KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS

Subject: COMBINATORICS **Subject Code: 16MMP305B**

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4	0	0	4

PO: After completing this course, the learner gain a clear knowledge on various combinatorial numbers and the applications of combinatorial techniques in real life problems.

PLO: To be familiar with the Stirling numbers, Bell's formula, Multinomial theorem, Euler function and be exposed with the Necklace problem.

UNIT I
Basic Combinatorial Numbers – Stirling numbers of the second kind – Recurrence formula for Pnm.

UNIT II
Generating functions – Recurrence relations- Bell's formula.

UNIT III
Multinomial – Multinomial theorem- Inclusion and Exclusion principle.

UNIT IV
Euler function –Permutations with forbidden positions –the Menage Problem.

UNIT V
Problem of Fibonacci –Necklace problem – Burnside's lemma.

TEXT BOOK
1. Krishnamurthy, V. (2002), Combinatorics: Theory and Applications, East West Press Pvt. Ltd.

REFERENCES
1. Balakrishnan V.K., (1995). Theory and problems of Combinatorics, Schaums outline series,McGraw Hill Professional.
2. Alan tucker, (2002). Applied Combinatorics, 4e, John wiley & Sons, New York.

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Lecture Plan

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

S. No	Lecture Duration Hour	Topics To Be Covered	Support Materials
		UNIT-I	
1.	1	Basic Combinatorial Numbers	R4: Ch 6: Pg.No: 314-316
2.	1	Continuation of Basic Combinatorial Numbers	R3: Ch:1: Pg.No: 43-45
3.	1	Stirling numbers of the second kind	R3: Ch:2: Pg.No: 117-120
4.	1	Continuation of Stirling numbers of the second kind	R3: Ch:2: Pg.No: 120-123
5.	1	Continuation of Stirling numbers of the second kind	R3: Ch: 2: Pg.No: 124-127
6.	1	Recurrence formula for Pnm.	R5: Ch 14: pg.No: 129-131
7.	1	Problems of Recurrence formula for Pnm.	R5: Ch 14: pg.No: 131-133
8.	1	Problems of Recurrence formula for Pnm.	R5: Ch 14: pg.No: 134-137
9.	1	Continuation of problems of Recurrence formula for Pnm.	R5: Ch 14: pg.No: 138-140
10.	1	Basic Combinatorial Numbers	R4: Ch 6: Pg.No: 314-316
11.	1	Continuation of Basic Combinatorial Numbers	R4: Ch 6: Pg.No: 316-318
12	1	Recapitulation and Discussion of possible	

		questions	
Total	**12Hours**		

Reference Book:
R3:. Russell Merris, (2003).Combinatorics, Second edition, John wiley & Sons, New York.
R4:. Veerarajan. T, (2007), Discrete Mathematics with Graph Theory and Combinatorics, Mc-Graw Hill companies,New Delhi.
R5. Sebastian M. Cioaba and M. Ram Murty, A First Course in graph Theory and Combinatorics, Hindhustan Book Agency Pvt. Ltd.

UNIT-II

1.	1	Generating functions	R1: Ch 3: Pg.No: 104-105
2.	1	Problems using Generating functions	R1: Ch 3: Pg.No: 111-114
3.	1	Continuation of Problems using Generating functions	R1: Ch 3: Pg.No: 114-116
4.	1	Continuation of Problems using Generating functions	R1: Ch 3: Pg.No: 117-120
5.	1	Continuation of Problems using Generating functions	R1: Ch 3: Pg.No: 120-123
6.	1	Recurrence relations	R1: Ch 3: Pg.No:107-110
7.	1	Problems using Recurrence relations	R1: Ch 3: Pg.No:128-130
8.	1	Continuation of Problems using Recurrence relations	R1: Ch 3: Pg.No:131-133
9	1	Continuation of Problems using Recurrence relations	R1: Ch 3: Pg.No:134-138
10	1	Bell's formula.	R1: Ch 3: Pg.No:139-140

11	1	Continuation of Bell’s formula.	R1: Ch 3: Pg.No:140-142
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

Reference Book:

1. Balakrishnan V.K., (1995). Theory and problems of Combinatorics, Schaums outline series,McGraw Hill Professional.

UNIT-III

1	1	Multinomial	R3: Ch 1: Pg.No: 69
2	1	Multinomial theorem	R3: Ch 1: Pg.No: 70-71
3	1	Examples of Multinomial theorem	R3: Ch 2: Pg.No:72-74
4	1	Continuation of Inclusion and Exclusion principle.	R3: Ch 2: Pg.No:75-76
5	1	Inclusion and Exclusion principle.	R1: Ch 2: Pg.No: 47
6	1	Examples of Inclusion and Exclusion principle	R2: Ch 8: 328-330
7	1	Examples of Inclusion and Exclusion principle	R2: Ch 8: 330-333
8	1	Continuation of Examples of Inclusion and Exclusion principle	R2: Ch 8: 333-335
9	1	Continuation of Examples of Inclusion and Exclusion principle	R1: Ch 2: Pg.No: 54-56
10	1	Multinomial	R3: Ch 1: Pg.No: 69
11	1	Examples on multinomial	T1: Ch5: Pg.No. 55-58
12	1	Recapitulation and Discussion of possible questions	
Total	**12Hours**		

Textbook:

1. Krishnamurthy, V. (2002), Combinatorics: Theory and Applications, East West Press Pvt. Ltd.

References:

1. Balakrishnan V.K., (1995). Theory and problems of Combinatorics, Schaums outline series,McGraw Hill Professional.
2. Alan tucker, (2002). Applied Combinatorics, 4e, John wiley & Sons, New York.
3. Russell Merris, (2003).Combinatorics, Second edition, John wiley & Sons, New York.

UNIT-IV

1	1	Euler function	R6: Ch:10.Pg.No:92
2	1	Problems related to Euler function	R6: Ch:10.Pg.No:93-94
3	1	Permutations with forbidden positions	R3:Ch 3: Pg.No: 183-185
4	1	Continuation of Permutations with forbidden positions	R3:Ch 3: Pg.No: 186-187
5	1	The Menage Problem	R6: Ch:10.Pg.No:95
6	1	Continuation of Menage Problem	R6: Ch:10.Pg.No:96-97
7	1	Continuation of Menage Problem	R6: Ch:10.Pg.No:98-99
8	1	Continuation of Menage Problem	R6: Ch:10.Pg.No: 100-103
9	1	Continuation of Menage Problem	R6: Ch:10.Pg.No:103-105
10	1	Euler function	R6: Ch:10.Pg.No:92
11	1	Continuation of Euler function	R6: Ch:10.Pg.No:93-95
12	1	Recapitulation and Discussion of possible questions	
Total	**12 Hours**		

Reference Book:

R3. Russell Merris, (2003).Combinatorics, Second edition, John wiley & Sons, New York.

R6. J. H. Van Lint and R.M. Wilson ,(2001) A Course in Combinatorics, Second Edition, Cambridge University Press, New Delhi.

UNIT-V

1	1	Problem of Fibonacci	R5: Ch 2: Pg.No: 47

2	1	Continuation of Problem of Fibonacci	R5: Ch 2: Pg.No: 48-51
3	1	Necklace problem	R3: Ch: 3: Pg.No: 191-193
4	1	Continuation of Necklace problem	R3: Ch: 3: Pg.No: 194-196
5	1	Burnside's lemma.	R6: Ch:10.Pg.No: 94
6	1	Continuation of Burnside's lemma.	R6: Ch:10.Pg.No: 95-98
7	1	Theorems and examples of Burnside's lemma.	R3: Ch: 3: Pg.No:197-200
8	1	Theorems and examples of Burnside's lemma.	R3: Ch: 3: Pg.No:200-203
9	1	Recapitulation and Discussion of possible questions	
10	1	Discussion on Previous ESE Question Papers	
11	1	Discussion on Previous ESE Question Papers	
12	1	Discussion on Previous ESE Question Papers	
Total	**12 Hours**		

Text Book:

T1: Herstein.I. N.,(2010). Topics in Algebra, Second edition, Wiley and sons Pvt Ltd, Singapore.

Reference Book:

R3. Russell Merris, (2003).Combinatorics, Second edition, John wiley & Sons, New York.

R5. Sebastian M. Cioaba and M. Ram Murty, A First Course in graph Theory and Combinatorics, Hindhustan Book Agency Pvt. Ltd.

R6. J. H. Van Lint and R.M. Wilson ,(2001) A Course in Combinatorics, Second Edition, Cambridge University Press, New Delhi

Total no. of Hours for the Course: 60 hours

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DEPARTMENT OF MATHEMATICS

Subject: COMBINATORICS Semester :III L T P C
Subject Code: 16MMU305B Class :II-M.Sc Mathematics 4 0 0 4

UNIT I

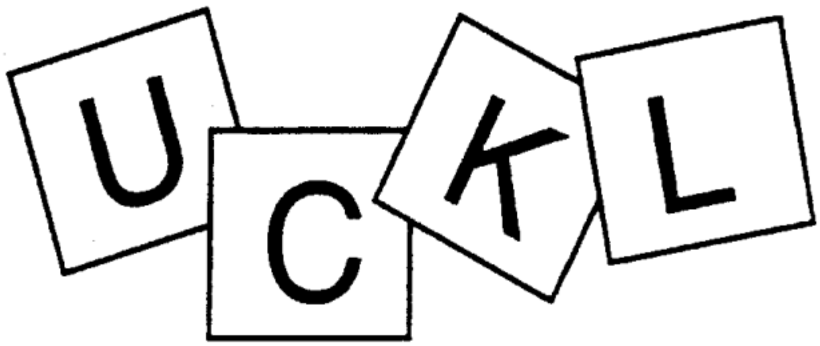
Basic Combinatorial Numbers – Stirling numbers of the second kind – Recurrence formula for Pnm.

REFERENCES:

1. Russell Merris, (2003).Combinatorics, Second edition, John wiley & Sons, New York.
2. Veerarajan. T, (2007), Discrete Mathematics with Graph Theory and Combinatorics, Mc- Graw Hill companies,New Delhi.
3. Sebastian M. Cioaba and M. Ram Murty, A First Course in graph Theory and Combinatorics, Hindhustan Book Agency Pvt. Ltd.

UNIT-I

THE FUNDAMENTAL COUNTING PRINCIPLE



How many different four-letter words, including nonsense words, can be produced by rearranging the letters in LUCK? In the absence of a more inspired approach,there is always the brute-force strategy: Make a systematic list. Once we become convinced that Fig. 1.1.1 accounts for every possible rearrangement and that no ‘‘word’’ is listed twice, the solution is obtained by counting the 24 words on the list.

While finding the brute-force strategy was effortless, implementing it required some work. Such an approach may be fine for an isolated problem, the like of which one does not expect to see again. But, just for the sake of argument, imagine yourself in the situation of having to solve a great many thinly disguised variations of this same problem. In that case, it would make sense to invest some effort in finding a strategy that requires less work to implement. Among the most powerful tools in this regard is the following commonsense principle.

Fundamental Counting Principle: Consider a (finite) sequence of decisions. Suppose the number of choices for each individual decision is independent of decisions made previously in the sequence. Then the number of ways to make the whole sequence of decisions is the product of these numbers of choices.

To state the principle symbolically, suppose ci is the number of choices for decision i. If, for 1 i < n, ciþ1 does not depend on which choices are made in

LUCK	LUKC	LCUK	LCKU	LKUC	LKCU
ULCK	ULKC	UCLK	UCKL	UKLC	UKCL
CLUK	CLKU	CULK	CUKL	CKLU	CKUL
KLUC	KLCU	KULC	KUCL	KCLU	KCUL

Figure 1.1.1. The rearrangements of LUCK

decisions 1 ; . . . ; i, then the number of different ways to make the sequence of decisions is $c_1 x\ c_2 x \ldots x\ c_n$.

Let's apply this principle to the word problem we just solved. Imagine yourself in the midst of making the brute-force list. Writing down one of the words involves a sequence of four decisions. Decision 1 is which of the four letters to write first, so $c_1 = 4$. (It is no accident that Fig. 1.1.1 consists of four rows!) For each way of making decision 1, there are $c_2 = 3$ choices for decision 2, namely which letter to write second. Notice that the specific letters comprising these three choices depend on how decision 1 was made, but their number does not. That is what is meant by the number of choices for decision 2 being independent of how the previous decision is made. Of course, $c_3 = 2$, but what about c4? Facing no alternative, is it correct to say there is ''no choice'' for the last decision? If that were literally true, then c4 would be zero. In fact, c4 ¼ 1. So, by the fundamental counting principle, the number of ways to make the sequence of decisions, i.e., the number of words on the final list, is $c_1 \text{ x } c_2 \text{ x } c_3 \text{ x } c_4 = 4\text{x } 3 \text{ x } 2 \text{ x } 1$:

The product n x (n 1) x (n- 2) x….x2 x 1 is commonly written n! and read n-factorial: The number of four-letter words that can be made up by rearranging the letters in the word LUCK is 4! =24.

What if the word had been LUCKY? The number of five-letter words that can be produced by rearranging the letters of the word LUCKY is 5! =120. A systematic list might consist of five rows each containing 4! = 24 words.

Suppose the word had been LOOT? How many four-letter words, including nonsense words, can be constructed by rearranging the letters in LOOT? Why not apply the fundamental counting principle? Once again, imagine yourself in the midst of making a brute-force list. Writing down one of the words involves a sequence of four decisions. Decision 1 is which of the three letters L, O, or T to write first. This time, c_1 = 3. But, what about c_2? In this case, the number of choices for decision 2 depends on how decision 1 was made! If, e.g., L were chosen to be the first letter, then there would be two choices for the second letter, namely O or T. If, however, O were chosen first, then there would be three choices for the second decision, L, (the second) O, or T. Do we take c_2 = 2 or c_2 = 3? The answer is that the fundamental counting principle does not apply to this problem (at least not directly).

The fundamental counting principle applies only when the number of choices for decision i + 1 is independent of how the previous i decisions are made.

To enumerate all possible rearrangements of the letters in LOOT, begin by distinguishing the two O's. maybe write the word as LOoT. Applying the fundamental counting principle, we find that there are 4! = 24 different-looking four-letter words that can be made up from L, O, o, and T.

LOoT	LOTo	LoOT	LoTO	LTOo	LToO
OLoT	OLTo	OoLT	OoTL	OTLo	OToL
oLOT	oLTO	oOLT	oOTL	oTLO	oTOL
TLOo	TLoO	TOLo	TOoL	ToLO	ToOL

Figure 1.1.2. Rearrangements of LOoT.

Among the words in Fig. 1.1.2 are pairs like OLoT and oLOT, which look different only because the two O's have been distinguished. In fact, every word in the list occurs twice, once with ''big O'' coming before ''little o'', and once the other way around. Evidently, the number of different words (with indistinguishable O's) that can be produced from the letters in LOOT is not 4! but 4 !=2 ¼ 12.

What about TOOT? First write it as TOot. Deduce that in any list of all possible rearrangements of the letters T, O, o, and t, there would be 4 ! ¼ 24 different-looking words. Dividing by 2 makes up for the fact that two of the letters are O's. Dividing by 2 again makes up for the two T's. The result, 24=ð2 2Þ ¼ 6, is the number of different words that can be made up by rearranging the letters in TOOT. Here they are

TTOO TOTO TOOT OTTO OTOT OOTT

All right, what if the word had been LULL? How many words can be produced by rearranging the letters in LULL? Is it too early to guess a pattern? Could the number we're looking for be 4 !=3 ¼ 8? No. It is easy to see that the correct answer must be 4. Once the position of the letter U is known, the word is completely determined. Every other position is filled with an L. A complete list is ULLL, LULL, LLUL, LLLU.

To find out why 4!/3 is wrong, let's proceed as we did before. Begin by distinguishing the three L's, say L1, L2, and L3. There are 4! different-looking words that can be made up by rearranging the four letters L1, L2, L3, and U. If we were to make a list of these 24 words and then erase all the subscripts, how many times would, say, LLLU appear? The answer to this question can be obtained from the fundamental counting principle! There are three decisions: decision 1 has three choices, namely which of the three L's to write first. There are two choices for decision 2 (which of the two remaining L's to write second) and one choice for the third decision, which L to put last. Once the subscripts are erased, LLLU would appear 3! times on the list. We should divide 4 ! ¼ 24, not by 3, but by 3 ! ¼ 6. Indeed, 4 !=3 ! ¼ 4 is the correct answer.

Whoops! if the answer corresponding to LULL is 4!/3!, why didn't we get 4!/2! for the answer to LOOT? In fact, we did: 2 ! = 2.

Are you ready for MISSISSIPPI? It's the same problem! If the letters were all different, the answer would be 11!. Dividing 11! by 4! makes up for the fact that there are four I's. Dividing the quotient by another 4! compensates for the four S's. Dividing that quotient by 2! makes up for the two P's. In fact, no harm is done if that quotient is divided by 1 ! = 1 in honor of the single M. The result is

11 !/(4 ! 4 ! 2 ! 1 !)= 34, 650

(Confirm the arithmetic.) The 11 letters in MISSISSIPPI can be (re)arranged in 34,650 different ways.*

There is a special notation that summarizes the solution to what we might call the ''MISSISSIPPI problem.''

Definition. The *multinomial coefficient*

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!r_2!\cdots r_k!},$$

where $r_1 + r_2 + \cdots + r_k = n$.

So, "multinomial coefficient" is a *name* for the answer to the question, how many n-letter "words" can be assembled using r_1 copies of one letter, r_2 copies of a second (different) letter, r_3 copies of a third letter, ..., and r_k copies of a kth letter?

Example. After cancellation,

$$\binom{9}{4,3,1,1} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 1 \times 1}$$
$$= 9 \times 8 \times 7 \times 5 = 2520.$$

Therefore, 2520 different words can be manufactured by rearranging the nine letters in the word SASSAFRAS. □

Example.

Suppose you wanted to determine the number of positive integers that exactly divide $n = 12$. That isn't much of a problem; there are six of them, namely, 1, 2, 3, 4, 6, and 12. What about the analogous problem for $n = 360$ or for $n = 360,000$? Solving even the first of these by brute-force list making would be a lot of work. Having already found another strategy whose implementation requires a lot less work, let's take advantage of it.

Consider $360 = 2^3 \times 3^2 \times 5$, for example. If $360 = dq$ for positive integers d and q, then, by the uniqueness part of the *fundamental theorem of arithmetic*, the prime factors of d, together with the prime factors of q, are precisely the prime factors of 360, multiplicities included. It follows that the prime factorization of d must be of the form $d = 2^a \times 3^b \times 5^c$, where $0 \le a \le 3$, $0 \le b \le 2$, and $0 \le c \le 1$. Evidently, there are four choices for a (namely 0, 1, 2, or 3), three choices for b, and two choices for c. So, the number of possibile d's is $4 \times 3 \times 2 = 24$. □

COMBINATORIAL IDENTITIES

$C(n,r) = \binom{n}{r}$ is the same as multinomial coefficient $\binom{n}{r, n-r}$. In fact, $C(n,r)$ is commonly called a *binomial* coefficient.* Given that binomial coefficients are special cases of multinomial coefficients, it is natural to wonder whether we still need a separate name and notation for n-choose-r. On the other hand, it turns out that multinomial coefficients can be expressed as products of binomial coefficients. Thus, one could just as well argue for discarding the multinomial coefficients!

Theorem. *If $r_1 + r_2 + \cdots + r_k = n$, then*

$$\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}.$$

Proof. Multinomial coefficient $\binom{n}{r_1, r_2, \ldots, r_k}$ is the number of n-letter "words" that can be assembled using r_1 copies of one "letter", say A_1; r_2 copies of a second, A_2; and so on, finally using r_k copies of some kth character, A_k. The theorem is proved by counting these words another way and setting the two (different-looking) answers equal to each other.

Think of the process of writing one of the words as a sequence of k decisions. Decision 1 is which of n spaces to fill with A_1's. Because this amounts to selecting r_1 of the n available positions, it involves $C(n, r_1)$ choices. Decision 2 is which of the remaining $n - r_1$ spaces to fill with A_2's. Since there are r_2 of these characters, the second decision can be made in any one of $C(n - r_1, r_2)$ ways. Once the A_1's and A_2's have been placed, there are $n - r_1 - r_2$ positions remaining to be filled, and A_3's can be assigned to r_3 of them in $C(n - r_1 - r_2, r_3)$ ways, and so on. By the fundamental counting principle, the number of ways to make this sequence of decisions is the product

$$C(n, r_1) \times C(n - r_1, r_2) \times C(n - r_1 - r_2, r_3) \times \cdots \times C(n - r_1 - r_2 - \cdots - r_{k-1}, r_k).$$

(Because $r_1 + r_2 + \cdots + r_k = n$, the last factor in this product is $C(r_k, r_k) = 1$.)

Chu's Theorem.* *If $n \geq r$, then*

$$\sum_{k=0}^{n} C(k,r) = C(r,r) + C(r+1,r) + C(r+2,r) + \cdots + C(n,r)$$
$$= C(n+1, r+1)$$

(*where* $\sum_{k=0}^{n} C(k,r) = \sum_{k=r}^{n} C(k,r)$ *because* $C(k,r) = 0$, $k < r$).

Proof. Replace $C(r,r)$ with $C(r+1, r+1)$ and use Pascal's relation repeatedly to obtain

$$C(r+1, r+1) + C(r+1, r) = C(r+2, r+1),$$
$$C(r+2, r+1) + C(r+2, r) = C(r+3, r+1),$$

and so on, ending with

$$C(n, r+1) + C(n, r) = C(n+1, r+1).$$ ■

FOUR WAYS TO CHOOSE

From its combinatorial definition, n-choose-r is the number of different r-element subsets of an n-element set. Because two subsets are equal if and only if they contain the same elements, $\binom{n}{r}$ depends on *what* elements are chosen, not when. In

computing $C(n, r)$, the *order* in which elements are chosen is irrelevant. The $C(5, 2) = 10$ two-element subsets of $\{L, U, C, K, Y\}$ are

$$\{L, U\}, \{L, C\}, \{L, K\}, \{L, Y\}, \{U, C\}, \{U, K\}, \{U, Y\}, \{C, K\}, \{C, Y\}, \{K, Y\},$$

where, e.g., $\{L, U\} = \{U, L\}$. There are, of course, circumstances in which order is important.

Example. Consider all possible "words" that can be produced using two

letters from the word LUCKY. By the fundamental counting principle, the number of such words is 5×4, twice $C(5, 2)$, reflecting the fact that order is important. The 20 possibilities are

$$\begin{array}{l} LU, LC, LK, LY, UC, UK, UY, CK, CY, KY, \\ UL, CL, KL, YL, CU, KU, YU, KC, YC, YK. \end{array} \qquad \square$$

Definition. Denote by $P(n, r)$ the number of *ordered* selections of r elements chosen from an n-element set.

By the fundamental counting principle,

$$\begin{aligned} P(n, r) &= n(n-1)(n-2)\cdots(n-[r-1]) \\ &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \\ &= r!C(n, r). \end{aligned}$$

There is another way to arrive at this last identity: We may construe $P(n,r)$ as the number of ways to make a sequence of just two decisions. Decision 1 is which of the r elements to select, without regard to order, a decision having $C(n,r)$ choices. Decision 2 is how to order the r elements once they have been selected, and there are $r!$ ways to do that. By the fundamental counting principle, the number of ways to make the sequence of two decisions is $C(n,r) \times r! = P(n,r)$.

Example. Suppose nine members of the Alameda County School Boards Association meet to select a three-member delegation to represent the association at a statewide convention. There are $C(9,3) = 84$ different ways to choose the delegation from those present. If the bylaws stipulate that each delegation be comprised of a delegate, a first alternate, and a second alternate, the nine members can comply from among themselves in any one of $P(9,3) = 3!C(9,3) = 504$ ways. □

Example. Door prizes are a common feature of fundraising luncheons. Suppose each of 100 patrons is given a numbered ticket, while its duplicate is placed in a bowl from which prize-winning numbers will be drawn. If the prizes are \$10, \$50, and \$150, then (assuming winning tickets are not returned to the bowl) a total of $P(100,3) = 970,200$ different outcomes are possible. If, on the other hand, the three prizes are each \$70, then the order in which the numbers are drawn is immaterial. In this case, the number of different outcomes is $C(100,3) = 161,700$. □

Both $C(n,r)$ and $P(n,r)$ involve situations in which an object can be chosen at most once. We have been choosing *without replacement*. What about choosing *with* replacement? What if we recycle the objects, putting them back so they can be chosen again? How many ways are there to choose r things from n things with replacement? The answer depends on whether order matters. If it does, the answer is easy. The number of ways to make a sequence of r decisions each of which has n choices is n^r.

Example. How many different two-letter "words" can be produced using the "alphabet" $\{L, U, C, K, Y\}$? If there are no restrictions on the number of times a letter can be used, then $5^2 = 25$ such words can be produced; i.e., there are 25 ways to choose 2 things from 5 with replacement if order matters. In addition to the 20 words from Example 1.6.1, there are five new ones, namely, LL, UU, CC, KK, and YY. □

Theorem. *The number of different ways to choose r things from n things with replacement if order doesn't matter is* $C(r+n-1, r)$.

Proof. As in Example 1.6.6, there is a one-to-one correspondence between selections and $[r+(n-1)]$-letter words consisting of r tally marks and $n-1$ dashes. The number of such words is $C(r+n-1, r)$. ■

Possible Questions

Name of the Faculty : Pavithra. K

Class : II – M.Sc. Mathematics

Subject Name : Combinatorics

Subject Code : 16MMP305B

UNIT-I

1. State and Prove the Pascal's Identity.

2. From a club consistiong of 6 men and 7 womwn, in how many ways can we select a committee of
   a) 3 men and 4 women
   b) 4 persons which has atleast one women
   c) 4 persons that has atmost one man
   d) 4 persons that has persons of both sexes
   e) 4 persons so that two specific members are not included.

3.The number of different permutations of n objects which include n_1 identical objects of type I, n_2 identical objects of type II,… and n_k identical objects of type k is equal to $\frac{n!}{n1!n2!n3!\ldots nk!}$, where $n_1+n_2+n_3+\ldots+n_k= n$.

4. When repetition of n elements contained in a set is permitted in r-permutations, then prove that the number of r-permutations is n^r.
5. There are 3 Piles of identical red, blue and green Balls, where each pile contains atleast 10 balls. In how many ways can 10 balls be selected.
   i) if there is no restriction
   ii) if atleast one red ball must be selected
   iii) if atleast one red ball, atleast 2 blue balls and atleast 3 green balls must be selected.
   iv) if exactly one red ball and atleast one blue ball must be selected.
   v) if at most one red ball is selected.
6. State and prove the pigeonhole principle.
7. Let A be a set consisting of n elements (n≥2). Then prove that there are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd Permutations.
8. State and Prove the Vandermonde's Identity.
9. Prove the number of onto functions in $F_{m,n}$ is $n!S(m,n)$.
10. The number of circular permutations of n objects is (n-1)!.

11. Prove that i) $P(n,n) = P(n,n-1)$

    ii) $P(n,r) = (n-r+1)\, P(n, r-1)$

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

UNIT-I

Question	Option-1	Option-2	Option-3	Option-4	Answer
$^6P_1$ is equal to	18	12	6	0	6
$^6P_4$ is equal to	36	360	6	4	360
If $^nC_{12} = {}^nC_6$ value of n is	12	14	16	18	18
An arrangement of a finite number of objects taken some or all at a time is called their	A.P	Combination	Sequence	permutation	permutation
Letters of SAP taken all at a time can be written in	2 ways	6 ways	24 ways	120 ways	6 ways
6ι/8ι	23743	65	56	$^1/_{56}$	$^1/_{56}$
Factorial of a positive integer n is nι =	n(n-1)(n-2)(n-3)…3.2.1	(n-1)(n-2)(n-3)…3.2.1	(n-1)n(n-1)(n-2)(n-3)…3.2.1	(n-2)(n-3)…3.2.1	n(n-1)(n-2)(n-3)…3.2.1
$^nP_2 = 30 \rightarrow n =$	6	4	5	720	6
Number of word that can be formed out of letters of word BOTSWANA is	8ι	2ι	8ι.2ι	8ι/2ι	8ι/2ι
1/20.19.18.17 =	20ι/16ι	16ι/20ι	1/16ι	20ι	16ι/20ι
Value of $^{10}C_4$ x $^8C_3$ is	12760	11760	10760	9760	11760
For a negative integer n, factorial nι	is unique	is 0	does not exist	is 1	does not exist
1/12.11.10 =	1/12ι	9ι/12ι	12ι/9ι	12ι	9ι/12ι
^nC_r .rι =	$^{n+1}P_r$	$^nP_{r+1}$	$^{n-1}P_r$	^nP_r	^nP_r
Letters of CHORD taken all at a time can be written in	2 ways	6 ways	24 ways	120 ways	120 ways
$^5C_2 + {}^5C_1 =$	$^6C_2$	$^6C_1$	$^5C_2$	$^5C_1$	$^6C_1$
10.9/2.1 =	1/10ι	2ι8ι/10ι	10ι/2ι8ι	10ι	10ι/2ι8ι
Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?	210	1050	25200	21400	25200
In how many ways can the letters of the word 'LEADER' be arranged?	72	144	360	720	360
In how many ways a committee, consisting of 5 men and 6 women can be formed from 8 men and 10 women?	266	5040	11760	86400	11760
In how many ways can a group of 5 men and 2 women be made out of a total of 7 men and 3 women?	63	90	126	45	63
In how many different ways can the letters of the word 'MATHEMATICS' be arranged so that the vowels always come together?	10080	4989600	120960	13546	120960
How many 4-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?	40	400	5040	2520	5040
In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?	159	194	205	209	209
How many 3-digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9, which are divisible by 5 and none of the digits is repeated?	5	10	15	20	20
A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw?	32	48	96	64	64
In how many different ways can the letters of the word 'DETAIL' be arranged in such a way that the vowels occupy only the odd positions?	32	36	48	60	36

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DEPARTMENT OF MATHEMATICS
Subject: COMBINATORICS Semester :III L T P C
Subject Code: 16MMU305B Class :II-M.Sc Mathematics 4 0 0 4

UNIT II

Generating functions – Recurrence relations- Bell's formula.

REFERENCES:

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Generating Functions

On a superficial level, a generating function is simply a way to exhibit a sequence of numbers $a_0, a_1, a_2, \ldots$. However, the act of writing

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

has some surprising consequences. Because the left-hand side of this expression *looks* like a function, it is tempting to treat the right-hand side as if it were one, a "mistake" having some interesting implications.

Those sequences $a_0, a_1, a_2, \ldots$ with the property that a_n is a polynomial function of n are characterized in the first section. *Ordinary* generating functions and some of their properties are discussed in Section 4.2. Applications, e.g., to Newton's binomial theorem, are the focus of Section 4.3. Section 4.4 deals with some variations on the generating function idea. Techniques for solving recurrences occupy the final section.

Definition. The notation $\{a_n\}$ is used to denote the sequence $a_0, a_1, a_2, \ldots$.

Note that the first *number* in the sequence $\{a_n\}$ is the zeroth *term*, a_0. The 4th number in Sequence (4.1) is $27 = a_3$. (While this system may seem awkward now, it will simplify our work later on.)

Definition. The sequence $\{a_n\}$ is *arithmetic* if, for all $n \geq 0$, the difference

$a_{n+1} - a_n = d$ is a constant, independent of n.

An arithmetic sequence satisfies the pattern, or *recurrence*, $a_{n+1} = a_n + d$, $n \geq 0$. *Given* that Sequence (4.1) comprises an arithmetic sequence, then $d = 7$, and there can be no ambiguity about the 5th number. It is $a_4 = 27 + 7 = 34$. So far, so good. Now you know how to exhibit intelligence by the standards of the last century.

What if you were asked to determine, not a_4, but a_{400}? Using the recurrence $a_{400} = a_{399} + 7$ is not much help. The key to *solving* Sequence (4.1) is to think of it symbolically, as

$$6, 6+7, (6+7)+7, (6+7+7)+7, \ldots$$

From this perspective, it is clear that a_n is a sum of $n+1$ numbers, one 6 and n 7's, i.e., $a_n = 7n+6$. So, $a_{400} = 7 \times 400 + 6 = 2806$. This solution illustrates the tension between mathematics and computation. Doing the arithmetic at each step leads to $a_{400} = a_{399} + 7$. Not doing the arithmetic reveals a pattern leading to the mathematical abstraction $a_n = 7n+6$.

More generally, every arithmetic sequence takes the form

$$a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \ldots$$

So, the nth term of an arithmetic sequence (the $(n+1)$st number in the sequence) is

$$a_n = dn + a_0. \tag{4.2}$$

An expression like Equation (4.2), in which a_n is given as an explicit function of n, is called a *closed formula*, or *solution*, for $\{a_n\}$.

Associated with the sequence $\{a_n\}$ is a natural function of the nonnegative integers, namely, $f(n) = a_n$, $n \geq 0$. Conversely, to any function f of the nonnegative integers, there corresponds a natural sequence, namely, $\{f(n)\}$. Informally, a closed formula for $\{a_n\}$ is a "nice" description of the corresponding function, e.g., $\{a_n\}$ is arithmetic if and only if it corresponds to a function of the form $f(n) = dn + a_0$, i.e., to a polynomial of degree (at most) 1.

Consider the sequence $\{n^2\}$, i.e.,

$$0, 1, 4, 9, 16, 25, \ldots$$

It is *not* arithmetic. For one thing, the closed formula $f(n) = n^2$ is a nonlinear polynomial. For another, while a_{n+1} is obtained from a_n by adding an odd number, that number changes. The difference, $a_{n+1} - a_n = (n+1)^2 - n^2 = 2n+1$, is not constant.

Definition. Let $\{a_n\}$ be a fixed but arbitrary sequence. Its *difference sequence*, denoted $\{\Delta a_n\}$, is defined by $\Delta a_n = a_{n+1} - a_n$, $n \geq 0$.

Perhaps $\Delta(a_n)$ would be a better notation. Certainly, Δa_n should not be confused with a product of Δ and a_n. Whatever the notation, $\{a_n\}$ is an arithmetic sequence if and only if its difference sequence $\{\Delta a_n\}$ is constant, that is, $\Delta a_n = d$, $n \geq 0$. When $a_n = n^2$, $\Delta a_n = 2n+1$. In other words, $\{\Delta n^2\} = \{2n+1\}$.

If $f(n) = a_n$, $n \geq 0$, then $\Delta a_n = \Delta f(n) = f(n+1) - f(n)$. It seems that

$$\Delta f(n) = \frac{f(n+1) - f(n)}{1} \tag{4.3}$$

is a kind of *discrete derivative*.

It can be revealing to look at a sequence and its difference sequence (*also called sequence of differences) side by side. In the case of* $\{n^2\}$, *the side-by-side comparison looks like this:*

$$\begin{array}{ccccccccc} 0, & 1, & 4, & 9, & 16, & 25, & 36, & 49, & \ldots \\ 1, & 3, & 5, & 7, & 9, & 11, & 13, & \ldots & \end{array}$$

Evidently, the difference sequence of the sequence of perfect squares is the sequence of odd numbers. More useful to our present objective is the fact that the difference sequence is arithmetic. This suggests looking at the difference sequence of a difference sequence. The following *difference array* gives two generations of difference sequences for $\{n^2\}$:

$$\begin{array}{ccccccccc} 0, & 1, & 4, & 9, & 16, & 25, & 36, & 49, & \ldots \\ 1, & 3, & 5, & 7, & 9, & 11, & 13, & \ldots & \\ 2, & 2, & 2, & 2, & 2, & 2, & \ldots & & \end{array}$$

Denote by $\{\Delta^2 a_n\}$ the difference sequence of the difference sequence, Then, e.g., $\{\Delta^2 n^2\} = \{2\}$, the constant sequence each of whose terms is 2. In general,

$$\begin{aligned} \Delta^2 a_n &= \Delta a_{n+1} - \Delta a_n \\ &= a_{n+2} - 2a_{n+1} + a_n, \end{aligned} \tag{4.4}$$

$$\begin{array}{cccccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & \cdots \\ \Delta a_0, & \Delta a_1, & \Delta a_2, & \Delta a_3, & \Delta a_4, & \Delta a_5, & \cdots & \\ \Delta^2 a_0, & \Delta^2 a_1, & \Delta^2 a_2, & \Delta^2 a_3, & \Delta^2 a_4, & \cdots & & \\ \Delta^3 a_0, & \Delta^3 a_1, & \Delta^3 a_2, & \Delta^3 a_3, & \cdots & & & \\ & & \cdots & & & & & \end{array}$$

Figure 4.1.1. A generic difference array.

Letting $\Delta^0 a_n = a_n$ and $\Delta^1 a_n = \Delta a_n$, we can define $\Delta^{r+1} a_n = \Delta(\Delta^r a_n)$ for all $r \geq 1$, i.e.,

$$\Delta^{r+1} a_n = \Delta^r a_{n+1} - \Delta^r a_n, \qquad r \geq 1. \tag{4.5}$$

Successive generations of difference sequences are displayed in Fig. 4.1.1.

4.1.4 Example. The difference array for $\{n^3\}$ is

$$\begin{array}{ccccccccc} 0, & 1, & 8, & 27, & 64, & 125, & 216, & 343, & \dots \\ 1, & 7, & 19, & 37, & 61, & 91, & 127, & \dots & \\ 6, & 12, & 18, & 24, & 30, & 36, & \dots & & \\ 6, & 6, & 6, & 6, & 6, & \dots & & & \end{array}$$

While one could write out additional rows, there isn't much point in doing so. If the fourth row, corresponding to $\{\Delta^3 n^3\}$, is constant, then each row after the fourth consists entirely of zeros. But, is the fourth row really constant? Let's see.

If $\{a_n\}$ is any sequence, then $\Delta a_n = a_{n+1} - a_n$. From Equation (4.4), $\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n$. From Equation (4.5),

$$\begin{aligned} \Delta^3 a_n &= \Delta^2 a_{n+1} - \Delta^2 a_n \\ &= (a_{n+3} - 2a_{n+2} + a_{n+1}) - (a_{n+2} - 2a_{n+1} + a_n) \\ &= a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n. \end{aligned} \tag{4.6}$$

Substituting $a_n = n^3$ into Equation (4.6) yields

$$\begin{aligned} \Delta^3 n^3 &= (n+3)^3 - 3(n+2)^3 + 3(n+1)^3 - n^3 \\ &= (n^3 + 9n^2 + 27n + 27) - 3(n^3 + 6n^2 + 12n + 8) + 3(n^3 + 3n^2 + 3n + 1) - n^3 \\ &= 6 \end{aligned}$$

for all n. □

Is it too early to guess a pattern? Might $\{\Delta^4 a_n\}$ be constant when $a_n = n^4$? More generally, might $\{\Delta^r a_n\}$ be constant when $\{a_n\} = \{n^r\}$. If so, can the constant be predicted in advance? Before we can answer such questions, we need to know a little more about $\{\Delta^r a_n\}$.

4.1.5 Lemma. *If $\{a_n\}$ is a sequence then, for all $n \geq 0$,*

$$\Delta^r a_n = \sum_{t=0}^{r} (-1)^{r+t} C(r,t) a_{n+t}.$$

Proof. The identity has already been established for small r (see, e.g., Equations (4.4) and (4.6)). From Equation (4.5) and induction on r,

$$\begin{aligned}
\Delta^{r+1} a_n &= \Delta^r a_{n+1} - \Delta^r a_n \\
&= \sum_{t=0}^{r} (-1)^{r+t} C(r,t) a_{n+1+t} - \sum_{t=0}^{r} (-1)^{r+t} C(r,t) a_{n+t} \\
&= \sum_{t=1}^{r+1} (-1)^{r+t-1} C(r,t-1) a_{n+t} + \sum_{t=0}^{r} (-1)^{r+t-1} C(r,t) a_{n+t} \\
&= a_{n+r+1} + \sum_{t=1}^{r} (-1)^{r+t-1} [C(r,t-1) + C(r,t)] a_{n+t} + (-1)^{r-1} a_n \\
&= \sum_{t=0}^{r+1} (-1)^{r+1+t} C(r+1,t) a_{n+t}.
\end{aligned}$$

■

With the help of Lemma 4.1.5, we can answer our questions about $\{\Delta^r n^r\}$.

4.1.6 Theorem. *Suppose r is a fixed but arbitrary positive integer. Let* $a_n = n^r$, $n \geq 0$. *Then* $\Delta^r a_n = r!$, $n \geq 0$.

Proof. By Lemma 4.1.5,

$$\begin{aligned}\Delta^r n^r &= \sum_{t=0}^{r}(-1)^{r+t}C(r,t)(n+t)^r \\ &= \sum_{t=0}^{r}(-1)^{r+t}C(r,t)\sum_{m=0}^{r}C(r,m)n^{r-m}t^m \\ &= \sum_{m=0}^{r}C(r,m)n^{r-m}\sum_{t=0}^{r}(-1)^{r+t}C(r,t)t^m \\ &= \sum_{m=0}^{r}C(r,m)n^{r-m}r!S(m,r)\end{aligned}$$

by Stirling's identity. Because the Stirling number of the second kind, $S(m,r)$, is equal to 0 when $m < r$ and equal to 1 when $m = r$, the only surviving term in the final summation is $C(r,r)n^{r-r}r! = r!$. ■

4.1.7 Corollary. *Suppose m is a fixed but arbitrary positive integer. Then* $\Delta^{r+1}n^m = 0$ *for all* $n \geq 0$ *and all* $r \geq m$.

Proof. From Theorem 4.1.6, $\Delta^{m+1}n^m = \Delta(\Delta^m n^m) = \Delta m! = m! - m! = 0$. If $r > m$, then $\Delta^{r+1}n^m = \Delta^{r-m}(\Delta^{m+1}n^m) = \Delta^{r-m}0 = 0$. ■

Corollary 4.1.7 remains valid when n^m is replaced by any polynomial in n of degree m.

4.1.8 Theorem. *Let m be a fixed but arbitrary positive integer. Suppose f is a polynomial of degree m. If* $a_n = f(n)$, $n \geq 0$, *then* $\Delta^{r+1}a_n = 0$ *for all* $n \geq 0$ *and all* $r \geq m$.

Proof. Suppose $\{y_n\}$ and $\{z_n\}$ are sequences. Let b and c be numbers. Then

$$\begin{aligned}\Delta(by_n + cz_n) &= (by_{n+1} + cz_{n+1}) - (by_n + cz_n)\\ &= b(y_{n+1} - y_n) + c(z_{n+1} - z_n)\\ &= b\,\Delta y_n + c\,\Delta z_n.\end{aligned}$$

So, Δ is linear. Therefore,

$$\begin{aligned}\Delta^2(by_n + cz_n) &= \Delta(\Delta(by_n + cz_n))\\ &= \Delta(b\,\Delta y_n + c\,\Delta z_n)\\ &= b\,\Delta^2 y_n + c\,\Delta^2 z_n,\end{aligned}$$

and, more generally, $\Delta^k(by_n + cz_n) = b\,\Delta^k y_n + c\,\Delta^k z_n$ for all $k \geq 1$. If $f(x) = c_0x^m + c_1x^{m-1} + \cdots + c_m$ and $a_n = f(n)$, $n \geq 0$, then

$$\begin{aligned}\Delta^{r+1}a_n &= \Delta^{r+1}f(n)\\ &= \Delta^{r+1}(c_0n^m + c_1n^{m-1} + \cdots + c_m)\\ &= c_0\,\Delta^{r+1}n^m + c_1\,\Delta^{r+1}n^{m-1} + \cdots + c_m\,\Delta^{r+1}(1)\\ &= 0\end{aligned}$$

by linearity and Corollary 4.1.7. ■

4.1.10 Theorem. *Let $\{a_n\}$ be a sequence. If the mth difference sequence $\{\Delta^m a_n\}$ is constant, i.e., if $\Delta^{m+1}a_n = 0$ for all $n \geq 0$, then there exists a polynomial f of degree at most m such that $a_n = f(n)$ for all $n \geq 0$. Moreover,*

$$f(n) = \sum_{r=0}^{m} C(n,r)\,\Delta^r a_0. \tag{4.10}$$

Proof. Equation (4.10) follows either by replacing $n^{(r)}/r!$ with $C(n, r)$ in Equation (4.9) or by replacing a_n with $f(n)$ in Equation (4.7). ■

Theorem 4.1.10 is a "strong" converse of Theorem 4.1.8 because it does more than establish the existence of f. Equation (4.10) is an explicit formula; it is the "easy way" to find f (short of solving a linear system of equations). Note, in particular, that if $\{\Delta^m a_n\}$ is a constant sequence then f, hence $\{a_n\}$, is completely determined by the $m+1$ numbers $a_0, \Delta a_0, \ldots, \Delta^m a_0$ from the first column (or *leading edge* of the difference array for $\{a_n\}$.

4.1.11 Example. Suppose $\{a_n\}$ is a sequence the first column of whose difference array is 1, 5, 4, 6, with zeros thereafter. Compute a_{100}. Solution: Let $f(n) = a_n$, $n \geq 0$. Because $\Delta^r a_0 = 0$, $r \geq 4$, Equation (4.10) yields

$$\begin{aligned} a_n &= \sum_{r=0}^{3} C(n, r)\, \Delta^r a_0 \\ &= C(n, 0) \times 1 + C(n, 1) \times 5 + C(n, 2) \times 4 + C(n, 3) \times 6 \\ &= 1 + 5n + 4n(n-1)/2 + 6n(n-1)(n-2)/6 \\ &= 1 + 5n + 2n^2 - 2n + n^3 - 3n^2 + 2n \\ &= n^3 - n^2 + 5n + 1, \end{aligned}$$

so $a_{100} = 10^6 - 10^4 + 500 + 1 = 990{,}501$. □

4.1.12 Example. Let m be a fixed positive integer and $\{a_n\}$ be the sequence whose nth term is $a_n = n^m$, $n \geq 0$. From Equation (4.10) (and Corollary 4.1.7), we obtain

$$n^m = \sum_{r=0}^{m} C(n, r)\, \Delta^r a_0.$$

On the other hand, from Corollary 2.2.3,

$$n^m = \sum_{r=1}^{m} r! S(m, r) C(n, r),$$

4.1.13 Example. Perhaps the techniques of this section can be made to yield additional new insights about Stirling numbers of the second kind. Consider, e.g., the sequence

$$S(k,0),\ S(k+1,1),\ S(k+2,2),\ S(k+3,3),\ldots,$$

where k is fixed but arbitrary. (The previous example involved $S(m,r)$ where m was fixed. This time, $m-r=k$ is fixed.) When $k=2$, the first few terms of the sequence are

$$0,\ 1,\ 7,\ 25,\ 65,\ 140,\ 266,\ 462,\ldots$$

The initial portion of the difference array for this sequence is illustrated in Fig. 4.1.3. If the fourth difference sequence, corresponding to the fifth row of the difference array, really is the constant sequence $\{3\}$ then, from Equation (4.10), there is some polynomial f_2 of degree 4 such that $S(2+n,n)=f_2(n)$ for all $n\geq 0$. Moreover, from the leading edge of Fig. 4.1.3,

$$\begin{aligned} f_2(n) &= C(n,1)+5C(n,2)+7C(n,3)+3C(n,4)\\ &= [C(n,1)+C(n,2)]+4[C(n,2)+C(n,3)]+3[C(n,3)+C(n,4)]\\ &= C(n+1,2)+4C(n+1,3)+3C(n+1,4)\\ &= [C(n+1,2)+C(n+1,3)]+3[C(n+1,3)+C(n+1,4)]\\ &= C(n+2,3)+3C(n+2,4). \end{aligned}$$

4.2.1 Definition. The sequence $\{a_n\}$ is *geometric* if it satisfies a recurrence of the form $a_{n+1}=da_n$, $n\geq 0$, where d is a constant, independent of n.

Evidently, the nth term of a generic geometric sequence is given by the closed formula $a_n = a_0 \times d^n$, $n \geq 0$.

Consider the sequence

$$3,4,22,46,178,454,\ldots \tag{4.11}$$

defined by $a_0 = 3$, $a_1 = 4$, and $a_n = a_{n-1} + 6a_{n-2}$, $n \geq 2$. This one is neither arithmetic nor geometric. While there is a simple closed formula for a_n, its discovery requires either an inspired guess or a new approach.

4.2.2 Definition. The (ordinary) *generating function* for the sequence $\{a_n\}$ is

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \tag{4.12}$$

Generating functions come in assorted sizes, shapes, and flavors. The pattern inventory* $W_G(x_1, x_2, \ldots, x_n)$ is one kind of generating function; Equation (4.12) is another. The name "generating function" is more than a little curious. The pattern inventory doesn't generate anything; it is *generated by* the cycle index polynomial.† Moreover, as we are about to see, it is useful to view $g(x)$ as something *other* than a function!

If $g(x)$ is the generating function for Sequence (4.11), then

$$\begin{array}{rcl} g(x) & = & 3 + 4x + 22x^2 + 46x^3 + 178x^4 + \cdots + a_nx^n + \cdots \\ -xg(x) & = & \quad - 3x - 4x^2 - 22x^3 - 46x^4 - \cdots - a_{n-1}x^n - \cdots \\ -6x^2g(x) & = & \quad\quad - 18x^2 - 24x^3 - 132x^4 - \cdots - 6a_{n-2}x^n - \cdots \end{array}$$

Summing these three equations produces

$$g(x)(1 - x - 6x^2) = 3 + x.$$

(The recurrence guarantees that $[a_n - a_{n-1} - 6a_{n-2}]x^n = 0$, $n \geq 2$.) Evidently,

$$g(x) = 3 + 4x + 22x^2 + 46x^3 + 178x^4 + 454x^5 + \cdots \tag{4.13a}$$

$$= \frac{3 + x}{1 - x - 6x^2}. \tag{4.13b}$$

A typical backpacker will sacrifice many things to decrease weight. Freeze-dried food is a good example. Why carry water (even as a constituent of food) if it is available at campsites? Equation (4.13b) might be viewed as a freeze-dried version of Equation (4.13a). (If you had to stuff $g(x)$ into a backpack, which version would you prefer?)

Okay. Imagine yourself at a campsite. What is the easy way to resurrect (or *generate*) the sequence $\{a_n\}$ from $g(x) = (3+x)/(1-x-6x^2)$? One perfectly acceptable alternative is long division. Another is to factor the denominator as $(1+2x)(1-3x)$, so that

$$g(x) = (3+x)\left(\frac{1}{1+2x}\right)\left(\frac{1}{1-3x}\right).$$

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad (4.14)$$

so

$$\frac{1}{1+2x} = 1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots \qquad (4.15)$$

and

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \cdots. \qquad (4.16)$$

Therefore, $g(x)$ can be expressed as the (formidable *looking*) product

$$g(x) = (3+x)(1 - 2x + 4x^2 - 8x^3 + \cdots)(1 + 3x + 9x^2 + 27x^3 + \cdots).$$

A third, easier approach is to make use of the method of partial fractions*, i.e., to write

$$g(x) = \frac{3+x}{1-x-6x^2} = \frac{3+x}{(1+2x)(1-3x)} = \frac{1}{1+2x} + \frac{2}{1-3x}.$$

Together with Equations (4.15) and (4.16), this yields

$$\begin{aligned} g(x) &= [1 + (-2x) + (-2x)^2 + \cdots] + 2[1 + 3x + (3x)^2 + \cdots] \\ &= [1 - 2x + 4x^2 - 8x^3 + \cdots] + [2 + 6x + 18x^2 + 54x^3 + \cdots] \\ &= 3 + 4x + 22x^2 + 46x^3 + \cdots, \end{aligned}$$

and

$$g(x) = [1 + (-2x) + (-2x)^2 + \cdots] + 2[1 + 3x + (3x)^2 + \cdots]$$

yields

$$a_n = (-2)^n + 2(3^n), \qquad n \geq 0. \tag{4.17}$$

It is striking, but is it right? Without checking for convergence, what justifies manipulating the generating "function" just as if it were an honest-to-goodness function? It would appear that our derivation may have some holes in it. On the other hand, *independently of where it came from*, we can prove that Equation (4.17) is a valid identity.

Define a sequence $\{b_n\}$ by $b_n = 2(3^n) + (-2)^n$, $n \geq 0$. Then $b_0 = 2(3^0) + (-2)^0 = 3 = a_0$ and $b_1 = 2(3) - 2 = 4 = a_1$. So, the first two numbers in the sequences $\{a_n\}$ and $\{b_n\}$ are the same. If we could prove that the sequences satisfy the same recurrence, i.e., if $b_n = b_{n-1} + 6b_{n-2}$, $n \geq 2$, it would follow that $b_n = a_n$ for all n.

Observe that

$$2(3^n) = 6(3^{n-1}) = 2(3^{n-1}) + 4(3^{n-1}) = 2(3^{n-1}) + 6[2(3^{n-2})]$$

and

$$(-2)^n = -2(-2)^{n-1} = (-2)^{n-1} - 3(-2)^{n-1} = (-2)^{n-1} + 6(-2)^{n-2}.$$

and the generating function has been reassembled. There is more. Obscured by the rush to compute is a closed formula for a_n. Comparing the coefficients of x^n in

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

4.2.3 Definition. A *formal power series* in x is an infinite sum of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, where the *coefficients* $a_0, a_1, a_2, a_3, \ldots$ are fixed constants. It is sometimes convenient to give a shorthand name to a power series, writing, e.g.,

$$\begin{aligned} g(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= \sum_{n\geq 0} a_n x^n. \end{aligned}$$

Multiplication of polynomials also extends to formal power series:

$$\begin{aligned} &(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots) \\ &\quad = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots. \end{aligned}$$

In general,

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n, \tag{4.19a}$$

where

$$c_n = \sum_{r=0}^{n} a_r b_{n-r}. \tag{4.19b}$$

Most of the algebraic manipulations associated with polynomials extend naturally to formal power series. (If all but finitely many of its coefficients are zero, a formal power series *is* a polynomial.) If

$$f(x) = \sum_{n\geq 0} a_n x^n \qquad \text{and} \qquad g(x) = \sum_{n\geq 0} b_n x^n,$$

then $f(x) = g(x)$ if and only if $a_n = b_n$ for all $n \geq 0$. If c and d are constants, then $h(x) = cf(x) + dg(x)$ is the formal power series defined by

$$h(x) = c\sum_{n\geq 0} a_n x^n + d\sum_{n\geq 0} b_n x^n = \sum_{n\geq 0}(ca_n + db_n)x^n. \tag{4.18}$$

4.2.4 Example. Observe that

$$(1 + x + x^2 + x^3 + x^4 + \cdots)(1 - x) = 1. \tag{4.20}$$

In fact, this product is just a variation of Equation (4.14). □

It is instructive to turn Example 4.2.4 around. How *do* we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots ?$$

One justification comes from calculus:

$$\begin{aligned} g(x) &= 1 + x + x^2 + x^3 + x^4 + \cdots \\ &= \lim_{n\to\infty} 1 + x + x^2 + \cdots + x^{n-1} \\ &= \lim_{n\to\infty} \frac{1 - x^n}{1 - x} \\ &= \frac{1}{1-x}, \end{aligned}$$

$x \in (-1, 1)$, because $\lim_{n\to\infty} x^n = 0$ whenever $|x| < 1$. But, this argument depends upon viewing $g(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$ as a function, precisely the perspective we are trying to avoid. What we want is a justification that depends only on the algebra of formal power series.

Possible Questions

Name of the Faculty : Pavithra. K

Class : II – M.Sc. Mathematics

Subject Name : Combinatorics

Subject Code : 16MMP305B

UNIT-II

1. Solve the recurrence relation $a_n=4a_{n-1}-4a_{n-2}+(n+1)2^n$.
2. Use the method of generating function to solve the recurrence relation

   $a_{n+1} - 8a_n +16a_{n-1}=4^n;n\geq1;\ a_0=1,a_1=8.$
3. Form a recurrence relation satisfied by $a_n= \sum_{k=1}^{n} k^2$ and find the value of $\sum_{k=1}^{n} k^2$.
4. Use the method of generating function to solve the recurrence relation $a_n=4a_{n-1}+3n.2^n$;$n\geq1$, given that $a_0= 4$.
5. Use the method of generating function to solve the recurrence relation $a_n=3a_{n-1}+1;n\geq1$, given that

   $a_0= 1$.
6. Solve the recurrence relation $a_r-7a_{r-1}+10a_{r-2}= 3^r$ given that $a_0=0$ and $a_1=1$.
7. Find a formula for the general term F_n of the Fibonacci sequence 0,1,1,2,3,5,8,13,…
8. Use the method of generating function to solve the recurrence relation $a_{n+1}-a_n=3^n;n\geq0$, given that $a_0= 1$.
9. Solve the recurrence relation $a_n= 2a_{n-1}+2_n$; $a_0=2$.
10. State and Prove the Bells formula.
11. Solve the recurrence relation $a_{n+2}-6a_{n+1}+9a_n= 3(2^n)+7(3^n),n\geq 0$, given that $a_0=1,a_1=4$

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

UNIT-II

Question	Option-1	Option-2	Option-3	Option-4	Answer
There are 30 people in a group. If all shake hands with one another , how many handshakes are possible?	870	435	30!	29! + 1	435
In how many ways can we arrange the word 'FUZZTONE' so that all the vowels come together?	1440	6	2160	4320	2160
In Cricket League, in first round every team plays a match with every other team. 9 teams participated in the Cricket league. How many matches were played in the first round?	36	9!	9!-1	72	36
How many combinations are possible while selecting four letters from the word 'SMOKEJACK' with the condition that 'J' must appear in it?	81	41	8!/2!	3!/2!	41
In a room there are 2 green chairs, 3 yellow chairs and 4 blue chairs. In how many ways can Raj choose 3 chairs so that at least one yellow chair is included?	3	30	84	64	64
In a room there are 2 green chairs, 3 yellow chairs and 4 blue chairs. In how many ways can Raj choose 3 chairs so that at least one yellow chair is included?	${}^{17}C_9$ x 9! X 8!	${}^{17}C_9$ x 8! X 7!	8! X 7!	${}^{17}C_8$ x 8! X 9!	${}^{17}C_9$ x 8! X 7!
On a railway line there are 20 stops. A ticket is needed to travel between any 2 stops. How many different tickets would the government need to prepare to cater to all possibilities?	760	190	72	380	380
In Daya's bag there are 3 books of History, 4 books of Science and 2 books of Maths. In how many ways can Daya arrange the books so that all the books of same subject are together?	9	6	8640	1728	1728
Mayur travels from Mumbai to Jammu in 7 different ways. But he is allowed to return to Mumbai by any way except the one he used earlier. In how many ways can he complete his journey?	49	42	48	6	42
Without repetition, using digits 2, 3, 4, 5, 6, 8 and 0, how many numbers can be made which lie between 500 and 1000?	70	147	60	90	90
If Suraj doesn't want three vowels together, then in how many, can he arrange letters of the word 'MARKER'?	500	720	240	360	240
How many words can be formed by using all letters of word ALIVE.	86	95	105	120	120
How many 3-letter words can be formed out of the letters of the word 'CORPORATION', if repetition of letters is not allowed?	990	336	720	504	336
In how many different ways can the letters of the word 'GEOMETRY' be arranged so that the vowels always come together?	720	4320	2160	40320	4320
In how many ways can the letters of the word ENCYCLOPAEDIA be arranged such that vowels only occupy the even positions?	453600	128000	478200	635630	453600
In how many ways can the letters of the word INDIA be arranged, such that all vowels are never together?	48	42	28	36	42
Evaluate 30!28!	970	870	770	670	870
Evaluate permutation equation 59P3	195052	195053	195054	185054	195054
Evaluate permutation 5P5	120	110	98	24	120
Evaluate permutation equation 75P2	5200	5300	5450	5550	5550
Evaluate combination 100C97=100!(97)!(3)!	161700	151700	141700	131700	161700
Evaluate combination 100C100	10000	1000	100	1	1
How many words can be formed by using all letters of TIHAR	100	120	140	160	120
In how many words can be formed by using all letters of the word BHOPAL	420	520	620	720	720
In how many way the letter of the word "APPLE" can be arranged	20	40	60	80	60
In how many ways can the letters of the CHEATER be arranged	20160	2520	360	80	2520
In how many way the letter of the word "RUMOUR" can be arranged	2520	480	360	180	180
How many words can be formed from the letters of the word "SIGNATURE" so that vowels always come together.	17280	4320	720	80	17280
In how many ways can the letters of the word "CORPORATION" be arranged so that vowels always come together.	5760	50400	2880	80	50400
In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there	109	128	138	209	209
How many words can be formed from the letters of the word "AFTER", so that the vowels never comes together.	48	52	72	100	72
In a Cricket cup total 153 matches were played and every two teams played exactly one match with each other. So what were the total number of teams participating in Cricket Cup ?	15	16	17	18	18
A box contains 4 red, 3 white and 2 blue balls. Three balls are drawn at random. Find out the number of ways of selecting the balls of different colours	12	24	48	168	24
A bag contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the bag, if at least one black ball is to be included in the draw	64	128	132	222	64
From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done	456	556	656	756	756
The Permutations of {a,b,c,d,e,f,g} are listed in lex order. What permutations are just before and just after bacdefg?	Before:agedbc, After:bacdfge	Before:agf edcb, After:badcef g	Before:agf ebcd, After: bacedg	Before:agf edcb, After:bacdeg	Before:agf edcb, After:bacdeg
The number of four letter words that can be formed from the let ters in BUBBLE (each letter occurring at most as many times as it occurs in BUBBLE) is	72	74	76	78	72
The number of ways to seat 3 boys and 2 girls in a row if each boy must sit next to atleast one girl is	36	48	148	184	36
How many different rearrangements are there of the letters in the word BUBBLE?	40	120	50	70	120

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DEPARTMENT OF MATHEMATICS

Subject: COMBINATORICS	**Semester :III**	**L T P C**
Subject Code: 16MMU305B	**Class :II-M.Sc Mathematics**	**4 0 0 4**

UNIT III
Multinomial – Multinomial theorem- Inclusion and Exclusion principle.

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THE PRINCIPLE OF INCLUSION AND EXCLUSION

Suppose $f : A \to A$ is a function from a set A to itself, i.e., suppose the domain and range of f are equal. If A is the set of real numbers, it is not difficult to find functions like $f(x) = e^x$ that are one-to-one but not onto and functions like

$f(x) = x^3 - x$ that are onto but not one-to-one. This kind of thing cannot, happen if A is finite. Specifically, $f \in F_{n,n}$ is one-to-one if and only if it is onto. (The same thing cannot be said about functions in $F_{m,n}$ when $m \neq n$. There are $P(5,3) = 60$ one-to-one functions in $F_{3,5}$, but $F_{3,5}$ contains no onto functions at all; there are $3!S(5,3) = 150$ onto functions in $F_{5,3}$, but $F_{5,3}$ does not contain a single one-to-one function.)

2.3.1 Definition. A one-to-one function in $F_{n,n}$ is called a *permutation*. The subset of $F_{n,n}$ consisting of the one-to-one (onto) functions is denoted S_n.

Of the n^n functions in $F_{n,n}$, $P(n,n) = n!$ are one-to-one, so $o(S_n) = n!$. (The same conclusion follows by counting the $n!S(n,n) = n!$ onto functions in $F_{n,n}$.) Recognizing the permutations in $F_{n,n}$ is easy. They are the sequences in which no integer occurs twice.

2.3.2 Example. $F_{2,2} = \{(1,1),(1,2),(2,1),(2,2)\}$ and $S_2 = \{(1,2),(2,1)\}$. Of the $3^3 = 27$ functions in $F_{3,3}$, only $3! = 6$ are permutations: $S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$. □

A *fixed point* of $f \in F_{n,n}$ is an element $i \in \{1,2,\ldots,n\}$ such that $f(i) = i$. Some of the deepest theorems in mathematics involve fixed points. Fixed points of permutations comprise the foundation of Pólya's theory of enumeration (discussed in Chapter 3). For the present, we will focus on permutations that have no fixed points.

2.3.3 Definition. A permutation with no fixed points is called a *derangement*. The number of derangements in S_n is denoted $D(n)$.

There is only one permutation $p \in S_1$, and it is completely defined by $p(1) = 1$. Because 1 is a fixed point of p, there are no derangements in S_1, i.e., $D(1) = 0$. There is one derangement in S_2, namely (2, 1), so $D(2) = 1$. In S_3 (see Example 2.3.2), the derangements are (2, 3, 1) and (3, 1, 2), so $D(3) = 2$. While one can tell at a glance whether a sequence represents a permutation, it usually takes more than a glance to recognize a derangement. Identification of functions with sequences has many advantages, but picking out derangements is not one of them.

The easiest (and most illuminating) way to evaluate $D(n)$ involves a new idea. Let's begin by recalling our discussion of the second counting principle: If A and B are disjoint, then $o(A \cup B) = o(A) + o(B)$. If A and B are not disjoint, then $o(A \cup B) < o(A) + o(B)$, because $o(A) + o(B)$ counts every element of $A \cap B$ twice. (See Fig. 2.3.1.) Compensating for this double counting yields the formula

$$o(A \cup B) = o(A) + o(B) - o(A \cap B).$$

What if there are three sets? Then

$$\begin{aligned} o(A \cup B \cup C) &= o(A \cup [B \cup C]) \\ &= o(A) + o(B \cup C) - o(A \cap [B \cup C]). \end{aligned}$$

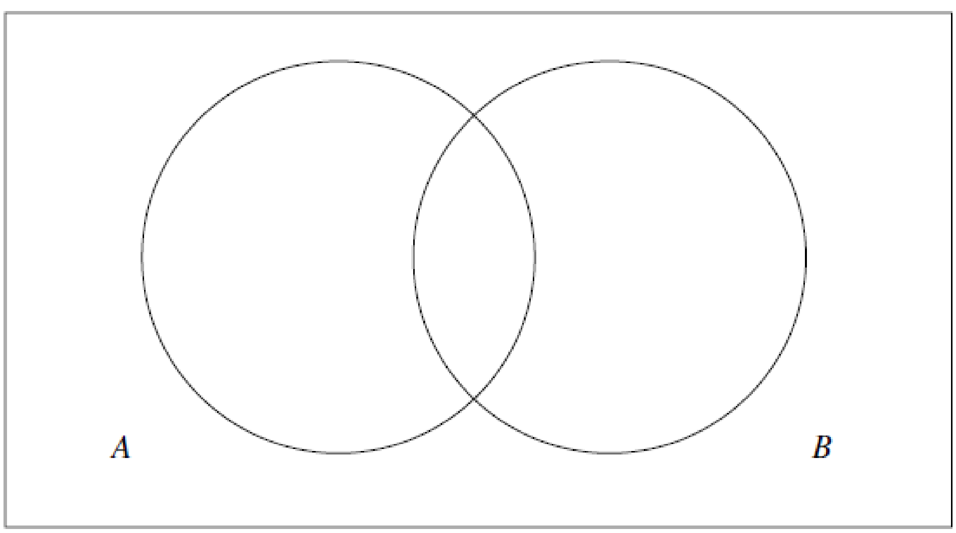


Figure 2.3.1

Applying Equation (2.11) to $o(B \cup C)$ gives

$$o(A \cup B \cup C) = o(A) + [o(B) + o(C) - o(B \cap C)] - o(A \cap [B \cup C]). \qquad (2.12)$$

Because $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we can apply Equation (2.11) again to obtain

$$o(A \cap [B \cup C]) = o(A \cap B) + o(A \cap C) - o(A \cap B \cap C). \qquad (2.13)$$

Finally, a combination of Equations (2.12) and (2.13) produces

$$o(A \cup B \cup C) = [o(A) + o(B) + o(C)] - [o(A \cap B) + o(A \cap C) + o(B \cap C)] + o(A \cap B \cap C). \qquad (2.14)$$

Adding back $o(A \cap B \cap C)$ is, perhaps, the most interesting part of Equation (2.14). It seems the subtracted term *over* compensates for elements that belong to all three sets. An element of $A \cap B \cap C$ is counted seven times in Equation (2.14), the first three times with a plus sign, then three time with a minus sign, and then once more with a plus. (See Fig 2.3.2.)

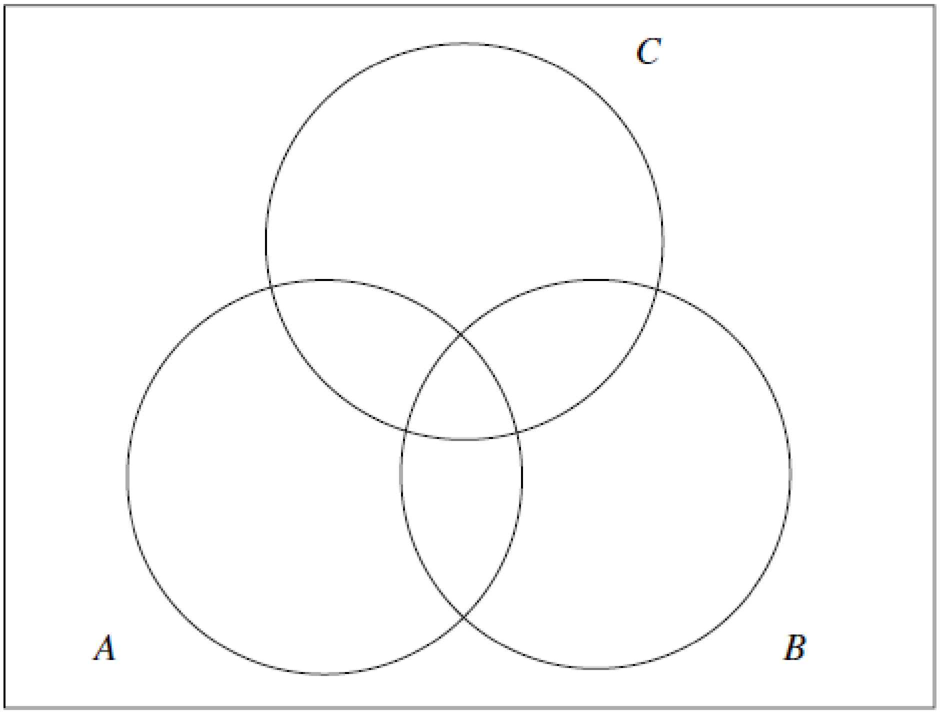
2.3.4 Example. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{2, 4, 6, 7\}$, then $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7\}$, a set of seven elements. Let's see what Equation (2.14) produces. Because $o(A) = o(B) = o(C) = 4$,

$$o(A) + o(B) + o(C) = 12.$$

In this case, it just so happens that $o(A \cap B) = o(A \cap C) = o(B \cap C) = 2$, so

$$o(A \cap B) + o(A \cap C) + o(B \cap C) = 6.$$

Finally, $A \cap B \cap C = \{4\}$, so $o(A \cap B \cap C) = 1$. Substituting these values into Equation (2.14) yields $o(A \cup B \cup C) = 12 - 6 + 1 = 7$.



Don't misunderstand. No one is suggesting that Equation (2.14) is the easiest way to solve *this* problem. The point of the example is merely to confirm that Equation (2.14) generates the correct solution! □

2.3.5 Principle of Inclusion and Exclusion (PIE). If $A_1, A_2 \ldots, A_n$ are finite sets, the cardinality of their union is

$$o\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} N_r, \tag{2.15}$$

Because $f \in Q_{r,n}$ if and only if f is a strictly increasing function, N_r is the sum of the cardinalities of the intersections of the sets taken r at a time. That is,

$$N_1 = \sum_{i=1}^{n} o(A_i), \qquad N_2 = \sum_{\substack{i,j=1 \\ i<j}}^{n} o(A_i \cap A_j), \qquad N_3 = \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} o(A_i \cap A_j \cap A_k),$$

and so on. Written out, Equations (2.15)–(2.16) look like this:

$$o(A_1 \cup \cdots \cup A_n) = \sum_{i} o(A_i) - \sum_{i<j} o(A_i \cap A_j) + \sum_{i<j<k} o(A_i \cap A_j \cap A_k) - \cdots.$$

where

$$N_r = \sum_{f \in Q_{r,n}} o\left(\bigcap_{i=1}^{r} A_{f(i)}\right).$$

Proof. Let x be a fixed but arbitrary element of $A_i \cup A_2 \cup \cdots \cup A_n$. Then x belongs to some k of the n sets. Without loss of generality, we may assume that x belongs to the *first* k sets, i.e., $x \in A_i$, $1 \leq i \leq k$, and $x \notin A_i$, $k < i \leq n$. Let's compute the contribution of x to N_r. For any $f \in Q_{r,n}$, $x \in \cap_{i=1}^{r} A_{f(i)}$ if and only if $f(r) \leq k$ if and only if $f \in Q_{r,k}$. Hence, the contribution of x to N_r is $o(Q_{r,k}) = C(k,r)$, $1 \leq r \leq k$. So, the contribution of x to the right-hand side of Equation (2.15) is

$$\sum_{r=1}^{k}(-1)^{r+1}C(k,r) = 1 - \sum_{r=0}^{k}(-1)^{r}C(k,r)$$
$$= 1$$

(because $\sum_{r=0}^{k}(-1)^{r}C(k,r) = [-1+1]^{k} = 0$). In other words, the right-hand side of Equation (2.15) counts every element of the union exactly once. ■

It may seem hard to believe that PIE could ever be *useful*. In fact, it is exactly the right tool for counting problems like the one in Example 2.3.4, where, for $1 \leq r \leq n$, "it just so happens" that

$$o\left(\bigcap_{i=1}^{r} A_{f(i)}\right)$$

is the same for all $f \in Q_{r,n}$. Let's illustrate with the derangement numbers. If $A_i = \{p \in S_n : p(i) = i\}$, $1 \leq i \leq n$, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is the set of permutations having at least one fixed print, so

$$D(n) = n! - o(A_1 \cup A_2 \cup \cdots \cup A_n).$$

Using the Principle of Inclusion and Exclusion,

$$D(n) = n! - \sum_{r=1}^{n}(-1)^{r+1}N_r.$$

To evaluate N_r on the right-hand side of Equation (2.17), let $f \in Q_{r,n}$. Then $p \in A_{f(1)} \cap A_{f(2)} \cap \cdots \cap A_{f(r)}$ if and only if the numbers $f(1), f(2) \ldots, f(r)$ are all fixed points of p. Because there are no restrictions on how p might permute the remaining $n-r$ numbers among themselves, there are exactly $(n-r)!$ permutations $p \in S_n$ that fix $f(i)$, $1 \leq i \leq r$, i.e.,

$$o(A_{f(1)} \cap A_{f(2)} \cap \cdots \cap A_{f(r)}) = (n-r)!,$$

for all $f \in Q_{r,n}$. It follows that $N_r = (n-r)!C(n,r) = n!/r!$. Thus, from Equation (2.17),

$$\begin{aligned} D(n) &= n! - \sum_{r=1}^{n} \frac{(-1)^{r+1} n!}{r!} \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + \frac{(-1)^n n!}{n!} \\ &= n! \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right]. \end{aligned} \tag{2.18}$$

Recall that the power series expansion

$$e^x = \sum_{n \geq 0} \frac{x^n}{n!}$$

is absolutely convergent for all x. Setting $x = -1$, we obtain the alternating series

$$\frac{1}{e} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots.$$

By the alternating-series test, the error in the estimate

$$\frac{1}{e} \doteq \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$$

is at most $1/(n+1)!$. (The notation "$\doteq$" means "approximately equal".) It follows that the error in the estimate

$$D(n) \doteq \frac{n!}{e} \tag{2.19}$$

is at most $1/(n+1)$, which is enough to prove the following.

2.3.6 Theorem. *The nth derangement number, $D(n)$, is the integer closest to $n!/e$.*

2.3.7 Example. From Equation (2.18),

$$\begin{aligned} D(4) &= 4!\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24}\right) \\ &= 24 - 24 + 12 - 4 + 1 \\ &= 9, \end{aligned}$$

whereas $4!/e \doteq 8.8291$. Similarly,

$$\begin{aligned} D(5) &= 5!\left(1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120}\right) \\ &= 120 - 120 + 60 - 20 + 5 - 1 \\ &= 44, \end{aligned}$$

while $5!/e \doteq 44.1455$. (It turns out that $D(n) > n!/e$ if n is even and $D(n) < n!/e$ if n is odd.) □

How many permutations $p \in S_n$ have exactly k fixed points? This is a job for the fundamental counting principle. There are $C(n,k)$ ways to choose the numbers to be fixed and $D(n-k)$ ways to derange the remaining $n-k$ "points". So, among the $n!$ permutations of S_n, $C(n,k) \times D(n-k)$ have exactly k fixed points.

Denote by $P(k)$ the fraction of permutations in S_n that have exactly k fixed points.* If we assume that n is enough larger than k for the estimate $D(n-k) \doteq (n-k)!/e$ to be valid, then

$$P(k) = \frac{C(n,k)D(n-k)}{n!} \doteq \frac{1}{k!e}. \tag{2.20}$$

It is proved in Section 3.3 that the *average* of the numbers of fixed points of the permutations in S_n is 1. Setting $k=1$ in Equation (2.20) shows that the fraction of permutations in S_n that have exactly 1 fixed point is $P(1) \doteq 1/e$.

2.3.8 Example. Let $F(p)$ be the number of fixed points of $p \in S_3 = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$. Then $F(1,2,3) = 3$, $F(1,3,2) = F(2,1,3) = F(3,2,1) = 1$, and $F(2,3,1) = F(3,1,2) = 0$. From these data, it is easy to see that the average number of fixed points is $[3+1+1+1+0+0]/6 = 1$, and easy to confirm that the fraction of permutations in S_3 having exactly one fixed point is $C(3,1)D(2)/6 = \frac{3}{6} = 0.5$. (The estimate $0.5 \doteq 1/e = 0.3678794\ldots$ afforded by Equation (2.20) when $n=3$ and $k=1$ is evidently not very good.)

It follows from Theorem 2.3.6 that $D(9) = 133,496$. From Equation (2.20), the fraction of permutations in S_{10} having exactly one fixed point is $C(10,1)D(9)/10! = D(9)/9! \doteq 0.3678792$, which compares more favorably with $1/e$. □

Let's see how the Principle of Inclusion and Exclusion might be used to produce new information about Stirling numbers of the second kind. Let $A_s = \{f \in F_{m,n} : f^{-1}(s) = \varnothing\}$, $1 \le s \le n$. Observe that no $f \in A_s$ can be onto. In fact, $g \in F_{m,n}$ is onto if and only if

$$g \notin A_1 \cup A_2 \cup \cdots \cup A_n.$$

2.3. The Principle of Inclusion and Exclusion

Therefore,

$$\begin{aligned} n!S(m,n) &= n^m - o(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= n^m - \sum_i o(A_i) + \sum_{i<j} o(A_i \cap A_j) \\ &\quad - \sum_{i<j<k} o(A_i \cap A_j \cap A_k) + \cdots . \end{aligned}$$

Now, A_n is the set of functions in $F_{m,n}$ that do not map anything to n. In fact, it would be very easy to confuse A_n with $F_{m,n-1}$. Certainly, $o(A_n) = (n-1)^m$. But, the number of functions in $F_{m,n}$ that map nothing to n is the same as the number of functions that map nothing to 1 or nothing to 2. In other words, $o(A_i) = (n-1)^m$, $1 \le i \le n$. Similarly, there is a one-to-one correspondence between the functions in $A_n \cap A_{n-1}$ and $F_{m,n-2}$. Thus, $o(A_n \cap A_{n-1}) = (n-2)^m$. Hence, $o(A_i \cap A_j) = (n-2)^m$, $1 \le i < j \le n$. Similarly, $o(A_i \cap A_j \cap A_k) = (n-3)^m$, $1 \le i < j < k \le n$, and so on. Substituting these values into Equation (2.21) yields

$$\begin{aligned} n!S(m,n) &= n^m - n(n-1)^m + C(n,2)(n-2)^m - C(n,3)(n-3)^m + \cdots \\ &= \sum_{s=0}^{n-1} (-1)^s C(n,s)(n-s)^m. \end{aligned} \tag{2.22}$$

Because $C(n, n-t) = C(n,t)$, replacing s with $n-t$ in Equation (2.22) yields

$$n!S(m,n) = \sum_{t=1}^{n} (-1)^{n-t} C(n,t) t^m.$$

It seems we have done nothing more than rediscover Stirling's identity (Corollary 2.2.4)!

Let's try something else, maybe an example from the intersection of combinatorics and number theory.

2.3.9 Definition. Let n be a positive integer. The *Euler totient function* $\varphi(n)$ is the number of positive integers $m \leq n$ such that m and n are relatively prime.

2.3.11 Theorem. *Suppose* $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, *where* $r_i > 0$, $1 \leq i \leq k$, *and* $p_1, p_2, \ldots, p_k$ *are distinct primes. Then*

$$\varphi(n) = n \prod_{i=1}^{k} \frac{p_i - 1}{p_i}.$$

Proof. Let $S = \{1, 2, \ldots, n\}$. Define

$$A_i = \left\{ p_i, 2p_i, 3p_i, \ldots, \left(\frac{n}{p_i}\right) p_i \right\}, \qquad 1 \leq i \leq k.$$

Then A_i is the subset of S consisting of the multiples of p_i. Moreover (just count its elements), $o(A_i) = n/p_i$. If $i \neq j$, then $A_i \cap A_j$ consists of those elements of S that are multiples of p_i and p_j and, therefore, of $p_i p_j$. So,

$$A_i \cap A_j = \{p_i p_j, 2p_i p_j, 3p_i p_j, \ldots, \left(\frac{n}{p_i p_j}\right) p_i p_j\}.$$

In particular, for $i < j$, $o(A_i \cap A_j) = n/(p_i p_j)$. If $i < j < k$, then $o(A_i \cap A_j \cap A_k) = n/(p_i p_j p_k)$, and so on.

If $1 \leq m \leq n$ (i.e., if $m \in S$), then the greatest common divisor of m and n is greater than 1 if and only if m and n have a common prime divisor if and only if $m \in A_1 \cup A_2 \cup \cdots \cup A_k$. So,

$$\begin{aligned}
\varphi(n) &= n - o(A_1 \cup A_2 \cup \cdots \cup A_k) \\
&= n - \sum_i o(A_i) + \sum_{i<j} o(A_i \cap A_j) - \sum_{i<j<k} o(A_i \cap A_j \cap A_k) + \cdots \\
&= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots\right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots\right) - \left(\frac{n}{p_1 p_2 p_3} + \cdots\right) + \cdots \\
&= \frac{n}{p_1 p_2 \cdots p_k}(E_k - E_{k-1} + E_{k-2} - \cdots + [-1]^k E_0),
\end{aligned}$$

where $E_t = E_t(p_1, p_2, \ldots, p_k)$ is the tth elementary symmetric function, $1 \leq t \leq k$. Because $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1) = E_k - E_{k-1} + E_{k-2} - \cdots + [-1]^k E_0$,

$$\varphi(n) = \frac{n}{p_1 p_2 \cdots p_k}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1). \qquad \blacksquare$$

2.3.12 Example. A favorite number of the Babylonians was $60 = 2^2 \times 3 \times 5$. By Theorem 2.3.11,

$$\begin{aligned}\varphi(60) &= 60\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{5-1}{5}\right)\\ &= 16.\end{aligned}$$

The 16 numbers less than 60 and relatively prime to 60 are 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, and 59. □

POSSIBLE QUESTIONS

1. State and prove the Inclusion –Exclusion Principle.

2. Find the number of integers between 1 and 2000 inclusive that are not divisible by 2,3, 5 or 7.

3. Let |A| = n and |B|= m and n≥ m. The number of onto functions f: A →B is given by m^n-[$n(m-1)^n - {}^nC_2(m-2)^n + {}^nC_3(m-3)^n + \ldots (-1)^m m$].

4. Using the principle of inclusion and exclusion find the number of prime numbers not exceeding 100.
5. State and prove the Binomial Theorem.

6. Use the principle of inclusion –exclusion to derive a formula for φ(n) when the prime factorization of n is $n = p_1^{a1} p_2^{a2} \ldots p_m^{am}$.
7. Show that the number of dearrangements of a set of n elements is given by, $D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}]$.
8. Using principle of inclusion –exclusion find the number of onto functions from a set with m elements to a set with n elements where m and n are positive integers with m≥n.

9. A survey of 150 college students reveals that 83 own automobiles, 97 own bikes, 28 own motorcycles, 53 own a car and a bike, 14 own a car and motorcycle, 7m own a bike and a motorcycle and 2 all three.

How many students own a bike and nothing else.

How many students do not own any of the three.

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DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

UNIT-III

Question	Option-1	Option-2	Option-3	Option-4	Answer
How many different rearrangements are there of the letters in the word TATARS if the two A's are never adjacent?	24	120	144	180	120
The number of partitions of X= {a,b,c,d}with a and b in the same block is	4	5	6	7	5
The number of partitions of X= {a,b,c,d,e,f,g}with a,b and c in the same block and c,d and e in the same block is	2	5	10	15	5
A class of 15 students is visiting the Louvre and the teacher wants to take a photograph of 5 of them lined up under the Mona Lisa. How many such photographs are possible?	P(15,5)	2	15	C(15,5)	P(15,5)
If n is an integer and n2 is odd ,then n is:	even	odd	even or odd	prime	odd
In how many ways can 5 balls be chosen so that 2are red and 3 are black	910	970	980	990	970
Pigeonhole principle states that A → B and A> Bthen:	f is not onto	f is not one-one	f is neither one-one nor onto	f may be one-one	f is not one-one
The number of distinct relations on a set of 3 elements is	8	9	18	512	512
In how many ways can a party of 7 persons arrange themselves around a circular table?	6!	5!	7!	8!	6!
In how many ways can a hungry student choose 3 toppings for his prize from a list of 10delicious possibilities?	100	120	130	110	120
A debating team consists of 3 boys and 2 girls.Find the number of ways they can sit in a row?	120	30	50	60	120
How many different words can be formed out of theletters of the word VARANASI?	64	120	403	720	720
How many permutations are there for the 8 letters a,b,c,d,e,f,g,h start with a.	8!	6!	7!	2!	8!
How many permutations are there for the 8 letters a,b,c,d,e,f,g,h end with h.	8!	6!	7!	2!	7!
How many permutations are there for the 8 letters a,b,c,d,e,f,g,h start with a and end with h.	8!	6!	7!	2!	6!
In how many ways can the symbols a,b,c,d,e,e,e,e,e be arranged so that no e is adjacent to another e?	14	24	36	72	24
What is the number of arrangements of all the six letters in the word PEEPER?	90	60	40	20	60
How many distinct four- digit integers can one make from the digits 1,3,3,7,7 and 8	90	60	40	20	90
How many different outcomes are possible when 5 dice are rolled?	522	252	520	220	252
In a group of 100people, several will have birth days in the same month. Atleast how many must have birth days in the same month.	10	9	8	7	9
How many positive integers not exceeding 1000 are divisible by 7 or 11?	221	223	220	229	220
In how many ways can five letters be choosen from the list A,B,…I? In how many ways can five letters be chosen.	9C5	9C6	5C9	6C9	9C5
A wife wants to present three shirts to her husband. At the shop the husband finds seven shirts of his liking. What is the number of choices available to the wife?	39	36	34	35	35
How many matricesof order 2x3can be formed,in which the digits from 0 to 9 occur not more than once.	10P5	10P6	10P10	1P10	10P6
How many four digit numbers can be formed using the seven digits 0,1,2…6 if repetitions are not allowed?	720	630	780	480	720
thenumber of circular permutations of n objects is	n!	(n-1)!	n!/2	n/2	(n-1)!
if $\lvert A\rvert$= n, then $\lvert P(A)\rvert$=	2n	n	n!/2	n!	2n
How many numbers are there between 1 and 65, which are divisibleby any one of 2,3 and 5	45	46	47	48	48
How many ways can we draw a club or a diamond from a pack of 2 cards.	26	15	13	25	26
In how many ways one candraw an ace or a king from an ordinary deck of playing cards.	4	8	6	2	8
How many ways can we get a sum of 7 or 11 when two distinguishable dice arerolled	6	2	4	8	8
How many ways can we get an even sum when two distinguishable dice arerolled	6	8	18	12	18
How many possible outcomes are there when we roll a pair of dice one red and one green.	6	30	36	23	36
In how many different ways one can answerall the true or false test consisting of 4 question.	2	4	8	16	16
Find the number of licenceplates that can be made where each plate contains two distinguish letters followed by three different digits.	4,68,000	6,84,000	8,64,000	6,48,000	4,68,000
In a railway compartment, 6 seats are vacant on a bench.In how many ways can3 passengers can sit on them.	210	230	120	150	120
If there are 12 boys and 16 girls in a class, find the number of ways of selecting one student as class representative.	12	16	26	28	28
How many different four letter words can be formed out of the word LOGARITHMS if repetition of letters is not allowed.	5010	5040	4010	4050	5040
How many different 8-bit strings are there that begin and end with 1.	36	64	32	62	64
How many different 8-bit string are there that end with 0 1 1 1 .	2	4	8	16	16
How many different 2-digit numbers can be made using the digits 0 to 9 when repetition is allowed.	90	80	100	120	100
How many different 2-digit numbers can be made using the digits 0 to 9 when repetition is not allowed.	90	80	100	120	90
How many words an be constructed with three English alphabet with repetition.	17576	17570	15676	15346	17576
How many words an be constructed with three English alphabet without repetition.	15600	16500	12600	13600	15600
There are 10-true false questions on a examination. In how many ways all the questionsbe answered.	1000	1024	2410	2100	1024
In how many ways can we geta sum of 4 or 8 when two distinguishable dice arerolled.	8	6	5	3	8
The value of 0! Is	1	0	2 n		1
The value of 1! Is	1	0	2 n		1
the value of 5!is	100	120	160	150	120
The value of 10!/8! Is	100	120	90	60	90

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DEPARTMENT OF MATHEMATICS

Subject: COMBINATORICS	Semester :II	L T P C
Subject Code: 16MMU305B	Class :II-M.Sc Mathematics	4 0 0 4

UNIT IV
Euler function –Permutations with forbidden positions –the Menage Problem.

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FUNCTION COMPOSITION

Let $f : D \to R$ be a function. While there is general agreement that D should be called the *domain* of f, not everyone concurs that *range* is the proper name for R; some authors use "range" to denote the set $\{f(x) : x \in D\}$.

3.1.1 Definition. Let $f : D \to R$ be a function. The *image* of f is the set $f(D) = \{f(x) : x \in D\}$, sometimes denoted image(f).

Note that image(f) $= f(D) \subset R$, with equality if and only if f is onto. If $f \in F_{m,n}$, then $f(D)$ is the set of numbers that appear in the sequence $(f(1), f(2), \ldots, f(m))$.

Suppose $f : D \to R$ and $g : A \to B$ are functions. If $f(D) \subset A$, then the *composition* of g and f is the function $g \circ f : D \to B$ defined by $g \circ f(x) = g(f(x))$. (In calculus, the derivative of a composition of functions is described by the *chain rule*.)

There is an awkward "backwardness" about the standard notation for function composition. It is occasioned by the fact that we read from left to right but evaluate a composition from right to left: The rule of assignment $g \circ f$ is determined by first applying f and then applying g. The French school has eliminated the difficulty by putting the function on the right, i.e., writing xf rather than $f(x)$. In the French scheme, cumbersome expressions like $g \circ f(x)$ and $g(f(x))$ become xfg. Because this right-handed notation has not been widely accepted in the United States, we will stick with the familiar $f(x)$.

3.1.2 Example. If $f \in F_{2,5}$ and $g \in F_{5,3}$, where might $g \circ f$ be found? Because f is applied first, $g \circ f$ shares the domain of f. Because g is applied second, image($g \circ f$) $\subset$ image(g); so $g \circ f$ shares the range of g. Therefore, $g \circ f \in F_{2,3}$. To take a specific example, let $f = (3, 4) \in F_{2,5}$ and $g = (3, 3, 2, 1, 3) \in F_{5,3}$. Then

$$g \circ f(1) = g(f(1)) = g(3) = 2,$$
$$g \circ f(2) = g(f(2)) = g(4) = 1,$$

so $g \circ f = (2, 1)$.

What about $f \circ g$? Because that little circle looks like multiplication, one might be tempted to conclude that $g \circ f = f \circ g$. Let's check it out. Observe that $f \circ g(1) = f(g(1)) = f(3)$. Given that $f = (3, 4)$, what is $f(3)$? (Don't say $f(3) = 4$. This is no time to confuse sequences with cycles. The cycle idea is valid only in the context of permutations. While $f \in F_{2,5}$ may be one-to-one, it most certainly is *not* onto.) Because $3 \notin \{1, 2\}$, the domain of f, "$f(3)$" is nonsense; there is no third component in the sequence $(3, 4) = (f(1),\ f(2))$. Since $f(3)$ doesn't exist, $f \circ g$ doesn't exist either. In other words, it doesn't make sense even to write $f \circ g$, much less expect that it should equal $g \circ f = (2, 1)$. □

3.1.3 Example. Suppose $f = (3, 2, 1, 1, 2) \in F_{5,3}$ and $g = (2, 1, 1) \in F_{3,2}$. Then $\text{image}(f) = \text{range}(f) = \{1, 2, 3\} = \text{domain}(g)$, so there is a function $g \circ f \in F_{5,2}$. To determine which function it is requires a little work:

$$\begin{aligned}
g \circ f(1) &= g(f(1)) = g(3) = 1,\\
g \circ f(2) &= g(f(2)) = g(2) = 1,\\
g \circ f(3) &= g(f(3)) = g(1) = 2,\\
g \circ f(4) &= g(f(4)) = g(1) = 2,\\
g \circ f(5) &= g(f(5)) = g(2) = 1,
\end{aligned}$$

so $g \circ f = (1, 1, 2, 2, 1)$. What about $f \circ g$? This time $\text{image}(g) = \{1, 2\} \subset \{1, 2, 3, 4, 5\} = \text{domain}(f)$, so $f \circ g$ is a legitimate function. Maybe now $f \circ g = g \circ f$? Let's see. The domain of $f \circ g$ is $\text{domain}(g) = \{1, 2, 3\}$;

$$\begin{aligned}
f \circ g(1) &= f(g(1)) = f(2) = 2,\\
f \circ g(2) &= f(g(2)) = f(1) = 3,\\
f \circ g(3) &= f(g(3)) = f(1) = 3,
\end{aligned}$$

so $f \circ g = (2, 3, 3) \in F_{3,3}$, which is not hard to distinguish from $g \circ f = (1, 1, 2, 2, 1) \in F_{5,2}$. □

What is the easy way to compute function compositions? Unfortunately, there are no shortcuts. With a little experience, one can find $g \circ f$ without taking up so much space, but only by keeping track of all the steps in one's head. Give it a try. Let $f, g \in F_{4,4}$ be defined by $f = (1,1,2,2)$ and $g = (4,3,1,1)$. If you can, confirm in your head that $g \circ f = (4,4,3,3)$ and $f \circ g = (2,2,1,1)$. If you can't manage to do it in your head, that's not a problem, *provided* you work it out with pencil and paper!

What about composing three functions? The only really good news here is that function composition is *associative*. If the domains and images match up so that $f \circ (g \circ h)$ makes sense, then $(f \circ g) \circ h$ also makes sense, and

$$f \circ (g \circ h) = (f \circ g) \circ h. \tag{3.1}$$

This is useful for two reasons. It means $f \circ g \circ h$ is unambiguous, and it means that $f \circ g \circ h$ can be evaluated, one composition at a time.

Suppose $f \in F_{m,m}$ is a permutation. Then $f \in S_m$ is one-to-one (and onto). So, f has an inverse. It might be helpful at this point to recall the definition of "inverse".

3.1.4. Definition. Suppose $f : D \to R$ and $g : R \to D$ are functions. Then g is the *inverse* of f if

$$g \circ f(d) = d \quad \text{for every } d \in D, \tag{3.2}$$

and

$$f \circ g(r) = r \quad \text{for every } r \in R. \tag{3.3}$$

If f has an inverse, then its rule of assignment is uniquely determined by Equation (3.2). In other words, if f has an inverse, it is unique. The inverse of f is typically written, not g, but f^{-1}. Two things about this notation deserve comment. The first is that f^{-1} is just a name for the unique function g that, along with f, satisfies Equations (3.2) and (3.3). The second is that Equations (3.2) and (3.3) are symmetric, i.e., $f^{-1} = g$ if and only if $g^{-1} = f$. (In particular, $[f^{-1}]^{-1} = f$.)

3.1.5 Example. Focusing on permutations does not affect function composition, but disjoint cycle notation changes the way it looks! If $p_1 = (1473)(2)(56)$ and $p_2 = (167)(24)(35)$, then

$$p_1 \circ p_2 = (1473)(2)(56) \circ (167)(24)(35) \tag{3.4}$$
$$= (15)(274)(36), \tag{3.5}$$

and

$$p_2 \circ p_1 = (167)(24)(35) \circ (1473)(2)(56)$$
$$= (124)(36)(57).$$

There is a purely mechanical way to produce the disjoint cycle factorization of $p_1 \circ p_2$. Write "(1". Then place your finger at the *right-hand* end of Equation (3.4) and start moving it to the left, searching for the number 1. When your finger comes to 1, stop. The number immediately to the right of 1 is $p_2(1) = 6$. (So far, so good: $p_1 \circ p_2(1) = p_1(p_2(1)) = p_1(6)$. It remains to find $p_1(6)$.) Resume the leftward motion of your finger, but with a new objective. Instead of searching for 1, look for (another occurrence of) 6. When you come to 6, stop. (Having already determined that $p_2(1) = 6$, we are about to find $p_1(6)$.) Because 6 is the last number in its cycle, move your finger leftward to the first number of that same cycle. In this case, that number is 5. Write 5 next to 1 in "(1", obtaining "(15".

Now, return your finger to the far right-hand end of Equation (3.4) and repeat the process, this time beginning your search with 5. Because 5 is the first number encountered, the search is brief. As 5 is at the end of its cycle, move your finger to the 3 at the beginning of the (same) cycle. (You have just determined that $p_2(5) = 3$. The next step is to determine $p_1(3)$.) Without writing anything down, resume your leftward movement, looking for the next occurrence of 3. Since it is found at the end of its cycle, move your finger to the front of that same cycle, bringing it to rest on 1. Evidently, $1 = p_1(3) = p_1(p_2(5))$. In the disjoint cycle factorization of $p_1 \circ p_2$, 1 follows 5. Since we opened the cycle with 1, it is time to close the cycle, i.e., change "(15" to "(15)".

Next, find the smallest number that has not yet been used. In this case it is 2. Replace "(15)" with "(15) (2". Place your finger at the far right-hand end of Equation (3.4) and repeat the process, searching for 2. Continue in this way until you've obtained Equation (3.5). □

3.1.6 Definition. Let $e_m \in S_m$ be the function defined by $e_m(i) = i, 1 \leq i \leq m$. The permutation e_m is called the *identity* of S_m. In disjoint cycle notation, $e_m = (1)(2)\cdots(m)$.

Before reading on, convince yourself that

$$f \circ e_m = f = e_m \circ f \tag{3.6}$$

for every $f \in S_m$. A more significant application of Definition 3.1.6 is the following useful alternative to the definition of inverse, one that is special to permutations.

3.1.7 Theorem. *Suppose $f, g \in S_m$. Then $g = f^{-1}$ if and only if $g \circ f = e_m$ and $f \circ g = e_m$.*

Proof. This is just a restatement of Definition 3.1.4 using e_m. ■

We now come to an important technical observation.

3.1.8 Lemma. *If $p, q \in S_m$, then, while they may not be equal, both $p \circ q$ and $q \circ p$ exist, and both are permutations in S_m.*

Proof. Because $S_m \subset F_{m,m}$, both $p \circ q$ and $q \circ p$ exist as functions in $F_{m,m}$. It remains to prove that they are permutations. By definition, S_m consists of those functions $f \in F_{m,m}$ that are one-to-one (and onto), i.e., S_m consists (precisely) of the invertible functions in $F_{m,m}$. It follows from $[f^{-1}]^{-1} = f$ that the inverse of an invertible function is invertible, so $p^{-1}, q^{-1} \in S_m$. To see that $q \circ p$ is invertible, observe that

$$\begin{aligned}(q \circ p) \circ (p^{-1} \circ q^{-1}) &= q \circ (p \circ p^{-1}) \circ q^{-1} \\ &= q \circ e_m \circ q^{-1} \\ &= q \circ q^{-1} \\ &= e_m\end{aligned}$$

by associativity, Theorem, 3.1.7, and Equation (3.6). The identity $(p^{-1} \circ q^{-1}) \circ (q \circ p) = e_m$ can be proved similarly. Thus, by Theorem 3.1.7,

$$p^{-1} \circ q^{-1} = (q \circ p)^{-1}, \tag{3.7}$$

the inverse of $q \circ p$. In particular, $q \circ p$ *has* an inverse, which is the criterion that must be met to guarantee that $q \circ p \in S_m$. Interchanging p and q in Equation (3.7) yields $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$, proving that $p \circ q \in S_m$. ■

3.1.9 Example. Let $p = (1524)(3)$ and $q = (143)(25)$. Then $p^{-1} = (4251)(3) = (1425)(3)$ and $q^{-1} = (341)(52) = (134)(25)$. Let's confirm Equation (3.7) by comparing $p^{-1} \circ q^{-1}$ with $(q \circ p)^{-1}$. Observe that

$$\begin{aligned} p^{-1} \circ q^{-1} &= (1425)(3) \circ (134)(25) \\ &= (132)(4)(5). \end{aligned}$$

Next, compute

$$\begin{aligned} q \circ p &= (143)(25) \circ (1524)(3) \\ &= (123)(4)(5), \end{aligned}$$

from which it follows that $(q \circ p)^{-1} = (321)(4)(5) = (132)(4)(5)$.

3.1.13 Definition. A nonempty subset G of S_m is *closed* if $fg \in G$ for all $f, g \in G$.

We have already proved that $f, g \in G$ implies $fg \in S_m$. That's not the point. The issue is whether the composition is an element of the subset G.

3.1.14 Example. Of the 63 nonempty subsets of S_3, only six are closed. Apart from S_3, itself, the other five are $\{e_3\}, \{e_3, (12)(3)\}, \{e_3, (13)(2)\}, \{e_3, (1)(23)\}$, and $\{e_3, (123), (132)\}$. If S is one of the remaining 57 nonempty subsets of S_3, there exist permutations $f, g \in S$ such that $fg \notin S$.

From our perspective, there is a kind of aristocracy among the subsets of S_m. The closed sub*sets* are called sub*groups*. □

3.1.15 Definition. Let G be a (nonempty) closed subset of S_m. Then G is a *subgroup* of S_m, or a *permutation group* of *degree m*.

In biology, a *riparian habitat* is found at the boundary of water and land. Life occurs in its richest diversity in the vicinity of such natural boundaries. A similar richness may frequently be found near the boundaries of mathematical disciplines. That is where we are now, at the boundary between combinatorics and algebra. Because every finite group is *isomorphic* to a permutation group, the case is sometimes made that combinatorial group theory embraces all finite group theory. At best, that viewpoint is misleading. Two permutation groups that are isomorphic as abstract groups may have very different combinatorial properties. It is the combinatorial properties of permutation groups that are of interest in this chapter.

One final pedagogical issue needs to be discussed. The group S_m has been defined in terms of the permutations of $V = \{1, 2, \ldots, m\}$. The fact that V is a set of *numbers* is beside the point. We have used V because it is convenient. We might just as well have discussed the set of permutations of $Y = \{y_1, y_2, \ldots, y_m\}$, denoting it S_Y. (In that notation, S_m becomes S_V.) Strictly speaking, elements of S_Y permute the $y's$, whereas elements of S_m permute their subscripts. But, the "action" is the same. For our purposes, S_m and S_Y are clones. When the time comes to talk about permutations of Y, we will talk about S_m *acting* on Y.

3.2.2 Definition. A cycle is *nontrivial** if its length is greater than 1. A permutation having just one nontrivial cycle in its disjoint cycle factorization will, itself, be referred to as a *cycle*. A *k-cycle* in S_m is any permutation of cycle type $[k, 1^{m-k}]$.

3.2.3 Definition. If $p \in S_m$, let $p^0 = e_m$ and $p^n = p \circ p^{n-1}$, $n \geq 1$. Denoted $o(p)$, the *order** of p is the smallest positive integer k such that $p^k = e_m$.

Observe that $o(e_m) = 1$ for all m. (In particular, *order* is independent of *degree*.) Before getting to a proof of the existence of $o(p)$, let's see some examples.

3.2.4 Example. Let $p = (123456) \in S_m$ (where $m \geq 6$). Then (check the computations)

$$
\begin{aligned}
p^1 &= pe_m = p = (123456),\\
p^2 &= pp^1 = (123456)(123456) = (135)(246),\\
p^3 &= pp^2 = (123456)(135)(246) = (14)(25)(36),\\
p^4 &= pp^3 = (123456)(14)(25)(36) = (153)(264),\\
p^5 &= pp^4 = (123456)(153)(264) = (165432),\\
p^6 &= pp^5 = (123456)(165432) = e_m,
\end{aligned}
$$

so $o(p) = 6$. (It follows from Lemma 2.4.1 that $o(g) = k$ for any k-cycle $g \in S_m$.) Observe that the next few *powers* of p are

$$p^7 = pp^6 = pe_m = p, \qquad p^8 = pp^7 = pp = p^2, \qquad p^9 = pp^8 = pp^2 = p^3,$$

and so on. In particular, $p^{12} = p^6 = e_m$.

If $f = (12)(3456) \in S_7$, then f is a permutation of degree 7. To find its order, observe that

$$
\begin{aligned}
f^1 &= f = (12)(3456),\\
f^2 &= (12)(3456)(12)(3456) = (35)(46)\\
f^3 &= (12)(3456)(35)(46) = (12)(3654)\\
f^4 &= (12)(3456)(12)(3654) = e_7,
\end{aligned}
$$

so $o(f) = 4$. (Does $f^{12} = e_7$?) □

3.2.5 Lemma. *Let n be a positive integer. Suppose $p \in S_m$ has order $o(p) = k$. Then $p^n = e_m$ if and only if k is a factor of n.*

Proof. Dividing n by k yields a quotient q and remainder $r = n - kq$, where $0 \leq r < k$. Because function composition is associative, $p^n = p^{kq+r} = (p^k)^q p^r = (e_m)^q p^r = e_m p^r = p^r$. In particular, $p^n = e_m$ if and only if $p^r = e_m$. Because $r < k = o(p), p^r = e_m$ if and only if $r = 0$ if and only if $n = kq$. ■

3.2.6 Theorem. *If $p \in S_m$, then $o(p)$ is the least common multiple of the lengths of the cycles in the disjoint cycle factorization of p. (In particular, $o(p)$ exists.)*

Proof. If $p = e_m$, there is nothing to prove. So, suppose $p \neq e_m$. Then

$$p = C_p(i_1)C_p(i_2)\cdots C_p(i_r),$$

where $C_p(i_t)$, $1 \leq t \leq r$, are the nontrivial inequivalent cycles of p. In the aftermath of Definition 3.2.2, this means $p = p_1 p_2 \cdots p_r$, where the cycle $p_t \in S_m$ differs from $C_p(i_t)$ at most by some fixed points. Because inequivalent cycles of p are disjoint, and disjoint cycles commute, $p^n = p_1^n p_2^n \cdots p_r^n$.

Observe that $e_m = p^n = p_1^n(p_2^n \cdots p_r^n)$ if and only if

$$(p_1^n)^{-1} = p_2^n \cdots p_r^n. \tag{3.8}$$

If $p_1^n \neq e_m$, then $p_1^n(i) = j$ for some $j \neq i$. Because any fixed point of p_1 is a fixed point of p_1^n, this can happen only if $i, j \in C_p(i_1)$, only if both i and j are fixed points of $p_2, p_3, \ldots, p_r$. So, the left-hand side of Equation (3.8) sends j to i, but the right-hand side fixes j. This contradiction proves that $p_1^n = e_m$. Since any one of the cycles could have been first, $p^n = e_m$ if and only if $p_t^n = e_m$, $1 \leq t \leq r$. By Lemma 3.2.5 (and Lemma 2.4.1), $p_t^n = e_m$ if and only if n is a multiple of $o(p_t)$, the length of $C_p(i_t)$. Thus, $p^n = e_m$ if and only if n is a common multiple of these lengths, the least of which is $o(p)$. ■

3.2.7 Example. Let $f = (3,8,5,6,7,2,9,4,1) \in S_9$. Apart from establishing that $o(f)$ exists, Theorem 3.2.6 illustrates one of the benefits of disjoint cycle notation. From the expression $f = (13579)(2846)$, it is easy to see that $o(f) = 20$.

What about $p = (2,3,1,5,4)$? Can you see that $o(p) = 6$ *without* expressing it in the form $p = (123)(45)$? Let's confirm that $o(p) = 6$. (Check the computations.)

$$\begin{aligned}
p^2 &= (123)(45)(123)(45) = (132),\\
p^3 &= (123)(45)(132) = (45),\\
p^4 &= (123)(45)(45) = (123),\\
p^5 &= (123)(45)(123) = (132)(45),\\
p^6 &= (123)(45)(132)(45) = e_5.
\end{aligned}$$

3.2.8 Theorem. *Let $p \in S_m$. If $o(p) = k > 1$, then $p^{-1} = p^{k-1}$.*

Proof. By Exercises 16 and 19 of Section 3.1, p^{-1} is a name for the unique permutation $f \in S_m$ that solves the equation $pf = e_m$. So, the theorem is a consequence of $pp^{k-1} = p^k = e_m$. ■

3.2.9 Definition. *Let $p \in S_m$. The cyclic group generated by p is $\langle p \rangle = \{p^n : 0 \le n < o(p)\}$.*

3.2.10 Example. If $o(p) = k$, then $p^k = e_m$, so

$$\begin{aligned}
\langle p \rangle &= \{e_m, p, p^2, \ldots, p^{k-1}\}\\
&= \{p, p^2, \ldots, p^{k-1}, p^k\}.
\end{aligned}$$

Observe that $o(\langle p \rangle) = k = o(p)$; the number of elements in the subgroup $\langle p \rangle$ is equal to the smallest positive integer k such that $p^k = e_m$. In particular, calling k the *order* of p is no great abuse of language after all.

As in Example 3.2.4,

$$\begin{aligned}
p^{k+1} &= pp^k = pe_m = p,\\
p^{k+2} &= pp^{k+1} = pp = p^2,\\
p^{k+3} &= pp^{k+2} = pp^2 = p^3,
\end{aligned}$$

and so on. Evidently, the infinite sequence

$$p^0, p^1, p^2, \ldots = e_m, p^1, \ldots, p^{k-1}, e_m, p^1, \ldots, p^{k-1}, e_m, p^1, \ldots, p^{k-1}, e_m, \ldots$$

is *cyclic* with period k. In particular,

$$\begin{aligned}\{p^n : n \geq 0\} &= \{p^n : 0 \leq n < k\} \\ &= \{e_m, p, p^2, \ldots, p^{k-1}\} \\ &= \langle p \rangle, \end{aligned} \tag{3.9}$$

which explains why $\langle p \rangle$ is a *cyclic* group. □

We now justify the word *group* in Definition 3.2.9.

3.2.11 Theorem. *If $p \in S_m$, then $\langle p \rangle$ is a subgroup of S_m.*

Proof. Because (associativity and induction) $p^r p^s = p^{r+s}, r, s \geq 0$, the nonempty subset of S_m on the left-hand side of Equation (3.9) is closed. ■

3.2.12 Corollary. *Let G be a permutation group of degree m. Then*

1. $e_m \in G$ *and*

2. $p \in G \Rightarrow p^{-1} \in G$.

Proof. Because G cannot be empty, it contains a permutation that may as well be denoted p. Suppose $o(p) = k$. If $k = 1$, then $p^{-1} = e_m = p \in G$. Otherwise, by Implication (3.10), $\langle p \rangle = \{e_m, p, \ldots, p^{k-1}\} \subset G$. Thus, $e_m \in G$ and, by Theorem 3.2.8, $p^{-1} = p^{k-1} \in G$. ■

POSSIBLE QUESTIONS

UNIT-IV

1. For A ={1,2,3,4} and B= {u,v,w,x,y,z}, determine the number of one to one functions f: A→ B where f(1) ≠u,v , f(2) ≠w; f(3) ≠w,x and f(4)≠ x,y,z.
2. Let f(n) and g(n) be functions defined for every positive integer n satisfying f(n) = $\sum_{d|n} g(d)$ Then g satisfies g(n) = $\sum_{d|n} \mu(d)$ f(n /d).
3. In making seating arrangements for their son's wedding reception, Grace and Nick are down to four relatives, denoted by R_i for $1\leq i \leq 4$ who do not get along with one another. There is a single open seat at each of the five tables T_j where $1 \leq j\leq 5$. Because of family differences
   a. R_1 will not sit at T_1 or T_2.
   b. R_2 will not sit at T_2
   c. R_3 will not sit at T_3 or T_4.
   d. R_4 will not sit at T_4 or T_5

   Find the number of ways these four relations can be seated at four different tables satisfying the above stated conditions.
4. State and prove the Euler function
5. Let A= {1,2,3,4,5,6,7,8}. Find the number of one to one functions f: A→B where f(i)≠ I, for all i∈ A.
6. A pair of dice, one red and the green is rolled six times. We know that the ordered pairs (1,2), (2,1),(2,5),(3,4), (4,1),(4,5) and (6,6) did not come up what is the probability every value came up on both the red die and the green one.
7. Prove the Menage problem.
8. For A={ 1,2,3,4,5} and B= {u,v,w,x,y,z}, determine the number of one to one functions f: A→ B where f(1) ≠v , w; f(2) ≠u,w; f(3) ≠x and f(4)≠ v, x,y.
9. Prove the $\sum_{d|n} \varphi(d) = n$.
10. Find the closed form expression for the Fibonacci sequence defined by $F_n= F_{n-1}+F_{n-2}$, n≥2.
11. Obtain Fractional Decomposition and identify the sequence having the expression $\frac{3-5z}{1-2z-3z2}$

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DEPARTMENT OF MATHEMATICS

Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

UNIT-IV

Question	Option-1	Option-2	Option-3	Option-4	Answer
A ----------------is an arrangement ofa number ofobjects in a definite order, taken s	Permutation	Combination	combinatorics	Factorial	Permutation
The number of permutations of n things taken all at a time is	n!	(n-1)!	(n+1)!	(n/2)!	n!
In how many different ways can the lettersof the word HEXAGON be permutted.	5040	4050	4150	5150	5040
How many permutations of the characters in the word COMPUTER are there? Ho	15120	12150	14520	13620	15120
In how many ways can first, second and third prize in pie-baking contest be givent	2730	3720	7230	3450	2730
The arrangements of n objects in a circle is -----------	permutation	combination	factorial	circular permutation	circular permutation
The number of arrangements of n objects in a circle is -----------	n!	(n-1)!	(n+1)!	(n/2)!	(n-1)!
A ---------------- is a selection of some or all of a number of different objects wher	permutation	combination	factorial	circular permutation	combination
The valueof nCn is -------------------	n!	1	n!	n!/2	1
How many 16 bit strings are there containing exactly 5 zeroes?	348	4368	538	5632	4368
Three travellers arrive at a town where there are five hotels. In how many ways car	56	60	62	64	60
In how many ways can 6 differentlycoloured marbles be arranged in a row.	720	70	60	26	720
In how many ways can 8 people be seated on a bench if only 3 seats are available.	330	320	336	332	336
Find the number of permutations of letters in the word STATISTICS	40500	41500	50400	51400	50400
In how many ways can a committee of a persons be chosen out f 10.	200	220	240	210	210
In how many ways can 4 red balls be drawn from a bag containing 10 red balls.	200	220	240	210	210
In how many ways can a random sample of 5 citiesbe drawn from a total of 20.	15504	2456	34567	12897	15504
In how many ways can a committee of 6 menand 2 women be formed out of 10 me	2000	2100	2200	2300	2100
Find the number of permutations of letters in the word QUEUE	30	60	90	120	30
Find the number of permutations of letters in the word COMMITTEE	45360	53480	44360	42350	45360
Find the number of permutations of letters in the word PROPOSITION	1,66,3200	1,553,200	1,44,3200	1,33,3200	1,66,3200
Find the number of permutations of letters in the word BASEBALL	5050	5040	5060	5070	5040
How many permutations of the letterw A B C D E F G H contain the string ED	5040	4050	4150	5150	5040
How many permutations of the letterw A B C D E F G H contain the string CDE	730	720	760	780	720
How many permutations of the letterw A B C D E F G H contain the string BA an	120	130	140	150	120
How many permutations of the letterw A B C D E F G H contain the string AB,DI	120	130	140	150	120
How many permutations of the letterw A B C D E F G H contain the string CAB a	12	24	26	30	24
Find the number of permutations of the lettersof the word KAPIL beginning with I	12	24	6	32	6
Find the number of permutations of the lettersof the word KAPIL vowels always b	24	48	36	42	48
Find the number of arrangements of the letters of the words MATHEMATICS	4989600	456700	457600	482300	4989600
Find the number of arrangements of the letters of the words COMMISSION	226800	236800	267300	234600	226800
How many bit strings of length 12 contain exactly three 0's	210	220	230	250	220
How many bit strings of length 12 contain atleast three 1's	4017	4027	4016	4026	4017
How many bit strings of length 12 contain atmost three 1's	928	968	978	948	968
How many bit strings of length 12 contain an equal number of 0's and 1's	928	968	924	948	924
There are 6gentlemen and 4 ladies to dine at a round table. In how many can they b	43200	43500	43600	43100	43200
From 6 gentlemen and 4 ladies, a committee of five is to be selected. In how many	240	246	236	226	246
Ravi has 5 friends. In how many ways can he invite one or more of them to a party	31	32	36	39	31
How many bytes contain exactly two 1's	25	50	28	100	28
How many bytes contain exactly four 1's	60	40	30	70	70
How many bytes contain exactly six 1's	25	50	28	100	28
How many bytes contain atleast six 1's	35	37	32	43	37
In how many can we distribute seven apples and six oranges among four childrens	1860	1680	1540	1450	1680
A student has to answer 10 out of 13 questions in an examination. How many choi	240	246	286	226	286
A student has to answer 10 out of 13 questions in an examination. How many choi	156	15	165	172	165
A student has to answer 10 out of 13 questions in an examination. How many choi	10	110	120	130	110
A student has to answer 10 out of 13 questions in an examination. How many choi	60	70	80	90	80
A student has to answer 10 out of 13 questions in an examination. How many choi	226	276	256	236	276
Find the number of 4 combinations of 5 objects with unlimited repetitions.	30	50	60	70	70
Find the number of ways of placing 8 similar balls in 5 numbered boxes.	456	495	465	432	495
Find the number of binary numbers with five 1's and three 0's.	36	56	42	52	56
How many outcomes are possible by rolling six faced die 10 times.	C(5,10)	C(15,10)	C(25,10)	C(10,10)	C(15,10)
How many different outcomes are possible from tossing 10 similar dice.	2003	3003	4003	5003	3003
Find the number of 3 combinations of 5 objects with unlimited repetitions.	43	35	36	46	35

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DEPARTMENT OF MATHEMATICS

Subject: COMBINATORICS	**Semester :II**	**L T P C**
Subject Code: 16MMU305B	**Class :II-M.Sc Mathematics**	**4 0 0 4**

UNIT V

Problem of Fibonacci –Necklace problem – Burnside's lemma.

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BURNSIDE'S LEMMA

Getting from point a to point b can sometimes be a problem. Consider the case in which $a, b \in V = \{1, 2, \ldots, m\}$. Let G be subgroup of S_m, and suppose the only way to get from a to b is via some permutation $p \in G$ that maps a to b. If G were a transportation system, the ideal situation would be one in which, for any $a, b \in V$, there is a $p \in G$ such that $p(a) = b$. But, few real-life systems are ideal. Take the San Francisco Bay Area, for example, where public transportation is relatively good. If a and b are both in Oakland, an AC-Transit bus will take passengers from point a to point b. If a and b are in San Francisco, MUNI will do the job. Getting from point a in Oakland to point b in San Francisco, however, is another matter. If the system were enlarged to include BART,[*] there would be no problem. But, anyone restricted to AC-Transit or MUNI would be out of luck.

3.3.1 Definition. If G is a permutation group of degree m, then $x, y \in V = \{1, 2, \ldots, m\}$ are *equivalent modulo* G, written

$$x \equiv y \pmod{G} \qquad (3.15)$$

if there is a permutation $p \in G$ such that $p(x) = y$.

For the case modeled by Bay Area buses, any two points in Oakland are equivalent, as are any two points in "the City". Without BART, however, no point of Oakland is equivalent to any point in San Francisco. The two cities are in different transit districts or *equivalence classes*, language that depends on the next result.

3.3.2 Theorem. *If G is a permutation group of degree m, then equivalence modulo G is an equivalence relation.*

To prove the theorem, it will be necessary to verify the following: For all $x, y, z \in V = \{1, 2, \ldots, m\}$,

1. $x \equiv x \pmod{G}$.

2. $x \equiv y \pmod{G} \Rightarrow y \equiv x \pmod{G}$.

3. $x \equiv y \pmod{G}$ and $y \equiv z \pmod{G} \Rightarrow x \equiv z \pmod{G}$.

Proof of Theorem 3.3.2. By Corollary 3.2.12, $e_m \in G$. Because $e_m(x) = x, 1 \leq x \leq m$, criterion 1 is verified.

If $x \equiv y \pmod{G}$, there is a permutation $p \in G$ such that $p(x) = y$. By Corollary 3.2.12, $p^{-1} \in G$. Because $p(x) = y$ if and only if $p^{-1}(y) = x$, criterion 2 is proved.

If $x \equiv y \pmod{G}$ and $y \equiv z \pmod{G}$, there are permutations $f, g \in G$ such that $f(x) = y$ and $g(y) = z$. Because G is closed, $p = gf \in G$. Since $p(x) = gf(x) = g(f(x)) = g(y) = z$, criterion 3 is established. ■

Equivalence classes arising from the action of a permutation group are of fundamental importance in combinatorial enumeration.

3.3.3 Definition. Let G be a permutation group of degree m. Equivalence classes modulo G are called *orbits* of G. The orbit of G containing x is

$$O_x = \{p(x) : p \in G\}. \tag{3.16}$$

In this definition, x and $p(x)$ are numbers. In particular, the orbits of G are subsets, not of G, but of $V = \{1, 2, \ldots, m\}$. From the general theory of equivalence relations, if O_x and O_y overlap at all, they are identical, i.e., *the different orbits of G comprise a partition of V*. In the bus metaphor, the orbit of a point in San Francisco is the entire city, and the San Francisco orbit is disjoint from the Oakland orbit.

rem 3.2.16, if $f(x) = y$, then $\{p \in G : p(x) = y\} = fG_x$. Hence, as p runs through G, y occurs as the value of $p(x)$ exactly $o(fG_x)$ times. Moreover, by Equation (3.14), the multiplicity $o(fG_x) = o(G_x)$ is the same for every $y \in O_x$. ■

$2 \in O_1$, it follows from the general theory that $O_2 = O_1$. This can, of course, be confirmed directly: $O_2 = \{p(2) : p \in G\} = \{2, 1, 2, 1\}$, multiplicities included.

3.3.6 Example. While the group

$$H = \{e_4, (12)(34), (13)(24), (14)(23)\},$$

from Example 3.3.4, is transitive, the group

$$K = \{e_5, (12)(34), (13)(24), (14)(23)\}$$

is not. The difference, of couse, is a matter of degree. Being of degree 4, the single orbit of H is $O_1 = O_2 = O_3 = O_4 = \{1,2,3,4\}$. Because it is of degree 5, the orbits of K are $O_1 = O_2 = O_3 = O_4 = \{1,2,3,4\}$ and $O_5 = \{5\}$.

Perhaps the easiest way to see that S_m is transitive is via sequence notation. Suppose $i,j \in V = \{1,2,\ldots,m\}$. If $f = (f(1), f(2),\ldots,f(m)) \in F_{m,m}$, then $f(i)$ is the number in the ith component of the sequence. With j occupying that position, there are $(m-1)!$ permutations $f \in S_m$ map i to j. □

3.3.7 Lemma. *Let G be a permutation group of degree m. If $x \in \{1,2,\ldots,m\}$, then the number of elements in the orbit to which x belongs is*

$$o(O_x) = \frac{o(G)}{o(G_x)}. \tag{3.17}$$

Proof. The set $O_x = \{p(x) : p \in G\}$ appears to contain $o(G)$ elements but, as we saw in Example 3.3.4, this includes the multiplicities that arise when $p_1(x) = y = p_2(x)$ for two different permutations $p_1, p_2 \in G$. However, from Theorem 3.2.18, if $f(x) = y$, then $\{p \in G : p(x) = y\} = fG_x$. Hence, as p runs through G, y occurs as the value of $p(x)$ exactly $o(fG_x)$ times. Moreover, by Equation (3.14), the multiplicity $o(fG_x) = o(G_x)$ is the same for every $y \in O_x$. ■

Having counted the elements in each orbit, how hard can it be to count the number of orbits? If every orbit had the same size, counting them would be as easy as dividing m by $o(O_x)$ for some fixed but arbitrary $x \in \{1, 2, \ldots, m\}$. However, orbits need not have the same size. (See, e.g., Example 3.3.6, where the orbits of K are $O_1 = \{1, 2, 3, 4\}$ and $O_5 = \{5\}$.)

There is, in fact, a *beautiful* way to calculate the number of orbits of a permutation group, a method that is as powerful as it is unexpected. The significance of this result may justify a brief anecdote about its history.

William Burnside (1852–1927) published the lemma in his 1897 book on finite groups, along with a footnote citing an 1887 article by Georg Frobenius (1849–1917) as its source. When the footnote was inadvertently omitted from the book's second edition, the result came to be known as "Burnside's lemma". In fact, the same idea had appeared even earlier in an 1847 article by Cauchy (1789–1857).* Before we can state this famous result, one more bit of notation is needed.

3.3.8 Definition. Denote by $F(p)$ *the number of fixed points* of $p \in S_m$.

3.3.9 Burnside's Lemma. *Let G be a permutation group with a total of t orbits. Then t is the average of the numbers of fixed points of the permutations in G. That is,*

$$\frac{1}{o(G)} \sum_{g \in G} F(g) = t. \tag{3.18}$$

3.3.10 Example. For the group $H = \{e_4, (12)(34), (13)(24), (14)(23)\}$, from Example 3.3.6, $F(e_4) = 4$, and $F((12)(34)) = F((13)(24)) = F((14)(23)) = 0$. Because the average of these four numbers is $1, H$ has just one orbit, confirming that it is transitive.

If $K = \{e_5, (12)(34), (13)(24), (14)(23)\}$, then $F(e_5) = 5$, and $F((12)(34)) = F((13)(24)) = F((14)(23)) = 1$. (This would be a natural time to have misgivings about suppressing 1-cycles.) The average of these numbers of fixed points is $(5 + 1 + 1 + 1)/4 = 2$, consistent with our observation in Example 3.3.6 that K partitions $\{1, 2, 3, 4, 5\}$ into two orbits. □

3.3.11 Example. Because S_m is transitive, it has just one orbit. It follows from Brunside's lemma that, on average, the permutations of S_m have one fixed point. (Recall from Section 2.3 that the fraction of permutations in S_m having exactly one fixed point is something else entirely.)

In $S_3, F(e_3) = 3, F(12) = F(13) = F(23) = 1$, and $F(123) = F(132) = 0$. So (as predicted),

$$[3 + 1 + 1 + 1 + 0 + 0]/6 = 1.$$ □

Proof of Burnside's Lemma. Define $S = \{(g,j) : g \in G \text{ and } g(j) = j\}$. Then S is the set of all ordered pairs (g,j) in which j is a fixed point of g. Because $F(g)$ of these ordered pairs begin with g,

$$o(S) = \sum_{g \in G} F(g). \tag{3.19}$$

On the other hand, exactly $o(G_j)$ permutations of G fix j. Therefore,

$$\begin{aligned} o(S) &= \sum_{j=1}^{m} o(G_j) \\ &= \sum_{j=1}^{m} \frac{o(G)}{o(O_j)}, \end{aligned} \tag{3.20}$$

by a rearrangement of Equation (3.17).

Let $C_1, C_2, \ldots, C_t$ be the distinct orbits of G, so that $O_j \in \{C_1, C_2, \ldots, C_t\}, 1 \leq j \leq m$. Then, continuing from Equation (3.20),

$$o(S) = o(G) \sum_{i=1}^{t} \sum_{j \in C_i} \frac{1}{o(C_i)}.$$

Note that, in the second of these summations, $1/o(C_i)$ is added to itself $o(C_i)$ times, i.e.,

$$\begin{aligned} o(S) &= o(G) \sum_{i=1}^{t} \frac{o(C_i)}{o(C_i)} \\ &= t\,o(G). \end{aligned} \tag{3.21}$$

Comparing Equations (3.19) and (3.21) completes the proof. ■

3.3.12 Corollary. *If G is a subgroup of S_m, then*

$$\frac{1}{o(G)}\sum_{g\in G}F(g)\geq 1$$

with equality if and only if G is transitive.

Proof. Because $t = 1$ if and only if G is transitive, the result is an immediate consequence of Equation (3.18). ■

3.3.13 Example. From Example 3.3.4, the orbits of $G = \{e_4, (12), (34), (12)(34)\}$ are $\{1,2\}$ and $\{3,4\}$. Averaging the fixed points of the permutations in G yields $\frac{1}{4}(4+2+2+0) = 2 > 1$, confirming that G is not transitive. □

A subgroup G of S_m is *doubly* transitive if, for all $x_1, x_2, y_1, y_2 \in \{1,2,\ldots,m\}$, where $x_1 \neq x_2$ and $y_1 \neq y_2$, there is a permutation $p \in G$ such that $p(x_1) = y_1$ *and* $p(x_2) = y_2$.

This definition looks complicated, in part, because of technical considerations: If $x_1 \neq x_2$ but $y_1 = y_2$, then *no* one-to-one function could send x_1 to y_1 and x_2 to y_2; if $x_1 = x_2$ but $y_1 \neq y_2$, then *no function* could send x_1 to y_1 and x_2 to y_2. Informally, G is doubly transitive if, for all appropriate sequences $x = (x_1, x_2)$ and $y = (y_1, y_2)$, there is a permutation $p \in G$ that maps x to y.

Would it surprise you to learn that, if $m \geq 2$, then

$$\frac{1}{o(G)}\sum_{g\in G}F(g)^2\geq 2 \qquad (3.23)$$

with equality if and only if G is doubly transitive? It is hard to look at Inequalities (3.22)–(3.23) and not conjecture that, if $m \geq 3$, then the average over $g \in G$ of $F(g)^3$ is not less than 3 with equality if and only if G is *triply* transitive.

Let's test this hypothesis. The numbers of fixed points of the permutations in S_3 are listed in Example 3.3.11. The average of their third powers is $\frac{1}{6}(3^3 + 1^3 + 1^3 + 1^3 + 0^3 + 0^3) = \frac{30}{6} = 5$. Five? What happened to 3? Maybe we glided too nimbly over the details of what "triply transitive" might mean. If S_3 turns out not to be triply transitive, there is still hope for the conjecture. On the other hand, maybe the correct lower bound is not 3 but 5. (After all, $1, 2, 3, \ldots$ is not the only sequence of positive integers.) Before doing anything else, let's give a proper definition of multiple transitivity.

3.3.14 Definition. Let G be a subgroup of S_m. Suppose $1 \leq r \leq m$. Then G is *r-fold transitive* if, for all one-to-one functions $f, g \in F_{r,m}$, there exists a permutation $p \in G$ such that $pf = g$.

Using one-to-one functions enormously simplifies the *statement* of Definition 3.3.14. To see what it *means*, recall that $f = (x_1, x_2, \ldots, x_r) \in F_{r,m}$ is one-to-one if and only if the x's are all different. Thus, G is r-fold transitive if and only if, for each of the $P(m,r)^2$ ways to choose one-to-one functions $f = (x_1, x_2, \ldots, x_r)$ and $g = (y_1, y_2, \ldots, y_r)$ from $F_{r,m}$, there is a permutation $p \in G$ such that

$$p(x_i) = p(f(i)) = pf(i) = g(i) = y_i, \qquad 1 \leq i \leq r.$$

In other words, G is r-fold transitive if and only if, for any of the $P(m,r)^2$ ways to choose (without replacement, where order matters) sequences of distinct integers $(x_1, x_2, \ldots, x_r)$ and $(y_1, y_2, \ldots, y_r)$ from $\{1, 2, \ldots, m\}$, there exists a $p \in G$ such that, simultaneously, $p(x_1) = y_1, p(x_2) = y_2, \ldots,$ and $p(x_r) = y_r$.

Evidently, "transitive" is the same as "1-fold transitive" and "doubly transitive" is the same as "2-fold transitive". Moreover, every $(r+1)$-fold transitive group is r-fold transitive.

3.3.15 Example. Recall that $H = \{e_4, (12)(34), (13)(24), (14)(23)\}$ is transitive. Suppose $(x_1, x_2) = (1, 2)$ and $(y_1, y_2) = (2, 3)$. The only permutation in H that maps $x_1 = 1$ to $y_1 = 2$ is $p = (12)(34)$. Because $p(2) \neq 3$, no permutation in H simultaneously sends x_1 to y_1 *and* x_2 to y_2, i.e., H is not doubly transitive.

What about S_4? Any function in $F_{4,4}$ of the form $(2, 3, r, s)$ maps $x_1 = 1$ to $y_1 = 2$ and $x_2 = 2$ to $y_2 = 3$. Two of these functions are permutations, namely, $p_1 = (2, 3, 1, 4)$ and $p_2 = (2, 3, 4, 1)$. (In disjoint cycle notation, $p_1 = (123)$ and $p_2 = (1234)$.) More generally, if $f, g \in F_{r,m}$ are fixed but arbitrary one-to-one functions, then $(m - r)!$ permutations $p \in S_m$ satisfy $pf = g$. In particular, S_m is r-fold transitive, $1 \leq r \leq m$. (Compare with the last part of Example 3.3.6.) □

Consider another example. Suppose G is permutation group of degree $m \geq 2$. Let $j \in V = \{1, 2, \ldots, m\}$ be fixed but arbitrary. Because $p(j) = j$ for all p in the stabilizer subgroup G_j, the set $\{j\}$ is an orbit of G_j. Thus, G_j is not transitive. Suppose, however, we ignore the orbit $\{j\}$ and think of G_j as a permutation group of degree $m - 1$ acting on

$$\begin{aligned} V_j &= V \backslash \{j\} \\ &= \{1, 2, \ldots, j-1, j+1, \ldots, m\}. \end{aligned}$$

If G is $(r+1)$-fold transitive on V, then G_j is r-fold transitive on V_j. This observation even has a partial converse.

3.3.16 Lemma. *Let G be a permutation group of degree $m \geq 3$. Let $V = \{1, 2, \ldots, m\}$, and suppose $1 \leq r < m$. If the stabilizer subgroup G_j is r-fold transitive on $V_j = V \backslash \{j\}, 1 \leq j \leq m$, then G is $(r+1)$-fold transitive on V.*

Proof. Let $(x_1, x_2, \ldots, x_{r+1})$ and $(y_1, y_2, \ldots, y_{r+1})$ be two one-to-one functions in $F_{r+1,m}$. Because $m \geq 3$, there is some $t \in V$ such that $x_1 \neq t \neq y_1$. By hypothesis, there is a permutation $f \in G_t$ such that $f(x_1) = y_1$. Suppose $f(x_k) = z_k, 2 \leq k \leq r+1$. Since f is one-to-one, and the y's are all different, $z_k \neq y_1 \neq y_k$, $2 \leq k \leq r+1$. So, another application of the hypothesis yields a permutation $g \in G_{y_1}$ such that $g(z_k) = y_k, 2 \leq k \leq r+1$. If $p = gf$, then $p(x_1) = g(f(x_1)) = g(y_1) = y_1$, and $p(x_k) = g(f(x_k)) = g(z_k) = y_k, 2 \leq k \leq r+1$, i.e., $p \in G$ and $p(x_k) = y_k, 1 \leq k \leq r+1$. ■

3.3.17 Example. Let's see what we get when we average the fourth powers of the numbers of fixed points of the permutations in a 4-fold transitive group, e.g.,

$$\frac{1}{4!}\sum_{g\in S_4} F(g)^4.$$

The cycle types of the permuations in S_4 are $[4], [3,1], [2^2], [2,1^2]$, and $[1^4]$. Permutations with cycle types $[4]$ and $[2^2]$ don't have fixed points. There are $P(4,3)/3 = [4\times 3\times 2]/3 = 8$ permutations of cycle type [3,1] each of which has one fixed point. Permutations of type $[2,1^2]$ have two fixed points, and there are $C(4,2) = 6$ of these. Finally, e_4 has four fixed points. So,

$$\begin{aligned}\frac{1}{4!}\sum_{g\in S_4} F(g)^4 &= \tfrac{1}{24}[8\times 1^4 + 6\times 2^4 + 4^4]\\ &= \tfrac{1}{24}[8+96+256]\\ &= \tfrac{360}{24} = 15.\end{aligned}$$

□

3.3.18 Theorem. *Let G be a permutation group of degree m. If* $1 \le r \le m$, *then*

$$\frac{1}{o(G)}\sum_{g\in G} F(g)^r \ge B_r,$$

the rth Bell number, with equality if and only if G is r-fold transitive.

Proof. The proof is by induction on r. The $r = 1$ case having already been established in Corollary 3.3.12, we may assume $r \ge 2$. If $m = 2$, then $G = S_2$ or $G = \{e_2\}$. As the result is easily seen to be valid in both of these cases, we may assume $m \ge 3$.

As in the proof of Burnside's lemma, a certain set is counted in two different ways. Let

$$T = \{(g, i_1, i_2, \ldots, i_r) : g \in G \quad \text{and} \quad g(i_k) = i_k, 1 \leq k \leq r\}.$$

By the fundamental counting principle, $F(g)^r$ of the elements of T begin with g. Thus,

$$o(T) = \sum_{g \in G} F(g)^r.$$

Any element of T that ends with $j = i_r$ must begin with a permutation $g \in G_j$. By the fundamental counting principle, there are $F(g)^{r-1}$ ways to choose the intermediate $r - 1$ entries. Therefore,

$$o(T) = \sum_{j=1}^{m} \sum_{g \in G_j} F(g)^{r-1}.$$

Of course, every $g \in G_j$ has at least one fixed point, namely j. Let $F_1(g) = F(g) - 1$. Then, for $g \in G_j, F_1(g)$ is the number of fixed points of the restriction of g to

$$V_j = \{1, 2, \ldots, j-1, j+1, \ldots, m\}.$$

Substituting $F(g) = F_1(g) + 1$ in Equation (3.24) produces

$$\begin{aligned}
\sum_{g \in G} F(g)^r &= \sum_{j=1}^{m} \sum_{g \in G_j} [F_1(g) + 1]^{r-1} \\
&= \sum_{j=1}^{m} \sum_{g \in G_j} \sum_{k=0}^{r-1} C(r-1, k) F_1(g)^k \\
&= \sum_{j=1}^{m} \sum_{k=0}^{r-1} C(r-1, k) \sum_{g \in G_j} F_1(g)^k \\
&\geq \sum_{j=1}^{m} o(G_j) \sum_{k=0}^{r-1} C(r-1, k) B_k \\
&= B_r \sum_{j=1}^{m} o(G_j) \qquad (3.25)
\end{aligned}$$

by the binomial theorem, induction, and the Bell recurrence relation (Theorem 2.2.7). Moreover, by the induction hypothesis, equality holds in Equation (3.25) if and only if G_j is $(r-1)$-fold transitive for all j, if and only if (Lemma 3.3.16) G is r-fold transitive. Finally, by Equations (3.20) and (3.21), $\sum_{j=1}^{m} o(G_j) = to(G) \geq o(G)$, with equality if and only if $t = 1$, if and only if G is transitive. Because an r-fold transitive group is transitive, the proof is complete. ■

POSSIBLE QUESTIONS

1. Let G be a permutation group with a total of t orbits.Then prove that t is the average of the numbers of fixed points of the permutations in G. That is, $\frac{1}{\phi(G)}\sum_{g\epsilon G} f(G) = t.$

2. Let G be a permutation group acting on a set X. For g ∈ G let ψ(g) denote the number of points of X fixed by g. Then the number of orbits of G is equal to $\frac{1}{|G|}\sum_{g\in G} \psi(g)$.
3. State and prove the Burnside's Lemma.
4. Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?
5. Given the set S consisting of the first n positive integers and a fixed integer v satisfying 0 < v≤n, how many different subsets A of S including the empty subset can be formed with the property that a'-a'' ł v for any two elements a',a'' of A(that is subsets A such that integers I and i+v do not both appear in A for any 1=1,2,..,n-v)?

6.How many different 3 colourings of the bands of an n-band baton are there if the baton is unoriented.

7. Suppose a necklace can be made from beads of three colors, black, white and red. How many different necklaces with n beads are there?
8. Find the pattern inventory for Edge 2 colourings of a tetrahedron.

9. Determine the pattern inventory for 3-beadnecklaces distinct under rotations using black and white beads. Repeat using black, white and red balls.

10.Determine the pattern inventory for 7-Bead necklaces distinct under rotations using three black and four white beads.

11. Find the pattern inventory for corner 2 colourings of a cube.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021
DEPARTMENT OF MATHEMATICS
Multiple Choice Questions (Each Question Carries One Mark)

Subject Name: COMBINATORICS **Subject Code: 16MMP305B**

UNIT-V

Question	Option-1	Option-2	Option-3	Option-4	Answer
Find the value of the combinatorial numbers C(8,3)	43	35	56	46	56
Find the value of the combinatorial numbers C(4,1)	5	6	3	4	4
Find the value of the combinatorial numbers C(7,2)	22	45	21	41	21
Find the value of the combinatorial numbers C(12,7)	762	782	792	800	792
Find the value of the combinatorial numbers C(15,10)	2003	4003	5003	3003	3003
Determine the number of integers between 1 and 250 that are not divisible by 2,3 or 5.	22	33	44	66	66
How many positive integers not exceeding 1000 are divisible by 7 or 11?	120	100	125	110	110
A permutation of objects such that no objects is in its position is called ----------	arrangement	dearrangement	permutation	combination	dearrangement
In a ------------- nothing is in its right place.	arrangement	dearrangement	permutation	combination	dearrangement
The dearrangement of 1 2 3 is	3 2 1	1 3 2	2 1 3	2 3 1	2 3 1
___________denotes the number of dearrangments of n objects.	Fn	Dn	Sn	Kn	Dn
The number of dearrangements of 1 2 3 4 is------------		7	8	9	9
How many dearrangements are there of a set with seven elements.	1654	1854	1236	3421	1854
How many dearrangement of {1,2,3,4,5,6} begin with the integer 1, 2 and 3 in some order.	4	5	6	7	4
How many dearrangement of {1,2,3,4,5,6} end with the integer 1, 2 and 3 in some order.	6	32	36	66	36
The ---------------- is used to find the arrangement with forbidden positions together with principle of inclusion-exclusion.	root polynomial	cube polynomial	rook polynomial	polynomial	rook polynomial
The rook polynomial is used to find the ------------- with forbidden positions together with principle of inclusion-exclusion.	arrangement	dearrangement	permutation	combination	arrangement
The rook polynomial is used to find the arrangement with forbidden positions together with --------------------	principle of inclusion-exclusion.	arrangement	dearrangement	permutation	principle of inclusion-exclusion.
If $\{a_n\}$, $n>0$ represents a sequence of numbers, then an expression that relates a term of the sequence to one or more of its preceeding terms is called a --------------	generating function	recurrence relation	exponential function	dearrangement	recurrence relation
Order of a recurrence relation =	higher subscript- lower subscript	lower subscript- higher subscript	both a and b	none of these	higher subscript- lower subscript
When f(n) =0 , the recurrence relation is said to be ---------------	Homogenous	non homogenous	linear	non linear	Homogenous
In how many ways can we geta sum of 4 or 8 when two distinguishable dice arerolled.	8	6	5	3	8
A debating team consists of 3 boys and 2 girls.Find the number of ways they can sit in a row?	120	30	50	60	120

Reg No----------------------

[16MMP305B]

KARPAGAM ACADEMY OF HIGHER EDUCATION
Karpagam University
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
Third SEMESTER
I INTERNAL TEST-Jul'17
Combinatorics

Date : .07.2017 **Time: 2 Hours**
Class : II M.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)

Answer all the questions

1. $^6P_1$ is equal to
   a. 18 b. 12 c. 6 d. 0
2. An arrangement 0f a finite number of objects taken some or all at a time is called their ----------
   a.A.P b. Combination c. Sequence d. permutation
3. $^nP_2 = 30 \rightarrow n =$ -----------
   a. 6 b. 4 c. 5 d.720
4. 1/20.19.18.17 =
   a.20ι/16ι b. 16ι/20ι c. 1/16ι d. 20ι
5. For a negative integer n, factorial nι
   a. is unique b. is 0 c. does not exist d. is 1
6. Letters of SAP taken all at a time can be written in
   a. 2 ways b. 6 ways c. 24 ways d. 120 ways
7. Value of $^{10}C_4$ x $^8C_3$ is ---------
   a. 12760 b. 11760 c. 10760 d. 9760

8. 10.9/2.1 =
   a. 1/10ι b. 2ι8ι/10ι c. 10ι/2ι8ι d. 10ι

9. Letters of CHORD taken all at a time can be written in
   a. 2 ways b. 6 ways c. 24 ways d. 120 ways

10. $^5C_2 + {}^5C_1 =$

    a. $^6C_2$ b. $^6C_1$ c. $^5C_2$ d. $^5C_1$

11. ^nC_r .rι =

    a. $^{n+1}P_r$ b. · $^nP_{r+1}$ c. $^{n-1}P_r$ d. ^nP_r

12. If $^nC_{12} = {}^nC_6$ value of n is

    a.12 b.14 c.16 d.18

13. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

    a. 564 b. 645 c. 735 d. 756

14. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

    a. 210 b. 1050 c. 25200 d. 21400

15. The number of ways to seat 3 boys and 2 girls in a row if each boy must sit next to at least one girl is

    a. 36 b.48 c.148 d. 184

16. A pair of fair dice is tossed. Find the probability that the greatest common divisor of the two numbers is one.

    a. 12/36 b.15/36 c. 17/36 d. 23/36

17. Suppose the odds of A occurring are 1:2, the odds of B occurring are 5:4, and the odds of both A and B occurring are 1:8. The odds of (A ∩ Bc) ∪ (B ∩ Ac) occurring are
    a.2:3 b. 4:3 c.5:3 d. 6:3

18. The number of partitions of X = {a, b, c, d} with a and b in the same block is -----------

a. 4 b. 5 c.6 d. 7

19. The number of partitions of X = {a, b, c, d, e, f, g} with a, b, and c in the same block and c, d, and e in the same block is

a.2 b. 5 c.10 d. 15

20. How many different rearrangements are there of the letters in the word BUBBLE?

a. 40 b.50 c. 70 d.12

PART – B (3x2 = 6 Marks)

Answer all the questions

21.Define Permutation and Combination.
22. Define Striling number of second kind with example.
23. Define Generating function.

PART –C(3x8= 24 Marks)

Answer all the questions

24. **a**. State and Prove the i) Vandermonde's Identity and ii) Pascal's Identity.

(OR)

b. There are 3 Piles of identical red, blue and green Balls, where each pile contains atleast 10 balls. In how many ways can 10 balls be selected.
i) if there is no restriction
ii) if atleast one red ball must be selected
iii) if atleast one red ball, atleast 2 blue balls and atleast 3 green balls must be selected.
iv) if exactly one red ball and atleast one blue ball must be selected.
v) if at most one red ball is selected.

25. a) The number of onto functions in $F_{m,n}$ is n!S(m,n).

(OR)

b) Let A be a set consisting of n elements (n≥2). Then there are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd Permutations.

26. a) Solve the recurrence relation $a_n = 2a_{n-1}+2_n$; $a_0=2$.

(OR)

b) Solve the recurrence relation $a_r-7a_{r-1}+10a_{r-2}= 3^r$ given that $a_0=0$ and $a_1=1$.

Reg No----------------------

[16MMP305B]

KARPAGAM UNIVERSITY
Karpagam Academy of Higher Education
COIMBATORE –21
DEPARTMENT OF MATHEMATICS
Third SEMESTER
II INTERNAL TEST-Sep'17
Combinatorics

Date : .09.2017 **Time: 2 Hours**
Class : II M.Sc Mathematics **Maximum: 50 Marks**

PART – A(20X1=20 Marks)
Answer all the questions

1. The number of partitions of X= {a,b,c,d,e,f,g}with a,b and c in the same block and c,d and e in the same block is ------

a. 2 b. 4 c.5 d.15

2. How many dearrangements are there of a set with seven elements?

a. 1654 b. 1854 c. 1236 d. 3421

3. Pigeonhole principle states that A → B and A> B then --------

a. f is not onto b. f is not one-one
c. f is neither one-one nor onto d. f may be one-one

4. How many permutations are there for the 8 letters a,b,c,d,e,f,g,h end with h.
a. 8! b.6! c.7! d.2!

5. If there are 12 boys and 16 girls in a class, find the number of ways of selecting one student as class representative.
a. 12 b.16 c.26 d.28

6. The rook polynomial is used to find the ------------- with forbidden positions together with principle of inclusion-exclusion.

a. arrangement b. de arrangement
c. permutation d. combination

7. Three travellers arrive at a town where there are five hotels. In how many ways can they select their room, each at a different hotel.
a. 56 b.60 c. 62 d. 64

8. Find the number of binary numbers with five 1's and three 0's.
a. 36 b.56 c.32 d.52

9. How many dearrangement of {1,2,3,4,5,6} begin with the integer 1, 2 and 3 in some order.
a. 4 b.5 c.6 d.7

10. Find the number of 3 combinations of 5 objects with unlimited repetitions?
a. 43 b. 35 c. 36 d. 46

11. A permutation of objects such that no objects is in its position is called -----------
a. arrangement b. dearrangement
c. permutation d. combination

12. The rook polynomial is used to find the arrangement with forbidden positions together with -----------
a. principle of inclusion-exclusion. b. arrangement
c. dearrangement d. permutation

13. A debating team consists of 3 boys and 2 girls.Find the number of ways they can sit in a row?
a. 120 b. 30 c. 60 d. 50

14. Order of a recurrence relation = ----------------
a. higher subscript- lower subscript
b. higher subscript+ lower subscript
c. lower subscript-higher subscript
d. both a and c

15. Find the value of the combinatorial numbers C(7,2)?
   a. 22   b. 45   c. 41   d.21

16. Find the number of ways of placing 8 similar balls in 5 numbered boxes?
   a. 456   b. 495   c. 465   d. 432

17. How many bytes contain exactly four 1's?
   a. 60   b. 40   c. 30   d. 70

18. A student has to answer 10 out of 13 questions in an examination. How many choices has he if he must answer the first or second question but not both?
   a. 10   b. 110   c. 130   d. 120

19. Find the number of 3 combinations of 5 objects with unlimited repetitions?
   a. 43   b. 35   c. 36   d. 46

20. In a ------------- nothing is in its right place?
   a. arrangement   b. de arrangement
   c. combination   d. permutation

PART – B (3x2 = 6 Marks)
Answer all the questions

21. Define dearrangements with examples.
22. List all the dearrangements of {1,2,3,4}
23. Write the Inclusion-Exclusion Principle for any four sets.

PART –C(3x8= 24 Marks)
Answer all the questions

24. a) Find the number of integers between 1 and 2000 inclusive that are not divisible by 2,3, 5 or 7.

(OR)

b) How many solutions does $x_1+x_2+x_3 = 11$ have where x_1, x_2 and x_3 are non –negative integers with $x_1 \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

25. a) Let $|A| = n$ and $|B| = m$ and $n \geq m$. The number of onto functions f: A →B is given by $m^n-[n(m-1)^n-{}^nC_2(m-2)^n+{}^nC_3(m-3)^n+\ldots(-1)^m m]$.

(OR)

b) Using the principle of inclusion and exclusion find the number of prime numbers not exceeding 100.

26. a) Show that the number of dearrangements of a set of n elements is given by, $D_n = n![1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}]$.

(OR)

b) Find the closed form expression for the Fibonacci sequence defined by $F_n= F_{n-1}+F_{n-2}$, $n\geq2$.