KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: Mechanics	**Subject Code: 17MMP106**	**L T P C**
Class:I M.Sc	**Semester:III**	**4 0 0 4**

PO: This course provides a strong foundation in understanding the concepts of mechanics and to know how the friction is regulating the motion of objects**,** deep knowledge about the motion of particles under the influence of various forces like gravitational force, central force, impulsive force etc., which plays a vital role in Applied Mathematics.

PLO: To be familiar with D'Alembert's principle, Lagrange's equations, Extension of Hamilton's principle, Cyclic coordinates, and Canonical transformations and to be exposed with Hamilton Jacobi Theory**.**

UNIT I

Survey of Elementary principles: Constraints - Generalized coordinates, Holonomic and non-holonomic systems, Scleronomic and Rheonomic systems. D'Alembert's principle and Lagrange's equations – Velocity – dependent potentials and the dissipation function – some applications of the Lagrange formulation.

UNIT II

Variation principles and Lagrange's equations: Hamilton's principle – Some techniques of calculus of variations – Derivation of Lagrange's Equations from Hamilton's principle – Extension of Hamilton's principle to non-holonomic systems – Conservation theorems and symmetry properties.

UNIT III

Hamilton Equations of motion: Legendre Transformations and the Hamilton Equations of motion-canonical equations of Hamilton – Cyclic coordinates and conservation theorems – Routh's procedure - Derivation of Hamilton's equations from a variational principle – The principle of least action.

UNIT IV

Canonical transformations: The equations of canonical transformation – Examples of Canonical transformations – Poission Brackets and other Canonical invariants – integral invariants of Poincare, Lagrange brackets.

UNIT V

Hamilton Jacobi Theory: Hamilton Jacobi equations for Hamilton's principle function – Harmonic oscillator problem - Hamilton Jacobi equation for Hamilton's characteristic function – Separation of variables in the Hamilton-Jacobi equation.

SUGGESTED READINGS

TEXT BOOK

T1: Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1: Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.

R2: Gelfand, I. M., and Fomin, S. V., (2003),Calculus of Variations, Prentice Hall, New Delhi.

R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

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DEPARTMENT OF MATHEMATICS

Subject: Mechanics **Subject Code: 17MMP106** **L T P C**

Class:II M.Sc **Semester:I** **4 0 0 4**

S.No	Lecture Duration (Hr)	Topics to be covered	Support Materials
	UNIT-I		
1	1	Survey of Elementary principles	T1: Ch1: P.No: 1-3
2	1	Continuation of Survey of Elementary principles	T1:Ch2: P.No: 4-6
3	1	Continuation of Survey of Elementary principles	T1:Ch2: P.No: 7-10
4	1	Constraints	T1:Ch2: P.No:11-12
5	1	Generalized coordinates	T1:Ch2: P.No: 12-13
6	1	Holonomic and non-holonomic systems, Scleronomic and Rheonomic systems	T1:Ch2: P.No: 13-15
7	1	D'Alembert's principle	T1:Ch2: P.No: 16-17
8	1	Lagrange's equations	T1:Ch3: P.No: 17-21
9	1	Velocity- dependent potentials and the dissipation function	T1:Ch3: P.No: 21-22
10	1	Continuation of Velocity- dependent potentials and the dissipation function	T1:Ch3: P.No: 23-24
11	1	some applications of the Lagrange formulation.	T1:Ch3: P.No:25-26
12	1	Continuation of some applications of the Lagrange formulation.	T1:Ch3: P.No: 27-29

13	1	Recapitulation and Discussion of possible questions	
Total	**13 hrs**		

TEXT BOOK

T_1. Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1:Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
R2:Gelfand, I. M., and Fomin, S. V., (2003),Calculus of Variations, Prentice Hall, New Delhi.
R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

UNIT-II			
1	1	Variation principles and Lagrange's equations Hamilton's principle	T1:Ch2: P.No: 35-37
2	1	Some techniques of calculus of variations	T1:Ch2: P.No: 37-39
3	1	Continuation of Some techniques of calculus of variations	T1:Ch2: P.No: 40-41
4	1	Continuation of Some techniques of calculus of variations	T1:Ch2: P.No: 41-43
5	1	Derivation of Lagrange's Equations from Hamilton's principle	T1:Ch2: P.No: 43-45
6	1	Extension of Hamilton's principle to non-holonomic systems	T1:Ch2: P.No: 45-47
7	1	Continuation of Extension of Hamilton's principle to non-holonomic systems	T1:Ch2: P.No: 47-48
8	1	Continuation of Extension of Hamilton's principle to non-holonomic systems	T1:Ch2: P.No: 48-50
9	1	Conservation theorems and symmetry properties.	T1:Ch2: P.No: 54-56
10	1	Continuation of Conservation theorems and symmetry properties.	T1:Ch2: P.No: 56-58
11	1	Continuation of Conservation theorems and symmetry properties.	T1:Ch2: P.No: 58-60

12	1	Continuation of Conservation theorems and symmetry properties.	T1:Ch2: P.No: 61-63
13	1	Recapitulation and Discussion of possible questions	
Total	**13 hrs**		

TEXT BOOK

T_1. Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1:Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
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R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

UNIT-III			
1	1	Legendre Transformations	T1:Ch8: P.No: 339-341
2	1	Continuation of Legendre Transformations	T1: Ch8: P.No: 341-343
3	1	Hamilton Equations of motion	T1: Ch8: P.No: 343-345
4	1	Continuation of Hamilton Equations of motion	T1: Ch8: P.No: 345-347
5	1	canonical equations of Hamilton	T1:Ch8 P.No: 347-348
6	1	Continuation of canonical equations of Hamilton	T1: Ch8: P.No: 349-350
7	1	Cyclic coordinates and conservation theorems	T1: Ch8: P.No: 351-353
8	1	Routh's procedure	T1: Ch8: P.No: 353-356
9	1	Derivation of Hamilton's equations from a variational principle	T1: Ch8: P.No: 362-363
10	1	Continuation of Derivation of Hamilton's equations from a variational principle	T1: Ch8: P.No: 364-365
11	1	The principle of least action.	T1: Ch8: P.No: 365-367
12	1	Continuation of The principle of least action.	T1: Ch8: P.No: 368-371

13	1	Recapitulation and Discussion of possible questions	
Total	**13 hrs**		

TEXT BOOK

T_1. Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1:Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
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R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

UNIT-IV			
1	1	The equations of canonical transformation	T1:Ch9: P.No: 373-375
2	1	Continuation of the equations of canonical transformation	T1:Ch9: P.No: 375-377
3	1	Examples of Canonical transformations	T1:Ch9: P.No: 379-381
4	1	Examples of Canonical transformations	T1:Ch9: P.No: 382-384
5	1	Poission Brackets and other Canonical invariants	R1:Ch7: P.No: 292-294
6	1	Continuation of Poission Brackets and other Canonical invariants	T1:Ch9: P.No: 385-387
7	1	integral invariants of Poincare	T1:Ch9: P.No: 388-390
8	1	Continuation of integral invariants of Poincare	R3:Ch2: P.No: 56-58
9	1	Lagrange brackets	T1:Ch9: P.No: 391-393
10	1	Continuation of Lagrange brackets	T1:Ch9: P.No: 395-397
11	1	Recapitulation and Discussion of possible questions	
Total	**9 hrs**		

TEXT BOOK

T_1. Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1:Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
R2:Gelfand, I. M., and Fomin, S. V., (2003),Calculus of Variations, Prentice Hall, New Delhi.
R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

UNIT-V

1	1	Hamilton Jacobi Theory	T1:Ch10: P.No:400-401
2	1	Hamilton Jacobi equations for Hamilton's principle function	T1:Ch10: P.No: 402-404
3	1	Continuation of Hamilton Jacobi equations for Hamilton's principle function	T1:Ch10: P.No: 406-408
4	1	Harmonic oscillator problem	R2:Ch5: P.No: 76-78
5	1	Continuation of Harmonic oscillator problem	R2:Ch5: P.No: 79-81
6	1	Continuation of Harmonic oscillator problem	T1:Ch10: P.No: 410-412
7	1	Hamilton Jacobi equation for Hamilton's characteristic function	T1:Ch10: P.No: 413-414
8	1	Separation of variables in the Hamilton-Jacobi equation.	
9	1	Recapitulation and discussion of possible questions on unit V	
10	1	Discussion of Previous year ESE question paper	
11	1	Discussion of Previous year ESE question paper	
12	1	Discussion of Previous year ESE question paper	
Total	**12 hrs**		

TEXT BOOK

T_1. Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1:Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.
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DEPARTMENT OF MATHEMATICS

Subject: Mathematical Modeling	**Subject Code: 16MMP303**	**L T P C**
Class:II M.Sc	**Semester:III**	**4 1 0 4**

UNIT I

Mathematical Modeling through Ordinary Differential Equations of First order: Linear Growth and Decay Models – Non-Linear Growth and Decay Models – Compartment Models – Dynamics problems – Geometrical problems.

SUGGESTED READINGS

TEXT BOOK

T1: J.N. Kapur, (2015). Mathematical Modeling, Wiley Eastern Limited, New Delhi.

REFERENCES

R1: Kapur, J. N., (1985). Mathematical Models in Biology and Medicine, Affiliated East –West Press Pvt Limited, New Delhi.

R3: Frank. R. Giordano, Maurice. D.Weir, WilliamP. Fox, (2003). A first course in Mathematical Modelling, Vikash Publishing House, UK.

MATHEMATICAL MODELLING THROUGH DIFFERENTIAL EQUATIONS

Mathematical Modelling in terms of differential equations arises when the situation modelled involves some *continuous* variable(s) varying with respect to some other continuous variable(s) and we have some reasonable hypotheses about the *rates of change* of dependent variable(s) with respect to independent variable(s).

When we have one dependent variable x (say population size) depending on one independent variable (say time t), we get a mathematical model in terms of an *ordinary differential equation* of the *first order*, if the hypothesis is about the rate of change dx/dt. The model will be in terms of an *ordinary differential equation of the second order* if the hypothesis involves the rate of change of dx/dt.

If there are a number of dependent continuous variables and only one independent variable, the hypothesis may give a mathematical model in terms of a *system of first or higher order ordinary differential equations.*

If there is one dependent continuous variable (say velocity of fluid u) and a number of independent continuous variables (say space coordinates x, y, z and time t), we get a mathematical model in terms of a *partial differential equation.* If there are a number of dependent continuous variables and a number of independent continuous variables, we can get a mathematical model in terms of systems of *partial differential equations.*

LINEAR GROWTH AND DECAY MODELS

Populational Growth Models

Let $x(t)$ be the population size at time t and let b and d be the birth and death rates, i.e. the number of individuals born or dying per individual per unit time, then in time interval $(t, t + \Delta t)$, the numbers of births and deaths would be $bx\,\Delta t + 0(\Delta t)$ and $dx\,\Delta t + 0(\Delta t)$ where $0(\Delta t)$ is an infinitesimal which approaches zero as Δt approaches zero, so that

$$x(t + \Delta t) - x(t) = (bx(t) - dx(t))\Delta t + 0(\Delta t), \qquad (1)$$

so that dividing by Δt and proceeding to the limit as $\Delta t \to 0$, we get

$$\Rightarrow \frac{dx}{dt} = (b - d)x = ax \quad \text{(say)} \tag{2}$$

Integrating (2), we get

$$x(t) = x(0) \exp(at), \tag{3}$$

so that the population grows exponentially if $a > 0$, decays exponentially if $a < 0$ and remains constant if $a = 0$ (Figure 2.1)

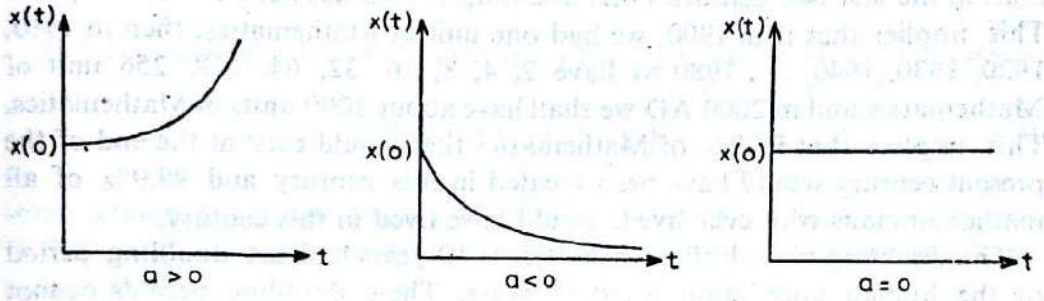

Figure 2.1

(i) If $a > 0$, the population will become double its present size at time T, where

$$2x(0) = x(0) \exp(aT) \quad \text{or} \quad \exp(aT) = 2$$

or
$$T = \frac{1}{a} \ln 2 = (0.69314118)a^{-1} \tag{4}$$

T is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$. It depends only on a and is such that greater the value of a (i.e. greater the difference between birth and death rates), the smaller is the doubling period.

(ii) If $a < 0$, the population will become half its present size in time T' when

$$\frac{1}{2}x(0) = x(0) \exp(aT') \quad \text{or} \quad \exp(aT') = \frac{1}{2}$$

or
$$T' = \frac{1}{a} \ln \frac{1}{2} = -(0.69314118)\, a^{-1} \tag{5}$$

It may be noted that T' is also independent of $x(0)$ and since $a < 0$, $T' > 0$. T' may be called the half-life (period) of the population and it decreases as the excess of death rate over birth rate increases.

Growth of Science and Scientists

Let $S(t)$ denote the number of scientists at time t, $bS(t)\Delta t + 0(\Delta t)$ be the number of new scientists trained in time interval $(t\ t + \Delta t)$ and let

$dS(t)\Delta t + 0(\Delta t)$ be the number of scientists who retire from science in the same period, then the above model applies and the number of scientists should grow exponentially.

The same model applies to the growth of Science, Mathematics and Technology. Thus if $M(t)$ is the amount of Mathematics at time t, then the rate of growth of Mathematics is proportional to the amount of Mathematics, so that

$$dM/dt = aM \quad \text{or} \quad M(t) = M(0) \exp(at) \tag{6}$$

Thus according to this model, Mathematics, Science and Technology grow at an exponential rate and double themselves in a certain period of time. During the last two centuries this doubling period has been about ten years. This implies that if in 1900, we had one unit of Mathematics, then in 1910, 1920, 1930, 1940, . . . 1980 we have 2, 4, 8, 16, 32, 64, 128, 256 unit of Mathematics and in 2000 AD we shall have about 1000 units of Mathematics. This implies that 99.9% of Mathematics that would exist at the end of the present century would have been created in this century and 99.9% of all mathematicians who ever lived, would have lived in this century.

The doubling period of mathematics is 10 years and the doubling period of the human population is 30-35 years. These doubling periods cannot obviously be maintained indefinitely because then at some point of time, we shall have more mathematicians than human beings. Ultimately the doubling period of both will be the same, but hopefully this is a long way away.

This model also shows that the doubling period can be shortened by having more intensive training programmes for mathematicians and scientists and by creating conditions in which they continue to do creative work for longer durations in life.

Effects of Immigration and Emigration on Population Size

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing the birth rate. Similarly if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate.

If however immigration and emigration take place at constant rate i and e respectively, equation (3) is modified to

$$\frac{dx}{dt} = bx - dx + i - e = ax + k \tag{7}$$

Integrating (7) we get

$$x(t) + \frac{k}{a} = \left(x(0) + \frac{k}{a}\right)e^{at} \tag{8}$$

The model also applies to growth of populations of bacteria and micro-organisms, to the increase of volume of timber in forest, to the growth of malignant cells etc. In the case of forests, planting of new plants will correspond to immigration and cutting of trees will correspond to emigration.

Radio-Active Decay

Many substances undergo radio-active decay at a rate proportional to the amount of the radioactive substance present at any time and each of them has a half-life period. For uranium 238 it is 4.55 billion years. For potassium it is 1.3 billion years. For thorium it is 13.9 billion years. For rubidium it is 50 billion years while for carbon 14, it is only 5568 years and for white lead it is only 22 years.

In radiogeology, these results are used for radioactive dating. Thus the ratio of radio-carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. Radioactive dating has also been used to estimate the age of the solar system and of earth as 45 billion years.

2.2.7 Diffusion

According to Fick's law of diffusion, the time rate of movement of a solute across a thin membrane is proportional of the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at a and the concentration of the solution on the other side initially is $c_0 < a$, then Fick's law gives

$$\frac{dc}{dt} = k(a - c), \quad c(0) = c_0, \tag{15}$$

so that

$$a - c(t) = (a - c(0))e^{-kt} \tag{16}$$

and $c(t) \to a$ as $t \to \infty$, whatever be the value of c_0.

Change of Price of a Commodity

Let $p(t)$ be the price of a commodity at time t, then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the commodity in the market so that

$$\frac{dp}{dt} = k(d(t) - s(t)), \tag{17}$$

where $k > 0$, since if demand is more than the supply, the price increases. If $d(t)$ and $s(t)$ are assumed linear functions of $p(t)$, i.e. if

$$d(t) = d_1 + d_2p(t), \quad s(t) = s_1 + s_2p(t), \quad d_2 < 0, s_2 > 0 \tag{18}$$

we get

$$\frac{dp}{dt} = k(d_1 - s_1 + (d_2 - s_2)p(t)) = k(a - \beta p(t)), \quad \beta > 0 \tag{19}$$

or

$$\frac{dp}{dt} = K(p_e - p(t), \tag{20}$$

where p_e is the equilibrium price, so that

$$p_e - p(t) = (p_e - p(0))e^{-kt} \tag{21}$$

and

$$p(t) \to p_e \quad \text{as} \quad t \to \infty$$

Logistic Law of Population Growth

(i) $x(0) < a/b \Rightarrow x(t) < a/b \Rightarrow dx/dt > 0 \Rightarrow x(t)$ is a monotonic increasing function of t which approaches a/b as $t \to \infty$.

(ii) $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Rightarrow x(t)$ is a monotonic decreasing function of t which approaches a/b as $t \to \infty$.

Now from (23)

$$\frac{d^2x}{dt^2} = a - 2bx, \tag{25}$$

so that $d^2x/dt^2 \gtreqless 0$ according as $x \gtreqless a/2b$. Thus in case (i) the growth curve is convex if $x < a/2b$ and is concave if $x > a/2b$ and it has a point of inflexion at $x = a/2b$. Thus the graph of $x(t)$ against t is as given in Figure 2.2.

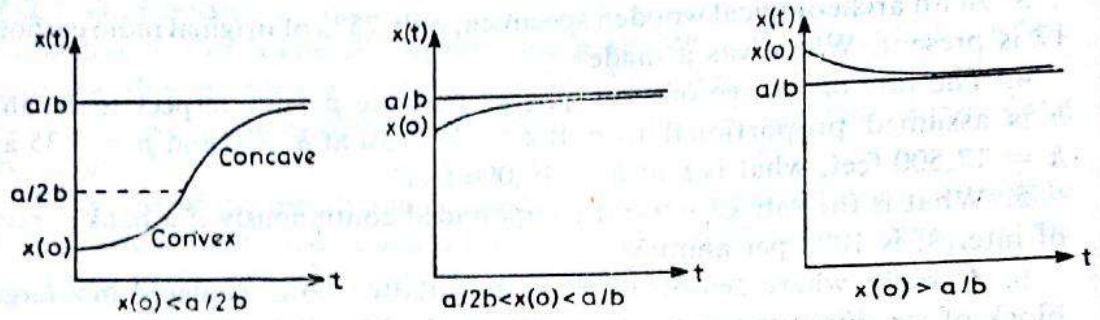

Figure 2.2

—If $x(0) < a/2b$, $x(t)$ increases at an increasing rate till $x(t)$ reaches $a/2b$ and then it increases at a decreasing rate and approaches a/b at $t \to \infty$

—If $a/2b < x(0) < a/b$, $x(t)$ increases at a decreasing rate and approaches a/b as $t \to \infty$

—If $x(0) = a/b$, $x(t)$ is always equal to a/b

—If $x(0) > a/b$, $x(t)$ decreases at a decreasing absolute rate and approaches a/b as $t \to \infty$

Spread of Technological Innovations and Infectious Diseases

Let $N(t)$ be the number of companies which have adopted a technological innovation till time t, then the rate of change of the number of these companies depends both on the number of companies which have adopted this innovation and on the number of those which have not yet adopted it, so that if R is the total number of companies in the region

$$\frac{dN}{dt} = kN(R - N), \tag{26}$$

which is the logistic law and shows that ultimately all companies will adopt this innovation.

Similarly if $N(t)$ is the number of infected persons, the rate at which the number of infected persons increases depends on the product of the numbers of infected and susceptible persons. As such we again get (26), where R is the total number of persons in the system.

It may be noted that in both the examples, while $N(t)$ is essentially an integer-valued variable, we have treated it as a continuous variable. This can be regarded as an idealisation of the situation or as an approximation to reality.

Rate of Dissolution

Let $x(t)$ be the amount of undissolved solute in a solvent at time t and let c_0 be the maximum concentration or saturation concentration, i.e. the maximum amount of the solute that can be dissolved in a unit volume of the solvent. Let V be the volume of the solvent. It is found that the rate at which the solute is dissolved is proportional to the amount of undissolved solute and to the difference between the concentration of the solute at time t and the maximum possible concentration, so that we get

$$\frac{dx}{dt} = kx(t)\left(\frac{x(0) - x(t)}{V} - c_0\right) = \frac{kx(t)}{V}((x_0 - c_0V) - x(t)) \tag{27}$$

Law of Mass Action: Chemical Reactions

Two chemical substances combine in the ratio $a:b$ to form a third substance Z. If $z(t)$ is the amount of the third substance at time t, then a proportion $az(t)/(a+b)$ of it consists of the first substance and a proportion $bz(t)/(a+b)$ of it consists of the second substance. The rate of formation of the third substance is proportional to the product of the amount of the two component substances which have not yet combined together. If A and B are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right) \tag{28}$$

This is the non-linear differential equation for a second order reaction. Similarly for an nth order reaction, we get the non-linear equation

$$\frac{dz}{dt} = k(A_1 - a_1z)(A_2 - a_2z)\ldots(A_n - a_nz), \tag{29}$$

where $a_1 + a_2 + \ldots + a_n = 1$.

EXERCISE

1. If in (24), $a = 0.03134$, $b = (1.5887)(10)^{-10}$, $x(0) = 39\times10^6$, show that

$$x(t) = \frac{313{,}400{,}000}{1.5887 + 78{,}7703^{-0.03134t}}$$

This is Verhulst model for the population of USA when time zero corresponds to 1790. Estimate the population of USA in 1800, 1850, 1900 and 1950. Show that the point of inflexion should have occurred in about 1914. Find also the limiting population of USA on the basis of this model.

2. In (26) $k = 0.007$, $R = 1000$, $N(0) = 50$, find $N(10)$ and find when $N(t) = 500$.

COMPARTMENT MODELS

In the last two sections, we got mathematical models in terms of ordinary differential equations of the first order, in all of which variables were separable. In the present section, we get models in terms of linear differential equations of first order.

We also use here the principle of continuity i.e. that the gain in amount of a substance in a medium in any time is equal to the excess of the amount that has entered the medium in the time over the amount that has left the medium in this time.

A Simple Compartment Model

Let a vessel contain a volume V of a solution with concentration $c(t)$ of a substance at time t (Figure 2.3) Let a solution with constant concentration C in an overhead tank enter the vessel at a constant rate R and after mixing thoroughly with the solution in the vessel, let the mixture with concentration $c(t)$ leave the vessel at the same rate R so that the volume of the solution in the vessel remains V.

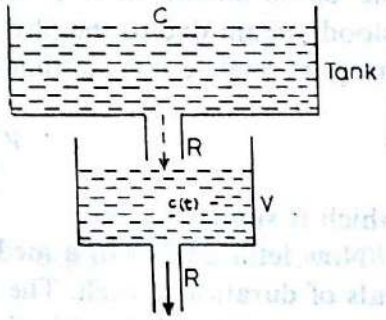

Figure 2.3

Using the principle of continuity, we get

$$V(c(t + \Delta t) - c(t)) = RC\Delta t - Rc(t)\Delta t + 0(\Delta t)$$

giving

$$V\frac{dc}{dt} + Rc = RC \tag{30}$$

Integrating

$$c(t) = c(0)\exp\left(-\frac{R}{V}t\right) + C\left(1 - \exp\left(-\frac{R}{V}t\right)\right) \tag{31}$$

As $t \to \infty$, $c(t) \to C$, so that ultimately the vessel has the same concentration as the overhead tank. Since

$$c(t) = C - (C - c_0)\exp\left(-\frac{R}{V}t\right), \tag{32}$$

if $C > c_0$, the concentration in the vessel increases to C; on the other hand if $C < c_0$, the concentration in the vessel decreases to C (Figure 2.4).

If the rate R' at which the solution leaves the vessel is less than R, the equations of continuity gives

$$\frac{d}{dt}[(V_0 + (R - R')t)c(t)] = RC - R'(ct) \qquad (33)$$

where V is the initial volume of the solution in the vessel. This is also a linear differential equation of the first order.

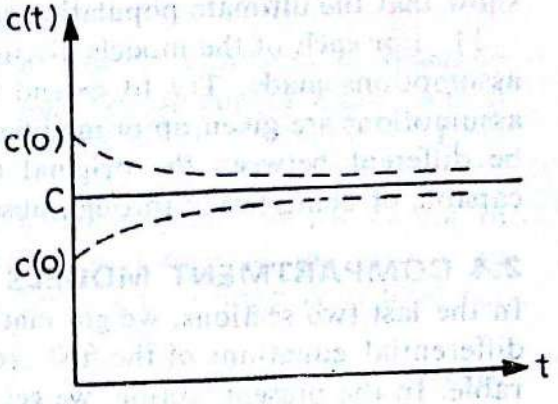

Figure 2.4

Diffusion of Glucose or a Medicine in the Blood Stream

Let the volume of blood in the human body be V and let the initial concentration of glucose in the blood stream be $c(0)$. Let glucose be introduced in the blood stream at a constant rate I. Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate proportional to $c(t)$, so that the continuity principle gives

$$V\frac{dc}{dt} = I - kc \qquad (34)$$

which is similar to (30).

Now let a dose D of a medicine be given to a patient at regular intervals of duration T each. The medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the blood stream, then the differential equation given by the continuity principle is

$$V\frac{dc}{dt} = -kc \qquad (35)$$

Integrating

$$c(t) = D\exp\left(-\frac{k}{V}t\right), \quad 0 \leqslant t < T \qquad (36)$$

At time T, the residue of the first dose is $D\exp\left(-\frac{k}{V}T\right)$ and now another dose D is given so that we get

$$c(t) = \left(D \exp\left(-\frac{k}{V}T\right) + D\right) \exp\left(-\frac{k}{V}(t-T)\right), \quad (37)$$

$$= D \exp\left(-\frac{k}{V}t\right) + D \exp\left(-\frac{k}{V}(t-T)\right), \quad (38)$$

$$T \leqslant t < 2T$$

The first term gives the residual of the first dose and the second term gives the residual of the second dose. Proceeding in the same way, we get after n doses have been given

$$c(t) = D \exp\left(-\frac{k}{V}t\right) + D \exp\left(-\frac{k}{V}(t-T)\right) + D \exp\left(-\frac{k}{V}(t-2T)\right) + \ldots + D \exp\left(-\frac{k}{V}(t-\overline{n-1}T)\right) \quad (39)$$

$$= D \exp\left(-\frac{k}{V}t\right)\left(1 + \exp\left(\frac{k}{V}T\right) + \exp\left(\frac{2k}{V}T\right) + \ldots + \exp\left((n-1)\frac{k}{V}T\right)\right)$$

$$= D \exp\left(-\frac{k}{V}t\right)\frac{\exp\left(n\frac{k}{V}T\right) - 1}{\exp\left(\frac{k}{V}T\right) - 1}, \quad (n-1)T \leqslant t < nT \quad (40)$$

$$c(nT - 0) = D\frac{1 - \exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right) - 1} \quad (41)$$

$$c(nT + 0) = D\frac{\exp\left(\frac{kT}{V}\right) - \exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right) - 1} \quad (42)$$

Thus the concentration never exceeds $D/\left(1 - \exp\left(-\frac{kT}{V}\right)\right)$. The graph of $c(t)$ is shown in Figure 2.5.

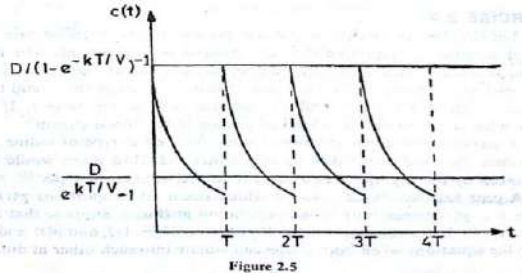

Figure 2.5

Thus in each interval, concentration decreases. In any interval, the concentration is maximum at the beginning of this interval and thus maximum concentration at the beginning of an interval goes on increasing as the number of intervals increases, but the maximum value is always below $D/(1 - e^{-kT/V})$. The minimum value in an interval occurs at the end of each interval. This also increases, but it lies below $D/(\exp(kT/V) - 1)$.

The concentration curve is piecewise continuous and has points of discontinuity at $T, 2T, 3T, \ldots$

By injecting glucose or penicillin in blood and fitting curve (36) to the data, we can estimate the value of k and V. In particular this gives a method for finding the volume of blood in the human body.

2.4.3 The Case of a Succession of Compartments

Let a solution with concentration $c(t)$ of a solute pass successively into n tanks in which the initial concentrations of the solution are $c_1(0), c_2(0), \ldots, c_n(0)$. The rates of inflow in each tank is the same as the rate of outflow from the tank. We have to find the concentrations $c_1(t), c_2(t) \ldots c_n(t)$ at time t. We get the equations

$$\begin{aligned} V\frac{dc_1}{dt} &= Rc - Rc_1 \\ V\frac{dc_2}{dt} &= Rc_1 - Rc_2 \\ &\cdots\cdots\cdots\cdots \\ V\frac{dc_n}{dt} &= Rc_{n-1} - Rc_n \end{aligned} \tag{43}$$

By solving the first of these equations, we get $c_1(t)$. Substituting the value of $c_1(t)$ and proceeding in the same way, we can find $c_3(t), \ldots, c_n(t)$.

MATHEMATICAL MODELLING IN DYNAMICS THROUGH ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

Let a particle travel a distance x in time t in a straight line, then its velocity v is given by dx/dt and its acceleration is given by

$$dv/dt = (dv/dx)(dx/dt) = vdv/dx = d^2x/dt^2$$

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
UNIT-I

Subject: Mechanics **Subject Code: 17MMP106**

Survey of Elementary principles
Part-A(20X1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Multiple Choice Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If the total external force is zero,then the total lionear momentum of the system is _____	Zero	non-zero	conserved	rigid	conserved
Conservation of total angular momentu in the absence of_____requires strong law of action and reaction	Applied force	torque	force	applied torque	applied torque
The equation of motion of the form_____is always a suitable way to construct a lagrangian for a conservative system	L=T+V	L=V-T	L=T-V	L=T	L=T-V
The _____- doesn't deform under the action of loads	Force	Rigid body	varied path	applied torque	rigid body
A bead sliding on a rigid wire on space is subject to_____constraint	Scheleronomous	Rheonomous	holonomic	non holonomic	Scheleronomous
The space around the magnet is called the_____	magnetic field	electric field	magnetic induction	flux	magnetic field
The number of coordinates minus the number of independent equations of constraint is_____	units	number of degrees of freedom	dimensions	number of generalized coordinates	number of degrees of freedom
____ have no effect on the motion of the centre of mass	applied torque	internal force	external force	force	internal force
The total kinetic energy of the system can be wriiten as_____	$T=T_o+T_1$	$T=T_1+T_2$	$T=T_o+T_2$	$T=T_o+T_1+T_2$	$T=T_o+T_1+T_2$
the equation of motion is the differential equation of _____order	Fourth	first	second	third	second
To linearize the motion for small oscillation,we assume that_____	$\cos(\phi-\theta)\approx1$	$\sin(\phi-\theta)\approx1$	$\cos(\phi-\theta)\approx0$	$\sin(\phi-\theta)\approx0$	$\cos(\phi-\theta)\approx1$
The total amount of electric field lines or the magnetic field lines passing through an area is called a _____	Flux	density	torque	current	flux
The motion of a body in a straight line is known as_____	linear motion	linear displacement	rotational motion	moment of force	linear motion
_____is the quantity of matter contained in the body	velocity	mass	force	weight	mass
Kinetic energy of a body is th energy possessed by the body of virtue of its___	position	energy	constant	motion	motion
_____of a body is defined as the product of mass of a body and its velocity	density	momentum	force	acceleration	momentum
Physical forces like gravity obeys_____of gravitational force	weak law	strong law	Maxwell's law	none of the above	Strong law
____of a body is the energy it posses by the virtue of its position	kinetic energy	total energy	potential energy	energy	potential energy
____of a body is the energy possessed by the virtue of its motion	potential energy	energy	kinetic energy	total energy	kinetic energy
The lagrangian L=T-U where U is called a_____	generalise force	generalised potential	generalised momentum	generalised acceleration	generalised potential
The _____is denoted by the basic field vector B	flux	magnetic induction	electric field	magnetic field	magnetic induction
The _____is a measure of the number of magnetic flux lines passing per unit are through a surface normal to the lines	magnetic flux density	electric field	magnetic induction	magnetic field	magnetic flux density
If the external torque applied is zero,then_____is conserved	N	L	P	F	L
The lagrangian for a charged particle in an electromagentic field can be written as_____	L=V-T	L=T-U	L=T+V	L=V	L=T-U
A sytem of particle is called a _____	statical system	dynamical system	kinematics	mechanics	dynamical system
The set of positions of all the particles is known as _____of the dynamical sytem	torque	linear motion	configuration	acceleration	configuration
The angular momentum of the particle about the point O denoted by vector L is defined as _____	r X P	r X L	r X M	r X N	r X P
Any restriction on the motion of a sytem is known as _____	energy	constraint	moment of force	torque	constraint
_____are used in electrical and magnetic field	equation of motion	maxwell equation	conservation equation	lagrange equation	maxwell equation
A particle is constrained to move along a curve or on a surface is example of _____	Holonomic	Scheleronomous	non holonomic	rheonomous	holonomic
the walls of a gas container constitute a _____constraint	scheleronomous	rheonomous	holonomic	non holonomic	non holonomic
A_____is one which has only fixed constraints	rheonomous	non holonomic	Scheleronomous	holonomic	Scheleronomous
A_____has moving constraints	non holonomic	scheleronomous	holonomic	rheonomous	rheonomous
The work due to virtual displacement is known as _____	maxwell's law	acceleration	virtual work	dynamical system	virtual work

The number of generalised coordinates required to desribe the configuration of a system is called the _____	number of degrees of freedom	number of generalisd coordinates	linear displacement	moment of force	number of degrees of freedom
A particle is constrained to move on a surface,the force of constraint is perpendicular to the surface,while the virtual displacement must be tangent to it,and hence the _____vanishes	Force	virtual displacement	virtual work	acceleration	virtual work
In a simple dynamicl system T+V=_____	constant	zero	one	conserved	constant
_____is used to solve the holonomic problem	Lagrange's method of multiplier	velocity dependent potential	D'Alemberts principle	principle of virtual work	Lagrange's method of multiplier
_____is defined as the force with which a body is attracted towards the centre of the earth	mass	acceleration	weight	velocity	weight
A dynamical system is called _____if it is possible to give arbitrary and independent variations to the generalised coordinates of the system without violating constraints	rheonomous	non holonomic	holonomic	scheleronomous	holonomic
the degrees of freedom is given by _____	3N-K	N-3K	N+3K	3N+K	3N-K
Potential energy of a body is the energy it possess by the virtue of its_____	constant	motion	position	energy	position

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DEPARTMENT OF MATHEMATICS

Subject: Mechanics	**Subject Code: 17MMP106**	**L T P C**
Class:I M.Sc	**Semester:I**	**4 0 0 4**

UNIT II

Variation principles and Lagrange's equations: Hamilton's principle – Some techniques of calculus of variations – Derivation of Lagrange's Equations from Hamilton's principle – Extension of Hamilton's principle to non-holonomic systems – Conservation theorems and symmetry properties.

SUGGESTED READINGS

TEXT BOOK:

T1: Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

2.1 ■ HAMILTON'S PRINCIPLE

The derivation of Lagrange's equations presented in Chapter 1 started from a consideration of the instantaneous state of the system and small virtual displacements about the instantaneous state, i.e., from a "differential principle" such as D'Alembert's principle. It is also possible to obtain Lagrange's equations from a principle that considers the entire motion of the system between times t_1 and t_2, and small virtual variations of this motion from the actual motion. A principle of this nature is known as an "integral principle."

Before presenting the integral principle, the meaning attached to the phrase "motion of the system between times t_1 and t_2" must first be stated in more precise language The instantaneous configuration of a system is described by the values of the n generalized coordinates $q_1, \ldots, q_n$, and corresponds to a particular point in a Cartesian hyperspace where the q's form the n coordinate axes. This n-dimensional space is therefore known as configuration space. As time goes on, the state of the system changes and the system point moves in configuration space tracing out a curve, described as "the path of motion of the system." The "motion of the system," as used above, then refers to the motion of the system point along this path in *configuration space*. Time can be considered formally as a parameter of the curve; to each point on the path there is associated one or more values of the time. Note that configuration space has no necessary connection with the physical three-dimensional space, just as the generalized coordinates are not necessarily position coordinates. The path of motion in configuration space has no resemblance to the path in space of any actual particle; each point on the path represents the *entire* system configuration at some given instant of time.

The integral *Hamilton's principle* describes the motion of those mechanical systems for which all forces (except the forces of constraint) are derivable from a generalized scalar potential that may be a function of the coordinates, velocities, and time. Such systems will be denoted as *monogenic*. Where the potential is an explicit function of position coordinates only, then a monogenic system is also conservative (cf. Section 1.2).

For monogenic systems, Hamilton's principle can be stated as

> *The motion of the system from time t_1 to time t_2 is such that the line integral (called the* action *or the* action integral*),*

$$I = \int_{t_1}^{t_2} L\,dt, \tag{2.1}$$

where $L = T - V$, has a stationary value for the actual path of the motion.

That is, out of all possible paths by which the system point could travel from its position at time t_1 to its position at time t_2, it will actually travel along that path for which the value of the integral (2.1) is stationary. By the term "stationary value" for a line integral, we mean that the integral along the given path has the same value to within first-order infinitesimals as that along all neighboring paths (i.e., those that differ from it by infinitesimal displacements). (Cf. Fig. 2.1.) The notion of a stationary value for a line integral thus corresponds in ordinary function theory to the vanishing of the first derivative.

We can summarize Hamilton's principle by saying that the motion is such that the *variation* of the line integral I for fixed t_1 and t_2 is zero:

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,dt = 0. \tag{2.2}$$

Where the system constraints are holonomic, Hamilton's principle, Eq. (2.2), is both a necessary and sufficient condition for Lagrange's equations, Eqs. (1.57). Thus, it can be shown that Hamilton's principle follows directly from Lagrange's equations. Instead, however, we shall prove the converse, namely, that Lagrange's equations follow from Hamilton's principle, as being the more important theorem. That Hamilton's principle is a sufficient condition for deriving the equations of motion enables us to construct the mechanics of monogenic systems from Hamilton's principle as the basic postulate rather than Newton's laws of motion. Such a formulation has advantages; e g , since the integral I is obviously invariant to the system of generalized coordinates used to express L, the equations of motion must always have the Lagrangian form no matter how the generalized coordinates

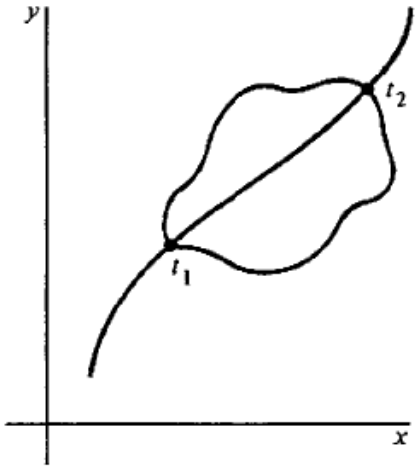

FIGURE 2.1 Path of the system point in configuration space.

are transformed. More important, the formulation in terms of a variational principle is the route that is generally followed when we try to describe apparently nonmechanical systems in the mathematical clothes of classical mechanics, as in the theory of fields.

2.2 ■ SOME TECHNIQUES OF THE CALCULUS OF VARIATIONS

Before demonstrating that Lagrange's equations do follow from (2.2), we must first examine the methods of the calculus of variations, for a chief problem of this calculus is to find the curve for which some given line integral has a stationary value.

Consider first the problem in an essentially one-dimensional form: We have a function $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2, where $\dot{y}$ is the derivative of y with respect to x. We wish to find a particular path $y(x)$ such that the line integral J of the function f between x_1 and x_2,

$$\dot{y} \equiv \frac{dy}{dx},$$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx, \tag{2.3}$$

has a stationary value relative to paths differing infinitesimally from the correct function $y(x)$. The variable x here plays the role of the parameter t, and we consider only such varied paths for which $y(x_1) = y_1$, $y(x_2) = y_2$. (Cf. Fig. 2.2.) Note that Fig. 2.2 does *not* represent configuration space. In the one-dimensional configuration space, both the correct and varied paths are the segment of the straight line connecting y_1 and y_2; the paths differ only in the functional relation between y and x. The problem is one-dimensional, y is a function of x not a coordinate.

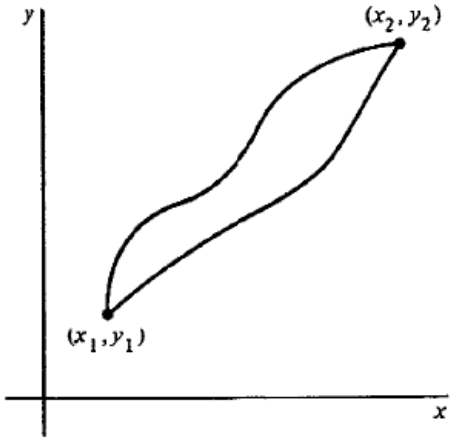

FIGURE 2.2 Varied paths of the function of $y(x)$ in the one-dimensional extremum problem.

We put the problem in a form that enables us to use the familiar apparatus of the differential calculus for finding the stationary points of a function. Since J must have a stationary value for the correct path relative to *any* neighboring path, the variation must be zero relative to *some* particular set of neighboring paths labeled by an infinitesimal parameter α. Such a set of paths might be denoted by $y(x, \alpha)$, with $y(x, 0)$ representing the correct path. For example, if we select any function $\eta(x)$ that vanishes at $x = x_1$ and $x = x_2$, then a possible set of varied paths is given by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x). \tag{2.4}$$

For simplicity, it is assumed that both the correct path $y(x)$ and the auxiliary function $\eta(x)$ are well-behaved functions—continuous and nonsingular between x_1 and x_2, with continuous first and second derivatives in the same interval. For any such parametric family of curves, J in Eq. (2.3) is also a function of α:

$$J(\alpha) = \int_{x_1}^{x_2} f\,(y(x, \alpha), \dot{y}(x, \alpha), x)\; dx. \tag{2.5}$$

and the condition for obtaining a stationary point is the familiar one that

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0. \tag{2.6}$$

By the usual methods of differentiating under the integral sign, we find that

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right) dx. \tag{2.7}$$

Consider the second of these integrals.

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\, \partial \alpha}\, dx.$$

Integrating by parts, the integral becomes

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\, \partial \alpha}\, dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\, dx. \tag{2.8}$$

The conditions on all the varied curves are that they pass through the points (x_1, y_1), (x_2, y_2), and hence the partial derivative of y with respect to α at x_1 and x_2 must vanish. Therefore, the first term of (2.8) vanishes and Eq. (2.7) reduces to

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}\, dx.$$

The condition for a stationary value, Eq. (2.6), is therefore equivalent to the equation

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\left(\frac{\partial y}{\partial \alpha}\right)_0 dx = 0. \tag{2.9}$$

Now, the partial derivative of y with respect to α occurring in Eq. (2.9) is a function of x that is arbitrary except for continuity and end point conditions. For example, for the particular parametric family of varied paths given by Eq. (2.4), it is the arbitrary function $\eta(x)$. We can therefore apply to Eq (2.9) the so-called "fundamental lemma" of the calculus of variations, which says if

$$\int_{x_1}^{x_2} M(x)\eta(x)\,dx = 0 \tag{2.10}$$

for all arbitrary functions $\eta(x)$ continuous through the second derivative, then $M(x)$ must identically vanish in the interval (x_1, x_2). While a formal mathematical proof of the lemma can be found in texts on the calculus of variations, the validity of the lemma is easily seen intuitively. We can imagine constructing a function η that is positive in the immediate vicinity of any chosen point in the interval and zero everywhere else. Equation (2.10) can then hold only if $M(x)$ vanishes at that (arbitrarily) chosen point. which shows M must be zero throughout the interval. From Eq. (2.9) and the fundamental lemma, it therefore follows that J can have a stationary value only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \tag{2 11}$$

The differential quantity,

$$\left(\frac{\partial y}{\partial \alpha}\right)_0 d\alpha \equiv \delta y, \tag{2 12}$$

represents the infinitesimal departure of the varied path from the correct path $y(x)$ at the point x and thus corresponds to the virtual displacement introduced in Chapter 1 (hence the notation δy). Similarly, the infinitesimal variation of J about the correct path can be designated

$$\left(\frac{dJ}{d\alpha}\right)_0 d\alpha \equiv \delta J. \tag{2.13}$$

The assertion that J is stationary for the correct path can thus be written

$$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\delta y\,dx - 0.$$

requiring that $y(x)$ satisfy the differential equation (2.11). The δ-notation, introduced through Eqs. (2.12) and (2.13), may be used as a convenient shorthand for treating the variation of integrals, remembering always that it stands for the manipulation of parametric families of varied paths such as Eq. (2.4).

2.3 ■ DERIVATION OF LAGRANGE'S EQUATIONS FROM HAMILTON'S PRINCIPLE

The fundamental problem of the calculus of variations is easily generalized to the case where f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$. (Of course, all these quantities are considered as functions of the parametric variable x.) Then a variation of the integral J,

$$\delta J = \delta \int_1^2 f(y_1(x); y_2(x), \ldots, \dot{y}_1(x); \dot{y}_2(x), \ldots, x)\, dx, \tag{2.14}$$

is obtained, as before, by considering J as a function of parameter α that labels a possible set of curves $y_1(x, \alpha)$. Thus, we may introduce α by setting

$$\begin{aligned} y_1(x, \alpha) &= y_1(\lambda, 0) + \alpha\eta_1(x), \\ y_2(x, \alpha) &= y_2(x, 0) + \alpha\eta_2(x), \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \end{aligned} \tag{2.15}$$

where $y_1(x, 0)$, $y_2(x, 0)$, etc., are the solutions of the extremum problem (to be obtained) and η_1, η_2, etc., are independent functions of x that vanish at the end points and that are continuous through the second derivative, but otherwise are completely arbitrary.

The calculation proceeds as before. The variation of J is given in terms of

$$\frac{\partial J}{\partial \alpha}\, d\alpha = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha}\, d\alpha + \frac{\partial f}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \alpha}\, d\alpha \right) dx. \tag{2.16}$$

Again we integrate by parts the integral involved in the second sum of Eq. (2.16):

$$\int_1^2 \frac{\delta f}{\partial \dot{y}_i} \frac{\partial^2 y_i}{\partial \alpha\, \partial x}\, dx = \left. \frac{\partial f}{\partial \dot{y}_i} \frac{\partial y_i}{\partial \alpha} \right|_1^2 - \int_1^2 \frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_i} \right) dx,$$

where the first term vanishes because all curves pass through the fixed end points. Substituting in (2.16), δJ becomes

$$\delta J = \int_1^2 \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \right) \delta y_i\, dx, \tag{2.17}$$

where, in analogy with (2.12), the variation δy_i is

$$\delta y_i = \left(\frac{\partial y_i}{\partial \alpha} \right)_0 d\alpha.$$

Since the y variables are independent, the variations δy_i are independent (e.g., the functions $\eta_i(x)$ will be independent of each other). Hence, by an obvious extension of the fundamental lemma (cf. Eq. (2.10)), the condition that δJ is zero

requires that the coefficients of the δy_i separately vanish:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{2.18}$$

Equations (2.18) represent the appropriate generalization of (2.11) to several variables and are known as the *Euler-Lagrange differential equations*. Their solutions represent curves for which the variation of an integral of the form given in (2.14) vanishes. Further generalizations of the fundamental variational problem are easily possible. Thus, we can take f as a function of higher derivatives $\ddot{y}$, $\dddot{y}$, etc., leading to equations different from (2.18). Or we can extend it to cases where there are several parameters x_j and the integral is then multiple, with f also involving as variables derivatives of y_i with respect to each of the parameters x_j. Finally, it is possible to consider variations in which the end points are *not* held fixed.

For present purposes, what we have derived here suffices, for the integral in Hamilton's principle,

$$I = \int_1^2 L(q_i, \dot{q}_i, t)\, dt, \tag{2.19}$$

has just the form stipulated in (2.14) with the transformation

$$x \to t$$

$$y_i \to q_i$$

$$f(y_i, \dot{y}_i, x) \to L(q_i, \dot{q}_i, t).$$

In deriving Eqs. (2.18), we assumed that the y_i variables are independent. The corresponding condition in connection with Hamilton's principle is that the generalized coordinates q_i be independent, which requires that the constraints be holonomic. The Euler-Lagrange equations corresponding to the integral I then become the Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n,$$

and we have accomplished our original aim, to show that Lagrange's equations follow from Hamilton's principle—for monogenic systems with holonomic constraints.

2.4 ■ EXTENSION OF HAMILTON'S PRINCIPLE TO NONHOLONOMIC SYSTEMS

It is possible to extend Hamilton's principle, at least in a formal sense, to cover certain types of nonholonomic systems. In deriving Lagrange's equations from

either Hamilton's or D'Alembert's principle, the requirement of holonomic constraints does not appear until the last step, when the variations q_i are considered as independent of each other. With nonholonomic systems the generalized coordinates are not independent of each other, and it is not possible to reduce them further by means of equations of constraint of the form $f(q_1, q_2, \ldots, q_n, t) = 0$. Hence, it is no longer true that the q_i's are all independent.

Another difference that must be considered in treating the variational principle is the manner in which the varied paths are constructed. In the discussion of Section 2.2, we pointed out that δy (or δq) represents a virtual displacement from a point on the actual path to some point on the neighboring varied path. But, with independent coordinates it is the final varied path that is significant, not how it is constructed. When the coordinates are not independent, but subject to constraint relations, it becomes important whether the varied path is or is not constructed by displacements consistent with the constraints. Virtual displacements, in particular, may or may not satisfy the constraints.

It appears that a reasonably straightforward treatment of nonholonomic systems by a variational principle is possible only when the equations of constraint can be put in the form

$$f_\alpha(q_1, \ldots, q_n;\ \dot{q}_1 \ldots, \dot{q}_n) = 0, \tag{2.20}$$

when this can be done the constraints are called semi-holonomic. The index α indicates that there may be more than one such equation. We will assume there are m equations in all, i.e., $\alpha = 1, 2, \ldots, m$. Equation (2.20) commonly appears in the restricted form

$$\sum_k a_{ik}\, dq_k + a_{it}\, dt = 0. \tag{2.20'}$$

We might expect that the varied paths, or equivalently, the displacements constructing the varied path, should satisfy the constraints of Eq. (2.20). However, it has been proven that no such varied path can be constructed unless Eqs. (2.20) are integrable, in which case the constraints are actually holonomic. A variational principle leading to the correct equations of motion can nonetheless be obtained when the varied paths are constructed from the actual motion by virtual displacements.

The procedure for eliminating these extra virtual displacements is the method of *Lagrange undetermined multipliers*. If Eqs. (2.20) hold, then it is also true that

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0, \tag{2.21}$$

where the λ_α, $\alpha = 1, 2, \ldots, m$, are some undetermined quantities, functions in general of the coordinates and of the time t. In addition, Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{2.2}$$

is assumed to hold for this semiholonomic system. Following the development of Section 2.3, Hamilton's principle then implies that

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0. \tag{2.22}$$

The variation cannot be taken as before since the q_k are not independent; however, combining (2.21) with (2.2) gives

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt = 0 \tag{2.23}$$

The variation can now be performed with the n δq_i and m λ_α for $m+n$ independent variables. For the simplifying assumption that $\lambda_\alpha = \lambda_\alpha(t)$, the resulting equations from δq_i become*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k, \tag{2.24}$$

where

$$Q_k = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial f_\alpha}{\partial \dot{q}_k} \right) \right] - \frac{d\lambda_\alpha}{dt} \frac{\partial f_\alpha}{\partial \dot{q}_k} \right\}, \tag{2.25}$$

while the $\delta\lambda_\alpha$ give the equations of constraint (2.20). Equations (2.24) and (2.20) together constitute $n + m$ equations for $n + m$ unknowns. The system can now be interpreted as an $m + n$ holonomic system with generalized forces Q_k. The generalization to $\lambda_\alpha = \lambda_\alpha(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is straightforward.

As an example, let us consider a particle whose Lagrangian is

$$L = \tfrac{1}{2}m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(x, y, z) \tag{2.26}$$

subject to the constraint

$$f(\dot{x}, \dot{y}, y) = \dot{x}\dot{y} + ky = 0 \tag{2.27}$$

with k a constant. The resulting equations of motion are

$$m\ddot{x} + \lambda\ddot{y} + \dot{\lambda}\dot{y} + \frac{\partial V}{\partial x} = 0, \tag{2.28}$$

$$m\ddot{y} + \lambda\ddot{x} - k\lambda + \dot{\lambda}\dot{x} + \frac{\partial V}{\partial y} = 0, \tag{2.29}$$

$$m\ddot{z} + \frac{\partial V}{\partial z} = 0, \tag{2.30}$$

and the equation of constraint, (2.20), becomes

$$\dot{y}\dot{x} + ky = 0.$$

In this process we have obtained more information than was originally sought. Not only do we get the q_k's we set out to find, but we also get $m\lambda_l$'s. What is the physical significance of the λ_l's? Suppose we remove the constraints on the system, but instead apply external forces Q'_k in such a manner as to keep the motion of the system unchanged. The equations of motion likewise remain the same. Clearly these extra applied forces must be equal to the forces of constraint, for they are the forces applied to the system so as to satisfy the condition of constraint. Under the influence of these forces Q'_k, the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q'_k. \tag{2.31}$$

But these must be identical with Eqs. (2.24). Hence, we can identify (2.25) with Q'_k, the generalized forces of constraint. In this type of problem we really do not eliminate the forces of constraint from the formulation. They are supplied as part of the answer.

Although it is not obvious, the version of Hamilton's principle adopted here for semiholonomic systems also requires that the constraints do no work in virtual displacements. This can be most easily seen by rewriting Hamilton's principle in the form

$$\delta \int_{t_1}^{t_2} L\,dt = \delta \int_{t_1}^{t_2} T\,dt - \delta \int_{t_1}^{t_2} U\,dt = 0. \tag{2.32}$$

If the variation of the integral over the generalized potential is carried out by the procedures of Section 2.3, the principle takes the form

$$\delta \int_{t_1}^{t_2} T\,dt = \int_{t_1}^{t_2} \sum_k \left[\frac{\partial U}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right)\right] \delta q_k dt; \tag{2.33}$$

or, by Eq. (1.58),

$$\delta \int_{t_1}^{t_2} T\,dt = -\int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt. \tag{2.34}$$

In this dress, Hamilton's principle says that the difference in the time integral of the kinetic energy between two neighboring paths is equal to the negative of the time integral of the work done in the virtual displacements between the paths. The work involved is that done only by the forces derivable from the generalized potential. The same Hamilton's principle holds for both holonomic and semiholonomic systems, it must be required that the additional forces of semiholonomic constraints do no work in the displacements δq_k. This restriction parallels the earlier condition that the virtual work of the forces of holonomic constraint also be

zero (cf. Section 1.4). In practice, the restriction presents little handicap to the applications, as many problems in which the semiholonomic formalism is used relate to rolling without slipping, where the constraints are obviously workless.

Note that Eq. (2.20) is not the most general type of nonholonomic constraint; e.g., it does not include equations of constraint in the form of inequalities. On the other hand, it does include holonomic constraints. A holonomic equation of constraint,

$$f(q_1, q_2, q_3, \ldots, q_n, t) = 0, \tag{2.35}$$

is equivalent to (2.20) with no dependence on $\dot{q}_k$. Thus, the Lagrange multiplier method can be used also for holonomic constraints when (1) it is inconvenient to reduce all the q's to independent coordinates or (2) we might wish to obtain the forces of constraint.

As another example of the method, let us consider the following somewhat trivial illustration—a hoop rolling, without slipping, down an inclined plane. In this example, the constraint of "rolling" is actually holonomic, but this fact will be immaterial to our discussion. On the other hand, the holonomic constraint that the hoop be on the inclined plane will be contained implicitly in our choice of generalized coordinates.

The two generalized coordinates are x, θ, as in Fig. 2.5, and the equation of rolling constraint is

$$r\,d\theta = dx.$$

The kinetic energy can be resolved into kinetic energy of motion of the center of mass plus the kinetic energy of motion about the center of mass:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}Mr^2\dot{\theta}^2.$$

The potential energy is

$$V = Mg(l - x)\sin\phi,$$

where l is the length of the inclined plane and the Lagrangian is

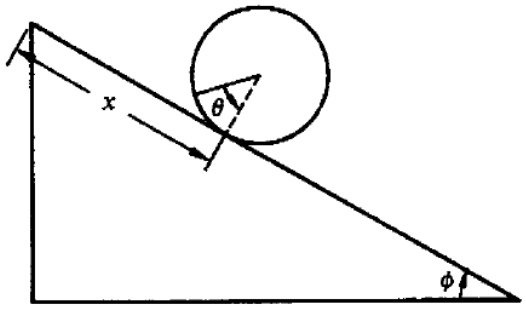

FIGURE 2.5 A hoop rolling down an inclined plane.

$$L = T - V$$
$$= \frac{M\dot{x}^2}{2} + \frac{Mr^2\dot{\theta}^2}{2} - Mg(l - x)\sin\phi. \tag{2.36}$$

Since there is one equation of constraint, only one Lagrange multiplier λ is needed. The coefficients appearing in the constraint equation are:

$$a_\theta = r,$$
$$a_x = -1.$$

The two Lagrange equations therefore are

$$M\ddot{x} - Mg\sin\phi + \lambda = 0, \tag{2.37}$$
$$Mr^2\ddot{\theta} - \lambda r = 0, \tag{2.38}$$

which along with the equation of constraint,

$$r\dot{\theta} = \dot{x}, \tag{2.39}$$

constitutes three equations for three unknowns, θ, x, λ.

Differentiating (2.39) with respect to time, we have

$$r\ddot{\theta} = \ddot{x}.$$

Hence, from (2.38)

$$M\ddot{x} = \lambda,$$

and (2.37) becomes

$$\ddot{x} = \frac{g\sin\phi}{2},$$

along with

$$\lambda = \frac{Mg\sin\phi}{2}$$

and

$$\ddot{\theta} = \frac{g\sin\phi}{2r}.$$

Thus, the hoop rolls down the incline with only one-half the acceleration it would have slipping down a frictionless plane, and the friction force of constraint is $\lambda = Mg\sin\phi/2$.

Although we can extend the original formulation of Hamilton's principle (2.2) to include some nonholonomic constraints, in practice this formulation of mechanics is most useful when a Lagrangian of independent coordinates can be set up for the system. The variational principle formulation has been justly described as "elegant," for in the compact Hamilton's principle is contained all of the mechanics of holonomic systems with forces derivable from potentials. The principle has the further merit that it involves only physical quantities that can be defined without reference to a particular set of generalized coordinates, namely, the kinetic and potential energies. The formulation is therefore automatically invariant with respect to the choice of coordinates for the system.

From the variational Hamilton's principle, it is also obvious why the Lagrangian is always uncertain to a total time derivative of any function of the coordinates and time, as mentioned at the end of Section 1.4. The time integral of such a total derivative between points 1 and 2 depends only on the values of the arbitrary function at the end points. As the variation at the end points is zero, the addition of the arbitrary time derivative to the Lagrangian does not affect the variational behavior of the integral.

Another advantage is that the Lagrangian formulation can be easily extended to describe systems that are not normally considered in dynamics—such as the elastic field, the electromagnetic field, and field properties of elementary particles. Some of these generalizations will be considered later, but as three simple examples of its application outside the usual framework of mechanics, let us consider the cases of an RL circuit, an LC circuit, and coupled circuits.

We consider the physical system of a battery of voltage V in series with an inductance L and a resistance of value R and choose the electric charge q as the dynamical variable. The inductor acts as the kinetic energy term since the inductive effect depends upon the time rate of change of the charge. The resistor provides a dissipative term and the potential energy is qV. The dynamic terms in Lagrange's equation with dissipation (1.70) are

$$T = \tfrac{1}{2}L\dot{q}^2, \quad \mathcal{F} = \tfrac{1}{2}R\dot{q}^2,$$

and potential energy $= qV$. The equation of motion is

$$V = L\ddot{q} + R\dot{q} = L\dot{I} + RI. \tag{2.40}$$

where $I = \dot{q}$ is the electric current. A solution for a battery connected to the circuit at time $t = 0$ is

$$I = I_0(1 - e^{-Rt/L}),$$

where $I_0 = V/R$ is the final steady-state current flow.

The mechanical analog for this is a sphere of radius a and effective mass m' falling in a viscous fluid of constant density and viscosity η under the force of

gravity. The effective mass is the difference between the actual mass and the mass of the displaced fluid, and the direction of motion is along the y axis. For this system,

$$T = \tfrac{1}{2}m'\dot{y}^2, \quad \mathcal{F} = 3\pi\eta a\dot{y}^2,$$

and potential energy $= m'gy$, where the frictional drag force, $F_f = 6\pi\eta a y$, called Stokes' law, was given at the end of Section 1.5.

The equation of motion is given by Lagrange's equations (1.70) as

$$m'g = m'\ddot{y} + 6\pi\eta a\dot{y}.$$

Using $v = \dot{y}$, the solution (if the motion starts from rest at $t = 0$), is

$$v = v_0(1 - e^{-t/\tau})$$

where $\tau = m'/(6\pi\eta a)$ is a measure of the time it takes for the sphere to reach $1/e$ of its terminal speed of $v_0 = m'g/6\pi\eta a$.

Another example from electrical circuits is an inductance, L, in series with a capacitance, C. The capacitor acts as a source of potential energy given by q^2/C where q is the electric charge. The Lagrangian produces the equation of motion,

$$L\ddot{q} + \frac{q}{C} = 0, \tag{2.41}$$

which has the solution

$$q = q_0 \cos\omega_0 t,$$

where q_0 is the charge stored in the capacitor at $t = 0$, and the assumption is that no charge is flowing at $t = 0$. The quantity

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

is the resonant frequency of the system.

The mechanical analog of this system is the simple harmonic oscillator described by the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$, which gives an equation of motion,

$$m\ddot{x} + kx = 0,$$

whose solution for the same boundary conditions is

$$x = x_0 \cos\omega_0 t \qquad \text{with} \quad \omega_0 = \sqrt{k/m}.$$

These two examples show that an inductance is an inertial term, the electrical analog of mass. Resistance is the analog of Stokes' law type of frictional drag, and the capacitance term $1/C$ represents a Hooke's law spring constant. With this

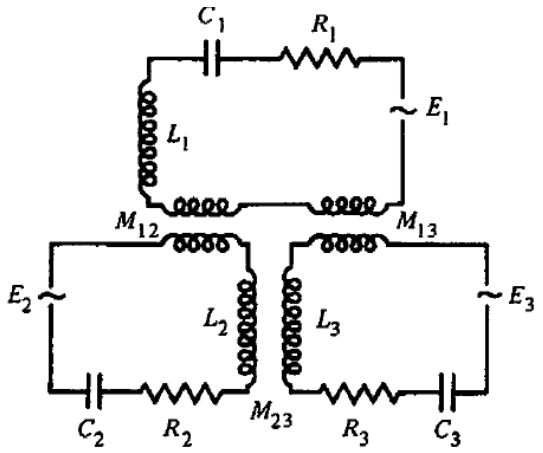

FIGURE 2.6 A system of coupled circuits to which the Lagrangian formulation can be applied.

background, a system of coupled electrical circuits of the type shown in Fig. 2.6 has a Lagrangian of the form

$$L = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{\substack{jk \\ j\neq k}} M_{jk}\dot{q}_j\dot{q}_k - \sum_j \frac{q_j^2}{2C_j} + \sum_j e_j(t)q_j,$$

and a dissipation function

$$\mathcal{F} = \frac{1}{2}\sum_j R_j \dot{q}_j^2.$$

where the mutual inductance terms, $M_{jk}\dot{q}_j\dot{q}_k$, are added to take into account the coupling between inductors. The Lagrange equations are

$$L_j\frac{d^2q_j}{dt^2} + \sum_{\substack{k \\ j\neq k}} M_{jk}\frac{d^2q_k}{dt^2} + R_j\frac{dq_j}{dt} + \frac{q_j}{C_j} = E_j(t). \tag{2.42}$$

where the $E_j(t)$ terms are the external emf's.

This description of two different physical systems by Lagrangians of the same form means that all the results and techniques devised for investigating one of the systems can be taken over immediately and applied to the other. In this particular case, the study of the behavior of electrical circuits has been pursued intensely and some special techniques have been developed; these can be directly applied to the corresponding mechanical systems. Much work has been done in formulating equivalent electrical problems for mechanical or acoustical systems, and vice versa. Terms hitherto reserved for electrical circuits (reactance, susceptance, etc.) are now commonly found in treatises on the theory of vibrations of mechanical systems.

Additionally, one type of generalization of mechanics is due to a subtler form of equivalence. We have seen that the Lagrangian and Hamilton's principle together form a compact invariant way of obtaining the mechanical equations of motion. This possibility is not reserved for mechanics only; in almost every field of physics variational principles can be used to express the "equations of motion," whether they be Newton's equations, Maxwell's equations, or the Schrödinger equation. Consequently, when a variational principle is used as the basis of the formulation, all such fields will exhibit, at least to some degree, a *structural analogy*. When the results of experiments show the need for alterating the physical content in the theory of one field, this degree of analogy has often indicated how similar alterations may be carried out in other fields. Thus, the experiments performed early in this century showed the need for quantization of both electromagnetic radiation and elementary particles. The methods of quantization, however, were first developed for particle mechanics, starting essentially from the Lagrangian formulation of classical mechanics. By describing the electromagnetic field by a Lagrangian and corresponding Hamilton's variational principle, it is possible to carry over the methods of particle quantization to construct a quantum electrodynamics (cf. Sections 13.5 and 13.6).

CONSERVATION THEOREMS AND SYMMETRY PROPERTIES

Thus far, we have been concerned primarily with obtaining the equations of motion, but little has been said about how to solve them for a particular problem once they are obtained. In general, this is a question of mathematics. A system of n degrees of freedom will have n differential equations that are second order in time. The solution of each equation will require two integrations resulting, all told, in $2n$ constants of integration. In a specific problem these constants will be determined by the initial conditions, i.e., the initial values of the nq_j's and the $n\dot{q}_j$'s. Sometimes the equations of motion will be integrable in terms of known functions, but not always. In fact, the majority of problems are not completely integrable. However, even when complete solutions cannot be obtained, it is often possible to extract a large amount of information about the physical nature of the system motion. Indeed, such information may be of greater interest to the physicist than the complete solution for the generalized coordinates as a function of time. It is important, therefore, to see how much can be stated about the motion of a given system without requiring a complete integration of the problem.*

In many problems a number of first integrals of the equations of motion can be obtained immediately; by this we mean relations of the type

$$f(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots, t) = \text{constant}, \tag{2.43}$$

which are first-order differential equations. These first integrals are of interest because they tell us something physically about the system. They include, in fact, the conservation laws obtained in Chapter 1.

Let us consider as an example a system of mass points under the influence of forces derived from potentials dependent on position only. Then

$$\frac{\partial L}{\partial \dot{x}_i} \equiv \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right)$$
$$= m_i \dot{x}_i = p_{ix},$$

which is the x component of the linear momentum associated with the ith particle. This result suggests an obvious extension to the concept of momentum. The generalized momentum associated with the coordinate q_j shall be defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{2.44}$$

The terms *canonical momentum* and *conjugate momentum* are often also used for p_j. Notice that if q_j is not a Cartesian coordinate, p_j does not necessarily have the dimensions of a linear momentum. Further, if there is a velocity-dependent potential, then even with a Cartesian coordinate q_j the associated *generalized* momentum will not be identical with the usual *mechanical* momentum. Thus, in the case of a group of particles in an electromagnetic field, the Lagrangian is (cf. 1.63)

$$L = \sum_i \frac{1}{2} m_i \dot{r}_i^2 - \sum_i q_i \phi(x_i) + \sum_i q_i \mathbf{A}(x_i) \cdot \dot{\mathbf{r}}_i$$

(q_i here denotes charge) and the generalized momentum conjugate to x_i is

$$p_{ix} = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i + q_i A_x, \tag{2.45}$$

i.e., mechanical momentum plus an additional term.

If the Lagrangian of a system does not contain a given coordinate q_j (although it may contain the corresponding velocity $\dot{q}_j$), then the coordinate is said to be *cyclic* or *ignorable*. This definition is not universal, but it is the customary one and will be used here. The Lagrange equation of motion,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0,$$

reduces, for a cyclic coordinate, to

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0$$

or

$$\frac{dp_j}{dt} = 0,$$

which mean that

$$p_j = \text{constant.} \tag{2.46}$$

Hence, we can state as a general conservation theorem that *the generalized momentum conjugate to a cyclic coordinate is conserved.*

Note that the derivation of Eq. (2.46) assumes that q_j is a generalized coordinate; one that is linearly independent of all the other coordinates. When equations of constraint exist, all the coordinates are not linearly independent. For example, the angular coordinate θ is not present in the Lagrangian of a hoop rolling without slipping in a horizontal plane that was previously discussed, but the angle appears in the constraint equations $rd\theta = dx$. As a result, the angular momentum, $p_\theta = mr^2\dot{\theta}$, is not a constant of the motion.

Equation (2.46) constitutes a first integral of the form (2.43) for the equations of motion. It can be used formally to eliminate the cyclic coordinate from the problem, which can then be solved entirely in terms of the remaining generalized coordinates. Briefly, the procedure, originated by Routh, consists in modifying the Lagrangian so that it is no longer a function of the generalized velocity corresponding to the cyclic coordinate, but instead involves only its conjugate momentum. The advantage in so doing is that p_j can then be considered one of the constants of integration, and the remaining integrations involve only the noncyclic coordinates. We shall defer a detailed discussion of Routh's method until the Hamiltonian formulation (to which it is closely related) is treated.

Note that the conditions for the conservation of generalized momenta are more general than the two momentum conservation theorems previously derived. For example, they furnish a conservation theorem for a case in which the law of action and reaction is violated, namely, when electromagnetic forces are present. Suppose we have a single particle in a field in which neither ϕ nor $\mathbf{A}$ depends on x. Then x nowhere appears in L and is therefore cyclic. The corresponding canonical momentum p_x must therefore be conserved. From (1.63) this momentum now has the form

$$p_x = m\dot{x} + qA_x = \text{constant.} \tag{2.47}$$

In this case, it is not the mechanical linear momentum $m\dot{x}$ that is conserved but rather its sum with qA_x.* Nevertheless, it should still be true that the conservation theorems of Chapter 1 are contained within the general rule for cyclic coordinates; with proper restrictions (2.46) should reduce to the theorems of Section 1.2.

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Define canonical momentum. Show that the generalized momentum conjugate to a cyclic co-ordinate is conserved
2. Write a short note on physical significance of the lagrangian undeterminant multiplier
3. Derive the Brachistochrome problem
4 Find the curve for which some line integral has a stationary value
5. State and prove Euler Lagrange differential equation
6. Find the equation of motion of a hoop or disc rolling without slipping down on the inclined plane.

Part C (10 Marks)

1. Find the minimum surface of revolution
2. Derive the conservation theorem for total energy of system
3. Show that the shortest curve between any two points in the plane is a straight line
4. Derive the conservation theorem for dissipation function

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UNIT-II

Subject: Mechanics **Subject Code: 17MMP106**

Variation principles and Lagrange's equations
Part-A(20X1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Multiple Choice Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
UNIT II					
A principle where smalll variations of the entire motion from the actual motion take place is known as an_____	Hamilton's principle	Lagrange's principle	cyclic coordinates	D'Alemberts principle	Hamilton's principle
In total kinetic energy $T=T_o+T_1+T_2$ wher T_o is the function of _____ only	generalised coordinates	cyclic coordinates	conjugate momentum	canonical momentum	generalised coordinates
The instantaneous configuration of a system is described by the values of n generalised coordinates and corresponds to a particular point in the n-dimensional space known as the _____	cyclic coordinates	moment of force	configuration space	virtual displacement	configuration space
As time goes on,the state of the system changes and the system point moves in configuration space tracing out a curve known as the_____	projectory	trajectory	motion	acceleration	trajectory
The procedure for eliminating extra virtual displacement is the method of_____	monogenic	lagrange's undetermined multipliers	euler method	Routhian	lagrange's undetermined multipliers
The line integral may also be denoted as_____	J	L	X	K	J
The shortest distance between any two points in space are called_____	varied path	geodesics	curve	circle	geodesics
In the derivation of hamilton's principle for holonomic constraints,the variation q_i are considered_____of each other	dependent	independent	neighbouring	significant	independent
All the generalized coordinates cannot be_____	linear	non cyclic	non linear	cyclic	cyclic
If the lagrangian of the system does not contain the given co-ordinate q_i then it is said to be_____	linear momentum	conjugate momentum	angular momentum	cyclic	cyclic
A_____is a path traced by a point on the circumference of the disc rolling with a constant speed	straight line	catenary	varied path	cycloid	catenary
One of the first integrals of motion is referred to as _____integral	Jacobi	Poisson	Lagrange	Euler	Jacobi
In total kinetic energy $T=T_o+T_1+T_2$ where T_1 is the _____generalised velocities	constant	quadratic	linear	nonl inear	linear
A function f(x) is said to be statinary at x=a when_____	f(a)=constant	f(a)=0	f'(a)=constant	f'(a)=0	f'(a)=0
In the derivation of hamilton's principle for non-Holonomic constraints,the generalised co-ordinates q_i are considered_____of each other	neighboring	significant	dependent	independent	dependent
A sytem of mass points under the influence of force derived from potentials depend on_____only	energy	constant	position	motion	position
The generalized momentum associated with the co-ordinate q_i shall be defined as the term_____	linear momentum	canonical momentum	angular momentum	momentum	canonical momentum
The canonical momentum is otherwise known as_____	angular momentum	constant momentum	conjugate momentum	linear momentum	conjugate momentum
Lagrange's equationfollows from the _____principle	Jacobi's	Routh's	Hamilton's	D'Alemberts	Hamilton's
If the system is invariant under the_____along the given direction then the corresponding angular momentum is conserved	translation	irrotation	rotation	constant	rotation
The generalised momentum conjugate to the cyclic coordinate is_____	Zero	motion	conserved	non zero	conserved
A cyclic coordinate is also known as_____	constraint	ignorable	constant	motion	ignorable
In hamilton formulation,the 2n equations of motion describe the behaviour of the system point in _____	cyclic co-ordinates	projectory	configuration space	virtual displacement	configuration space
The lagrangian multiplier method is used for _____constraints	Rheonomous	non holonomic	monogenic	holonomic	holonomic
In variation principle the integral must be evaluated over the trajectory of the system in _____	phase space	geometric space	configuration space	space	phase space
In total kinetic energy $T=T_o+T_1+T_2$ where T_2 is _____function of the generalised velocities	quadratic	non linear	linear	constant	quadratic
If a component of the total applied forces_____then the corresponding component of the total linear momentum is conserved	cyclic	vanishes	constant	ignorable	vanishes
qj a rotation coordinates a generalised force is the component of the total _____about the axis of rotation	applied force	force	torque	applied torque	applied torque
If the system is _____ under the translation along the given direction then the corresponding linear momentum is conserved	constant	invariant	linear	zero	invariant

_____is used to find the curve joining two points along which a particle falling from rest under the influence of gravity in the least time	geodesics	projectory problem	brachistochrone problem	Holonomic problem	brachistochrone problem
In conservation theorem for total energy of system the energy function h is_____	zero	non zero	conserved	motion	conserved
The _____which is pertained to the function h to be the total energy of the system	surface	circumstence	volume	distance	circumstence
If qj is not a cartesian co-ordinate then pj does not necessarily have the dimensions of_____	canonical momentum	linear momentum	angular momentum	conjugate momentum	linear momentum
Hamilton's principle is also known as _____	monogenic principle	D'Alemberts principle	Integral principle	Lagrange's principle	Integral principle
The generalised momentum conjugate to the _____ coordinate is conserved	momentum	cyclic	linear	differential	cyclic
If L=T-U,the enrgy function h depends on the_____and functional form of the specific set of generalised co-ordinates	force	magnitude	direction	torque	magnitude
The procedure for _____ extra virtual displacement is the method of Lagrange's undetermined multipliers	eliminating	adding	multiplying	subtracting	eliminating
The fundamental problem of the calculus of variations is generalised to the case where f is a function of many independent variables and their_____	integrals	dependent variables	derivatives	components	derivatives
The energy function h=_____	L2-L0	L1-L0	L2-L1	L2+L1-L0	L2-L0
L2 is the ____function of the generalised velocity	cyclic	cubic	linear	quadratic	quadratic
A cyclic coordinate qj is the one which does not appear explicitly in the _____	Hamiltonian	Lagrangian	Eulerian	routhian	Lagrangian
A function f(x) is said to be_____at x=a when f'(a)=0	ignorable	applied torque	stationary	constant	stationary
Applied to the total energy function h,the lagrangian takes the form_____	h=T+V	h=2L2+L1-L	h=2L2+L1+L	h=T-V	h=2L2+L1-L

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DEPARTMENT OF MATHEMATICS

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Class:I M.Sc	**Semester:I**	**4 0 0 4**

UNIT III

Hamilton Equations of motion: Legendre Transformations and the Hamilton Equations of motion-canonical equations of Hamilton – Cyclic coordinates and conservation theorems – Routh's procedure - Derivation of Hamilton's equations from a variational principle – The principle of least action.

SUGGESTED READINGS

TEXT BOOK

T1: Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi

LEGENDRE TRANSFORMATIONS AND THE HAMILTON EQUATIONS OF MOTION

In the Lagrangian formulation (nonrelativistic), a system with n degrees of freedom possesses n equations of motion of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{8.1}$$

As the equations are of second order, the motion of the system is determined for all time only when $2n$ initial values are specified, for example, the n q_i's and n $\dot{q}_i$'s at a particular time t_1, or then n q_i's at *two* times, t_1 and t_2. We represent the state of the system by a point in an n-dimensional *configuration space* whose coordinates are the n generalized coordinates q_i and follow the motion of the system point in time as it traverses its path in configuration space. Physically, in the Lagrangian viewpoint a system with n independent degrees of freedom is a

problem in n independent variables $q_i(t)$, and $\dot{q}_i$ appears only as a shorthand for the time derivative of q_i. All n coordinates must be independent. In the Hamiltonian formulation there can be no constraint equations among the coordinates. If the n coordinates are not independent, a reduced set of m coordinates, with $m < n$, must be used for the formulation of the problem before proceeding with the following steps.

The Hamiltonian formulation is based on a fundamentally different picture. We seek to describe the motion in terms of *first-order* equations of motion. Since the number of initial conditions determining the motion must of course still be $2n$, there must be $2n$ independent first-order equations expressed in terms of $2n$ *independent variables*. Hence, the $2n$ equations of the motion describe the behavior of the system point in a *phase space* whose coordinates are the $2n$ independent variables. In thus doubling our set of independent quantities, it is natural (though not inevitable) to choose half of them to be the n generalized coordinates q_i. As we shall see, the formulation is nearly symmetric if we choose the other half of the set to be the generalized or *conjugate momenta* p_i already introduced by the definition (cf. Eq. (2.44)):

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i} \qquad \text{(no sum on } j\text{)} \tag{8.2}$$

where the j index shows the set of q's and $\dot{q}$'s. The quantities (q, p) are known as the *canonical variables*.*

From the mathematical viewpoint, it can however be claimed that the q's and $\dot{q}$'s have been treated as distinct variables. In Lagrange's equations, Eq. (8.1), the partial derivative of L with respect to q_i means a derivative taken with all other q's and all $\dot{q}$'s constant. Similarly, in the partial derivatives with respect to $\dot{q}$, the q's are kept constant. Treated strictly as a mathematical problem, the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables in our mechanical functions from $(q, \dot{q}, t)$ to (q, p, t), where p is related to q and $\dot{q}$ by Eqs. (8.2). The procedure for switching variables in this manner is provided by the *Legendre transformation*, which is tailored for just this type of change of variable.

Consider a function of only two variables $f(x, y)$, so that a differential of f has the form

$$df = u\,dx + v\,dy, \tag{8.3}$$

where

$$u = \frac{\partial f}{\partial x}, \qquad v = \frac{\partial f}{\partial y}. \tag{8.4}$$

We wish now to change the basis of description from x, y to a new distinct set of variables u, y, so that differential quantities are expressed in terms of the differentials du and dy. Let g be a function of u and y defined by the equation

$$g = f - ux. \tag{8.5}$$

A differential of g is then given as

$$dg = df - u\,dx - x\,du,$$

or, by (8.3), as

$$dg = v\,dy - x\,du,$$

which is exactly in the form desired. The quantities x and v are now functions of the variables u and y given by the relations

$$x = -\frac{\partial g}{\partial u}, \qquad v = \frac{\partial g}{\partial y}, \tag{8.6}$$

which are the analogues of Eqs. (8.4).

The Legendre transformation so defined is used frequently in thermodynamics. The first law of thermodynamics relates the differential change in energy, dU, to the corresponding change in heat content, dQ, and the work done, dW:

$$dU = dQ - dW. \tag{8.7}$$

For a gas undergoing a reversible process, Eq. (8.7) can be written as

$$dU = T\,dS - P\,dV, \tag{8.8}$$

where $U(S, V)$ is written as a function of the entropy, S, and the volume, V, where the temperature, T, and the gas pressure, P, are given by

$$T = \frac{\partial U}{\partial S} \qquad \text{and} \qquad P = -\frac{\partial U}{\partial V}. \tag{8.9}$$

The enthalpy, $H(S, P)$ is generated by the Legendre transformation

$$H = U + PV, \tag{8.10}$$

which gives

$$dH = T\,dS + V\,dP. \tag{8.11}$$

where

$$T = \frac{\partial H}{\partial S} \qquad \text{and} \qquad V = \frac{\partial H}{\partial P}.$$

Additional Legendre transformations,

$$\left.\begin{aligned} F &= U - TS \\ G &= H - TS, \end{aligned}\right\} \tag{8.12}$$

generate the Helmholtz free energy, $F(T, V)$, and the Gibbs free energy, $G(T, P)$.

The transformation from $(q, \dot{q}, t)$ to (q, p, t) differs from the type considered in Eqs. (8.3) to (8.12) only in that more than one variable is to be transformed. We begin by writing the differential of the Lagrangian, $L(q, \dot{q}, t)$, as

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{8.13}$$

The canonical momentum was defined in Eq. (2.44) as $p_i = \partial L / \partial \dot{q}_i$; substituting this into the Lagrange equation (8.1), we obtain

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \tag{8.14}$$

so Eq. (8.13) can be written as

$$dL = \dot{p}_i \, dq_i + p_i \, d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{8.13'}$$

The Hamiltonian $H(q, p, t)$ is generated by the Legendre transformation

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t), \tag{8.15}$$

which has the differential

$$dH = \dot{q}_i \, dp_i - \dot{p}_i \, dq_i - \frac{\partial L}{\partial t}, \tag{8.16}$$

where the term $p_i \, d\dot{q}_i$ is removed by the Legendre transformation. Since dH can also be written as

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \tag{8.17}$$

we obtain the $2n + 1$ relations

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}\right\} \tag{8.18}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{8.19}$$

Equations (8.18) are known as the *canonical equations of Hamilton*; they constitute the desired set of $2n$ first-order equations of motion replacing the n second-order Lagrange equations.*

The first half of Hamilton's equations give the $\dot{q}_i$'s as functions of (q, p, t). They form therefore the inverse of the constitutive equations (8.2), which define the momenta p_i as functions of $(q, \dot{q}, t)$. It may therefore be said that they provide no new information. In terms of solving mechanical problems by means of the canonical equations, the statement is correct. But within the framework of the Hamiltonian picture, where $H(q, p, t)$ is some given function obtained no matter how, the two halves of the set of Hamiltonian equations are equally independent and meaningful. The first half says how $\dot{q}$ depends on q, p, and t; the second says the same thing for $\dot{p}$.

Of course, the Hamiltonian H is constructed in the same manner, and has identically the same value, as h, the energy function defined in Eq. (2.53). But they are functions of different variables: Like the Lagrangian, h is a function of q, $\dot{q}$ (and possibly t), while H must always be expressed as a function of q, p (and possibly t). It is to emphasize this difference in functional behavior that different symbols have been given to the quantities even though they have the same numerical values.

Nominally, the Hamiltonian for each problem must be constructed via the Lagrangian formulation. The formal procedure calls for a lengthy sequence of steps:

1. With a chosen set of generalized coordinates, q_i, the Lagrangian $L(q_i, \dot{q}_i, t) = T - V$ is constructed.
2. The conjugate momenta are defined as functions of q_i, $\dot{q}_i$, and t by Eqs. (8.2).
3. Equation (8.15) is used to form the Hamiltonian. At this stage we have some mixed function of q_i, $\dot{q}_i$, p_i, and t.
4. Equations (8.2) are then inverted to obtain $\dot{q}_i$ as functions of (q, p, t). Possible difficulties in the inversion will be discussed below.
5. The results of the previous step are then applied to eliminate $\dot{q}$ from H so as to express it solely as a function of (q, p, t).

Now we are ready to use the Hamiltonian in the canonical equations of motion.

For many physical systems it is possible to shorten this drawn-out sequence quite appreciably. As has been described in Section 2.7, in many problems the Lagrangian is the sum of functions each homogeneous in the generalized veloc-

ities of degree 0, 1, and 2, respectively. In that case, H by the prescription of Eq. (8.15) is given by (cf. Eqs. (2.53) and (2.55))

$$H = \dot{q}_i p_i - L = \dot{q}_i p_i - [L_0(q_i, t) + L_1(q_i, t)\dot{q}_k + L_2(q_i, t)\dot{q}_k\dot{q}_m] \tag{8.20}$$

(no sum on i in the square brackets) where L_0 is the part of the Lagrangian that is independent of the generalized velocities, L_1 represents the coefficients of the part of the Lagrangian that is homogeneous in $\dot{q}_i$ in the first degree, and L_2 is the part that is homogeneous in $\dot{q}_i$ in the second degree. Further, if the equations defining the generalized coordinates don't depend on time explicitly, then $L_2\dot{q}_k\dot{q}_m = T$ (the kinetic energy), and if the forces are derivable from a conservative potential V (that is, work is independent of the path), then $L_0 = -V$. When both these conditions are satisfied, the Hamiltonian is *automatically* the total energy:

$$H = T + V = E. \tag{8.21}$$

If either Eq. (8.20) or (8.21) holds, then much of the algebra in steps 3 and 4 above is eliminated.

We can at times go further. In large classes of problems, it happens that L_2 is a quadratic function of the generalized velocities and L_1 is a linear function of the same variables with the following specific functional dependencies:

$$L(q_i, \dot{q}_i, t) = L_0(q, t) + \dot{q}_i a_i(q, t) + \dot{q}_i^2 T_i(q, t), \tag{8.22}$$

where the a_i's and the T_i's are functions of the q's and t.

The algebraic manipulations required in steps 2–5 can then be carried out, at least formally, once and for all. To show this, let us form the $\dot{q}_i$'s into a single column matrix $\dot{\mathbf{q}}$. Under the given assumptions the Lagrangian can be written as

$$L(q, \dot{q}, t) = L_0(q. t) + \tilde{\dot{\mathbf{q}}}\mathbf{a} + \tfrac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}}, \tag{8.23}$$

where the single row matrix $\tilde{\dot{\mathbf{q}}}$ has been written explicitly as the transpose of a single column matrix, $\dot{\mathbf{q}}$. Here $\mathbf{a}$ is a column matrix, and $\mathbf{T}$ is a square $n \times n$ matrix (much like the corresponding matrix introduced in Section 6.2). The elements of both are in general functions of q and t. To illustrate this formalism, let us consider the special case where $q_i = \{x, y, z\}$ and $\mathbf{T}$ is diagonal. We would then write

$$\frac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} = \frac{1}{2}(\dot{x}\dot{y}\dot{z})\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{8.24a}$$

and

$$\tilde{\dot{\mathbf{q}}}\mathbf{a} = (\dot{x}\dot{y}\dot{z})\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x\dot{x} + a_y\dot{y} + a_z\dot{z} = \mathbf{a} \cdot \dot{\mathbf{r}}. \tag{8.24b}$$

In this notation the Hamiltonian, $H = \tilde{\dot{\mathbf{q}}}\mathbf{p} - L$, becomes

$$H = \tilde{\dot{\mathbf{q}}}(\mathbf{p} - \mathbf{a}) - \tfrac{1}{2}\tilde{\dot{\mathbf{q}}}\mathbf{T}\dot{\mathbf{q}} - L_0. \tag{8.24c}$$

The conjugate momenta, considered as a column matrix $\mathbf{p}$, is then, by Eq. (8.2), given as

$$\mathbf{p} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{a}, \tag{8.25}$$

which can be inverted (step 4) to the column vector $\dot{\mathbf{q}}$

$$\dot{\mathbf{q}} = \mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}). \tag{8.26a}$$

This step presupposes that $\mathbf{T}^{-1}$ exists, which it normally does by virtue of the positive definite property of kinetic energy.

The corresponding equation for $\tilde{\dot{\mathbf{q}}}$ is

$$\tilde{\dot{\mathbf{q}}} = (\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}. \tag{8.26b}$$

To obtain the correct functional form for the Hamiltonian, Eqs. (8.26) must be used to replace $\dot{\mathbf{q}}$ and $\tilde{\dot{\mathbf{q}}}$, yielding the final form for the Hamiltonian:

$$H(q, p, t) = \tfrac{1}{2}(\tilde{\mathbf{p}} - \tilde{\mathbf{a}})\mathbf{T}^{-1}(\mathbf{p} - \mathbf{a}) - L_0(q, t). \tag{8.27}$$

If the Lagrangian can be written in the form of Eq. (8.23), then we can immediately skip the intervening steps and write the Hamiltonian as Eq. (8.27). The inverse matrix $\mathbf{T}^{-1}$ can usually most easily be obtained straightforwardly as

$$\mathbf{T}^{-1} = \frac{\tilde{\mathbf{T}}_c}{|\mathbf{T}|}, \tag{8.28}$$

where $\mathbf{T}_c$ is the cofactor matrix whose elements $(\mathbf{T}_c)_{jk}$ are $(-1)^{j+k}$ times the determinant of the matrix obtained by striking out the jth row and the kth column of $\mathbf{T}$.

In the example Eq. (8.24a), these three matrices are given explicitly by

$$\mathbf{T} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix}, \qquad \text{and}$$

$$\tilde{\mathbf{T}}_c = \begin{bmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & m^2 \end{bmatrix},$$

and the determinant $|\mathbf{T}| = m^3$. It is easy to see that for the usual case when $\mathbf{T}$ is diagonal, then $\mathbf{T}^{-1}$ is also diagonal with elements that are just the reciprocals of the corresponding elements of $\mathbf{T}$.

CYCLIC COORDINATES AND CONSERVATION THEOREMS

According to the definition given in Section 2.6, a cyclic coordinate q_j is one that does not appear explicitly in the Lagrangian; by virtue of Lagrange's equations

its conjugate momentum p_j is then a constant. But comparison of Eq. (8.14) with Eq. (8.16) has already told us that

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}.$$

A coordinate that is cyclic will thus also be absent from the Hamiltonian.* Conversely if a generalized coordinate does not occur in H, the conjugate momentum is conserved. The momentum conservation theorems of Section 2.6 can thus be transferred to the Hamiltonian formulation with no more than a substitution of H for L. In particular, the connection between the invariance or symmetry properties of the physical system and the constants of the motion can also be derived in terms of the Hamiltonian. For example, if a system is completely self-contained, with only internal forces between the particles, then the system can be moved as a rigid ensemble without affecting the forces or subsequent motion. The system is said to be invariant under a rigid displacement. Hence, a generalized coordinate describing such a rigid motion will not appear explicitly in the Hamiltonian, and the corresponding conjugate momentum will be conserved. If the rigid motion is a translation along some particular direction, then the conserved momentum is the corresponding Cartesian component of the total linear (canonical) momentum of the system. Since the direction is arbitrary, the total vector linear momentum is conserved. The rigid displacement may be a rotation, from whence it follows that the total angular momentum vector is conserved. Even if the system interacts with external forces, there may be a symmetry in the situation that leads to a conserved canonical momentum. Suppose the system is symmetrical about a given axis so that H is invariant under rotation about that axis. Then H obviously cannot involve the rotation angle about the axis and the particular angle variable must be a cyclic coordinate. It follows, as in Section 2.6, that the component of the angular momentum about that axis is conserved.†

The considerations concerning h in Section 2.7 have already shown that if L (and in consequence of Eq. (8.15), also H) is not an explicit function of t, then H is a constant of motion. This can also be seen directly from the equations of motion (8.18) by writing the total time derivative of the Hamiltonian as

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}.$$

In consequence of the equations of motion (8.18), the first two sums on the right cancel each other, and it therefore follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{8.41}$$

Thus if t doesn't appear explicitly in L, it will also not be present in H, and H will be constant in time.

Further, it was proved in Section 2.7 that if the equations of transformation that define the generalized coordinates (1.38),

$$\mathbf{r}_m = \mathbf{r}_m(q_1, \ldots, q_n; t),$$

do not depend explicitly upon the time, and if the potential is velocity independent, then H is the total energy, $T+V$. The identification of H as a constant of the motion and as the total energy are two separate matters, and the conditions sufficient for the one are not enough for the other. It can happen that the Eqs. (1.38) do involve time explicitly but that H does not. In this case, H is a constant of the motion but it is *not* the total energy. As was also emphasized in Section (2.6), the Hamiltonian is dependent both in magnitude and in functional form upon the initial choice of generalized coordinates. For the Lagrangian, we have a specific prescription, $L = T - V$, and a change of generalized coordinates within that prescription may change the functional appearance of L but cannot alter its magnitude. On the other hand, use of a different set of generalized coordinates in the definition for the Hamiltonian, Eq. (8.15), may lead to an entirely different quantity for the Hamiltonian. It may be that for one set of generalized coordinates H is conserved, but that for another it varies in time.

To illustrate some of these points in a simple example, we may consider a somewhat artificial one-dimensional system. Suppose a point mass m is attached to a spring, of force constant k, the other end of which is fixed on a massless cart that is being moved uniformly by an external device with speed v_0 (cf. Fig. 8.1). If we take as generalized coordinate the position x of the mass particle in the stationary system, then the Lagrangian of the system is obviously

$$L(x, \dot{x}, t) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2. \tag{8.42}$$

(For simplicity, the origin has been chosen so that the cart passes through it at $t = 0$.) The corresponding equation of motion is clearly

$$m\ddot{x} = -k(x - v_0 t).$$

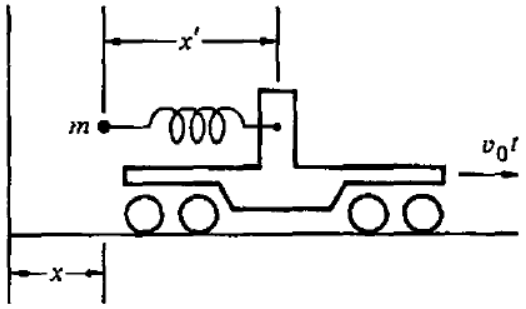

FIGURE 8.1 A harmonic oscillator fixed to a uniformly moving cart.

An obvious way of solving this equation is to change the unknown to $x'(t)$ defined as

$$x' = x - v_0 t, \tag{8.43}$$

and noting that $\ddot{x}' = \ddot{x}$, the equation of motion becomes

$$m\ddot{x}' = -kx'. \tag{8.44}$$

From Eq. (8.43), x' is the displacement of the particle relative to the cart; Eq. (8.44) says that to an observer on the cart the particle exhibits simple harmonic motion, as would be expected on the principle of equivalence in Galilean relativity.

Having looked at the nature of the motion, let us consider the Hamiltonian formulation. Since x is the Cartesian coordinate of the particle, and the potential does not involve generalized velocities, the Hamiltonian relative to x is the sum of the kinetic and potential energies, that is, the total energy. In functional form the Hamiltonian is given by

$$H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2. \tag{8.45}$$

The Hamiltonian *is* the total energy of the system, but since it is explicitly a function of t, it is *not* conserved. Physically this is understandable; energy must flow into and out of the "external physical device" to keep the cart moving uniformly against the reaction of the oscillating particle.*

Suppose now we formulated the Lagrangian from the start in terms of the relative coordinate x'. The same prescription gives the Lagrangian as

$$L(x', \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}. \tag{8.46}$$

In setting up the corresponding Hamiltonian, we note there is now a term linear in $\dot{x}'$, with the single component of **a** being mv_0. The new Hamiltonian is now

$$H'(x', p') = \frac{(p' - mv_0)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_0^2}{2}. \tag{8.47}$$

Note that the last term is a constant involving neither x' nor p'; it could, if we wished, be dropped from H' without affecting the resultant equations of motion. Now H' is *not* the total energy of the system, but it *is* conserved. Except for the last term, it can be easily identified as the total energy of motion of the particle relative to the moving cart. The two Hamiltonian's are different in magnitude.

ROUTH'S PROCEDURE

It has been remarked that the Hamiltonian formulation is not particularly helpful in the direct solution of mechanical problems. Often we can solve the $2n$ first-order equations only by eliminating some of the variables, for example, the p variables, which speedily leads back to the second-order Lagrangian equations of motion. But an important exception should be noted. The Hamiltonian procedure is especially adapted to the treatment of problems involving cyclic coordinates.

Let us consider the situation in Lagrangian formulation when some coordinate, say q_n, is cyclic. The Lagrangian as a function of q and $\dot{q}$ can then be written

$$L = L(q_1, \ldots, q_{n-1};\ \dot{q}_1, \ldots, \dot{q}_n;\ t).$$

All the generalized velocities still occur in the Lagrangian and in general will be functions of the time. We still have to solve a problem of n degrees of freedom, even though one degree of freedom corresponds to a cyclic coordinate. A cyclic coordinate in the Hamiltonian formulation, on the other hand, truly deserves its alternative description as "ignorable," for in the same situation p_n is some constant α, and H has the form

$$H = H(q_1, \ldots, q_{n-1};\ p_1, \ldots, p_{n-1};\ \alpha; t).$$

In effect, the Hamiltonian now describes a problem involving only $n - 1$ coordinates, which may be solved completely ignoring the cyclic coordinate except as it is manifested in the constant of integration α, to be determined from the initial conditions. The behavior of the cyclic coordinate itself with time is then found by integrating the equation of motion

$$\dot{q}_n = \frac{\partial H}{\partial \alpha}.$$

The advantages of the Hamiltonian formulation in handling cyclic coordinates may be combined with the Lagrangian conveniences for noncyclic coordinates by a method devised by Routh. Essentially, we carry out a mathematical transformation from the $q, \dot{q}$ basis to the q, p basis only for those coordinates that are cyclic, obtaining their equations of motion in the Hamiltonian form, while the remaining coordinates are governed by Lagrange equations. If the cyclic coordinates are labeled $q_{s+1}, \ldots, q_n$, then a new function R (known as the Routhian) may be introduced, defined as

$$R(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_s;\ p_{s+1}, \ldots, p_n;\ t) = \sum_{i=s+1}^{n} p_i \dot{q}_i - L, \qquad (8.48)$$

which is equivalent to writing

$$R(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_s;\ p_{s+1}, \ldots, p_n;\ t) =$$
$$H_{cycl}(p_{s+1}, \ldots, p_n) - L_{noncycl}(q_1, \ldots, q_s;\ \dot{q}_1, \ldots, \dot{q}_s). \qquad (8.49)$$

It is easy to show for the s nonignorable coordinates, the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \qquad i = 1, \ldots, s, \qquad (8.50)$$

are satisfied. while for the $n-s$ ignorable coordinates, Hamilton's equations apply as

$$\frac{\partial R}{\partial q_i} = -\dot{p}_i = 0, \qquad \text{and} \qquad \frac{\partial R}{\partial p_i} = \dot{q}_i, \qquad i = s+1, \ldots, n. \qquad (8.51)$$

A simple, almost trivial, example may clarify Routh's procedure and the physical significance of the quantities involved. Consider the Kepler problem investigated in Section 3.7, that of a single particle moving in a plane under the influence of the inverse-square central force $f(r)$ derived from the potential $V(r) = -k/r^n$. The Lagrangian is then

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r^n}.$$

As noted before, the ignorable coordinate is θ, and if the constant conjugate momentum is denoted by p_θ, the corresponding Routhian (8.49) is

$$R(r, \dot{r}, p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{1}{2}m\dot{r}^2 - \frac{k}{r^n}.$$

Physically we see that the Routhian is the equivalent one-dimensional potential $V'(r)$ *minus* the kinetic energy of radial motion.

Applying the Lagrange equation (8.50) to the noncyclic radial coordinate r, we obtain the equation of motion (3.11)

$$\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{nk}{r^{n+1}} = 0. \tag{8.52}$$

Applying Hamilton's equation (8.51) to the cyclic variable θ, we obtain the pair of equations

$$\dot{p}_\theta = 0 \qquad \text{and} \qquad \frac{p_\theta}{mr^2} = \dot{\theta}. \tag{8.53}$$

whose solution is the same as Eq. (3.8),

$$p_\theta = mr^2\dot{\theta} = l = \text{constant}.$$

Typically, Routh's procedure does not add to the physics of the analysis presented earlier in Chapter 3, but it makes the analysis more automatic. In complicated problems with many degrees of freedom, this feature can be a considerable advantage. it is not surprising therefore that Routh's procedure finds its greatest usefulness in the direct solution of problems relating to engineering applications. But as a fundamental entity, the Routhian is a sterile hybrid, combining some of the features of both the Lagrangian and the Hamiltonian pictures. For the development of various formalisms of classical mechanics, the complete Hamiltonian formulation is more fruitful.

DERIVATION OF HAMILTON'S EQUATIONS FROM A VARIATIONAL PRINCIPLE

Lagrange's equations have been shown to be the consequence of a variational principle, namely, the Hamilton's principle of Section 2.1. Indeed, the variational method is often the preferable one for deriving Lagrange's equations, for it is applicable to types of systems not usually included within the scope of mechanics. It would be similarly advantageous if a variational principle could be found that

leads directly to the Hamilton's equations of motion. Hamilton's principle,

$$\delta I \equiv \delta \int_{t_1}^{t_2} L\, dt = 0, \tag{8.64}$$

lends itself to this purpose, but as formulated originally it refers to paths in configuration space. The first modification therefore is that the integral must be evaluated over the trajectory of the system point in phase space, and the varied paths must be in the neighborhood of this phase space trajectory. In the spirit of the Hamiltonian formulation, both q and p must be treated as independent coordinates of phase space, to be varied independently. To this end the integrand in the action integral, Eq. (8.64), must be expressed as a function of both q and p, and their time derivatives, through Eq. (8.15). Equation (8.64) then appears as

$$\delta I = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t)\right) dt = 0. \tag{8.65}$$

As a variational principle in phase space, Eq. (8.65) is sometimes referred to as the *modified Hamilton's principle*. Although it will be used most frequently in connection with transformation theory (see Chapter 9), the main interest in it here is to show that the principle leads to Hamilton's canonical equations of motion.

The modified Hamilton's principle is exactly of the form of the variational problem in a space of $2n$ dimensions considered in Section 2.3 (cf. Eq. (2.14)):

$$\delta I = \delta \int_{t_1}^{t_2} f(q, \dot{q}, p, \dot{p}, t)\, dt = 0, \tag{8.66}$$

for which the $2n$ Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_j}\right) - \frac{\partial f}{\partial q_j} = 0 \qquad j = 1, \ldots, n \tag{8.67}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_j}\right) - \frac{\partial f}{\partial p_j} = 0 \qquad j = 1, \ldots, n. \tag{8.68}$$

The integrand f as given in Eq. (8.65) contains $\dot{q}_j$ only through the $p_i\dot{q}_i$ term, and q_j only in H. Hence, Eqs. (8.67) lead to

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0. \tag{8.69}$$

On the other hand, there is no explicit dependence of the integrand in Eq. (8.65) on $\dot{p}_j$. Equations (8.68) therefore reduce simply to

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0. \tag{8.70}$$

Equations (8.69) and (8.70) are exactly Hamilton's equations of motion, Eqs. (8.18). The Euler–Lagrange equations of the modified Hamilton's principle are thus the desired canonical equations of motion.

This derivation of Hamilton's equations from the variational principle is so brief as to give the appearance of a sleight-of-hand trick. One wonders whether something extra has been sneaked in while we were being misdirected by the magician's patter. Is the modified Hamilton's principle equivalent to Hamilton's principle, or does it contain some additional physics? The question is largely irrelevant; the primary justification for the modified Hamilton's principle is that it leads to the canonical equations of motion in phase space. After all, no further argument was given for the validity of Hamilton's principle than that it corresponded to the Lagrangian equations of motion. So long as Hamiltonian can be constructed, the Legendre transformation procedure shows that the Lagrangian and Hamiltonian formulations, and therefore their respective variational principles, have the same physical content.

Nonetheless, there are advantages to requiring that the varied paths in the modified Hamilton's principle return to the same end points in both q and p, for we then have a more generalized condition for Hamilton's equations of motion. As with Hamilton's principle, if there is no variation at the end points we can add a total time derivative of any arbitrary (twice-differentiable) function $F(q, p, t)$ to the integrand without affecting the validity of the variational principle. Suppose, for example, we subtract from the integrand of Eq. (8.65) the quantity

$$\frac{d}{dt}(q_i p_i).$$

The modified Hamilton's principle would then read

$$\delta \int_{t_1}^{t_2} (-\dot{p}_i q_i - H(q, p, t))\, dt = 0. \tag{8.71}$$

Here the f integrand of Eq. (8.66) is a function of $\dot{p}$, and it is easily verified that the Euler–Lagrange equations (8.67) and (8.68) with this f again correspond to Hamilton's equations of motion, Eqs. (8.18). Yet the integrand in Eq. (8.71) is not the Lagrangian nor can it in general be simply related to the Lagrangian by a point transformation in configuration space. By restricting the variation of both q and p to be zero at the end points, the modified Hamilton's principle provides an independent and general way of setting up Hamilton's equations of motion without a prior Lagrangian formulation. If you will, it does away with the necessity of a linkage between the Hamiltonian canonical variables and a corresponding Lagrangian set of generalized coordinates and velocities. This will be very important to us in the next chapter where we examine transformations of phase space variables that preserve the Hamiltonian form of the equations of motion.

The requirement of independent variation of q and $\dot{p}$, so essential for the above derivation, highlights the fundamental difference between the Lagrangian and Hamiltonian formulations. Neither the coordinates q_i nor the momenta p_i are to be considered there as the more fundamental set of variables; both are equally independent. Only by broadening the field of independent variables from n to $2n$ quantities are we enabled to obtain equations of motion that are of first order. In a sense, the names "coordinates" and "momenta" are unfortunate, for they bring to mind pictures of spatial coordinates and linear, or at most, angular momenta. A wider meaning must now be given to the terms. The division into coordinates and momenta corresponds to no more than a separation of the independent variables describing the motion into two groups having an almost symmetrical relationship to each other through Hamilton's equations.

THE PRINCIPLE OF LEAST ACTION

Another variational principle associated with the Hamiltonian formulation is known as *the principle of least action*. It involves a new type of variation, which we shall call the Δ-variation, requiring detailed explanation. In the δ-variation process used in the discussion of Hamilton's principle in Chapter 2, the varied path in configuration space always terminated at end points representing the system configuration at the same time t_1 and t_2 as the correct path. To obtain Lagrange's equations of motion, we also required that the varied path return to the same end points in configuration space, that is, $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The Δ-variation is less constrained; in general, the varied path over which an integral is evaluated may end at different times than the correct path, and there

may be a variation in the coordinates at the end points. We can however use the same parameterization of the varied path as in the δ-variation. In the notation of Section 2.3, a family of possible varied paths is defined by functions (cf. Eq. (2.15))

$$q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i(t), \tag{8.72}$$

where α is an infinitesimal parameter that goes to zero for the correct path. Here the functions η_i do not necessarily have to vanish at the end points, either the original or the varied. All that is required is that they be continuous and differentiable. Figure 8.3 illustrates the correct and varied path for a Δ-variation in configuration space.

Let us evaluate the Δ-variation of the action integral:

$$\Delta \int_{t_1}^{t_2} L\, dt \equiv \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L(\alpha)\, dt - \int_{t_1}^{t_2} L(0)\, dt, \tag{8.73}$$

where $L(\alpha)$ means the integral is evaluated along the varied path and $L(0)$ co
spondingly refers to the actual path of motion. The variation is clearly compo
of two parts. One arises from the change in the limits of the integral; to first-or
infinitesimals, this part is simply the integrand on the actual path times the dif
ence in the limits in time. The second part is caused by the change in the integr
on the varied path, but now between the same time limits as the original integ
We may therefore write the Δ-variation of the action integral as

$$\Delta \int_{t_1}^{t_2} L\,dt = L(t_2)\Delta t_2 - L(t_1)\Delta t_1 + \int_{t_1}^{t_2} \delta L\,dt. \qquad (8.$$

Here the variation in the second integral can be carried out through a parar
terization of the varied path, exactly as for Hamilton's principle except that

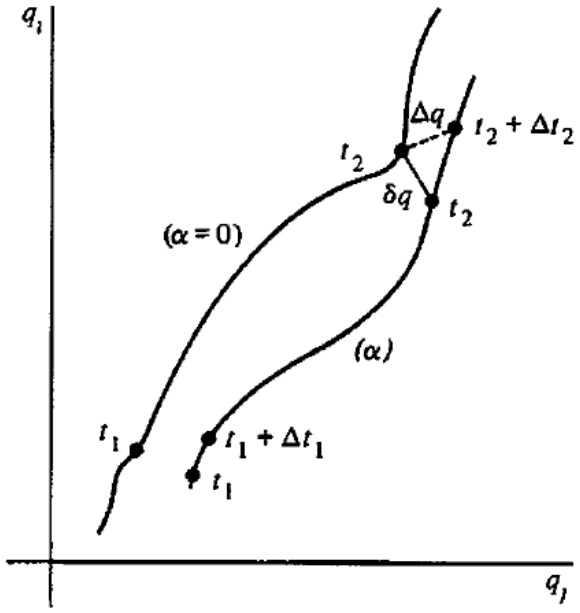

FIGURE 8.3 The Δ-variation in configuration space.

variation in q_i does not vanish at the end points. The end point terms arising in the integration by parts must be retained, and the integral term on the right appears as

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i\, dt + \frac{\partial L}{\partial \dot{q}_i}\, \delta q_i \Big|_1^2.$$

By Lagrange's equations the quantities in the square brackets vanish, and the Δ-variation therefore takes the form

$$\Delta \int_{t_1}^{t_2} L\, dt = (L\Delta t + p_i \delta q_i)\Big|_1^2. \tag{8.75}$$

In Eq. (8.75), δq_i refers to the variation in q_i at the original end point times t_1 and t_2. We would like to express the Δ-variation in terms of the change Δq_i between q_i at the end points of the actual path and q_i at the end points of the varied path, including the change in end point times. It is clear from Fig. 8.3 that these two variations are connected by the relation*

$$\Delta q_i = \delta q_i + \dot{q}_i \Delta t. \tag{8.76}$$

Hence, Eq. (8.75) can be rewritten as

$$\Delta \int_{t_1}^{t_2} L\, dt = (L\Delta t - p_i \dot{q}_i \Delta t + p_i \Delta q_i)\Big|_1^2$$

or

$$\Delta \int_{t_1}^{t_2} L\, dt = (p_i \Delta q_i - H\, \Delta t)\Big|_1^2. \tag{8.77}$$

To obtain the principle of least action, we restrict our further considerations by three important qualifications:

1. Only systems are considered for which L, and therefore H, are not explicit functions of time, and in consequence H is conserved.
2. The variation is such that H is conserved on the varied path as well as on the actual path.
3. The varied paths are further limited by requiring that Δq_i vanish at the end points (but not Δt).

The nature of the resultant variation may be illustrated by noting that the varied path satisfying these conditions might very well describe the same curve in configuration space as the actual path. The difference will be the speed with which the system point traverses this curve; that is, the functions $q_i(t)$ will be altered in the varied path. In order then to preserve the same value of the Hamiltonian at all points on the varied path, the times of the end points must be changed. With these three qualifications satisfied, the Δ-variation of the action integral, Eq. (8.77), reduces to

$$\Delta \int_{t_1}^{t_2} L\,dt = -H(\Delta t_2 - \Delta t_1). \tag{8.78}$$

But under the same conditions, the action integral itself becomes

$$\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(t_2 - t_1),$$

the Δ-variation of which is

$$\Delta \int_{t_1}^{t_2} L\,dt = \Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt - H(\Delta t_2 - \Delta t_1). \tag{8.79}$$

Comparison of Eqs. (8.78) and (8.79) finally gives the *principle of least action*:*

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i\,dt = 0. \tag{8.80}$$

By way of caution, note that the modified Hamilton's principle can be written in a form with a superficial resemblance to Eq. (8.80). If the trajectory of the system point is described by a parameter θ, as in Sections 7.10 and 8.4, the modified Hamilton's principle appears as

$$\delta \int_{\theta_1}^{\theta_2} (p_i \dot{q}_i - H)t'\,d\theta = 0. \tag{8.81}$$

It will be recalled (cf. footnote on p. 351) that the momenta p_i do not change under the shift from t to θ, and that $\dot{q}_i t' = q_i'$. Further, the momentum conjugate to t is $-H$. Hence, Eq. (8.81) can be rewritten as

$$\delta \int_{\theta_1}^{\theta_2} \sum_{i=1}^{n+1} p_i q_i'\,d\theta = 0, \tag{8.82}$$

where t has been denoted by q_{n+1}. There should however be no confusion between Eq. (8.82) and the principle of least action. Equations (8.82) involve phase

space of $(2n + 2)$ dimensions, as is indicated by the explicit summation to $i = n + 1$, whereas Eq. (8.80) is in the usual configuration space. But most important, the principle of least action is in terms of a Δ-variation for constant H, while Eq. (8.82) employs the δ-variation, and H in principle could be a function of time. Equation (8.82) is nothing more than the modified Hamilton's principle, and the absence of a Hamiltonian merely reflects the phenomenon that the Hamiltonian vanishes identically for the "homogeneous problem."

The least action principle itself can be exhibited in a variety of forms. In non-relativistic mechanics, if the defining equations for the generalized coordinates do not involve the time explicitly, then the kinetic energy is a quadratic function of the $\dot{q}_l$'s (cf. Eq. (1.71)):

$$T = \tfrac{1}{2} M_{jk}(q)\dot{q}_j\dot{q}_k. \tag{8.83}$$

When in addition the potential is not velocity dependent, the canonical momenta are derived from T only, and in consequence

$$p_l\dot{q}_l = 2T.$$

The principle of least action for such systems can therefore be written as

$$\Delta \int_{t_1}^{t_2} T\, dt = 0. \tag{8.84}$$

If, further, there are no external forces on the system, as, for example, a rigid body with no net applied forces, then T is conserved along with the total energy H. The least action principle then takes the special form

$$\Delta(t_2 - t_1) = 0. \tag{8.85}$$

Equation (8.85) states that of all paths possible between two points, consistent with conservation of energy, the system moves along that particular path for which the time of transit is the least (more strictly, an extremum). In this form the principle of least action recalls Fermat's principle in geometrical optics that a light ray travels between two points along such a path that the time taken is the least. We discussed these considerations in Section 10–8 of the Second Edition when we considered the connection between the Hamiltonian formulation and geometrical optics.

In Section 7.4 we discussed the infinitesimal interval in a metric space giving the interval as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{7.32'}$$

where $g_{\mu\nu}$ was the metric of a possibly curvilinear space and ds^2 was the interval traversed for displacements given by dx^μ. We can do something entirely similar here whenever T is of the form of Eq. (8.83). A configuration space is therefore constructed for which the M_{jk} coefficients form the metric tensor. In general, the

space will be curvilinear and nonorthogonal. The element of path length in the space is then defined by (cf. Eq. (7.33′))

$$(d\rho)^2 = M_{jk}\, dq_j\, dq_k \tag{8.86}$$

so that the kinetic energy has the form

$$T = \frac{1}{2}\left(\frac{d\rho}{dt}\right)^2, \tag{8.87}$$

or equivalently

$$dt = \frac{d\rho}{\sqrt{2T}}. \tag{8.88}$$

Equation (8.88) enables us to change the variable in the abbreviated action integral from t to ρ, and the principle of least action becomes

$$\Delta \int_{t_1}^{t_2} T\, dt = 0 = \Delta \int_{\rho_1}^{\rho_2} \sqrt{T/2}\, d\rho,$$

or, finally

$$\Delta \int_{\rho_1}^{\rho_2} \sqrt{H - V(q)}\, d\rho = 0. \tag{8.89}$$

Equation (8.89) is often called *Jacobi's form of the least action principle*. It now refers to the path of the system point in a special curvilinear configuration space characterized by a metric tensor with elements M_{jk}. The system point traverses the path in this configuration space with a speed given by $\sqrt{2T}$. If there are no forces acting on the body, T is constant, and Jacobi's principle says the system point travels along the shortest path length in the configuration space. Equivalently stated, the motion of the system is then such that the system point travels along the geodesics of the configuration space.

Note that the Jacobi form of the principle of least action is concerned with the *path* of the system point rather than with its motion in *time*. Equation (8.89) is a statement about the element of path length $d\rho$; the time nowhere appears, since H is a constant and V depends upon q_i only. Indeed, it is possible to use the Jacobi form of the principle to furnish the differential equations for the path, by a procedure somewhat akin to that leading to Lagrange's equations. In the form of Fermat's principle, the Jacobi version of the principle of least action finds many fruitful applications in geometrical optics and in electron optics. To go into any detail here would lead us too far afield.

A host of other similar, variational principles for classical mechanics can be derived in bewildering variety. To give one example out of many, the principle of least action leads immediately to *Hertz's principle of least curvature*, which states that a particle not under the influence of external forces travels along the

path of least curvature. By Jacobi's principle such a path must be a geodesic, and the geometrical property of minimum curvature is one of the well-known characteristics of a geodesic. It has been pointed out that variational principles in themselves contain no new physical content, and they rarely simplify the practical solution of a given mechanical problem. Their value lies chiefly as starting points for new formulations of the theoretical structure of classical mechanics. For this purpose, Hamilton's principle is especially fruitful, and to a lesser extent, so also is the principle of least action.

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Define Cyclic coordinates and Explain conservation theorems.
2.Derive Hamilton's canonical equation of motion
3.Obtain the hamilton's equation of motion considering a single non relativistic particle moving in an electromagnetic field
4.Explain the principle of least action
5.Explain the construction of Hamilton through Lagrangian.

Part C (10 Marks)

1.Explain Routh's procedure.
2.Obtain the hamilton's equation of motion using spherical polar co-ordinates considering the spatial motion of a particle in the central force field
4.Derivation of Hamilton equation from a variational principle
5.Explain the principle of least action

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UNIT-III

Subject: Mechanics **Subject Code: 17MMP106**

Hamilton Equations of motion

Part-A(20X1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Multiple Choice Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
If the rigid motion is a rotation then the total_____is conserved	canonical momentum	linear momentum	angular momentum	conjugate momentum	angular momentum
The hamiltonian is the total energy of the system_____	$H^2=T^2+V^2$	H=T-V	H=T+V	$T^2=H^2+V^2$	H=T+V
In hamiltonian formulation,we have a set of 2n __equation of motion	second order	fourth order	first order	third order	first order
The harmonic oscillator problem is an example of __equation	Euler's	hamilton lagrange's	hamilton jacobi's	Routh's	Hamilton jacobi's
Tthe variational principle associated with the hamilton's formulation is known as the _____	holonomic problem	principle of least action	monogenic principle	Brachistrone problem	principle of least action
Two particles connected bya rigid rod of length 'L' is given by the equation(x2-x1)^2+(y2-y1)^2-L^2=0 the constraint is _____	Rheonomous	non holonomic	holonomic	scheleronomous	holonomic
The hamilton principle functiondiffers at most from the indefinite integral of the lagrangian by_____	the invariants	time	coordinates	a constant	a constant
In Lagrangian formulation,we have a set of n ___equation of motion	second order	fourth order	first order	third order	second order
The important variational principle associated with hamiltonian formulation is the _____	Integral principle	principle of least action	Lagrange's principle	monogenic principle	principle of least action
If H is not an explicit function of t, then H is a _____ of the motion	energy	constant	position	linear	constant
If the equations of transformation do not depend explicitly on time and if the potential energy is velocity independent,then H is the _____ of the system	total force	total work	total energy	total momentum	total energy
The principle of least action states that the variation in A with time vanishes on the actual path as compared with some neighbouring paths, provided H is _____ throughout the actual path	inclined	linear	constant	dependent	constant
The end points are _____ in both Hamilton's principle and principle of least action	fixed	changed	inclined	ignorable	fixed
All coordinates of a dynamical system of n degrees of freedom are _____	ignorable	stationary	non zero	zero	ignorable
In _____ formulation, we have a set of n second order equations of motion	Hamiltonian	Lagrangian	Eulerian	Lorentz relation	Lagrangian
Routhian is the equivalent one -dimensional potential V'(r)minus the _____ energy of the radial motion	potential	scalar	kinetic	proportionate	kinetic
The _____ vanishes identically for the homogeneous problem	Hamiltonian	Lagrangian	Eulerian	Lorentz relation	Hamiltonian
In Hamiltonian formulation,the momentum are also _____ variables	dependent	inclined	independent	single	independent
For a conservative system, the Hamiltonian is _____ and the potential energy is independent of time	energy	constant	the invariants	linear	constant
In Hamiltonian formulation the _____ variables are the generalised coordinates and the generalised momenta	independent	dependent	constant	fixed	independent
The function H is known as the _____ of the system	Lorentz relation	Eulerian	Hamiltonian	Lagrangian	Hamiltonian
The function qi(t) will be changed in the _____	actual path	varied path	phase path	configuration path	varied path
The enthalpy H(S,P) is generated by the _____	contact transformation	Legendre transformation	canonical transformation	point transformation	Legendre transformation
The advantage of the Hamiltonian formulation in handling cyclic coordinates may be combined with the Lagrangian for non cyclic coordinates by the method devised by _____	Lorentz	routh	Eulerian	Hamiltonian	Routh
Variational method is often the preferable one for deriving _____ equation	Lagrange's	Hamiltonian	monogenic principle	Integral principle	Lagrange's

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DEPARTMENT OF MATHEMATICS

Subject: Mechanics	**Subject Code: 17MMP106**	**L T P C**
Class:I M.Sc	**Semester:I**	**4 0 0 4**

UNIT IV

Canonical transformations: The equations of canonical transformation – Examples of Canonical transformations – Poission Brackets and other Canonical invariants – integral invariants of Poincare, Lagrange brackets.

SUGGESTED READINGS

TEXT BOOK

T1: Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R1: Gantmacher, F., (2013). Lectures in Analytic Mechanics, MIR Publishers, Moscow.

R3: Loney, S. L., (1979). An elementary treatise on Statics, Kalyani Publishers, New Delhi.

When applied in a straightforward manner, the Hamiltonian formulation usually does not materially decrease the difficulty of solving any given problem in mechanics. We wind up with practically the same differential equations to be solved as are provided by the Lagrangian procedure. The advantages of the Hamiltonian formulation lie not in its use as a calculational tool, but rather in the deeper insight it affords into the formal structure of mechanics. The equal status accorded to coordinates and momenta as independent variables encourages a greater freedom in selecting the physical quantities to be designated as "coordinates" and "momenta." As a result we are led to newer, more abstract ways of presenting the physical content of mechanics. While often of considerable help in practical applications to mechanical problems, these more abstract formulations are primarily of interest to us today because of their essential role in constructing the more modern theories of matter. Thus, one or another of these formulations of classical mechanics serves as a point of departure for both statistical mechanics and quantum theory. It is to such formulations, arising as outgrowths of the Hamiltonian procedure, that this and the next chapter are devoted.

THE EQUATIONS OF CANONICAL TRANSFORMATION

There is one type of problem for which the solution of the Hamilton's equations is trivial. Consider a situation in which the Hamiltonian is a constant of the motion, and where *all* coordinates q_i are cyclic. Under these conditions, the conjugate momenta p_i are all constant:

$$p_i = \alpha_i,$$

and since the Hamiltonian cannot be an explicit function of either the time or the cyclic coordinates, it may be written as

$$H = H(\alpha_1, \ldots, \alpha_n).$$

Consequently, the Hamilton's equations for $\dot{q}_i$ are simply

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i, \tag{9.1}$$

where the ω_i's are functions of the α_i's only and therefore are also constant in time. Equations (9.1) have the immediate solutions

$$q_i = \omega_i t + \beta_i, \tag{9.2}$$

where the β_i's are constants of integration, determined by the initial conditions.

It would seem that the solution to this type of problem, easy as it is, can only be of academic interest, for it rarely happens that all the generalized coordinates are cyclic. But a given system can be described by more than one set of generalized coordinates. Thus, to discuss motion of a particle in a plane, we may use as generalized coordinates either the Cartesian coordinates

$$q_1 = x, \qquad q_2 = y,$$

or the plane polar coordinates

$$q_1 = r, \qquad q_2 = \theta.$$

Both choices are equally valid, but one of the other set may be more convenient for the problem under consideration. Note that for central forces neither x nor y is cyclic, while the second set does contain a cyclic coordinate in the angle θ. The number of cyclic coordinates can thus depend upon the choice of generalized coordinates, and for each problem there may be one particular choice for which all coordinates are cyclic. If we can find this set, the remainder of the job is trivial. Since the obvious generalized coordinates suggested by the problem will not normally be cyclic, we must first derive a specific procedure for *transforming* from one set of variables to some other set that may be more suitable.

The transformations considered in the previous chapters have involved going from one set of coordinates q_i to a new set Q_i by transformation equations of the form

$$Q_i = Q_i(q, t). \tag{9.3}$$

For example, the equations of an orthogonal transformation, or of the change from Cartesian to plane polar coordinates, have the general form of Eqs. (9.3). As has been previously noted in Derivation 10 of Chapter 1, such transformations are known as *point transformations*. But in the Hamiltonian formulation the momenta are also independent variables on the same level as the generalized coordinates. The concept of transformation of coordinates must therefore be widened to include the simultaneous transformation of the independent *coordinates* and *momenta*, q_i, p_i, to a new set Q_i, P_i, with (invertible) equations of transformation:

$$\begin{aligned} Q_i &= Q_i(q, p, t), \\ P_i &= P_i(q, p, t). \end{aligned} \tag{9.4}$$

Thus, the new coordinates will be defined not only in terms of the old coordinates but also in terms of the old momenta. Equations (9.3) may be said to define

a *point transformation of configuration space*; correspondingly Eqs. (9.4) define a *point transformation of phase space*.

In developing Hamiltonian mechanics, only those transformations can be of interest for which the new Q, P are canonical coordinates. This requirement will be satisfied provided there exists some function $K(Q, P, t)$ such that the equations of motion in the new set are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \tag{9.5}$$

The function K plays the role of the Hamiltonian in the new coordinate set.* It is important for future considerations that the transformations considered be problem-independent. That is to say, (Q, P) must be canonical coordinates not only for some specific mechanical systems, but for all systems of the same number of degrees of freedom. Equations (9.5) must be the form of the equations of motion in the new coordinates and momenta no matter what the particular initial form of H. We may indeed be incited to develop a particular transformation from (q, p) to (Q, P) to handle, say, a plane harmonic oscillator. But the same transformation must then also lead to Hamilton's equations of motion when applied, for example, to the two-dimensional Kepler problem.

As was seen in Section 8.5, if Q_i and P_i are to be canonical coordinates, they must satisfy a modified Hamilton's principle that can be put in the form

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))\, dt = 0, \tag{9.6}$$

(where summation over the repeated index i is implied). At the same time the old canonical coordinates of course satisfy a similar principle:

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0. \tag{9.7}$$

The simultaneous validity of Eqs. (9.6) and (9.7) does not mean of course that the integrands in both expressions are equal. Since the general form of the modified Hamilton's principle has zero variation at the end points, both statements will be satisfied if the integrands are connected by a relation of the form

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \frac{dF}{dt}. \tag{9.8}$$

Here F is any function of the phase space coordinates with continuous second derivatives, and λ is a constant independent of the canonical coordinates and the time. The multiplicative constant λ is related to a particularly simple type of transformation of canonical coordinates known as a *scale transformation*.

Suppose we change the size of the units used to measure the coordinates and momenta so that in effect we transform them to a set (Q', P') defined by

$$Q_i' = \mu q_i, \qquad P_i' = \nu p_i. \tag{9.9}$$

Then it is clear Hamilton's equations in the form of Eqs. (9.5) will be satisfied for a transformed Hamiltonian $K'(Q', P') = \mu\nu H(q, p)$. The integrands of the corresponding modified Hamilton's principles are, also obviously, related as

$$\mu\nu(p_i\dot{q}_i - H) = P_i'\dot{Q}_i' - K', \tag{9.10}$$

which is of the form of Eq. (9.8) with $\lambda = \mu\nu$. With the aid of suitable scale transformation, it will always be possible to confine our attention to transformations of canonical coordinates for which $\lambda = 1$. Thus, if we have a transformation of canonical coordinates $(q, p) \rightarrow (Q', P')$ for some $\lambda \neq 1$, then we can always find an intermediate set of canonical coordinates (Q, P) related to (Q', P') by a simple scale transformation of the form (9.9) such that $\mu\nu$ also has the same value λ. The transformation between the two sets of canonical coordinates (q, p) and (Q, P) will satisfy Eq. (9.8), but now with $\lambda = 1$:

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF}{dt}. \tag{9.11}$$

Since the scale transformation is basically trivial, the significant transformations to be examined are those for which Eq. (9.11) holds.

A transformation of canonical coordinates for which $\lambda \neq 1$ will be called an *extended canonical transformation*. Where $\lambda = 1$, and Eq. (9.11) holds, we will speak simply of a *canonical transformation*. The conclusion of the previous paragraph may then be stated as saying that any extended canonical transformation can be made up of a canonical transformation followed by a scale transformation. Except where otherwise stated, all future considerations of transformations between canonical coordinates will involve only canonical transformations. It is also convenient to give a specific name to canonical transformations for which the equations of transformation Eqs. (9.4) do not contain the time explicitly; they will be called *restricted canonical transformations*.

The last term on the right in Eq. (9.11) contributes to the variation of the action integral only at the end points and will therefore vanish if F is a function of (q, p, t) or (Q, P, t) or any mixture of the phase space coordinates since these have zero variation at the end points. Further, through the equations of transformation, Eqs. (9.4) and their inverses F can be expressed partly in terms of the old set of variables and partly of the new. Indeed, F is useful for specifying the exact form of the canonical transformation only when half of the variables (beside the time) are from the old set and half are from the new. It then acts, as it were, as a bridge between the two sets of canonical variables and is called the *generating function* of the transformation.

To show how the generating function specifies the equations of transformation, suppose F were given as a function of the old and new generalized space

coordinates:

$$F = F_1(q, Q, t). \tag{9.12}$$

Equation (9.11) then takes the form

$$\begin{aligned} p_i \dot{q}_i - H &= P_i \dot{Q}_i - K + \frac{dF_1}{dt} \\ &= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i. \end{aligned} \tag{9.13}$$

Since the old and the new coordinates, q_i and Q_i, are separately independent, Eq. (9.13) can hold identically only if the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ each vanish:

$$p_i = \frac{\partial F_1}{\partial q_i}, \tag{9.14a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}, \tag{9.14b}$$

leaving finally

$$K = H + \frac{\partial F_1}{\partial t}. \tag{9.14c}$$

Equations (9.14a) are n relations defining the p_i as functions of q_j, Q_j, and t. Assuming they can be inverted, they could then be solved for the n Q_i's in terms of q_j, p_j, and t, thus yielding the first half of the transformation equations (9.4). Once the relations between the Q_i's and the old canonical variables (q, p) have been established, they can be substituted into Eqs. (9.14b) so that they give the n P_i's as functions of q_j, p_j, and t, that is, the second half of the transformation equations (9.4). To complete the story, Eq. (9.14c) provides the connection between the new Hamiltonian, K, and the old one, H. We must be careful to read Eq. (9.14c) properly. First q and p in H are expressed as functions of Q and P through the inverses of Eqs. (9.4). Then the q_i in $\partial F_1/\partial t$ are expressed in terms of Q, P in a similar manner and the two functions are added to yield $K(Q, P, t)$.

The procedure described shows how, starting from a given generating function F_1, the equations of the canonical transformation can be obtained. We can usually reverse the process: Given the equations of transformation (9.4), an appropriate generating function F_1 may be derived. Equations (9.4) are first inverted to express p_i and P_i as functions of q, Q, and t. Equations (9.14a, b) then constitute a coupled set of partial differential equations than can be integrated, in principle, to find F_1 providing the transformation is indeed canonical. Thus, F_1 is always uncertain to within an additive arbitrary function of t alone (which doesn't affect the equations of transformation), and there may at times be other ambiguities.

It sometimes happens that it is not suitable to describe the canonical transformation by a generating function of the type $F_1(q, Q, t)$. For example, the transformation may be such that p_i cannot be written as functions of q, Q, and t, but

rather will be functions of q, P, and t. We would then seek a generating function that is a function of the old coordinates q and the new momenta P. Clearly Eq. (9.13) must then be replaced by an equivalent relation involving $\dot{P}_i$ rather than $\dot{Q}_i$. This can be accomplished by writing F in Eq. (9.11) as

$$F = F_2(q, P, t) - Q_i P_i. \tag{9.15}$$

Substituting this F in Eq. (9.11) leads to

$$p_i \dot{q}_i - H = -Q_i \dot{P}_i - K + \frac{d}{dt} F_2(q, P, t). \tag{9.16}$$

Again, the total derivative of F_2 is expanded and the coefficients of $\dot{q}_i$ and P_i collected, leading to the equations

$$p_i = \frac{\partial F_2}{\partial q_i}, \tag{9.17a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}, \tag{9.17b}$$

with

$$K = H + \frac{\partial F_2}{\partial t}. \tag{9.17c}$$

As before, Eqs. (9.17a) are to be solved for P_i as functions of q_j, p_j, and t to correspond to the second half of the transformation equations (9.4). The remaining half of the transformation equations is then provided by Eqs. (9.17b).

The corresponding procedures for the remaining two basic types of generating functions are obvious, and the general results are displayed in Table 9.1.

It is tempting to look upon the four basic types of generating functions as being related to each other through Legendre transformations. For example, the

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i, \quad Q_i = p_i, \quad P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i, \quad Q_i = q_i, \quad P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i, \quad Q_i = -q_i, \quad P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i, \quad Q_i = p_i, \quad P_i = -q_i$

transition from F_1 to F_2 is equivalent to going from the variables q, Q to q, P with the relation

$$-P_i = \frac{\partial F_1}{\partial Q_i}. \tag{9.18}$$

This is just the form required for a Legendre transformation of the basis variables, as described in Section 8.1, and in analogy to Eq. (8.5) we would set

$$F_2(q, P, t) = F_1(q, Q, t) + P_i Q_i, \tag{9.19}$$

which is equivalent to Eq. (9.15) combined with Eq. (9.12). All the other defining equations for the generating functions can similarly be looked on, in combination with Eq. (9.12) as Legendre transformations from F_1, with the last entry in Table 9.1 describing a double Legendre transformation. The only drawback to this picture is that it might erroneously lead us to believe that any given canonical transformation can be expressed in terms of the four basic types of Legendre transformations listed in Table 9.1. This is not always possible. Some transformations are just not suitable for description in terms of these or other elementary forms of generating functions, as has been noted above and as will be illustrated in the next section with specific examples. If we try to apply the Legendre transformation process, we are then led to generating functions that are identically zero or are indeterminate. For this reason, we have preferred to define each type of generating function relative to F, which is some unspecified function of $2n$ independent coordinates and momenta.

Finally, note that a suitable generating function doesn't have to conform to one of the four basic types for *all* the degrees of freedom of the system. It is possible, and for some canonical transformations necessary, to use a generating function that is a mixture of the four types. To take a simple example, it may be desirable for a particular canonical transformation with two degrees of freedom to be defined by a generating function of the form

$$F'(q_1, p_2, P_1, Q_2, t). \tag{9.20}$$

This generating function would be related to F in Eq. (9.11) by the equation

$$F = F'(q_1, p_2, P_1, Q_2, t) - Q_1 P_1 + q_2 p_2, \tag{9.21}$$

and the equations of transformation would be obtained from the relations

$$\begin{aligned} p_1 &= \frac{\partial F'}{\partial q_1}, \qquad Q_1 = \frac{\partial F'}{\partial P_1}, \\ q_2 &= -\frac{\partial F'}{\partial p_2}, \qquad P_2 = -\frac{\partial F'}{\partial Q_2}, \end{aligned} \tag{9.22}$$

with

$$K = H + \frac{\partial F'}{\partial t}. \tag{9.23}$$

EXAMPLES OF CANONICAL TRANSFORMATIONS

The nature of canonical transformations and the role played by the generating function can best be illustrated by some simple yet important examples. Let us consider, first, a generating function of the second type with the particular form

$$F_2 = q_i P_i \tag{9.24}$$

found in column 3 of Table 9.1. From Eqs. (9.17), the transformation equations are

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_i, \\ Q_i &= \frac{\partial F_2}{\partial P_i} = q_i, \\ K &= H. \end{aligned} \tag{9.25}$$

The new and old coordinates are the same; hence F_2 merely generates the *identity transformation* (cf. Table 9.1). We also note, referring to Table 9.1, that the particular generating function $F_3 = p_i Q_i$ generates an identity transformation with negative signs; that is, $Q_i = -q_i$, $P_i = -p_i$.

A more general type of transformation is described by the generating function

$$F_2 = f_i(q_1, \ldots, q_n;\ t) P_i, \tag{9.26}$$

where the f_i may be any desired set of independent functions. By Eqs. (9.17b), the new coordinates Q_i are given by

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \ldots, q_n;\ t) \tag{9.27}$$

Thus, with this generating function the new coordinates depend only upon the old coordinates and the time and do not involve the old momenta. Such a transformation is therefore an example of the class of point transformations defined by Eqs. (9.3). In order to define a point transformation, the functions f_i must be independent and invertible, so that the q_j can be expressed in terms of the Q_i. Since the f_i are otherwise completely arbitrary, we may conclude that *all point transformations are canonical*. Equation (9.17c) furnishes the new Hamiltonian in terms of the old and of the time derivatives of the f_i functions.

Note that F_2 as given by Eq. (9.26) is not the only generating function leading to the point transformation specified by the f_i. Clearly the same point transformation is implicit in the more general form

$$F_2 = f_i(q_1, \ldots, q_n;\ t)P_i + g(q_1, \ldots, q_n;\ t), \tag{9.28}$$

where $g(q, t)$ is any (differentiable) function of the old coordinates and the time. Equations (9.27), the transformation equations for the coordinates, remain unaltered for this generating function. But the transformation equations of the momenta differ for the two forms. From Eqs. (9.17a), we have

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j}, \tag{9.29}$$

using the form of F_2 given by Eq. (9.28). These equations may be inverted to give P as a function of (q, p), most easily by writing them in matrix notation:

$$\mathbf{p} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{P} + \frac{\partial g}{\partial \mathbf{q}}. \tag{9.29'}$$

Here $\mathbf{p}$, $\mathbf{P}$, and $\partial g/\partial \mathbf{q}$ are n-elements of single-column matrices, and $\partial \mathbf{f}/\partial \mathbf{q}$ is a square matrix whose ijth element is $\partial f_i/\partial q_j$. In two dimensions, Eq. (9.29') can be written as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} \\ \dfrac{\partial f_2}{\partial q_1} & \dfrac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \dfrac{\partial g}{\partial q_1} \\ \dfrac{\partial g}{\partial q_2} \end{bmatrix}.$$

It follows that $\mathbf{P}$ is a linear function of $\mathbf{p}$ given by

$$\mathbf{P} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right]^{-1}\left[\mathbf{p} - \frac{\partial g}{\partial \mathbf{q}}\right]. \tag{9.30}$$

In two dimensions, (9.30) becomes

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial q_1} & \dfrac{\partial f_1}{\partial q_2} \\ \dfrac{\partial f_2}{\partial q_1} & \dfrac{\partial f_2}{\partial q_2} \end{bmatrix}^{-1} \left[\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} \dfrac{\partial g}{\partial q_1} \\ \dfrac{\partial g}{\partial q_2} \end{bmatrix}\right]. \tag{9.31}$$

Thus, the transformation equations (9.27) for Q are independent of g and depend only upon the $f_i(q, t)$, but the transformation equations (9.29) for P do depend upon the form of g and are in general functions of both the old coordinates and momenta. The generating function given by Eq. (9.26) is only a special case of Eq. (9.28) for which $g = 0$, with correspondingly specialized transformation equations for P.

An instructive transformation is provided by the generating function of the first kind, $F_1(q, Q, t)$, of the form

$$F_1 = q_k Q_k.$$

The corresponding transformation equations, from (9.14a, b) are

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \tag{9.32a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i. \tag{9.32b}$$

In effect, the transformation interchanges the momenta and the coordinates; the new coordinates are the old momenta and the new momenta are essentially the old coordinates. Table 9.1 shows that the particular generating function of type $F_4 = p_i P_i$ produces the same transformation. These simple examples should emphasize the independent status of generalized coordinates and momenta. They are both needed to describe the motion of the system in the Hamiltonian formulation. The distinction between them is basically one of nomenclature. We can shift the names around with at most no more than a change in sign. There is no longer present in the theory any lingering remnant of the concept of q_i as a spatial coordinate and p_i as a mass times a velocity. Incidentally, we may see directly from Hamilton's equations,

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i},$$

that this exchange transformation is canonical. If q_i is substituted for p_i, the equations remain in the canonical form only if $-p_i$ is substituted for q_i.

A transformation that leaves some of the (q, p) pairs unchanged, and interchanges the rest (with a sign change), is obviously a canonical transformation of a "mixed" form. Thus, in a system of two degrees of freedom, the transformation

$$Q_1 = q_1, \qquad P_1 = p_1,$$
$$Q_2 = p_2, \qquad P_2 = -q_2,$$

is generated by the function

$$F = q_1 P_1 + q_2 Q_2, \tag{9.33}$$

which is a mixture of the F_1 and F_2 types.

POISSON BRACKETS AND OTHER CANONICAL INVARIANTS

The *Poisson bracket* of two functions u, v with respect to the canonical variables (q, p) is defined as

$$[u, v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}. \tag{9.67}$$

In this bilinear expression we have a typical symplectic structure, as in Hamilton's equations, where q is coupled with p, and p with $-q$. The Poisson bracket thus lends itself readily to being written in matrix form, where it appears as

$$[u, v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial v}{\partial \boldsymbol{\eta}}. \tag{9.68}$$

The transpose sign is used on the first matrix on the right-hand side to indicate explicitly that this matrix must be treated as a single-row matrix in the multiplication. On most occasions this specific reminder will not be needed and the transpose sign may be omitted.

Suppose we choose the functions u, v out of the set of canonical variables (q, p) themselves. Then it follows trivially from the definition, either as Eq. (9.67) or (9.68), that these Poisson brackets have the values

$$[q_j, q_k]_{q,p} = 0 = [p_j, q_k]_{q,p},$$

and

$$[q_j, p_k]_{q,p} = \delta_{jk} = -[p_j, q_k]_{q,p}. \tag{9.69}$$

We can summarize the relations of Eqs. (9.69) in one equation by introducing a *square matrix Poisson bracket*, $[\boldsymbol{\eta}, \boldsymbol{\eta}]$, whose lm element is $[\eta_l, \eta_m]$. Equations (9.69) can then be written as

$$[\boldsymbol{\eta}, \boldsymbol{\eta}]_{\eta} = \mathbf{J}. \tag{9.70}$$

Now let us take for u, v the members of the transformed variables (Q, P), or $\boldsymbol{\zeta}$, defined in terms of (q, p) by the transformation equations (9.59). The set of all the Poisson brackets that can be formed out of (Q, P) comprise the matrix Poisson bracket defined as

$$[\zeta, \zeta]_{\eta} = \frac{\widetilde{\partial \zeta}}{\partial \boldsymbol{\eta}} \mathbf{J} \frac{\partial \zeta}{\partial \boldsymbol{\eta}}.$$

But we recognize the partial derivatives as defining the square Jacobian matrix of the transformation, so that the Poisson bracket relation is equivalent to

$$[\zeta, \zeta]_\eta = \tilde{\mathbf{M}}\mathbf{J}\mathbf{M}. \tag{9.71}$$

If the transformation $\eta \rightarrow \zeta$ is canonical, then the symplectic condition holds and Eq. (9.71) reduces to (cf. Eq. (9.58))

$$[\zeta, \zeta]_\eta = \mathbf{J}, \tag{9.72}$$

and conversely, if Eq. (9.72) is valid, then the transformation is canonical.

Poisson brackets of the canonical variables themselves, such as Eqs. (9.70) or (9.72), are referred to as the *fundamental Poisson brackets*. Since we have from Eq. (9.70) that

$$[\zeta, \zeta]_\zeta = \mathbf{J}, \tag{9.73}$$

Eq. (9.72) states that the fundamental Poisson brackets of the ζ variables have the same value when evaluated with respect to *any* canonical coordinate set. In other words, the *fundamental Poisson brackets are invariant under canonical transformation*. We have seen from Eq. (9.71) that the invariance is a necessary and sufficient condition for the transformation matrix to be symplectic. The invariance of the fundamental Poisson brackets is thus in all ways equivalent to the symplectic condition for a canonical transformation.

It does not take many more steps to show that *all* Poisson brackets are invariant under canonical transformation. Consider the Poisson bracket of two functions u, v with respect to the η set of coordinates, Eq. (9.68). In analogy to Eq. (9.53), the partial derivative of v with respect to η can be expressed in terms of partial derivatives with respect to ζ as

$$\frac{\partial v}{\partial \eta} = \tilde{\mathbf{M}} \frac{\partial v}{\partial \zeta}$$

(that is, the partial derivative transforms as a 1-form). In a similar fashion,

$$\widetilde{\frac{\partial u}{\partial \eta}} = \widetilde{\tilde{\mathbf{M}} \frac{\partial u}{\partial \zeta}} = \widetilde{\frac{\partial u}{\partial \zeta}}\mathbf{M}.$$

Hence the Poisson bracket Eq. (9.68) can be written

$$[u, v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}}\mathbf{J}\frac{\partial v}{\partial \eta} = \widetilde{\frac{\partial u}{\partial \zeta}}\mathbf{M}\mathbf{J}\tilde{\mathbf{M}}\frac{\partial u}{\partial \zeta}.$$

If the transformation is canonical, the symplectic condition in the form of Eq. (9.55) holds, and we then have

$$[u, v]_\eta = \widetilde{\frac{\partial u}{\partial \zeta}}\mathbf{J}\frac{\partial v}{\partial \zeta} \equiv [u, v]_\zeta. \tag{9.74}$$

Thus, the Poisson bracket has the same value when evaluated with respect to any canonical set of variables—*all Poisson brackets are canonical invariants*. In writing the symbol for the Poisson bracket, we have so far been careful to indicate by the subscript the set of variables in terms of which the brackets are defined. So long as we use only canonical variables that practice is now seen to be unnecessary, and we shall in general drop the subscript.*

The hallmark of the canonical transformation is that Hamilton's equations of motion are invariant in form under the transformation. Similarly, the canonical invariance of Poisson brackets implies that equations expressed in terms of Poisson brackets are invariant in form under canonical transformation. As we shall see, we can develop a structure of classical mechanics, paralleling the Hamiltonian formulation, expressed solely in terms of Poisson brackets. Historically this Poisson bracket formulation, which has the same form in all canonical coordinates, was especially useful for carrying out the original transition from classical to quantum mechanics. There is a simple "correspondence principle" that says that the classical Poisson bracket is to be replaced by a suitably defined commutator of the corresponding quantum operators.

The algebraic properties of the Poisson bracket are therefore of considerable interest. We have already used the obvious properties

$$[u, u] = 0, \tag{9.75a}$$

$$[u, v] = -[v, u]. \qquad \text{(antisymmetry)} \tag{9.75b}$$

Almost equally obvious are the characteristics

$$[au + bv, w] = a[u. w] + b[v, w], \qquad \text{(linearity)} \tag{9.75c}$$

where a and b are constants, and

$$[uv, w] = [u, w]v + u[v, w]. \tag{9.75d}$$

One other property is far from obvious, but is very important in defining the nature of the Poisson bracket. It is usually given in the form of *Jacobi's identity*, which states that if u, v, and w are three functions with continuous second derivatives, then

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0; \tag{9.75e}$$

that is, the sum of the cyclic permutations of the double Poisson bracket of three functions is zero. There seems to be no simple way of proving Jacobi's identity for the Poisson bracket without lengthy algebra. However, it is possible to mitigate the complexity of the manipulations by introducing a special nomenclature. We

shall use subscripts on u, v, w (or functions of them) to denote partial derivatives by the corresponding canonical variable. Thus,

$$u_i \equiv \frac{\partial u}{\partial \eta_i}, \qquad \text{and} \qquad v_{ij} \equiv \frac{\partial v}{\partial \eta_i \, \partial \eta_j}.$$

In this notation the Poisson bracket of u and v can be expressed as

$$[u, v] = u_i J_{ij} v_j.$$

Here J_{ij}, as usual, is simply the ijth element of $\mathbf{J}$. In the proof, the only property of $\mathbf{J}$ that we shall need is its antisymmetry.

Now let us consider the first double Poisson bracket in Eq. (9.75e):

$$[u, [v, w]] = u_i J_{ij} [v, w]_j = u_i J_{ij} (v_k J_{kl} w_l)_j.$$

Because the elements J_{kl} are constants, the derivative with resect to η doesn't act on them, and we have

$$[u, [v, w]] = u_i J_{ij} (v_k J_{kl} w_{lj} + v_{kj} J_{kl} w_l). \tag{9.76}$$

The other double Poisson brackets can be obtained from Eq. (9.76) by cyclic permutation of u, v, w. There are thus six terms in all, each being a fourfold sum over dummy indices i, j, k, and l. Consider the term in Eq. (9.76) involving a second derivative of w:

$$J_{ij} J_{kl} u_i v_k w_{lj}.$$

The only other second derivative of w will appear in evaluating the second double Poisson bracket in (Eq. 9.75e):

$$[v, [w, u]] = v_k J_{kl} (w_j J_{ji} u_i)_l.$$

Here the term in the second derivative in w is

$$J_{ji} J_{kl} u_i v_k w_{jl}.$$

Since the order of differentiation is immaterial, $w_{lj} = w_{jl}$, and the sum of the two terms is given by

$$(J_{ij} + J_{ji}) J_{kl} u_i v_k w_{ij} = 0,$$

by virtue of the antisymmetry of J. The remaining four terms are cyclic permutations and can similarly be divided in two pairs, one involving second derivatives of u and the other of v. By the same reasoning, each of these pairs sums to zero, and Jacobi's identity is thus verified.

If the Poisson bracket of u, v is looked on as defining a "product" operation of the two functions, then Jacobi's identity is the replacement for the associa-

tive law of multiplication. Recall that the ordinary multiplication of arithmetic is associative; that is, the order of a sequence of multiplications is immaterial:

$$a(bc) = (ab)c.$$

Jacobi's identity says that the bracket "product" is not associative and gives the effect of changing the sequence of "multiplications." Brackets that satisfy Eqs. (9.75), together with the expression

$$[u_i, u_j] = \sum_k c_{ij}^k u_k, \tag{9.77}$$

constitute a generally noncommunitive algebra called a Lie algebra. For Poisson brackets in three-dimensional space, either the structure constants c_{ij}^k are all zero or only one term in the right-hand side of Eq. (9.77) exists for any pair of indices. Examples of this will be given later, and a more detailed discussion of Lie algebras is given in Appendix B.

Poisson bracket operation is not the only type of "product" familiar to physicists that satisfies the conditions for a Lie algebra. It will be left to the exercises to show that that vector product of two vectors,

$$_{\mathrm{V}}[\mathrm{A}, \mathrm{B}] \rightarrow \mathrm{A} \times \mathrm{B}, \tag{9.78a}$$

and the commutator of two matrices,

$$_{\mathrm{M}}[\mathrm{A}, \mathrm{B}] \rightarrow \mathrm{AB} - \mathrm{BA}, \tag{9.78b}$$

satisfy the same Lie algebra conditions as the Poisson bracket. It is this last that makes it feasible to replace the classical Poisson bracket by the commutator of the quantum mechanical operators. In other words, the "correspondence principle" can work only because both the Poisson bracket and commutator are representations of a Lie algebra "product."*

There are other canonical invariants besides the Poisson bracket. One, mainly of historical interest now, is the *Lagrange bracket*, denoted by $\{u, v\}$. Suppose u and v are two functions out of a set of $2n$ independent functions of the canonical variables. By inversion, the canonical variables can then be considered as functions of the set of $2n$ functions. On this basis, the Lagrange bracket of u and v with respect to the (q, p) variables is defined as

$$\{u, v\}_{q,p} = \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u}\frac{\partial q_i}{\partial v}, \tag{9.79}$$

or, in matrix notation,

$$\{u, v\}_{\eta} = \frac{\widetilde{\partial \eta}}{\partial u}\mathbf{J}\frac{\partial \eta}{\partial v}. \tag{9.80}$$

Proof of the canonical invariance of the Lagrange bracket parallels that for the Poisson bracket.

If for u and v we take two members of the set of canonical variables, then we obtain the *fundamental Lagrange brackets*:

$$\{q_i, q_j\}_{qp} = 0 = \{p_i, p_j\}_{qp} \qquad \{q_i, p_j\}_{qp} = \delta_{ij}, \tag{9.81}$$

or, in matrix notation,

$$\{\eta, \eta\} = \mathbf{J}. \tag{9.82}$$

The Lagrange and Poisson brackets clearly stand in some kind of inverse relationship to each other, but the precise form of this relation is somewhat complicated to express. Let u_i, $i = 1, \ldots, 2n$, be a set of $2n$ independent functions of the canonical variables, to be represented by a column (or row) matrix $\mathbf{u}$. Then $\{\mathbf{u}, \mathbf{u}\}$ is the $2n \times 2n$ matrix whose ijth element is $\{u_i, u_j\}$, with a similar description for $[\mathbf{u}, \mathbf{u}]$. The reciprocal character of the two brackets manifests itself in the relation

$$\{\mathbf{u}, \mathbf{u}\}[\mathbf{u}, \mathbf{u}] = -\mathbf{1}. \tag{9.83}$$

If for u we choose the canonical set itself, η, then Eq. (9.83) obviously follows from the fundamental bracket formulas, Eqs. (9.70) and (9.82), and the properties of $\mathbf{J}$. The proof for arbitrary u is not difficult if written in terms of the matrix definitions of the brackets and is reserved for the exercises. While the properties of the Lagrange and Poisson brackets parallel each other in many aspects, note that the Lagrange brackets do *not* obey Jacobi's identity. Lagrange brackets therefore do not qualify as a "product" operation in a Lie algebra.

Another important canonical invariant is the magnitude of a volume element in phase space. A canonical transformation $\eta \to \zeta$ transforms the $2n$-dimensional phase space with coordinates η_i to another phase space with coordinates ζ_i. The volume element

$$(d\eta) = dq_1 dq_2 \ldots dq_n dp_1 \ldots dp_n$$

transforms to a new volume element

$$(d\zeta) = dQ_1 dQ_2 \ldots dQ_n dP_1 \ldots dP_n.$$

As is well known, the sizes of the two volume elements are related by the absolute value of the Jacobian determinant $\|\mathbf{M}\|$;

$$(d\zeta) = \|\mathbf{M}\|(d\eta).$$

For example, in the two-dimensional transformation from $\eta_i = q, p$ to $\zeta_i = Q, P$, this expression becomes

$$dQ\,dP = \begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{vmatrix} dq\,dp = [q, p]_\zeta\, dq\,dp. \tag{9.84}$$

But, by taking the determinant of both sides of the symplectic condition, Eq. (9.58), we have

$$|\mathbf{M}|^2|\mathbf{J}| = |\mathbf{J}|. \tag{9.85}$$

Thus, in a real canonical transformation the Jacobian determinant is ± 1, and the absolute value is always unity, proving the canonical invariance of the volume element in phase space. It follows, also, that the volume of any arbitrary region in phase space,

$$J_n = \int \cdots \int (d\eta), \tag{9.86}$$

is a canonical invariant. In our two-dimensional example, the invariant is $d\eta = dq\,dp$ and $J_1 = \int dq\,dp$.

The volume integral in Eq. (9.86) is the final member of a sequence of canonical invariants known as the *integral invariants of Poincaré*, comprising integrals over subspaces of phase space of different dimensions. The other members of the sequence cannot be stated as simply as J_n, and because they are not needed for the further development of the theory, they will not be discussed here.

Finally, the invariance of the fundamental Poisson brackets now enables us to outline a proof that the symplectic condition implies the existence of a generating function, as mentioned at the conclusion of the previous section. To simplify considerations, we shall examine only a system with one degree of freedom; the general method of the proof can be directly extended to systems with many degrees of freedom.* We suppose that the first of the equations of transformation,

$$Q = Q(q, p), \qquad P = P(q, p),$$

is invertable so as to give p as a function q and Q, say

$$p = \phi(q, Q). \tag{9.87}$$

Substitution in the second equation of transformation gives P as some function of q and Q, say

$$P = \psi(q, Q). \tag{9.88}$$

In such a case, we would expect the transformation to be generated by a generating function of the first kind,* F_1, with Eqs. (9.87) and (9.88) appearing as

$$p = \frac{\partial F_1(q, Q)}{\partial q}, \qquad P = -\frac{\partial F_1}{\partial Q}(q, Q). \tag{9.89}$$

If Eq. (9.89) holds, then it must be true that

$$\frac{\partial \phi}{\partial Q} = -\frac{\partial \psi}{\partial q}. \tag{9.90}$$

Conversely, if we can show that Eq. (9.90) is valid, then there must exist a function F_1 such that p and P are given by Eqs. (9.89).

To demonstrate the validity of Eq. (9.90), we try to look on all quantities as functions of q and Q. Thus, we of course have the identity

$$\frac{\partial Q}{\partial Q} = 1,$$

but if Eq. (9.87) be substituted in the first transformation equation,

$$Q = Q(q, \phi(q, Q)), \tag{9.91}$$

the partial derivative can also be written

$$\frac{\partial Q}{\partial Q} = \frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q},$$

so that we have the relation

$$\frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q} = 1. \tag{9.92}$$

In the same spirit we evaluate the Poisson bracket

$$[Q, P] \equiv \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1.$$

The derivatives of P are derivatives of ψ from Eq. (9.88) considered as a function of q and $Q(q, p)$. Hence, the Poisson bracket can be written

$$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\left(\frac{\partial \psi}{\partial q} + \frac{\partial \psi}{\partial Q}\frac{\partial Q}{\partial q}\right),$$

or, consolidating terms, as

$$[Q, P] = \frac{\partial \psi}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right) - \frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q},$$

and therefore

$$1 = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}. \tag{9.93}$$

Combining Eqs. (9.92) and (9.93), we have

$$\frac{\partial Q}{\partial p}\frac{\partial \phi}{\partial Q} = -\frac{\partial Q}{\partial p}\frac{\partial \psi}{\partial q}.$$

Since the partial derivative of Q with respect to p is the same on both sides of the equation, that is, the other variable being held constant is q in both cases, and since the derivative doesn't vanish (else the Q equation could not be inverted), it follows that Eq. (9.90) must be true. Thus, from the value of the fundamental Poisson bracket $[Q, P]$, which we have seen is equivalent to the symplectic condition, we are led to the existence of a generating function. The two approaches to canonical transformations, though arrived at independently, are fully equivalent.

POSSIBLE QUESTIONS

Part B (6 Marks)

1.Explain the canonical transformation with an example
2. Explain Jacobi's identity
3. Explain the simple harmonic oscillator problem
4. Explain the integral invariants of poincare
5. Derive the Lagrange's equation from Hamilton's principle for holonomic system
6. Derive an expression for $\delta\varphi$ and δ
7. Show that how the generating function specifies the equations of transformations

Part C (10 Marks)

1. Explain that the fundamental poisson brackets are invariant under canonical transformat
2. Show that the transformation P=q cot p , Q = log($\frac{\sin p}{q}$) is canonical. Also find the generating function.
3. Explain the Lagrange's bracket
4. Show that the transformation P=1/2 * $(p^2 + q^2)$, Q = $\tan^{-1}(-1)$ (q/p) is canonical.

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UNIT-IV

Subject: Mechanics **Subject Code: 17MMP106**

Canonical transformations

Part-A(20X1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Multiple Choice Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
canonical transformation is otherwise known as _____	contact transformation	point transformatiion	generating function	legendre transformation	contact transformation
Generalising the form,transformation from one set of co-ordinate qj to a new set Qj such transformation is called _____	legendre transformation	generating function	point transformatiion	contact transformation	point transformatiion
point transformation are the transformations of _____space	Configuration	position	phase	co-ordinate	Configuration
Canonical transformation are the transformations of _____space	position	co-ordinate	Configuration	phase	phase
While deducing lagrangian equations,no _was give to any particularly choice of co-ordinate system	force	stress	work	strain	stress
_____equations of motion are invariant in form with respect to the choice of the set of any generalised co-ordinates	lagrangian	Hamiltonian	canonical	legendre	lagrangian
Lagrangian equations are_____with respect to point transformation	invariant	constant	covariant	vanishes	covariant
The _____canonical equations can also be covariant	lagrange's	Hamilton's	point	Legendre's	Hamilton's
_____transformation is extended to hamiltonian formulation	contact	point	canonical	legendre	point
In hamiltonian formulation,we admit the existence of one more independent variable called_____	velocity	force	momentum	acceleration	momentum
Qj,pj are to be canonical co-ordinates,they must also satisfy the_____principle	modified hamilton's	legendre's	Hamilton's	extended hamilton's	modified hamilton's
Old coordinates qj,pj are already_____	zero	non zero	constant	canonical	canonical
Which is not a possible forms of function F_____	F4(p,P,t)	F1(q,Q,t)	F3(p,q,t)	F2(q,P,t)	F3(p,q,t)
___is a function of old and new set of co-ordinates	F	R	H	v	F
Transformation relations can be derived by the knowledge of the function F.It is thus termed as the _____	generating velocity	generating force	generating function	generating acceleration	generating function
F4 and F1 can be connected by_____	conatct transformation	legendre transformation	canonical transformation	point transformation	legendre transformation
The way of _____to obtain solution of a mechanical problem is to transform old set of co-ordinates into new set of co-ordinates	point transformation	canonical transformation	legendre transformation	conatct transformation	canonical transformation
Canonical transformations are all_____	cyclic	point	vanish	constant	cyclic
In a new set of co-ordinates Pj,Qj all co-ordinates Qj are cyclic so that all momenta Pj are_____	point	cyclic	constant	vanish	constant
If t cannot occur in K explicitly provided generating function F doesn't contain__explicitly	time	mass	point	force	time
The function f generates the _____if k=H	contact transformation	identiity transformation	canonical transformation	point transformation	identiity transformation
_____constitutes a special case of canonical transformation	point inversion	path inversion	space inversion	inversion	space inversion
If the expression to be an _then transformation from(qj,pj) to (Qj,Pj)set is canonical_____	exact differential	integral	exact integral	differential	exact differential
_____is the exact differential of F	df	dF	Df	DF	dF
_____an exact differential	pdq-PdQ=dF	PdQ-pdq=dF	pdq-Pdq=df	pdq+PdQ=dF	pdq-PdQ=dF
_____are the transformation in which old and new co-ordinates differ only slightly	legendre transformation	contact transformation	infinitesimal contact transformation	identiity transformation	infinitesimal contact transformation
a transformation that leaves some of the (q,p) pairs unchanged and interchanges the rst with the sign change is called a _transformation	extended canonical	canonical	restricted canonical	simple	canonical
The lagrangian and poisson brackets stand in _realtionship to each other	direct	converse	inverse	proportionate	inverse
the property that [u,v]=[v,u]is known as _____	symmetry	anti symmetry	transient	reflexive	anti symmetry
point transformation are the transformations of _____space	(p,q)	(q,p)	(q,t)	(p,t)	(q,p)
Always q_i's are expressed as the linear function of the p's in the form _____	(q,p,t)	(Q,p,t)	(q,P,t)	(Q,p,t)	(q,p,t)
The function $q_i(t)$will be changed in the _____	actual path	varied path	phase space	configuration space	varied path
the point transformation is _____	$Q_i=Q_i(q,t)$	$Q_i=Q_i(Q,t)$	$Q_i=Q_i(p,t)$	$Q_i=Q_i(P,t)$	$Q_i=Q_i(q,t)$

A transformation of canonical coordinates for which lambda not equal to one is calles_____	simple transformation	canonical transformation	extended canonical transformation	restricted canonical transformation	extended canonical transformation
The fundamental poisson brackets are _____under canonical invariants	real	constant	imaginary	invariant	invariant
[u,u]=	0	1	-1	2	0
[u,v]=	[negative(u),negative(v)]	[negative(u),v]	negative[v,u]	[u,v]	negative[v,u]
[uv,w]=	[u,v]w	[u,w]v	[u,w]v+u[v,w]	[u,w]v-u[v,w]	[u,w]v+u[v,w]
the reciprocal character of lagrange's and poisson brackets is_____	{u,u}[u,u]=-1	{u,v}[u,v]=-1	{u,v}[v,u]=1	{u,u}[v,v]=1	{u,u}[u,u]=-1
In a real canonical transformations the jacobian determinant is _____	IMI=i	IMI=0	IMI^2=1	IMI=1	IMI^2=1
The generating function for canonical transformation is denoted by_____	H(q,p)	H(q,Q)	W(q,p)	W(Q,p)	W(q,p)
K plays the role of _____	legendre's	Lagrangian	Hamiltonian	point	Hamiltonian
are treated_____	single	independent	dependent	inclined	independent
_____provided that only q_i,P_i are trated independent	F2(Q,P,t)	F2(q,p,t)	F2(q,P,t)	F2(Q,p,t)	F2(q,P,t)
F3(p,Q,t) provided that only _____are treated independent	p_i,Q_i	P_i,Q_i	p_i,q_i	P_i,q_i	p_i,Q_i
F4(p,P,t) provided that p_i,P_i are treated _____	independent	single	inclined	dependent	independent
_____is a 2n dimensional space having coordinatesq1,q2,.....qn and p1,p2,.....pn	actual path	configuration space	phase space	varied path	phase space
The _____have the property of preserving the form of hamilton's equations of motion under the transformation	point transformation	conatct transformation	legendre transformation	canonical transformation	canonical transformation

UNIT IV

canonical transformation is otherwise known as _____	contact transformation	point transformatiion	generating function	legendre transformation	contact transformation
Generalising the form,transformation from one set of co-ordinate qj to a new set Qj such transformation is called _____	legendre transformation	generating function	point transformatiion	contact transformation	point transformatiion
point transformation are the transformations of _____space	Configuration	position	phase	co-ordinate	Configuration
Canonical transformation are the transformations of _____space	position	co-ordinate	Configuration	phase	phase
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Lagrangian equations are_____with respect to point transformation	invariant	constant	covariant	vanishes	covariant
The _____canonical equations can also be covariant	lagrange's	Hamilton's	point	Legendre's	Hamilton's
_____transformation is extended to hamiltonian formulation	contact	point	canonical	legendre	point
In hamiltonian formulation,we admit the existence of one more independent variable called_____	velocity	force	momentum	acceleration	momentum
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Which is not a possible forms of function F_____	F4(p,P,t)	F1(q,Q,t)	F3(p,q,t)	F2(q,P,t)	F3(p,q,t)
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Transformation relations can be derived by the knowledge of the function F.It is thus termed as the _____	generating velocity	generating force	generating function	generating acceleration	generating function
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The way of _____to obtain solution of a mechanical problem is to transform old set of co-ordinates into new set of co-ordinates	point transformation	canonical transformation	legendre transformation	conatct transformation	canonical transformation
Canonical transformations are all_____	cyclic	point	vanish	constant	cyclic
In a new set of co-ordinates Pj,Qj all co-ordinates Qj are cyclic so that all momenta Pj are_____	point	cyclic	constant	vanish	constant
If t cannot occur in K explicitly provided generating function F doesn't contain__explicitly	time	mass	point	force	time
The function f generates the _____if k=H	contact transformation	identiity transformation	canonical transformation	point transformation	identiity transformation
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_____an exact differential	pdq-PdQ=dF	PdQ-pdq=dF	pdq-Pdq=df	pdq+PdQ=dF	pdq-PdQ=dF
_____are the transformation in which old and new co-ordinates differ only slightly	legendre transformation	contact transformation	infinitesimal contact transformation	identiity transformation	infinitesimal contact transformation
a transformation that leaves some of the (q,p) pairs unchanged and interchanges the rst with the sign change is called a _transformation	extended canonical	canonical	restricted canonical	simple	canonical
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point transformation are the transformations of _____space	(p,q)	(q,p)	(q,t)	(p,t)	(q,p)
Always q_i's are expressed as the linear function of the p's in the form _____	(q,p,t)	(Q,p,t)	(q,P,t)	(Q,p,t)	(q,p,t)
The function $q_i(t)$will be changed in the _____	actual path	varied path	phase space	configuration space	varied path
the point transformation is _____	$Q_i=Q_i(q,t)$	$Q_i=Q_i(Q,t)$	$Q_i=Q_i(p,t)$	$Q_i=Q_i(P,t)$	$Q_i=Q_i(q,t)$

A transformation of canonical coordinates for which lambda not equal to one is calles_____	simple transformation	canonical transformation	extended canonical transformation	restricted canonical transformation	extended canonical transformation
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[u,u]=	0	1	-1	2	0
[u,v]=	[negative(u),negative(v)]	[negative(u),v]	negative[v,u]	[u,v]	negative[v,u]
[uv,w]=	[u,v]w	[u,w]v	[u,w]v+u[v,w]	[u,w]v-u[v,w]	[u,w]v+u[v,w]
the reciprocal character of lagrange's and poisson brackets is_____	{u,u}[u,u]=-1	{u,v}[u,v]=-1	{u,v}[v,u]=1	{u,u}[v,v]=1	{u,u}[u,u]=-1
In a real canonical transformations the jacobian determinant is _____	IMI=i	IMI=0	IMI^2=1	IMI=1	IMI^2=1
The generating function for canonical transformation is denoted by_____	H(q,p)	H(q,Q)	W(q,p)	W(Q,p)	W(q,p)
K plays the role of _____	legendre's	Lagrangian	Hamiltonian	point	Hamiltonian
are treated_____	single	independent	dependent	inclined	independent
_____provided that only q_i,P_i are trated independent	F2(Q,P,t)	F2(q,p,t)	F2(q,P,t)	F2(Q,p,t)	F2(q,P,t)
F3(p,Q,t) provided that only _____are treated independent	p_i,Q_i	P_i,Q_i	p_i,q_i	P_i,q_i	p_i,Q_i
F4(p,P,t) provided that p_i,P_i are treated _____	independent	single	inclined	dependent	independent
_____is a 2n dimensional space having coordinatesq1,q2,.....qn and p1,p2,.....pn	actual path	configuration space	phase space	varied path	phase space
The _____have the property of preserving the form of hamilton's equations of motion under the transformation	point transformation	conatct transformation	legendre transformation	canonical transformation	canonical transformation

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DEPARTMENT OF MATHEMATICS

Subject: Mechanics	**Subject Code: 17MMP106**	**L T P C**
Class:I M.Sc	**Semester:I**	**4 0 0 4**

UNIT V

Hamilton Jacobi Theory: Hamilton Jacobi equations for Hamilton's principle function – Harmonic oscillator problem - Hamilton Jacobi equation for Hamilton's characteristic function – Separation of variables in the Hamilton-Jacobi equation.

SUGGESTED READINGS

TEXT BOOK

T1: Goldstein, H. (2001),Classical Mechanics Second Edition, Narosa Publishing House, New Delhi.

REFERENCES

R2: Gelfand, I. M., and Fomin, S. V., (2003),Calculus of Variations, Prentice Hall, New Delhi.

Hamilton Jacobi Theory:

It has already been mentioned that canonical transformations may be used to provide a general procedure for solving mechanical problems. Two methods have been suggested. If the Hamiltonian is conserved, then a solution could be obtained by transforming to new canonical coordinates that are all cyclic, thereby providing new equations of motion with trivial solutions. An alternative technique is to seek a canonical transformation from the coordinates and momenta, (q, p), at the time t, to a new set of constant quantities, which may be the $2n$ initial values, (q_0, p_0), at $t = 0$. With such a transformation, the equations of transformation relating the old and new canonical variables are exactly the desired solution of the mechanical problem:

$$q = q(q_0, p_0, t),$$

$$p = p(q_0, p_0, t).$$

They give the coordinates and momenta as a function of their initial values and the time. This last procedure is the more general one, especially as it is applicable, in principle at least, even when the Hamiltonian involves the time. We shall therefore begin our discussion by considering how such a transformation may be found.

THE HAMILTON–JACOBI EQUATION FOR HAMILTON'S PRINCIPAL FUNCTION

We can automatically ensure that the new variables are constant in time by requiring that the transformed Hamiltonian, K, shall be identically zero, for then the equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0,$$
$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0. \tag{10.1}$$

As we have seen, K must be related to the old Hamiltonian and to the generating function by the equation

$$K = H + \frac{\partial F}{\partial t},$$

and hence will be zero if F satisfies the equation

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0. \tag{10.2}$$

It is convenient to take F as a function of the old coordinates q_i, the new constant momenta P_i, and the time; in the notation of the previous chapter we would designate the generating function as $F_2(q, P, t)$. To write the Hamiltonian in Eq. (10.2) as a function of the same variables, use may be made of the equations of transformation (cf. Eq. (9.17a)),

$$p_i = \frac{\partial F_2}{\partial q_i},$$

so that Eq. (10.2) becomes

$$H\left(q_1, \ldots, q_n; \frac{\partial F_2}{\partial q_i}, \ldots, \frac{\partial F_2}{\partial q_n}; t\right) + \frac{\partial F_2}{\partial t} = 0. \tag{10.3}$$

Equation (10.3), known as the *Hamilton–Jacobi equation*, constitutes a partial differential equation in $(n+1)$ variables, $q_1, \ldots, q_n;\ t$, for the desired generating function. It is customary to denote the solution F_2 of Eq. (10.3) by S and to call it *Hamilton's principal function.*

Of course, the integration of Eq. (10.3) only provides the dependence on the old coordinates and time; it would not appear to tell how the new momenta are contained in S. Indeed, the new momenta have not yet been specified except that we know they must be constants. However, the nature of the solution indicates how the new P_i's are to be selected.

Mathematically Eq. (10.3) has the form of a first-order partial differential equation in $n + 1$ variables. Suppose there exists a solution to Eq. (10.3) of the form

$$F_2 \equiv S = S(q_1, \ldots, q_n;\ \alpha_1, \ldots, \alpha_{n+1}; t), \tag{10.4}$$

where the quantities $\alpha_1, \ldots, \alpha_{n+1}$ are $n + 1$ *independent* constants of integration. Such solutions are known as *complete solutions* of the first-order partial differential equation.* One of the constants of integration, however, is in fact irrelevant to the solution, for it will be noted that S itself does not appear in Eq. (10.3); only its partial derivatives with respect to q or t are involved. Hence, if S is some solution of the differential equation, then $S + \alpha$, where α is any constant, must also be a solution. One of the $n + 1$ constants of integration in Eq. (10.4) must therefore appear only as an additive constant tacked on to S. But by the same token, an additive constant has no importance in a generating function, since only partial derivatives of the generating function occur in the transformation equations.

Hence, for our purposes a complete solution to Eq. (10.3) can be written in the form

$$S = S(q_1, \ldots, q_n;\ \alpha_1, \ldots, \alpha_n; t), \tag{10.5}$$

where none of the n independent constants is solely additive. In this mathematical garb, S tallies exactly with the desired form for an F_2 type of generating function, for Eq. (10.5) presents S as a function of N coordinates, the time t, and n independent quantities α_i. We are therefore at liberty to take the n constants of integration to be the new (constant) momenta:

$$P_i = \alpha_i. \tag{10.6}$$

Such a choice does not contradict the original assertion that the new momenta are connected with the initial values of q and p at time t_0. The n transformation equations (9.17a) can now be written as

$$p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i}, \tag{10.7}$$

where q, α stand for the complete set of quantities. At the time t_0, these constitute n equations relating the n α's with the initial q and p values, thus enabling us to evaluate the constants of integration in terms of the specific initial conditions of the problem. The other half of the equations of transformation, which provide the new constant coordinates, appear as

$$Q_i = \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}. \tag{10.8}$$

The constant β's can be similarly obtained from the initial conditions, simply by calculating the value of the right side of Eq. (10.8) at $t = t_0$ with the known initial values of q_i. Equations (10.8) can then be "turned inside out" to furnish q_j in terms of α, β, and t:

$$q_j = q_j(\alpha, \beta, t), \tag{10.9}$$

which solves the problem of giving the coordinates as functions of time and the initial conditions.* After the differentiation in Eqs. (10.7) has been performed, Eqs. (10.9) may be substituted for the q's, thus giving the momenta p_i as functions of the α, β, and t:

$$p_i = p_i(\alpha, \beta, t). \tag{10.10}$$

Equations (10.9) and (10.10) thus constitute the desired complete solution of Hamilton's equations of motion.

Hamilton's principal function is thus the generator of a canonical transformation to constant coordinates and momenta; *when solving the Hamilton–Jacobi equation, we are at the same time obtaining a solution to the mechanical problem*. Mathematically speaking, we have established an equivalence between the $2n$ canonical equations of motion, which are first-order differential equations, to the first-order partial differential Hamilton–Jacobi equation. This correspondence is not restricted to equations governed by the Hamiltonian; indeed, the general theory of first-order partial differential equations is largely concerned with the properties of the equivalent set of first-order ordinary differential equations. Essentially, the connection can be traced to the fact that both the partial differential equation and its canonical equations stem from a common variational principle, in this case Hamilton's modified principle.

To a certain extent, the choice of the α_i's as the new momenta is arbitrary. We could just as well choose any n quantities, γ_i, which are independent functions of the α_i constants of integration:

$$\gamma_i = \gamma_i(\alpha_i, \ldots, \alpha_n). \tag{10.11}$$

By means of these defining relations, Hamilton's principal function can be written as a function of q_i, γ_i, and t, and the rest of the derivation then goes through unchanged. It often proves convenient to take some particular set of γ_i's as the new momenta, rather than the constants of integration that appear naturally in integrating the Hamilton–Jacobi equation.

Further insight into the physical significance of Hamilton's principal function S is furnished by an examination of its total time derivative, which can be computed from the formula

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t},$$

since the P_i's are constant in time. By Eqs. (10.7) and (10.3), this relation can also be written

$$\frac{dS}{dt} = p_i\dot{q}_i - H = L, \tag{10.12}$$

so that Hamilton's principal function differs at most from the indefinite time integral of the Lagrangian only by a constant:

$$S = \int L\,dt + \text{constant}. \tag{10.13}$$

Now, Hamilton's principle is a statement about the definite integral of L, and from it we obtained the solution of the problem via the Lagrange equations. Here the same action integral, in an indefinite form, furnishes another way of solving the problem. In actual calculations, the result expressed by Eq. (10.13) is of no help, because we cannot integrate the Lagrangian with respect to time until q_i and p_i are known as functions of time, that is, until the problem is solved.

When the Hamiltonian does not depend explicitly upon the time, Hamilton's principle function can be written in the form

$$S(q, \alpha, t) = W(q, \alpha) - at, \tag{10.14}$$

where $W(q, \alpha)$ is called *Hamilton's characteristic function.* The physical significance of W can be understood by writing its total time derivative

$$\frac{dW}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i.$$

Comparing this expression to the results of substituting Eq. (10.14) into Eq. (10.7), it is clear that

$$p_i = \frac{\partial W}{\partial q_i}, \tag{10.15}$$

and hence,

$$\frac{dW}{dt} = p_i\dot{q}_i. \tag{10.16}$$

This can be integrated to give

$$W = \int p_i\dot{q}_i\, dt = \int p_i\, dq_i, \tag{10.17}$$

which is just the abbreviated action defined by Eq. (8.80).

THE HARMONIC OSCILLATOR PROBLEM AS AN EXAMPLE OF THE HAMILTON–JACOBI METHOD

To illustrate the Hamilton–Jacobi technique for solving the motion of mechanical systems, we shall work out in detail the simple problem of a one-dimensional harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \equiv E, \tag{10.18}$$

where

$$\omega = \sqrt{\frac{k}{m}}, \tag{10.19}$$

k being the force constant. We obtain the Hamilton–Jacobi equations for S by setting p equal to $\partial S/\partial q$ and substituting in the Hamiltonian; the requirement that the new Hamiltonian vanishes becomes

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0. \tag{10.20}$$

Since the explicit dependence of S on t is present only in the last term, Eq. (10.14) can be used to eliminate the time from the Hamilton–Jacobi equation (10.20)

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha. \tag{10.21}$$

The integration constant α is thus to be identified with the total energy E. This can also be recognized directly from Eq. (10.14) and the relation (cf. Eq. (10.3))

$$\frac{\partial S}{\partial t} + H = 0,$$

which then reduces to

$$H = \alpha.$$

Equation (10.21) can be integrated immediately to

$$W = \sqrt{2m\alpha} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}, \tag{10.22}$$

so that

$$S = \sqrt{2m\alpha} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - \alpha t. \tag{10.23}$$

While the integration involved in Eq. (10.23) is not particularly difficult, there is no reason to carry it out at this stage, for what is desired is not S but its partial derivatives. The solution for q arises out of the transformation equation (10.8):

$$\beta' = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}} \int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t,$$

which can be integrated without trouble to give

$$t + \beta' = \frac{1}{\omega} \arcsin q \sqrt{\frac{m\omega^2}{2\alpha}}. \tag{10.24}$$

Equation (10.24) can be immediately "turned inside out" to furnish q as a function of t and the two constants of integration α and $\beta = \beta'\omega$:

$$q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t + \beta), \tag{10.25}$$

which is the familiar solution for a harmonic oscillator. Formally, the solution for the momentum comes from the transformation equation (10.7), which, using Eq. (10.22), can be written

$$p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}. \tag{10.26}$$

In conjunction with the solution for q, Eq. (10.25), this becomes

$$p = \sqrt{2m\alpha(1 - \sin^2(\omega t + \beta))},$$

or

$$p = \sqrt{2m\alpha}\cos(\omega t + \beta) \tag{10.27}$$

Of course, this result checks with the simple identification of p as $m\dot{q}$.

To complete the story, the constants α and β must be connected with the initial conditions q_0 and p_0 at time $t = 0$. By squaring Eqs. (10.25) and (10.27), it is clearly seen that α is given in terms of q_0 and p_0 by the equation

$$2m\alpha = p_0^2 + m^2\omega^2 q_0^2. \tag{10.28}$$

The same result follows immediately of course from the previous identification of α as the conserved total energy E. Finally, the phase constant β is related to q_0 and p_0 by

$$\tan\beta = m\omega\frac{q_0}{p_0}. \tag{10.29}$$

The choice $q_0 = 0$ and hence $\beta = 0$ corresponds to starting the motion with the oscillator at its equilibrium position $q = 0$.

Thus, Hamilton's principle function is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy.

If the solution for q is substituted into Eq. (10.23), Hamilton's principal function can be written as

$$S = 2\alpha\int \cos^2(\omega t + \beta)\,dt - \alpha t = 2\alpha\int (\cos^2(\omega t + \beta) - \tfrac{1}{2})\,dt. \tag{10.30}$$

Now, the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2m}(p^2 - m^2\omega^2 q^2) \\ &= \alpha(\cos^2(\omega t + \beta) - \sin^2(\omega t + \beta)) \\ &= 2\alpha(\cos^2(\omega t + \beta) - \tfrac{1}{2}), \end{aligned}$$

so that S is the time integral of the Lagrangian, in agreement with the general relation (10.13). Note that the identity could not be proved until *after* the solution to the problem had been obtained.

As another illustration for the Hamilton–Jacobi method, it is instructive to consider the two-dimensional anisotropic harmonic oscillator. If we let m be the mass of the oscillating body and k_x and k_y be the spring constants in the x- and y-directions, respectively, the Hamiltonian is

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2),$$

where

$$\omega_x = \sqrt{\frac{k_x}{m}} \quad \text{and} \quad \omega_y = \sqrt{\frac{k_y}{m}}.$$

Since the coordinates and momenta separate into two distinct sets, the principal function can be written as a sum of the characteristic function for each pair. Assuming that we solve the y-functional dependency first, this means

$$S(x, y, \alpha, \alpha_y, t) = F_x(x, \alpha) + F_y(y, \alpha_y) - \alpha t, \tag{10.31}$$

and the Hamilton–Jacobi equation assumes the form

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2\omega_x^2 x^2 + \left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha \tag{10.32}$$

in analogy with Eq. (10.18). Since the variables are separated, the y-part of the Eq. (10.32) must be equal to a constant, which we call α_y, so

$$\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{2}m\omega_y^2 y^2 = \alpha_y, \tag{10.33}$$

and we replace the y-term in (10.32) with α_y from (10.33), yielding

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}m\omega_x^2 x^2 = \alpha_x, \tag{10.34}$$

where we write $\alpha - \alpha_y = \alpha_x$ showing the symmetry of Eqs. (10.33) and (10.34).

Each equation has a solution analogous to Eqs. (10.25) and (10.27), so

$$\begin{aligned} x &= \sqrt{\frac{2\alpha_x}{m\omega_x^2}}\sin(\omega_x t + \beta_x) \\ p_x &= \sqrt{2m\alpha_x}\cos(\omega_x t + \beta_x) \\ y &= \sqrt{\frac{2\alpha_y}{m\omega_y^2}}\sin(\omega_y t + \beta_y) \\ p_y &= \sqrt{2m\alpha_y}\cos(\omega_y t + \beta_y), \end{aligned} \tag{10.35}$$

where the β_i's are phase constants and the total energy is given by

$$E = \alpha_x + \alpha_y = \alpha.$$

As a third example of Hamilton–Jacobi theory, we again consider the two-dimensional harmonic oscillator; only we will assume the oscillator is isotropic, so

$$k_x = k_y = k \qquad \text{and} \qquad \omega_x = \omega_y = \omega,$$

and use polar coordinates to write

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta & \theta &= \tan^{-1}\frac{y}{x} \\ p_x &= m\dot{x} & p_r &= m\dot{r} \\ p_y &= m\dot{y} & p_\theta &= mr^2\dot{\theta}. \end{aligned} \tag{10.36}$$

The Hamiltonian now written as

$$E = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2 r^2\right) \tag{10.37}$$

is cyclic in the angular coordinate θ. The principle function can then be written as

$$\begin{aligned} S(r, \theta, \alpha, \alpha_\theta) &= W_r(r, \alpha) + W_\theta(\theta, \alpha_\theta) - \alpha t \\ &= W_r(r, \alpha) + \theta\alpha_\theta - \alpha t, \end{aligned} \tag{10.38}$$

where, as we show later, a cyclic coordinate q_i always has the characteristic function component $W_{q_i} = q_i\alpha_i$. The canonical momentum p_θ associated with the cyclic coordinate, θ, is calculated from the generating function

$$p_\theta = \frac{\partial F_\theta}{\partial\theta} = \alpha_\theta$$

has its expected constant value.

When this p_θ is substituted into Eqs. (10.37) and (10.38), $W_r(r, \alpha)$ satisfies

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 = \alpha. \tag{10.39}$$

Rather than solving this equation directly for W_r, we shall write the Cartesian coordinate solution for these conditions as

$$\begin{aligned} x &= \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta) \qquad & p_x &= \sqrt{2m\alpha}\cos(\omega t + \beta) \\ y &= \sqrt{\frac{2\alpha}{m\omega^2}}\sin \omega t & p_y &= \sqrt{2m\alpha}\cos \omega t, \end{aligned} \tag{10.35'}$$

and use these to get the polar counterparts,

$$r = \sqrt{\frac{2\alpha}{m\omega^2}}\sqrt{\sin^2 \omega t + \sin^2(\omega t + \beta)}, \qquad p_r = m\dot{r},$$

and (10.40)

$$\theta = \tan^{-1}\left[\frac{\sin \omega t}{\sin(\omega t + \beta)}\right], \qquad p_\theta = mr^2\dot{\theta}.$$

There are two limiting cases. The linear case is when $\beta = 0$, for which

$$r = \sqrt{\frac{4\alpha}{m\omega^2}}\sin \omega t, \qquad p_r = \sqrt{2m\alpha}\cos \omega t,$$

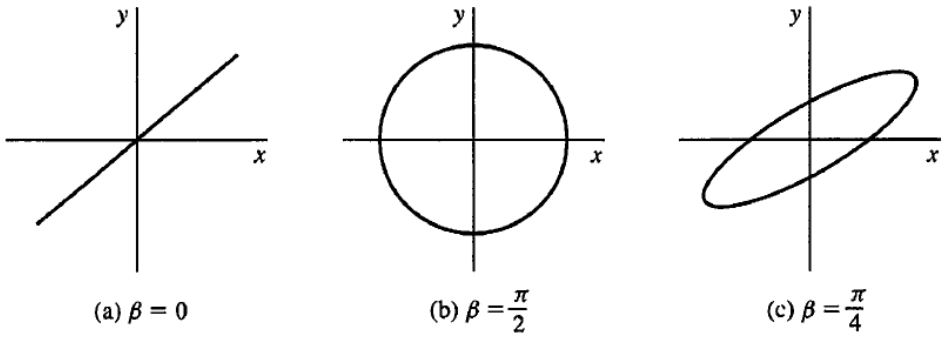
and (10.41)

$$\theta = \frac{\pi}{4}, \qquad p_\theta = 0.$$

The motion in an x-y plot will be an oscillation along a diagonal line as shown in Fig. 10.1a. The other limiting case is when $\beta = \pi/2$, for which

$$\begin{aligned} r &= r_0 = \sqrt{\frac{2\alpha}{m\omega^2}}, \qquad & p_r &= 0 \\ \theta &= \omega t, & p_\theta &= mr_0^2\omega. \end{aligned} \tag{10.42}$$

The motion in an x-y plot for this limiting case is a circle of radius r_0 as is shown in Figure 10.1b. For other values of β ($0 < \beta < \pi/2$), the orbit in coordinate space is an ellipse. The case for $\beta = \pi/4$ is shown in Fig. 10.1c. The plots shown in Fig. 10.1 are further examples of *Lissajous figures*.



(a) $\beta = 0$ (b) $\beta = \frac{\pi}{2}$ (c) $\beta = \frac{\pi}{4}$

FIGURE 10.1 The two limiting cases (a) and (b) for the harmonic oscillator and an intermediate example (c).

THE HAMILTON–JACOBI EQUATION FOR HAMILTON'S CHARACTERISTIC FUNCTION

It was possible to integrate the Hamilton–Jacobi equation for the simple harmonic oscillator primarily because S could be separated into two parts, one involving q only and the other only time. Such a separation of variables using Hamilton's characteristic function $W(q, \alpha)$ (Eq. (10.14)) is always possible *whenever the old Hamiltonian does not involve time explicitly*. This provides us with the restricted Hamilton–Jacobi equation

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1, \tag{10.43}$$

which no longer involves the time. One of the constants of integration, namely α_1, is thus equal to the constant value of H. (Normally H will be the energy, but remember that this need not always be the case, cf. Section 8.2.)

The time-independent function, Hamilton's characteristic function W, appears here merely as a part of the generating function S when H is constant. It can also be shown that W separately generates its own contact transformation with properties quite different from that generated by S. Let us consider a canonical transformation in which the new momenta are all constants of the motion α_i, and where α_1 in particular is the constant of motion H. If the generating function for this transformation be denoted by $W(q, P)$, then the equations of transformation are

$$p_i = \frac{\partial W}{\partial q_i}, \qquad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}. \tag{10.44}$$

While these equations resemble Eqs. (10.7) and (10.8) respectively for Hamilton's principal function S, the condition now determining W is that H is the new canonical momentum α_1:

$$H(q_i, p_i) = \alpha_1.$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0, \qquad P_i = \alpha_i. \tag{10.45}$$

Because the new Hamiltonian depends upon only one of the momenta α_i, the equations of motion for $\dot{Q}_i$ are

$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = 1, \quad i = 1,$$
$$= 0, \quad i \neq 1,$$

with the immediate solutions

$$\begin{aligned} Q_1 &= t + \beta_i \equiv \frac{\partial W}{\partial \alpha_1}, \\ Q_i &= \beta_i \equiv \frac{\partial W}{\partial \alpha_i} \quad i \neq 1. \end{aligned} \tag{10.46}$$

The only coordinate that is not simply a constant of the motion is Q_1, which is equal to the time plus a constant. We have here another instance of the conjugate relationship between the time as a coordinate and the Hamiltonian as its conjugate momentum.

The dependence of W on the old coordinates q_i is determined by the partial differential equation (10.43), which, like Eq. (10.3), is also referred to as the Hamilton–Jacobi equation. There will now be n constants of integration in a complete solution, but again one of them must be merely an additive constant. The $n-1$ remaining independent constants, $\alpha_2, \ldots, \alpha_n$, together with α_1 may then be taken as the new constant canonical momenta. When evaluated at t_0 the first half of Eqs. (10.44) serve to relate the n constants α_i with the initial values of q_i and p_i. Finally, Eqs. (10.45) and (10.46) can be solved for the q_i as a function of α_i, β_i, and the time t, thus completing the solution of the problem. It will be noted

that $(n-1)$ of the Eqs. (10.46) do not involve the time at all. One of the q_i's can be chosen as an independent variable, and the remaining coordinates can then be expressed in terms of it by solving only these time-independent equations. We are thus led directly to the *orbit equations* of the motion. In central force motion, for example, this technique would furnish r as a function of θ, without the need for separately finding r and θ as functions of time.

It is not always necessary to take α_1 and the constants of integration in W as the new constant canonical momenta. Occasionally it is desirable rather to use some particular set of n independent functions of the α_i's as the transformed momenta. Designating these constants by γ_i the characteristic function W can then be expressed in terms of q_i and γ_i as the independent variables. The Hamiltonian will in general depend upon more than one of the γ_i's and the equations of motion for $\dot{Q}_i$ become

$$\dot{Q}_i = \frac{\partial K}{\partial \gamma_i} = v_i,$$

where the v_i's are functions of γ_i. In this case, all the new coordinates are linear functions of time:

$$Q_i = v_i t + \beta_i. \tag{10.47}$$

The form of W cannot be found a priori without obtaining a complete integral of the Hamilton–Jacobi equation. The procedures involved in solving a mechanical problem by either Hamilton's principal or characteristic function may now by summarized in the following tabular form:

The two methods of solution are applicable when the Hamiltonian

is any general function of q, p, t: $H(q, p, t)$.	is conserved: $H(q, p) = \text{constant}$.

We seek canonical transformations to new variables such that

all the coordinates and momenta Q_i, P_i are constants of the motion.	all the momenta P_i are constants.

To meet these requirements it is sufficient to demand that the new Hamiltonian

shall vanish identically: $K = 0$.	shall be cyclic in all the coordinates: $K = H(P_i) = \alpha_1$.

Under these conditions, the new equations of motion become

$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0,$	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = v_i,$
$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0,$	$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0,$

with the immediate solutions

$$\begin{aligned} Q_i &= \beta_i, \\ P_i &= \gamma_i, \end{aligned} \qquad \begin{aligned} Q_i &= v_i t + \beta_i \\ P_i &= \gamma_i \end{aligned}$$

which satisfy the stipulated requirements.

The generating function producing the desired transformation is Hamilton's

Principal Function:	Characteristic Function:
$S(q, P, t)$,	$W(q, P)$,

satisfying the Hamilton–Jacobi partial differential equation:

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0. \qquad H\left(q, \frac{\partial W}{\partial q}\right) - \alpha_1 = 0.$$

A complete solution to the equation contains

n nontrivial constants of integration $\alpha_1, \ldots, \alpha_n$.	$n - 1$ nontrivial constants of integration, which together with α_1 form a set of n independent constants $\alpha_1, \ldots, \alpha_n$.

The new constant momenta, $P_i = \gamma_i$, can be chosen as any n independent functions of the n constants of integration:

$$P_i = \gamma_i(\alpha_1, \ldots, \alpha_n), \qquad P_i = \gamma_i(\alpha_1, \ldots, \alpha_n),$$

so that the complete solutions to the Hamilton–Jacobi equation may be considered as functions of the new momenta:

$$S = S(q_i, \gamma_i, t). \qquad W = W(q_i, \gamma_i).$$

In particular, the γ_i's may be chosen to be the α_i's themselves. One-half of the transformations equations,

$$p_i = \frac{\partial S}{\partial q_i}, \qquad p_i = \frac{\partial W}{\partial q_i},$$

are fulfilled automatically, since they have been used in constructing the Hamilton–Jacobi equation. The other half,

$$Q_i = \frac{\partial S}{\partial \gamma_i} = \beta_i, \qquad Q_i = \frac{\partial W}{\partial \gamma_i} = v_i(\gamma_j)t + \beta_i,$$

can be solved for q_i in terms of t and the $2n$ constants β_i, γ_i. The solution to the problem is then completed by evaluating these $2n$ constants in terms of the initial values, (q_{i0}, p_{i0}), of the coordinates and momenta.

When the Hamiltonian does not involve time explicitly, both methods are suitable, and the generating functions are then related to each other according to the formula

$$S(q, P, t) = W(q, P) - \alpha_1 t.$$

SEPARATION OF VARIABLES IN THE HAMILTON–JACOBI EQUATION

It might appear from the preceding section that little practical advantage has been gained through the introduction of the Hamilton–Jacobi procedure. Instead of solving the $2n$ ordinary differential equations that make up the canonical equations of motion, we now must solve the partial differential Hamilton–Jacobi equation, and partial differential equations can be notoriously complicated to solve. Under certain conditions, however, it is possible to separate the variables in the Hamilton–Jacobi equation, and the solution can then always be reduced to quadratures. In practice, the Hamilton–Jacobi technique becomes a useful computational tool only when such a separation can be effected.

A coordinate q_j is said to be separable in the Hamilton–Jacobi equation when (say) Hamilton's principal function can be split into two additive parts, one of which depends only on the coordinate q_j and the other is entirely independent of q_j. Thus, if q_1 is taken as a separable coordinate, then the Hamiltonian must be such that one can write

$$S(q_1, \ldots, q_n;\ \alpha_1, \ldots, \alpha_n;\ t) = S_1(q_1;\ \alpha_1, \ldots, \alpha_n;\ t) + S'(q_2, \ldots, q_n;\ \alpha_1, \ldots, \alpha_n;\ t), \quad (10.48)$$

and the Hamilton–Jacobi equation can be split into two equations—one separately for S_1 and the other for S'. Similarly the Hamilton–Jacobi equation is described as *completely separable* (or simply, *separable*) if all the coordinates in the problem are separable. A solution for Hamilton's principal function of the form

$$S = \sum_i S_i(q_i;\ \alpha_1, \ldots, \alpha_n;\ t) \quad (10.49)$$

will then split the Hamilton–Jacobi equation into n equations of the type

$$H_i\left(q_j;\ \frac{\partial S_j}{\partial q_j}; \alpha_1, \ldots, \alpha_n; t\right) + \frac{\partial S_j}{\partial t} = 0. \quad (10.50)$$

If the Hamiltonian does not explicitly depend upon the time, then, for each S_i we have

$$S_i(q_j;\ \alpha_1, \ldots, \alpha_n; t) = W_i(q_j;\ \alpha_1, \ldots, \alpha_n; t) - \alpha_i t, \quad (10.51)$$

which provide n restricted Hamilton–Jacobi equations,

$$H_i\left(q_i;\ \frac{\partial W_i}{\partial q_i}; \alpha_1, \ldots, \alpha_n\right) = \alpha_i. \quad (10.52)$$

(No summation in Eqs. (10.50) to (10.52)!)

The functions H_i in Eqs. (10.50) and (10.52) may or may not be Hamiltonians, and the α_i may be an energy, an angular momentum squared, or some other quantity depending on the nature of q_i.

POSSIBLE QUESTIONS

Part B (6 Marks)

1. Derive Hamilton Jacobi equation from Hamilton's characteristic function
2. Explain the physical significance of Hamilton Jacobi equation
3. Derive Jacobi's theorem
4. Derive kepler's problem solution by Hamilton Jacobi method
5. Derive the separation of variables in the Hamilton Jacobi equation

Part C (10 Marks)

1.Derive Harmonic oscillator problem by Hamilton Jacobi method

2. Derive Hamilton Jacobi equation from Hamilton's characteristic function

3.Derive the Lagrange's equation from Hamilton's principle for holonomic system

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
UNIT-V

Subject: Mechanics **Subject Code: 17MMP106**

Hamilton Jacobi Theory
Part-A(20X1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)
Multiple Choice Questions

Question	Opt 1	Opt 2	Opt 3	Opt 4	Answer
The harmonic oscilllator problem is an example of____equation	Euler's	Hamilton lagrange's	hamilton jacobi's	Routh's	hamilton jacobi's
The variation principle in the phases space is reffered to as the _____	modified jacobi's	modified lagrange's	modified Hamilton's	modified euler	modified Hamilton's
The solution of the Hamilton-jcobi eqaution is called _____function	hamilton principle	Hamilton's charecteristics	jacobi	Routhian	hamilton principle
The hamilton principle function differs atmost from the indefinite time integral of the lagrangian by_____	The invariants	time	coordinates	a constant	a constant
The hamilton jacobi equation can be_____	seperable	quadratures	dependent	completely seperable	completely seperable
The hamilton principle function can be split into_____	Finitely many parts	infinite parts	two additive parts	two multiplicative parts	two additive parts
The Hamilton jacobi equation can hold if the two terms are constant with_____	equal value	trial solution	opposite value	equal and opposite value	equal and opposite value
The orthogonal coordinate system can be used to _____	seperable condition	staeckel condition	quadratic function	generating function	staeckel condition
hamilton's principle function S and Hamilton's charecteristic function W for conservative system related as _____where E is the total energy and t is the time	S=W	S=W-Et	S=W+Et	S is not relateed to W	S=W-Et
For a one-dimensional harmonic oscillator.the representative point in two-dimensional phase space traces_____	an ellipse	a parabola	a hyperbola	always a straight line	an ellipse
If a function F doesnot depend on time explicitly and is a constant of motion,its poisson bracket with hamiltonian_____	non zero	conserved	vanishes	fixed	vanishes
Poisson bracket and lagrange's bracket doesn't obey the _____Law of algebra	inverse	commutative	identity	associative	commutative
the _____of two constants of motion is itself a constant of motion	Lagrange's bracket	jacobi's inverse	poisson bracket	jacobi's identity	poisson bracket
The lagrange's bracket is _____under canonical transformation	real	constant	imaginary	invariant	invariant
The changing state of the system may be described by a curve r(t) in the phase space is _____	imaginary path	circular path	phase path	constant path	phase path
Poisson bracket of twoo dynamical variables is invariant under infinitesimal_____	point transformation	canonical transformation	contact transformation	legendre transformation	canonical transformation
The generaloized coordinates conjugate to Jj are called_____	angle variables	constant variables	fixed variables	action variables	angle variables
__which is a partial differential equation of first order in (n+1) variablesq1,q2,q,3,,,,,qn,t	hamilton jacobi's equation	hamilton lagrange's equation	Euler's equation	Routh's equation	hamilton jacobi's equation
_____function S is the generator of a canonical transformation to constant co-ordinates and momenta	hamilton principle	Hamilton's charecteristic	jacobi	Routhian	hamilton principle
In solving the hamilton jacobi equation,we obtain simultaneously a solution to the _____problem	dynamical	geometrical	mechanical	physical	mechanical
the force acting on the oscillator at a displacement q is _____	F=-kq	F=p	F=q	F=-Kp	F=-kq
Additive constant C will__the transformation,because to obtain the new position coordinate only partial derivative of S with respect tio a is required	increase	not effect	decrease	vanish	not effect
The constant alpha and beta are to be known from_____	final condition	fixed value	initial condition	stationary value	initial condition
For the Hamilton's principle function S,first part is the function of a and q is called _____Function	Hamilton Principle	Hamilton's characteristic	Jacobi	Routhian	Hamilton's characteristic
Hamilton's characteristic function is denoted by _____	H(q,a)	D(q,a)	E(q,a)	W(q,a)	W(q,a)
The new canonical momentum is identified as the _____	harmonic oscillator	total energy of the oscillator	small oscillator	lagrange oscillator	total energy of the oscillator
For harmonic oscillator the hamilton's principle function is the_____Integral of lagrangian	mass	velocity	time	distance	time
Separation of variables is always possible,if the Hamiltonian H does not involve time t explicitly is called_____	method of separation of variable	method of point of variable	method of hamilton of variable	method of lagrange of variable	method of separation of variable

UNIT V

The harmonic oscilllator problem is an example of____equation	Euler's	Hamilton lagrange's	hamilton jacobi's	Routh's	hamilton jacobi's
The variation principle in the phases space is reffered to as the _____	modified jacobi's	modified lagrange's	modified Hamilton's	modified euler	modified Hamilton's
The solution of the Hamilton-jcobi eqaution is called _____function	hamilton principle	Hamilton's charecteristics	jacobi	Routhian	hamilton principle
The hamilton principle function differs atmost from the indefinite time integral of the lagrangian by_____	The invariants	time	coordinates	a constant	a constant
The hamilton jacobi equation can be_____	seperable	quadratures	dependent	completely seperable	completely seperable
The hamilton principle function can be split into_____	Finitely many parts	infinite parts	two additive parts	two multiplicative parts	two additive parts
The Hamilton jacobi equation can hold if the two terms are constant with_____	equal value	trial solution	opposite value	equal and opposite value	equal and opposite value
The orthogonal coordinate system can be used to _____	seperable condition	staeckel condition	quadratic function	generating function	staeckel condition
hamilton's principle function S and Hamilton's charecteristic function W for conservative system related as _____where E is the total energy and t is the time	S=W	S=W-Et	S=W+Et	S is not relateed to W	S=W-Et
For a one-dimensional harmonic oscillator.the representative point in two-dimensional phase space traces_____	an ellipse	a parabola	a hyperbola	always a straight line	an ellipse
If a function F doesnot depend on time explicitly and is a constant of motion,its poisson bracket with hamiltonian_____	non zero	conserved	vanishes	fixed	vanishes
Poisson bracket and lagrange's bracket doesn't obey the _____Law of algebra	inverse	commutative	identity	associative	commutative
the _____of two constants of motion is itself a constant of motion	Lagrange's bracket	jacobi's inverse	poisson bracket	jacobi's identity	poisson bracket
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The generaloized coordinates conjugate to Jj are called_____	angle variables	constant variables	fixed variables	action variables	angle variables
__which is a partial differential equation of first order in (n+1) variablesq1,q2,q,3,,,,,qn,t	hamilton jacobi's equation	hamilton lagrange's equation	Euler's equation	Routh's equation	hamilton jacobi's equation
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In solving the hamilton jacobi equation,we obtain simultaneously a solution to the _____problem	dynamical	geometrical	mechanical	physical	mechanical
the force acting on the oscillator at a displacement q is _____	F=-kq	F=p	F=q	F=-Kp	F=-kq
Additive constant C will__the transformation,because to obtain the new position coordinate only partial derivative of S with respect tio a is required	increase	not effect	decrease	vanish	not effect
The constant alpha and beta are to be known from_____	final condition	fixed value	initial condition	stationary value	initial condition
For the Hamilton's principle function S,first part is the function of a and q is called _____Function	Hamilton Principle	Hamilton's characteristic	Jacobi	Routhian	Hamilton's characteristic
Hamilton's characteristic function is denoted by _____	H(q,a)	D(q,a)	E(q,a)	W(q,a)	W(q,a)
The new canonical momentum is identified as the _____	harmonic oscillator	total energy of the oscillator	small oscillator	lagrange oscillator	total energy of the oscillator
For harmonic oscillator the hamilton's principle function is the_____Integral of lagrangian	mass	velocity	time	distance	time
Separation of variables is always possible,if the Hamiltonian H does not involve time t explicitly is called_____	method of separation of variable	method of point of variable	method of hamilton of variable	method of lagrange of variable	method of separation of variable

KARPAGAM UNIVERSITY
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COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

M.Sc., DEGREE EXAMINATION, APRIL 2016
Second Semester

MATHEMATICS

MECHANICS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. Derive the equation of motion of a single particle using plane polar co-ordinates

Or

b. i)Show that $T = T_0 + T_1 + T_2$
ii) Derive the Principle of virtual work

22. a. Find the minimum surface of revolution

Or

b. Derive the conservation theorem for total energy of system

23. a) Obtain the hamilton's equation of motion considering a single non relativistic particle moving in an electromagnetic field

Or

b) Explain the principle of least action

24. a) Explain the simple harmonic oscillator problem

Or

b) Explain the integral invariants of poincare

Or

b) Derive the Hamilton Jacobi equation for Hamilton's pri

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Derive the Lagrange's equation from Hamilton's principl

Reg. No.....

[16MMP206]

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M.Sc., DEGREE EXAMINATION, APRIL 2017
Second Semester

MATHEMATICS

MECHANICS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a) Derive the Lagrange's equation of motion for holonomic constraint.
Or
b) Obtain the equation of motion of one particle using Cartesian co-ordinates

22. a) Show that the shortest curve between any two points in the plane is a straight line
Or
b) Derive the conservation theorem for dissipation function

23. a) Define Cyclic coordinates and Explain conservation theorems.
Or
b) Derive Hamilton's canonical equation of motion

24. a) Explain the canonical transformation with an example
Or
b) Explain Jacobi's identity

25. a) Derive the Hamilton Jacobi equation for Hamilton's principle function.
Or
b) Derive Kepler's problem solution by Hamilton Jacobi method

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Derive the D' Alembert's principle

Reg. No.....

[15MMU301]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
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COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

B.Sc. DEGREE EXAMINATION, NOVEMBER 2016

Third Semester

MATHEMATICS

MECHANICS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 8 = 40 Marks) (2 ½ Hours)
Answer ALL the Questions

21. a. (i) State and prove parallelogram of forces.
(ii) State and prove triangle of forces.

Or

b. Find the resultant of two like parallel forces acting on a rigid body.

22. a. State and prove Varigon's theorem of moments.

Or

b. Prove the equilibrium of a particle on a rough inclined plane.

23. a. Find the components of the acceleration of a particle in the tangential and normal directions.

Or

b. Obtain the differential equation of a central orbit in polar coordinates.

24. a. Prove that the composition of two simple harmonic motion of the same period in two perpendicular directions.

Or

b. Explain the seconds pendulum.

25. a. Eight centimetres of rainfall in a certain district in 24 hours. Assuming that the drops fall freely from a height of 109 metres, find the pressure on the ground per square kilometer of the district.

Or

b. A gun of mass M fires a shell of mass m, the elevation of the gun being α. If the gun can recoil freely in the horizontal direction, show that the angle θ which is the path of the shell initially makes with the horizontal is given by equation $\tan\theta = (1+ m/M)\tan\alpha$.

Reg. No ----------

(17MMP106)

KARPAGAM UNIVERSITY

Karpagam Academy Of Higher Education

COIMBATORE-21

DEPARTMENT OF MATHEMATICS

Third Semester

I INTERNAL TEST- JUL '17

MECHANICS

Date: . 08.17 ( ) **Time:2 hours**

Class: IM.Sc(Mathematics) **Maximum: 50 Marks**

PART- A (20x1 = 20 Marks)

ANSWER THE FOLLOWING

1. Point transformation are the transformations of _____space
   a) Configuration b) position
   c) phase d) co-ordinate
2. A function f(x) is said to be statinary at x=a when_____
   a) f(a)=constant b) f(a)=0 c) f'(a)=constant d) f'(a)=0
3. The number of coordinates minus the number of independent equations of constraint is_____
   a) units b) number of degrees of freedom
   c) dimensions d) number of generalized coordinates
4. The total kinetic energy of the system can be written as_____
   a) $T=T_0+T_1$ b) $T=T_1+T_2$ c) $T=T_0+T_2$ d) $T=T_0+T_1+T_2$
5. The sign is _____ if F rotates in anticlockwise direction.
   a) Zero b) Positive c) Negative d) Unity
6. The equation of motion is the differential equation of _____order
   a) Fourth b) first c) second d) third
7. The generalised momentum conjugate to the cyclic coordinate is_____
   a) Zero b) motion c) non zero d) conserved
8. The _____ doesn't deform under the action of loads
   a) Force b) Rigid body
   c) varied path d) applied torque
9. The space around the magnet is called the_____
   a) magnetic field b) electric field
   c) magnetic induction d) flux
10. Friction is a _____ force.
    a) Sliding b) Self adjusting c) Rolling d) Resultant
11. The space around the magnet is called the_____
    a) Force b) Rigid body
    c) varied path d) applied torque
12. If the total external force is zero,then the total linear momentum of the system is _____
    a) Zero b) non-zero
    c) conserved d) rigid
13. The equation of motion is the differential equation of _____order
    a) Fourth b) first c) second d) third
14. The force which opposes the sliding of one body over another is called the ____
    a) Sliding friction b) Coefficient of friction
    c) Rolling friction d) Cone of friction
15. The line integral may also be denoted as_____
    a) J b) L c) X d) K
16. All the generalized coordinates cannot be_____
    a) linear b) non cyclic c) non linear d) cyclic
17. Friction is a _____ force.

a) Sliding b) Self adjusting c) Rolling d) Resultant

18. A cyclic coordinate is also known as_____
a) constraint b) ignorable c) constant d) motion

19._____ is defined as the line along which the force acts.
a) line of action of force b) Components of force
c) Resultant of force d) Direction of force

20.Force of friction is also called _____
a) Massive force b) Passive force
c) Coplanar force d) Colinear force

PART- B (3 x 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21.Explain the Rayleighs dissipation function .

22.Explain degrees of freedom.

23.Explain Hamiton's principle.

PART- C (3 x 8 = 24 Marks)

ANSWER ALL THE QUESTIONS

24.a) Derive the Lagrange's equation of motion for holonomic constraint.

(OR)

b) Derive the D' Alembert's principle

25. a)i)Show that $T = T_0 + T_1 + T_2$
ii)Derive the Principle of virtual work

(OR)

b) Show that the shortest curve between any two points in the plane is a straight line

26. a) State and prove Euler Lagrange differential equation.

(OR)

b) Define Cyclic coordinates and Explain conservation theorems.

Reg No : ----------

[17MMP106]

KARPAGAM UNIVERSITY
Karpgam Academy of Higher Education
COIMBATORE – 641021
DEPARTMENT OF MATHEMATICS
First Semester
II Internal Test Sep ' 17
MECHANICS

Class : I M.Sc (MATHEMATICS **Time:2 hours**

Date : .09.17 ( ) **Maximum Marks : 50**

PART-A (20x 1 =20 Marks)

ANSWER ALL THE QUESTIONS :

1.The generalised momentum conjugate to the cyclic coordinate is_____

a) Zero b) motion c) non zero d) conserved

2. All the generalized coordinates cannot be_____

a) linear b) non cyclic c) non linear d) cyclic

3. The hamiltonian is the total energy of the system_____

a) $H^2=T^2+V^2$ b) H=T-V c) H=T+V d) $T^2=H^2+V^2$

4. The variational principle associated with the hamilton's formulation is known as the _____

a) holonomic problem b) principle of least action

c) monogenic principle d) Brachistrone problem

5. K plays the role of _____

a) Legendre's b) Lagrangian

c) Hamiltonian d) point

6. A cyclic coordinate is also known as_____

a) constraint b) ignorable c) constant d) motion

7. All the generalized coordinates cannot be_____

a) linear b) non cyclic c) non linear d) cyclic

8. To linearize the motion for small oscillation,we assume that_____

a) $\cos(_{\varphi}-\theta)\approx1$ b) $\sin(_{\varphi}-\theta)\approx1$

c) $\cos(_{\varphi}-\theta)\approx0$ d) $\sin(_{\varphi}-\theta)\approx0$

9. Lagrange's equationfollows from the _____principle

a) Jacobi's b) Routh's c) Hamilton's d) D'Alemberts

10. The polar equation of a conic is _____

a) $l/r=1+e\cos\theta$ b) $r= 1+e\cos\theta$

c) $r= 1/(1+e\cos\theta)$ d) $r= 1- e\cos\theta$

11. The velocity of Simple Harmonic Motion is zero when _____

a) amplitude b) phase

c) period d) acceleration

12. A _____ is a position of matter occupying finite space .

a) Impact b) Body

c) Momentum d) Elastic body

13. The sign is _____ if F rotates in anticlockwise direction.

a) Zero b) Positive c) Negative d) Unity

14. The important variational principle associated with hamiltonian formulation is the _____

a) Integral principle b) principle of least action .

c) Lagrange's principle d) monogenic principle

15. A cyclic coordinate is also known as_____

a) constraint b) ignorable c) constant d) motion

16. Hamilton's principle function S and Hamilton's charecteristic function W for conservative system related as _____.where E is the total energy and t is the time

a) S=W b) S=W-Et

c) S=W+Et d) S is not relateed to W

17. The hamilton jacobi equation can be_____
a) seperable b) quadratures
c)dependent d) completely seperable

18 . The equation of motion is the differential equation of _____order
a) Fourth b) first c) second d) third

19.Friction is a _____force.
a) Sliding b) Self adjusting c) Rolling d) Resultant

20. Poisson bracket and lagrange's bracket doesn't obey the _____law of algebra
. a) inverse b) commutative
c) identity d) associative

PART B (3 X 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21. Define canonical momentum.

22. Write short notes on construction of Hamilton through Lagrangian

23. Obtain the hamilton's equation of motion using spherical polar co-ordinates .

PART-C (3X8=24 Marks)

ANSWER ALL THE QUESTIONS

24. (a) Explain the canonical transformation with an example

(OR)

(b) Derive Hamilton's canonical equation of motion

25. (a) Show that the transformation P=1/2 $\times(p^2+q^2)$, Q = tan^{-1}(q/p) is canonical

(OR)

(b) Derive Jacobi's theorem

26. (a) Show that the transformation P=q cot p , Q = log($\frac{sin\, p}{q}$) is canonical. Also find the generating function.

(OR)

(b) Derive Harmonic oscillator problem by Hamilton Jacobi method