KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT Of MATHEMATICS

Subject: FUNCTIONAL ANALYSIS **Subject Code: 16MMP301**

L	T	P	C
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PO: This course provides a systematic study of linear, topological or metric structures and it also deals with spaces and operators acting on them.

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PLO: To be thorough with Banach spaces, related theorems, orthonormal sets, normal and unitary operators and to be familiar with Banach algebras. **UNIT I**

Banach Spaces- Normed linear space – Definitions and Examples-Theorems. Continuous Linear Transformations – Some theorems- Problems. The Hahn- Banach Theorem –Lemma and Theorems. The Natural imbedding of N in N**-Definitions and Theorems.

UNIT II

The Open Mapping Theorem- Theorem and Examples –Problems. The closed graph theorem. The conjugate of an operation- The uniform boundedness theorem- Problems.

UNIT III

Hilbert Spaces- The Definition and Some Simple Properties – Examples and Problems. Orthogonal Complements - Some theorems .Ortho-normal sets – Definitions and Examples-Bessel's inequality- The conjugate space H*.

UNIT IV

The Adjoint of an operator – Definitions and Some Properties-Problems. Self- adjoint operators – and Unitary operators –Theorems and Problems. Some Theorems and Problems. Normal Projections - Theorems and Problems

UNIT V

Banach algebras: The definition and some examples of Banach algebra – Regular and singular elements – Topological divisors of zero – The spectrum – The formula for the spectral radius.

TEXT BOOK

1. Balmohan V., and Limaye.**,** 2004. Functional Analysis, New Age International Pvt. Ltd, Chennai.

REFERENCES

1. Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi.
2. Chandrasekhara Rao.K., 2006. Functional Analysis, Narosa Publishing House, Chennai.

3. Choudhary .B,and Sundarsan Nanda., 2003. Functional Analysis with Applications, New Age International Pvt. Ltd, Chennai.

4. Ponnusamy.S., 2002. Foundations of functional analysis, Narosa Publishing House, Chennai.

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DEPARTMENT OF MATHEMATICS

Lecture Plan

Subject Name: Functional Analysis **Subject Code: 16MMP301**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-I	
1	1	Introduction to linear Spaces	T: Ch 2, 33
2	1	Definitions and Examples on normed linear space	R1: Ch 9, 212
3	1	Definitions and Examples on normed linear space	R1: Ch 9, 213
4	1	Theorems on normed linear spaces	R1: Ch 9, 214
5	1	Theorems on normed linear spaces	R1: Ch 9, 215
6	1	Theorems on continuous Linear Transformations	R1: Ch 9, 216
7	1	Theorems on continuous Linear Transformations	R1: Ch 9, 217
8	1	The Hahn- Banach Theorem.	R1: Ch 9, 218
9	1	The Hahn- Banach Theorem.	R1: Ch 9, 219
10	1	The Natural imbedding of N in N	R1: Ch 9, 220
11	1	The Natural imbedding of N in N	R1: Ch 9, 221-222
12	1	Recapitulation and Discussion of possible questions	
Total	**12Hours**		

TEXT BOOK

T Balmohan V., and Limaye., 2004. Functional Analysis, New Age International Pvt. Ltd, Chennai.

REFERENCES

R1 Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi.

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-II	
1	1	The Open Mapping Theorem	R1: Ch 9, 235
2	1	Continuation of The Open Mapping Theorem	R1: Ch 9, 236
3	1	Continuation of The Open Mapping Theorem	R1: Ch 9, 237
4	1	Continuation of The Open Mapping Theorem	R1: Ch 9, 238

5	1	The closed graph theorem	R1: Ch 9, 239
6	1	The closed graph theorem	R1: Ch 9, 240
7	1	Theorems on conjugate of an operation	R1: Ch 9, 241
8	1	Theorems on conjugate of an operation	R1: Ch 9, 241
9	1	Problems on conjugate of an operation	R1: Ch 9, 242
10	1	Problems on conjugate of an operation	R1: Ch 9, 243
11	1	The uniform boundedness theorem	R1: Ch 9, 244
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

REFERENCES

R1 Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi

UNIT-III

1	1	Introduction to Hilbert Spaces	R1: Ch 10, 244
2	1	Introduction to Hilbert Spaces	R1: Ch 10, 245
3	1	Some Simple Properties	R1: Ch 10, 246
4	1	Some Simple Properties.	R1: Ch 10, 247-248
5	1	Theorems on orthogonal Complements	R1: Ch 10, 248-249
6	1	Theorems on orthogonal Complements	R1: Ch 10, 250
7	1	Examples and Problems on orthogonal Complements	R1: Ch 10, 251
8	1	Examples and Problems on orthogonal Complements.	R1: Ch 10, 252-253
9	1	Theorems on orthonormal sets	R1: Ch 10, 254
10	1	Definitions and Examples-Bessel's inequality	R1: Ch 10, 255
11	1	The conjugate space *H*	R1: Ch 10, 256-257
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

REFERENCES

R1 Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi.

UNIT-IV

1	1	Introduction to Adjoint of an operator	R1: Ch 10, 262
2	1	Introduction to Adjoint of an operator	R1: Ch 10, 263
3	1	Theorems on Adjoint of an operator	R1: Ch 10, 265
4	1	Theorems on Adjoint of an operator	R1: Ch 10, 266
5	1	Theorems on Adjoint of an operator	R1: Ch 10, 267
6	1	Examples and Problems on Adjoint of an operator	R1: Ch 10, 268

7	1	Examples and Problems on Adjoint of an operator	R1: Ch 10, 269-270
8	1	Theorems on normal Projections	R1: Ch 10, 271
9	1	Theorems for Nilpotent Form	R1: Ch 10, 272
10	1	Theorems on normal Projections	R1: Ch 10, 273
11	1	Theorems on normal Projections	R1: Ch 10, 274
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

REFERENCES

R1 Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi.

UNIT-V

1	1	The definition of Banach algebra	R1: Ch 12, 302-303
2	1	some examples of Banach algebra	R1: Ch 12, 304
3	1	Continuation of Banach algebras	R1: Ch 12, 305
4	1	Some Theorems on Banach algebras	R1: Ch 12, 306
5	1	Problems on Banach algebras	R1: Ch 12, 307
6	1	Problems on Banach algebras	R1: Ch 12, 308
7	1	Theorems on regular and singular elements	R1: Ch 12, 309
8	1	Theorems on regular and singular elements	R1: Ch 12, 309
9	1	Theorems on topological divisors of zero	R1: Ch 12, 310-311
10	1	Recapitulation and Discussion of possible questions	
11	1	Discussion on Previous ESE Question Papers	
12	1	Discussion on Previous ESE Question Papers	
Total	**12 Hours**		

REFERENCES

R1 Simmons. G.F., 1963. Introduction to Topology & Modern Analysis, Tata McGraw-Hill Publishing Company Ltd, New Delhi..

Total no. of Hours for the Course: 60 hours

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DEPARTMENT OF MATHEMATICS

Subject: Functional Analysis	**Semester: III**	**L T P C**
Subject Code: 16MMP301	**Class: II-M.Sc. Mathematics**	**4 0 0 4**

UNIT I

Banach Spaces- Normed linear space – Definitions and Examples-Theorems. Continuous Linear Transformations – Some theorems- Problems. The Hahn- Banach Theorem –Lemma and Theorems. The Natural imbedding of N in N**-Definitions and Theorems.

TEXT BOOK

1. Balmohan V., and Limaye., 2004. Functional Analysis, New Age International Pvt. Ltd, Chennai.

REFERENCES

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3. Choudhary .B,and Sundarsan Nanda., 2003. Functional Analysis with Applications, New Age International Pvt. Ltd, Chennai.

4. Ponnusamy.S., 2002. Foundations of functional analysis, Narosa Publishing House, Chennai.

1 Linear spaces

The readers will recall that by the scalars we mean either the sytem of real numbers or the system of complex numbers.

Definition 1 *A linear space is an additive abelian group L with the property that any scalar α and any $x \in L$ can be combined by an operation with $\alpha x \in L$ in such a manner that*

i $\alpha(x + y) = \alpha x + \alpha y$

ii $(\alpha + \beta)x = \alpha x + \beta x$

iii $(\alpha\beta)x = \alpha(\beta y)$

iv $1 \cdot x = x$

Remark 1 *The elements of L are called vectors*

Remark 2 *The operation newly defined is called scalar mulyiplication*

Remark 3 *The linear space is called real linear space or a complex space according as the scalars are the real numbers or the complex numbers.*

Example 1 *The simplest example of a vector space is the trivial space* $\{0\}$*, which contains only the zero vector and with scalar multiplication defined by* $\alpha \cdot 0 = 0$

Example 2 *The set* $\mathbb{R}$ *of real numbers* $\mathbb{R}$ *is a vector space over* $\mathbb{R}$*.*

Example 3 *The set* $\mathbb{R}^2$ *of all ordered pairs of real numers is a vector space over* $\mathbb{R}$*.*

Example 4 *The set of all functions f defined at 1 with* $f(1) = 0$ *.The number* 0 *is essential in this example. If we replace* 0 *by a nonzero number c, we violate the closure axioms.*

2 Subspaces

Definition 2 *A nonempty subset M of a vector space L is called a subspace of L if M is a vector space under the operations addition and scalar multiplication defined in L.*

Theorem 1 *Suppose L is a linear space over $\mathbb{R}$ and $L \subset L$ is a nonempty subset of L. Then M is a subspace of L if and only if the following two closure conditions hold:*

1 *If u, v are in M, then $u + v$ is in W.*

2 *If u is in M and c is a scalar, then cu is in M.*

3 Linear transformation

Definition 3 *Let X and Y be vecto,rspaces. A linear operator or linear function or linear transformation from X into Y is a function $T : X \to Y$such that the following two conditions are satisfied whenever $x, x_1, x_2 \in X$ and a $\alpha \in \mathbb{F}$*

(1) $T(x_1 + x_2) = T(x_1) + T(x_2)$

(2) $T(\alpha x) = \alpha T(x)$

4 Cosets

Definition 4 *Let M be a subspace of a vector space L. Then the cosets of M are the sets*

$$f + M = \{f + m : m \in M\}, f \in L$$

Theorem 2 *Let L be a vector space, and let M be a subspace of L.*

(i) *If $x, y \in M$ then either $x + M = y + M$ or $(x + M) \cap (y + M) = \emptyset$.*

(ii) $x + M = M$ *iff* $x \in M$

(iii) $x + M = y + M$ *iff* $x - y \in M$

(iv) *If* $x \in L$ *and* $m \in M$ *then* $x + M = x + m + M$.

(v) *The set of distinct cosets of M is a partition of L.*

Remark 4 *The elements of L are called vectors*

Remark 5 *The new operation is called scalar multiplication*

5 Quotient space

Definition 5 *If X is a vector space and S a subspace, we may define the quotient space* X/S *of cosets.*

Remark 6 *Since two cosets of M are either identical or disjoint, the quotient space* L/M *is the set of all the distinct cosets of M.*

Definition 6 *Let M be a subspace of a vector space L. Given* $x, y \in L$, *define addition of cosets by*

$$(x + M) + (y + M) = (x + y) + M$$

. *Given* $x \in L$ *and* $c \in \mathbb{F}$, *define scalar multiplication by*

$$c(x + M) = cx + M$$

.

6 Normed linear space

Definition 7 *Given a linear space L over* $\mathbb{R}$, *a mapping* $\|\cdot\| : L \to \mathbb{R}$ *is a nann for L if it satisfies the following properties: For all* $x \in L$,

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0$ *if and only if* $x = 0$

(iii) $||\alpha x|| = |\alpha| ||x||$ *for all scalar* α

(iv) $||x + y|| \leq ||x|| + ||y||$

The pair $(L, || \cdot ||)$ *is called a normed linear space.*

Remark 7 *The norm generates a special metric on the linear space. Given a normed linear space* $(L, || \cdot ||)$*, a function* $d : N \times N \to \mathbb{R}$ *defined by*

$$d(x, y) = ||x - y||$$

is a metric for N and is called the metric generated by the norm.

Remark 8 *Properties (iii) and (iv) tell us that the norm is a convex function on L; that is, for any* $x, y \in L$ *we have*

$$\begin{aligned} ||\lambda x + (1 - \lambda)y|| &\leq ||\lambda x|| + ||(1 - \lambda)y|| \\ &= |\lambda| \, ||x|| + |(1 - \lambda)| \, ||y|| \\ &= \lambda \, ||x|| + (1 - \lambda) \, ||y|| \end{aligned}$$

for all $0 \leq \lambda \leq 1$.

Problem 1

$$| \, ||x|| - ||y|| \, | \leq ||x - y||$$

Solution Now

$$\begin{aligned} ||x|| &= ||x - y + y|| \\ &\leq ||x - y|| + ||y|| \\ ||x|| - ||y|| &\leq ||x - y|| \qquad (1) \end{aligned}$$

Hence

$$\begin{aligned} -(||x|| - ||y||) &= ||y|| - ||x|| \\ &\leq ||y - x|| \quad by (1.1) \\ &= ||x - y|| \\ ||x|| - ||y|| &\geq -||x - y|| \end{aligned}$$

Therefore

$$-||x-y|| \leq ||x|| - ||y|| \leq ||x-y||$$

i.e

$$|\ ||x|| - ||y||\ | \leq ||x-y||$$

Problem 2 *The norm is a continuous function*

Solution Suppose $x_n \to x$.
Now

$$\begin{aligned} |\ ||x_n|| - ||x||\ | &\leq ||x_n - x|| \\ &\to 0 \end{aligned}$$

Hence

$$||x_n|| \to ||x||$$

7 Banach space

Definition 8 *A normed linear space which is complete as a metric space with its metric generated by the norm, is called a Banach space.*

Theorem 3 *In a normed linear space, we have*

$$|\ ||x|| - ||y||\ | \leq ||x-y||$$

Proof.

$$\begin{aligned} ||x|| &= ||x-y+y|| \\ &\leq ||x-y|| + ||y|| \\ ||x|| - ||y|| &\leq ||x-y|| \end{aligned}$$

Hence

$$\begin{aligned} -(||x|| - ||y||) &= ||y|| - ||x|| \\ &\leq ||y-x|| \quad by(1.1) \\ &= ||x-y|| \\ ||x|| - ||y|| &\geq -||x-y|| \end{aligned}$$

Therefore

$$-||x-y|| \leq ||x|| - ||y|| \leq ||x-y||$$

i.e

$$|\ ||x|| - ||y||\ | \leq ||x-y||$$

Theorem 4 *The norm is a continuous function*

proof. Suppose $x_n \to x$.
Now

$$\begin{aligned} |\ ||x_n|| - ||x||\ | &\leq ||x_n - x|| \\ &\to 0 \end{aligned}$$

Hence

$$||x_n|| \to ||x||$$

Norm is a contiuous function.

Example 5 *The set of real number and complex are the simplest of all normed linear spaces. The norm of a number x is defined by*

$$||x|| = |x|$$

. Since $\mathbb{R}$*and*$\mathbb{C}$ *are complete,* $\mathbb{R}$*and*$\mathbb{C}$ *are banach spaces.*

Example 6 *The linear spaces* $\mathbb{R}$ *and* $\mathbb{C}$ *of all n-tuples* $x = (x_1, x_2, ...x_n)$ *of real and complex number are normed linear space with norm*

$$||x|| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$$

. Since $\mathbb{R}$ *and* $\mathbb{C}$ *are complete,*$\mathbb{R}$ *and* $\mathbb{C}$ *are Banach spaces.*

Possible Questions
8 marks

1. Prove that if N is a normed linear space and x_0 is non-zero vector in N then there exist functional $f_0 \in N*$ such that $f(x_0) = ||x_0||$ and $||f_0|| = 1$

2. Let M be a closed linear subspace of a normed linear space N and let x_0 be a point not in M. If d is the distance from x_0 to M. Show that there exist a functional $f \in N*$ such that $f(M) = 0$, $f(x_0) = 1$ and $||f_0|| = \frac{1}{d}$

3. State and prove Hahn Banach theorem.

4. Prove that let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by $||x + M|| = \inf\{||x + M|| : m \in M\}$ then N/M is a normed linear space. Also, if N is a Banach space then N/M is also a Banach space

5. Prove that if M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exist a functional f_0 in $N*$ such that $f_0(M) = 0$, $f_0(x) \neq 0$.

6. If N is a normed linear space then each vector x induces a functional F_x on $N*$ by $F_x(f) = f(x)$ for every $f \in N*$ such that $||F_x|| = ||x||$. Moreover there is an isometric isomorphism from N into $N**$

7. Prove that if N is a normed linear space and x_0 is non-zero vector in N then there exist functional f_0 in $N*$ such that $f_0(x_0) = ||x0||$ and $||f_0|| = 1$.

8. Prove that let N & N' be a normed linear space the set $\mathcal{B}(N, N')$ of all continuous linear transformation of N into N" is a normed linear space with respect to the pointwise linear operations

i $(T + U)(x) = T(x) + U(x)$

ii $(\alpha T)(x) = \alpha T(x)$

and the norm defined by $\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}$.Also if N' is a Banach space then $\mathcal{B}(N, N')$ is also a Banach space.

9. Prove that let M be a linear subspace of a normed linear space N. Let f be a functional defined on M of x_0 is a vector not in M and if $M_0 = M + x_0$. Then f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$

10. Let N and $N^{'}$ be normed linear spaces and T be a linear transformation of N into $N^{'}$. Then the following condition on T are all equivalent to one snother.

- T is continous.
- T is continnuous at the origin.
- There exist a real number $K \geq 0$ with the probability that

$$\|T(x)\| \leq K\|(x)\|$$

  for every x in N.
- If $S = x : \|x\| \leq 1$ isa closed unit sphere then its image $T(s)$ is a bounded set in $N^{'}$.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Functional analysis **Subject Code: 16MMP301**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
The norm of x is called as the ……….. of the vector	direction	length	weight	scalar	Length
Every …………. normed linear space is a banach space	complete	metric	compact	connected	complete
A banach space is a normed linear space which is complete as a ………..	complete	connected	compact	metric	metric
The metric space arise on norm as d(x,y)=……………	$\|x+y\|$	$\|x\|$	$\|x-y\|$	$\|xy\|$	$\|x-y\|$
The linear operation is denoted by……….	R	N	L	K	L
The two primary operation in alinear space is called ……………	Linear operation	Arithmetic operation	Operators	Operations	Linear operation
The size of an element x is a real number denoted by ……………	norm x	real x	banach x	complex x	norm x
A linear space is called real linear space when its scalar is ……..	norm	real	banach	complex	real
A linear space is called ……... linear space when its scalar is complex	norm	real	banach	complex	complex
………………… is called the distance between x and y	c(x,y)	r(x,y)	d(x,y)	p(x,y)	d(x,y)
Every cauchy sequence has a convergent………….	sequence	subsequence	series	serial	subsequence
The real part of Z is denoted by ………….	Re(Z)	Re(x+y)	Im(Z)	Im(x+y)	Re(Z)
The imaginary part of Z is denoted by ………….	Re(Z)	Re(x+y)	Im(Z)	Im(x+y)	Im(Z)
The ……………. of A is the lub of the distance between pair of its points.	direction	distance	weight	scalar	distance
If f is a……………. function if there is areal number K such that $\|f(x)\| \le k$.	norm	finite	bounded	unbounded	bounded
A…………..set is one whose diameter is finite.	complete	connected	metric	bounded	bounded
Every sequentially compact metric space is…………….	complete	connected	compact	metric	compact
Every sequentially ………………. metric space is totally bounded.	complete	connected	compact	metric	compact
A mapping of a nonempty set b in to a metric space is called a……….. mapping	norm	finite	bounded	unbounded	bounded
A metric space is compact iff it is ……………. and totally bounded.	complete	connected	metric	bounded	complete
A closed subspace of a complete metric space is ……… iff it is totally bounded	complete	connected	compact	metric	compact
A metric space is said to be sequentially compact if every sequence in it has a convergent …………………	sequence	subsequence	space	subspace	subsequence
The ……… is called second conjugate space of N.	N**	N	N''	N*	N**
A complete metric space is ametric space in which every cauchy sequence is………….	complete	connected	compact	convergent	convergent
If N is a banach space then the product N/M is a…………………..	Banach space	hilbert space	Inner product space	linear space	Banach space
The elements of N* are called continuous linear functional or………………	continuous	functional	linear space	convergent	functional
The identity transformation I is an ……for the algebra B(N)	continuous	functional	linear space	identity	identity
N is said to be isometrically isomorphic to N' if ther exist an ……………of N into N'	isomorphic	isometric	isometric isomorphism	isomorphism	isometric isomorphism
If T is continuous at the origin , then $Xn \rightarrow 0$ implies ……………..	$Xn \rightarrow 0$	$T(Xn) \rightarrow 0$	$Xn \rightarrow 1$	$T(Xn) \rightarrow \infty$	$T(Xn) \rightarrow 0$
The set of all …………. for T equals the set of all radii of closed sphere centered on the origin which contain T(S)	bounds	convex set	continuous	functional	bounds
Any infinite set which is numerically equivalent to N is said to be ……………	Countable	uncountable	uncountably infinite	countably finite	countably finite
A set is …………… if it is nonempty and finite	Countable	uncountable	uncountably infinite	countably finite	countable
Any countable union of countable set is……….	countably finite	not countable	Countable	uncountable	Countable
Uncountable is otherwise called as…………………	countably finite	not countable	Countable	uncountably infinite	uncountably infinite
The absolute value is the …………… between the numbers.	direction	distance	weight	scalar	distance
The triangle inequality for metric space is ………………	d(x,y) <d(x,z)+d(y,z)	d(x,y) <d(x,z)+d(y,s)	d(x,y) <d(x,y)+d(y,z)	d(x,y) >d(x,z)+d(y,z)	d(x,y) <d(x,z)+d(y,z)
The elements of x are called the points of………space (x,d)	Banach space	hilbert space	Metric space	linear space	Metric space
Let x be a metric space then it is ……. property is d(x,y)=d(y,x)	asymmetry	symmetry	abelian	commutate	symmetry
Let x be a metric space then it is symmetry Property is d(x,y)= …….	d(x,y) <d(x,z)+d(y,z)	d(x,y) <d(x,z)+d(y,s)	d(x,y)	d(y,x)	d(y,x)
f is said to be continuous if it is …………… at each point of x.	continuous	functional	linear space	convergent	continuous

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DEPARTMENT OF MATHEMATICS

Subject: Functional Analysis	**Semester: III**	**L T P C**
Subject Code: 16MMP301	**Class: II-M.Sc. Mathematics**	**4 0 0 4**

UNIT II

The Open Mapping Theorem- Theorem and Examples –Problems. The closed graph theorem. The conjugate of an operation- The uniform boundedness theorem- Problems.

TEXT BOOK

1. Balmohan V., and Limaye.**,** 2004. Functional Analysis, New Age International Pvt. Ltd, Chennai.

REFERENCES

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4. Ponnusamy.S., 2002. Foundations of functional analysis, Narosa Publishing House, Chennai.

Definition 1 *A normed linear space which is complete as a metric space with its metric generated by the norm, is called a Banach space.*

Theorem 1 *In a normed linear space, we have*

$$| \, ||x|| - ||y|| \, | \leq ||x - y||$$

Proof.

$$\begin{aligned} ||x|| &= ||x - y + y|| \\ &\leq ||x - y|| + ||y|| \\ ||x|| - ||y|| &\leq ||x - y|| \end{aligned}$$

Hence

$$\begin{aligned} -(||x|| - ||y||) &= ||y|| - ||x|| \\ &\leq ||y - x|| \quad by(1.1) \\ &= ||x - y|| \\ ||x|| - ||y|| &\geq -||x - y|| \end{aligned}$$

Therefore

$$-||x - y|| \leq ||x|| - ||y|| \leq ||x - y||$$

i.e

$$| \, ||x|| - ||y|| \, | \leq ||x - y||$$

Theorem 2 *The norm is a continuous function*

proof. Suppose $x_n \to x$.
Now

$$\begin{aligned} | \, ||x_n|| - ||x|| \, | &\leq ||x_n - x|| \\ &\to 0 \end{aligned}$$

Hence

$$||x_n|| \to ||x||$$

Norm is a contiuous function.

Example 1 *The set of real number and complex are the simplest of all normed linear spaces. The norm of a number x is defined by*

$$||x|| = |x|$$

. Since $\mathbb{R} and \mathbb{C}$ are complete, $\mathbb{R} and \mathbb{C}$ are banach spaces.

Example 2 *The linear spaces $\mathbb{R}$ and $\mathbb{C}$ of all n-tuples $x = (x_1, x_2, ...x_n)$ of real and complex number are normed linear space with norm*

$$||x|| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$$

. Since $\mathbb{R}$ and $\mathbb{C}$ are complete,$\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

Definition 2 *subspace A non empty subset M of L is called a subspace (or a linear subspace)of L, if M is a linear space with respect to the linear operations defined in L.*

Remark 1 *If the subspace M is a proper of L, then it is called a proper subspace of L.*

Remark 2 *The zero spaces* 0 *and the full spaces L itself are always subspaces of L.*

Theorem 3 *Let M be a closed linear subspace ofa normed linear space N, if the norm of a coset $x + M$ in the quotient space N/M is defined by*

$$||x + M|| = inf(||x + M|| : m \in M)$$

then N/M is a normed linear space further,if N is Banach space then so N/M .

proof. Since,each $||x + M|| \geq 0$, we use

$$inf(||x + M|| : m \in M) \geq 0$$

Given

$$
\begin{aligned}
||x + M|| \geq o \\
\text{Suppose}||x + M|| &= 0 \\
inf(||x + M|| : m \in M) &= 0
\end{aligned}
$$

There is a sequence $[m_k]$ in M.
such that $||x + M_k|| \to 0$
$\Leftrightarrow x \in M$ since M is closed.
$\Leftrightarrow x + M = M$ the zero element of N/M.
now,

$$
\begin{aligned}
||(x = M) + (y + M)|| &= ||(x + y) + M|| \\
&= inf[||(x + y) + m|| : m \in M] \\
&= inf[||x + y + m + m^{'}|| : m, m^{'} \in M] \\
&= inf[||(x + m)|| + ||(y + m^{'})|| : m, m^{'} \in M] \\
&= inf[||(x + m)|| : m \in M] + inf[(y + m^{'})|| : m^{'} \in M] \\
&= ||x + M|| + ||y + M||. \\
||\alpha(x + m)|| &= inf[||\alpha(x + m)|| : m \in M] \\
&= inf[|\alpha|.||x + m|| : m \in M] \\
&= |\alpha|.inf[||x + m|| : m \in M] \\
&= |\alpha|.||x + m||
\end{aligned}
$$

The N/M is a normed linear space.
Suppose N is complete
To prove: N/M is complete
Let us consider a cauchy sequence in N/M it is enough to show that cauchy seequence has a convergent subsequence
Let $y_1 \in x_1 + M$ and $y_2 \in x_2 + M$.
Such that

$$||y_1 - y_2|| < \frac{1}{2}$$

.

Next select $y_3 \in x_3 + M$.

Such that

$$\|y_2 - y_3\| < \frac{1}{4}$$

.

Continuing in this way, we obtain a sequence y_n in N.

Such that

$$\|y_n - y_{n+1}\| < \frac{1}{2^n}$$

.

Suppose$m < n$

$$\begin{aligned}
\|y_m - y_n\| &= \|y_m - y_{m+1} + y_{m+1} - y_{m+2} + ... + y_{n-1} - y_n\| \\
&\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + ... + \|y_{n-1} - y_n\| \\
&< \frac{1}{2^m} + \frac{1}{2^{m+1}} + ... + \frac{1}{2^{n_1}} \\
&= \frac{1}{2^m}\left[1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1-m}}\right] \\
&= \frac{1}{2^m}\left[\frac{1 - \frac{1}{2^{n-m}}}{1 - \frac{1}{2}}\right] \\
&= \frac{1}{2^m}\left[1 - \frac{1}{2^{n-m}}\right] \\
&= \frac{1}{2^{m-1}}\left[1 - \frac{1}{2^{n-m}}\right] \\
&< \frac{1}{2^{m-1}}.
\end{aligned}$$

y_n is a cauchy sequence

Since N is acomplete, we have $\{y_n\}$ is convergent.

Let y be the limit of sequence y_n.

$\|y_n - y\| \to 0$.

$\|(x_n + M) - (y + M)\| \to 0$.

$\{x_n + M\}$ is a convergence.

Hence N/M is complete.
since N/M is a normed linear space, N/M is a Banach Space.

Example 3 *Let Pbe real number such that $1 \leq P < \infty$. We denote by $(l)_p^n$ the space of all n-tuples $x = (x_1, x_2.....x_n)$ of scalars with the norm defined by*

$$\|x\|_p = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}}$$

Problem 1 *The l_p^n space is the normed linear space with norm defined by*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

solution: Since each $|x_i| \geq 0$,we have $\|x\|_p \geq 0$.
Suppose $\|x\|_p = 0$ then

$$\begin{aligned}
\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}} &= 0. \\
\Rightarrow \sum_{i=1}^{n} \|x_i\|^p &= 0 for all i = 1, 2, ...n \\
\Rightarrow |x_i|^p &= 0 for all i = 1, 2, ...n \\
\Rightarrow |x_i| &= 0 for all i = 1, 2, ...n \\
\Rightarrow x_i &= 0 for all i = 1, 2, ...n \\
\Rightarrow x_i &= (x_1, x_2, ...x_n) = (0, 0, ..., 0) \\
\Rightarrow x = 0.
\end{aligned}$$

Suppose$\|x + y\|_p = 0$. then the triangle inequality is trivial.
Suppose $\|x + y\|_p \neq o$.

$$
\begin{aligned}
||x+y||_p &= \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \\
\left(||x+Y||_p\right)^p &= \sum_{i=1}^{n} |x_i + y_i|^p \\
&= \sum_{i=1}^{n} |x_i + y_i|.|x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} [|x_i + y_i|].[|x_i + y_i|^{p-1}] \\
&= \left(\sum_{i=1}^{n} |x_i|[|x_i + y_i|]^{p-1}\right) + \left(\sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}\right)
\end{aligned}
$$

Now consider the first sum in the R.H.S

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i|[|x_i + y_i|]^{p-1} &\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} . \left(\sum_{i=1}^{n} |x_i + y_i|^{p-1}\right)^{q} . \\
\text{where} \frac{1}{p} + \frac{1}{q} = 1 & \\
&= ||x||_p \left(\sum_{i=1}^{n} |x_i + y_i|^{p-1} q\right)^{1-\frac{1}{p}} . \\
&= ||x||_p \left[\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}}\right]^{p-1} . \\
&= ||x||_p \left(||x + y||_p\right)^{p-1} \\
\text{Hence, } ||x + y||^p &\leq ||x||_p \left(||x + y||_p\right)^{p-1} + ||y||_p \left(||x + y||_p\right)^{p-1} \\
&= \left(||x + y||_p\right)^{p-1} . \left[||x||_p + ||y||_p\right] \\
||x + y||_p &\leq ||x||_p + ||y||_p \\
||\alpha x||_p &= \left(\sum_{i=1}^{\infty} |\alpha x_i|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p\right)^{\frac{1}{p}} \\
&= |\alpha| \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \\
&= |\alpha| ||x||_p \\
(l_p^n, ||.||_p) &
\end{aligned}
$$

Hence the proof.

Problem 2 *The l_p^n space is a banach space with norm defined by*

$$
||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} .
$$

solution: By previous problem, l_p^n is a normed linear space. It is enough to show that l_p^n space is complete.
Let f_m be a cauchy sequence and $f_m(i) = i$ cordinate of x_m.
For given $\Sigma \geq 0$ there exist a positive integer N.
Such that $\|f_m - f_m^{'}\|_p \leq \Sigma$ for all $m, m^{'} \geq N$

$$\begin{aligned}
\Rightarrow \left(\sum_{i=1}^{n} |f_m - f_m^{'}|^p\right)^{\frac{1}{p}} &\leq \Sigma^p \\
\Rightarrow \left(\sum_{i=1}^{n} |f_m(i) - f_m(i)^{'}|^p\right) &< \Sigma^p \\
\text{J}|f_m(i) - f_m^{'}(i)|^p &< \Sigma^p foralli = 1, 2, ..n \\
\text{J}|f_m(i) - f_m^{'}(i)| &< \Sigma \\
\text{J}|x_{mi} - f_{mi}^{'}| &< \Sigma
\end{aligned}$$

$$x_{mi}$$

is a cauchys sequence of real number
Since $\mathbb{R}$ is a complete,x_{mi} is convergence to x_i
$x_{mi} \rightarrow x_i$ for all i.
Let $f = (x_1, x_2..., x_n)$ then $f \in l_p^n$ and $f_m \rightarrow f$.
l_p^n space is complete.
l_p^n space is a Banach space.

Definition 3 *Consider a real number p with* $1 \leq p < \infty$ *and we denote* l_p *be space of all sequence* $x = (x_1, x_2 ... x_n)$ *of scalars. Such that*

$$\sum_{i=1}^{\infty} |x_i|^p \leq \infty$$

with the norm defined by

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \quad (1)$$

Problem 3 *The l_p is a Banach space.*

solution. To prove s is a convex.
Let $x, y \in s$ be arbitrary.
Let α, β be non-negative real numbers with $\alpha + \beta = 1$.
$z = (\alpha x + \beta y)$
where $z \in s$.
Now,

$$\begin{aligned}
||z|| &= ||\alpha x + \beta y|| \\
&= |\alpha|||x|| + |\beta|||y|| \\
&= \alpha||x|| + \beta||y|| \\
&\leq \alpha.1 + \beta.1 \\
||z|| &\leq \alpha + \beta \\
||z|| &\leq 1.
\end{aligned}$$

s is a convex

Remark 3 *Consider the real linear space $\mathbb{R}$ of all ordered pairs $x = (a, b)$ of real numbers.*
There are many different norm can be defined on $\mathbb{R}^2$.

$$\begin{aligned}
||x||_1 &= |a| + |b| \\
||x||_2 &= |a|^2 + |b|^2 \\
||x||_\infty &= max(|a| + |b|)
\end{aligned}$$

Remark 4 *Considering the norm defined by*

$$||x||_p = (|a|^p + |b|^p)^{\frac{1}{p}}$$

where $1 \leq p\infty$.
If $p < 1$, then $s = [x : ||x||_p \leq 1]$ need not be a convex set.
$||x||_p$ need not be a norm.

Possible Questions
8 marks

1. Prove that a nonempty subset X of a normed linear space N is bounded iff $f(X)$ is a bounded set of numbers for each f in $N*$
2. Show that E is a projection on M along N iff $I - E$ is a projection on N along M
3. State and prove closed graph theorem
4. Let T be an operator, a normed linear space N, $T*$ be its conjugate defined by $T*(f) = f_0 T$ (or) $(T^* f)x = f(T(x))$ for every $f \in N^*$ and $x \in N$ then prove that T^* is an operator on N^* and the mapping $\phi : B(N) \to B(N*)$ such that $\phi(T) = T^*$ for every $T \in B(N)$ is an isometric isomorphism of $B(N)$ into $B(N^*)$ and also preserves the identity transformation.
5. Prove that let B be a Banach space and let M and N be a closed linear subspace of B such that $B = M \oplus N$ .If $Z = x + y$ is the unique representation of a vector in B as a sum of vectors in M and N , then the mapping P defined by $P(Z) = x$ is a projection on B whose range and null spaces are M & N.
6. State and prove open mapping theorem.
7. State and prove uniform boundedness theorem.
8. Prove that if B and B' are Banach space and if T is a continuous linear transformation of B on to B' then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'
9. Prove that if P is a projection on a Banach space B and if M and N are its range and null space then M and N are closed linear subspace of B such that $B = M \oplus N$

10. Prove that T is a closed linear transformation iff its graph T_G is closed subspace.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Functional analysis **Subject Code: 16MMP301**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
The centre of some open sphere contained in A is called the ………..	closed	open	interior	exterior	interior
Each operator T on a normed linear space N induces a corresponding operator denoted by …….	T'	T**	T*	T	T*
The M is the null space for the projection on ……………..	P	I-P	Space	subspace	I-P
If P is a projection on a Banach space B and if M and N are its ………	dense sets	range and null space	subspaces	projection	range and null space
A projection on E determines a pair of linear subspace M and N then ……..	L= m+N	L=M+N	L=M-N	L= M⊕ N	L= M⊕ N
The image of open sphere centered on the origin in B contains an …………. Sphere centered on the origin in B and B'	closed	open	interior	exterior	open
The …... is the null space of the operator on the projection on I-P	M	N	L=M-N	L= m+N	M
The …... is the null space of the operator on the projection on P	M	N	L=M-N	L= m+N	N
The …... is the range of the operator on the projection on I-P	M	N	L=M-N	L= m+N	N
The …... is the range of the operator on the projection on P	M	N	L=M-N	L= m+N	M
A pair of linear subspace M and N such L= M⊕N determines a………. on E.	dense sets	range and null space	subspaces	projection	projection
If T is continuous , then its graph is ………. as a subset of BxB'	closed	open	interior	exterior	closed
A closed set in a topological space in a set whose compliment is………..	closed	open	interior	exterior	open
A is ………… iff A = Int(A)	closed	open	interior	exterior	open
Int(A) equals the union of all …………….. of A.	closed	open	open subset	open set	open subset
The interior of A is denoted by …………………	Int(A)	Cl(A)	Ext(A)	Im(A)	Int(A)
Int(A) is an open subset of A which contains every ……………. of A	closed	open	open subset	open set	open subset
Let x be any metric space then any union of open set in x is ……..	closed	open	open subset	open set	open
Let x be any metric space then any finite intersection of …….in x is open.	closed	open	open subset	open set	open set
In any metric space x, each open sphere is an ………………..	closed	open	open subset	open set	open set
The open sphere $Sr(x_0)$ with center x_0 and radius r is the subset of x define by ……………..	d(x,y)	d(y,x)	$d(x,x_0) < r$	$d(x,x_0) = r$	$d(x,x_0) < r$
An open sphere is always non empty for it contain its………….	center	radius	distance	length	center
An ………… sphere with radius 1 contain only its center.	closed	open	open subset	open set	open
If the open sphere is bounded open interval $(x_0 - r, x_0 + r)$ with midpoint x_0 and total length…...	r	2r	3r	0	2r
$Sr(x_0)$ is an open sphere with radius …...centered on x_0	r	2r	3r	0	r
In the linear space the………. transformation I defined by I(x)=x	identity	linear	one to one	onto	identity
The mapping P(Z) = x is a ……………… on B.	dense sets	range and null space	subspaces	projection	projection
B and B' have same topology means they are …………….	homomorphic	homeomorphic	linear	connected	homeomorphic
B and B' have same …….. means they are homeomorphic	strong topology	nullspace	topology	weak topology	topology
The identity mapping of B' to B is ………. for $\Vert T(x)\Vert = \Vert x\Vert$.	continuous	functional	linear space	convergent	continuous
If T is continuous linear transformation of B onto B' then T is an ……. mapping.	closed	open	open subset	open set	open
A 1-1 linear transformation T of abanach space onto itself is continuous then its inverse is automatically …………	continuous	functional	linear space	convergent	continuous
The mapping T →T* is thus anorm preserving map onf B(N) into ……….	B(N)*	B(N')	B(N)	B(N)**	B(N')

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DEPARTMENT OF MATHEMATICS

Subject: Functional Analysis	**Semester: III**	**L T P C**
Subject Code: 16MMP301	**Class: II-M.Sc. Mathematics**	**4 0 0 4**

UNIT III

Hilbert Spaces- The Definition and Some Simple Properties – Examples and Problems. Orthogonal Complements - Some theorems .Ortho-normal sets – Definitions and Examples-Bessel's inequality- The conjugate space H*

TEXT BOOK

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2. Chandrasekhara Rao.K., 2006. Functional Analysis, Narosa Publishing House, Chennai.

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1 Hilbert spaces

Definition 1 *Let N and $N^{'}$ be a normed linear spaces withthe same scalars and T be a linear transformation of N and $N^{'}$.We say that T is continuous if $x_n \to x \in N$.*

$$T(x_n) \to T(x)$$

in $N^{'}$.

Theorem 1 *Let N and $N^{'}$ be normed linear spaces and T be a linear transformation of N into $N^{'}$. Then the following condition on T are all equivalent to one snother.*

- *T is continous.*
- *T is continnuous at the origin.*
- *There exist a real number $K \geq 0$ with the probability that*

  $$\|T(x)\| \leq K\|(x)\|$$

  for every x in N.
- *If $S = x : \|x\| \leq 1$ isa closed unit sphere then its image $T(s)$ is a bounded set in $N^{'}$.*

proof. 1=2
Assume that T is continuous.
Suppose $x_n \to 0$.
Then,

$$\begin{aligned} T(x_n) \to T(0) &= 0 \\ T(x_n) \to &= 0 \end{aligned}$$

T is continuous at 0.
Conversely, assume that T is continuous at 0.
Suppose,

$$\begin{aligned} x_n &\to x \\ x_n - x &\to 0 \\ T(x_n - x) &\to 0 \\ T(x_n) - T(x) &\to 0 \\ T(x_n) &\to T(x) \end{aligned}$$

Hence, T is continuous.
Next,2=3
Suppose 3 is true.
Since ||.|| is continuous,

$$||x_n|| \to ||0||$$

$$||x_n|| \to 0$$

Now,

$$||T(x_n)|| \leq K(x_n)$$

for all x_n in N.
T is continuous at 0.
Conversely,assume that T is continuous at origin.
To prove a real number $K \geq 0$.

$$||T(x)|| \quad leq \quad K||x|| forall x \in N. \tag{1}$$

Suppose there is a no such K.
For each positive integer n, wwe can find a vector

$$x_n \ni ||T(x_n)|| \geq n||x_n||$$

.

$$\begin{aligned} y_n &= \frac{x_n}{n||x_n||} \in N \\ \text{now, } ||T(y_n)|| &= ||T(\frac{x_n}{n||x_n||})|| \\ &= |\frac{1}{n||x_n||}|.||T(x_n)|| \\ &= frac1n||x_n||.||T(x_n)|| \\ &\geq 1 \end{aligned}$$

Clearly,$y_n \to 0$, but $T(x_n \to)0$

$$||T(x)|| \leq K||x||forallxinN.$$

For 3=4,
suppose 3 is true
let

$$\begin{aligned} S &= [x : ||x|| \leq 1] \\ T(S) &= [y : y = T(x)forsomexinS] \\ ||y|| &= ||T(x)|| \\ &\leq k.||x|| \\ &\leq k \end{aligned}$$

T(S) is bounded
conversely, Assume that 4 is true.
Suppose $x = 0$ then

$$\begin{aligned} T(x) &= T(0) \\ &= 0 \\ ||T(x)|| &= 0 \end{aligned}$$

3 is trivially true.
suppose $x \neq 0$ then $||x|| > 0$ then

$$y = \frac{x}{||x||} \in s$$

$$\begin{aligned} ||y|| &= ||\frac{x}{||x||}|| \\ &= \frac{1}{||x||}.||x|| \\ &= 1 \end{aligned}$$

$$y \in S$$

then $T(y) \in T(S)$.
since T(S) is a bounded set, we have

$$\begin{aligned} ||T(y)|| &\leq k \\ ||T(\frac{x}{||x||}) &\leq k \\ \frac{1}{||x||}.||T(x)|| &\leq k \\ ||T(x)|| &\leq k.||x|| \end{aligned}$$

Definition 2 *suppose T is a linear transformation and there exist a real number* $k \geq 0$ *with the property that* $||T(x)|| \leq k||x||$ *for every x , then k is called a bound for T and such a linear transformation T is called bounded linear transformation.*

Remark 1 *By previous theorem,T is bounded linear transformation iff T is continous*

Definition 3 *suppose T is continuous, we define its norm by* $||T|| = sup[||T(x)|| : ||X|| \leq 1]$

Remark 2 *suppose* $\|x\| = 1$.*Then we have another espression for the norm of T.*

$$\begin{aligned} \|T\| &= inf[k : k \geq 0] \\ \|T(x)\| &\leq k.\|x\| forallx \end{aligned}$$

Theorem 2 *If N and* N' *are normed linear spaces,then the set* $\mathbb{B}(N, N')$ *of all continuous linear transformation of N into* N' *is itself normed linear space with respect to the pointwise linear operations and the norm defined by*

$$\|T\| : sup(\|T(x)\| : \|x\| \leq 1)$$

.Further if N' *is a banach space then* $\mathbb{B}(N, N')$ *is also a Banach space.*

Definition 4 *Let N be a normed linear space .A continuous linear transformation of N and* N' *itself is called an 'Operator on N' and we denote the normed linear space of all operators on N by* $\mathbb{B}(N)$.

Remark 3 *Suppose N is a banach space, then by previous therom* $\mathbb{B}$ *is a Banach space.*

Remark 4 *If T and U are any two linear transformation on their product*$(TU)(x) = T(U(x))$

Remark 5 $\mathbb{B}(N)$ *is an algebra in accordance with the mutliplication defined by*

$$(TU(x) = T(U(x)$$

. for all $T, U \in \mathbb{B}$

Remark 6 *Suppose* $T_n \to T$ *and* $T_n \to T'$. *Then* $T_n T_n' \to TT'$.

$$\begin{aligned} \|T_n T_n' - TT'\| &= \|T_n(T_n' - T') + (T_n - T)T'\| \\ &\leq \|T_n(T_n' - T') + (T_n - T)T'\| \\ &\leq \|T_n\|\|(T_n' - T') + (T_n - T)\|\|T'\| \\ &\to 0 \\ T_n T_n' &\to TT' \end{aligned}$$

Remark 7 *Suppose $N \neq 0$.Then the identity transformation T is an identity for the algebra $\mathbb{B}(N)$.*

$$\begin{aligned} \|I\| &= 1, for \\ \|I\| &= sup[\|I(x)\| : \|x\| \leq 1] \\ &= sup[\|x\| : \|x\| \leq 1] \\ &= 1 \end{aligned}$$

Possible Questions
8 marks

1. Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.
2. State and prove Schwartz inequality.
3. Show that inner product function is jointly continuous.i.e., $x_n \to x$, $y_n \to y$ the $(x_n, y_n) \to (x, y)$.
4. Prove that if B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$
5. Prove that let M be a closed linear subspace of a Hilbert space H. Let x be a vector not in M and let d be the distance from x to M. Then there exist a unique vector $y_0 \in M$ such that $\|x - y_0\| = d$.
6. Prove that if M is a proper closed linear subspace of a Hilbert space then there exist non-zero vector $z_0 \in H$, $z_0 \perp M$
7. If the $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H then $\langle x \sum\langle x, e_i\rangle, e_i\rangle \perp e_j$ for every j.
8. State and prove Bessel's inequality .
9. Prove that if M is a linear subspace of a Hilbert space , Show that it is closed iff $M = M^{\perp\perp}$
10. Prove that if $\{e_i\}$ is an orthonormal set in a Hilbert space, H and if X is any vector in H then the set $S = \{e_i : \langle x, e_i\rangle \neq 0\}$ is either empty or countable.
11. Prove that if M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M + N$ is closed.
12. State and prove Bessel?s inequality for finite orthonormal set.

13. State and prove Riesz representation theorem.

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Functional analysis **Subject Code: 16MMP301**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
Every inner product space is a……………	normed linear space	hilbert space	banach space	continuous	normed linear space
The ……….. is orthogonal to any vector	product	scalar	zero vector	real value	zero vector
The relation of orthogonality in a Hilbert space is …………	asymmetry	symmetry	abelian	commutate	symmetry
The zero vector is the only vector which is ………… to itself.	asymmetry	symmetry	orthogonal	direction	orthogonal
A complex banach space is said to be a ………….. if there is an inner product which satisfies the three conditions.	Banach space	hilbert space	Inner product space	linear space	hilbert space
For the space l_2^n we use cauchy inequality to prove …………inequality	minkowski's	schwartz	triangle	cauchy triangle	schwartz
Two vectors x and y in a hilbert space H are said to be orthogonal if ……	(x,y)>1	(x,y)=0	(x,y)=1	(x,y)<1	(x,y)=0
If x is orthogonal to y then every scalar multiple is ………….. to y.	parallel	symmetry	orthogonal	perpendicular	perpendicular
The ………………. is orthogonal to every vector.	product	scalar	zero vector	real value	zero vector
The d is the distance from ……….. to c.	center	vertices	edges	origin	origin
If M is a closed linear subspace of ahilbert space H then H is the ………… of M and M perp	product	scalar	zero vector	Direct sum	Direct sum
If M is a proper closed linear subspace of H, there exist a $Z_0 \neq 0$ in H such that……….	M+Mperp	Mperp+z_0	$Z_0 \perp M$	$Z_0 \perp P$	$Z_0 \perp M$
If M and N are closed linear subspace of ahilbert space H such that $M \perp N$ then the linear subspace M+N is………	closed	open	open subset	open set	closed
The scalars in a Hilbert space are usually ……………. numbers.	Irrational	algebraic	complex	rational	complex
The distance property in inner product space is (ax+by, Z) =…………	a(x,z)+b(y,z)	a(x,x)+b(y,x)	a(x,z)-b(y,z)	a(x,z)+b(x,z)	a(x,z)+b(y,z)
The distance property in inner product space is (ax-by, Z) =…………	a(x,z)+b(y,z)	a(x,x)+b(y,x)	a(x,z)-b(y,z)	a(x,z)+b(x,z)	a(x,z)-b(y,z)
If $s_1 \subseteq s_2$ then $s_1^{\perp} \supseteq$ ……………	$s_1^{\perp}$	$s_2^{\perp}$	$s_1^{\perp} - s_2^{\perp}$	$s_1^{\perp} s_2^{\perp}$	$s_2^{\perp}$
An orthonormal set cannot has an …………	product	scalar	zero vector	real value	zero vector
The set S is finite or…………………..	Countable	uncountable	countably infinite	countably finite	countably infinite
The orthonormal set is either ………………. or countable.	Countable	uncountable	finite	empty	empty
The orthonormal set is either empty or ……………..	Countable	uncountable	finite	empty	Countable
If M is linear subspace of a hilbert space then it is closed iff M=………	$s^{\perp}$	$M^{\perp}$	$M^{\perp}M$	$M^{\perp\perp}$	$M^{\perp\perp}$
A nonempty set $\{e_i\}$ of a hilbert space H is said to be an orthonormal set if …………… for all i=j	$(e_i, e_j) > 0$	$(e_i, e_j) = 0$	$(e_i, e_j) = 1$	$(e_i, e_j) < 1$	$(e_i, e_j) = 1$
A nonempty set $\{e_i\}$ of a hilbert space H is said to be an orthonormal set if …………… for all $i \neq j$	$(e_i, e_j) > 0$	$(e_i, e_j) = 0$	$(e_i, e_j) = 1$	$(e_i, e_j) < 1$	$(e_i, e_j) = 0$
If H contains only the zero vector then it has no …………………	orthonormal set	orthonormal basis	Banach space	hilbert space	orthonormal set
If H contains a nonzero vector and if we normalised x then $\|e\|$=……	zero	four	five	one	one
If (x,y) are any two vectors in a Hilbert space then $\|(x,y)\|$ <=……….	$\|x\|\|y\|$	$\|x\|/\|y\|$	$\|x\|-\|y\|$	$\|x\|+\|y\|$	$\|x\|\|y\|$
If (x,y) are any two vectors in a Hilbert space then $\|x+y\|^2+\|x-y\|^2$ =……….	$2\|x\|^2/2\|y\|^2$	$2\|x\|^2-2\|y\|^2$	$2\|x\|^2+\|y\|^2$	$2\|x\|^2+2\|y\|^2$	$2\|x\|^2+2\|y\|^2$
The sum of Z and Z conjugate is equal to………	2 im Z	2 Re z	2 z	Re z	2 Re z
If (x,y) are any two vectors in a Hilbert space then $\|x+y\|^2-\|x-y\|^2$ =……….	$2\|x\|^2/2\|y\|^2$	2(x,y)+2(y,x)	$2\|x\|^2+\|y\|^2$	2(x,y)-2(y,x)	2(x,y)+2(y,x)
The product of α and conjugate of α is …………..	$\|\alpha\|$	$\|\alpha\|n$	$\|\alpha\|^2$	α	$\|\alpha\|^2$
If (x,y) are any two orthogonal vectors in a Hilbert space then $\|x+y\|^2$ =……….	$2\|x\|^2/2\|y\|^2$	$2\|x\|^2-2\|y\|^2$	$\|x\|^2+\|y\|^2$	$2\|x\|^2+2\|y\|^2$	$\|x\|^2+\|y\|^2$
If (x,y) are any two orthogonal vectors in a Hilbert space then $\|x-y\|^2$ =……….	$2\|x\|^2/2\|y\|^2$	$\|x\|^2-\|y\|^2$	$\|x\|^2+\|y\|^2$	$2\|x\|^2+2\|y\|^2$	$\|x\|^2+\|y\|^2$
Every inner product space is expressed as $a\|x\|^2$……………	(x,y)>1	(x,x)	(y,y)	(y,x)	(x,x)
A close convex subset of a hilbert space H contains a unique vector of smallest ………………	metric	space	subset	norm	norm
A close ………. subset of a hilbert space H contains a unique vector of smallest norm.	concave	convex	linear	metric	convex
Parseval's equation is otherwise called as parseval's …………	transform	fourier	identity	subscript	identity
Let x be anarbitrary vector in H the numbers (x,ei) are called the ,,,,,,,,, coefficient of x.	parseval	fourier	schwartz	bessels	fourier
The set of all continuous linear functional on H is denoted by ……………	H	H**	H*	T*	H*
The expression x= $\sum$(x,ei)ei is called the ………… expansion of x.	parseval	fourier	schwartz	bessels	fourier

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DEPARTMENT OF MATHEMATICS

Subject: Functional Analysis	**Semester: III**	**L T P C**
Subject Code: 16MMP301	**Class: II-M.Sc. Mathematics**	**4 0 0 4**

UNIT IV

The Adjoint of an operator – Definitions and Some Properties-Problems. Self- adjoint operators – and Unitary operators –Theorems and Problems. Some Theorems and Problems. Normal Projections - Theorems and Problems

TEXT BOOK

1. Balmohan V., and Limaye., 2004. Functional Analysis, New Age International Pvt. Ltd, Chennai.

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1 The Adjoint of an operator

Definition 1 *Let N and N' be normed linear spaces. An isometric isomorphism of N and N' is a one to one linear transformation T of N into N'. Such that $||T(x)|| = ||x||$ for every x in N and N is said to be isometrically isomorphic to N' if there exist an isometric isomorphism of N into N'.*

Definition 2 *The set $\mathbb{B}(N, R)$ or $\mathbb{B}(N, C)$ is denoted by N^* and is called the 'conjugate space of N' and the elements of N^* are called 'continuous linear functionals (or) functionals '.*

Definition 3 *Let N and N' be normed linear spaces and let $T : N \to N'$ be a linear transformation. The kernal of T is the*

$$ker(T) = [x : T(x) = 0]$$

Problem 1 *ker T is a linear manifold and that ker T is closed if T is continuous.*

solution A non-empty subset M of a linear space L is a linear suspace L (a linear manifold in L) iff

$$\begin{aligned} x.y \in M &\Rightarrow x + y \in M \\ x \in M and \alpha is scalar &\Rightarrow \alpha.x \in M \\ \alpha, \beta any scalars &\Rightarrow \alpha.x + \beta.y \in M \end{aligned}$$

Let $x, y \in ker\ T$
Let α, β be any scalars
Now,

$$\begin{aligned} T(\alpha x + \beta y) &= T(\alpha x) + T(\beta y) \\ &= \alpha T(x) + \beta T(y) \\ &= \alpha.0 + \beta.0 \\ &= 0 \end{aligned}$$

$\alpha x + \beta y \in ker\ T$.
Let x be a limit point of $ker\ T$.
Then there exist a sequence x_n. Such that $x_n \to x$.
Since T is continuous

$$T(x_n) \to T(x)$$

. Since x_n is a sequence in $ker\ T$, we have

$$\begin{aligned} T(x_n) &= 0 \\ T(x) &= 0 \\ x &\in ker\ T \end{aligned}$$

Hence $ker\ T$ is closed.

Definition 4 *Let S be any non-empty subset of L and let M be the set of all finite linear combination of element of S. Then M is called as the 'linear monifold spanned by S'.*

Remark 1 *M is the intersection of all linear manifold containing S.*

Remark 2 *M is the smallest linear manifold which contain S.*

Definition 5 *Suppose M is linear manifold in a linear space L. Two elements $x_1.x_2 \in L$ are called 'Equavalent Modulo M' if $x_1 - x_2 \in M$ and we write*

$$x_1 \equiv x_2(modM)$$

Remark 3 *L is divided into mutual disjoint equivalence classes. We denote the set of all such equivalence classes by L/M.*

Remark 4 *Let $[x]$ denote the equivalence class which contains the element x.Then equivalence class of*

$$\begin{aligned} [x] &= [y \in L : x \equiv y(modM)] \\ &= [y : x - y \in M] \\ &= [y : x - y \in m] forsomem \in M \\ &= [y : y = x + m] forsomem \in M. \end{aligned}$$

Theorem 1 *Let N be a non-wero normed linear space and let* $S = [x \in N : ||x|| = 1]$ *be a linear subspace of N. Then N is a Banachspace* $\Longleftrightarrow S$ *is complete.*

solution. Suppose N is a Banach space
Let x_n be a cauchy sequence in S. Then

$$||x_n|| = 1$$

for every n.
Since N is a Banach space and x_n is a cauchy sequence in N.

$$x_n \to x$$

Since $||.||$ is a continuous function.

$$\begin{aligned} ||x_n|| &\to ||x|| \\ ||x_n|| &= \lim_{n\to\infty} ||x|| \\ &= \lim_{n\to\infty} 1 \\ &= 1 \\ x &\in S. \end{aligned}$$

S is complete. Conversely, assume that S is complete.
Since N is a normal linear space, it is enough to show that N is complete.
Let y_n be a cauchys sequence in N.
For given $\Sigma > 0$ a positive integer M.
Such that

$$||y_m - y_n|| < \Sigma$$

for all $n, m \geq M$.

Let ,

$$\begin{aligned}
x_n &= \frac{y_n}{||y_n||} \in N \\
||x_n|| &= ||\frac{y_n}{||y_n||}|| \\
&= |\frac{1}{||y_n||}|.||y_n|| \\
&= 1 \\
x_n &\in S.
\end{aligned}$$

Hence x_n is a cauchyy sequece in S.

$$\begin{aligned}
||x - m - x_n|| &= ||\frac{y_m}{||y_m||} - \frac{y_n}{||y_n||}|| \\
&= ||\frac{y_m}{||y_m||} - \frac{y_n}{||y_m||} + \frac{y_n}{||y_m||} - \frac{y_n}{||y_n||}|| \\
&\leq ||\frac{y_m}{||y_m||} - \frac{y_n}{||y_m||}|| + ||\frac{y_n}{||y_m||} - \frac{y_n}{||y_n||}|| \\
&= \frac{1}{||y_m||}||y_m - y_n|| + |\frac{1}{||y_m||} - \frac{1}{y_n}|||y_n|| \\
&= ||\frac{y_m - y_n}{||y_m||}|| + ||\frac{y_n - y_m}{||y_m||}|| \\
&\leq 2||\frac{y_m - y_n}{||y_m||}|| \\
&< \frac{2}{||y_m||} if n, m \geq M.
\end{aligned}$$

x_n is a cauchy sequence in S.
Since S is complete,$x_n \to x$ and $x \in S$.

$$\begin{aligned}
\frac{y_n}{||y_n||} &\to x \\
y_n &\to ||y_n||.x \\
|||y_m|| - ||y_n||| &\leq ||y_m y_n|| \\
&< \Sigma if n, m \geq M.
\end{aligned}$$

$\|y_n\|$ is a cauchys sequence in $\mathbb{R}$
Since $\mathbb{R}$ is complete,

$$\begin{aligned} \|y_n\| &\rightarrow \alpha \in \mathbb{R} \\ y_n &\rightarrow \alpha x \in N \end{aligned}$$

N is a Banach space.
Hence proved.

Definition 6 *Let S be set then a partial order on S is a binary operation* $\leq$ *on S.*
That satisfies,

$$\begin{aligned} a &\leq a forall a \in S. \\ a &\leq band b \leq a \\ \Rightarrow a &= b. \\ a &\leq band b \leq c \\ \Rightarrow a &= c. \end{aligned}$$

The pair $(S, \leq)$ *is called partially ordered set (or) po-set.*

Definition 7 *Let* $(S, \leq)$ *be a partially ordered set. A subset T of S is called totally ordered set if* $a, b \in T \Rightarrow a \leq b$
A totally ordered subset is also called a chain.

Definition 8 *An element* $U \in S$ *is said to be an upper bound for a subset T of S if*

$$a \leq U$$

Definition 9 *A maximal element of S is an element* $m \in S$ *such that* $m \leq x \Rightarrow m = x$

Example 1 *The set of all real number* $\mathbb{R}$ *is a partially ordered set with usually ordered on* $\mathbb{R}$

Example 2 *The set of all integer $\mathbb{Z}$ is a totally ordered set (or) chain.*

Example 3 *Let A be an arbitrary set and let $\mathbb{P}(\mathbb{A})$ be the set of all subset of A. Then $\mathbb{P}(\mathbb{A}), \leq$ is a po-set (or) paetially ordered set.*

Definition 10 **Zorn's lemma :** *Let $(S, \leq)$ be a partially ordered set in which every chain has an upper bounded then $(S, \leq)$ contains maximal element.*

Theorem 2 **The Hahn Banach theorem** *Let M be a linear subspace of a normed linear space N and let f be a functional defined on M then f can be extended to a fractional f_0 defined on N such that,*

$$||f|| = ||f_0||.$$

proof. Let P be the set of all ordered pairs $(M - \lambda, f_\lambda)$. Such that

- M_λ is a linear subspace of N contains M.
- f_λ is a bounded linear functional on M_λ
- f_λ is the extension of f.
- $||f_\lambda|| = ||f||$

define a relaion on P as follows:

$$(M_\lambda, f_\lambda) \leq (M_\mu, f_\mu)$$

iff $M_\lambda < f_\mu$ and $f_\lambda < f_\mu$ on M_λ
Clearly ,P is a partially ordered set . Let Q be a ordered chain of P.
$Q = (M_i, f_i)$ be a chain in P.
Consider (UM_i, Φ) where $\Phi(x) = f_i(x)$ for all $x \in M_i$
Let $(x, y) \in UM_i$, and α, β be any scalars. Then $x \in m_i y \in m_j$
Since Q is totally ordered set ,$M_i \in m_j$
Without loss of generality assume that $M_i \in m_j$

Then $x, y \in M_j$
Since M_j is a subspace of N, we have

$$\alpha x + \beta y$$

. Since M_j is a subset of UM_i.
UM_i is a subspace of N.
Suppose $x \in UM_i$ such that $x \in M_i$ and $x \in M_j$. Then by definition of Φ,

$$\Phi(x) = f_i(x)$$

and

$$\Phi(x) = f_j(x)$$

.

$$f_i(x) = f_j(x)$$

. Φ is well defined.
(UM_i, Φ) is an upper bound of Q.
P satisfies the all condition of 'Zorns lemma'.
Hence thereexist a maximal element (H, F) in P.
Suppose $H \in N$ then there exist $x_0 \in N - H$ by previous theorem,F can be extended to a functional F_0 on $H_0 = HU[x_0]$ which contains H properly.Whic is contradiction to the maximal of (H, F).
Hence $N = H$

Theorem 3 *Let N be a normed linear space and x_0 a non-zero vector in N, then exist a functional F in N^* such that*

$$\begin{aligned} F(x_0) &= \|x_0\| \\ \|F\| &= 1. \end{aligned}$$

proof. Let $M = [\alpha x_0]$ be a linear subspaces of N spanned by x_0.
Define f_0 on M by

$$f_0(\alpha x_0) = \alpha\|x_0\|$$

Let $y_1, y_0 \in M$
Then $y_1 = \alpha x_0$ and $y_2 = \beta x_0$ for some scalars $\alpha, \beta \in F$.
Suppose γ and S are scalars

$$\begin{aligned}
f_0(\gamma y_1 + S y_[2]) &= f_0(\gamma\alpha x_0 + \gamma\beta x_0) \\
&= f_0[(\gamma\alpha + S\beta)x_0] \\
&= (\gamma\alpha + S\beta)||x_0|| \\
&= (\gamma\alpha||x_0||(+(S\beta)||x_0|| \\
&= \gamma f_0(\alpha x_0) + S f_0(\beta x_0)
\end{aligned}$$

f_0 is a linear on M.
Let $y = \alpha x_0$

$$\begin{aligned}
||y|| &= ||\alpha x_0|| \\
&= |\alpha|||x_0|| \\
||f_0(y)|| &= ||f_0(\alpha x_0)|| \\
&= ||\alpha||x_0|||| \\
&= |\alpha|||x_0|| \\
&= ||y||
\end{aligned}$$

f_0 is bounded on M.
$f_0$is functional on M.
Now,

$$\begin{aligned}
||f_0|| &= sup[||f_0(y)|| : y \in M, ||y|| = 1] \\
&= sup[||y|| : y \in Mand||y|| \leq 1] \\
&= 1. \\
\text{Also} f_0(x_0) &= f_0(1.x_0) \\
&= ||x_0||
\end{aligned}$$

By Hahn Banach f_0 can be extended to a functional $F \in N^*$ such that

$$\begin{aligned} F(x_0) &= f_0(x_0) \\ &= ||x_0|| \\ and ||F|| &= ||f_0|| \\ &= 1. \end{aligned}$$

Hence the proof.

Theorem 4 *Let M be a closed linear subsoace of a normed linear space N and x_0 a vector not in M,then there exist a functional F in N^*. Such that $F(M) = 0$ and $F(x_0) \neq 0$*

proof. Consider $\phi : N \to \frac{N}{M}$ defined by
$\phi(x) = x + M$
ϕ is a continuous linear transformation and if $m \in M$.
$\phi(m) = m + M = 0$
$x_0 \in M$
$\phi(x_0) = x_0 + M \neq 0$
$\phi(x_0)$ is not a zero vector in N/M. By Hahn Banach theorem there exist a functional $f \in (N/M)^*$ such that

$$\begin{aligned} f(x_0 + M) &= ||x_0 + M|| \neq 0 \\ F(\alpha x + \beta y) &= f(\phi(\alpha x + \beta y)) \\ &= f((\alpha x + \beta y) + M) \\ &= f(\alpha x + M) + f(\beta y + M) \\ &= \alpha f(x + M) + \beta f(y + M) \\ &= \alpha F(x) + \beta F(y) \end{aligned}$$

F is linaer

$$\begin{aligned}
Now|F(x)| &= |f(\phi(x))| \\
&\leq ||f||.||\phi(x)|| \\
&\leq ||f||.||\phi||||(x)|| \\
&\leq ||f||.||x||
\end{aligned}$$

F is bounded
F is a functional on N
$F \in N^*$
Suppose $m \in M$ then

$$\begin{aligned}
F(m) &= f(\phi(m)) \\
&= f(m + M) \\
&= f(0) \\
&= 0 \\
Also F(x_0) &= f(\phi(x_0)) \\
&= f(x_0 + m) \\
&= 0 \\
&\neq 0
\end{aligned}$$

Hence the proof.

Possible Questions
8 marks

1. Prove that the self adjoint operators in $B(H)$ form a closed real linear subspace of $B(H)$ and Therefore a real Banach space which contains the identity transformation

2. Prove that if A_1 & A_2 are self adjoint operator on H then their product A_1A_2 is self Adjoint iff $A_1A_2 = A_2A_1$.

3. Prove that an operator T on a Hilbert space H is self adjoint if $\langle Tx, x\rangle$ is real $\forall x$.

4. If N_1 and N_2 are normed operators on H with the property that either with the adjoint of the other then $N_1 + N_2$ and N_1N_2 are normal

5. Prove that if A is a positive operator on H then $I+A$ is non-singular. In particular $I+T^*T$ and $I+TT^*$ are non-singular for any arbitrary operator T on H.

6. Prove that an operator on Hilbert space H is unitary iff T is unitary. It is an isometric isomorphism of H onto itself.

7. If P is a projection on Hilbert space H with range M and null space N , then $M \perp N \leftrightarrow P$ is self adjoint. In this case $N = M^{\perp}$

8. If P is a projection on a closed linear subspace M of H, then M is invariant under T iff $TP = PTP$

9. If P is a projection on a closed linear subspace $M(H)$, then M reduces to an operator T iff $TP = PT$

10. Prove that the set of all normal operators on H is a closed subset of $B(H)$ which contains the set of all self adjoint operators and is closed under scalar multiplication

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Functional analysis **Subject Code: 16MMP301**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
The null space of any continuous linear transformation is …………	closed	open	open subset	open set	closed
Let M ={x / f(x)=0} then M is the ……………….. of f.	range	linear	nullspace	open subset	nullspace
Let the adjoint of T denoted by ………… on H.	H	H**	H*	T*	T*
The adjoint of an operator is (Tx,y) =………….	(Tx,y)	(x,T*y)	(T*x,y)	(Tx,Ty)	(x,T*y)
The adjoint of operator T to T* on B(H) is $(T_1+T_2)^*$=…….	$T_1^*-T_2^*$	$T_1+T_2^*$	$T_1^*+T_2$	$T_1^*+T_2^*$	$T_1^*+T_2^*$
The adjoint of operator T to T* on B(H) is $(\alpha T)^*$ =…………….	$\alpha(T)^*$	Conjugate of α (T)*	$T1^*+\alpha$	T*	Conjugate of α (T)*
The adjoint of operator T to T* on B(H) is $(T_1T_2)^*$=……..	$T_1^*-T_2^*$	$T_1+T_2^*$	$T_1^*T_2^*$	$T_2^*T_1^*$	$T_2^*T_1^*$
The adjoint of operator T to T* on B(H) is T** = ……………	$\alpha(T)^*$	T1+T2*	T	T*	T
The adjoint of operator T to T* on B(H) is $\lVert T\rVert$=…………...	$\lVert T\rVert^*$	$\lVert T^*\rVert$	T*	T	$\lVert T^*\rVert$
The adjoint of operator T to T* on B(H) is $\lVert T^*T\rVert$=…………	$\lVert T\rVert^*$	$\lVert T^*\rVert$	T*	$\lVert T\rVert^2$	$\lVert T\rVert^2$
If T is nonsingular operator on H then T* is also nonsingular then $(T^*)^{-1}$=…….	$\lVert T\rVert^*$	$\lVert T^*\rVert$	$(T^{-1})^*$	(T^{-1})	$(T^{-1})^*$
If T = T* then 0 and I are ……………. operators	adjoint	commutate	self adjoint	symmetric	self adjoint
If A_1 and A_2 are self adjoint operators on H then their product A_1A_2 is self adjoint iff A_1A_2 =……………..	A_2A_1	A_1A_2	A_1	A_2	A_2A_1
If T is an arbitrary operator on a hilbert space H then T=0 iff ………………	(Tx,y)	(x,T*y)	(T*x,y)	(Tx,y)=0	(Tx,y)=0
If T is an arbitrary operator on a hilbert space H then (Tx,x)=0 iff ………………	T=1	T=0	T=T*	T= Tx	T=0
If $(A_1A_2)^*$=…………if A_1 and A_2 are self adjoint operator	A_2A_1	A*2	A*1	A_2	A_2A_1
The adjoint operator 0*= …………..	6	2	0	1	0
The adjoint operator 1*= …………..	6	2	0	1	1
If A is a positive operator on a H then I+A is ……………….	singular	nonsingular	commutate	self adjoint	nonsingular
I+T*T are…………. for any arbitrary oprator on T on H .	singular	nonsingular	commutate	self adjoint	nonsingular
The self adjoint operator A is said to be positive if ………………..	(Ax,x) =0	(Ax,x) ≥ 0	(A*x,y)	(Ax,y)=0	(Ax,x) >= 0
Every complete subspace of a complete space is …………….	closed	open	open subset	open set	closed
An operator N on H is said to be ………… if it commutes with its adjoint.	complete	closed	normal	open	normal
An operator N on H is said to be normal If it ……………. with its adjoint.	singular	nonsingular	commutes	self adjoint	commutes
The normal operator is NN*=……………	N*	nonsingular	N	N*N	N*N
Every ………………. operator is normal	adjoint	commutate	self adjoint	symmetric	self adjoint
An operator T on H is ……………. Iff $\lVert T^*x\rVert=\lVert Tx\rVert$	complete	closed	normal	open	normal
If is a …………..operation on H then $\lVert N^2\rVert=\lVert N\rVert^2$	complete	closed	normal	open	normal
If T is an operator on H then T is normal iff its real and imaginary parts ……	singular	nonsingular	commutes	self adjoint	commutes
An operator U on H is said to be ……….. If UU*= U*U= I	complete	closed	normal	unitary	unitary
An operator U on H is said to be unitary If ………………….	UU*= U*U= I	U*U=0	U=1	U=0	UU*= U*U= I
Every unitary opeartor is a ……………… operator	complete	closed	normal	unitary	normal
…………. operators are precisely nonsingular operators.	complete	closed	normal	unitary	unitary
Unitary operator inverse equals theirs …………	adjoint	commutate	self adjoint	symmetric	adjoint
Unitary operators are precisely ………………. operators.	singular	nonsingular	commutes	self adjoint	nonsingular
A projection on a banch space B is an ……………. operator when T^2 =T.	adjoint	commutate	self adjoint	idempotent	idempotent
An operator P on a hilbert space H is a ………… on H which satisfies the condition P^2= P and P* = P	dense sets	range and null space	subspaces	projection	projection
A closed linear subsapce M(H) is ………… under T if T(M) $\subseteq$ M	invariant	commutate	self adjoint	idempotent	invariant
Two projectionP and Q on ahilbert space H are said to be……... if PQ=0	invariant	commutate	orthogonal	idempotent	orthogonal
If P is a…………… on a closed linear subspace M of H then M reduces an operator T iff TP=PT	projection	commutate	self adjoint	idempotent	projection

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DEPARTMENT OF MATHEMATICS

Subject: Functional Analysis	**Semester: III**	**L T P C**
Subject Code: 16MMP301	**Class: II-M.Sc. Mathematics**	**4 0 0 4**

UNIT V

Banach algebras: The definition and some examples of Banach algebra – Regular and singular elements – Topological divisors of zero – The spectrum – The formula for the spectral radius

TEXT BOOK

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1 Banach algebra

Definition 1 The natural embedding on N in N^*
The conjugate space N^* of a normed linear space N is itself a normed linear space . we can form the conjugate space $(N^*)^*$ of N^*. We denote this space by N^{**} and called second conjuate.

Theorem 1 *Let N an arbitrary normed linear space then each vector x in N includes a functional F_x on N^* defined by $F_x(f) = f(x)$ for all $x \in N$. such that*

$$||F_x|| = ||x||$$

.

Proof: Let $f, g \in N^*$ be arbitrary and α, β be any scalars

$$\begin{aligned} F_x(\alpha f + \beta g) &= (\alpha f + \beta g)(x) \\ &= (\alpha f(x) + \beta g(x)) \\ &= \alpha.F_x(f) + \beta F_x(g) \end{aligned}$$

F_x is linear.
Now,

$$\begin{aligned} ||F_x(f)|| &= ||f(x)|| \\ &\leq ||f||||x|| \end{aligned}$$

F_x is bounded.
F_x is linear functional on N^*
Next,we have to prove that

$$\begin{aligned} ||F_x|| &= ||x|| \\ Now, ||F_x|| &= sup[||F_x(f)|| : ||f|| \leq 1] \\ &\leq sup[||f||||x|| : ||f|| \leq 1] \\ &\leq ||x|| \end{aligned}$$

Suppose $x = 0$ then $||f_x|| = ||x||$.
It is true for $x = 0$
Such that

$$\begin{aligned} g(x) &= ||x|| \\ ||g|| &= 1 \end{aligned}$$

By defn of F_x we have,

$$\begin{aligned} g(x) &= F_x(g) \\ F_x(g) &= g(x) = ||x|| \\ Now, ||x|| &= g(x) \\ &\leq sup[|g(x)| : ||g|| \leq 1] \\ &= ||F_x|| \\ Hence||x|| &\leq ||F_x|| \\ ||x|| &= ||F_x|| \end{aligned}$$

Theorem 2 *The mapping $J : N \to N^{**}$ defined by*

$$J(x) = F_x$$

for all $x \in N$ defines an isomorphic of N into N^{**}.

proof: Now,

$$\begin{aligned} F_{(ax+by))}(f) &= f(ax + by) \\ &= a.f(x) + b.f(y) \\ &= a.F_x(f) + b.F_y(f) \\ &= (a.F_x + b.F_y)(f) \\ F_{(ax+by))} &= a.F_x(f) + b.F_y \\ J(ax + by) &= aJ(x) + bJ(y) \end{aligned}$$

J is linear.

Also,

$$\begin{aligned}
||J(x)|| &= ||F_x|| \\
&= ||x|| \\
\text{Suppose} J(x) &= J(y) \\
\Rightarrow J(x) - J(y) &= 0 \\
\Rightarrow J(x-y) &= 0 \\
\Rightarrow ||J(x-y)|| &= ||0|| \\
\Rightarrow ||x-y|| &= 0 \\
\Rightarrow x-y &= 0 \\
\Rightarrow x &= y
\end{aligned}$$

J is one-one.
Hence J is isometric isomorphism.

Remark 1 *J is an isometric isomorphism of N into N^{**} and therefore we may say N as a part of N^{**} without changing any of its structure as a normed linear space.*
N is a subset of $N^{**}(N \in N^{**})$. Hence the map J is called natural imbedding of N into N^{**}.

Remark 2 *Suppose J is onto $N = N^{**}$ the sign of equality is the same sence of isomorphism under the map J. The map J is called reflexive.*

Banach spaces 2

2 Open mapping

Theorem 3 The open mapping theorem
Let B and $B^{'}$ are Bananch spaces, the symbols $S(x,r)$ and $S^{'}(x,r)$ denote the open sphere with centre x and radius r

$$\begin{aligned}
S(x,r) &= \{y \in B : ||y-x|| < r\} \\
S^{'}(x,r) &= \{y \in B^{'} : ||y-x|| < r\}
\end{aligned}$$

Remark 3 *S_r and $S_{r'}$ denote the open sphere centered at origin and the radius r.*

$$\begin{aligned} S_r &= \{y \in B : \|y\| < r\} \\ S_{r'} &= \{y \in B' : \|y\| < r\} \end{aligned}$$

Remark 4 $S_r = r.S_r$

proof:

$$\begin{aligned} S_r &= \{y \in B : \|y\| < r\} \\ &= \{y \in B : \frac{\|y\|}{r} < 1\} \\ \text{let} y = rx & \\ S_r &= \{rx \in B : \|x\| < 1\} \\ &= r\{x \in B : \|x\| < 1\} \\ &= r.S_1 \end{aligned}$$

Theorem 4 *Let B and B' be the Banach space and T be the linear transformation of B and B'. Then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.*

proof: Let S_r and $S_{r'}$ denote the open sphere centered at origin and the radius r in B and B' respectively
By previous note, $S_r = r.S$

$$\begin{aligned} T(s_r) &= T(rS_1) \\ &= r.T(S_1) \end{aligned}$$

For each positive integer n,consider an open sphere S_n in B.
Then

$$B = \bigcup_{n=1}^{\infty} S_n$$

Since T is onto, we have

$$\begin{aligned} B' &= T(B) \\ &= T\left(B = \bigcup_{n=1}^{\infty} S_n\right) \\ &= \bigcup_{n=1}^{\infty} T(S_n) \end{aligned}$$

Since B' is a Banach space, we have B' is complete.
B' is of second category (Baire's category theorem).
If complete metric space is the union of sequence of its subset then the closure atleast one set in the sequence must have non-empty interior.

$$(\overline{T(S_{no})})^o \neq 0$$

for some n_0
For all y such that

$$y \in (\overline{T(S_{no})})^0$$

y is an interior point of $\overline{T(S_{no})}$.
G of y such that$y \in G \subset \overline{T(S_{no})}$
Since $y \in \overline{T(S_{no})}$, $y \in T(S_{no})$ or y is a limit point of $T(S_{no})$.
y is an interior point of $\overline{T(S_{no})}$.
Such that $y \in T(S_{no})$.
Suppose y is a limit point of $T(S_{no})$.
Since G is a nbd of y,G has atleast one point other than y.
For a point $y_0 \in G$ and $y_0 \in T(S_{no})$.
Define a map,$f : B' \to B$ by $f(y) = y - y_0$
Suppose

$$\begin{aligned} f(y_1) &= f(y_2) \\ y_1 - y_2 &= y_2 - y_0 \\ y_1 &= y_2 \end{aligned}$$

f is one-one.
f is onto.
Suppose $y_n \to y$
Then

$$\begin{aligned} f(y_n) &= y_n - y_0 \\ &= y - y0 \\ &= f(y) \\ f(y_n) &\to f(y) \end{aligned}$$

f is continuous.
Now

$$\begin{aligned} f^{-1}(y_n) &= y_n + y_0 \\ y + y_0 &= f^{-1}(y) \\ f^{-1}(y_n) &\to f^{-1}(y) \end{aligned}$$

f is a homeomorphism.
claim: 0 is the interior point of $\overline{T(S_{no})} - y_0$.
Since y_0 is a interior point of $\overline{T(S_{no})}$,a neighbourhood G of

$$y_0 \ni y_0 \in G \subset \overline{T(S_{no})}$$

.

$$f(y_0) \in f(G) \subset \overline{f(T(S_{no}}))$$

.

$$0 \in f(G) \subset \overline{T(S_{no})} - y_0$$

.
0 is a interior point of $\overline{T(S_{no})} - y_0$.
claim: $T(S_{n0}) - y_0 \subset T(S_{2n0})$
Let $y \in T(S_{n0}) - y0$

for all $x \in S_{n0}$ $y = T(x) - y_0$
Since $y_0 \in T(S_{n0})$, for all $x_0 \in S_{n0}$

$$y_0 = T(x_0)$$

$$\begin{aligned} y_0 &= T(x_0) \\ y &= T(x) - T(x_0) \\ &= T(x - x_0) \end{aligned}$$

Now $x, x_0 \in s_{no}$

$$\begin{aligned} \Rightarrow \|x\| &< n_0, \|x_0\| < n_0 \\ \Rightarrow \|x - x_0\| &\leq \|x\| + \|x_0\| \\ \Rightarrow \|x - x_0\| &\leq n_0 + n_0 \\ \Rightarrow \|x - x_0\| &\leq 2n_0 \\ \Rightarrow x - x_0 &\in S_{2n0} \end{aligned}$$

$T(x - x_0) \in T(S_{2n0})$
$y \in T(S_{2n0})$
$T(S_{n0}) - y_0 \subset T(S_{2n0}) = 2n0T(s_1)$

$$\overline{T(S_{n0}) - y_0} \subset \overline{2n0T(s_1)}$$

Since f is continuous.

$$\begin{aligned} f(\overline{T(S_{n0})}) &\subset \overline{f(T(S_{n0}))} \\ \overline{T(S_{n0})} - y_0 &\subset \overline{T(S_{n0}) - y_0} \\ &\subset \overline{2n_0T(S_1)} \end{aligned}$$

Define a map $g : B' \to B'$ by $g(x) = 2n_0x$
Clearly g is homeomorphism.

$$\begin{aligned}
\overline{g(T(S_{n0}))} &= g\overline{(T(S_{n0}))} \\
\overline{2n_0T(S_1} &= 2n_0\overline{T(S_1)} \\
\overline{T(S_{n0}) - y_0} &\subset 2n_0\overline{T(s_1)}
\end{aligned}$$

0 is a interior point of $\overline{T(S_1)}$.
Hence there exist $\epsilon > 0$ such that

$$S'_\epsilon(\overline{T(S_1)}) \tag{1}$$

Claim: $S'_\epsilon \subset (T(S_3))$.
Let $y \in S'_\epsilon$ be arbitrary
Then $\|y\| < \epsilon$
From (2.1), $y \in \overline{T(S_1)}$, then y is a limit point of (S_1).
Therefore a vector $y_1 \in T(S_1)$.
Such that $y_1 \in T(S_1)$ and $\|y - y_1\| < \epsilon/2$.
Then $y_1 = T(x_1)$ for some $x_1 \in S$ with $\|x_1\| < 1$.
We have $S'_\epsilon/2 \subset \overline{T(S_{\frac{1}{2}})}$.
Since $\|y - y_1\| < \epsilon/2$,$y - y_1 \in \epsilon/2$.

$$y - y_1 \in \overline{T(S_{\frac{1}{2}})}$$

Vector y_2 in $T(S_{\frac{1}{2}})$ such that

$$\|(y - y_1) - y_2\| < \epsilon/2$$

when $y_2 \in T(S_{\frac{1}{2}})$,$y_2 = T(x_2)$
for some $x_2 \in S_{\frac{1}{2}}$
$\|x_2\| < \frac{1}{2}$.
Continuing in this way, we get a sequence x_n in B such that $\|x_n\| < \frac{1}{2^{n-1}}$ and

$$\|y - (y_1 + y_2 + ... + y_n)\| < \frac{\epsilon}{2^n}. \tag{2}$$

Let

$$\begin{aligned} S_n &= x_1 + x_2 + ... + x_n. \\ \|S_n\| &= \|x_1 + x_2 + ... + x_n\| \\ &\leq \|x_1\| + \|x_2\| + ... + \|x_n\| \\ &< 1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}} \\ &= \frac{1(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} \\ &< 2 \end{aligned} \tag{3}$$

For $n > m$ we have

$$\begin{aligned} \|s_n - s_m\| &= \|x_{m+1} + ... + x_n\| \\ &\leq \|x_{m+1}\| + ... + \|x_n\| \\ &< \frac{1}{2^m} + ... + \frac{1}{2^{n-1}} \\ &= \frac{\frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)}{1 - \frac{1}{2}} \\ &= \frac{1}{2^{m-1}} - \frac{1}{2^{n-m-1}} \to 0 as m, n \to \infty \end{aligned}$$

S_n is a cauchy's sequence in B.
Sice B is complete,S_n converges to some vectors $x \in B$. Now

$$\begin{aligned} \|x\| &= \|lim S_n\| \\ &= lim\|S_n\| \\ &\leq 2 (by 2.2) \\ &< 3 \end{aligned}$$

$x \in S_3$
Since T is continuous, $x = lim S_n$

$$\begin{aligned}
\Rightarrow T(x) &= limT(S_[n]) \\
&= lim[T(x_1 + ... + x_n)] \\
&= lim[T(x_1) + ... + T(x_n)] \\
&= lim[y_1 + ... + y_n] \\
&= y(by2.2)
\end{aligned}$$

$y = T(x)$ where $\|x\| < 3$.
$y \in T(S_3)$
Hence $S'_\epsilon \subset T(S_3)$.
Hence the proof.

Theorem 5 Open Mapping Theorem: *Let B and B' be Banach space, T is a continuous linear transformation of B and B' then T is open mapping.*

proof: Given that $T : B \rightarrow B'$ is continuous onto
To Prove: T is an open map.
Let G be an open set in B
Let $y \in T(G)$ be arbitrary
Since T is onto ,$x \in G \ni T(x) = y$
Since G is an open set, x is a interior point of G.
For an open sphere $S(x, r) \subset G$.
W.K.T $S(x, r) = x + S_r$.
where S_r is an open sphere centre at origin with radius r.

$$x + S_r \subset G$$

By previous theorem, an open sphere $S'_\epsilon inB'$ such that $S'_\epsilon \subset T(S_r)$.

$$
\begin{aligned}
y + S'_{\epsilon} &\subset y + T(S_r) \\
&= T(x) + T(S_r) \\
&= T(x + S_r) \\
&= T(s(x, r)). \\
S'(y, \epsilon) &\subset T(S(x, r))
\end{aligned}
$$

y is a n interior point on T(G).
T(G) is an open set (or) open sphere.
Hence the proof.

Theorem 6 *Let B and B' be Banach spaces and let T be a one-one continuous linear transformation of B nto B'. Then T is homeomorphism.*

proof:
Since T is one-one, onto, continuous we have to prove that T^{-1} is continuous.
By prrevious theorem ,T is an open map .
T(G) is open if G is open.
Since $T(^{-1})^{-1} = T$, we have T^{-1} is continuous.
Hence T is homeomorphism.

Definition 2 Closed Graph Theorem:
Let x and Y be any non-empty set and let $f : X \to Y$ be an imaginary mapping.Then the graph of f is a subset of XxY which consist of all ordered pairs of the form(x,f(x)).

Remark 5 *Let N and N' be the normed liinear spaces.Then NxN' is a normed linear space with co-ordinate wise linear operation and the norm*

- $||x, y|| = max[||x||, ||y||]$

- $||x, y|| = max[||x|| + ||y||]$

Definition 3 *Let N and $N^{'}$ be the normed liinear spaces and D be the subspace of N. Then a linear transformation $T : D \to N^{'}$ is said to be closed iff $x_n \in D, x_n \in x, T(x_n \in y)$and $T(x) = y$.*

Theorem 7 *Let N and $N^{'}$ be the normed liinear spaces and D be the subspace of N. Then a linear transformation $T : D \to N^{'}$ is said to be closed iff T_G is closed.*

proof:
Suppose T is a Closed linear transformation
To prove:T_G is closed
Suppose T_G has limit point (x,y). Then there exist a sequence $(x_n, T(x_n))$ where $x_n \in D$ and $(x_n, T(x_n) \in (x, y)$.

$$\begin{aligned} ||(x_n, T(x_n) - (x, y)|| \to 0 \\ &\Rightarrow ||(x_n - x, T(x_n) - y)|| \to 0 \\ &\Rightarrow ||(x_n - x|| + ||T(x_n) - y)|| \to 0 \\ &\Rightarrow ||(x_n - x|| \to 0 \\ and &\Rightarrow ||T(x_n) - y)|| \to 0 \end{aligned}$$

therefore

$$x_n \to x$$

and

$$T(x_n) \to y$$

. Since T is a closed linear transformation $T(x) = y$ and $x \in D$.
$(x, y) \in T_(G)$.
$T_(G)$ is closed. Conversely assume that the graph of T is closed,$T(G)$ is closed.
To prove: T is a closed linear transformstion.

Let $x_n \in D, x_n \in x, T(x_n) \in y$ then (x,y) is a limit point of $T_(G)$.

$$(x, y) \in \overline{T_G}$$

.
Since $T_(G)$ is closed ,we have $\overline{T_G} = T_G$.
$x \in G$ and $T(x) = y$
Hence T is a closed linear transformation.

Theorem 8 *Let B and $B^{'}$ be Banach spaces and T be a linear transformation of BintoB$^{'}$. Then T is a continuous mapping iff its graph is closed.*

proof:
Let $T : B \to B^{'}$ be continuous.
To prove:
T_G is closed.
$T_G = \overline{T_G}$
Since $T_G \subset \overline{T_G}$, it is enough to show that $T_G \supset \overline{T_G}$
Let $x, y \in \overline{T_G}$
Then there exist a sequence $(x_n, T(x_n))$ in T_G. Such that $(x_n, T(x_n)) \to (x, y)$
Hence $(x_n \to x, T(x_n) \to y)$
Since T is a continuous $T(x_n) \to y$

$$\begin{aligned} T(x) &= y \\ (x, T(x)) &\in T_G \\ (x, y) &\in T_G \\ T_G &\supset \overline{T_G} \end{aligned}$$

T_G is closed.
Conversely,assume that T_G is closed.

Denote B_1 the linear space B renormed by

$$\begin{aligned} ||x_1|| &= ||x|| + ||T(x)|| \\ Now, ||T(x)|| &\leq ||T(x)|| + ||x|| \\ &\leq ||x||_1 \end{aligned} \tag{4}$$

T is bounded by $||x||_1$
T is cotinuous.
It is sufficient to prove B_1 and B are homeomorphic.
Consider the identity map
$I : B_1 \to B$ by $I(x) = x$
Clearly I is one-one and onto.
Also

$$\begin{aligned} ||I(x)|| &= ||x|| \\ &\leq ||x|| + ||T(x)|| \\ &= ||x|| \end{aligned}$$

I is bounded by $||x||_1$.
I is continuous. Let x_n be any cauchy sequence in B_1.
Then $||x_n - x_m|| \to 0$ as $n, m \to \infty$.

$$\begin{aligned} &\Rightarrow ||x_n - x_m|| + ||T(x_n - x_m)|| \to 0 as n, m \to \infty \\ &\Rightarrow ||x_n - x_m|| + ||T(x_n) - T(x_m)|| \to 0 as n, m \to \infty \\ &\Rightarrow ||x_n - x_m|| \to 0, ||T(x_n - x_m)|| \to 0 as n, m \to \infty \end{aligned}$$

$x_n, T(x_n)$ are cauchy sequence in B, B' respectively.
Since $BandB'$ are complete,$x_n \to B$ and $T(x_n) \to B'$.
SinceT_G is closed, $(x, y) \in T_G$.
Such that $T(x) = y$

Now,

$$\begin{aligned} ||x_n - x||_1 &= ||x_n - x|| + ||T(x_n - x)|| \\ &= ||x_n - x|| + ||T(x_n) - T(x)|| \\ &= ||x_n - x|| + ||T(x_n - y)|| \\ &= \to 0 asn \to \infty \end{aligned}$$

B_1 is complete.
B and B_1 are Banach spaces and I is a one-one conitinuous linear transformation of B and B_1.
By previous theorem, I is homeomorphism.
B and B_1 are homeomorphic.
Hence the proof.

Possible Questions
8 marks

1. Prove that every element x for which $\|x - 1\| = 1$ is regular and the inverse of such an element is given by the formula, $x^{-1} = 1 + \sum(1 - x)^n$.
2. Prove that the boundary of S is a subset of Z
3. Prove that if G is an open set therefore S is a closed set.
4.) If T is a operator on a finite dimension Hilbert space H, then prove the following
   i T is singular $\leftrightarrow 0 \in \sigma(T)$
   ii If T is not singular, $\alpha \in \sigma(T) \leftrightarrow \alpha \in \sigma(T)$
   iii If A is non-singular, then $\sigma(ATA^-1) = \sigma(T)$
   iv If $\alpha \in \sigma(T)$ and if P is any polynomial then $P(\alpha) \in \sigma(P(T))$
5. If T be an arbitrary operator on a finite dimension Hilbert space H and N be a normal operator on H. Show that if T commutes with N then T commutes with N^*.
6. Prove that the mapping $x \to x^{-1}$ of G into G is continuous and its therefore a homeomorphism of G onto itself.
7. Prove that for every element x in a Banach algebra A, $\sigma(x)$ is non empty and compact.
8. Prove that if 0 is the only topological divisor of zero in a Banach algebra A then $A = C$.
9. Prove that $\Gamma(x) = \lim \|x^n\|^{\frac{1}{n}}$

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DEPARTMENT OF MATHEMATICS
PART-A Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: Functional analysis **Subject Code: 16MMP301**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
A non zero vector x such that Tx=λ x is true for some scalar λ is called an ………….. of T.	eigen value	eigen vector	scalar	idempotent	eigen vector
A scalar λ such that Tx=λx holds for some nonzero x is called an……… of T.	eigen value	eigen vector	scalar	idempotent	eigen value
Each eigen vector corresponds precisely to one…………………..	eigen value	eigen vector	scalar	idempotent	eigen value
Each eigen value has one or more ……………. associated with it.	eigen value	eigen vector	scalar	idempotent	eigen vector
Eigen value are otherwise called as ………………………..	characterestic value	characterestic vector	eigen vector	scalar	characterestic value
Eigen vector are otherwise called as ………………………..	characterestic value	characterestic vector	eigen value	scalar	characterestic vector
If T is an operator on hilbert space H, then T to a vector x is to transform it into a scalar multiple ………………	Tx=λx	Tx =0	Tx=1	λ x=1	Tx=λx
The image of the zero operator under the mapping T$\rightarrow$ [T] is the………………	scalar	zero vector	real value	zero matrix	zero matrix
If T has different Eigen values then each one is ……….. to one another	corresponding	same	distinct	identity	distinct
The expression for T=$\lambda_1 P_1+\lambda_2 P_2+$…………….+$\lambda_m P_m$ when it exist is called the ………….. resolution	resolvement	spectral	distinct	identity	spectral
The image of the identity operator is the ………… matrix	singular	identity	nonsingular	null	identity
The ……….. Matrix is 1's down the main diagonal and zero elsewhere.	singular	identity	nonsingular	null	identity
Two matrices in A_n are ………… iff they are the matrices of a single operator on H relative to different bases	similar	asimilar	vary	distinct	similar
The …………. of S is a subset of Z.	boundary	resolvement	spectral	distinct	boundary
The set of all …………… divisor of zero by z.	identical	topological	boundary	resolvement	topological
The set of all complex number is a…………….. algebra	Ring	hardy	banach	functional	banach
The regular element is the compliment of …………… element	singular	identity	nonsingular	null	singular
A banach algebra is acomplex ………….. which is also an algebra with identity 1.	Banach space	hilbert space	Inner product space	linear space	Banach space
Let A be a …………. algebra then the set of all reular elements in A by G.	Ring	hardy	banach	functional	banach
Let A be a …………. algebra then the set of all reular elements in A by S.	singular	identity	nonsingular	null	singular
The set of all values in a banach algebra is ……….. number	complex	real	inverse	scalar	complex
G is an open set and therefore s is an …………. set.	closed	open	open subset	open set	closed
The compliment of spectrum is called the ……………… of x.	resolvement	spectral	distinct	identity	resolvement
For every element x in a banach algebra A the ……….. of x is nonempty and compact.	resolvement	spectrum	distinct	identity	spectrum
A division algebra is an algebra with identity in which each non zero element is ……………..	singular	nonsingular	commutate	regular	regular
0 is the only …………… divisor of zero in a banach algebra then A=C.	identical	topological	boundary	resolvement	topological
0 is the only topological divisor of zero in a banach algebra then ………..	A=C	A=1	A=0	A=V	A=C
A banach algebra is called a banach* algebra if it has an …………	involution	topological	boundary	resolvement	involution
The element x* is called the ………… of x and so asubalgebra of A is said to be self adjoint if it contains the adjoint of each of its elements.	adjoint	commutate	self adjoint	idempotent	adjoint
A banach* algebra in which $\lVert x^* x\rVert=\lVert x\rVert^2$ for all x is called ………………	B* algebra	resolvement	spectrum	distinct	B* algebra
If x is anormal element in a …………… then $\lVert x^2\rVert=\lVert x\rVert^2$	B* algebra	resolvement	spectrum	distinct	B* algebra
An element x$\in$A is ………… if there exist an element y such that xy=yx=1	singular	left regular	right regular	regular	regular
An element x$\in$A is ………… if there exist an element y such that yx=1	singular	left regular	right regular	regular	left regular
An element x$\in$A is ………… if there exist an element y such that xy=1	singular	left regular	right regular	regular	right regular
Every maximal left ideal in A is ……………..	closed	open	open subset	open set	closed
If x is not right regular then it is called……………..	right singular	left regular	right regular	regular	right singular
If x is not left regular then it is called……………..	left singular	left regular	right regular	regular	left singular
If x is both right and left regular then it is called …………	left singular	left regular	right regular	regular	regular
A is the intersection of all its ……….. left ideal	maximal	minimal	right regular	regular	maximal
A maximal left ideal in A is a proper left ideal which is not properly contained if their …………… left ideal	maximal	minimal	proper	regular	proper

Reg. No..................

16MMP301

Karpagam Academy of Higher Education
Karpagam University
Coimbatore-21
Department of Mathematics
Third Semester- I Internal test
Functional Analysis

Date: Time: 2 hours
Class: II M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. The metric space arise on norm as $d(x, y) =$

   a. $||x||$ b. $||x + y||$
   c. $||x - y||$ d. $||y||$

2. Every ——normed linear space is a banach space

   a. compact b. connected
   c. complete d. closed

3. Every linear is an abelian group under

   a. multiplication b. addtition
   c. neither a nor b d. both a and b

4. $\big| ||x|| - ||y|| \big| \leq$

   a. $||x||$ b. $||y||$
   c. $||x|| - ||y||$ d. $||x|| + ||y||$

5. The norm is a —— function

   a. real valued b. continuous
   c. neither a nor b d. both a and b

6. The space $\mathbb{R}$ is a normed linear space with $||x|| =$

   a. x b. $|x|$
   c. $\sqrt{x}$ d. $\frac{x}{2}$

7. The set $S = \{x : ||x|| \leq 1\}$ is

   a. closed sphere b. unit sphere
   c. neither a nor b d. both a and b

8. $||x||_p = \left(\sum_{n=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ is norm if

   a. $p < 1$ b. $p > 1$
   c. $1 \leq p < \infty$ d. neither a nor b

9. T is continuous if T is

   a. unbounded b. bounded
   c. neither a nor b d. both a and b

10. If u is a unit and is idempotent, then $u =$

    a. 0 b. 1
    c. either 0 or 1 d. neither 0 nor 1

11. T is bounded linear transformation if

    a.$||T(x)|| < K||x||$ b.$||T(x)|| \leq K||x||$
    c.$||T(x)|| > K||x||$ d. $||T(x)|| \geq K||x||$

12. $||T|| =$

    a. $\sup\{||T(x)|| : ||x|| < 1\}$ b. $\sup\{||T(x)|| : ||x|| \leq 1\}$
    c. $\inf\{||T(x)|| : ||x|| < 1\}$ d. $\inf\{||T(x)|| : ||x|| \leq 1\}$

13. $||T(x)|| \leq$

a. $||T||$ b. $||x||$
c. neither a nor b d. both a and b

14. Isometric isomorphism is a —— function

a. one-one b. onto
c. neither a nor b d. both a and b

15. $||I|| =$

a.0 b. 1
c. neither a nor b d. both a and b

16. Which of the following is true?

a. $||TT'|| \leq ||T||||T'||$ b. $||TT'|| < ||T||||T'||$
c. $||TT'|| \geq ||T||||T'||$ d.$||TT'|| > ||T||||T'||$

17. A linear space is called —— linear space when its scalar is complex

a. norm b. real
c. complex d.Banach

18. N/M is Banach space if N is —-

a. norm b. real
c. complex d.Banach

19. In a Banach space every Cauchy sequence is

a. bounded b. convergent
c. neither a nor b d. both a and b

20. Consider the set of all bounded sequences of scalars with $||x|| = \sup |x_n|$ is

a. normed linear space b. Banach space
c. neither a nor b d. both a and b

Part B-($3 \times 2 = 6$ marks)

21. Prove that norm is convex function
22. Define an operator
23. Prove that the closed unit sphere is convex

Part C-($3 \times 8 = 24$ marks)

24. a) Prove that norm is a continuous function

OR

b) Prove that l_p^n space is a normed linear space

25. a) Prove that l_p^n space is a Banach space

OR

b) Let N and N' be normed vector spaces and let $T : N \to N'$ be a linear transformation. Prove that the following statements are equivalent.

(i) T is continuous.
(ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$
(iii) there exists a real number $K \geq 0$ with the property that $||T(x)|| \leq K||x||$ for every $x \in N$
(iv) if $S = \{x : ||x|| \leq 1\}$ is the closed unit sphere in N, then its image $T(S)$ is a bounded set in N'

26. a) Let M be a closed linear subspace of a normed linear space N. Prove that N/M is a normed linear space

OR

b) Let N be a non-zero normed linear space. Prove that N is a Banach space iff $\{x : \|x\| = 1\}$ is complete

Reg. No..................

16MMP301

Karpagam Academy of Higher Education
Karpagam University
Coimbatore-21
Department of Mathematics
Third Semester- II Internal test
Functional Analysis

Date: Time: 2 hours
Class: II M.Sc Mathematics Max Marks: 50

Answer ALL questions
PART - A ($20 \times 1 = 20$ marks)

1. The conjugate space of N is the set of all ——- transformations form N into $\mathbb{R}$

   a. linear b. bounded
   c. neither a nor b d. both a and b

2. N^{**} is the

   a. second conjugate of N b. dual space of N
   c. neither a nor b d. both a and b

3. Which of the following is true?

   a. $||F_x|| \leq ||x||$ b. $||F_x|| \geq ||x||$
   c. neither a nor b d. both a and b

4. $T(S_r)$

   a. $T(S_1)$ b. $rT(S_1)$
   c. S_r d. S_1

5. Each operator T on a normed linear space N induces a corresponding operator denoted by

   a. T' b. $T*$
   c. neither a nor b d. both a and b

6. An open sphere is always non empty for it contain its

   a. centre b. distance
   c. length d. radius

7. $S_r(x_0)$ is an open sphere with radius — centered on x_0

   a. 1 b. r
   c. neither a nor b d. both a and b

8. A 1-1 linear transformation T of abanach space onto itself is continuous then its inverse is automatically

   a. linear b. continuous
   c. neither a nor b d. both a and b

9. If T is continuous linear transformation of B onto B′ then T is an — mapping.

   a. closed b.open
   c. neither a nor b d. both a and b

10. Which of the following is true?

    a. $||T^*|| \leq ||T||$ b. $||T^*|| \geq ||T||$
    c. neither a nor b d. both a and b

11. Which of the following is true?

    a. $I^* \leq I$ b. $I^* \geq I$
    c. neither a nor b d. both a and b

12. A complex banach space is said to be

a. inner product space b. Banach space
c. Hilbert space d. all the above

13. In a Hilbert space $< x, 0 >$

a. x b. 0
c. neither a nor b d. both a and b

14. In a Hilbert space $\overline{< x, y >}$

a. $< x, y >$ b. $< y, x >$
c. $< x, x >$ d. both a and b

15. $||I|| =$

a.0 b. 1
c. neither a nor b d. both a and b

16. In a Hilbert space $< x, x >$

a. $||x||$ b. $||x||^2$
c. neither a nor b d. both a and b

17. $< x, x >$

a. ≤ 0 b. ≥ 0
c. =1 d.=0

18. If $F = ---$ then the inner product space is called Hermitaian

a. $\mathbb{R}$ b. $\mathbb{C}$
c. neither a nor b d. both a and b

19. If $F = ---$ then the inner product space is called Euclidean

a. $\mathbb{R}$ b. $\mathbb{C}$
c. neither a nor b d. both a and b

20. If x and y are orthogonal then $< x, y >=$

a.1 b. 0
c. neither a nor b d. both a and b

Part B-($3 \times 2 = 6$ marks)

21. Define graph of a linear transformation
22. Define an inner product space
23. Define conjugate of T

Part C-($3 \times 8 = 24$ marks)

24. a) State and prove two properties of a inner product space

OR

b) State and prove closed graph theorem

25. a) State and prove Hahn Banach theorem

OR

b) State and prove open mapping theorem

26. a) Prove that $\mathbb{R}$ is a Hilbert space

OR

b) State and prove Schwarz inequality

KARPAGAM UNIVERSITY
(Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021

(For the candidates admitted from 2009 onwards)

M.Sc DEGREE EXAMINATION, NOVEMBER 2010

Third Semester

MATHEMATICS (COMPUTER APPLICATIONS)

FUNCTIONAL ANALYSIS

Time : 3 hours Maximum : 60 marks

PART – A (20 x ½ = 10 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 X 4 = 20 Marks) (2 ½ Hours)
Answer ALL the Questions

21. a. Define normed linear space and give two examples.

Or

b. Prove that if N is a normed linear space, then the closed unit sphere S* in N* is a compact Hausdorff space in the weak* topology.

22. a. If B and B' are Banach spaces, and if T is a continuous linear transformation of B on to B', then T is an open mapping.

Or

b. Let B be a Banach space and N a normed linear space. If $\{T_n\}$ is a sequence in B(B,N) such that $T(x) = \lim T_n(x)$ exists for each x in B, prove that T is a continuous linear transformation.

23. a. Define Hilbert space and state any two properties.

Or

b. If $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is any vector in H, then the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

24. a. Show that $\|TT^*\| = \|T\|^2$

Or

b. Prove that an operator T on H is normal if and only if $\|T^* x\| = \|Tx\|$ for every x

25. a. Prove that if T is normal, then x is an eigenvector of T with eigenvalue $\lambda \Leftrightarrow$ x is an eigenvector of T* with eigenvalue $\bar{\lambda}$.

Or

b. Let T be an operator on H, Prove that T is singular if and only if $O \in \sigma$ (T).

PART C (3 X 10 = 30 Marks)
Answer any THREE Questions

26. Let N and N' be normed linear spaces and T a linear transformation of N in to N'. Then the following conditions on T are all equivalent to one another. 1. T is continuous 2. T is continuous at the origin, in the sense that $X_n \to 0 \Rightarrow T(x_n) \to 0$.

27. Prove that if B and B' are Banach spaces, and if T is a linear transformation of B into B' then T is continuous $\Leftrightarrow$ its graph is closed.

28. If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then $\sum |(x, e_i)|^2 \leq \|x\|^2$ for every vector x in H.

29. The adjoint operation $T \to T^*$ on B(H).
Prove that 1) $(T_1 + T_2)^* = T_1^* + T_2^*$ 2) $(\alpha T)^* = \bar{\alpha} T^*$ 3) $(T_1 T_2)^* = T_2^* T_1^*$ 4) $T^{**} = T$

30. Let B be a basis for H, and T an operator whose matrix relative to B is $[\alpha_{ij}]$. Then prove that T is non-singular if and only if $[\alpha_{ij}]$ is non singular, and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$.

Reg. No................................

[15MMP301]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2015 onwards)

M.Sc., DEGREE EXAMINATION, NOVEMBER 2016

Third Semester

MATHEMATICS

FUNCTIONAL ANALYSIS

Time: 3 hours

Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

(Part - B & C 2 ½ Hours)

PART B (5 x 6 = 30 Marks)
Answer ALL the Questions

21. a. Let X be a normed space. Show that the following are equivalent:
    i. Every closed and bounded subset of x is compact
    ii. The Subset $\{x \in X : \|x\| \leq 1\}$ of X is compact and
    iii. X is finite dimensional

    Or

    b. State the prove Hahn- Banach extension theorem.

22. a. Let X be a Banach space Y be a normed space and F be a subset of BL (X, Y) such that for each $x \in X$, the set $\{f(x) : f \in F\}$ is bounded in Y. Then for each bounded subset E of X, the show that $\{f(x) : x \in E, f \in F\}$ is bounded in Y, that is , F is Uniformly bounded on E. In Particular show that $\sup \{\|f\| : f \in F\} < \infty$.

    Or

    b. Let X be a normed space and E be a subset of X. Prove that E is bounded in X if and only if f(E) is bounded in K for every $f \in X'$.

23. a. State and prove Schwarz inequality.

    Or

b. Let $\{u_1, u_2, u_3 \ldots\ldots\ldots\}$ be a countable orthonormal set in an inner product space X and xεX. Prove that $\sum_n |(x, u_n)|^2 \leq \|x\|^2$, where equality hold if and only if $x = \sum_n (x, u_n) u_n$

24. a. Let H be a Hilbert space and A ε BL (H). Show that there is a unique B ε BL (H) such that for all x, y ε H $\langle A(x), y\rangle = \langle x, B(y)\rangle$

Or

b. Let H be a Hilbert space and A ε BL (H). Show that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all x ε H. In that case show that $\|A^2\| = \|A^*A\| = \|A\|^2$.

25. a. Let X be a normed space and AεBL(X). Then A is invertible if and only of A is bounded below and surjective. Let X be a Banach space . Show that A is invertible if and only if A is bijective.

Or

b. Let X be a normed space and AεCL(X). If X is infinite dimensional prove that $0 \in \sigma_n(A)$.

PART C (1 x 10 = 10 Marks)
(Compulsory)

26. Consider an infinite dimensional separable Banach space $Y = l^p$ for some p with $1 \leq p < \infty$. Let X be denote the set of all scalar valued functions on S which vanish at all but a finite number of elements of S,S being he uncountable set and let $\{y_n : s \in S\}$ be a (Hamel) basis for Y with $\|y\| = 1$ for all $s \in S$. Show that X is not Banach space and open mapping theorem fails.
