can also define Boolean algebra as follows:

Definition: A finite lattice is called a **Boolean Algebra** if it is isomorphic with B_n for some non-negative integer n.

For example, D_{30} is isomorphic to B_3. In fact, the mapping f: $D_{30} \rightarrow B_3$ defined by

$$f(1) = 000, \quad f(2) = 100, \quad f(3) = 010, \quad f(5) = 001,$$

$$f(6) = 110, \quad f(10) = 101, \quad f(15) = 011, \quad f(30) = 111$$

is an isomorphism. Hence D_{30} is a Boolean algebra.

If a finite L does not contain 2^n elements for some non-negative integer n, then L cannot be a Boolean Algebra.

For example, consider $D_{20} = \{1, 2, 4, 5, 10, 20\}$ that has 6 elements and $6 \neq 2^n$ for any integer $n \geq 0$. Therefore, D_{20} is not a Boolean algebra.

If | L | = 2^n, then L may or not be a Boolean Algebra. If L is isomorphic to B_n, then it is Boolean algebra, otherwise it is not.

For large value of n, we use the following theorem for determining whether D_n is a Boolean Algebra or not.

Theorem: Let

$$n = p_1\, p_2 \ldots\ldots p_k,$$

where p_i are distinct primes, known as set of atoms. Then D_n is a Boolean algebra.

Proof: Let $A = \{p_1, p_2, \ldots., p_k\}$. If $B \subseteq A$ and a_B is the product of primes in B, then ${}^{a}{}_{B}|n$. Also any divisor of n must be of the form ${}^{a}{}_{B}$ for some subset B of A, where we assume that $a_\varphi = 1$. Further, if C and B are subsets of A, then $C \subseteq B$ if and only if ${}^{a}{}_{C}|{}^{a}{}_{B}$. Also

$${}^{a}{}_{C\cap B} = {}^{a}{}_{C} \wedge {}^{a}{}_{B} = \gcd({}^{a}{}_{C}\ , {}^{a}{}_{B})$$

and

$${}^{a}{}_{C\cup B} = {}^{a}{}_{C} \vee {}^{a}{}_{B} = \text{lcm}\ ({}^{a}{}_{C}\ , {}^{a}{}_{B})$$

Thus the function $f : P(A) \to D_n$ defined by

$$f(B) = {}^{a}{}_{B}$$

is an isomorphism. Since P(A) is a Boolean algebra, it follows that D_n is also a Boolean algebra.

For example, consider D_{20}, D_{30}, D_{210}, D_{66}, D_{646}. We notice that

(i) 20 cannot be represented as product of distinct primes and so D_{20} is not a Boolean algebra.

(ii) 30 = 2.3.5, where 2, 3, 5 are distinct primes. Hence D_{30} is a Boolean Algebra.

(iii) 210 = 2.3.5.7 (all distinct primes) and so D_{210} is a Boolean algebra.

(iv) 66 = 2.3.11 (product of distinct primes) and so D_{66} is a Boolean algebra.

(v) 646 = 2.17.19 (product of distinct primes) and so D_{646} is a Boolean Algebra.

Duality: The **dual of any statement** in a Boolean algebra B is obtained **by interchanging** $\vee$ and $\wedge$ and interchanging the zero element and unit element in the original statement.

For example, the dual of $a \wedge 0 = 0$ is $a \wedge I = I$

Principle of duality: The dual of any theorem in a Boolean Algebra is also a theorem.
(Thus, dual theorem is proved by using the **dual of each step of the proof of the original statement).**

Properties of a Boolean Algebra

Theorem: Let a, b and c be any elements in a Boolean algebra $(B, \vee, \wedge, ', 0, I)$. Then

1. **Idempotent Laws:**

   (i) $a \vee a = a$ (ii) $a \wedge a = a$

2. **Boundedness Laws:**

   (i) $a \vee I = I$ (ii) $a \wedge 0 = 0$

3. **Absorption Laws:**

   (i) $a \vee (a \wedge b) = a$ (ii) $a \wedge (a \vee b) = a$

4. **Associative Laws:**

   (i) $(a \vee b) \vee c = a \vee (b \vee c)$ (ii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

Proof: It is sufficient to prove first part of each law since second part follows from the first by principle of duality.

1. (i). We have

$$a = a \vee 0 \text{ (by identity law in a Boolean algebra)}$$
$$= a \vee (a \wedge a') \text{ (by complement law)}$$
$$= (a \vee a) \wedge (a \vee a') \text{ (by distributive law)}$$
$$= (a \vee a) \wedge I \text{ (complement law)}$$
$$= a \vee a \text{ (identity law) ,}$$

which proves 1(i).

2(i) : We have

$$a \vee I = (a \vee I) \wedge I \text{ (identity law)}$$

$$= (a \vee I) \wedge (a \vee a') \text{ (complement law)}$$
$$= a \vee (I \wedge a') \text{ (Distributive law)}$$
$$= a \vee a' \text{ (identity law)}$$
$$= I \text{ (complement law).}$$

3(i) : we note that

$$a \vee (a \wedge b) = (a \wedge I) \vee (a \wedge b) \text{ (identity law)}$$
$$= a \wedge (I \vee b) \text{ (distributive law)}$$
$$= a \wedge (b \vee I) \text{ (commutativity)}$$
$$= a \wedge I \text{ (Identity law)}$$
$$= a \text{ (identity law)}$$

4(i) Let

$$L = (a \vee b) \vee c, \qquad R = a \vee (b \vee c)$$

Then

$$a \wedge L = a \wedge [(a \vee b) \vee c]$$
$$= [a \wedge (a \vee b)] \vee (a \wedge c) \text{ (distributive Law)}$$
$$= a \vee (a \wedge c) \text{ (absorption law)}$$
$$= a \text{ (absorption law)}$$

and

$$a \wedge R = a \wedge [a \vee (b \vee c)]$$

$$= (a \wedge a) \vee (a \wedge (b \vee c)] \text{ (distributive law)}$$

$$= a \vee (a \wedge (b \vee c)] \text{ (idempotent law)}$$

$$= a \text{ (absorption Law)}$$

Thus $a \wedge L = a \wedge R$ and so, by duality, $a \vee L = a \vee R$ **.**

Further,

$$a' \wedge L = a' \wedge [(a \vee b) \vee c]$$

$$= [a' \wedge (a \vee b)] \vee (a' \wedge c) \text{ (distributive law)}$$

$$= [(a' \wedge a) \vee (a' \wedge b)] \vee (a' \wedge c) \text{ (distributive law)}$$

$$= [0, \vee (a' \wedge b)] \vee (a' \wedge c) \text{ (complement Law)}$$

$$= (a' \wedge b)] \vee (a' \wedge c) \text{ (Identity law)}$$

$$= a' \wedge (b \vee c) \text{ (distributive law)}$$

On the other hand,

$$a' \wedge R = a' \wedge [a \vee (b \vee c)]$$

$$= (a' \wedge a) \vee [a' \wedge (b \vee c)] \text{ (distributive law)}$$

$$= 0 \vee [a' \wedge (b \vee c)] \text{ (complement law)}$$

$$= a' \wedge (b \vee c)] \text{ (identity law)}$$

Hence

$a' \wedge L = a' \wedge R$ and so by duality $a' \vee L = a' \vee R$

Therefore

$$L = (a \vee b) \vee c$$

$$= 0 \vee [(a \vee b) \vee c] = 0 \vee L \text{ (identity law)}$$

$$= (a \wedge a') \vee [(a \vee b) \vee c] = (a \wedge a') \vee L \text{ (complement law)}$$

$= (a \vee L) \wedge (a' \vee L)$ (distributive law)

$= (a \vee R) \wedge (a' \vee R)$ (using $A \vee L = a \vee R$ and $a' \vee L = a' \vee R$]

$= (a \wedge a') \vee R$ (distributive law)

$= 0 \vee R$ (complement law)

$= R$ (identity law)

Hence

$$(a \vee b) \vee c = a \vee (b \vee c),$$

which completes the proof of the theorem.

Theorem: Let a be any element of a Boolean algebra B. Then

(i) Complement of a is unique (**uniqueness of complement**)

(ii) $(a')' = a$ (**Involution law**)

(iii) $0' = 1$ and $1' = 0$

Proof: (i) Let a' and x be two complements of $a \varepsilon B$. Then

$$a \vee a' = I \quad \text{and} \quad a \wedge a' = 0 \qquad (i)$$

$$a \vee x = I \quad \text{and} \quad a \wedge x = 0 \qquad (ii)$$

and we have

$a' = a' \vee 0$ (Identity law)

$= a' \vee (a \wedge x)$ by (ii)

$= (a' \vee a) \wedge (a' \vee x)$ (Distributive law)

$= I \wedge (a' \vee x)$ by (i)

$= a' \vee x$ [Identity law]

Also

$$x = x \vee 0 \text{ (Identity law)}$$

$$= x \vee (a \wedge a') \; , \qquad \text{by (i)}$$

$$= (x \vee a) \wedge (x \vee a') \quad \text{[Distributive law]}$$

$$= I \wedge (x \vee a') \; , \qquad \text{(by (ii))}$$

$$= x \vee a' = a' \vee x \qquad \text{(Identity and commutative law)}$$

Hence $a' = x$ and so complement of any element in B is unique.

(ii) Let a' be a complement of a. Then

$$a \vee a' = I \qquad \text{and} \qquad a \wedge a' = 0$$

or , by commutativity ,

$$a' \vee a = I \qquad \text{and} \qquad a' \wedge a = 0$$

This implies that a is complement of a', that is,

$$a = (a')'.$$

(iii) By boundedness law,

$$0 \vee 1 = 1$$

and by identity law

$$0 \wedge 1 = 0$$

These two relations imply that 1 is the complement of 0, that is $1 = 0'$.

By principle of duality, we have then

$$0 = 1'.$$

Theorem: Let a, b be elements of a Boolean Algebra. Then $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$.

Proof: we have

$$(a \vee b) \vee (a' \wedge b') = (b \vee a) \vee (a' \wedge b') \quad \text{(commutative)}$$

$$= b \vee (a \vee (a' \wedge b')) \quad \text{(associative)}$$

$$= b \vee [(a \vee a' \wedge (a \vee b')] \quad \text{(distributive)}$$

$$= b \vee [I \wedge (a \vee b') \quad \text{(complement)}$$

$$= b \vee (a \vee b') \quad \text{(identity)}$$

$$= b \vee (b' \vee a) \quad \text{(commutative)}$$

$$= (b \vee b') \vee a \quad \text{(associative law)}$$

$$= I \vee a \quad \text{(complement law)}$$

$$= I \quad \text{(Identity law)}$$

Also

$$(a \vee b) \wedge (a' \wedge b') = [(a \vee b) \wedge a'] \wedge b' \quad \text{(associativity)}$$

$$= [a \wedge a') \vee (b \wedge a')] \wedge b' = [0 \vee (b \wedge a')] \wedge b'$$

(complement) (distributive)

$$= (b \wedge a') \wedge b' \quad \text{(identity)}$$

$$= b \wedge b' \wedge a' = 0 \wedge a' = 0$$

Hence $a' \wedge b'$ is complement of $a \vee b$, i.e. $(a \vee b)' = a' \wedge b'$.

The second part follows by principle of duality.

We have proved already that Boolean algebra (B, $\vee$, $\wedge$, $'$, 0, I) satisfies associative laws, commutative law and absorption law. Hence every Boolean algebra is a lattice with join as $\vee$ and meet as $\wedge$. Also boundedness law hold in a Boolean algebra. Thus Boolean algebra becomes a bounded lattice. Also Boolean algebra obeys distributive law and is complemented. Conversely, every bounded, distributive and complemented lattice satisfied all the axiom of a Boolean algebra. Hence we can define a Boolean algebra as

Definition: A Boolean Algebra is a **bounded distributive and complemented lattice.**

Now, being a lattice, a Boolean algebra must have a partial ordering. Recall that in case of lattice we had defined partial ordering $\leq$ by $a \leq b$ if $a \vee b = b$ or $a \wedge b = a$.

The following result yields much more than these required conditions:

Theorem: If a, b are in a Boolean algebra, then the following are equivalent:

$$(1)\ a \vee b = b$$

$$(2)\ a \wedge b = a$$

$$(3)\ a' \vee b = I$$

$$(4)\ a \wedge b' = 0$$

Proof: (1) $\Leftrightarrow$ (2) already proved.

(1) $\Rightarrow$ (3) : Suppose $a \vee b = b$, then

$$\begin{aligned} a' \vee b &= a' \vee (a \vee b) \\ &= (a' \vee a) \vee b && \text{(associativity)} \\ &= I \vee b = I && \text{(complement \& boundedness)} \end{aligned}$$

Conversely, suppose $a' \vee b = I$, then

$$\begin{aligned} a \vee b = 1 \wedge (a \vee b) &= (a' \vee b) \wedge (a \vee b) && \text{(by assumption of (3))} \\ &= (a' \wedge a) \vee b && \text{(distributivity)} \\ &= 0 \vee b = b && \text{(complement \& identity)} \end{aligned}$$

Thus (1) ⇔ (3).

Now we show that (3) ⇔ (4).

Suppose first that (3) holds. Then, using De-Morgan Law and involution, we have

$$0 = I' = (a' \vee b')' = a'' \wedge b'$$

$$= a \wedge b' \quad \text{(Involution)}$$

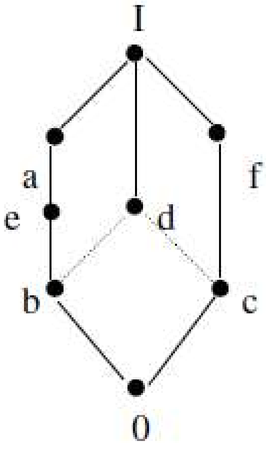
Conversely, if (4) holds, then

$$1 = 0' = (a \wedge b')' = a' \vee b'' = a' \vee b$$

Thus (3) ⇔ (4)

Hence all the four condition are equivalent.

Example: Show that the lattice whose diagram is



is not a Boolean algebra.

Solution: Elements a and e are both complements of c since $c \vee a = I$, $c \wedge a = 0$ and $c \vee e = I$, $c \wedge e = 0$

But in a Boolean algebra complement of an element is unique. Hence the given lattice is not a Boolean algebra.

Definition: Let (B, $\vee$, $\wedge$, $'$, 0, 1) be a Boolean algebra and S $\subseteq$ B. If S contains the elements 0 and 1 and is closed under the operation $\vee$, $\wedge$ and 1, then (S, $\wedge$, $\vee$, $'$, 0, 1) is called **Sub-Boolean Algebra**.

In practice, it is sufficient to check closure with respect to the set of operations ($\wedge$, $'$) or ($\vee$, $'$) for proving a subset S of B as the sub-Boolean algebra.

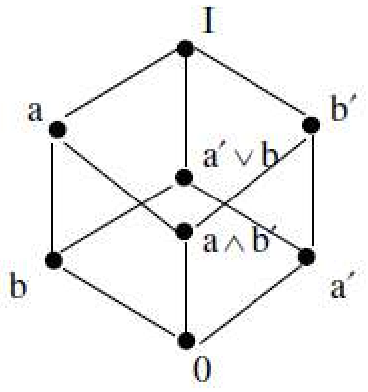
The definition of sub-Boolean implies that it is a Boolean algebra.

But a subset of Boolean algebra can be a Boolean algebra, but not necessarily a Boolean subalgebra because it is not closed with respect to the operations in B. For any Boolean algebra (B, $\wedge$, $\vee$, $'$, 0, 1), the subsets {0, 1} and the set B are both sub-Boolean algebras.
In addition to these sub-Boolean algebras, consider now any element a $\in$ B such that a $\neq$ 0 and a $\neq$ 1 and consider the set {a, a$'$, 0, 1}. Obviously this set is a sub-Boolean algebra of the given Boolean algebra.

For example D_{70} = {1, 2, 5, 7, 10, 14, 35, 70} is a Boolean algebra and {1, 2, 35, 70} is a subalgebra of D_{70}.

Every element of a Boolean algebra generates a sub-Boolean algebra, More generally, any subset of B generates a sub-Boolean algebra.

Example: Consider the Boolean algebra given in the diagram below:



Verify whether the following subsets are Boolean algebras or not :

S_1 = {a, a$'$, 0, 1}

S_2 = { a$'$ $\vee$ b , a $\wedge$ b$'$, 0, 1}

S_3 = {a $\wedge$ b$'$, b$'$, a, 1}

$$S_4 = \{b', a \wedge b', a', 0\}$$

$$S_5 = \{a, b', 0, 1\}$$

Solution: The subset S_1 and S_2 are sub-Boolean algebras. The subsets S_3 and S_4 are Boolean algebras but not sub-Boolean algebras of the given Boolean algebra. The subset S_5 is not even a Boolean algebra.

LATTICES OF DIRECT PRODUCT:

Definition: Let $(B_1, \wedge_1, \vee_1, ', 0_1, 1_1)$ and $(B_1, \wedge_2, \vee_2, '', 0_2, 1_2)$ be two Boolean algebras. The **Direct Product** of the two Boolean algebras is defined to be a Boolean algebra, denoted by, $(B_1 \times B_2, \wedge_3, \vee_3, ''', 0_3, 1_1)$ in which the operations are defined for any (a_1, b_1) and $(a_2, b_2) \in B_1 \times B_2$ as

$$(a_1, b_1) \wedge_3 (a_2 , b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2)$$

$$(a_1, b_1) \vee_3 (a_2 , b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$$

$$(a_1, b_1)''' = (a_1', b_1'')$$

$$0_3 = (0_1, 0_2) \text{ and } I_3 = (I_1, I_2)$$

Thus, from a Boolean algebra B, we can generate $B^2 = B \times B$, $B^3 = B \times B \times B$ etc.

Boolean Homomorphism

Definition: Let $(B, \wedge, \vee, ', 0, 1)$ and $(P, \cap, \cup, —, \alpha, \beta)$ be two Boolean Algebras. A mapping $f : B \to P$ is called a **Boolean Homomorphism** if all the operations of the Boolean Algebra are preserved , that is , for any $a, b \in B$

$$f(a \wedge b) = f(a) \cap f(b)$$

$$f(a \vee b) = f(a) \cup f(b)$$

$$f(a') = \overline{f(a)}$$

$$f(0) = \alpha$$

$$f(1) = \beta$$

Representation Theorem

Let B be a **finite** Boolean algebra. We know that an element a in B is called an **atom (or min term)** if a immediately succeed the **least element** 0 . Let A be the set of atoms of B and let P(A) be the Boolean algebra of all subsets of the set A of atoms. Then (as proved in chapter on lattices) each $x \neq 0$ in B can be expressed uniquely (except for order) as the join of atoms (i.e. elements of A). So, let

$$x = a_1 \vee a_2 \vee \ldots\ldots \vee a_n$$

Consider the function

$$f : B \rightarrow P(A)$$

defined by

$$f(x) = \{a_1, a_2, \ldots\ldots, a_n\}$$

for each $x = a_1 \vee a_2 \vee \ldots. \vee a_n$.

Stone's Representation Theorem: Any Boolean Algebra is isomorphic to a power set algebra $(P(S), \cap, \cup, \sim, \phi, S)$ for some set S.

Restricting our discussion to finite Boolean Algebra B, the representation theorem can be stated as :

Theorem: Let B be a finite Boolean Algebra and let A be the set of atoms of B. If P(A) is the Boolean Algebra of all subsets of the set A of atoms, then the mapping $f : B \to P(A)$ is an isomorphism.

Proof: Suppose B is finite Boolean algebra and P(A) is the Boolean algebra of all subsets of the set A of atoms of B. Consider the mapping

$$f : B \to P(A)$$

defined by

$$f(x) = \{a_1, a_2, \ldots, a_n\} \ ,$$

where $x = a_1 \vee a_2 \vee \ldots \vee a_n$ is the unique representation of $x \varepsilon B$ as the join of atoms $a_1, a_2, \ldots, a_n \in A$. If a_i are atoms, then we know that $a_i \wedge a_i = a_i$ but $a_i \wedge a_j = 0$ for $a_i \neq a_j$.

Let x and y are in the Boolean algebra B and suppose

$$x = a_1 \vee \ldots \vee a_r \vee b_1 \vee \ldots \vee b_s$$

$$y = b_1 \vee \ldots \vee b_s \vee c_1 \vee \ldots \vee c_t,$$

where

$$A = \{ a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, c_1, \ldots, c_t, d_1 \ldots, d_k\}$$

is the set of atoms of B. Then

$$x \vee y = a_1 \vee \ldots \vee a_r \vee b_1 \vee \ldots \vee b_s \vee c_1 \ldots \vee c_t$$

$$x \wedge y = b_1 \vee \ldots \vee b_s$$

Hence

$$f(x \vee y) = \{ a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, c_1, c_2 \ldots, c_t\}$$

$$= \{ a_1, \ldots, a_r, b_1, \ldots, b_s\} \cup \{b_1, b_2, \ldots, b_s, c_1, c_2 \ldots, c_t\}$$

$$= f(x) \cup f(y)$$

and

$$f(x \wedge y) = \{b_1,\ldots\ldots,b_s\}$$

$$= \{ a_1, a_2\ldots, a_r, b_1,\ldots..,b_s\} \cap \{b_1,\ldots..,b_s, c_1,\ldots.,c_t\}$$

$$= f(x) \cap f(y)$$

Let

$$y = c_1 \vee \ldots\ldots \vee c_t \vee d_1 \vee \ldots\ldots \vee d_k$$

Then

$$x \vee y = I \qquad \text{and } x \wedge y = 0$$

and so $y = x'$. Thus

$$f(x') = f(y) = \{c_1 \ldots\ldots c_t, d_1\ldots\ldots d_k \}$$

$$= \{ a_1, a_2\ldots, a_r, b_1, b_2\ldots..,b_s\}^c$$

$$= (f(x))^c .$$

Since the representation is unique, f is one-to-one and onto. Hence f is a Boolean algebra isomorphism. **Thus, every finite Boolean algebra is structurally the same as a Boolean algebra of sets.**

If a set A has n elements, then its power set P(A) has 2^n elements. Thus we have

JOIN IRREDUCIBLE:

Definition: Let $(L, \wedge, \vee)$ be a lattice. An element $a \in L$ is said to be **join-irreducible** if it cannot be expressed as the join of two distinct elements of L.

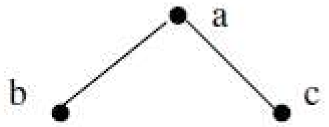
In other words, $a \in L$ is join-irreducible if for any $b, c \in L$

$$a = b \vee c \Rightarrow a = b \text{ or } a = c.$$

For example, prime number under multiplication have this property. In fact if p is a prime number, then $p = a\, b \Rightarrow p\, a$ or $p = b$.

Clearly 0 is join – irreducible.

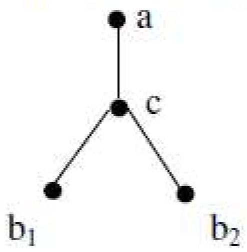
Further, if a has at least two immediate predecessors, say b and c as in the diagram below:



Then $a = b \vee c$ and so **a is not join – irreducible.**

On the other hand if a has a unique immediate predecessor c, then

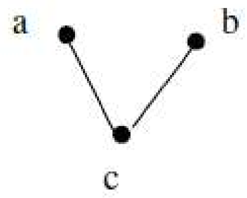
$a \neq \sup(b_1, b_2) = b_1 \vee b_2$ for any other elements b_1 and b_2 because c would lie between b_1, b_2 and a.



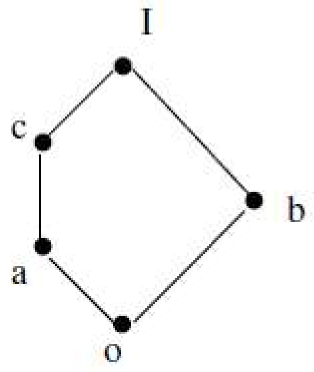
In other words, **$a \neq 0$ is join irreducible if and only if a has a unique predecessor.**

Definition: Those elements, which immediately succeed 0, are called **atoms.**

From the above discussion, it follows that the **atoms are join-irreducible**.



However, lattices can have other join-irreducible elements. For example, the element c in five-element lattice is not an atom, even then it is join irreducible because it has only **one immediate predecessor, namely a.**



Let a be an element of a finite lattice which is not join irreducible, then we can write

$$a = b \vee c$$

If b and c are not join irreducible, then we can write them as the join of other elements. Since L is finite we shall finally have

$$a = d_1 \vee d_2 \vee d_3 \vee \ldots\ldots \vee d_n \ , \qquad (1)$$

where d_i, $i = 1, 2, \ldots, n$ are join-irreducible. If d_i precedes d_j, then $d_i \vee d_j = d_j$, so we delete d_i from the expression. Thus d's are irredundant, i.e., no d precedes any other d.

The expression (1) need not be unique. For example, in lattice shown above

$$I = a \vee b \text{ and } I = b \vee c .$$

Theorem: Let (L, $\wedge$, $\vee$) be a finite distributive lattice. Then every a in L can written uniquely (except for order) as the join of irredundant join irreducible elements.

Proof: Let $a \in L$. Since L is finite, we can express a as the join of irredundant join irreducible elements (as discussed above). To prove uniqueness let

$$a = b_1 \vee b_2 \vee \ldots.. \vee b_n = c_1 \vee c_2 \vee \ldots.. \vee c_m ,$$

where b_i are irredundant join-irrducible and c_i are irrdundant and join-irreducible. For any given i, we have

$$b_i \leq (b_1 \vee b_2 \vee \ldots.. \vee b_n) = c_1 \vee c_2 \vee \ldots.. \vee c_m ,$$

Hence

$$b_i = b_i \wedge (c_1 \vee c_2 \vee \ldots.. \vee c_m)$$

$$= (b_i \wedge c_1) \vee (b_i \wedge c_2) \vee \ldots\ldots\ldots \vee (b_i \wedge c_m)$$

Since b_i is join-irreducible, there exists j such that $b_i = b_i \wedge c_j$ and so $b_i \leq c_j$.

Similarly, for c_i there exists a b_k such that $c_j \leq b_k$. Hence

$$b_i \leq c_j \leq b_k ,$$

which gives $b_i = c_j = b_k$ since b_i are irredundant. Hence b_i and c_i may be paired off. Hence the representation for a is unique except for order.

PART – B

POSSIBLE QUESTIONS – SIX MARKS

1.Define sublattice, lattice homomorphism, order isomorphic.

2.Show that in a bounded distributive lattice, the elements which have complements form sublattice.

3.Show that a lattice is distributive iff (a * b) + (b* c) + (c * a) = (a + b) * (b +c) * (c+ a).

4. Define complete, distributive lattice, Complemented lattice.

5.If (L, $\wedge$,$\vee$) is a complemented and distributive lattice , then the complement a of any element a$\in$L is unique.

6.Every chain is a distributive lattice.

7.Show that every distributive lattice is modular but not conversely.

8.Show that a lattice is distributive iff (a * b) + (b* c) + (c * a) = (a + b) * (b +c) * (c+ a).

9. In a distributive lattice, if an element has a complement then this complement is unique.

10.Show that a lattice homomorphism on a Boolean algebra which preserves 0 and 1 is a Boolean homomorphism.

11.The direct product of any two distributive lattices is a distributive lattice.

12.Prove that two bounded lattices A and B are complemented iff A $\times$B is complemented.

13.Prove that two lattices A and B are relatively complemented iff A$\times$B is relatively complemented.

PART – C

POSSIBLE QUESTIONS – TEN MARKS

1. If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. If the join operation is distributive over the meet operation, then the meet operation is also distributive over the join operation.

2. Let L be a finite distributive lattice. Then every a in L can be written uniquely (except for order) as the join of irredundant join irreducible elements.

3. If (A,$\leq$) and (B, $\leq$) are posets , then (AXB , $\leq$) is a poset with partial order defined by (a,b) $\leq$ ($\overline{a}$, $\overline{b}$) if a $\leq$ $\overline{a}$ and b $\leq$ $\overline{b}$.

4.Every finite lattice is complete.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Advanced Discrete Mathematics **Subject Code: 17MMP105A**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
If a variable x takes on only the values 0 and 1 it is called a ----------------------	simply variable	decision variable	boolean variable	dual variable	boolean variable
If C is a non empty subset of a boolean algebra such that C itself is a Boolean algebra with respect to the operations of B, then C is called a ------------	sub algebra	Boolean algebra	sub lattice	duals	sub algebra
A ----------- if n boolean variables is a boolean product of the n lierals in which each literal appears exactly once.	middle term	max term	min term	sub term	min term
A -------------- of n boolean variables is a boolean sum of the n literals in which each literal appears exactly once.	middle term	max term	min term	sub term	max term
When a boolean function is expressed as a product of maxterms, it is called its ---------	Product of sums expansion	Sums of product expansion	product of sums canonical form	sums of product canonical form	Product of sums expansion
When a boolean function is expressed as a product of minterms, it is called its ---------	Product of sums expansion	Sums of product expansion	product of sums canonical form	sums of product canonical form	Sums of product expansion
Boolean function expresses in the DNF or CNF are said to be in -------------	standard form	iterative form	canonical form	direct form	standard form
If a boolean function in n variables is expressed as the sum of all the 2 power n minterms it is said to be in -----------	Complete CNF	Complete DNF	Complete PDNF	complete PCNF	Complete DNF
Boolean function expressed in complete DNF or complete CNF are said to be in -------------	canonical form	standard form	complete standard form	complete canonical form	complete canonical form
In a boolean function sums of products expansion is said to be in the ---------	DNF	CNF	PDNF	PCNF	DNF
In a boolean function products of sums expansion is said to be in the ------------	DNF	CNF	PDNF	PCNF	CNF
The dual of a + a (b+1) = a is ----------	a ●(a + b ●0) = a	a + (a + b ●0) = a	a - (a + b ●0) = a	a ●(a - b ●0) = a	a ●(a + b ●0) = a
The dual of (a ●b)' = -----------	a'b'	a' -b'	a' + b'	a' / b'	a' + b'
In a boolean algebra, the operation + is called the ------------	boolean sum	boolean product	boolean variable	boolean symbol	boolean sum
In a boolean algebra, the operation ● is called the ------------	boolean sum	boolean product	boolean variable	boolean symbol	boolean product
Every lattices is a -------------	bounded	unbounded	poset	empty	poset
Every poset is ------------------	lattice	not lattice	bounded	unbounded	not lattice
Every finite lattice is ----------------	bounded	unbounded	poset	empty	bounded
If every element of L has atleast one complement then it is called -------------------	finite lattice	infinte lattice	distributive lattice	complemented lattice	complemented lattice
In any boolean algebra, the immediate successors of the 0 - element are called -----	atom	empty	unique	well defined	atom
If a + x = 1 and a ● x = 0, therefore x = -----------	a''	a	a'	0	a'
A lattice (L, ≤) which has both a least element denoted by 0 and the greast element by 1 is called a ----------- lattice	bounded	unbounded	poset	empty	bounded
Let D_{30} = {1, 2, 3, 5, 6, 10, 15, 30} and relation I be a partial ordering on D_{30}. The lub of 10 and 15 respectively is ----	30	15	10	6	30
Principle of duality is defined as ------	≤ is replaced by ≥	LUB becomes GLB	are all properties unaltered when ≤ is replaced by ≥	all properties are unaltered when ≤ is replaced by ≥ other than 0 and 1 element	all properties are unaltered when ≤ is replaced by ≥ other than 0 and 1 element
If lattice (C ,≤) is a complemented chain, then --------------	$\lvert C\rvert\leq1$	$\lvert C\rvert\leq2$	$\lvert C\rvert >1$	C doesn't exist	$\lvert C\rvert\leq2$
Different partially ordered sets may be represented by the same Hasse diagram if they are ------	same	lattices with same order	isomorphic	order - isomorphic	order - isomorphic
A self-complemented, distributive lattice is called -------	Boolean algebra	Modular lattice	Complete lattice	Self dual lattice	Boolean algebra
If a Boolean algebra, then the following is true ------------	B is a finite but not complemented lattice	B is a finite, complemented and distribtive lattice	B is a finite, distributive but not complemented lattice	B is not distributive lattice	B is a finite, complemented and distribtive lattice
Let L be a lattice. Then for every a and b in L which one of the following is correct?	a ∨b = a ^ b	a ∨ (b ∨ c) = (a ∨ b) v c	a v (b ^ c) = a	a ∨ (b ∨c) = a	a ∨ (b ∨ c) = (a ∨ b) v c
The Boolean expression XY + XY' +X'Z + XZ' is independent of the Boolean variable -----	Y	X	Z	X'	Y
The dual of (0 . a) + (b .1) = ----------	(1 . a) + (b+0) = b	(1+a).(b+0) = b	(1+a).(b.0)	(1.a).(b.0) = a	(1+a).(b+0) = b
A lattice (L, ≤) is said to be----------- if a ≤ c implies a ∨ (b ∧ c) = (a ∨ b) ∧ c	unbounded lattice	bounded lattice	modular lattice	complemented lattice	modular lattice
GLB {a, b} = -------- is called the meet of a and b	a ^ b	a & b	a v b	a = b	a ^ b
LUB {a,b} = ------------- is called the joint of a and b	a ^ b	a & b	a v b	a = b	a v b
An element a in a lattice (A, ≤) is called a ---------------------- if for every element b ε A, a ≤ b.	lower bound	universal lower bound	upper bound	universal upper bound	Universal lower bound

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DEPARTMENT OF MATHEMATICS

SUBJECT: ADVANCED DISCRETE MATHEMATICS	**SEMESTER: I**	**L T P C**
SUBJECT CODE: 17MMP105A	**CLASS:I PG (MATHEMATICS)**	**4 0 0 4**

UNIT IV

Graph Theory: Definition of a graph - applications, Incidence and degree - Isolated and pendant vertices - Null graph, Path and Circuits: Isomorphism - Subgraphs, Walks -Paths and circuits - Connected graphs, disconnected graphs – components - Euler graph.

TEXT BOOKS

1.Deo N., (2000). Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V)

REFERENCES

1. Liu C.L., (2000). Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.
2. Advance Discrete Mathematics Paperback – 2011 by G.C.Sharma (Author), Madhu Jain (Author) Publisher: Laxmi Publications; Second edition (2011)

INTRODUCTION : GRAPH THEORY

Graph theory is used to analyses problems of combinatorial nature that arise in computer science, operations research , physical science and economics . The term graph is familiar to you because it has been used in the context of straight lines and linear in equalities .In this chapter , first we will combine the concepts of graph theory with digraph of a relation to define a more general type of graph that has more than one edge between a pair of vertices. Second , we will identify basic components of a graph ,its features any many applications of graphs.

Definitions and Examples

Definition: A **graph** G = (V,E) is a mathematical structure consisting of two finite sets V and E. The elements of V are called **Vertices (or nodes)** and the elements of E are called Edges. Each edge
is associated with a set consisting of **either one** or **two vertices** called its **endpoints**.

The correspondence from edges to endpoints is called **edge-endpoint function**. This function is generally denoted by γ. Due to this function, some author denote graph by G = (V, E, γ).

Definition: A graph consisting of one vertex and no edges is called a **trivial graph**.

Definition: A graph whose vertex and edge sets are empty is called a **null graph**.

Definition: An edge with just one end point is called a **loop** or a **self loop**.
Thus, a loop is an edge that joins a single endpoint to itself.

Definition: An edge that is not a self-loop is called a **proper edge**.

Definition: If two or more edges of a graph G have the same vertices, then these edges are said to be
parallel or **multi-edges**.

Definition: Two vertices that are connected by an edge are called **adjacent**.

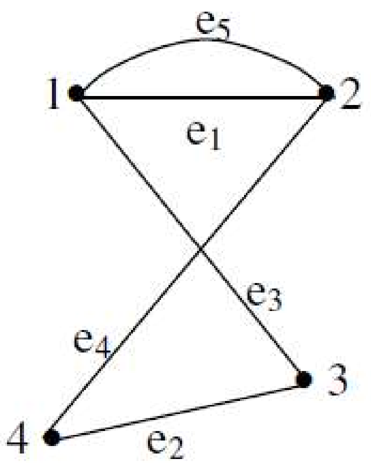
Definition: An endpoint of a loop is said to be **adjacent to itself**.

Definition: An edge is said to be **incident** on each of its endpoints.

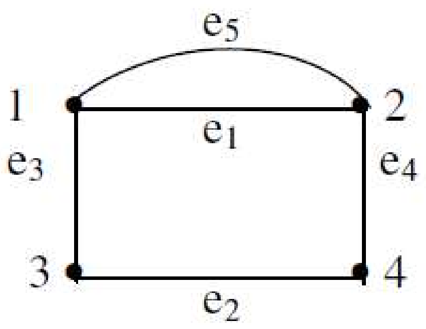
Definition: Two edges incident on the same endpoint are called **adjacent edges**.

Definition: The number of edges in a graph G which are incident on a vertex is called the degree of
that **vertex**.

Definition: A vertex of degree zero is called an **isolated vertex**.
Thus, a vertex on which no edges are incident is called isolated.
Definition: A graph without multiple edges **(parallel edges)** and loops is called **Simple graph**.
Notation: In pictorial representations of a graph, the vertices will be denoted by dots and edges by line segments.

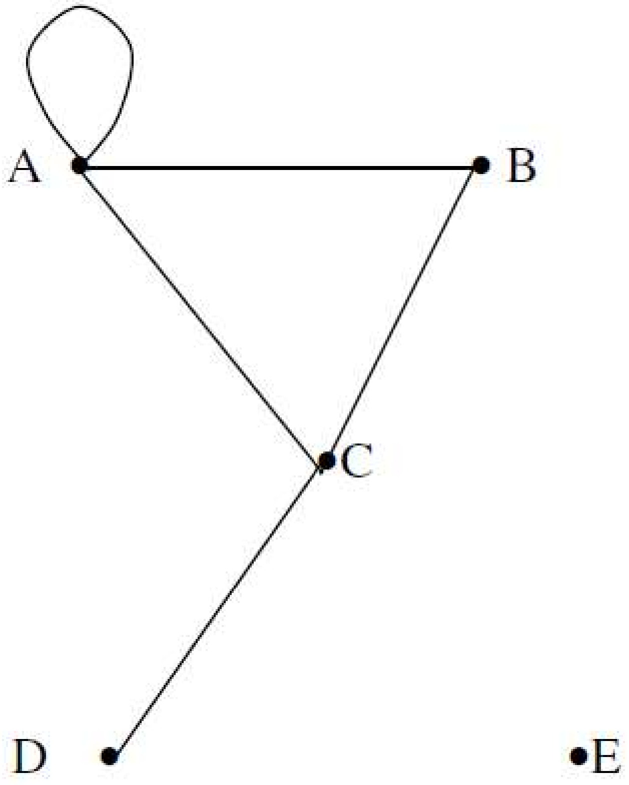


or



The edges e_2 and e_3 are adjacent edges because they are incident on the same vertex B.

2. Consider the graph with the vertices A, B , C, D and E pictured in the figure below.



In this graph, we note that

No. of edges = 5

Degree of vertex A = 4

Degree of vertex B = 2

Degree of vertex C = 3

Degree of vertex D = 1

Degree of vertex E = 0

Sum of the degree of vertices = 4 + 2 + 3 + 1 + 0 =10
Thus, we observe that

$$\sum_{i=1}^{5} \deg(v_i) = 2e \ ,$$

where $\deg(v_i)$ denotes the degree of vertex v_i and e denotes the number of edges.

Euler's Theorem: (The First Theorem of Graph Theory): The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G.

(Thus, total degree of a graph is even)

Proof: Each edge in a graph contributes a count of 1 to the degree of two vertices (end points of
the edge), That is, each edge contributes 2 to the degree sum. Therefore the sum of degrees of the
vertices is equal to twice the number of edges.

Corollary: There must be an even number of vertices of odd degree in a given graph G.
Proof: We know, by the Fundamental Theorem, that

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{no. of edges}$$

Thus the right hand side is an even number. Hence to make the left-hand side an even number there
can be only even number of vertices of odd degree.

Theorem: A non-trivial simple graph G must have at least one pair of vertices whose degrees are
equal.

Proof: Let the graph G has n vertices. Then there appear to be n possible degree values, namely 0, 1, ….,n − 1. But there cannot be both a vertex of degree 0 and a vertex of degree n − 1 because if there is a vertex of degree 0 then each of the remaining n − 1 vertices is adjacent to atmost n–2 other vertices. Hence the n vertices of G can realize atmost n−1 possible values for their degrees. Hence the pigeonhole principle implies that at least two of the vertices have equal degree.

Definition: A graph G is said to **simple** if it has no parallel edges or loops. In a simple graph, an edge with endpoints v and w is denoted by {v, w}.

Definition: For each integer $n \geq 1$, let D_n denote the graph with **n vertices** and **no edges**. Then D_n is called the **discrete graph on n vertices**.

For example, we have

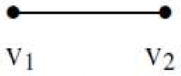
D_3

and

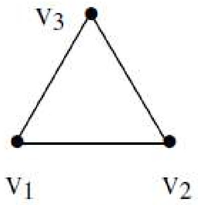
D_5

Definition: Let $n \geq 1$ be an integer. Then a simple graph with n vertices in which there is an edge between each pair of distinct vertices is called the **complete Graph** on n vertices. It is denoted by K_n.

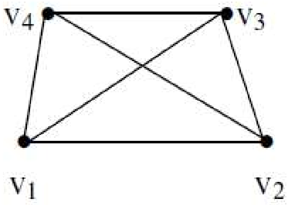
For example, the complete graphs K_2, K_3 and K_4 are shown in the figures below:



K_2



K_3



K_4

Definition: If each vertex of a graph G has the same degree as every other vertex, then G is called a **regular graph**.

A **k-regular graph** is a regular graph whose common degree is k.

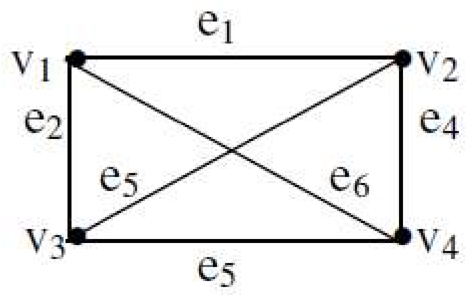
But this graph is not complete because v_2 and v_4 have not been connected through an edge. Similarly, v_1 and v_3 are not connected by any edge.

Thus

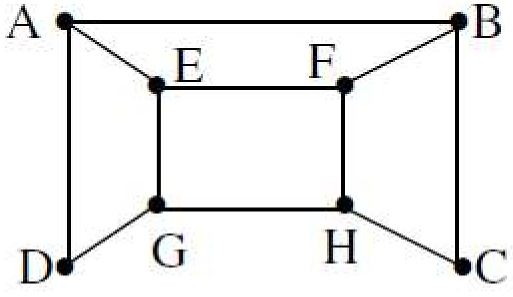
A Complete graph is always regular but **a regular graph need not be complete**.

Subgraphs

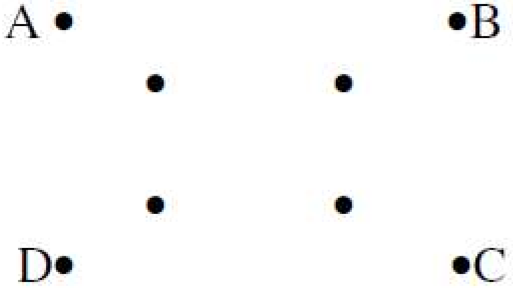
Definition: A graph H is said to be a subgraph of a graph G if and only if every vertex in H is also a vertex in G, every edge in H is also an edge in G and every edge in H has the same endpoints as in G.



Similarly, the graph



is a subgraph of the graph given below:

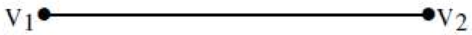


Definition: A subgraph H is said to be a **proper subgraph** of a graph G if vertex set V_H of H is a proper subset of the vertex set V_G of G or edge set E_H is a proper subset of the edge set E_G.

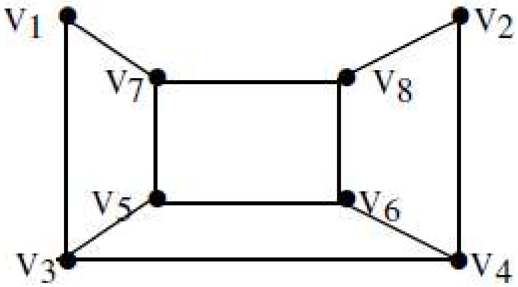
For example, the subgraphs in the above examples are proper subgraphs of the given graphs.

Definition: Let G = (V, E) be a graph. Then the **complement of a subgraph** G´ = (V´, E´) with respect to the graph G is another subgraph G´´ = (V´´, E´´) such that E´´ = E – E´ and V´´ contains only the vertices with which the edges in E´´ are incident.

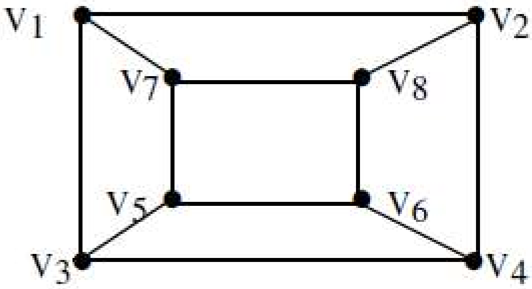
For example, the subgraph



is the complement of the subgraph



with respect to the graph G shown in the figure below:

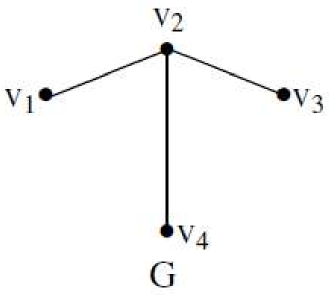


Definition: If G is a simple graph, the **complement of G**, (**Edge complement**), denoted by G′ or G^c is a graph such that

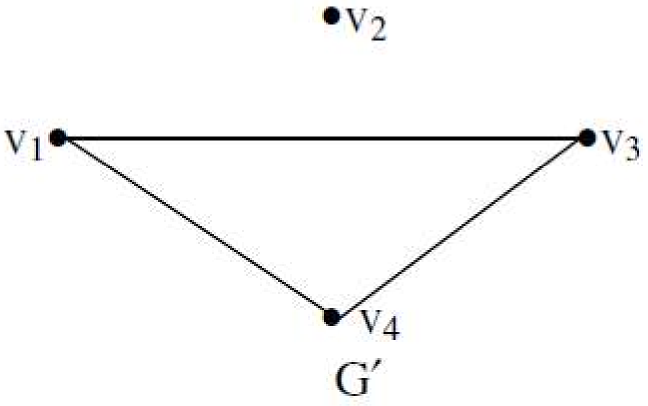
(i) The vertex set of G′ is identical to the vertex set of G, that is $V_{G'} = V_G$

(ii) Two distinct vertices v and w of G′ **are connected by** an edge if and only if v and w **are not connected** by an edge in G.

For example, consider the graph G

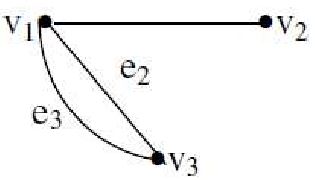


Then complement G′ of G is the graph

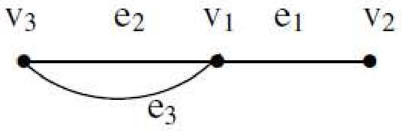


Isomorphisms of Graphs

We know that shape or length of an edge and its position in space are not part of specification of a graph. For example, the figures



and



represent the same graph.

Definition: Let G and H be graphs with vertex sets V(G) and v(H) and Edge sets E(G) and E(H) respectively. Then **G is said to isomorphic to H** iff there exist one-to-one correspondences $g : V(G) \to v(H)$ and $h : E(G) \to E(H)$ such that for all $v \in V(G)$ and $e \in E(G)$,

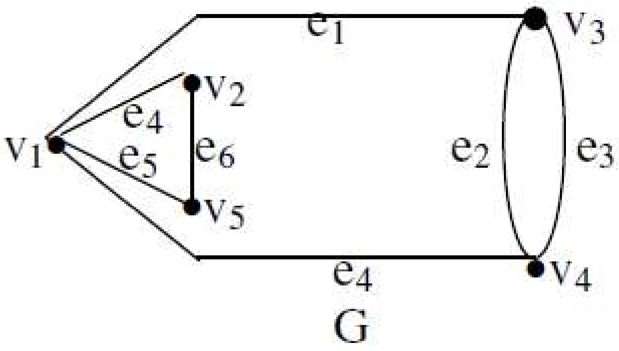
$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e).$$

Definition: The property of mapping endpoints to endpoints is called **preserving incidence** or **the continuity rule** for graph mappings.

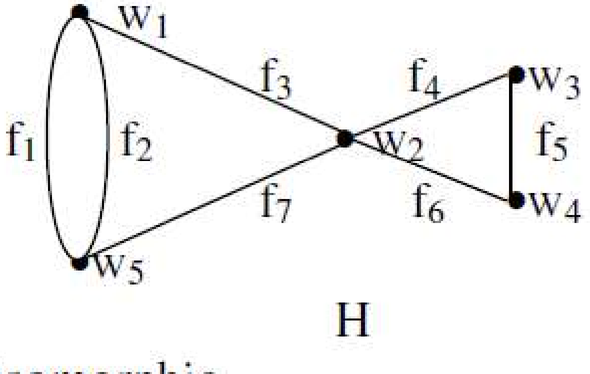
As a consequence of this property, a self-loop must map to a self-loop.

Thus, two isomorphic graphs are same except for the labeling of their vertices and edges.

Example: Show that the graphs



and



are isomorphic.

Solution: To solve this problem, we have to find g: V(G) → V(H) and h : E(G) → E(H) such that for all $v \in V(G)$ and $e \in E(G)$,

$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e).$$

Since e_2 and e_3 are parallel (have the same endpoints), $h(e_2)$ and $h(e_3)$ must also be parallel. Thus we have

$$h(e_2) = f_1 \text{ and } h(e_3) = f_2 \text{ or } h(e_2) = f_2 \text{ and } h(e_3) = f_1.$$

Also the endpoints of e_2 and e_3 must correspond to the endpoints of f_1 and f_2

and so

$$g(v_3) = w_1 \text{ and } g(v_4) = w_5 \text{ or } g(v_3) = w_5 \text{ and } g(v_4) = w_1.$$

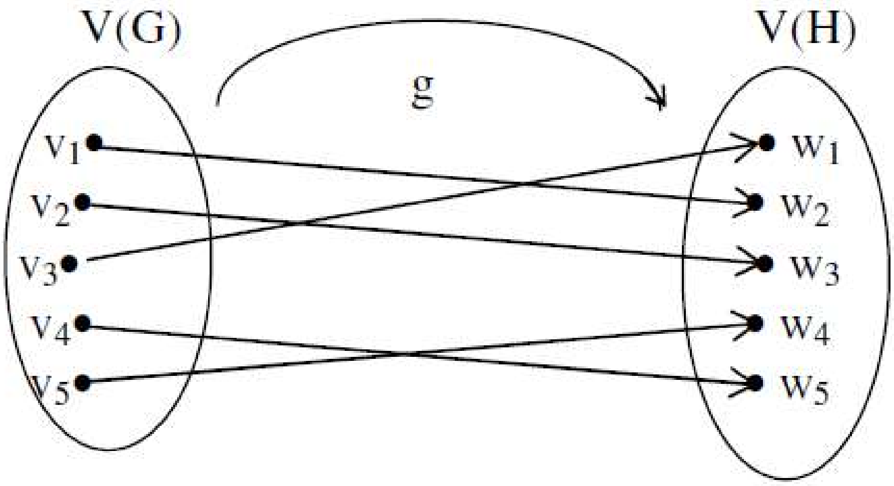
Further, we note that v_1 is the endpoint of four distinct edges e_1, e_7, e_5

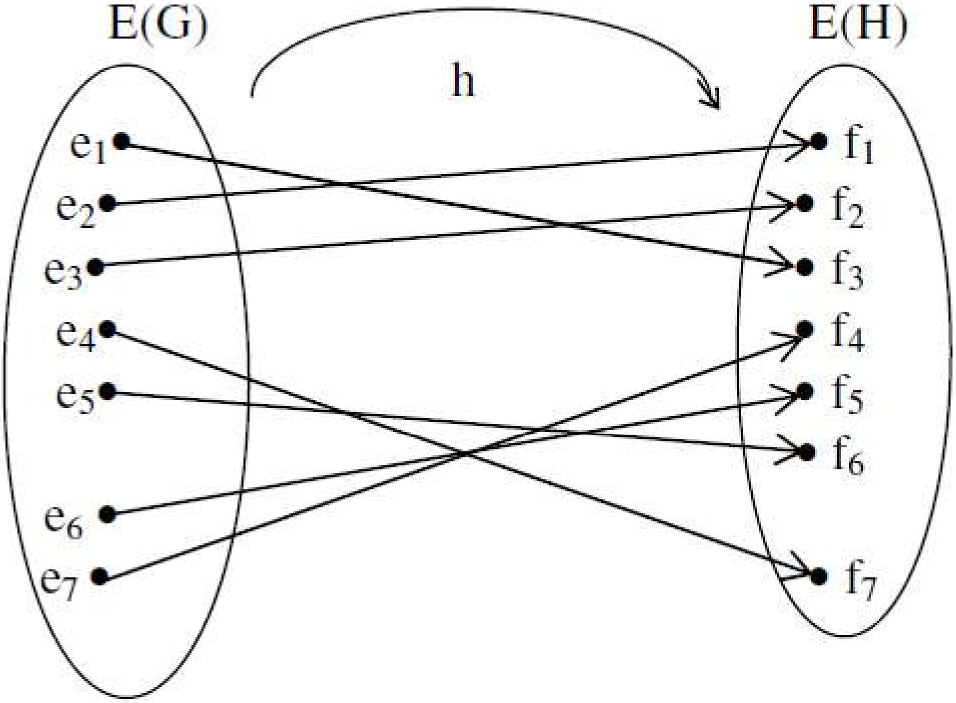
and e_4 -and so $g(v_1)$ should be the endpoint of form distinct edges. We observe that w_2 is the vertex having four edges and so $g(v_1) = w_2$. If $g(v_3) = w_1$, then since v_1 and v_3 are endpoints of e_1 in G, $g(v_1) = w_2$ and $g(v_3) = w_1$ must be endpoints of $h(e_1)$ in H. This implies that $h(e_1) = f_3$.

Continuing in this way we can find g and h to define the isomorphism between G and H.

One such pair of functions (of course there exist several) is shown

below:





Remark: Each of the following properties is invariant under graph isomorphism, where n, m and h are all non-negative integers:

1. has n vertices
2. has m edges
3. has a vertex of degree k
4. has m vertices of degree k

Walks, Paths and Circuits

Definition: In a graph G, **a walk from vertex v_0 to vertex v_n** is a finite alternating sequence:

$$\{v_0, e_1, v_1, e_2, \ldots.., v_{n-1}, e_n, v_n\}$$

of vertices and edges such that v_{i-1} and v_i are the endpoints of e_i.

The **trivial walk** from a vertex v to v consists of the single vertex v.

Definition: In a graph G, a **path** from the vertex v_0 to the vertex v_n is a walk from v_0 to v_n that does not contain a repeated edge.

Thus a **path** from v_0 to v_n is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots\ldots, v_{n-1}, e_n, v_n\},$$

where all the edges e_I are distinct.

Definition: In a graph, a simple path from v_0 to v_n is a path that does not contain a repeated vertex.

Thus a simple path is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots\ldots, v_{i-1}, e_n, v_n\},$$

where all the e_i are distinct and all the v_i are distinct.

Definition: A walk in a graph G that starts and ends at the same vertex is called a **closed walk**.

Definition: A closed walk that does not contain a repeated edge is called a **circuit**.

Thus, closed a closed path is called a circuit (or a cycle) and so a circuit is a walk of the form

$$\{v_0, e_1, v_1, e_2, v_2, \ldots\ldots, v_{n-1}, e_n, v_n\} ,$$

where $v_0 = v_n$ and all the e_i are distinct.

Definition: In a graph the number of edges in the path $\{v_0, e_1, v_1, e_2, \ldots\ldots, e_n, v_n\}$ from v_0 to v_n is called the **length of the path**.

Theorem: If there is a path from vertex v_1 to v_2 in a graph with n vertices, then there does not exist a path of more than n-1 edges from vertex v_1 to v_2.

Proof: Suppose there is a path from v_1 to v_2. Let

$$v_1,\ldots\ldots,v_i,\ldots\ldots\ldots,v_2$$

be the sequence of vertices which the path meets between the vertices v_1 and v_2. Let there be m edges in the path. Then there will be m + 1 vertices in the sequence. Therefore if $m > n-1$, then there will be more than n vertices in the sequence. But the graph is with n vertices. Therefore some vertex, say v_k, appears more than once in the sequence. So the sequence of vertices shall be

$$v_1,\ldots\ldots,v_i,\ldots\ldots,v_k,\ldots\ldots,v_k,\ldots\ldots,v_2.$$

Deleting the edges in the path that lead v_k back to v_k we have a path from v_1 to v_2 that has less edges than the original one. This argument is repeated untill we get a path that has n-1 or less edges.

CONNECTED AND DISCONNECTED GRAPHS :

Definition: Two vertices v_1 and v_2 of a graph G are said to be **connected** if and only if there is a walk from v_1 to v_2.

Definition: A graph G is said to be **connected** if and only if given any two vertices v_1 and v_2 in G, there is a walk from v_1 to v_2.

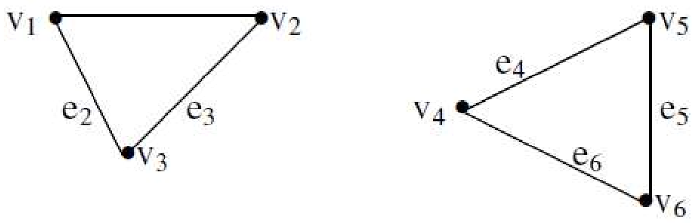
Thus, a graph G is connected if there exists a walk between every two vertices in the graph.

Definition: A graph which is not connected is called **Disconnected Graph.**

Example: Which of the graph below are connected?

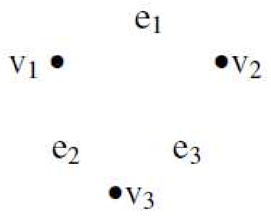
Definition: If a graph G is disconnected, then the various connected pieces of G are called the **connected components of the graph**.

Example: Consider the graph given below:



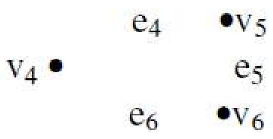
This graph is disconnected and have two connected components:

H_1 :



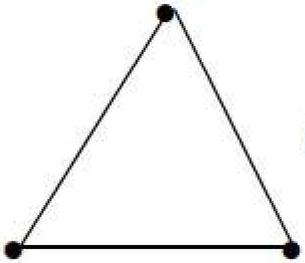
with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{e_1, e_2, e_3\}$

H_2 :

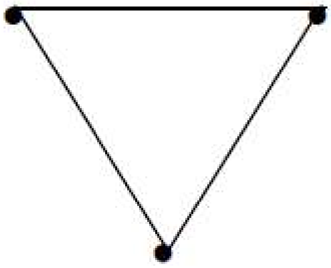


with vertex set $\{v_4, v_5, v_6\}$ and edge set $\{e_4, e_5, e_6\}$.

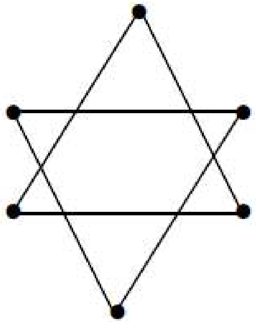
Solution: The connected components are :



and



Example: Find the number of connected components in the graph



Eulerian Paths And Circuits

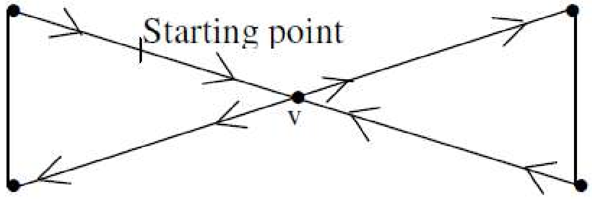
Definition: A path in a graph G is called an **Euler Path** if it includes **every edge exactly once**.

Definition: A graph is called **Eulerian graph** if there exists a Euler circuit for that graph.

Definition: A circuit in a graph G is called an **Euler Circuit** if it includes every edge exactly once. Thus, an Euler circuit (Eulerian trail) for a graph G is a sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses **every edge of G exactly once**.

Theorem 1. If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof: Let G be a graph which has an Euler circuit. Let v be a vertex of G. We shall show that degree of v is even. By definition, Euler circuit contains every edge of graph G. Therefore the Euler circuit contains all edges incident on v. We start a journey beginning in the middle of one of the edges adjacent to the start of Euler circuit and continue around the Euler circuit to end in the middle of the starting edge. Since Euler circuit uses every edge exactly once, the edges incident on v occur



in entry / exist pair and hence the degree of v is a multiple of 2. Therefore the degree of v is even. This completes the proof of the theorem.

We know that contrapositive of a conditional statement is logically equivalent to statement. Thus the above theorem is equivalent to:

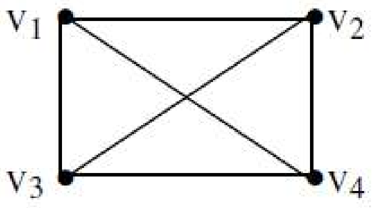
Theorem:2. If a vertex of a graph is not of even degree, then it does not have an Euler circuit.

or

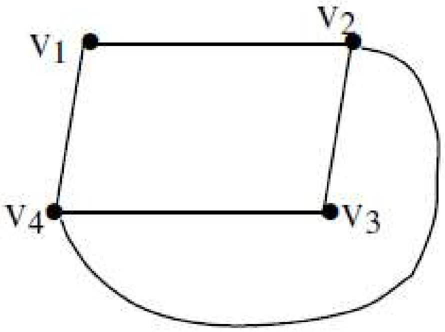
"If some vertex of a graph has odd degree, then that graph does not have an Euler circuit".

Example: Show that the graphs below do not have Euler circuits.

(a)



(b)



Solution: In graph (a), degree of each vertex is 3. Hence this **does not** have a Euler circuit.

In graph (b), we have

$$\deg(v_2) = 3$$

$$\deg(v_4) = 3$$

Since there are vertices of odd degree in the given graph, therefore it **does not** have an Euler circuit.

are graphs in which each vertex has degree 2 but these graphs do not have Euler circuits since there is no path which uses each vertex at least once.

Theorem 3. If G is a connected graph and every vertex of G has even degree, then G has an Euler circuit.

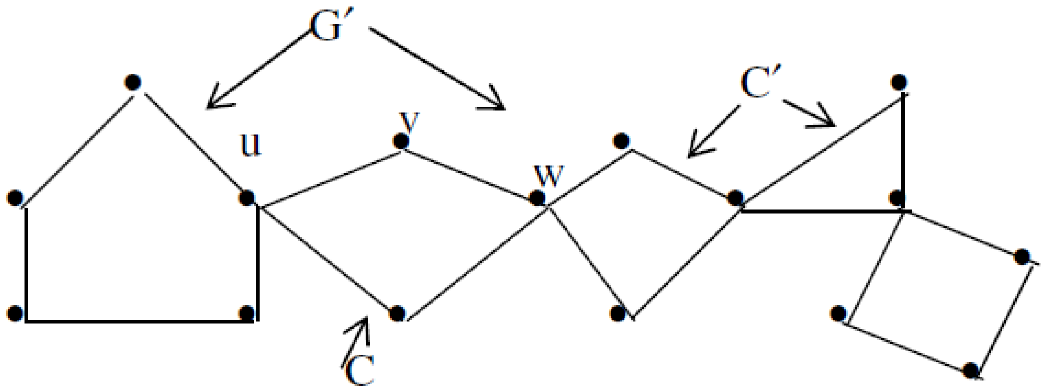
Proof: Let every vertex of a connected graph G has even degree. If G consists of a single vertex, the trivial walk from v to v is an Euler circuit. So suppose G consists of more than one vertices. We start from any verted v of G. Since the degree of each vertex of G is even, if we reach each vertex other than v by travelling on one edge, the same vertex can be reached by travelling on another previously unused edge. Thus a sequence of distinct adjacent edges can be produced indefinitely as long as v is not reached. Since number of edges of the graph is finite (by definition of graph), the sequence of distinct edges will terminate. Thus the sequence must return to the starting vertex. We thus obtain a sequence of adjacent vertices and edges starting and ending at v without repeating any edge. Thus we get a circuit C.

If C contains every edge and vertex of G, then C is an Eular circuit.

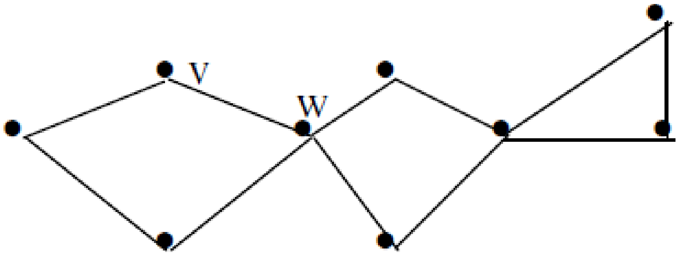
If C does not contain every edge and vertex of G, remove all edges of C from G and also any vertices that become isolated when the edges of C are removed. Let the resulting subgraph be G'. We note that when we removed edges of C, an even number of edges from each vertex have been removed. Thus degree of each remaining vertex remains even.

Further since G is connected, there must be at least one vertex common to both C and G'. Let it be w(in fact there are two such vertices). Pick any sequence of adjacent vertices and edges of G' starting and ending at w without repeating an edge. Let the resulting circuit be C'.

Join C and C' together to create a new circuit C''. Now, we observe that if we start from v and follow C all the way to reach w and then follow C' all the way to reach back to w. Then continuing travelling along the untravelled edges of C, we reach v.



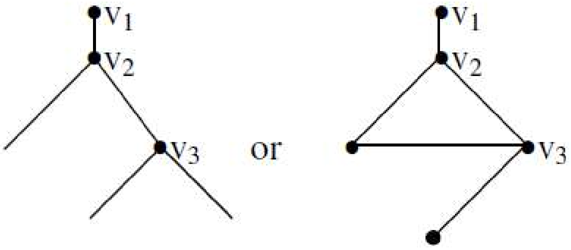
G



C″

Theorem 5. If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.

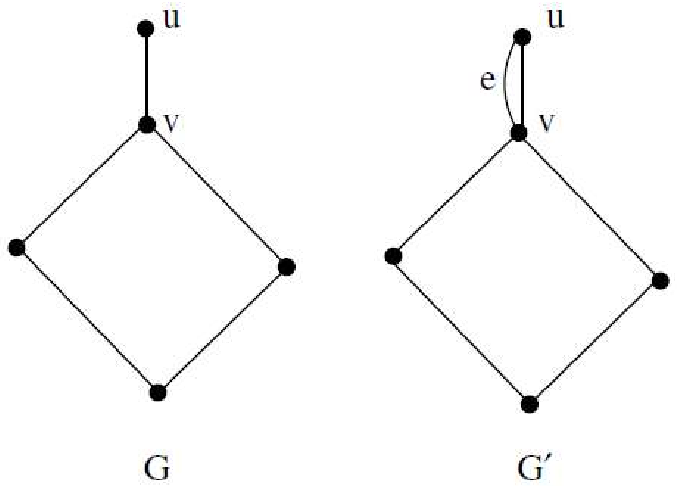
Proof : Let v_1, v_2 and v_3 be vertices of odd degree. Since each of these vertices had odd degree, any possible Euler path must leave (arrive at) each of v_1, v_2, v_3 with no way to return (or leave). One vertex of these three vertices may be the beginning of Euler path and another the end but this leaves the third vertex at one end of an untravelled edge. Thus there is no Euler path.



(Graphs having more than two vertices of odd degree).

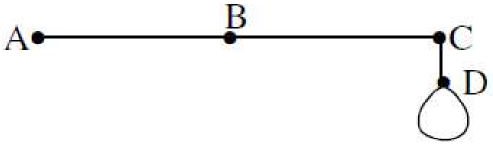
Theorem 6. If G is a connected graph and has exactly two vertices of odd degree, then there is an Euler path in G. Further, any Euler path in G must begin at one vertex of odd degree and end at the other.

Proof: Let u and v be two vertices of odd degree in the given connected graph G.



If we add the edge e to G, we get a connected graph G′ all of whose vertices have even degree. Hence there will be an Euler circuit in G′. If we omit e from Euler circuit, we get an Euler path beginning at u(or v) and edning at v(or u).

Examples. Has the graph given below an Eulerian path?



Solution: In the given graph,

$$\deg(A) = 1$$

$$\deg(B) = 2$$

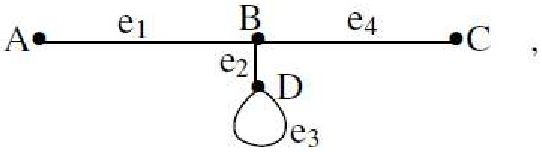
$$\deg(C) = 2$$

$$\deg(D) = 3$$

Thus the given connected graph has exactly two vertices of odd degree. Hence, it has an Eulerian path.

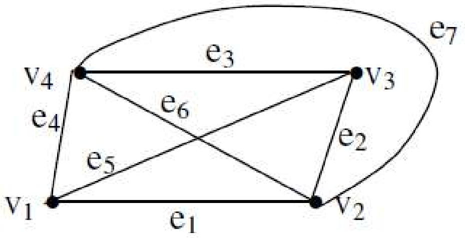
If it starts from A(vertex of odd degree), then it ends at D(vertex of odd degree). If it starts from D(vertex of odd degree), then it ends at A(vertex of odd degree).

But on the other hand if we have the graph as given below :



then deg(A) = 1, deg(B) = 3 deg(C) = 1, degree of D = 3 and so we have four vertices of odd degree. Hence it does not have Euler path.

Example: Does the graph given below possess an Euler circuit?



Solution: The given connected graph is connected. Further

$$\deg(v_1) = 3$$

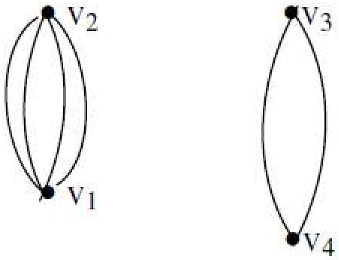
$$\deg(v_2) = 4$$

$$\deg(v_3) = 3$$

$$\deg(v_4) = 4$$

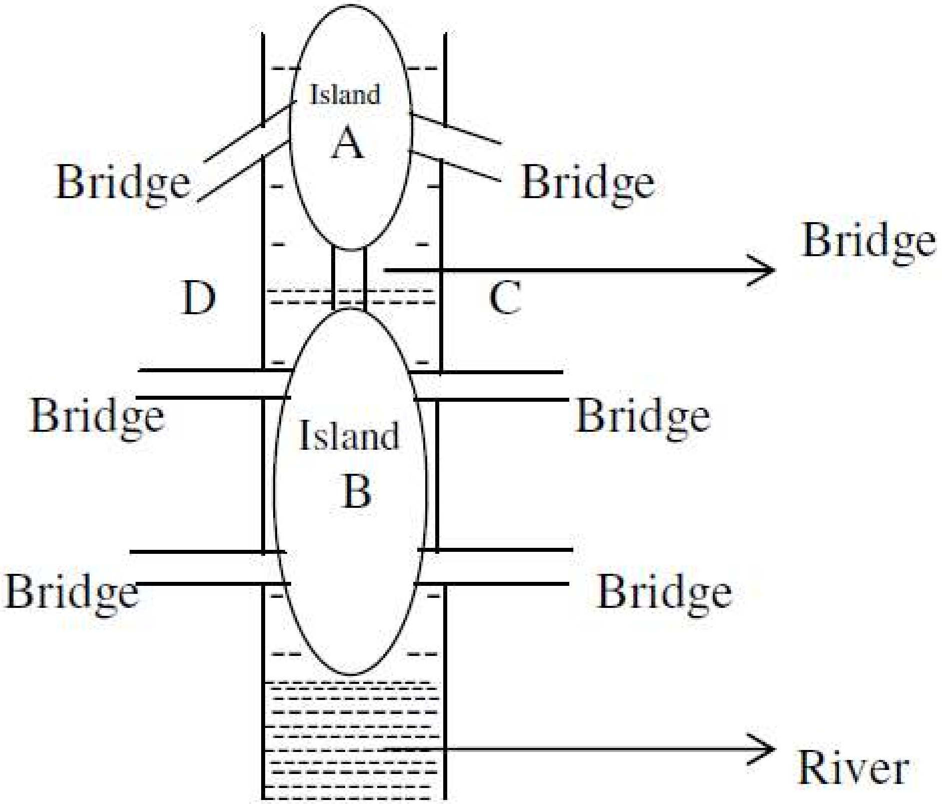
Since this connected graph has vertices with odd degree, it cannot have Euler circuit. But this graph has Euler path, since it has exactly two vertices of odd degree. For example, $v_3 e_2 v_2 e_7 v_4 e_6 v_2 e_1 v_1 e_4 v_4 e_3 v_3 e_5 v_1$

Example: Consider the graph



Here, $\deg(v_1) = 4$, $\deg(v_2) = 4$, $\deg(v_3) = 2$, $\deg(v_4) = 2$. Thus degree of each vertex is even. But the graph is not Eulerian since it is **not connected**.

Example 4:. The bridges of Konigsberg: The graph Theory began in 1736 when Leonhard Euler solved the problem of seven bridges on Pregel river in the town of Konigsberg in Prussia (now Kaliningrad in Russia). The two islands and seven bridges are shown below:



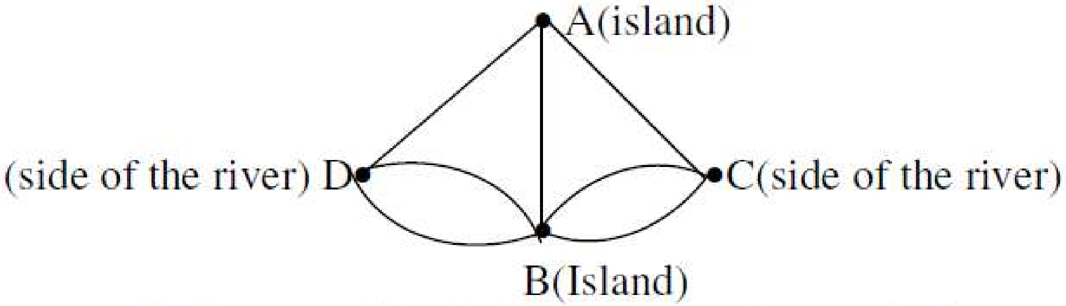
The people of Konigsgerg posed the following question to famous Swiss Mathematician Leonhard Euler:

"Beginning anywhere and ending any where, can a person walk through the town of Konigsberg crossing all the seven bridges exactly once?

Euler showed that such a walk is impossible. He replaced the islands A, B and the two sides (banks) C and D of the river by vertices and the bridges as edges of a graph. We note then that

$$\deg(A) = 3$$
$$\deg(B) = 5$$
$$\deg(C) = 3$$
$$\deg(D) = 3$$

Thus the graph of the problem is



(Euler's graphical representation of seven bridge problem)

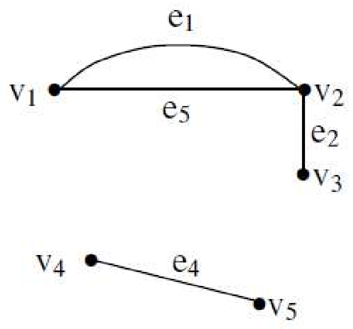
The problem then reduces to

"Is there any Euler's path in the above diagram?".

To find the answer, we note that there are more than two vertices having odd degree. Hence there exist no Euler path for this graph.

Definition: An edge in a connected graph is called a **Bridge** or a **Cut Edge** if deleting that edge creates a disconnected graph.

In this graph, if we remove the edge e_3, then the graph breaks into two Connected Component given below:

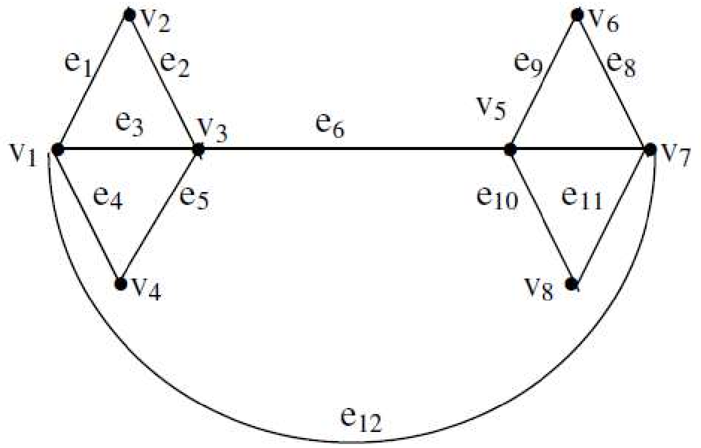


Hence the edge e_3 is a bridge in the given graph.

METHOD FOR FINDING EULER CIRCUIT

We know that if every vertex of a non empty connected graph has even degree, then the graph has an Euler circuit. We shall make use of this result to find an Euler path in a given graph.

Consider the graph



We note that

$$\deg(v_2) = \deg(v_4) = \deg(v_6) = \deg(v_8) = 2$$

$$\deg(v_1) = \deg(v_3) = \deg(v_5) = \deg(v_7) = 4$$

Hence all vertices have even degree. Also the given graph is connected. Hence the given has an Euler circuit. We start from the vertex v_1 and let C be

$$C : v_1\ v_2\ v_3\ v_1$$

Then C is not an Euler circuit for the given graph but C intersect the rest of the graph at v_1 and v_3. Let C′ be

$$C' : v_1 v_4\ v_3\ v_5\ v_7\ v_6\ v_5\ v_8\ v_7\ v_1$$

(In case we start from v_3, then C′ will be $v_3\ v_4\ v_1\ v_7\ v_6\ v_5\ v_7\ v_8\ v_5$)
Path C′ into C and obtain

$$C'' : v_1 v_2\ v_3\ v_1\ v_4\ v_3\ v_5\ v_7\ v_6\ v_5\ v_8\ v_7\ v_1$$

Or we can write

$$C'' : e_1 e_2\ e_3\ e_4\ e_5\ e_6\ e_7\ e_8\ e_9\ e_{10} e_{11}\ e_{12}$$

(If we had started from v_2, then C'' : $v_1 v_2\ v_3\ v_4\ v_1\ v_7\ v_6\ v_5\ v_7\ v_8\ v_5\ v_3\ v_1$ **or**

$e_1 e_2\ e_5\ e_4\ e_{12}\ e_8\ e_9\ e_7\ e_{11}\ e_{10} e_6\ e_3$)

In C″ all edges are covered exactly once. Also every vertex has been covered at least once. Hence C″ is a Euler circuit.

PART – B

POSSIBLE QUESTIONS – SIX MARKS

1.Show that if a graph G(either connected (or) disconnected) has exactly two vertices of odd degree there is a path joining these two vertices.

2.In a (directed or undirected) graph with n vertices, if there is a path from vertex v_1 to vertex v_2,then there is a path of no more than n-1 edges from vertex v_1 to vertex v_2.

3. Show that a simple graph with n vertices and k-components can have at most

$$\frac{(n-k)(n-k+1)}{2}$$

4.State and prove the Handshaking theorem.

5.Show that the sum of the degree of all vertices in a graph equal to twice in a number of edges incidence in G.

6.Show that if there is a (u, v)- walk in G, then there is also a (u, v)- path in G.

7.In a connected graph G with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

8.The number of vertices of odd degree in a graph is even.

9.Draw all possible simple graph of one, two, three, four, five vertices .

10. Prove that a connected graph is Euler graph iff it has even degree.

PART –C

POSSIBLE QUESTIONS – TEN MARKS

1. A non- empty connected graph G is Eulerian if and only if G is the union of some edges disjoint circuits.
2.Show that a connected graph G is an Euler graph if and only if the degree of every vertex in G is even.
3. A connected graph G is an Euler graph iff it can be decomposed into circuits.
4.If the intersection of two paths in a graph G disconnected then their union has atleast one circuit.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
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Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Advanced Discrete Mathematics **Subject Code: 17MMP105A**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
A _______________ consists of set of vertices and edges suchthat each edge is incident with vertices.	graph	tree	complete graph	network	graph
An edge having the same vertex as both its end vertices is called ____________.	graph	tree	self-loop	trival	self-loop
Edges that have the same end vertices are ________________.	same	parallel	null graph	connected	parallel
A graph is ___________ if it has no parallel edges or self-loops.	simple	directed	adjacent	self-loop	simple
A graph with no edges is __________.	null graph	trival	empty	parallel	empty
A graph with no verticesis a __________.	null graph	trival	empty	parallel	null graph
A graph with only one vertex is _________.	null graph	trival	empty	parallel	trival
A _________ is a vertex whose degree is _________.	one	two	three	four	one
A graph in which every vertex has the same degree is called___________.	null graph	regular graph	complete graph	simple graph	regular graph
A simple graph G with n vertices is said to be a____________ if the degree of every vertex is n-1.	null graph	regular graph	complete graph	simple graph	complete graph
A graph in which some edges are directed and some are undirected is called_______________.	mixed graph	regular graph	complete graph	simple graph	mixed graph
Every graph is its own ______________.	mixed graph	sub graph	complete graph	simple graph	sub graph
The maximum number of edges in a simple graph with n vertices is _________.	n	(n-2)/2	n-1	(n-1)/2	(n-1)/2
A vertex having no edge incident on it is called______________.	end vertex	isolated vertex	pendant vertex	null graph	isolated vertex
Any vertex having degree one is called a______________.	null vertex	isolated vertex	pendant vertex	null graph	pendant vertex
walk is also called___________.	chain	edge	vertex	graph	chain
A finite alternating sequence of vertices and edges of a graph G beginning and ending with vertices is called__________.	walk	sub graph	circuit	path	walk
Vertices with which a walk begins or ends are called its__________.	end vertex	isolated vertex	pendant vertex	terminal vertices	terminal vertices
If a walk begins and end with the same vertex then it is called a __________.	walk	closed walk	circuit	path	closed walk
If no vertex appears more than once in an open walk then it is called a __________.	walk	closed walk	circuit	path	path
The number of edges in a path is called the __________ of the path.	length	same	walk	trival	length
_____________ is defined as a closed walk in which no vertex and final vertex appears more than once.	walk	closed walk	circuit	path	circuit
A____________ is a closed , non intersecting walk.	circuit	closed walk	walk	path	circuit
___________ is also called cycle.	walk	closed walk	circuit	path	circuit
A graph in which weights or distance are assigned to each adge is called a _____________.	complete graph	simple graph	mixed graph	weighed graph	weighed graph
A _________ is a path in a digraph in which the edges are all distinct.	simple path	elementary path	node	self path	simple path
An ___________ is a path in which all the nodes through which it travels are distinct.	simple path	elementary path	node	self path	elementary path
A path orginates and ends in the same node is called a ______________.	walk	closed walk	circuit	path	circuit
A __________ is a graph whose components are all trees.	tree	graph	forest	walk	forest
A tree in which one vertex is distinguished from all others is called a __________.	tree	rooted tree	connected	pendant vertex	rooted tree
In adjacency matrix of graph all the entries along the leading diagonal are _______iff the grah has no self-loops.	zero	one	two	three	zero
The determinant of every square sub matrix of an incidence matrix is ___________.	1 or -1 or 0	1 or 2 or 3	2 or 3 or 4	1 or -2 or -1	1 or -1 or 0
The rank of an incidence matrix of a digraph with n vertices is _____________.	n	n+1	2n	n-1	n-1
In graph has 3 vertices then iit has ____________chromatic	3	atleast3	atmost3	more than3	atleast3
A graph consistng one circuit with greater than or equal to vertices is 2 chromatic if n is__________	even	odd	even and odd	equal to 3	even
A graph cooonsisting one circuit with n greatthan or equal to 3 vertices if n is odd then it has__________	1-chromatic	2-chromatic	3-chromatic	d.atleast 2-chromatic	3-chromatic
A graph consisting of only -------------vertices is 1-chromatics	isolated	pendenat	odd	even	isolated
A graph consisting of only isolated vertices is	4-chromatic	3-chromatic	2-chromatic	1-chromatic	1-chromatic
A graph with one or more edges is ------------2-chromatic	atleast	atmost	exactly	not	atleast3
A graph with one or more edges is atleast	4-chromatic	3-chromatic	2-chromatic	1-chromatic	2-chromatic
A complete graph with n vertices is--------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	n-chromatic
Every graph having ----------- is atleast 3-chromatic	triangle	square	odd vertices	even vertices	triangle
Every graph having triangle is atleast ------------------	4-chromatic	3-chromatic	2-chromatic	n-chromatic	3-chromatic
A complete graph with 5 vertices is--------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	5-chromatic
A graph consisting of simply one circuit with greater than or equal to 3 vertices is ------------- if n is even	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
A graph consisting of simply one circuit with greater than or equal to 3 vertices is 2-chromatic if n is ----------	even	odd	3	0	even
A graph consisting of simply one circuit with greater than or equal to 3 vertices is ------------- if n is odd	4-chromatic	3-chromatic	2-chromatic	5-chromatic	3-chromatic
A graph consisting of simply one circuit with greater than or equal to 3 vertices is 3-chromatic if n is ---------	even	odd	3	0	odd
A graph with ------- one edge is 2-chromatic if it has no circuits of odd length	atleast	atmost	exactly	3	atleast
A graph with atleast ---------- edge is 2-chromatic if it has no circuits of odd length	1	2	3	4	1
A graph with atleast one edge is 2-chromatic if it has no circuits of -------- length	odd	even	0	4	odd
A graph with atleast one edge is 2-chromatic if it has ---------- circuits of odd length	0	1	2	3	0

A graph with atleast one edge is ------------- if it has no circuits of odd length	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
A star graph is ----------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every ------------ graph is 2-chromatic	bipartiate	complete	regular	connected	bipartiate
Every biparitate graph is ------------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Two regions are said to be adjacent if they have a common -----------between them	edge	vertex	edge and vertex	neither edge nor vertex	edge
Two ---------------------- are said to be adjacent if they have a common egde between them	faces	regions	egdes	vertices	regions
Cover of a graph is a sub set of ----------------------	vertices	edges	both vertices and edges	neither edge nor vertex	vertices

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DEPARTMENT OF MATHEMATICS

SUBJECT: ADVANCED DISCRETE MATHEMATICS **SEMESTER: I**

		L	T	P	C
SUBJECT CODE:17MMP105A	**CLASS:I M.Sc (MATHEMATICS)**	4	0	0	4

UNIT V

Trees: Trees and its properties - minimally connected graph - Pendant vertices in a tree - distance and centers in a tree - rooted and binary tree. Levels in binary tree - height of a tree - Spanning trees - rank and nullity.

TEXT BOOKS

1. Tremblay J. P. and Manohar, R., (1997). Discrete Mathematical Structures with Applications to Computer Science, McGraw-Hill Book Co.(for unit I,II,III).
2. Deo N., (2000). Graph Theory with Applications to Engineering and Computer Sciences, Prentice Hall of India. (for unit IV,V)

REFERENCES

1. Liu C.L., (2000). Elements of Discrete Mathematics, McGraw-Hill Publishing Company Ltd, New Delhi.
2. Wiitala S., (2003),Discrete Mathematics- A Unified Approach, McGraw-Hill Book Co, New Delhi.
3. Seymour Lepschutz, (2007) ,Discrete Mathematics, Schaum Series, McGraw-Hill Publishing Company Ltd, New Delhi.
4..Advance Discrete Mathematics Paperback – 2011 by G.C.Sharma (Author), Madhu Jain (Author) Publisher: Laxmi Publications; Second edition (2011)

Introduction:

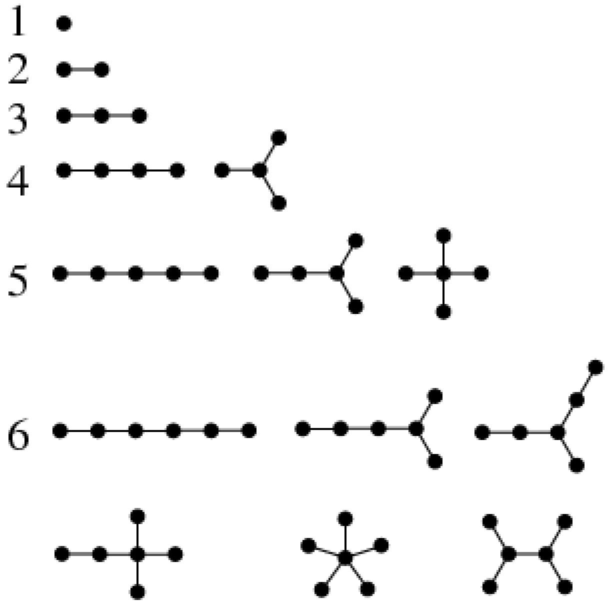
The graphs that we come across in most of the applications are connected. Among the connected graphs, trees are probably the most important ones. In this chapter ,We shall study trees and its properties. The relationships among circuits, trees and other associated concepts in a graph are also explored.

TREES:

Definition:

A connected graph without any circuits is called a **Tree.**

Example: Trees with one ,two three and four vertices are shown below



(Figure 5.1)

Since parallel edges and self – loops both form circuits , a tree can not have parallel edges and elf loops. Thus a tree has to be a simple graph.

Theorem 5.1 :

A graph G is a tree iff there is one and only one path between any two vertices of G.

Proof:

First suppose that the graph G is a tree. Then by definition of a tree ,G is a connected graph. Therefore ,there must exist atleast one path between any two vertices in G. Now suppose that there are two distinct paths between vertices a and b of G. Then the union of these two paths will contain a circuit and G can not be a tree. Thus there is one and only one path between any two vertices of G.

Conversely, suppose that there is one and only path between any two vertices of G. We shall show G is a tree. Since there exists a path between any two vertices of G, therefore G is connected. A circuit in a graph with two or more vertices implies that there exists a pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between any two vertices, G can have no circuits. Thus G is a tree.

Theorem 5.2:

A tree with n vertices has n-1 edges.

Proof:

We shall prove the theorem by induction on the number of vertices .Clearly, the theorem is true for trees with one or two vertices(see Fig.5.1).Assume that the theorem is true for all trees with fewer than n vertices.

Let us consider a tree G with n vertices .Let e_k be any edge in G with end vertices v_i and v_j.

According to theorem 1 above , the edge e_k is the only path between v_i and v_j. Hence deletion of e_k from G will disconnect the graph. Thus G-e_k is not connected. Further ,G-e_k will contain exactly two components ,for otherwise the graph G will not be connected. Let these two components of G-e_k be G_1 and G_2 respectively. Since $n_1 <n$ and $n_2<n$, we have by the induction hypothesis

$$\text{Number of edges in } G_1 = n_2 -1$$

and

$$\text{Number of edges in } G_2 = n_2 -1$$

Thus , number of edges in G – e_k is equal to $(n_1 -1)+(n_2 -1)= (n_2 + n_2) -2 = n-2$. Hence G has exactly n-1 edges.

Theorem 5.3:

Every connected graph with n vertices and n-1 edges is a tree.

Proof:

Let G be a connected graph with n vertices and n-1 edges. The theorems will be proved if we show that G has no circuit. Suppose that G contains atleast one circuit. Since removing an edge from a circuit does not disconnect a graph, we may remove edges, but no vertices from circuits in G until the resulting graph G^* is a circuit free.

Now G^* is a connected graph with n vertices and contains no circuit .Thus G^* is a tree with n vertices .Hence G^* has n-1 edges (by theorem 2).But now the graph G has more than n-1 edges, a contradiction.

Hence G has no circuit.This completes the proof.

Theorem 5.4: A graph G with n vertices ,n-1 edges and no circuit is tree.

Proof:

Let G be a graph with n vertices , n -1 edges and has no circuit. It wii be a tree if we show that it is connected .If possible, suppose that G is disconnected. Then G will consist of two or more circuitless components.Without loss of generality let G consist of two components G_1 and G_2 .

we add an edge e between a vertex v_1 in G_1 and v_2 in G_2. Since v_1 and v_2 are in different components of G , there is no path between v_1 and v_2 in G.Thus addition of edge e will not create a circuit.Thus G∪ e is a circuitless,connected graph (and therefore a tree)of n vertices and n edges,which is not possible because of theorem 2.This completes the proof.

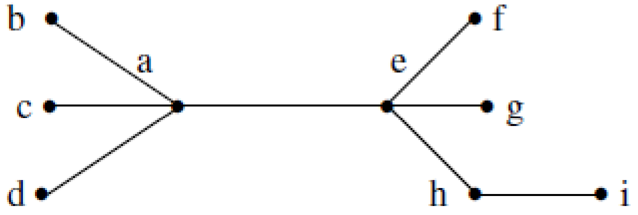
Definition: A collection of disjoint trees is called a **forest.**

Thus a graph is a forest if and only if it is circuit free.

Definition: A vertex of degree 1 in a tree is called a **leaf** or a **terminal node** or a **terminal vertex**.

Definition: A vertex of degree greater than 1 in a tree is called a **Branch node** or **Internal node** or **Internal vertex**.

Consider the tree shown below:



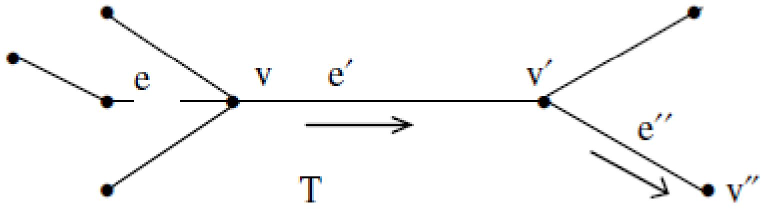
In this tree the vertices b, c, d, f, g, and i are leaves whereas the vertices a, e, h are branch nodes.

CHARACTERIZATION OF TREES

We have the following interesting characterization of trees:

Lemma 1: A tree that has more than one vertex has at least one vertex of degree 1.

Proof: Let T be a particular but arbitrary chosen tree having more than one vertex.



1. Choose a vertex v of T. Since T is connected and has at least two vertices, v is not isolated and there is an edge e incident on v.

2. If deg (v) > 1, there is an edge e′ ≠ e because there are, in such a case, at least two edges incident on v. Let v′ be the vertex at the other end of e′. This is possible because e′ is not a loop by the definition of a tree.

3. If deg(v′) > 1, then there are at least two edges incident on v′. Let e″ be the other edge different from e′ and v″ be the vertex at other end of e″. This is again possible because T is acyclic.

4. If deg(v″) > 1, repeat the above process. Since the number of vertices of a tree is finite and T is circuit free, the process must terminate and we shall arrive at a vertex of degree 1.

Remark: In the proof of the above lemma, after finding a vertex of degree 1, if we return to v and move along a path outward from v starting with e, we shall reach to a vertex of degree 1 again. Thus it follows that **"Any tree that has more than one vertex has at least two vertices of degree 1".**

Lemma 2: There is a unique path between every two vertices in a tree.

Proof: Suppose on the contrary that there are more than one path between any two vertices in a given tree T. Then T has a cycle which contradicts the definition of a tree because T is acyclic. Hence the lemma is proved.

Lemma 3: The number of vertices is one more than the number of edges in a tree.

Or

For any positive integer n, a tree with n vertices has n-1 edges.

Proof: We shall prove the lemma by mathematical induction.

Let T be a tree with **one** vertex. Then T has no edges, that is, T has 0 edge. But 0 = 1 – 1. Hence the lemma is true for n = 1.
Suppose that the lemma is true for k > 1. We shall show that it is then true for k + 1 also. Since the lemma is true for k, the tree has k vertices and k-1 edges. Let T be a tree with k +1 vertices. Since k is +ve, k+1 ≥ 2 and so T has more than one vertex. Hence, by Lemma 1, T has a vertex v of degree 1. Also there is another vertex w and so there is an edge e connecting v and w. Define a subgraph T′ of T so that

$$V(T') = V(T) - \{v\}$$

$$E(T') = E(T) - \{e\}$$

Then number of vertices in T′ = (k+1) – 1 = k and since T is circuit free and T′ has been obtained on removing one edge and one vertex, it follows that T′ is acyclic. Also T′ is connected. Hence T′ is a tree having k vertices and therefore by induction hypothesis, the number of edges in T′ is k-1. But then

No. of edges in T = number of edges in T′ + 1

$$= k - 1 + 1 = k$$

Thus the Lemma is true for tree having k + 1 vertices. Hence the lemma is true by mathematical induction.

Corollary 1. Let C(G) denote the number of components of a graph. Then a forest G on n vertices has n – C(G) edges.

Proof: Apply Lemma 3 to each component of the forest G.

Corollary 2. Any graph G on n vertices has at least n – C(G) edges.

Proof: If G has cycle-edges, remove them one at a time until the resulting graph G* is acyclic. Then G* has n – C(G*) edges by corollary 1. Since we have removed only circuit, C(G*) = C(G). Thus G* has n – C(G) edges. Hence G has at least n – C(G) edges.

Lemma 4: A graph in which there is a unique path between every pair of vertices is a tree

(This lemma is converse of Lemma 2).

Proof: Since there is a path between every pair of points, therefore the graph is connected. Since a path between every pair of points is unique, there does not exist any circuit because existence of circuit implies existence of distinct paths between pair of vertices. Thus the graph is connected and acyclic and so is a tree.

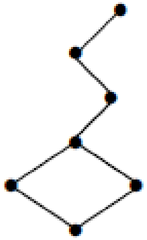
Lemma 5. (converse of Lemma 3) A connected graph G with e = v – 1 is a tree

Proof: The given graph is connected and

$$e = v - 1.$$

To prove that G is a tree, it is sufficient to show that G is acyclic. Suppose on the contrary that G has a cycle. Let m be the number of vertices in this cycle. Also, we know that **number of edges in a cycle is equal to number of vertices in that cycle**. Therefore number of edges in the present case is m.

Since the graph is connected, every vertex of the graph which is not in cycle must be connected to the vertices in the cycle.



Now each edge of the graph that is not in the cycle can connect only one vertex to the vertices in the cycle. There are v-m vertices that are not in the cycle. So the graph must contain at least v – m edges that are not in the cycle. Thus we have

$$e \geq v - m + m = v,$$

which is a contradiction to our hypothesis. Hence there is no cycle and so the graph in a tree.

ROOTED AND BINARY TREE :

Definition: A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is 0 and the incoming degrees of all other vertices are 1.
The vertex with incoming degree 0 is called the **root** of the rooted tree.
A tree T with root v_0 will be denoted by (T, v_0).

Definition: In a rooted tree, a vertex, whose outgoing degree is 0 is called a **leaf** or **terminal node**, whereas a vertex whose outgoing degree is non - zero is called a **branch node** or an **internal node**.

Definition: Let u be a branch node in a rooted tree. Then a vertex v is said to be **child** (**son** or **offspring**) of u if there is an edge from u to v. In this case u is called **parent** (**father**) of v.

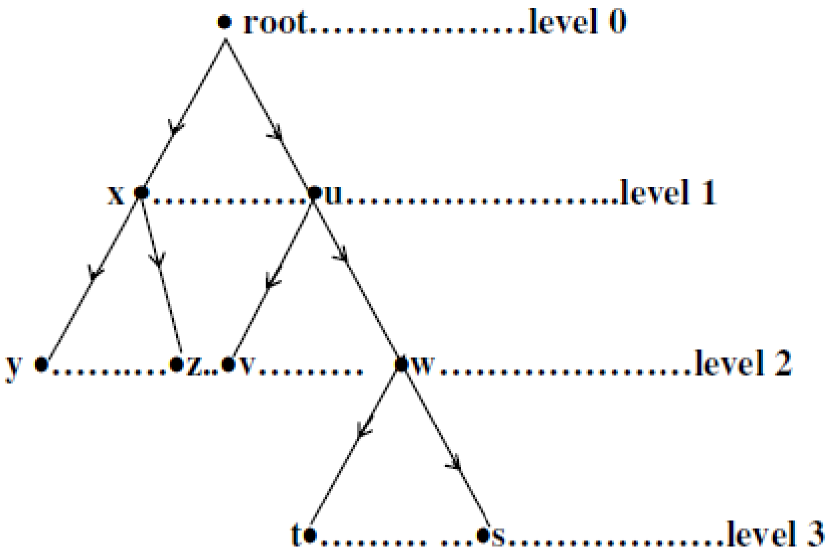
Definition: Two vertices in a rooted tree are said to be **siblings** (**brothers**) if they are both children of same parent.

Definition: A vertex v is said to be a **descendent** of a vertex u if there is a unique directed path from u to v.
In this case u is called the **ancestor** of v.

Definition: The **level** (or **path length**) of a vertex u in a rooted tree is the number of edges along the unique path between u and the root.

Definition: The **height** of a rooted tree is the maximum level to any vertex of the tree.

As an example of these terms consider the rooted tree shown below:


root.................level 0
x
u......................level 1
y
z..
v.........
w.....................level 2
t......... ...
s..................level 3

Here y is a child of x; x is the parent of y and z. Thus y and z are siblings. The descendents of u are v, w, t and s. Levels of vertices are shown in the figure. The height of this rooted tree is 3.

Definition: Let u be a branch node in the tree T = (V, E). Then the subgraph T′ = (V′, E′) of T such that the vertices set V′ contains u and all of its descendents and E′ contains all the edges in all directed paths emerging from u is called a **subtree** with u as the root.

Theorem: If T is a full binary tree with i internal vertices, then T has i+1 terminal vertices (leaves) and 2i+1 total vertices.

Proof: The vertices of T consists of the vertices that are children (of some parent) and the vertices that are not children (of any parent). There is nonchild – the root, Since there are i internal vertices, each having two children, there are 2i children. Thus the total number of vertices of T is 2i+1 and the number of terminal vertices is

$$(2i + 1) - i = i + 1$$

This completes the proof.

In the context of above example, we have

$$\text{No. of leaves} = p = i + 1$$

Or

$$i = p - 1$$

Remark: In case of full n-ary tree, if i denotes the number of branch nodes, then total number of vertices of T is $ni + 1$ and the number of terminal vertices is

$$n\,i + 1 - i = i(n - 1) + 1$$

If p is the number of terminal vertices, then

$$p = i(n - 1) + 1$$

or

$$(n - 1)\,i = p - 1$$

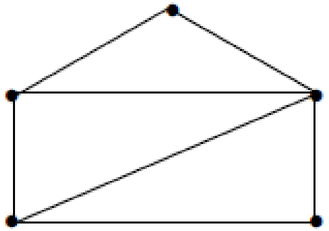
SPANNING TREE:

Definition: A **spanning tree** for a graph G is a subgraph of G that contains every vertex of G and is a tree.

Or

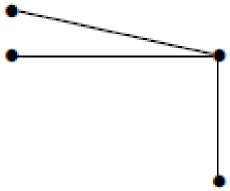
"A **spanning tree** for a graph G is a spanning subgroup of G which is a tree".

Example: Determine a tree and a spanning tree for the connected graph given below:

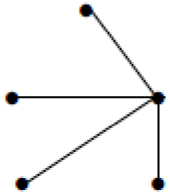


G

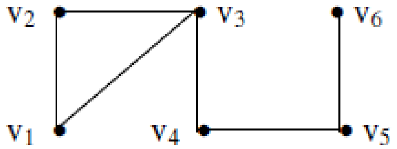
Solution: The given graph G contains circuits and we know that removal of the circuits gives a tree. So, we note that the figure below is a tree.



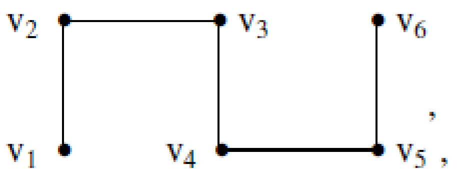
And the figure below is a spanning tree of the graph G.



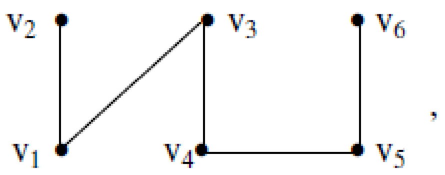
Example: Find all spanning trees for the graph G shown below:



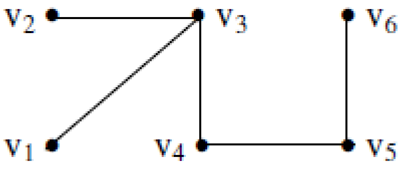
Solution: The given graph G has a circuit v_1 v_2 v_3 v_1. We know that removal of any edge of the circuit gives a tree. So the spanning trees of G are



T$_1$



T$_2$



T$_3$

Remark: We know that a tree with n vertices has exactly n – 1 edges. Therefore if G is a connected graph with n vertices and m edges, a spanning tree of G must have n – 1 edges. Hence the number of edges that must be removed before a spanning tree is obtained must be

$$m - (n - 1) = m - n + 1.$$

For Illustration, in the above example, n = 6, m = 6, so, we had to remove one edge to obtain a spanning tree.

Theorem: A graph G has a spanning tree if and only if G is connected.

Proof: Suppose first that a graph G has a spanning tree T. If v and w are vertices of G, then they are also vertices in T and since T is a tree there is a

path from v to w in T. This path is also a path in G. Thus every two vertices are connected in G. Hence G is connected.

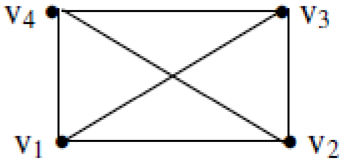
Conversely, suppose that G is connected. If G is acyclic, then G is its own spanning tree and we are done. So suppose that G contains a cycle C$_1$. If we remove an edge from the cycle, the subgraph of G so obtained is also connected. If it is acyclic, then it is a spanning tree and we are done. If not, it has at least one circuit, say C_2 . Removing one edge from C_2, we get a subgraph of G which is connected. Continuing in this way, we obtain a connected circuit free subgraph T of G. Since T contains all vertices of G, it is a spanning tree of G.

Cayley's Formula : The number of spanning trees of the complete graph K_n, $n \geq 2$ is n^{n-2}.

(Proof of this formula is out of scope of this book)

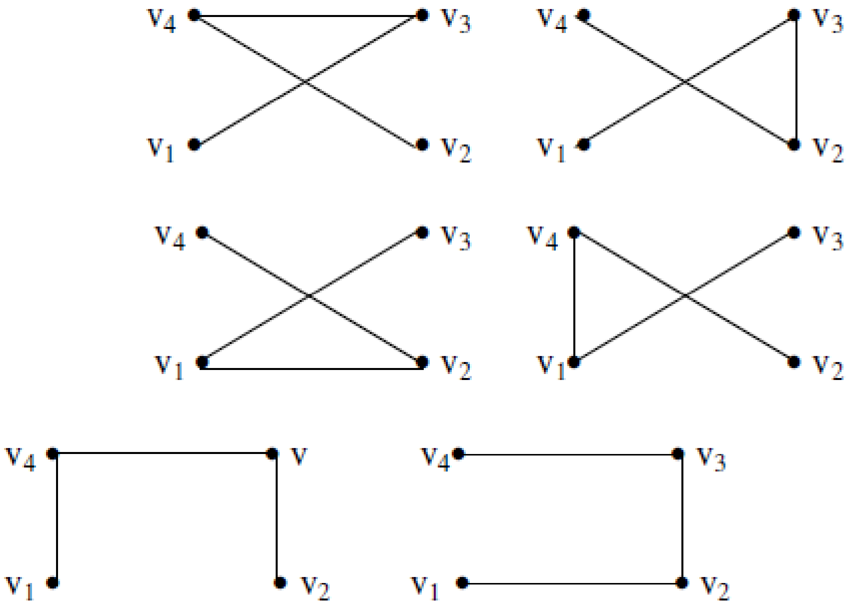
Example: Find all the spanning trees of K_4.

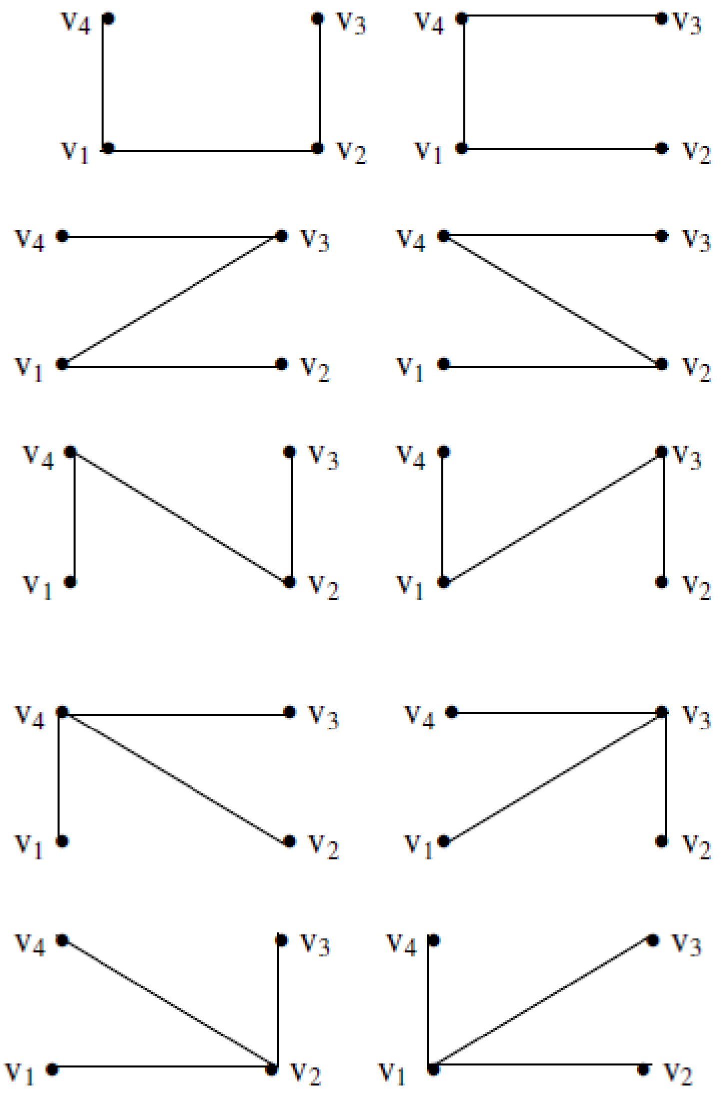
Solution: According to Cayley's formula, K_4 has $4^{4-2} = 4^2 = 16$ different spanning trees.



K_4

Here n = 4, so the number of edges in any tree should be $n - 1 = 4 - 1 = 3$. But here number of edges is equal to 6. So to get a tree, we have to remove three edges of K_4. The 16 spanning trees so obtained are shown below:





Minimal Spanning Tree

Definition : Let G be a weighted graph. A spanning tree of G with minimum weight is called **minimal spanning tree of G**.

Minimally connected graph :

A connected graph G is said to be minimally connected if removal of any edge from G disconnected the graph G.

Theorem :

A graph G is a tree iff it is minimally connected .

Proof:

Suppose that G is a tree.

We show G is minimally connected. Since G is a tree,it is connected .if G is not minimally connected then there must exist an edge e in G such that G-e is connected .

Therefore, e is an some circuit , which implies that G is not a tree, a contradiction.Thus G is minimally connected .

Conversely , suppose that G is a minimally connected graph.Then G is connected and cannot have a circuit; otherwise , we could remove one of the edge in the circuit and still leave the graph connected.Thus a minimally connected graph is a tree.

Minimum number of pendent vertices in a tree.

Recall that a pendent vertex in a graph is that vertex whose degree is one .In general,trees have several pendent vertices.The minimum number of pendent vertices ina tree is given by the following theorem .

Theorem

In any tree (with two or more vertices) there are atleast two pendent vertices .

Proof:

Let G be any tree having n vertices.Then G has n-1 edges.since each edge contributes two degrees,the sum of the degrees of all vertices in G is 2(n-1).

Now 2(n-1) degrees are to be divided amoung n vertices in G.

Let the number of vertices of degree one in G be x.

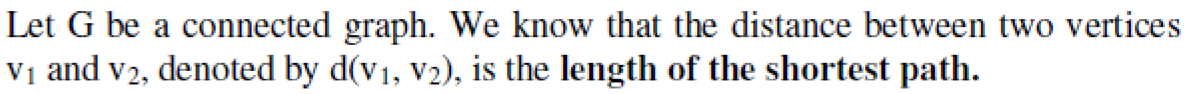
Since no vertex in a tree can be of zero degree,we have

$$\frac{2(n-1)-x}{n-x} \geq 2$$

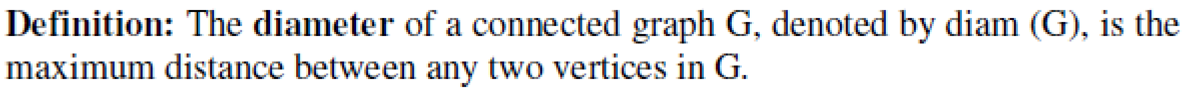
$$\rightarrow \quad x \geq 2$$

Thus , we must atleast two vertices of degree one is tree.

Distance and centre in a tree:

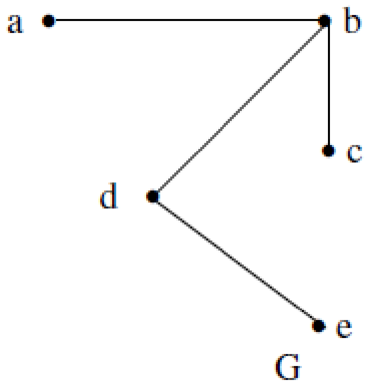


Let G be a connected graph. We know that the distance between two vertices v_1 and v_2, denoted by $d(v_1, v_2)$, is the **length of the shortest path.**



Definition: The **diameter** of a connected graph G, denoted by diam (G), is the maximum distance between any two vertices in G.

For example, in graph G shown below, we have



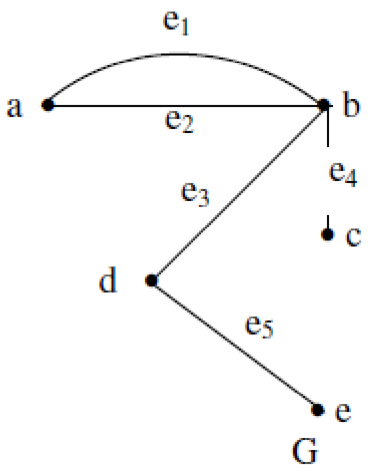
d(a, e) = 3, d(a, c) = 2, d(b, e) = 2 and diam (G) = 3.

Definition: A vertex in a connected graph G is called a **cut point** if G – v is disconnected, where G – v is the graph obtained from G by deleting v and all edges containing v.

For example, in the above graph, d is a cut point.

Definition: An edge e of a connected graph G is called a **bridge** (or cut edge) if G – e is disconnected, where G – e is the graph obtained by deleting the edge e.

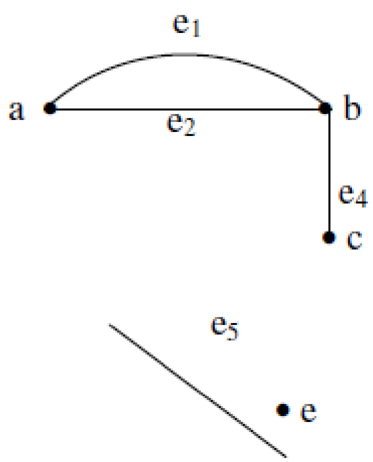
For example, consider the graph G shown below :



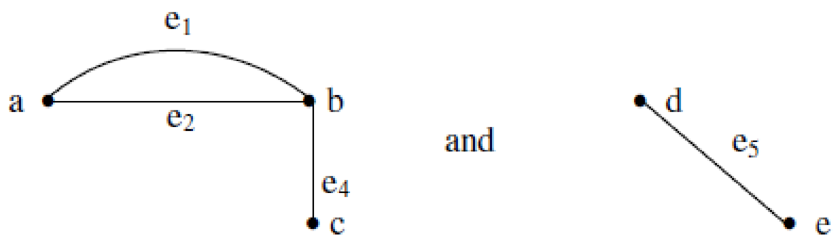
We observe that G – e_3 is disconnected. Hence the edge e_3 is a bridge.

Definition: A minimal set C of edges in a connected graph G is said to be a **cut set** (or **minimal edge – cut**) if the subgraph G – C has more connected components than G has.

For example, in the above graph, if we delete the edge (b, d) = e_3, the resulting subgraph G – e_3 is as shown below :



Thus G – e_3 has two connected components



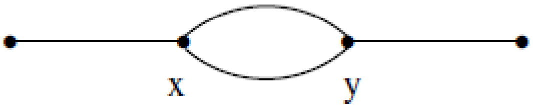
So, in this example, the cut set consists of single edge (b, d) = e_3, which is called edge or bridge.

Theorem: Let G be a connected graph with n vertices. Then G is a tree if and only if every edge of G is a bridge (cut edge).

(This theorem asserts that every edge in a tree is a bridge).

Proof: Let G be a tree. Then it is connected and has n – 1 edges (proved already). Let e be an arbitrary edge of G. Since G – e has n – 2 edges, and also we know that a graph G with n vertices has at least n – c(G) edges, it follows that n – 2 ≥ n – c(G – e). Thus G – e has at least two components. Thus removal of the edge e created more components than in the graph G. Hence e is a cut edge. This proves that every edge in a tree is a bridge.

Conversely, suppose that G is connected and every edge of G is a bridge. We have to show that G is a tree. To prove it, we have only to show that G is circuit – free. Suppose on the contrary that there exists a cycle between two points x and y in G. Then any edge on this cycle is



not a cut edge which contradicts the fact that every edge of G is a cut edge. Hence G has no cycle. Thus G is connected and acyclic and so is a tree.

Rank and Nullity:

Consider a graph G with n vertices , e edges and k components .The rank of graph G is defined as

Rank r = n-k

And the nullity of the graph G is defined as

Nullity µ=e-n+k

=e-r

We note that

Rank +nullity = no. of edges in a graph

The nullity of a graph is also called cyclomalic number or first Betti number.

If a graph G is connected then k=1 and therefore rank of a connected graph is n-1 and the nullity is e-n+1.

It follows from the definition of spanning tree that

Rank of a connected graph G = number of branches in any spanning tree of G

Nullity of connected graph G = number of chords in G

POSSIBLE QUESTIONS (SIX MARKS)

1.Show that a tree with n-vertices has (n-1) edges.

2.The number of pendent vertices (leaf) of a tree is equal to $\frac{n+1}{2}$

3. Show that every connected graph with n-vertices has (n-1) edges is a tree.

4.Show that a graph G is a tree if and only if it is minimally connected.

5.Show that an arborescence is a tree in which every vertex other than the root has an indegree of exactly one.

6. Show that a tree with n-vertices has (n-1) edges.

7.Define Centre and Eccentricity of vertex with example.

8. Show that a graph G is a tree if and only if there is one and only one path between any 2 vertices of G

9.Explain the properties of binary tree

10.Prove that in a tree, any two vertices are connected by exactly one path.

11.Show that every tree has one (or) two centre's**.**

POSSIBLE QUESTIONS (TEN MARKS)

1. In any tree (with two or more vertices), there are atleast two pendant vertices.

2.Prove that the number of labeled trees on 'n' vertices is n^{n-2}.

3.Show that the minimum height of a n-vertex binary tree is equal to [$\log_2$ (n+1)-1].

4.Show that in any tree with two (or) more vertices there are at least two pendent vertex

5.Show that every tree with two or more vertices is 2 chromatic**.**

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Advanced Discrete Mathematics **Subject Code: 17MMP105A**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
A tree with no nodes is a __________.	forest	graph	rooted tree	ordered tree	rooted tree
A single node with no children is a _____________.	forest	graph	rooted tree	ordered tree	rooted tree
A binary tree has a unique node is called the ____________ of the tree.	graph	edge	root	node	root
Two nodes having the same parent are called___________.	graph	siblings	root	node	siblings
A graph is said to be ___________ if there exists at least one path between every pair of vertices in G.	connected	disconnected	null graph	hamiltonian	connected
A tree with ________ vertices has at least two vertices of degree 1.	2	3	n	0	n
A tree with n vertices has __________ edges.	n	2	0	n-1	n-1
A ___________ is connected graph without circuit.	graph	directed graph	undirected graph	tree	tree
The total number of degrees of an isolated node is ___________.	0	1	2	3	0
The number of vertices of odd degree in a graph is always__________.	odd	even	zero	same number	even
The sum of the degrees of all vertices of a graph is equal to __________ the number of edges.	twice	thrice	same	any	twice
A tree is an ____________ graph.	cyclic	directed	null	acyclic	acyclic
The number of internal vertices in a binary tree with n vertices is _______________.	n	(n-2)/2	n-1	(n-1)/2	(n-1)/2
A ________ has at least two pendant vertices.	graph	tree	rooted tree	digraph	tree
A ________ can have more than one centre.	tree	graph	digraph graph	rooted tree	tree
Each column of an incidence matrix of a graph G has exactly________.	two 1s	three 1s	four 1s	five 1s	two 1s
A node with no children is called____________.	leaf	siblings	root	node	leaf
Every tree with two or more vertices is------------	4-chromatic	3-chromatic	2-chromatic	5-chromatic	2-chromatic
Every tree with -------------- vertics is 2-chromatic	greater than 2	less than 2	equal to 2	greater than or equal to 2	greater than or equal to 2
Every pendant edge in a graph included in ---------- covering of the graph	no	some	all	finite number of	all
A covering exists for a graph if the graph has no ------------------------	isolated vertex	odd vertex	even vertex	pendant vertex	isolated vertex
Every ------------------ in a graph included in every covering of the graph	pendant edge	odd vertex	even vertex	pendant vertex	pendant edge
Every --------with 2 or more vertics is 2-chromatic	tree	complete	connected	disconnected	tree

Reg No..............
(17MMP105A)

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE - 21
DEPARTMENT OF MATHEMATICS
First Semester
I INTERNAL TEST – AUG '17
Elective- I Advanced Discrete Mathematics

Date : **Time:2 hours**
Class :I M.Sc (Mathematics) **Maximum :50 Mark**

PART–A(20x1=20 Marks)

Answer All The Questions:

1**.**The Mathematical structure (G, *) is said to be a ---------------- if binary operation * satisfies closure property only.
a. Semigroup b. Quasigroup
c. Abelian Group d. Monoid

2. An equivalence relation R defined on the semigroup (S, *) is called -------if a R b and c R d then a*c R b*d.
a. congruence relation b. equivalence relation
c. Monoid d. Relation

3. The semigroup S/~ is called the --------- of S by ~.
a. equivalence class b. relation
c. quotient d. remainder

4. A function f: (S , *)→ (S',+) is called ------- if f(a*b) = f(a) + f(b)
a. Homomorphism b. automorphism
c. Isomorphism d. Relation

5. A set together with a number of operations on the set is called an -----------.
a. Monoid b. Group
c. Algebraic system d. Semigroup

6. The intersection of two congruence relation is -----------.
a. Not a congruence relation b. congruence relation
c. Direct product d. subalgebra

7. A function f: (S , *)→ (S',+) is called a homomorphism if
a. f(a*b) = f(a) + f(b) b. f(a+b) = f(a) + f(b)
c. f(a+b) = f(a) * f(b) d. f(a*b) = f(a) *f(b)

8. The algebraic structure (G, *) is said to be a semigroup if binary operation * satisfies -------.
a. closure and identity b. identity and inverse
c. closure and associativity d. associativity and identity

9. A lattice which has both a least element and a greatest element is called ------

a. sub lattice b. bounded lattice

c. complement lattice d. lattice homomorphism

10. The dual of $a + a (b+1) = a$ is ----------

a. $a \bullet(a + b \bullet 0) = a$ b. $a + (a + b \bullet 0) = a$

c. $a - (a + b \bullet 0) = a$ d. $a \bullet(a - b \bullet 0) = a$

11. A graph is ____________ if it has no parallel edges or self- loops.

a. simple b. directed

c. adjacent d. self-loop

12. The least member is denoted by -------

a. 0 b.1

c. -1 d.2

13. The greatest lower bound of $a,b \in L$ is denoted by $a*b$ and is also called -----

a. join b. sum

c. meet d. multiply

14. In a lattice, $\geq$ denotes --------------

a. addition of b. multiple of

c. divisor of d. subtraction of

15. A graph with only one vertex is -----------.

a. null graph b. trivial

c.empty d. Parallel

16. A simple graph G with n vertices is said to be a------------ if the degree of every vertex is n-1.

a. null graph b. regular graph

c. complete graph d. simple graph

17. If every element of L has atleast one complement then it is called -------------------

a. finite lattice b. infinte lattice

c. distributive lattice d. complemented lattice

18. Principle of duality is defined as ------

a. $\leq$ is replaced by $\geq$ b. LUB becomes GLB

c. are all properties unaltered when $\leq$ is replaced by $\geq$

d. all properties are unaltered when $\leq$ is replaced by $\geq$ other than 0 and 1 element

19. An edge having the same vertex as both its end vertices is called -----------.

a. graph b. tree

c. self-loop d. trival

20. A graph with no edges is ___________.

a. null graph b. trival

c. empty d. Parallel

Part – B (3x2= 6 Marks)

Answer All The Questions:

21.Define semigroup Homomorphism.

22. Show that in any Boolean algebra , (a+b)(a’+c)= ac+a’b+bc.

23. Write the some properties of lattices.

Part – C (3x8=24 Marks)

Answer All The Questions:

24. a) Show that the intersection of any two congruence relations on a set is also a congruence relation.

(OR)

b) Prove that under the semigroup homomorphism the properties associativity ,idempotency and commutative are preserved.

25. a) If a,b,c are elements of a distributive lattice (L, ⋀,⋁) then a⋁b =a⋁c and a⋀b=a⋀c show that b=c .

(OR)

b) Show that a sublattice of a

i) distributive lattice is distributive

ii)modular lattice is modular

26. Show that a lattice is distributive iff (a*b)+(b*c)+(c*a) = (a+b)*(b+c)*(c+a).

(OR)

b) Prove that every chain is a distributive lattice .

Reg No..............

(17MMP105A)

Karpagam Academy of Higher Education

COIMBATORE - 21

DEPARTMENT OF MATHEMATICS

First Semester

II INTERNAL TEST – SEP ‘17

Elective- I Advanced Discrete Mathematics

Date : **Time:2 hours**

Class :I M.Sc (Mathematics) **Maximum :50 Marks**

PART–A(20x1=20 Marks)

Answer All The Questions:

1.An element a in a lattice $(A, \leq)$ is called a ----------------------- if for every element $b \varepsilon A$, $a \leq b$.
a. lower bound b. Universal lower bound
c. Upper bound d. Universal upper bound

2. A lattice is said to be ------------------ if the meet operation distributes over the join operation.
a. Complemented b. Complete
c. Distributive d. Direct product

3. A lattice $(L, \leq)$ is said to be----------- if $a \leq c$ implies
$a \vee (b \wedge c) = (a \vee b) \wedge c$
a. unbounded lattice b. bounded lattice
c. modular lattice d. complemented lattice

4. A self-complemented, distributive lattice is called -------
a. Boolean algebra b. Modular lattice
c. Complete lattice d. Self dual lattice

5. A __________ is a graph whose components are all trees.
a. tree b. graph
c. forest d. walk

6. A graph is ___________ if it has no parallel edges or self-loops.
a. simple b. directed
c. adjacent d. self-loop

7. A graph in which some edges are directed and some are undirected is called_______________.
a. mixed graph b. regular graph
c. complete graph d. simple graph

8. Every graph is its own ______________.
a . mixed graph b. sub graph
c . complete graph d. simple graph

9. A tree in which one vertex is distinguished from all others is called a __________.
a. tree b. rooted tree
c. connected d. pendant vertex

10. ___________ is also called cycle
a. walk b. closed walk
c. circuit d. path

11. If no vertex appears more than once in an open walk then it is called a __________.
a. walk b. closed walk
c. circuit d. path

12. The number of edges in a path is called the __________ of the path
a. length b. same
c. walk d. circuit

13. A graph in which weights or distance are assigned to each edge is called a ______.

a. complete graph b. simple graph
c. mixed graph d. weighed graph

14. Two nodes having the same parent are called______

a. graph b. siblings
c. root d. node

15. The total number of degrees of an isolated node is _______

a. 0 b.1
c.2 d.3

16. A tree is an ______ graph.

a. cyclic b. directed
c. null d. Acyclic

17. A tree with ________ vertices has at least two vertices of degree

a. 2 b. 3
c. n d. 0

18. A ______ is connected graph without circuit.

a. graph b. directed graph
c. undirected graph d. tree

19. A graph is said to be ____ if there exists at least one path between every pair of vertices in G.

a. connected b. disconnected
c.Null graph d. Hamiltonian

20. A binary tree has a unique node is called the ___ of the tree.

a. graph b. edge
c.Root d. Node

Part – B (3x2= 6 Marks)

Answer All The Questions:

21.Define Graph Isomorphism.

22. Define Connected Graph with Example.

23. Define a tree.

Part – C (3x8=24 Marks)

Answer All The Questions:

24. a) If (L, ∧,∨) is a complemented and distributive lattice , then the complement a of any element a∈L is unique.

(OR)

b) Prove that join and meet operations are associative.

25. a) Show that a simple graph with n vertices and k-component can have at most

$$\frac{(n-k)(n-k+1)}{2}$$

(OR)

b) The number of vertices of odd degree in a graph is even.

26. a) Show that a tree with n-vertices has (n-1) edges.

(OR)

b). (i).Show that a graph G is a tree if and only if it is minimally connected.

ii)Define Centre and Eccentricity of vertex with example.