KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

SYLLABUS

Semester - II

		L	**T**	**P**	**C**
17MMU203	**REAL ANALYSIS**	**6**	**2**	**0**	**6**

Scope: On successful completion of course the learners gain about the real number system, sequences and series.

Objectives: To enable the students to learn and gain knowledge about suprema and infima points, Root test, Ratio test, alternating series, series of functions.

UNIT I

Finite and infinite sets, examples of countable and uncountable sets. Real line, bounded sets, suprema and infima, completeness property of R, Archimedean property of R, intervals.

UNIT II

Real Sequence, Bounded sequence, Cauchy convergence criterion for sequences. Limit of a sequence. Limit Theorems. Cauchy' stheorem on limits, order preservation and squeeze theorem, monotone sequences and their convergence (monotone convergence theorem without proof).

UNIT III

Infinite series. Cauchy convergence criterion for series, positive term series, geometric series,comparison test, convergence of p-series, Root test, Ratio test, alternating series, Leibnitz's test(Tests of Convergence without proof). Definition and examples of absolute and conditional convergence.

UNIT IV

Monotone Sequences, Monotone Convergence Theorem. Subsequences, Divergence Criteria, Monotone Subsequence Theorem (statement only), Bolzano Weierstrass Theorem for Sequences. Cauchy sequence, Cauchy's Convergence Criterion. Concept of cluster points and statement of Bolzano -Weierstrass theorem.

UNIT V

Sequence of functions, Series of functions, Pointwise and uniform convergence. Mn-test, M-test,Statements of the results about uniform convergence and integrability and differentiability of functions, Power series and radius of convergence.

SUGGESTED READINGS

TEXT BOOK

1. Bartle R.G. and Sherbert D. R., 2000. Introduction to Real Analysis, John Wiley and Sons (Asia) Pvt. Ltd.

REFERENCES

1. Fischer E., (2012). Intermediate Real Analysis, Springer Verlag.
2. Ross K.A., (2003).Elementary Analysis- The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag.
3. Apostol T. M., (2002). Calculus (Vol.I), John Wiley and Sons (Asia) P. Ltd.

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LESSON PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME: Y.SANGEETHA
SUBJECT NAME: REAL ANALYSIS SUB.CODE:17MMU203
SEMESTER: II CLASS: I B.SC MATHEMATICS

S.No	Lecture Duration Period	Topics to be covered	Support Materials/Page Nos
		Unit -I	
1	1	Introduction to set	R1: Ch :1, Pg.No:1-5
2	1	Set operations	R1: Ch :1, Pg.No:5-10
3	1	Finite sets	T1: Ch: 1, Pg.No:14-16
4	1	Tutorial-I	
5	1	Infinite sets	T1: Ch :1, Pg.No:16-17
6	1	Examples of finite and infinite set	T1: Ch :1, Pg.No: 17
7	1	Countable sets	T1:Ch :1,Pg.No:18-19
8	1	Tutorial-II	
9	1	Uncountable sets	T1:Ch: 1,Pg.No:19-20
10	1	Examples of countable and uncountable sets.	R1: Ch :1,Pg.No:18-19
11	1	Real line	R1: Ch: 2, Pg.No:33-34
12	1	Tutorial-III	
13	1	Bounded sets	R1: Ch :2, Pg.No:34

14	1	Theorems on supremum of a set	R1: Ch: 2, Pg.No:34-35
15	1	Theorems on infimum of a set	R1: Ch: 2,Pg.No: 34-35
16	1	Tutorial-IV	
17	1	Completeness property of R	R1: Ch :2,Pg.No: 37
18	1	Archimedean property of R	R1: Ch :2, Pg.No:40
19	1	Intervals	R1: Ch :2, Pg.No:44-47
20	1	Tutorial-V	
21	1	Continuation of intervals	R1: Ch: 2, Pg.No:50
22	1	Concept of cluster points	R1: Ch :2, Pg.No:50-51
23	1	Tutorial-VI	
24	1	Recapitulation and Disscussion of possible questions	
Total No of Hours Planned For Unit 1=24			
Unit-II			
1	1	Introduction to sequence	R1: Ch :3,Pg.No: 52
2	1	Real Sequence	R1: Ch: 3,Pg.NO: 53
3	1	Bounded sequence	R1: Ch :3, Pg.No:54
4	1	Tutorial-I	
5	1	Examples of real and bound Sequence	R1: Ch :3, Pg.No:54
6	1	Cauchy convergence criteria for sequences	R1: Ch: 3, Pg.No:54-56
7	1	Limit theorems	T1:Ch:3,Pg.No:60-63
8	1	Tutorial-II	
9	1	Cauchy’s theorem on limits	R1: Ch :3, Pg.NO:56-59
10	1	Continuation of Cauchy’stheorem on limits	R1: Ch: 3, Pg.No:60-62

11	1	Order preservation	R1: Ch: 3, Pg.No:62-64
12	1	Tutorial-III	
13	1	Squeeze theorem	R1: Ch: 3, Pg.No:64-67
14	1	Continuation of Squeeze theorem	R1: Ch: 3, Pg.No:68
15	1	Monotone sequences	R1: Ch :3, Pg.No:68-69
16	1	Tutorial-IV	
17	1	Convergence	R1: Ch: 3, Pg.No:69
18	1	Convergence of monotone sequences	R1: Ch :3, Pg.No:69-70
19	1	Continuation on convergence of monotone sequences	R1: Ch: 3,Pg.No: 70-72
20	1	Tutorial –V	
21	1	Monotone convergence theorem	R1: Ch :3, Pg.No:72-73
22	1	Continuation on monotone convergence theorem	R1: Ch: 3, Pg.No:73-75
23	1	Tutorial –VI	
24	1	Recapitulation and Disscussion of possible questions	
Total No of Hours Planned For Unit II =24			
Unit-III			
1	1	Introduction to Infinite series	R1: Ch :3,Pg.No: 89
2	1	Cauchy convergence thoorem	R1: Ch :3, Pg.No:90
3	1	Cauchy convergence criterion for series	R1: Ch :3,Pg.No: 90
4	1	Tutorial I	
5	1	Theorems on positive term series	R1: Ch :3, Pg.No:91
6	1	Theorems on geometric series	R3: Ch: 10, Pg.No:388
7	1	Comparison test	R3: Ch :10,Pg.No: 394
8	1	Tutorial II	

9	1	Convergence of p-series	R1: Ch: 3,Pg.No: 93-94
10	1	Root test	R3:Ch :10,Pg.NO:399-400
11	1	Continuation on Root test	R3:Ch :10,Pg.No:399-400
12		Tutorial III	
13	1	Ratio test	R1: Ch :10,Pg.No:401-402
14	1	Continuation of Ratio test	R1: Ch :10,Pg.No:401-402
15	1	Alternating series	R1: Ch :3, Pg.No:95-96
16	1	Tutorial IV	
17	1	Leibnitz's test	R2: Ch :2, Pg.No:105-109
18	1	Definition of absolute convergence	R3: Ch :10, Pg.No:406
19	1	Examples of absolute convergence	R3: Ch :10,Pg.No: 406
20	1	Tutorial V	
21	1	Definition of conditional convergence	R3: Ch: 10, Pg.No:407
22	1	Examples of conditional convergence	R3: Ch :10, Pg.No:407
23	1	Tutorial VI	
24	1	Recapitulation & discussion of possible questions	
Total No of Hours Planned For Unit III=24			

Unit-IV			
1	1	Introduction to Sequences	R3: Ch :4, Pg.No:257-260
2	1	Theorems on bounded sequence	T1: Ch: 3, Pg.No:66-67
3	1	Examples of bounded sequence	T1: Ch: 3, Pg.No:67
4	1	Tutorial I	

5	1	Convergent sequence	R3: Ch: 3,Pg.No: 81-86
6	1	Examples of Convergent sequence	R3: Ch: 3,Pg.No 84-86
7	1	Limit of a sequence	R3: Ch :4, Pg.No:294-297
8	1	Tutorial II	
9	1	Limit Theorems	T1: Ch: 4, Pg.No:125
10	1	Monotone Sequences	T1: Ch: 4, Pg.No:126
11	1	Monotone convergence Theorem	T1: Ch: 3, Pg.No:78
12	1	Tutorial III	
13	1	Subsequences	T1: Ch: 3,Pg.No:79
14	1	Divergence Criteria	T1: Ch: 3,Pg.No:79
15	1	Monotone subsequence Theorem	T1: Ch :3,Pg.No:79-80
16	1	Tutorial IV	
17	1	Bolzano Weierstrass Theorem for Sequences	T1: Ch: 3,Pg.No:80
18	1	Bolzano Weierstrass Theorem for Sequences	R3: Ch :3,Pg.No:80-81
19	1	Cauchy sequence	T1: Ch :3, Pg.No:81
20	1	Tutorial V	
21	1	Cauchy's Convergence Criterion	T1: Ch :3, Pg.No:81-82
22	1	Cauchy's Convergence Criterion	R3: Ch :3,Pg.No: 82-83
23	1	Tutorial VI	
24	1	Recapitulation and Disscussion of possible questions	
Total No of Hours Planned For Unit 1V =24			
Unit-V			
1	1	Introduction to Series of functions	R3: Ch :9, Pg.No:266

2	1	Pointwise convergence	R3: Ch :9, Pg.No:266
3	1	Tutorial-I	
4	1	Uniform convergence	R3: Ch: 9,Pg.No: 266-267
5	1	M-test	R3: Ch :9, Pg.No:267
6	1	Tutorial-II	
7	1	M-test	R3: Ch :9, Pg.No:267
8	1	Results about uniform convergence	R3: Ch: 9,Pg.No: 267-268
9	1	Tutorial-III	
10	1	Results about uniform convergence	R3: Ch :9, Pg.No:267-268
11	1	Subsequences	R3: Ch: 9, Pg.No:268
12	1	Continuation on subsequences	R3: Ch :9, Pg.No:268-269
13	1	Tutorial-IV	
14	1	Integrability of functions	R3: Ch: 9, Pg.No:269-270
15	1	Differentiability of functions	R3: Ch: 9,Pg.No: 270
16	1	Power series	R3: Ch :9, Pg.No:270-271
17	1	Tutorial-V	
18	1	Radius of convergence	R3: Ch: 9, Pg.No:271-272
19	1	Radius of convergence	R3: Ch :9, Pg.No:272
20	1	Tutorial-VI	
21	1	Recapitulation and Disscussion of possible questions	
22	1	Discussion of previous ESE Question papers	
23	1	Discussion of previous ESE Question papers	
24	1	Discussion of previous ESE Question papers	
Total No of Hours Planned For Unit V=24			

Text book:

1. Bartle, R.G. and Sherbert D. R., 2000. Introduction to Real Analysis,John Wiley and Sons (Asia) Pvt. Ltd.

References:

1. Fischer, E., (1983). Intermediate Real Analysis, Springer Verlag.
2. Ross, K.A., (2003). Elementary Analysis- The Theory of Calculus Series – Undergraduate Texts in Mathematics, Springer Verlag.
3. T Apostol T. M., (2002). Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd.

UNIT-I

SYLLABUS

Finite and infinite sets, examples of countable and uncountable sets. Real line, bounded sets, suprema and infima, completeness property of R, Archimedean property of R, intervals.

Sets:

A set is any collection of objects, for example, set of numbers. The objects of a set are called the elements of the set.

Finite Set

If a set contains a finite number of elements then we say that the set is finite. Otherwise we say that the set is infinite.
The cardinality of a finite set is the number of elements that it contains. We denote the cardinality of a set A by |A|.

Examples
•If A = {1, 4, 8, 10} then |A| = 4.
•If X = {x : x ∈ N and x < 7}, then |X| = 6.

Countable set

A set *A* is *countable*, if it can be put in a one to one correspon-dence with the set Z+ of positive integers

Uncountable sets

A set is *uncountable*, if it is infinite and is not countable.

A set S is said to be **countable** if it is either finite or denumerable.
A set S is said to be **uncountable** if it is not countable.

Examples **(a)** The set $E := \{2n : n \in \mathbb{N}\}$ of *even* natural numbers is denumerable.

(b) The set $\mathbb{Z}$ of *all* integers is denumerable.

Theorem

The following statements are equivalent

(a) *S is a countable set.*

(b) *There exists a surjection of $\mathbb{N}$ onto S.*

(c) *There exists an injection of S into $\mathbb{N}$.*

Proof. (a) $\Rightarrow$ (b) If S is finite, there exists a bijection h of some set $\mathbb{N}_n$ onto S and we define H on $\mathbb{N}$ by

$$H(k) := \begin{cases} h(k) & \text{for} \quad k = 1, \ldots, n, \\ h(n) & \text{for} \quad k > n. \end{cases}$$

Then H is a surjection of $\mathbb{N}$ onto S.

If S is denumerable, there exists a bijection H of $\mathbb{N}$ onto S, which is also a surjection of $\mathbb{N}$ onto S.

(b) $\Rightarrow$ (c) If H is a surjection of $\mathbb{N}$ onto S, we define $H_1 : S \to \mathbb{N}$ by letting $H_1(s)$ be the least element in the set $H^{-1}(s) := \{n \in \mathbb{N} : H(n) = s\}$. To see that H_1 is an injection of S into $\mathbb{N}$, note that if $s, t \in S$ and $n_{st} := H_1(s) = H_1(t)$, then $s = H(n_{st}) = t$.

(c) $\Rightarrow$ (a) If H_1 is an injection of S into $\mathbb{N}$, then it is a bijection of S onto $H_1(S) \subseteq \mathbb{N}$. By

Cantor's Theorem

If A is any set then there is no surjection of A onto the set P(A) of all subsets of A.

Proof. Suppose that $\varphi : A \to \mathcal{P}(A)$ is a surjection. Since $\varphi(a)$ is a subset of A, either a belongs to $\varphi(a)$ or it does not belong to this set. We let

$$D := \{a \in A : a \notin \varphi(a)\}.$$

Since D is a subset of A, if φ is a surjection, then $D = \varphi(a_0)$ for some $a_0 \in A$.

We must have either $a_0 \in D$ or $a_0 \notin D$. If $a_0 \in D$, then since $D = \varphi(a_0)$, we must have $a_0 \in \varphi(a_0)$, contrary to the definition of D. Similarly, if $a_0 \notin D$, then $a_0 \notin \varphi(a_0)$ so that $a_0 \in D$, which is also a contradiction.

Therefore, φ cannot be a surjection. Q.E.D.

Triangle Inequality: If a,b belongs to R then |a+b| ≤ |a| +|b|

Proof. From 2.2.2(d), we have $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$. On adding these inequalities, we obtain

$$-(|a| + |b|) \le a + b \le |a| + |b|.$$

we have $|a+b| \le |a| + |b|$.

Real line:

A convenient and familiar geometric interpretation of the real number system is the real line. In this interpretation, the absolute value $|a|$ of an element a in $\mathbb{R}$ is regarded as the distance from a to the origin 0. More generally, the **distance** between elements a and b in $\mathbb{R}$

is |a-b|.

Suprema and infima

Definition: Let S be a nonempty subset of R

(a) The set S is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \le u$ for all $s \in S$. Each such number u is called an **upper bound** of S.

(b) The set S is said to be bounded **below** if there exists a number $w \in \mathbb{R}$ such that $w \le s$ for all $s \in S$. Each such number w is called a **lower bound** of S.

(c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

Definition: Let S be a nonempty subset of R

(a) If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the conditions:

(1) u is an upper bound of S, and
(2) if v is any upper bound of S, then $u \le v$.

(b) If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the conditions:

(1′) w is a lower bound of S, and
(2′) if t is any lower bound of S, then $t \le w$.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup S \quad \text{and} \quad \inf S.$$

The Completeness Property of $\mathbb{R}$ *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Applications of the Supremum Property

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. We will prove that

$$\sup(a + S) = a + \sup S.$$

If we let $u := \sup S$, then $x \le u$ for all $x \in S$, so that $a + x \le a + u$. Therefore, $a + u$ is an upper bound for the set $a + S$; consequently, we have $\sup(a + S) \le a + u$.

Now if v is *any* upper bound of the set $a + S$, then $a + x \le v$ for all $x \in S$. Consequently $x \le v - a$ for all $x \in S$, so that $v - a$ is an upper bound of S. Therefore, $u = \sup S \le v - a$, which gives us $a + u \le v$. Since v is any upper bound of $a + S$, we can replace v by $\sup(a + S)$ to get $a + u \le \sup(a + S)$.

Combining these inequalities, we conclude that

$$\sup(a + S) = a + u = a + \sup S.$$

Archimedean Property *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$.*

Proof. If the assertion is false, then $n \le x$ for all $n \in \mathbb{N}$; therefore, x is an upper bound of $\mathbb{N}$. Therefore, by the Completeness Property, the nonempty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Subtracting 1 from u gives a number $u - 1$, which is smaller than the supremum u of $\mathbb{N}$. Therefore $u - 1$ is not an upper bound of $\mathbb{N}$, so there exists $m \in \mathbb{N}$ with $u - 1 < m$. Adding 1 gives $u < m + 1$, and since $m + 1 \in \mathbb{N}$, this inequality contradicts the fact that u is an upper bound of $\mathbb{N}$.

Corollary *If $S := \{1/n : n \in \mathbb{N}\}$, then* $\inf S = 0$.

Proof. Since $S \ne \emptyset$ is bounded below by 0, it has an infimum and we let $w := \inf S$. It is clear that $w \ge 0$. For any $\varepsilon > 0$, the Archimedean Property implies that there exists $n \in \mathbb{N}$ such that $1/\varepsilon < n$, which implies $1/n < \varepsilon$. Therefore we have

$$0 \le w \le 1/n < \varepsilon.$$

But since $\varepsilon > 0$ is arbitrary, $w = 0$.

Intervals

The Order Relation on $\mathbb{R}$ determines a natural collection of subsets called "intervals." The notations and terminology for these special sets will be familiar from earlier courses. If $a, b \in \mathbb{R}$ satisfy $a < b$, then the **open interval** determined by a and b is the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

The points a and b are called the **endpoints** of the interval; however, the endpoints are not included in an open interval. If both endpoints are adjoined to this open interval, then we obtain the **closed interval** determined by a and b; namely, the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

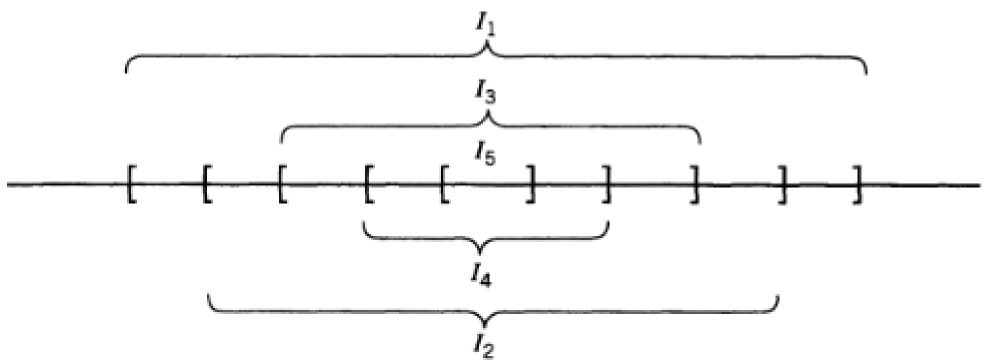
The two **half-open** (or **half-closed**) intervals determined by a and b are $[a, b)$, which includes the endpoint a, and $(a, b]$, which includes the endpoint b.

Each of these four intervals is bounded and has **length** defined by $b - a$.

$$(a, \infty) := \{x \in \mathbb{R} : x > a\} \quad \text{and} \quad (-\infty, b) := \{x \in \mathbb{R} : x < b\}.$$

$$[a, \infty) := \{x \in \mathbb{R} : a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$$

It is often convenient to think of the entire set $\mathbb{R}$ as an infinite interval; in this case, we write $(-\infty, \infty) := \mathbb{R}$. No point is an endpoint of $(-\infty, \infty)$.



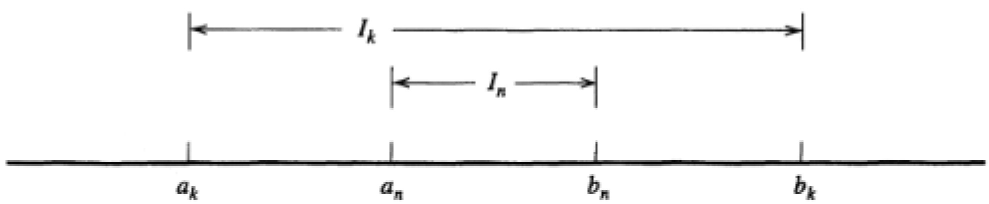
Nested intervals

Nested Intervals Property *If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.*

Proof. Since the intervals are nested, we have $I_n \subseteq I_1$ for all $n \in \mathbb{N}$, so that $a_n \leq b_1$ for all $n \in \mathbb{N}$. Hence, the nonempty set $\{a_n : n \in \mathbb{N}\}$ is bounded above, and we let ξ be its supremum. Clearly $a_n \leq \xi$ for all $n \in \mathbb{N}$.

We claim also that $\xi \leq b_n$ for all n. This is established by showing that for any particular n, the number b_n is an upper bound for the set $\{a_k : k \in \mathbb{N}\}$. We consider two cases. (i) If $n \leq k$, then since $I_n \supseteq I_k$, we have $a_k \leq b_k \leq b_n$. (ii) If $k < n$, then since $I_k \supseteq I_n$, we have $a_k \leq a_n \leq b_n$.

Thus, we conclude that $a_k \leq b_n$ for all k, so that b_n is an upper bound of the set $\{a_k : k \in \mathbb{N}\}$. Hence, $\xi \leq b_n$ for each $n \in \mathbb{N}$. Since $a_n \leq \xi \leq b_n$ for all n, we have $\xi \in I_n$ for all $n \in \mathbb{N}$.



If $k < n$, then $I_n \subseteq I_k$

Note: The set of real numbers can also be divided into two subsets of numbers called algebraic numbers and transcendental numbers. A real number is called *algebraic* if it is a solution of a polynomial equation $P(x) = 0$ where all the coefficients of the polynomial P are integers. A real number is called *transcendental* if it is not an algebraic number. It can be proved that the set of algebraic numbers is countably infinite, and consequently the set of transcendental numbers is uncountable. The numbers π and e are transcendental numbers

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define an uncountable set.
2. Give two examples for uncountable sets.
3. Define countable set.
4. Define bounded set.
5. Define unbounded set.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Prove that the set of all rational number is countable
2. If $a, b \in R$, ptove that $|a+b| \leq |a| + |b|$
3. State and prove Archimedean property.
4. Let S be a subset of R and $a \in R$. Prove that $a + \sup S = \sup(a + S)$
5. State and prove Uniqueness theorem on limits
6. State and prove Cantor's Theorem.
7. Prove that (0,1) is uncountable.
8. Suppose that A and B are non-empty subsets of R, suchthat $a \leq b$ for all $a \epsilon A$ and $b \epsilon B$ then sup A$\leq$ inf B .
9. Suppose that S and T are sets and that T is contained in S.
   a) If S is a finite set, then T is a finite set.
   b) If T is an infinite set, then S is an infinite set.
10. The following statements are equivalent:
    a) S is a countable set.
    b) There exists a surjection of N onto S.
    c) There exists an injection of S into N

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Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Real Analysis **Subject Code: 17MMU203**

Class : I - B.Sc. Mathematics **Semester : II**

Unit I

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
The set of all points between a and b is called ------------	integer	interval	elements	set	interval
The set {x: a < x < b} is --------------	(a, b)	[a, b]	(a, b]	[a, b)	(a, b)
A real number is called a positive integer if it belongs to -----------	interval	open interval	closed interval	inductive set	inductive set
Rational numbers is of the form -----------------	pq	p + q	p/q	p - q	p/q
e is ---------------	rational	irrational	prime	composite	irrational
An integer n is called ----------- if the only possible divisors of n are 1 and n	rational	irrational	prime	composite	prime
A set with no upper bound is called ------------	bounded above	bounded below	prime	function	bounded above
A set with no lower bound is called ------------	bounded above	bounded below	prime	function	bounded below
The least upper bound is called -----------	bounded above	bounded below	supremum	infimum	supremum
The greatest lower bound is called -----------	bounded above	bounded below	supremum	infimum	infimum
The supremum of {3, 4} is ----------	3	4	(3, 4)	[3, 4]	4
Every finite set of numbers is ----------	bounded	unbounded	prime	bounded above	bounded

A set S of real numbers which is bounded above and bounded below is called --------	bounded set	inductive set	super set	subset	bounded set
The set N of natural numbers is ----------	bounded	not bounded	irrational	rational	not bounded
The infimum of {3, 4} is ------------	3	4	(3, 4)	[3, 4]	3
Sup C = Sup A + Sup B is called -------------- property	approximation	additive	archimedean	comparison	additive
For any real x, there is a positive integer n such that -----------	$n > x$	$n < x$	$n = x$	$n = 0$	$n > x$
If x > 0 and if y is an arbitrary real number, there is a positive number n such that nx > y is -------------- property	approximation	additive	archimedean	comparison	archimedean
The set of positive integers is -------------	bounded above	bounded below	unbounded above	unbounded below	unbounded above
The absolute value of x is denoted by ---------------	$\|x\|$	$\|\|x\|\|$	$x < 0$	$x > 0$	$\|x\|$
If x < 0 then ---------------	$\|x\| = x$	$\|\|x\|\| = \|x\|$	$\|\|x\|\| = -x$	$\|x\| = -x$	$\|x\| = -x$
If S = [0, 1) then sup S = ----------------	0	1	(0, 1)	[0,1]	1
Triangle inequality is -------------------	$\|a\| + \|b\|$ greater than equal to $\|a + b\|$	$\|a\| > \|a + b\|$	$\|b\| > \|a + b\|$	$\|a + b\|$ less than equal to $\|a\| + \|b\|$	$\|a + b\|$ less than equal to $\|a\| + \|b\|$
$\|x + y\|$ greater than equal to -----------------	$\|x\| + \|y\|$	$\|x\|\ \|y\|$	$\|x\| - \|y\|$	$\|\ \|x\| - \|y\|\ \|$	$\|\ \|x\| - \|y\|\ \|$
If (x, y) belongs to F and (x, z) belongs to F, then -------------	$x = z$	$x = y$	$xy = z$	$y = z$	$y = z$
A mapping S into itself is called ------------	function	relation	domain	transformation	transformation
If F(x) = F(y) implies x =y is a --------------- function	one-one	onto	into	inverse	one-one
One-one function is also called -----------	injective	bijective	transformation	codomain	injective
S = {(a,b) : (b,a) is in S} is called ---------------	inverse	domain	codomain	converse	converse
If A and B are two sets andif there exists a one-one correspondence between them,then it is called ------------- set	denumerable	uncountable	finite	equinumerous	equinumerous
A set which is equinumerous with the set of all positive integers is called ---------------- set	finite	infinite	countably infinite	countably finite	countably infinite
A set which is either finite or countably infinite is called ------------ set	countable	uncountable	similar	equal	countable
Uncountable sets are also called ------------- set	denumerable	non-denumerable	similar	equal	non-denumerable

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Countable sets are also called --------------- set	denumerable	non-denumerable	similar	equal	denumerable
Every subset of a countable set is ------------	countable	uncountable	rational	irrational	countable
The set of all real numbers is ----------------	countable	uncountable	rational	irrational	uncountable
The cartesian product of the set of all positive integers is ----------	countable	uncountable	rational	irrational	countable
The set of those elements which belong either to A or to B or to both is called ---------	complement	intersection	union	disjoint	union
The set of those elements which belong to both A and B is called ------------	complement	intersection	union	disjoint	intersection
Union of sets is --------------	commutative	not commutative	not associative	disjoint	commutative
The complement of A relative to B is denoted by --------------	B - A	B	A	A - B	B - A
If A intersection B is the empty set, then A and B are called --------	commutative	not commutative	not associative	disjoint	disjoint
B - (union A) = -----------------	union (B -A)	B - (intersection A)	intersection (B - A)	{}	intersection (B - A)
B - (intersection A) = -----------------------	union (B -A)	B - (union A)	intersection (B - A)	{}	union (B -A)
Union of countable sets is -----------------	uncountable	infinite	countable	disjoint	countable
The set of all rational numbers is ---------------	uncountable	infinite	countable	disjoint	countable
The set S of intervals with rational end points is ---------- set	uncountable	infinite	countable	disjoint	countable
The product of two prime numbers will always be	even number	odd number	neither prime nor composite	composite	composite
Let A be the set of all prime numbers. Then number of elements in A is	countable	uncountable	finite	empty	countable

UNIT-II

SYLLABUS

Real Sequence, Bounded sequence, Cauchy convergence criterion for sequences. Limit of a sequence. Limit Theorems. Cauchy' s theorem on limits, order preservation and squeeze theorem, monotone sequences and their convergence (monotone convergence theorem without proof).

Sequences and their limits

Definition A **sequence of real numbers** (or a **sequence in $\mathbb{R}$**) is a function defined on the set $\mathbb{N} = \{1, 2, \cdots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

(a) If $b \in \mathbb{R}$, the sequence $B := (b, b, b, \cdots)$, all of whose terms equal b, is called the **constant sequence** b. Thus the constant sequence 1 is the sequence $(1, 1, 1, \cdots)$, and the constant sequence 0 is the sequence $(0, 0, 0, \cdots)$.

The celebrated **Fibonacci sequence** $F := (f_n)$ is given by the inductive definition

$$f_1 := 1, \quad f_2 := 1, \quad f_{n+1} := f_{n-1} + f_n \quad (n \geq 2).$$

The limit of a sequence

Definition A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

We will sometimes use the symbolism $x_n \to x$, which indicates the intuitive idea that the values x_n "approach" the number x as $n \to \infty$.

When a sequence has limit x, we will use the notation

$$\lim X = x \qquad \text{or} \qquad \lim(x_n) = x.$$

We will sometimes use the symbolism $x_n \to x$, which indicates the intuitive idea that the values x_n "approach" the number x as $n \to \infty$.

Uniqueness of Limits *A sequence in $\mathbb{R}$ can have at most one limit.*

Proof

Proof. Suppose that x' and x'' are both limits of (x_n). For each $\varepsilon > 0$ there exist K' such that $|x_n - x'| < \varepsilon/2$ for all $n \geq K'$, and there exists K'' such that $|x_n - x''| < \varepsilon/2$ for all $n \geq K''$. We let K be the larger of K' and K''. Then for $n \geq K$ we apply the Triangle Inequality to get

$$\begin{aligned} |x' - x''| &= |x' - x_n + x_n - x''| \\ &\leq |x' - x_n| + |x_n - x''| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $x' - x'' = 0$.

For $x \in \mathbb{R}$ and $\varepsilon > 0$, recall that the ε-neighborhood of x is the set

$$V_\varepsilon(x) := \{u \in \mathbb{R} : |u - x| < \varepsilon\}.$$

Since $u \in V_\varepsilon(x)$ is equivalent to $|u - x| < \varepsilon$, the definition of convergence of a sequence can be formulated in terms of neighborhoods. We give several different ways of saying that a sequence x_n converges to x in the following theorem.

Theorem *Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent.*

(a) *X converges to x.*

(b) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $|x_n - x| < \varepsilon$.*

(c) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $x - \varepsilon < x_n < x + \varepsilon$.*

(d) *For every ε-neighborhood $V_\varepsilon(x)$ of x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\varepsilon(x)$.*

Proof. The equivalence of (a) and (b) is just the definition. The equivalence of (b), (c), and (d) follows from the following implications:

$$|u - x| < \varepsilon \iff -\varepsilon < u - x < \varepsilon \iff x - \varepsilon < u < x + \varepsilon \iff u \in V_\varepsilon(x).$$

Examples **(a)** $\lim(1/n) = 0$.

If $\varepsilon > 0$ is given, then $1/\varepsilon > 0$. By the Archimedean Property 2.4.5, there is a natural number $K = K(\varepsilon)$ such that $1/K < \varepsilon$. Then, if $n \geq K$, we have $1/n \leq 1/K < \varepsilon$. Consequently, if $n \geq K$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

Therefore, we can assert that the sequence $(1/n)$ converges to 0.

(b) $\lim(1/(n^2 + 1)) = 0$.

Let $\varepsilon > 0$ be given. To find K, we first note that if $n \in \mathbb{N}$, then

$$\frac{1}{n^2 + 1} < \frac{1}{n^2} \leq \frac{1}{n}.$$

Now choose K such that $1/K < \varepsilon$, as in (a) above. Then $n \geq K$ implies that $1/n < \varepsilon$, and therefore

$$\left|\frac{1}{n^2 + 1} - 0\right| = \frac{1}{n^2 + 1} < \frac{1}{n} < \varepsilon.$$

Hence, we have shown that the limit of the sequence is zero.

(c) $\lim\left(\dfrac{3n+2}{n+1}\right) = 3.$

Given $\varepsilon > 0$, we want to obtain the inequality

(1) $$\left|\frac{3n+2}{n+1} - 3\right| < \varepsilon$$

when n is sufficiently large. We first simplify the expression on the left:

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}.$$

Now if the inequality $1/n < \varepsilon$ is satisfied, then the inequality (1) holds. Thus if $1/K < \varepsilon$, then for any $n \geq K$, we also have $1/n < \varepsilon$ and hence (1) holds. Therefore the limit of the sequence is 3.

(d) If $0 < b < 1$, then $\lim(b^n) = 0$.

We will use elementary properties of the natural logarithm function. If $\varepsilon > 0$ is given, we see that

$$b^n < \varepsilon \iff n \ln b < \ln \varepsilon \iff n > \ln \varepsilon / \ln b.$$

(The last inequality is reversed because $\ln b < 0$.) Thus if we choose K to be a number such that $K > \ln \varepsilon / \ln b$, then we will have $0 < b^n < \varepsilon$ for all $n \geq K$. Thus we have $\lim(b^n) = 0$.

For example, if $b = .8$, and if $\varepsilon = .01$ is given, then we would need $K > \ln .01 / \ln .8 \approx 20.6377$. Thus $K = 21$ would be an appropriate choice for $\varepsilon = .01$. □

Theorem *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have*

$$|x_n - x| \leq C a_n \quad \text{for all} \quad n \geq m,$$

then it follows that $\lim(x_n) = x$.

Proof. If $\varepsilon > 0$ is given, then since $\lim(a_n) = 0$, we know there exists $K = K(\varepsilon/C)$ such that $n \geq K$ implies

$$a_n = |a_n - 0| < \varepsilon / C.$$

Therefore it follows that if both $n \geq K$ and $n \geq m$, then

$$|x_n - x| \leq C a_n < C(\varepsilon/C) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $x = \lim(x_n)$.

If $0 < b < 1$, then $\lim(b^n) = 0$.

This limit was obtained earlier in Example 3.1.6(d). We will give a second proof that illustrates the use of Bernoulli's Inequality (see Example 2.1.13(c)).

Since $0 < b < 1$, we can write $b = 1/(1+a)$, where $a := (1/b) - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1+a)^n \geq 1 + na$. Hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

Thus, we conclude that $\lim(b^n) = 0$.

Squeeze Theorem *Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \quad \text{for all} \quad n \in \mathbb{N},$$

and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

Proof. Let $w := \lim(x_n) = \lim(z_n)$. If $\varepsilon > 0$ is given, then it follows from the convergence of X and Z to w that there exists a natural number K such that if $n \geq K$ then

$$|x_n - w| < \varepsilon \qquad \text{and} \qquad |z_n - w| < \varepsilon.$$

Since the hypothesis implies that

$$x_n - w \leq y_n - w \leq z_n - w \qquad \text{for all} \quad n \in \mathbb{N},$$

$$-\varepsilon < y_n - w < \varepsilon$$

for all $n \geq K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\lim(y_n) = w$.

Examples **(a)** The sequence (n) is divergent.

It follows from Theorem 3.2.2 that if the sequence $X := (n)$ is convergent, then there exists a real number $M > 0$ such that $n = |n| < M$ for all $n \in \mathbb{N}$.

$$\lim\left(\frac{2n+1}{n}\right) = 2.$$

If we let $X := (2)$ and $Y := (1/n)$, then $((2n+1)/n) = X + Y$. Hence it follows that $\lim(X+Y) = \lim X + \lim Y = 2 + 0 = 2$.

Definition Let $X = (x_n)$ be a sequence of real numbers. We say that X is **increasing** if it satisfies the inequalities

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots.$$

We say that X is **decreasing** if it satisfies the inequalities

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots.$$

We say that X is **monotone** if it is either increasing or decreasing.

Monotone Convergence Theorem *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*

(a) *If $X = (x_n)$ is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}.$$

(b) *If $Y = (y_n)$ is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}.$$

Proof. It was seen in Theorem 3.2.2 that a convergent sequence must be bounded.

Conversely, let X be a bounded monotone sequence. Then X is either increasing or decreasing.

(a) We first treat the case where $X = (x_n)$ is a bounded, increasing sequence. Since X is bounded, there exists a real number M such that $x_n \leq M$ for all $n \in \mathbb{N}$. According to the Completeness Property 2.3.6, the supremum $x^* = \sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$; we will show that $x^* = \lim(x_n)$.

If $\varepsilon > 0$ is given, then $x^* - \varepsilon$ is not an upper bound of the set $\{x_n : n \in \mathbb{N}\}$, and hence there exists a member of set x_K such that $x^* - \varepsilon < x_K$. The fact that X is an increasing sequence implies that $x_K \leq x_n$ whenever $n \geq K$, so that

$$x^* - \varepsilon < x_K \leq x_n \leq x^* < x^* + \varepsilon \qquad \text{for all} \quad n \geq K.$$

Therefore we have

$$|x_n - x^*| < \varepsilon \qquad \text{for all} \quad n \geq K.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (x_n) converges to x^*.

(b) If $Y = (y_n)$ is a bounded decreasing sequence, then it is clear that $X := -Y = (-y_n)$ is a bounded increasing sequence. It was shown in part (a) that $\lim X = \sup\{-y_n : n \in \mathbb{N}\}$. Now $\lim X = -\lim Y$ and also, by Exercise 2.4.4(b), we have

$$\sup\{-y_n : n \in \mathbb{N}\} = -\inf\{y_n : n \in \mathbb{N}\}.$$

Therefore $\lim Y = -\lim X = \inf\{y_n : n \in \mathbb{N}\}$.

Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

Lemma *If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.*

Proof. If $x := \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$ then $|x_n - x| < \varepsilon/2$. Thus, if $H(\varepsilon) := K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a Cauchy sequence.

Lemma *A Cauchy sequence of real numbers is bounded.*

Proof. Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup\left\{|x_1|, |x_2|, \cdots, |x_{H-1}|, |x_H| + 1\right\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_k})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\varepsilon/2)$ such that if $n, m \geq H(\varepsilon/2)$ then

$$(1) \qquad |x_n - x_m| < \varepsilon/2.$$

Since the subsequence $X' = (x_{n_k})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \cdots\}$ such that

$$|x_K - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_K| < \varepsilon/2 \qquad \text{for} \quad n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent.

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define supremum.
2. Define a convergent sequence.
3. Define a bounded sequence.
4. Define a sequence
5. Define a convergent sequence

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Prove that $\lim n^{\frac{1}{n}} = 0$
2. Let $X = (x_n)$ and $Y = (y_n)$ be sequence of real numbers that converges to x and y respectively.Prove that the sequences $X + Y$ and XY converge to $x + y$ and xy, respectively.
3. State and prove Squeeze theorem.
4. Prove that a convergent sequence of real numbers is bounded.
5. State and prove uniqueness theorem on limit.
6. State and prove Monotone convergence theorem.
7. If (x_n) is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \epsilon N$, then $x=\lim(x_n) \geq 0$
8. If $c>0$, then $\lim(c^{1/n}) = 1$
9. If $a > 0$, then prove that $\lim \left(\frac{1}{1+na}\right) = 0$
10. Prove that a convergent sequence of real numbers is bounded. Also prove that the converse is need not be true.

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Subject: Real Analysis **Subject Code: 17MMU203**

Class : I - B.Sc. Mathematics **Semester : II**

Unit II

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If A is the set of even prime numbers and B is the set of odd prime numbers. Then	A is a subset of B	B is a subset of A	A and B are disjoint	A and B are not disjoint	A and B are disjoint
which relation is not a function?	{(2,5),(3,6).(4,7)}	{(2,1),(3,2).(4,7)}	{(2,1),(2,3).(3,4),(4,1))}	{(2,1),(3,3),(4,1)}	{(2,1),(2,3).(3,4),(4,1))}
Given the relation A={(5,2),(7,4),(9,10),(x,5)}. Which of the following value for x will make relation on A as a function?	7	9	4	5	4
Let A be the set of letters in the word " trivial" and let B be the set of letters in the word difficult. Then A-B=	{a,r,v}	{d,f,c,u}	{I,l.t}	{a,I,l,r,t,v}	{a,r,v}
Let S be the set of of all 26 letters in the alphabet and let A be the set of letters in the word "trivial". Then the number of elements in is	19	20	21	22	21
Let A={1,2}. Then A X A =	{(1,1),(2,2)}	{(1,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}	{(1,1),(2,2),(2,1)}	{(1,1)(1,2),(2,1),(2,2)}
Let A={1,2} and B={a,b,c}. Then number of elements in A X B =	2	3	2*2*2	2*3	2*3
Suppose n(A)=a and n(B)=b. Then number of elements in A X B is	a	b	ab	a+b	ab

Let A={1,2} and B={a,b,c}. Then which of the following element does not belongs to A X B =	(1,a)	(3,c)	(c,2)	(1,c)	(c,2)
Let F be a function and (x,y) in F and (x,z) in F. Then we must have	x=y	y=z	z=x	x=x	y=z
If the number of elements in a set S are %. Then the number of elements of the power set P(S)=	5	6	16	32	32
If range of f is equal to codain set, then f is	into	onto	one-one	many to one	onto
Converse of function is a function only if f is	into	onto	one-one	bijection	bijection
Inverse function is always	into	onto	one-one	bijection	bijection
If A and B contains n elements then number bijection between A and B is	n!	n	n+1	n-1	n!
Let f be a function from A to B. Then we call f as a sequence only if A is a	set of positive integers	set of all real numbers	set of all rationals	set of irrationals	set of positive integers
Two sets A and B are said to be similar iff there is a function f exists such that f is	into	one-one	onto	bijection	bijection
If two sets A={1,2,...,m} and B={1,2,..,n} are smilar then	m<n	n<m	n=m	n>0	n=m
Which of the following is an example for countable?	set of real numbers	set of all irrationals	set of all rationals	(0,1)	set of all rationals
Number of elements in the set of all real numbers is	finite	countably infinite	1000000	uncountable	uncountable
The union of elements A and B is the set of elements belongs to	either A or B	neither A not B	both A and B	A and not in B	either A or B
The set of elements belongs A and not in B is	B	A	B-A	A-B	A-B
The set of elements belongs B and not in A is	B	A	B-A	A-B	B-A
Countable union of countable set is	uncountable	countable	finite	countably infinite	countable
N X N is	uncountable	countable	finite	countably infinite	countable
Z X R is	uncountable	countable	finite	countably infinite	uncountable
R x R is	uncountable	countable	finite	countably infinite	uncountable
The set of sequences consists of only 1 and 0 is	uncountable	countbale	finite	countably infinite	uncountable
Every subset of a countable set is	uncountable	countable	finite	countably infinite	countable
Every subset of a finite set is	uncountable	countable	finite	countably infinite	finite
Fibonnaci numbers is an example for	uncountable set	countable set	finite set	infinte set	countable
Suppose A and B is countable then A X B is	uncountable	countable	finite	infinite	countable
A X B is similar to	A	B	A XA	A X B	A X B

The set of all even integers is	uncountable	countable	finite	infinite	countable
(0,1] is	uncountable	countable	finite	countably infinite	uncountable
{1,2,.....,100000}	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y implies	f(x) is not equal to f(y)	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)
Suppose f is a one to one function. Then f(x)=f(y) implies	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatble then B is	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x)=x. Then f is	onto	one to one	bijection	neither one to one nor onto	bijection
Constant function is an example for	onto	one to one	many to one	bijection	many to one
Stricly increasing function is	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, evaluate *g* (2)	8	9	13	17	13
A = {x: x ≠ x }represents	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A is	n!	2n	2^n	n	2^n

UNIT-III

SYLLABUS

Infinite series. Cauchy convergence criterion for series, positive term series, geometric series, comparison test, convergence of p-series, Root test, Ratio test, alternating series, Leibnitz's test(Tests of Convergence without proof). Definition and examples of absolute and conditional convergence.

Series : Let (a_n) be a sequence of real numbers. Then an expression of the form $a_1 + a_2 + a_3 + \ldots\ldots$ denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

Examples : 1. $1 + \frac{1}{2} + \frac{1}{3} + \ldots.$ or $\sum_{n=1}^{\infty} \frac{1}{n}$ 2. $1 + \frac{1}{4} + \frac{1}{9} + \ldots.$ or $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Partial sums : $S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$,

Convergence or Divergence of $\sum_{n=1}^{\infty} a_n$

If $S_n \to S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S. If (S_n) does not converge then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges because $S_n = log(n+1)$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges because $S_n = 1 - \frac{1}{n+1} \to 1$.
3. If $0 < x < 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ because $S_n = \frac{1-x^{n+1}}{1-x}$.

Necessary condition for convergence

$$If \sum_{n=1}^{\infty} a_n \ converges\ then\ a_n \to 0.$$

Examples :

1. If $| x | \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges because $a_n \nrightarrow 0$.
2. $\sum_{n=1}^{\infty} sinn$ diverges because $a_n \nrightarrow 0$.
3. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges, however, $log(\frac{n+1}{n}) \to 0$.

Necessary and sufficient condition for convergence

Suppose $a_n \geq 0\ \forall\ n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded above.

Example : The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because

$$S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2}$$

for all k.

If $\sum_{n=1}^{\infty} \mid a_n \mid$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof : Since $\sum_{n=1}^{\infty} \mid a_n \mid$ converges the sequence of partial sums of $\sum_{n=1}^{\infty} \mid a_n \mid$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ satisfies the Cauchy criterion.

Remark : Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \geq 1$.

(**Comparison test**) *Suppose $0 \leq a_n \leq b_n$ for $n \geq k$ for some k. Then*

(1) The convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.

(2) The divergence of $\sum_{n=1}^{\infty} a_n$ implies the divergence of $\sum_{n=1}^{\infty} b_n$.

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges because $\frac{1}{(n+1)(n+1)} \leqslant \frac{1}{n(n+1)}$. This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leqslant \frac{1}{\sqrt{n}}$.
3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $n^2 < n!$ for $n \geqslant 4$.

(**Limit Comparison Test**) *Suppose $a_n, b_n \geq 0$ eventually. Suppose $\frac{a_n}{b_n} \to L$.*

1. *If $L \in \mathbb{R}, L > 0$, then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge or diverge together.*
2. *If $L \in \mathbb{R}, L = 0$, and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*
3. *If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.*

Examples :

1. $\sum_{n=1}^{\infty}(1 - n sin\frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.
2. $\sum_{n=1}^{\infty} \frac{1}{n} log(1 + \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$ in the previous result.

(Cauchy Test or Cauchy condensation test) *If $a_n \geq 0$ and $a_{n+1} \leq a_n\ \forall\ n$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.*

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(logn)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

(Ratio test) *Consider the series $\sum_{n=1}^{\infty} a_n,\ a_n \neq 0\ \forall\ n$.*

1. If $\mid \frac{a_{n+1}}{a_n} \mid \leq q$ eventually for some $0 < q < 1$, then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

2. If $\mid \frac{a_{n+1}}{a_n} \mid \geq 1$ eventually then $\sum_{n=1}^{\infty} a_n$ diverges.

Suppose $a_n \neq 0\ \forall\ n$, and $\mid \frac{a_{n+1}}{a_n} \mid \to L$ for some L.

1. If $L < 1$ then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

3. If $L = 1$ we cannot make any conclusion.

Examples :

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \to 0$.
2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n \to e > 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $\frac{a_{n+1}}{a_n} \to 1$.

(**Root Test**) *If* $0 \le a_n \le x^n$ *or* $0 \le {a_n}^{1/n} \le x$ *eventually for some* $0 < x < 1$ *then* $\sum_{n=1}^{\infty} | a_n |$ *converges.*

Suppose $| a_n |^{1/n} \to L$ *for some* L. *Then*

1. *If* $L < 1$ *then* $\sum_{n=1}^{\infty} | a_n |$ *converges.*
2. *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $L = 1$ *we cannot make any conclusion.*

Examples :

1. $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$ converges because $a_n^{1/n} = \frac{1}{logn} \to 0$.
2. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges because $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $a_n^{1/n} \to 1$.

(**Leibniz test**) *If* (a_n) *is decreasing and* $a_n \to 0$, *then* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}, \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=2}^{\infty} (-1)^n \frac{1}{logn}$ converge.

Definition Let $X := (x_n)$ be a sequence in $\mathbb{R}$. We say that the series $\sum x_n$ is **absolutely convergent** if the series $\sum |x_n|$ is convergent in $\mathbb{R}$. A series is said to be **conditionally** (or **nonabsolutely**) **convergent** if it is convergent, but it is not absolutely convergent.

Theorem *If a series in $\mathbb{R}$ is absolutely convergent, then it is convergent.*

Proof. Since $\sum |x_n|$ is convergent, the Cauchy Criterion 3.7.4 implies that, given $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq M(\varepsilon)$, then

$$|x_{n+1}| + |x_{n+1}| + \cdots + |x_m| < \varepsilon.$$

However, by the Triangle Inequality, the left side of this expression dominates

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m|.$$

Since $\varepsilon > 0$ is arbitrary, Cauchy's Criterion implies that $\sum x_n$ converges.

Theorem *If a series $\sum x_n$ is convergent, then any series obtained from it by grouping the terms is also convergent and to the same value.*

Proof. Suppose that we have

$$y_1 := x_1 + \cdots + x_{k_1}, \qquad y_2 := x_{k_1+1} + \cdots + x_{k_2}, \quad \cdots.$$

If s_n denotes the nth partial sum of $\sum x_n$ and t_k denotes the kth partial sum of $\sum y_k$, then we have

$$t_1 = y_1 = s_{k_1}, \qquad t_2 = y_1 + y_2 = s_{k_2}, \quad \cdots$$

Thus, the sequence (t_k) of partial sums of the grouped series $\sum y_k$ is a subsequence of the sequence (s_n) of partial sums of $\sum x_n$. Since this latter series was assumed to be convergent, so is the grouped series $\sum y_k$.

It is clear that the converse to this theorem is not true. Indeed, the grouping

$$(1-1) + (1-1) + (1-1) + \cdots$$

produces a convergent series from $\sum_{n=0}^{\infty}(-1)^n$

Definition A sequence $X := (x_n)$ of nonzero real numbers is said to be **alternating** if the terms $(-1)^{n+1}x_n$, $n \in \mathbb{N}$, are all positive (or all negative) real numbers. If the sequence $X = (x_n)$ is alternating, we say that the series $\sum x_n$ it generates is an **alternating series**.

In the case of an alternating series, it is useful to set $x_n = (-1)^{n+1}z_n$ [or $x_n = (-1)^n z_n$], where $z_n > 0$ for all $n \in \mathbb{N}$.

Alternating Series Test *Let $Z := (z_n)$ be a decreasing sequence of strictly positive numbers with $\lim(z_n) = 0$. Then the alternating series $\sum(-1)^{n+1}z_n$ is convergent.*

Proof. Since we have

$$s_{2n} = (z_1 - z_2) + (z_3 - z_4) + \cdots + (z_{2n-1} - z_{2n}),$$

and since $z_k - z_{k+1} \geq 0$, it follows that the subsequence (s_{2n}) of partial sums is increasing. Since

$$s_{2n} = z_1 - (z_2 - z_3) - \cdots - (z_{2n-2} - z_{2n-1}) - z_{2n},$$

it also follows that $s_{2n} \leq z_1$ for all $n \in \mathbb{N}$. It follows from the Monotone Convergence Theorem 3.3.2 that the subsequence (s_{2n}) converges to some number $s \in \mathbb{R}$.

We now show that the entire sequence (s_n) converges to s. Indeed, if $\varepsilon > 0$, let K be such that if $n \geq K$ then $|s_{2n} - s| \leq \frac{1}{2}\varepsilon$ and $|z_{2n+1}| \leq \frac{1}{2}\varepsilon$. It follows that if $n \geq K$ then

$$\begin{aligned} |s_{2n+1} - s| &= |s_{2n} + z_{2n+1} - s| \\ &\leq |s_{2n} - s| + |z_{2n+1}| \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Therefore every partial sum of an odd number of terms is also within ε of s if n is large enough. Since $\varepsilon > 0$ is arbitrary, the convergence of (s_n) and hence of $\sum(-1)^{n+1}z_n$ is established. Q.E.D.

Note It is an exercise to show that if s is the sum of the alternating series and if s_n is its nth partial sum, then

$$(2) \qquad |s - s_n| \leq z_{n+1}.$$

It is clear that this Alternating Series Test establishes the convergence of the two series already mentioned, in (1).

Abel's Lemma *Let $X := (x_n)$ and $Y := (y_n)$ be sequences in $\mathbb{R}$ and let the partial sums of $\sum y_n$ be denoted by (s_n) with $s_0 := 0$. If $m > n$, then*

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Proof. Since $y_k = s_k - s_{k-1}$ for $k = 1, 2, \cdots$, the left side of (3) is seen to be equal to $\sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$. If we collect the terms multiplying $s_n, s_{n+1}, \cdots, s_m$, we obtain the right side of (3). Q.E.D.

Dirichlet's Test *If $X := (x_n)$ is a decreasing sequence with $\lim x_n = 0$, and if the partial sums (s_n) of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.*

Proof. Let $|s_n| \le B$ for all $n \in \mathbb{N}$. If $m > n$, it follows from Abel's Lemma 9.3.3 and the fact that $x_k - x_{k+1} \ge 0$ that

$$\begin{aligned}\left|\sum_{k=n+1}^{m} x_k y_k\right| &\le (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B \\ &= [(x_m + x_{n+1}) + (x_{n+1} - x_m)]B \\ &= 2x_{n+1}B.\end{aligned}$$

Since $\lim(x_k) = 0$, the convergence of $\sum x_k y_k$ follows from the Cauchy Convergence Criterion 3.7.4. Q.E.D.

Abel's Test *If $X := (x_n)$ is a convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.*

Proof. If (x_n) is decreasing with limit x, let $u_n := x_n - x$, $n \in \mathbb{N}$, so that (u_n) decreases to 0. Then $x_n = x + u_n$, whence $x_n y_n = x y_n + u_n y_n$. It follows from the Dirichlet Test 9.3.4 that $\sum u_n y_n$ is convergent and, since $\sum x y_n$ converges (because of the assumed convergence of the series $\sum y_n$), we conclude that $\sum x_n y_n$ is convergent.

If (x_n) is increasing with limit x, let $v_n := x - x_n$, $n \in \mathbb{N}$, so that (v_n) decreases to 0. Here $x_n = x - v_n$, whence $x_n y_n = x y_n - v_n y_n$, and the argument proceeds as before.

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define a geometric series.
2. State the nth term test.
3. Define a harmonic sequence.
4. Define alternating harmonic series.
5. Define a harmonic sequence.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. Test the convergence of series $\sum_1^\infty \frac{n!(2n)}{n^n}$.
2. Prove the $p-series$ converges if $p > 1$.
3. Show that $\sum_1^\infty \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$
4. State & prove the n^{th} term test for series.
5. Let (x_n) be a sequence of non-negative real numbers.Then the series $\sum x_n$ converges iff the sequence $S= (s_k)$ of partial sums is bounded. In this case $\sum (x_n) = \lim(s_k) = \sup \{s_k : k \in N\}$
6. State and prove Cauchy criterion for series.
7. State and prove the comparison test for the series
8. Discuss about the series (i) $\sum \frac{1}{n^2+n}$ (ii) $\sum \frac{1}{n!}$
9. Prove that if $\sum x_n$ converges then $\lim(x_n) = 0$
10. Prove that the $2-series$ converges.

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Subject: Real Analysis **Subject Code: 17MMU203**

Class : I - B.Sc. Mathematics **Semester : II**

Unit III

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If an increasing sequence is bounded above then	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 1	sequence converges to 0	sequence converges to sup of its range
If an decreasing sequence is bounded below then	seqeunce converges to inf	sequence converges to sup	sequence converges to 2	sequence converges to 1	sequence converges to inf of its range
Fibonacci sequence is	an increasing sequence	a decresing sequence	constant sequence	bounded sequence	an incresing sequence
If an increasing sequence is bounded above then	seqeunce converges to inf of its range	sequence converges to sup of its range	sequence converges to 3	sequence converges to 2	sequence converges to sup of its range
Suppose a sequence in a metric space (S,d) converges to both a and b. Then we must have	a<b	a>b	a-b=1	a=b	a=b
In a metric space (S,d), a sequence converges to p. Then range of the sequence is	bounded	unbounded	finite	infinite	bounded
The range of a constant sequence is	infinite	countably infinite	uncountable	singlton set	singleton set
Suppose in a metric space (S,d), a sequence converges to p. Then the point p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an adherent point of S	an adherent point of S

Suppose in a metric space (S,d) , a sequence converges to p and the rnage of the sequence is infinite. Then p is	an adherent point of S	an accumulation point of S	an isolated point of S	not an accumulation point of S	an accumulation point of S
Suppose in a metric space, a sequence converges. Then	every sequence in a metric space converges	every subsequence of convergent sequence converges	some subsequence of convergent sequence converges	some sequence in a metric space converges	every subsequence of convergent sequence converges
A sequence is said to be bounded if if its range is	unbounded	bounded	countable	uncountable	bounded
The range of the sequence {1/n} is	finite	{1}	{}	infinite	infinite
The range of the sequence {1/n} is	unbounded	bounded	{}	{1,0}	bounded
The esequence {1/n}	converges	diverges	oscilates	converges to 1	converges
In Euclidean metric space every cauchy sequence is	convergent	divergent	oscilates	convergent to 0	converges
Every convergent sequence is a	constant seqeunce	cauchy sequence	increasing sequence	decreasing sequence	cauchy sequence
The sequence {n^2}	converges	diverges	oscilates	converges to 2	diverges
The range of the sequence {n^2} is	unbounded	bounded	{}	{0.1}	unbounded
The range of the sequence {n^2} is	finite	{1}	{}	infinite	infinite
The sequence {i^n}	converges	diverges	oscilates	converges to 0	diverges
The range of the sequence {i^n} is	unbounded	bounded	{}	{0,1}	bounded
The range of the sequence {i^n} is	finite	infinite	{}	{0,1}	finite
The sequence {1}	converges	diverges	oscilates	converges to 0	converges
The range of the sequence {1} is	{}	{1}	{1,0}	{1,2,3}	{1}
The range of the sequence {1} is	bounded	unbounded	{1,0}	{0}	bounded

UNIT-IV

SYLLABUS

Monotone Sequences, Monotone Convergence Theorem. Subsequences, Divergence Criteria, Monotone Subsequence Theorem (statement only), Bolzano Weierstrass Theorem for sequences.Cauchy sequence, Cauchy's Convergence Criterion. Concept of cluster points and statement of Bolzano -Weierstrass theorem.

Definition Let $X = (x_n)$ be a sequence of real numbers. We say that X is **increasing** if it satisfies the inequalities

$$x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots.$$

We say that X is **decreasing** if it satisfies the inequalities

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots.$$

We say that X is **monotone** if it is either increasing or decreasing.

The following sequences are increasing:

$$(1, 2, 3, 4, \cdots, n, \cdots), \qquad (1, 2, 2, 3, 3, 3, \cdots),$$
$$(a, a^2, a^3, \cdots, a^n, \cdots) \quad \text{if} \quad a > 1.$$

The following sequences are decreasing:

$$(1, 1/2, 1/3, \cdots, 1/n, \cdots), \qquad (1, 1/2, 1/2^2, \cdots, 1/2^{n-1}, \cdots),$$
$$(b, b^2, b^3, \cdots, b^n, \cdots) \quad \text{if} \quad 0 < b < 1.$$

The following sequences are not monotone:

$$\left(+1, -1, +1, \cdots, (-1)^{n+1}, \cdots\right), \qquad \left(-1, +2, -3, \cdots, (-1)^n n \cdots\right)$$

The following sequences are not monotone, but they are "ultimately" monotone:

$$(7, 6, 2, 1, 2, 3, 4, \cdots), \qquad (-2, 0, 1, 1/2, 1/3, 1/4, \cdots).$$

Monotone Convergence Theorem *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*

(a) *If $X = (x_n)$ is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}.$$

(b) *If $Y = (y_n)$ is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}.$$

Proof. It was seen in Theorem 3.2.2 that a convergent sequence must be bounded.

Conversely, let X be a bounded monotone sequence. Then X is either increasing or decreasing.

(a) We first treat the case where $X = (x_n)$ is a bounded, increasing sequence. Since X is bounded, there exists a real number M such that $x_n \leq M$ for all $n \in \mathbb{N}$. According to the Completeness Property 2.3.6, the supremum $x^* = \sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$; we will show that $x^* = \lim(x_n)$.

If $\varepsilon > 0$ is given, then $x^* - \varepsilon$ is not an upper bound of the set $\{x_n : n \in \mathbb{N}\}$, and hence there exists a member of set x_K such that $x^* - \varepsilon < x_K$. The fact that X is an increasing sequence implies that $x_K \leq x_n$ whenever $n \geq K$, so that

$$x^* - \varepsilon < x_K \leq x_n \leq x^* < x^* + \varepsilon \qquad \text{for all} \quad n \geq K.$$

Therefore we have

$$|x_n - x^*| < \varepsilon \qquad \text{for all} \quad n \geq K.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (x_n) converges to x^*.

(b) If $Y = (y_n)$ is a bounded decreasing sequence, then it is clear that $X := -Y = (-y_n)$ is a bounded increasing sequence. It was shown in part (a) that $\lim X = \sup\{-y_n : n \in \mathbb{N}\}$. Now $\lim X = -\lim Y$ and also, by Exercise 2.4.4(b), we have

$$\sup\{-y_n : n \in \mathbb{N}\} = -\inf\{y_n : n \in \mathbb{N}\}.$$

Therefore $\lim Y = -\lim X = \inf\{y_n : n \in \mathbb{N}\}$.

Examples (a) $\lim(1/\sqrt{n}) = 0$.

It is possible to handle this sequence by using Theorem 3.2.10; however, we shall use the Monotone Convergence Theorem. Clearly 0 is a lower bound for the set $\{1/\sqrt{n}: n \in \mathbb{N}\}$, and it is not difficult to show that 0 is the infimum of the set $\{1/\sqrt{n}: n \in \mathbb{N}\}$; hence $0 = \lim(1/\sqrt{n})$.

On the other hand, once we know that $X := (1/\sqrt{n})$ is bounded and decreasing, we know that it converges to some real number x. Since $X = (1/\sqrt{n})$ converges to x, it follows from Theorem 3.2.3 that $X \cdot X = (1/n)$ converges to x^2. Therefore $x^2 = 0$, whence $x = 0$.

Examples (a) Let $Y = (y_n)$ be defined inductively by $y_1 := 1$, $y_{n+1} := \frac{1}{4}(2y_n + 3)$ for $n \geq 1$. We shall show that $\lim Y = 3/2$.

Direct calculation shows that $y_2 = 5/4$. Hence we have $y_1 < y_2 < 2$. We show, by Induction, that $y_n < 2$ for all $n \in \mathbb{N}$. Indeed, this is true for $n = 1, 2$. If $y_k < 2$ holds for some $k \in \mathbb{N}$, then

$$y_{k+1} = \tfrac{1}{4}(2y_k + 3) < \tfrac{1}{4}(4 + 3) = \tfrac{7}{4} < 2,$$

so that $y_{k+1} < 2$. Therefore $y_n < 2$ for all $n \in \mathbb{N}$.

We now show, by Induction, that $y_n < y_{n+1}$ for all $n \in \mathbb{N}$. The truth of this assertion has been verified for $n = 1$. Now suppose that $y_k < y_{k+1}$ for some k; then $2y_k + 3 < 2y_{k+1} + 3$, whence it follows that

$$y_{k+1} = \tfrac{1}{4}(2y_k + 3) < \tfrac{1}{4}(2y_{k+1} + 3) = y_{k+2}.$$

Thus $y_k < y_{k+1}$ implies that $y_{k+1} < y_{k+2}$. Therefore $y_n < y_{n+1}$ for all $n \in \mathbb{N}$.

We have shown that the sequence $Y = (y_n)$ is increasing and bounded above by 2. It follows from the Monotone Convergence Theorem that Y converges to a limit that is at most 2. In this case it is not so easy to evaluate $\lim(y_n)$ by calculating $\sup\{y_n: n \in \mathbb{N}\}$. However, there is another way to evaluate its limit. Since $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for all $n \in \mathbb{N}$, the nth term in the 1-tail Y_1 of Y has a simple algebraic relation to the nth term of Y. Since, by Theorem 3.1.9, we have $y := \lim Y_1 = \lim Y$, it therefore follows from Theorem 3.2.3 (why?) that

$$y = \tfrac{1}{4}(2y + 3),$$

from which it follows that $y = 3/2$.

(b) Let $Z = (z_n)$ be the sequence of real numbers defined by $z_1 := 1$, $z_{n+1} := \sqrt{2z_n}$ for $n \in \mathbb{N}$. We will show that $\lim(z_n) = 2$.

Note that $z_1 = 1$ and $z_2 = \sqrt{2}$; hence $1 \leq z_1 < z_2 < 2$. We claim that the sequence Z is increasing and bounded above by 2. To show this we will show, by Induction, that

$1 \le z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$. This fact has been verified for $n = 1$. Suppose that it is true for $n = k$; then $2 \le 2z_k < 2z_{k+1} < 4$, whence it follows (why?) that

$$1 < \sqrt{2} \le z_{k+1} = \sqrt{2z_k} < z_{k+2} = \sqrt{2z_{k+1}} < \sqrt{4} = 2.$$

[In this last step we have used Example 2.1.13(a).] Hence the validity of the inequality $1 \le z_k < z_{k+1} < 2$ implies the validity of $1 \le z_{k+1} < z_{k+2} < 2$. Therefore $1 \le z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$.

Since $Z = (z_n)$ is a bounded increasing sequence, it follows from the Monotone Convergence Theorem that it converges to a number $z := \sup\{z_n\}$. It may be shown directly that $\sup\{z_n\} = 2$, so that $z = 2$. Alternatively we may use the method employed in part (a). The relation $z_{n+1} = \sqrt{2z_n}$ gives a relation between the nth term of the 1-tail Z_1 of Z and the nth term of Z. By Theorem 3.1.9, we have $\lim Z_1 = z = \lim Z$. Moreover, by Theorems 3.2.3 and 3.2.10, it follows that the limit z must satisfy the relation

$$z = \sqrt{2z}.$$

Hence z must satisfy the equation $z^2 = 2z$ which has the roots $z = 0, 2$. Since the terms of $z = (z_n)$ all satisfy $1 \le z_n \le 2$, it follows from Theorem 3.2.6 that we must have $1 \le z \le 2$. Therefore $z = 2$. □

Example Let $a > 0$; we will construct a sequence (s_n) of real numbers that converges to $\sqrt{a}$.

Let $s_1 > 0$ be arbitrary and define $s_{n+1} := \frac{1}{2}(s_n + a/s_n)$ for $n \in \mathbb{N}$. We now show that the sequence (s_n) converges to $\sqrt{a}$. (This process for calculating square roots was known in Mesopotamia before 1500 B.C.)

We first show that $s_n^2 \ge a$ for $n \ge 2$. Since s_n satisfies the quadratic equation $s_n^2 - 2s_{n+1}s_n + a = 0$, this equation has a real root. Hence the discriminant $4s_{n+1}^2 - 4a$ must be nonnegative; that is, $s_{n+1}^2 \ge a$ for $n \ge 1$.

To see that (s_n) is ultimately decreasing, we note that for $n \ge 2$ we have

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) = \frac{1}{2} \cdot \frac{(s_n^2 - a)}{s_n} \ge 0.$$

Hence, $s_{n+1} \le s_n$ for all $n \ge 2$. The Monotone Convergence Theorem implies that $s := \lim(s_n)$ exists. Moreover, from Theorem 3.2.3, the limit s must satisfy the relation

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right),$$

whence it follows (why?) that $s = a/s$ or $s^2 = a$. Thus $s = \sqrt{a}$.

For the purposes of calculation, it is often important to have an estimate of *how rapidly* the sequence (s_n) converges to $\sqrt{a}$. As above, we have $\sqrt{a} \le s_n$ for all $n \ge 2$, whence it follows that $a/s_n \le \sqrt{a} \le s_n$. Thus we have

$$0 \le s_n - \sqrt{a} \le s_n - a/s_n = (s_n^2 - a)/s_n \qquad \text{for} \quad n \ge 2.$$

Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$\left(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots\right)$$

is called a **subsequence** of X.

Example 1
Let (s_n) be the sequence defined by $s_n = n^2(-1)^n$. The positive terms of this sequence comprise a subsequence. In this case, the sequence (s_n) is

$$(-1, 4, -9, 16, -25, 36, -49, 64, \ldots)$$

and the subsequence is

$$(4, 16, 36, 64, 100, 144, \ldots).$$

More precisely, the subsequence is $(s_{n_k})_{k\in\mathbb{N}}$ where $n_k=2k$ so that $s_{n_k}=(2k)^2(-1)^{2k}=4k^2$. The selection function σ is given by $\sigma(k)=2k$. $\square$

Example 2
Consider the sequence $a_n = \sin(\frac{n\pi}{3})$ and its subsequence (a_{n_k}) of nonnegative terms. The sequence $(a_n)_{n\in\mathbb{N}}$ is

$$(\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}, 0, \frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, -\frac{1}{2}\sqrt{3}, -\frac{1}{2}\sqrt{3}, 0, \ldots)$$

and the desired subsequence is

$$(\frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, 0, \frac{1}{2}\sqrt{3}, \frac{1}{2}\sqrt{3}, 0, 0, \ldots).$$

It is evident that $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $n_4 = 6$, $n_5 = 7$, $n_6 = 8$, $n_7 = 9$, $n_8 = 12$, $n_9 = 13$, etc. We won't need a formula for n_k, but here is one: $n_k = k + 2\lfloor\frac{k}{4}\rfloor$ for $k \geq 1$, where $\lfloor x\rfloor$ is the "floor function," i.e., $\lfloor x\rfloor$ is the largest integer less than or equal to x, for $x \in \mathbb{R}$. $\square$

Theorem.

Let (s_n) be a sequence.

(i) *If t is in $\mathbb{R}$, then there is a subsequence of (s_n) converging to t if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$.*

(ii) *If the sequence (s_n) is unbounded above, it has a subsequence with limit $+\infty$.*

(iii) *Similarly, if (s_n) is unbounded below, a subsequence has limit $-\infty$.*

In each case, the subsequence can be taken to be monotonic.

Proof

The forward implications $\Longrightarrow$ in (i)–(iii) are all easy to check. For example, if $\lim_k s_{n_k} = t$ and $\epsilon > 0$, then all but finitely many of the n_ks are in $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$. We focus on the other implications.

(i) First suppose the set $\{n \in \mathbb{N} : s_n = t\}$ is infinite. Then there are subsequences $(s_{n_k})_{k\in\mathbb{N}}$ such that $s_{n_k} = t$ for all k. Such subsequences of (s_n) are boring monotonic sequences converging to t.

Henceforth, we assume $\{n \in \mathbb{N} : s_n = t\}$ is finite. Then

$$\{n \in \mathbb{N} : 0 < |s_n - t| < \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0.$$

Since these sets equal

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \cup \{n \in \mathbb{N} : t < s_n < t + \epsilon\},$$

and these sets get smaller as $\epsilon \to 0$, we have

$$\{n \in \mathbb{N} : t - \epsilon < s_n < t\} \quad \text{is infinite for all} \quad \epsilon > 0, \qquad (1)$$

or

$$\{n \in \mathbb{N} : t < s_n < t + \epsilon\} \quad \text{is infinite for all} \quad \epsilon > 0; \qquad (2)$$

otherwise, for sufficiently small $\epsilon > 0$, the sets in both (1) and (2) would be finite.

a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ satisfying $t-1 < s_{n_1} < t$ and

$$\max\left\{s_{n_{k-1}}, t-\frac{1}{k}\right\} \le s_{n_k} < t \quad \text{for} \quad k \ge 2. \tag{3}$$

Specifically, we will assume $n_1, n_2, \ldots, n_{k-1}$ have been selected satisfying (3) and show how to select n_k. This will give us an infinite increasing sequence $(n_k)_{k\in\mathbb{N}}$ and hence a subsequence (s_{n_k}) of (s_n) satisfying (3). Since we will have $s_{n_{k-1}} \le s_{n_k}$ for all k, this subsequence will be monotonically increasing. Since (3) also will imply $t-\frac{1}{k} \le s_{n_k} < t$ for all k, we will have $\lim_k s_{n_k} = t$:

Select n_1 so that $t-1 < s_{n_1} < t$; this is possible by (1). Suppose $n_1, n_2, \ldots, n_{k-1}$ have been selected so that

$$n_1 < n_2 < \cdots < n_{k-1} \tag{4}$$

and

$$\max\left\{s_{n_{j-1}}, t-\frac{1}{j}\right\} \le s_{n_j} < t \quad \text{for} \quad j = 2, \ldots, k-1. \tag{5}$$

Using (1) with $\epsilon = \max\{s_{n_{k-1}}, t-\frac{1}{k}\}$, we can select $n_k > n_{k-1}$ satisfying (5) for $j = k$, so that (3) holds for k. The procedure defines the sequence $(n_k)_{k\in\mathbb{N}}$. This completes the proof of (i), and is the crux of the full proof.

(ii) Let $n_1 = 1$, say. Given $n_1 < \cdots < n_{k-1}$, select n_k so that $s_{n_k} > \max\{s_{n_{k-1}}, k\}$. This is possible, since (s_n) is unbounded above. The sequence so obtained will be monotonic and have limit $+\infty$. A similar proof verifies (iii). ∎

The Bolzano-Weierstrass Theorem *A bounded sequence of real numbers has a convergent subsequence.*

First Proof. It follows from the Monotone Subsequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{n_k})$ that is monotone. Since this

subsequence is also bounded, it follows from the Monotone Convergence Theorem 3.3.2 that the subsequence is convergent. Q.E.D.

Second Proof. Since the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.

We now bisect I_1 into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:

$$A_1 := \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}, \qquad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}.$$

If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_1. (See 1.2.1.) If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I_1''$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I_2' and I_2'', and divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I_2'\}, \qquad B_2 := \{n \in \mathbb{N} : n > n_2, x_n \in I_2''\}$$

If A_2 is infinite, we take $I_3 := I_2'$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 := I_2''$ and let n_3 be the smallest natural number in B_2.

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \le (b-a)/2^{k-1},$$

whence it follows that the subsequence (x_{n_k}) of X converges to ξ. Q.E.D.

Theorem *Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.*

Proof. Suppose $M > 0$ is a bound for the sequence X so that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If X does not converge to x, then Theorem 3.4.4 implies that there exist $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

(1) $$|x_{n_k} - x| \geq \varepsilon_0 \qquad \text{for all} \quad k \in \mathbb{N}.$$

Since X' is a subsequence of X, the number M is also a bound for X'. Hence the Bolzano-Weierstrass Theorem implies that X' has a convergent subsequence X''. Since X'' is also a subsequence of X, it converges to x by hypothesis. Thus, its terms ultimately belong to the ε_0-neighborhood of x, contradicting (1). Q.E.D.

Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

Examples (a) The sequence $(1/n)$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \geq H$, we have $1/n \leq 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if $m, n \geq H$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

Lemma *A Cauchy sequence of real numbers is bounded.*

Proof. Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup \{|x_1|, |x_2|, \cdots, |x_{H-1}|, |x_H| + 1\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_k})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\varepsilon/2)$ such that if $n, m \geq H(\varepsilon/2)$ then

$$(1) \qquad |x_n - x_m| < \varepsilon/2.$$

Since the subsequence $X' = (x_{n_k})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \cdots\}$ such that

$$|x_K - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_K| < \varepsilon/2 \qquad \text{for} \quad n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent.

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. State monotone subsequence theorem.
3. Give an example for Cauchy sequence.
4. Define a subsequence.
5. Give an example of a bounded sequence that is not a Cauchy sequence.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. State and prove Bolzano- Weirstrass theorem.
2. Prove that a Cauchy sequence of real numbers is bounded
3. Test the convergence of the series $\sum (\cos n\pi)/(n^2+1)$
4. Prove that any subsequence of a convergent sequence is convergent. Also prove that the converse need not be true.
5. If a sequence $X=(x_n)$ of real numbers converges to a real number x,then any subsequence $X'=(x_{n_k})$ of X, also converges to x.
6. Prove that any convergent sequence is a Cauchy sequence.
7. State and prove Cauchy convergence criterion
8. Prove that any subsequence of a convergent sequence is convergent. Also prove that the converse need not be true.
9. State and prove monotone subsequence theorem.
10. Prove that a bounded sequence converges to x if every subsequence converges to x.

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Subject: Real Analysis **Subject Code: 17MMU203**

Class : I - B.Sc. Mathematics **Semester : II**

Unit IV

Part A (20x1=20 Marks) **(Question Nos. 1 to 20 Online Examinations)**

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
Constant sequence	converges	oscillates	diverges	converges to 1	Converges
The sequence {1,1,1,1,1,.....}	converges	oscillates	diverges	converges to 1	converges to 1
The sequence {1,0,1,0,1,0,...}	converges	oscillates	diverges	converges to 1	Oscillates
The harmonic series converges if	P=1	p>1	P<1	P=0	p>1
In limit comparison test both the series converges absolutely if	r=1	r=0	r is not equal to zero	R=2	r is not equal to zero
For the absolute convergence of the series, the ratio between n+1th term and nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
For the absolute convergence of the series, the nth root of nth term must be	Less than r	Greater than r	Less than or equal to r	Greater than equal to r	Less than or equal to r
The alternating harmonic series	converges	oscillates	diverges	converges to 1	Converges
If a series converges absolutely, the series	converges	oscillates	diverges	converges to 1	Converges
A series converges iff converges absolutely if the series consists of ----terms	positive	negative	Non zero	Either a or b	Positive
The series 1-1+1-1+1-1+...	converges	oscillates	diverges	converges to 1	Diverges
{1,2,.....,100000}	uncountable	countable	infinite	countably infinite	countable
Suppose f is a one to one function. Then x not eqaul y implies	f(x) is not equal to f(y)	f(x)=f(y)	f(x)<f(y)	f(x)>f(y)	f(x) is not equal to f(y)

Suppose f is a one to one function. Then f(x)=f(y) implies	x=-y	y=x+10	x=y	x is not eqaul y	x=y
Let f be a bijection between A and B and A is counatble then B is	uncountable	countable	finite	similar to R	countable
Let f be a function defined on A and itself such that f(x)=x. Then f is	onto	one to one	bijection	neither one to one nor onto	bijection
Constant function is an example for	onto	one to one	many to one	bijection	many to one
Stricly increasing function is	an onto function	one to one	many to one	bijection	one to one
Strictly decreasing function is	an onto function	one to one	many to one	bijection	one to one
If *g(x)* = 3x + x + 5, evaluate *g* (2)	8	9	13	17	13
A = {x: x ≠ x }represents	{1}	{}	{0}	{2}	{}
If a set A has n elements, then the total number of subsets of A is	n!	2n	2^n	n	2^n

UNIT-V

SYLLABUS

Sequence of functions, Series of functions, Pointwise and uniform convergence. M-test, Statements of the results about uniform convergence and integrability and differentiability of functions, Power series and radius of convergence.

Power Series

Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called a *power series*. Observe the variable x. Thus the power series is a function of x provided it converges for some or all x. Of course, it converges for $x = 0$; note the convention $0^0 = 1$. Whether it converges for other values of x depends on the choice of *coefficients* (a_n). It turns out that, given any sequence (a_n), one of the following holds for its power series:

(a) The power series converges for all $x \in \mathbb{R}$;
(b) The power series converges only for $x = 0$;
(c) The power series converges for all x in some bounded interval centered at 0; the interval may be open, half-open or closed.

Theorem.

For the power series $\sum a_n x^n$, let

$$\beta = \limsup |a_n|^{1/n} \quad \textit{and} \quad R = \frac{1}{\beta}.$$

[If $\beta = 0$ we set $R = +\infty$, and if $\beta = +\infty$ we set $R = 0$.] Then
(i) *The power series converges for $|x| < R$;*
(ii) *The power series diverges for $|x| > R$.*

R is called the *radius of convergence* for the power series. Note that (i) is a vacuous statement if $R = 0$ and that (ii) is a vacuous statement if $R = +\infty$. Note also that (a) above corresponds to the case $R = +\infty$, (b) above corresponds to the case $R = 0$, and (c) above corresponds to the case $0 < R < +\infty$.

Proof

The proof follows quite easily from the Root Test 14.9. Here are the details. We want to apply the Root Test to the series $\sum a_n x^n$. So for each $x \in \mathbb{R}$, let α_x be the number or symbol defined in 14.9 for the series $\sum a_n x^n$. Since the nth term of the series is $a_n x^n$, we have

$$\alpha_x = \limsup |a_n x^n|^{1/n} = \limsup |x||a_n|^{1/n} = |x| \cdot \limsup |a_n|^{1/n} = \beta|x|.$$

The third equality is justified by Exercise 12.6(a). Now we consider cases.

Case 1. Suppose $0 < R < +\infty$. In this case $\alpha_x = \beta|x| = \frac{|x|}{R}$. If $|x| < R$ then $\alpha_x < 1$, so the series converges by the Root Test. Likewise, if $|x| > R$, then $\alpha_x > 1$ and the series diverges.

Case 2. Suppose $R = +\infty$. Then $\beta = 0$ and $\alpha_x = 0$ no matter what x is. Hence the power series converges for all x by the Root Test.

Case 3. Suppose $R = 0$. Then $\beta = +\infty$ and $\alpha_x = +\infty$ for $x \neq 0$. Thus by the Root Test the series diverges for $x \neq 0$.

The series $\sum_{n=0}^{\infty} n! x^n$ has radius of convergence $R = 0$ because we have $\lim |\frac{(n+1)!}{n!}| = +\infty$. It diverges for every $x \neq 0$.

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n. \qquad (1)$$

The radius of convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} y^n$ is $R = 1$, so the interval of convergence for the series (1) is the interval $(0, 2)$ plus perhaps an endpoint or two. Direct substitution shows the series (1) converges at $x = 2$ [it's an alternating series] and diverges to $-\infty$ at $x = 0$. So the exact interval of convergence is $(0, 2]$. It turns out that the series (1) represents the function $\log_e x$ on $(0, 2]$.

Uniform Convergence

Definition.
Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges pointwise* [i.e., at each point] to a function f defined on S if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for all} \quad x \in S.$$

We often write $\lim f_n = f$ *pointwise* [*on* S] or $f_n \to f$ *pointwise* [*on* S].

Example

Let $f_n(x) = x^n$ for $x \in [0,1]$. Then $f_n \to f$ pointwise on $[0,1]$ where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$.

Now observe $f_n \to f$ pointwise on S means exactly the following:

$$\text{for each } \epsilon > 0 \text{ and } x \text{ in } S \text{ there exists } N \text{ such that} \\ |f_n(x) - f(x)| < \epsilon \text{ for } n > N. \tag{1}$$

Note the value of N depends on both $\epsilon > 0$ and x in S. If for each $\epsilon > 0$ we could find N so that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for } all \quad x \in S \quad \text{and} \quad n > N,$$

then the values $f_n(x)$ would be "uniformly" close to the values $f(x)$. Here N would depend on ϵ but not on x. This concept is extremely useful.

Definition.
Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges uniformly on* S to a function f defined on S if

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that} \\ |f_n(x) - f(x)| < \epsilon \text{ for all } x \in S \text{ and all } n > N. \tag{1}$$

We write $\lim f_n = f$ *uniformly on* S or $f_n \to f$ *uniformly on* S.

Theorem

The uniform limit of continuous functions is continuous. More precisely, let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, suppose $f_n \to f$ uniformly on S, and suppose $S = \mathrm{dom}(f)$. If each f_n is continuous at x_0 in S, then f is continuous at x_0. [So if each f_n is continuous on S, then f is continuous on S.]

Proof

This involves the famous "$\frac{\epsilon}{3}$ argument." The critical inequality is

$$|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|. \tag{1}$$

If n is large enough, the first and third terms on the right side of (1) will be small, since $f_n \to f$ uniformly. *Once such n is selected*, the continuity of f_n implies that the middle term will be small provided x is close to x_0.

For the formal proof, let $\epsilon > 0$. There exists N in $\mathbb{N}$ such that

$$n > N \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all} \quad x \in S.$$

In particular,

$$|f_{N+1}(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all} \quad x \in S. \tag{2}$$

Since f_{N+1} is continuous at x_0 there is a $\delta > 0$ such that

$$x \in S \quad \text{and} \quad |x - x_0| < \delta \quad \text{imply} \quad |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}; \tag{3}$$

see Theorem 17.2. Now we apply (1) with $n = N+1$, (2) twice [once for x and once for x_0] and (3) to conclude

$$x \in S \quad \text{and} \quad |x - x_0| < \delta \quad \text{imply} \quad |f(x) - f(x_0)| < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

This proves that f is continuous at x_0.

Remark

Uniform convergence can be reformulated as follows. *A sequence (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on*

S if and only if

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\}$$

(a) *If g and h are integrable on $[a,b]$ and if $g(x) \le h(x)$ for all $x \in [a,b]$, then $\int_a^b g(x)\,dx \le \int_a^b h(x)\,dx$.* See Theorem 33.4(i).

We also use the following corollary:

(b) *If g is integrable on $[a,b]$, then*

$$\left|\int_a^b g(x)\,dx\right| \le \int_a^b |g(x)|\,dx.$$

Definition.

A sequence (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is *uniformly Cauchy on S* if

$$\begin{array}{l}\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}\\ |f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in S \text{ and all } m, n > N.\end{array} \tag{1}$$

Theorem

If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy criterion uniformly on a set S, then the series converges uniformly on S.

Proof

Let $f_n = \sum_{k=0}^{n} g_k$. The sequence (f_n) of partial sums is uniformly Cauchy on S, so (f_n) converges uniformly on S

Weierstrass M-test.

Let (M_k) be a sequence of nonnegative real numbers where $\sum M_k < \infty$. If $|g_k(x)| \le M_k$ for all x in a set S, then $\sum g_k$ converges uniformly on S.

Example

Show that if the series $\sum g_n$ converges uniformly on a set S, then

$$\lim_{n\to\infty} \sup\{|g_n(x)| : x \in S\} = 0. \qquad (1)$$

Solution

Let $\epsilon > 0$. Since the series $\sum g_n$ satisfies the Cauchy criterion, there exists N such that

$$n \geq m > N \quad \text{implies} \quad \left|\sum_{k=m}^{n} g_k(x)\right| < \epsilon \quad \text{for all} \quad x \in S.$$

In particular,

$$n > N \quad \text{implies} \quad |g_n(x)| < \epsilon \quad \text{for all} \quad x \in S.$$

Therefore

$$n > N \quad \text{implies} \quad \sup\{|g_n(x)| : x \in S\} \leq \epsilon.$$

This establishes (1).

Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$ [possibly $R = +\infty$]. If $0 < R_1 < R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Proof

Consider $0 < R_1 < R$. A glance at shows the series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence, since β and R are defined in terms of $|a_n|$. Since $|R_1| < R$, we have $\sum |a_n| R_1^n < \infty$. Clearly we have $|a_n x^n| \leq |a_n| R_1^n$ for all x in

$[-R_1, R_1]$, so the series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$ by the Weierstrass M-test

The limit function is continuous at each point of $[-R_1, R_1]$ [1]

Corollary.
The power series $\sum a_n x^n$ converges to a continuous function on the open interval $(-R, R)$.

Proof
If $x_0 \in (-R, R)$, then $x_0 \in (-R_1, R_1)$ for some $R_1 < R$. The theorem shows the limit of the series is continuous at x_0.

Abel's Theorem
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$, then f is continuous at $x = -R$.

Proof of Abel's Theorem
The heart of the proof is in Case 1.

Case 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1 and the series converges at $x = 1$. We will prove f is continuous on $[0, 1]$. By subtracting a constant from f, we may assume $f(1) = \sum_{n=0}^{\infty} a_n = 0$. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and $s_n = \sum_{k=0}^{n} a_k = f_n(1)$ for $n = 0, 1, 2, \ldots$. Since $f_n(x) \to f(x)$ pointwise on $[0, 1]$ and each f_n is continuous,

For $m < n$, we have

$$\begin{aligned}
f_n(x) - f_m(x) &= \sum_{k=m+1}^{n} a_k x^k = \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k \\
&= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m+1}^{n} s_{k-1} x^{k-1} \\
&= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k,
\end{aligned}$$

and therefore

$$f_n(x) - f_m(x) = s_n x^n - s_m x^{m+1} + (1-x) \sum_{k=m+1}^{n-1} s_k x^k. \qquad (1)$$

Since $\lim s_n = \sum_{k=0}^{\infty} a_k = f(1) = 0$, given $\epsilon > 0$, there is an integer N so that $|s_n| < \frac{\epsilon}{3}$ for all $n \geq N$. Then for $n > m \geq N$ and x in $[0,1)$, we have

$$\begin{aligned} \left|(1-x) \sum_{k=m+1}^{n-1} s_k x^k\right| &\leq \frac{\epsilon}{3}(1-x) \sum_{k=m+1}^{n-1} x^k \\ &= \frac{\epsilon}{3}(1-x)x^{m+1}\frac{1-x^{n-m-1}}{1-x} < \frac{\epsilon}{3}. \qquad (2) \end{aligned}$$

The first term in inequality (2) is also less than $\frac{\epsilon}{3}$ for $x = 1$. Therefore, for $n > m \geq N$ and x in $[0,1]$, (1) and (2) show

$$|f_n(x) - f_m(x)| \leq |s_n|x^n + |s_m|x^{m+1} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus the sequence (f_n) is uniformly Cauchy on $[0,1]$, and its limit f is continuous.

Case 2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = R$. Let $g(x) = f(Rx)$ and note that

$$g(x) = \sum_{n=0}^{\infty} a_n R^n x^n \quad \text{for} \quad |x| < 1.$$

This series has radius of convergence 1, and it converges at $x = 1$. By Case 1, g is continuous at $x = 1$. Since $f(x) = g(\frac{x}{R})$, it follows that f is continuous at $x = R$.

Case 3. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$, and the series converges at $x = -R$. Let $h(x) = f(-x)$ and note that

$$h(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad \text{for} \quad |x| < R.$$

The series for h converges at $x = R$, so h is continuous at $x = R$ by Case 2. It follows that $f(x) = h(-x)$ is continuous at $x = -R$. ■

Lemma.
For $x \in \mathbb{R}$ and $n \geq 0$, we have

$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} = nx(1-x) \leq \frac{n}{4}. \qquad (1)$$

Proof
Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for $k \geq 1$, we have

$$\begin{aligned}\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k} &= n\sum_{k=1}^{n}\binom{n-1}{k-1}x^k(1-x)^{n-k}\\ &= nx\sum_{j=0}^{n-1}\binom{n-1}{j}x^j(1-x)^{n-1-j}\\ &= nx. \qquad (2)\end{aligned}$$

Since $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ for $k \geq 2$, we have

$$\begin{aligned}\sum_{k=0}^{n}k(k-1)\binom{n}{k}x^k(1-x)^{n-k} &= n(n-1)x^2\sum_{j=0}^{n-2}\binom{n-2}{j}x^j(1-x)^{n-2-j}\\ &= n(n-1)x^2. \qquad (3)\end{aligned}$$

Adding the results in (2) and (3), we find

$$\sum_{k=0}^{n}k^2\binom{n}{k}x^k(1-x)^{n-k} = n(n-1)x^2 + nx = n^2x^2 + nx(1-x). \quad (4)$$

Since $(nx-k)^2 = n^2x^2 - 2nx\cdot k + k^2$,
to obtain

$$\begin{aligned}\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} &= n^2x^2 - 2nx(nx) + [n^2x^2 + nx(1-x)]\\ &= nx(1-x).\end{aligned}$$

This establishes the equality in (1). The inequality in (1) simply reflects the inequality $x(1-x) \leq \frac{1}{4}$, which is equivalent to $4x^2 - 4x + 1 \geq 0$ or $(2x-1)^2 \geq 0$.

POSSIBLE QUESTIONS

PART-B (5 x 2 =10 Marks)

Answer all the questions

1. Define absolutely convergent of a series.
2. Define radius of convergence.
3. Define power series.
4. Define uniformly convergent of a series.
5. Define convergent of a series.

PART-C (5 x 6 =30 Marks)

Answer all the questions

1. State and prove Weierstrass M test
2. State and prove Cauchy Hadamard theorem.
3. If R is the radius of convergence of the power series $\sum a_n x^n$, prove that the series absolutely convergent if $|x| < R$ and divergent if $|x| > R$.
4. State and prove differentiation theorem
5. State and prove Cauchy criterion for series of functions.
6. If R is the radius of convergence of the power series $\sum a_n x^n$, prove that the series absolutely convergent if $|x| < R$ and divergent if $|x| > R$.
7. If $\sum a_n x^n$ and $\sum b_n x^n$ converges on some interval $(-r, r), r > 0$, to the same function f, then prove that $a_n = b_n$ for all $n \in N$.
8. State and prove M test
9. State and prove Cauchy criterion for series of functions.

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
Coimbatore –641 021

Subject: Real Analysis **Subject Code: 17MMU203**

Class : I - B.Sc. Mathematics **Semester : II**

Unit V

Part A (20x1=20 Marks)
(Question Nos. 1 to 20 Online Examinations)

Possible Questions

Question	Choice 1	Choice 2	Choice 3	Choice 4	Answer
If R is the radius of convergence of the series, the series converges absolutely if |x|	>R	=R	<R	Less than or equal to R	<R
If rho=infinity, the radius of convergence R is	0	1	2	3	0
If rho=0, the radius of convergence R is	0	1	2	infinity	Infinity
If rho is finite, the radius of convergence R is	0	Rho	Reciprocal of rho	infinity	Reciprocal of rho
If R is the radius of convergence of the series, the series diverges if |x|	>R	=R	<R	Less than or equal to R	>R
If R is the radius of convergence then the interval of convergence is	(-R,R]	[-R,R]	(-R,R)	[-R,R)	(-R,R)
The sequence of functions (x/n) converges to a function x=	0	1	2	3	0
The sequence of functions x power n converges to a function x=0 if x lies between	1 and 2	-1 and 1	0 and 1	-1 and 0	-1 and 1
A series of positive terms converges then the series	converges only	converges absolutely	both A and B	neither A nor B	both A and B
A convergent series contains only finite number of negative terms then it is	converges only	converges absolutely	both A and B	neither A nor B	converges absolutely

A convergent series contains only -------- number of negative terms then it is converges absolutely	$G \cong G^*$ infinite	$G \approx G^*$ 10	$G = G^*$ finite	$G \sim G^*$ countable	$G \approx G^*$ finite
A convergent series contains only finite number of --------- terms then it is converges absolutely	negative	positive	zero	1	negative

Reg. No................................

[16MMU103]

KARPAGAM UNIVERSITY

Karpagam Academy of Higher Education
(Established Under Section 3 of UGC Act 1956)
COIMBATORE – 641 021
(For the candidates admitted from 2016 onwards)

B.Sc., DEGREE EXAMINATION, NOVEMBER 2017

First Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

1. Which of the following is a rational number?
   a. e b. π c. $\sqrt{2}$ d. $\frac{1}{100}$
2. For every real x there is an integer n such that
   a. $n < x$ b. $x < n$ c. $n = x$ d. $x \leq n$
3. Which of the following is not true?
   a. $|a|+|b| \leq |a+b|$ b. $|a|-|b| \leq |a+b|$ c. $|a+b| \leq |a|+|b|$ d. $|ab| = |a||b|$
4. Let f: R $\rightarrow$ R be a function defined by $f(x) = x^2$. Then range of f is
   a. $[0,,\infty]$ b. $[-\infty,0]$ c. $[0,\infty]$ d. R
5. Fibonacci sequence is a ------- function
   a. bounded b. onto c. one-one d. unbounded
6. $\{x_n\}$ is a constant sequence if $x_n = c$, a constant for
   a. Some $n \in N$ b. all $n \in N$ c. no $n \in N$ d. only one $n \in N$
7. A sequence in R has ------- one limit
   a. atmost b. atleast c. no d. all the above
8. $\lim \frac{2}{n} =$
   a. -1 b. 1 c. 2 d. 0
9. Geometric series $\sum_{n=1}^{\infty} r^n$ converges if
   a. $r > 1$ b. $r < 1$ c. $r = 1$ d. $r \leq n$
10. For the series $\sum_{n}^{\infty}(-1)^n$, $S_n = 1$ if n is
    a. odd b. even c. prime d. composite
11. If the series $\sum_{n}^{\infty} x_n$ converges, $\lim x_n =$
    a. 0 b. -1 c. 1 d. 2
12. The series $\sum_{n}^{\infty} \frac{1}{np}$ converges if p
    a. p<1 b. p=1 c. p>1 d. $p \geq 1$
13. A convergent sequence is ------- sequence
    a. unbounded b. constant c. bounded d. non constant
14. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ is
    a. diverges b. converges c. converges to 1 d. converges to 2
15. The series $\frac{5}{2}-\frac{7}{4}+\frac{9}{6}-\frac{11}{8}+\dots$
    a. diverges b. converges c. oscillates d. converges to 0
16. The series $\frac{1}{2}-\frac{1}{\log 2}-\frac{1}{2}+\frac{1}{\log 3}+\frac{1}{2}-\frac{1}{\log 4}+\dots$
    a. converges b. diverges c. oscillates d. converges to 0
17. The series $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\dots$
    a. converges b. diverges c. oscillates d. converges to 1
18. $\sum x_n$ is absolutely convergent if $\exists$ ------- and $n \geq N$ such that $\left|\frac{x_{n+1}}{x_n}\right| \leq 1-\frac{a}{n}$
    a. $a > 1$ b. $a \leq 1$ c. $a \geq 1$ d. None of these

19. If $a = \lim \left(n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right)\right)$ exists the $\sum x_n$ converges absolutely when

a. $a > 1$ b. $a < 1$ c. $a = 1$ d. $a \geq 1$

20. If $\sum c_n \sin nx$ converges uniformly and (c_n) is a decreasing sequence then $\lim nc_n$

a. 1 b. 2 c. 3 d. 0

PART B (5 x 2 = 10 Marks) (2 ½ Hours)
Answer ALL the Questions

21. Define countable set.
22. Define a bounded sequence.
23. Define a harmonic sequence.
24. Give an example for Cauchy sequence.
25. Define power series.

PART C (5 x 6 = 30 Marks)
Answer ALL the Questions

26. a. Prove that the set of all rational number is countable.

Or

b. If $a,b \in R$, ptove that $|a + b| \leq |a| + |b|$

27. a. If $a > 0$, then prove that $\lim\left(\frac{1}{1+na}\right) = 0$

Or

b. Prove that a convergent sequence of real numbers is bounded. Also prove that the converse is need not be true.

28. a. State and prove Cauchy criterion for series

Or

b. Prove the p- series converges if $p > 1$.

29. a. State and prove Bolzano- Weirstrass theorem.

Or

b. Prove that every convergent sequence is Cauchy sequence. Also prove that the converse need not be true.

30. a. State and prove Weierstrass M test

Or

b. State and prove Cauchy Hadamard theorem.

Reg. No..................................

[17MMU203]

KARPAGAM ACADEMY OF HIGHER EDUCATION

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Pollachi Main Road, Eachanari Post, Coimbatore – 641 021.
(For the candidates admitted from 2017 onwards)

B.Sc., DEGREE EXAMINATION, APRIL 2018

Second Semester

MATHEMATICS

REAL ANALYSIS

Time: 3 hours | Maximum : 60 marks

PART – A (20 x 1 = 20 Marks) (30 Minutes)
(Question Nos. 1 to 20 Online Examinations)

PART B (5 x 2 = 10 Marks) (2 ½ Hours)
Answer ALL the Questions

21. Give two examples for uncountable sets.
22. Define a convergent sequence.
23. State the nth term test.
24. State monotone subsequence theorem.
25. Define radius of convergence.

PART C (5 x 6 = 30 Marks)
Answer ALL the Questions

26.a. Let S be a subset of R and $\alpha \in R$. Prove that $a + \sup S = Sup(a+S))$

Or

b. State and prove Archimedean property.

27.a. State and prove Squeeze theorem.

Or

b. Prove that a convergent sequence of real numbers is bounded

28.a. State and prove the comparison test for the series

Or

b. Discuss about the series (i) $\Sigma\frac{1}{n^2+n}$ (ii) $\Sigma\frac{1}{n!}$

29.a. State and prove Cauchy convergence criterion

Or

b. Prove that any subsequence of a convergent sequence is convergent. Also prove that the converse need not be true.

30.a. If R is the radius of convergence of the power series $\Sigma a_n x^n$, prove that the series absolutely convergent if $|x| < R$ and divergent if $|x| > R$.

Or

b. If $\Sigma a_n x^n$ and $\Sigma b_n x^n$ converges on some interval(-r ,r),r>0 to the same function f, thenprove that $a_n = b_n$ for all $n \in N$.

Real Analysis

Subject code : 17MMU203 Answer key

Part-B

21. i) The set of all real numbers R is uncountable
ii) The set of all irrational numbers is uncount

22. Convergent sequence:

A sequence $\{x_n\}$ in R is said to co to $x \in R$ (or) x is said to be a limit of (x_n) if for every $\epsilon > 0$ there exists a +ve integer N such that $|x_n - x| < \epsilon \quad \forall n \geq N$.

If a sequence has a limit, we say that the sequence is convergent.

23. The n^{th} term test:

If the series Σx_n converges then $\lim (x_n) = 0$.

24. Monotone subsequence theorem:

If $X = (x_n)$ is a sequence of real nu then there is a subsequence of X that is monotone.

25. Radius of convergence:

Let $\Sigma a_n x^n$ be a power series. If the sequence $|a_n|^{1/n}$ is bounded, we set

$$\rho = \lim\sup(|a_n|^{1/n})$$

If this sequence is not bounded we set $\rho = \infty$. We define the radius of convergence of $\Sigma a_n x^n$ to be given by

$$R = \begin{cases} 0 & \text{if } \rho = \infty \\ 1/\rho & \text{if } 0 < \rho < \infty \\ \infty & \text{if } \rho = 0 \end{cases}$$

Part-C

26)
a) Let S be a subset of $\mathbb{R}$. & $\alpha \in \mathbb{R}$.

To prove that $a + \sup S = \sup(a+S)$.

By completeness property, $\sup S$ exists.

Let $u = \sup S$.

Then $x \le u \quad \forall x \in S$

$\therefore a + x \le u + a, \ \forall x \in S$

$\Rightarrow u+a$ is an upper bound of $a+S$.

let $m = \sup(a+S)$

$\therefore m \le u + a$ — ①

$$a+x \le v$$

$$\Rightarrow x \le v-a$$

$\Rightarrow$ $v-a$ is an upper bound of S.

$$\Rightarrow a+u \le v$$

$$\therefore a+u \le m \quad - (2)$$

From (1) & (2)

$$a+u = m$$

$$a + \sup S = \sup(a+S)$$

b) Archimedean Property:

If $x \in R$, then there exists $n_x \in N \ni x < n_x$.

Proof:

T.P.T Atleast one $n_x \in N \ni x < n_x$

Suppose, $n \le x \ \forall n \in N$

$\therefore$ x is an upper bound of N.

let $u = \sup N$.

$$u-1 < m$$

$$\Rightarrow u < m+1$$

Since $m+1 \in N$, we must have a contradiction

$$m+1 \le u.$$

$\therefore$ There exists $n_x \in N$ suchthat $x < n_x$.

27) a) Squeeze theorem:

Suppose that (x_n), (y_n) and (z_n) are sequence of real numbers suchthat,

$$x_n \leq y_n \leq z_n \quad \forall\, n \in N,$$

and that $\lim (x_n) = \lim (z_n)$.

Then $\lim (x_n) = \lim (y_n) = \lim (z_n)$.

Proof:

Given that $\lim (x_n) = \lim (z_n)$

Then, $\lim (x_n) = \lim (z_n) = \omega$ (say)

let $\epsilon > 0$,

$$|x_n - \omega| < \epsilon \quad \text{if } n \geq N \quad \&$$

$$|z_n - \omega| < \epsilon, \quad \text{if } n \geq N$$

also

$$x_n \leq y_n \leq z_n$$

then

$$x_n - \omega \leq y_n - \omega \leq z_n - \omega$$

$$\therefore -\epsilon < x_n - \omega < y_n - \omega < z_n - \omega < \epsilon$$

$$\Rightarrow -\epsilon < y_n - \omega < \epsilon$$

$$\Rightarrow |y_n - \omega| < \epsilon \quad \text{if } n \geq N$$

$$\therefore \lim (y_n) = \omega.$$

ATP

27)
b) A convergent sequence of real numbers is bounded.

Proof:

Suppose that $\lim(x_n) = x$,

let $\epsilon = 1 > 0$

Then $\exists$ a natural +ve integer $N \ni$

$|x_n - x| < 1$ if $n \geq N$.

Now,

$$|x_n| = |x_n - x + x|$$
$$\leq |x_n - x| + |x|$$
$$< 1 + |x|$$

let $M = \sup\{|x_1|, |x_2|, |x_{N-1}|, \ldots \ldots 1 + |x|\}$

Then,

$$|x_n| \leq M \quad \forall \ n \geq 1$$

$\therefore (x_n)$ is bounded.

Comparison test:

Let $X = (x_n)$ and $Y = (y_n)$ be real sequences and suppose that for some $k \in N$, we have

$0 \leq x_n \leq y_n$ for $n \geq k$.

a) Then the convergence of Σy_n implies the convergence of Σx_n.

b) The divergence of Σx_n implies the divergence of Σy_n.

<u>Proof:</u>

a) Suppose that Σy_n converges

By cauchy criterion,

Given, $\epsilon > 0$ $\exists$ $M(\epsilon) \in N \ni$

$|y_{n+1} + y_{n+2} + \ldots + y_m| < \epsilon$ if $m > n \geq M(\epsilon)$.

$\therefore$ $y_{n+1} + y_{n+2} + \ldots + y_m < \epsilon$

Now,

$x_{n+1} + x_{n+2} + \ldots + x_m < y_{n+1} + y_{n+2} + \ldots + y_m < \epsilon$

$\therefore$ $x_{n+1} + x_{n+2} + \ldots + x_m < \epsilon$

ie) $|x_{n+1} + x_{n+2} + \ldots + x_m| < \epsilon$, if $m > n \geq M(\epsilon)$.

By cauchy criterion, Σx_n converges.

b) suppose, Σx_n diverges

To prove, Σy_n diverges

Suppose Σy_n converges, By (a), Σx_n converges.

which is contradiction to Σx_n diverges.

$\therefore$ Σy_n diverges.

i) $\sum \frac{1}{n^2+n}$

Clearly $0 < \frac{1}{n^2+n} < \frac{1}{n^2}$, $n \in N$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

ii) $\sum \frac{1}{n!}$

$a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1)!}$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)!}{n!} = (n+1)$$

$$\lim \frac{a_n}{a_{n+1}} = \lim (n+1)$$

$$\lim \frac{a_n}{a_{n+1}} \to \infty > 1$$

$\therefore$ By ratio test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

29) a) Cauchy convergence criterion

A sequence of real numbers is convergent iff it is a cauchy sequence.

Proof:

Suppose $X = (x_n)$ is a convergent sequence.

By the known theorem, X is a cauchy sequence

Converssely, suppose $X = (x_n)$ is a cauchy sequence.

$\Rightarrow X$ is a bounded sequence.

By Bolzano weierstrass theorem,

X has a convergent subsequence (x_{n_k}).

Let $x_{n_k} \to x$

Claim: $x_n \to x$

Since X is a cauchy sequence.

for given $\epsilon/2 > 0 \exists$ a +ve integer N, $\ni$

$$|x_n - x_m| < \epsilon/2, \text{ if } n, m \geq N.$$

Since $(x_{n_k}) \to x$,

$$|x_k - x| < \epsilon/2$$

Since $k \geq N$,

$$|x_n - x_k| < \epsilon/2 \text{ if } n \geq N$$

Now,

$$|x_n - x| = |x_n - x_k + x_k - x|$$

$$\leq |x_n - x_k| + |x_k - x|$$

$$< \epsilon/2 + \epsilon/2 \quad \text{if } n \geq N$$

$$= \epsilon$$

i.e.) $|x_n - x| < \epsilon$ if $n \geq N$

$\therefore x_n \rightarrow x$

i.e.) X is a convergent sequence.

2a) b) Given that $\lim x_n = x$

$\therefore$ For given $\epsilon > 0$, $\exists$ a +ve integer N $\ni$

$|x_n - x| < \epsilon$ if $n \geq N$.

let $X' = (x_{n_k})$ be a subsequence of X.

Then $n_1 < n_2 < n_3 < \dots$

Clearly $n_k \geq k$.

Suppose $k \geq N$

Then, $n_k \geq N$

$|x_{n_k} - x| < \epsilon$

$\therefore (x_{n_k})$ converges to x.

30) a) Proof:

Suppose $0 < R < \infty$

Suppose $|x| < R$.

There is a +ve real number $c < 1 \ni |x| < c.R$

$\therefore |x| < c.\, 1/\rho$

$\Rightarrow \rho < \frac{c}{|x|}$

$\Rightarrow \limsup \sqrt[n]{|a_n|} < c/|x|$

$\therefore |a_n| < \frac{c^n}{|x|^n}$

$\Rightarrow |a_n x^n| < c^n$

Since $c < 1$, the geometric series, Σc^n converges.

By comparison test,

$\Sigma |a_n x^n|$ converges

$\therefore \Sigma a_n x^n$ converges absolutely

Suppose $|x| > R$,

$|x| > 1/\rho$

$\rho > 1/|x|$

$\therefore \limsup \sqrt[n]{|a_n|} > \frac{1}{|x|}$

$\sqrt[n]{|a_n|} > \frac{1}{|x|}$ for infinitely many n,

$\Rightarrow |a_n x^n| \geq 1$ for infinitely many n.

By comparison test, $\sum a_n x^n$ diverges.

3(o) b) $\sum a_n x^n$ and $\sum b_n x^n$ converges on $(-r, r)$.

T.P.T $a_n = b_n \ \forall n \in N$.

By differentiated $\sum a_n x^n$ term-by-term, k times, obtain

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \quad -①$$

This series converges absolutely to $f^{(k)}(x)$ for $|x| < R$ and uniformly over any closed and bounded interval in the interval of convergence. If substitute $x=0$ in ①

$$f^{k}(0) = k! \, a_k$$

$$\Rightarrow n! \, a_n = f^{n}(0) = n! \, b_n \ \forall n \in N$$

$$\Rightarrow a_n = b_n \ \forall n \in N.$$

Reg. No -------------

(17MMU203)

KARPAGAM ACADEMY OF HIGHER EDUCATION
COIMBATORE – 641021
Department of Mathematics
SECOND SEMESTER
II Internal Test
REAL ANALYSIS

Date: 01.03.2018( ) **Time: 2 Hours**

Class: I B.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 marks)

ANSWER ALL THE QUESTIONS

1.Which of the following is a geometric series?

a. $\sum_{n=1}^{\infty} r$ b. $\sum_{n=1}^{\infty} r^{\frac{1}{n}}$ c. $\sum_{n=1}^{\infty} r^n$ d. $\sum_{n=1}^{\infty} \frac{n}{r}$

2. For the series $\sum_{n=0}^{\infty}(-1)^n$, $s_n = 0$ if n is ________

a. odd b. even c. prime d. composite

3. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to ______

a. 0 b. -1 c. 1 d. 2

4. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is ___________

a. converges b. diverges c. oscilates d. converges to 1

5. Geometric series $\sum_{n=1}^{\infty} r^n$ converges if ________

a. $r \geq 1$ b. $r < 1$ c. $r = 1$ d. $r \leq 1$

6. For the series $\sum_{n=0}^{\infty}(-1)^n$, $s_n = 1$ if n is ________

a. odd b. even c. prime d. composite

7. If the series $\sum x_n$ converges, $\lim x_n =$ __________

a. 0 b. -1 c. 1 d. 2/*

8. The series $\sum_{1}^{\infty} \frac{1}{n^p}$ converges if p ______

a. p<1 b. p=1 c. p>1 d. $p \geq 1$

9. Geometric series $\sum_{n=1}^{\infty} r^n$ diverges if _______

a. $r \geq 1$ b. $r < 1$ c. $r = 1$ d. $r \leq 1$

10. If $\lim |x_n| = 0$ then $\lim x_n =$_________

a. -1 b. 0 c. 1 d. 2

11. For the series $\sum_{0}^{\infty} r^n$, $s_n =$ ___________

a. $\frac{1}{1-r}$ b. $\frac{1-r^{n+1}}{1-r}$ c. $\frac{1-r^n}{1+r}$ d. $\frac{1-r^{n+1}}{1+r}$

12. The series $\sum_{1}^{\infty} \frac{1}{\sqrt{(n+1)}}$ is ____________

a. diverges b. oscillates c. converges d. converges to 0

13. $\lim \left(\frac{n}{n+2}\right) =$ ___________

a. -1 b. 1 c. 2 d. 0

14. Geometric series $\sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$ ________

a. diverges b. converges c. oscillates d. converges to 1

15. If $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots =$_______

a. diverges b. converges c. oscillates d. converges to 1

16. The series $\sum_{1}^{\infty} \frac{1}{\sqrt{(n+n^2)}}$ is _________

a. diverges b. oscillates c. converges d. converges to 0

17. The series $\sum_{2}^{\infty} \frac{1}{n \log n}$ is __________

a. converges b. diverges c. oscillates d. converges to 0

18. $\lim \left(1 + \frac{1}{n}\right)^n =$ __________

a. e b. π c. 1.6018 d. $\sqrt{2}$

19. $\lim \left(\frac{1}{n+2}\right) =$ ___________

a. -1 b. 1 c. 2 d. 0

20. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ … …________

a .diverges b. converges c. oscillates d. converges to 1

PART-B (3 X 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21. Define a bounded sequence.

22. Define a monotone sequence.

23. State the n^{th} term test.

PART-C (3 X 8 = 24 Marks)

ANSWER ALL THE QUESTIONS

24. a)State and prove Monotone convergence theorem.

(OR)

b)Prove that the $2 - series$ converges.

25 a) Let $X = (x_n)$ and $Y = (y_n)$ be sequence of real numbers that converges to x and y respectively. Prove that the sequences $X + Y$ and XY converge to $x + y$ and xy respectively.

(OR)

b) State and prove Squeeze theorem.

26. a) Show that $\sum_1^\infty \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

(OR)

b) Test the convergence of series $\sum_1^\infty \frac{n!(2n)}{n^n}$

Reg. No -------------

(17MMU203)

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE – 641021

Department of Mathematics

SECOND SEMESTER

I Internal Test - Jan' 18

REAL ANALYSIS

Date: 20.01.2018 (AN) **Time: 2 Hours**

Class: I B.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 marks)

ANSWER ALL THE QUESTIONS

1) A real number $\frac{p}{q}$, $(p,q \in Z)$ is a rational number if --------

a) $q > 0$ b) $q \neq 0$ c) $q < 0$ d) $q = 0$

2) For every real x there is an integer n such that ----------

a) $n < x$ b) $x < n$ c) $n = x$ d) $x \leq n$

3) Which of the following is the triangle inequality?

a) $|a| + |b| \leq |a+b|$ b) $|a| - |b| \leq |a-b|$

c) $|a+b| \leq |a| + |b|$ d) $|a| \leq |b|$

4) Let $f: R \rightarrow R$ be a function defined by $f(x) = |x|$. Then range of f is ---------

a) $[0,,\infty)$ b) $(-\infty,0)$ c) $(0,\infty)$ d) R

5. Let $f: R \rightarrow R$ be a function defined by $f(x) = x$. Then f is ------

a) onto b) one-one c) both a and b d) neither a nor b

6. A sequence in R has ------- one limit

a) atmost b) atleast c) no d) exactly

7. Let $f: A \rightarrow B$ be an onto function. Then we have -------

a) $R_f \subset A$ b) $R_f \subset B$ c) $R_f = A$ d) $R_f = B$

8. Which of the following is a rational number?

a) e b) π c) $\sqrt{2}$ d) $\frac{1}{100}$

9. Every nonempty set of real numbers that has an upper bound also has a --------- in R.

a) infimum b) supremum c) countable d) finite

10. A set S is said to be countably infinite if there is a -------- between N and S

a) one-to-one b) onto c) bijection d) injection

11. The set N of natural numbers is an -------- set.

a) uncountable b) finite c) infinite d) empty

12. The union of two disjoint denumerable sets is --------

a) denumerable b) uncountable c) finite d) infinite

13. A set S is said to be infinite if it is -------

a) finite b) not finite c) empty d) countable

14. Which of the following is a rational number?

a) e b) π c) $\sqrt{2}$ d) $\frac{1}{100}$

15. Let $f: R \rightarrow R$ be a function defined by $f(x) = x^2$. Then range of f is -----------

a) $[0,,\infty)$ b) $(-\infty,0)$ c) $(0,\infty)$ d) R

16. The nth (n≥2) term of the Fibonacci sequence is --------

a) $f_n = f_{n-2} + f_{n-1}$ b) $f_{n+1} = f_{n-1} + f_n$

c) $f_n = f_{n-2} \times f_{n-1}$ d) n

17. If $S=\{1/n: n \in N\}$, then inf S = -------

a) 0 b) 1 c)2 d) 3

18. The empty set φ is said to have --------- elements

a) 2 b) 8 c) 0 d) 5

19. $\lim(1/n)$ = ------

a) 2 b) 3 c) 0 d) 9

20. If $0<b<1$, then $\lim(b^n)$= ---------

a) 0 b) 6 c)3 d) 2

PART-B (3 X 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21. Define countable set.
22. Give two examples for uncountable sets.
23. Define a convergent sequence.

PART-C (3 X 8 = 24 Marks)

ANSWER ALL THE QUESTIONS

24. a) Prove that the set of all rational number is countable.

(OR)

b) If $a, b \in R$, then prove that $|a+b| \leq |a| + |b|$

25. a) State and prove uniqueness theorem on limit.

(OR)

b) State and prove Archimedean property.

26. a) State and prove Cantor 's Theorem

(OR)

b) Let S be a subset of R and $a \in R$. Prove that $a + \sup S = \sup(a + S)$

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Karpagam Academy of Higher Education

III Internal Test

Answer key

class : I B.Sc Mathematics

Subject Name : Real Analysis

subject code : 17MMU203

Part-A

1) a
2) d
3) b
4) b
5) c
6) a
7) a
8) a
9) c
10) b
11) c
12) c
13) b
14) b
15) a
16) d
17) b
18) a
19) c
20) a

Part-B

21) Monotone subsequence theorem

If $X=(x_n)$ is a sequence of real numbers then there is a subsequence of X that is monotone.

22) Radius of convergence:

let $\Sigma a_n x^n$ be a power series

$$\rho = \lim\sup\left(|a_n|^{1/n}\right), \quad R = \begin{cases} 0 & \text{if } \rho=\infty \\ 1/\rho & \text{if } 0<\rho<\infty \\ \infty & \text{if } \rho=0 \end{cases}$$

Part-c

26)(b) ~~Bolzano~~ Weierstrass M-test theorem

Statement:

Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D$, $n \in \mathbb{N}$.
If the series ΣM_n is convergent, then Σf_n is uniformly convergent on D.

Proof:

Suppose $m > n$,

$$|f_{n+1}(x) + f_{n+2}(x) + \dots f_m(x)|$$

$$\leq |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_m(x)|$$

$$\leq M_{n+1} + M_{n+2} + \dots + M_m$$

By cauchy criterion for series.

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \epsilon$$

Since ΣM_n converges,

$$|M_{n+1} + M_{n+2} + \dots + M_m| < \epsilon$$

$$\therefore |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \epsilon$$

By cauchy criterion for sequence of functions,

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \epsilon$$

$\therefore \Sigma f_n$ is uniformly convergent on D.

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24) b) Let $X=(x_n)$ be a cauchy sequence

let $\epsilon=1$.

Then $\exists$ a +ve integer $N \ni$

$$|x_n - x_m| < 1 \quad \text{if } n,m \geq N$$

In particular,

$$|x_n - x_N| < 1 \quad \text{if } n \geq N$$

Now, $|x_n| < 1 + |x_N|$, if $n \geq N$

Let $M = \sup\{|x_1|, |x_2|, \dots |x_{N-1}|, 1+|x_N|\}$

Then $|x_n| < M \quad \forall n$

$-M < x_n < M \quad \forall n$

$\therefore (x_n)$ is bounded

25) a) $\sum_1^\infty \frac{\cos n\pi}{n^2+1}$

$$\sum_1^\infty \frac{\cos n\pi}{n^2+1} = \frac{\cos\pi}{2} + \frac{\cos 2\pi}{2^2+1} + \frac{\cos 3\pi}{3^2+1} + \dots$$

$$= \frac{-1}{1^2+1} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \dots$$

$\therefore$ The given series is an alternating series

$a_n = \frac{1}{n^2+1}$, $a_{n+1} = \frac{1}{(n+1)^2+1}$

$$a_{n+1} - a_n = \frac{1}{(n+1)^2+1} - \frac{1}{n^2+1}$$

< 0

$a_{n+1} - a_n < 0$

$a_{n+1} < a_n$

$\therefore \{a_n\}$ is monotonically decreasing.

$$\lim_{n \to \infty} a_n = \frac{1}{n^2+1} = \frac{1}{\infty} = 0.$$

$\therefore$ The given series convergence.

25) b) Suppose $f_n \to f$ uniformly on A.

Then for $\epsilon > 0$, $\exists$ a +ve integer $N \ni$

$|f_n(x) - f(x)| < \epsilon$ if $n \geq N$.

$\Rightarrow \|f_n - f\| \to 0$

Conversely suppose $\|f_n - f\| \to 0$ on A.

$\|f_n - f\| < \epsilon$ if $n \geq N$

$\Rightarrow |f_n(x) - f(x)| < \|f_n - f\| < \epsilon$ on A

$\therefore f_n \to f$ uniformly

26) a) Given that, $\lim x_n = x$

$\therefore$ For given $\epsilon > 0$, $\exists$ a +ve integer $N \ni$

$|x_n - x| < \epsilon$ if $n \geq N$.

Let $X' = (x_{n_k})$ be a subsequence of X.

Then $n_1 < n_2 < n_3 < \ldots$

clearly $n_k \geq k$

Suppose, $k \geq N$.

Then $n_k \geq N$.

$\therefore |x_{n_k} - x| < \epsilon$

$\therefore (x_{n_k})$ converges to x.

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4) a) Bolzano- weiesstrass theorem:

A bounded sequence of real numbess has convergent subsequence.

Proof:

Let $X = (x_n)$ be a bounded sequence,

By monotone subsequence theorem, x has a monotone subsequence (x_{n_k}).

Since X is bounded, (x_{n_k}) is bounded.

$\therefore (x_{n_k})$ is monotone bounded sequence.

By monotone convergence theorem, (x_{n_k}) converges.

KARPAGAM ACADEMY Of Higher Education

I Internal Test

Answer key

class : I B.Sc Mathematics

Subject : Real Analysis

subject code : 17MMU203

Part-A

1. b
2. b
3. c
4. a
5. c
6. b
7. b
8. d
9. b
10. c
11. b
12. a
13. b
14. d
15. a
16. b
17. a
18. c
19. c
20. a

Part-B

21) A set which is either finite or countably infinite is called a countable set.

22) i) The set of all real numbers R is uncountable

ii) The set of all irrational numbers is uncountable

23) A sequence $\{x_n\}$ in R is said to converge to $x \in R$ if for every $\epsilon > 0$ there exists a +ve integer N such that $|x_n - x| < \epsilon$ $\forall$ $n \geq N$.

Part-C

24) a)

W.K.T $N \times N$ is countable

Then $\exists$ a surjection f of N onto $N \times N$.

The mapping $g : N \times N \rightarrow Q^+$

The composition $g \circ f$ is surjection of N onto Q^+.

$$Q = Q^- \cup \{0\} \cup Q^+$$

$\therefore$ Q is countable.

b) W.K.T $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$

By adding above inequalities,

$$-|a| - |b| \leq a+b \leq |a| + |b|$$

$$-(|a| + |b|) \leq a+b \leq |a| + |b|$$

By $|a| \leq c$,

$$\Rightarrow |a+b| \leq |a| + |b|$$

a) Uniqueness thm. on limit

A sequence (x_n) in R can have atmost one limit.

<u>Proof:</u>

Since $\epsilon/2 > 0$, & $x_n \to x'$ $\exists$ a N_1 s.t $|x_n - x'| < \epsilon/2$ for every $n \geq N_1$

Similarly, $|x_n - x''| < \epsilon/2 \; \forall \; n \geq N_2$

$$\therefore \; |x' - x''| \leq |x_n - x'| + |x_n - x''|$$

$$= \epsilon$$

$$\Rightarrow \; x' = x''$$

b) <u>Archimedean Property</u>

If $x \in R$, then there exists $n_x \in N$ suchthat $x < n_x$.

<u>Proof:</u>

let $u = \sup N$.

$$u - 1 < m$$

$$u < m+1$$

Since $m+1 \in N$, we must have contradiction.

$$m+1 \leq u$$

$\therefore$ there exists $n_x \in N$ s.t $x < n_x$.

26) a) Cantor's theorem.

If A is any set, then there is no surjection of A onto the set $P(A)$ of all subsets of A.

<u>Proof:</u> Suppose $\phi: A \to P(A)$ is a surjection, $D = \phi(a_0), a_0 \in A$

$D = \phi(a_0) \Rightarrow a_0 \in \phi(a_0) \Rightarrow \Leftarrow$ to the definition of D

$a_0 \in D$.

lly if $a_0 \notin D$, $a_0 \notin \phi(a_0)$ so that

which is also contradiction.

$\therefore \phi$ cannot be a surjection.

b) Let S be a nonempty bounded above subset of $\mathbb{R}$.

let $u = \sup S$

$\therefore a + x \leq u + a \quad \forall x \in S$

let $m = \sup(a+S)$

$\therefore m \leq u + a \quad - (1)$

Suppose v is an upperbound of $a+S$

$\therefore a + x \leq v \quad \forall x \in S$

$\therefore x \leq v - a$

Since v is an upper bound of $a+S$, $a + u \leq m - (2)$

From (1) & (2) $\quad a + u = m$

$a + \sup S = \sup(a+S)$

KARPAGAM ACADEMY of Higher Education

II Internal Test

Answer key

Class: I B.Sc Mathematics Subject code: 17MMU203

Subject: Real Analysis

Part A

1) c
2) a
3) c
4) a
5) b
6) b
7) a
8) c
9) a
10) b
11) a
12) a
13) b
14) b
15) a
16) a
17) b
18) a
19) d
20) a

Part-B

21) A sequence (x_n) of real numbers is said to bounded, if there exists a real number, $M>0$ suchthat

$$|x_n| \le M \quad \forall \text{ all } n \in N.$$

22) A sequence (x_n) is monotone if it is either increasing or decreasing.

23) The n^{th} term test:

If the series $\sum x_n$ converges then $\lim (x_n) = 0$.

Part-C

24) a) Monotone Convergence theorem

Statement:

A monotone sequence of real numbers is convergent iff it is bounded. Moreover

i) If $X=(x_n)$ is bounded increasing sequence, then

$\lim(x_n) = \sup\{x_n : n \in N\}$.

ii) If $Y=(y_n)$ is a bounded decreasing sequence, then

$\lim(y_n) = \inf\{y_n : n \in N\}$

Proof:

Suppose a monotone sequence is convergent, then the sequence is bounded.

Conversely, suppose a monotone sequence is bounded

i) Let 'X' be a increasing sequence & bounded.

$\therefore x_n \leq M \; \forall \; n \in N$.

$\therefore \{x_n : n \in N\}$ is bounded above.

let $x^* = \sup\{x_n, n \in N\}$

$x^* - \epsilon < x_k$

Hence $x^* - \epsilon < x_k \leq x_n \leq x^* < x^* + \epsilon$ if $n \geq k$

$x^* - \epsilon < x_n < x^* + \epsilon$

$\Rightarrow |x_n - x^*| < \epsilon$

$\lim(x_n) = x^*$

ii) Let $Y=(y_n)$ be a bounded decreasing sequence

Then $X = -Y = (-y_n)$

By (i), $\lim(-y_n) = \sup\{-y_n ; n \in N\}$

$\Rightarrow \lim Y = \inf\{y_n : n \in N\}$

24) b) P.T $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

let $K_1 = 2^1 - 1 = 2-1 = 1$

$S_{K_1} = S_1 = 1$

let $K_2 = 2^2 - 1 = 4-1 = 3$

$S_{K_2} = S_3$ = sum of first three terms

$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$< 1 + 1/2$

Illy $S_{K_3} < 1 + 1/2 + (1/2)^2$

$\therefore S_{K_j} < 1 + 1/2 + (1/2)^2 + \cdots + (1/2)^{j-1}$

since the terms on the right hand side is a partial sums of a geometric series $\sum_{n=0}^{\infty} r^n$ with $r = 1/2 < 1$

Also $\sum_{n=0}^{\infty} (1/2)^n = 2$

The partial sum of $\sum_{n=1}^{\infty} 1/n^2$ is bounded

also $S_1 \le S_2 \le \cdots$

$\therefore \sum_{n=1}^{\infty} 1/n^2$ converges.

25) a) Let $\epsilon > 0$ be given

$|x_n - x| < \epsilon/2 \quad \forall n \geq N_1$

$|y_n - y| < \epsilon/2 \quad \forall n \geq N_2$

$\therefore |(x_n + y_n) - (x+y)| \le |x_n - x| + |y_n - y|$

$< \epsilon$

$\therefore (x_n + y_n) \to x+y$

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consider, $|x_n y_n - xy| \le |x_n(y_n - y)| + |y(x_n - x)|$

$$= |x_n|\,|y_n - y| + |y|\,|x_n - x|$$

$$|x_n| \le M_1 \quad \forall n \geq 1.$$

$$\therefore |x_n y_n - xy| \le M(\epsilon/2M) + M(\epsilon/2M)$$

$$< \epsilon$$

$$\therefore x_n y_n \to xy$$

b) **Squeeze Theorem**

Suppose that (x_n), (y_n) and (z_n) are sequence of real numbers suchthat, $x_n \le y_n \le z_n \; \forall n \in \mathbb{N}$ and that $\lim(x_n) = \lim(z_n)$ Then

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Proof:

Given that $\lim(x_n) = \lim(z_n)$

Then, $\lim(x_n) = \lim(z_n) = \omega$ (Say)

Let $\epsilon > 0$ be given

then $\exists$ a +ve integer $N \ni |x_n - \omega| < \epsilon$ if $n \geq N$

and $|z_n - \omega| < \epsilon$ if $n \geq N$. also given that

$$x_n \le y_n \le z_n$$

Then $x_n - \omega \le y_n - \omega \le z_n - \omega$

$$-\epsilon < x_n - \omega < y_n - \omega < z_n - \omega < \epsilon$$

$$\Rightarrow |y_n - \omega| < \epsilon \quad \text{if } n \geq N.$$

$$\therefore \lim(y_n) = \omega.$$

26) a) S.T $\sum_{1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

Now, $\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$

$$\sum_{1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{1}^{\infty} \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+1} + \frac{1}{n+2}\right]$$

$$= 1/4$$

b) $\sum_{1}^{\infty} \frac{n!\,(2n)}{n^n}$

$a_n = \frac{n!\,(2n)}{n^n}$, $a_{n+1} = \frac{(n+1)!\; 2(n+1)}{(n+1)^{n+1}}$

$$\frac{a_n}{a_{n+1}} = \frac{\cancel{n^n}(1+1/n)^n\,(\cancel{n+1})}{(\cancel{n+1}).\,(1+1/n)\,\cancel{n^n}}$$

$$\lim \left(\frac{a_n}{a_{n+1}}\right) = e > 1$$

$\therefore$ By ratio test,

$\sum_{1}^{\infty} \frac{n!\,(2^n)}{n^n}$ converges.

Reg. No -------------

(17MMU203)

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE – 641021

Department of Mathematics

SECOND SEMESTER

II Internal Test

REAL ANALYSIS

Date: **Time: 2 Hours**

Class: I B.Sc Mathematics **Maximum: 50 Marks**

PART – A (20 × 1 = 20 marks)

ANSWER ALL THE QUESTIONS

1. Constant sequence is __________
   a) converges b)oscillates c)diverges d)converges to 1
2. The sequence {1,1,1,1,1,…..}________
   a)converges b)oscillates c)diverges d)converges to 1
3. The sequence {1,0,1,0,1,0,…} ____________
   a)converges b)oscillates c)diverges d)converges to 1
4. The harmonic series is converges if _______
   a)p=1 b)p>1 c)p<1 d)p=0
5. For the absolute convergence of the series, the nth root of nth term must be ________
   a)less than r b)greater than r c)less than or equal to r d)greater than or equal to r
6. The alternating harmonic series is ________
   a)converges b)oscillates c) diverges d)converges to 1
7. If a series converges absolutely, then the series ________
   a)converges b)oscillates c)diverges d)converges to 1
8. A series converges iff converges absolutely if the series consists of ______terms
   a)positive b)negative c)non zero d)either a or b
9. The series 1-1+1-1+1-1+… is __________
   a)converges b)oscillates c)diverges d)converges to 1
10. {1,2,…..,100000} is __________
    a)uncountable b)countable c)infinite d)countably infinite
11. Let f be a function defined on A and itself such that f(x)= x. Then f is _________
    a)onto b)one to one c)bijection d)neither one to one nor onto
12. Constant function is an example for ___________
    a)onto b)one to one c)many to one d)bijection
13. Strictly increasing function is an __________
    a) onto function b)one to one c)many to one d)bijection
14. Strictly decreasing function is an ___________
    a)onto function b)one to one c)many to one d)bijection
15. If rho=infinity, the radius of convergence R is ______
    a)0 b)1 c)2 d)3

16. If rho=0, the radius of convergence R is __________
a) 0 b)1 c)2 d)infinity
17. If rho is finite, the radius of convergence R is ________
a) 0 b)Rho c)Reciprocal of rho d)infinity
18. If R is the radius of convergence of the series, the series diverges if $|x|$ _____
a)>R b)=R c)<R d)less than or equal to R
19. If R is the radius of convergence then the interval of convergence is ________
a) (-R,R] b)[-R,R] c)(-R,R) d)[-R,R)
20."If the series converges at x = R, then f is continuous at __________"
a)x=R b)x<R c)x>R d)x≠R

PART-B (3 X 2 = 6 Marks)

ANSWER ALL THE QUESTIONS

21. State monotone subsequence theorem.
22. Define radius of convergence.
23. Give an example of a bounded sequence that is not a Cauchy sequence.

PART-C (3 X 8 = 24 Marks)

ANSWER ALL THE QUESTIONS

24. a) State and prove Bolzano- Weirstrass theorem.

(OR)

b) Prove that a Cauchy sequence of real numbers is bounded

25. a) Test the convergence of the series $\sum (\cos n\pi)/(n^2+1)$

(OR)

b)Prove that a sequence (f_n) of bounded function on A contained in R converges uniformly on A to f if and only if $\| f_n - f\| \to 0$.

26. a)Prove that any subsequence of a convergent sequence is convergent. Also prove that the converse need not be true.

(OR)

b) State and prove *M* test