UNIT V
SYLLABUS

The Riemann - Stieltjes integral : Introduction –Notation –The definition of Riemann –Stieltjes integral –linear properties –Integration by parts –change of variable in a Riemann –stieltjes integral – Reduction to a Riemann integral

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1.1.The Riemann-Stieltjes Integral.

Definitions:

Let [a, b] be a given interval. Then a set P = {x0 , x1, …, xn-1 , xn } of [a, b] such that a= x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b is said to be a Partition of [a, b]. The set of all partitions of [a, b] is denoted by P([a, b]). The intervals [x0 , x1], [x0 , x1], [x1 , x2], …, [xn-1 , xn] are called the subintervals of [a, b]. Write Δxi = xi – xi-1 is called the length of the interval [xi-1, xi] (i = 1, …, n) and max |Δxi | is called the norm of the partition P and is denoted by ||P|| or Q is called the refinement or finer of the partition P.⊂(P). A partition Q of [a, b] such that P ⊂ Q is called the refinement or finer of the partition P.⊂(P). A partition Q of [a, b] such that P Suppose f is a bounded real valued function defined on [a, b] and 2 P([a, b]). ThεMi = sup f(x) , m i = inf f(x) (xi-1 ≤ x ≤ xi) for each P

Suppose f is a bounded real valued function defined on [a, b] and P([a, b]). Thenε

Mi = sup f(x) , m i = inf f(x) (xi-1 ≤ x ≤ xi) for each P n n m i Δxi are called the Upper and Lower Riemann sumsΣ M i Δxi and L(P, f) = ΣU(P, f) = i =1 i =1 or Upper and Lower Darboux sums of f on [a, b] with respect to the partition P .

Further write - b b ∫ f dx = inf U(P, f) and ∫ f dx = sup L(P, f) a - a where the inf and the sup are taken over all partitions P of [a, b] are called the Upper and Lower Riemann integrals of f over [a, b], respectively.

If the upper and lower Riemann integrals are R[a, b] and we denoteεequal, we say that f is Riemann-integrable on [a, b] and we write f the common value of these integrals by b ∫ f d(x), a - b b b i.e., ∫ f dx = ∫ f dx = ∫ f dx .

R is bounded function then the upper and lower Riemann→1.1.1. Lemma . If f : [a, b] integrals of f are bounded. Since f is bounded, there exist two numbers m and M such that m ≤ f(x) ≤ M (a ≤ x ≤ b). Hence, for every partition P of [a, b] we have M≤ Mi ≤ mi ≤m miΔxiΣ ≤mΔxi Σ ⇒ MiΔxiΣ ≤ MΔxiΣ ≤ , i = 1, 2, 3, …, n. m(b –a) ≤ L(P,f) ≤ U(P,f) ≤ M(b-a),⇒ so that the numbers L(P,f) and U(P,f) form a bounded set. Therefore by the definition of lower and upper Riemann integrals this shows that the upper and lower integrals are defined for every bounded function f are bounded also. The question of their equality, and hence the question of the integrability of f,

R is bounded function, P is any partition of [a, b] and P* is the→

1.1.2. Lemma. If f : [a, b] refinement of P, then L(P, f) ≤ L(P*, f) and U(P*, f) ≤ U(P, f). R is bounded function and P1, P2 are any two partitions of [a, b]→

1.1.3. Lemma. R is bounded function and P1, P2 are any two partitions of [a, b]→If f : [a, b]

L(P1, f) ≤ U(P2, f) and L(P2, f) ≤ U(P1, f).

R are bounded functions and P is any partition of [a, b] then→1.1.4.

Lemma. If f, g : [a, b] (i) L(P, f + g) ≥ L(P, f) + L(P, g) (ii) U(P, f + g) ≤ U(P, f) + U(P, g). R is bounded function .

Theorem. If f : [a, b] - b b ∫ f dx ≥ ∫ f dx a - a εR is bounded function then for →1.1.2. Theorem (Darboux). If f : [a, b] δ> 0 there exists > 0 such that - b b U(P, f) < and L(P, f)ε∫ f + ε> ∫ f dx - a - a R is bounded function is Riemann Integrable if the oscillatory→

1.1.3. Theorem. If f : [a, b] sum < (P, f) = U(P,f) – L(P,f)ω, i.e. ε < ε , for ε > 0 and any partition P of [a, b]. R is Riemann Integrable.

1.1.4.Theorem. Every continuous function f : [a, b] R is Riemann Integrable.→1.1.5. Theorem. Every monotone function f : [a, b] Students you studied the properties given above and other properties of Riemann Integrals in previous classes therefore we are not interested to investigate these here. However we shall immediately consider a more general situation. be a monotonically increasingα R is bounded function and →

1.1.2 Definition. Let f : [a, b] function on [a, b]. Let P = {x0 , x1, …, xn-1 , xn } such that a= x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b be any Partition of [a, b] . We write (xi-1), i = 1, 2, 3, … , n.α(xi) - αi =αΔ is bounded on [a, b],α(b) are finite therefore α(a) and αBy the definition of monotone function i ≥ 0, i = 1, 2, 3, …, n.αΔ is monotonically increasing function then clearly αalso since P([a, b]). We defineεLet Mi = sup f(x) , mi = inf f(x) (xi-1 ≤ x ≤ xi) for each P n n i,α mi ΔΣ) = αi, and L(P, f, α Mi ΔΣ) = αU(P, f, i=1 i=1 are called the Upper and Lower Riemann Stieltjes sums respectively. Further we define -b b),α = sup L(P, f, α) and ∫ f dα = inf U(P, f, α∫ f d a -a where the inf and the sup are taken over all partitions P of [a, b], are called the Upper and Lower Riemann Stieltjes integrals of f over [a, b], respectively.

If the upper and lower Riemann Stieltjes integrals are equal, we say that f is Riemann Stieltjes integrable on [a, b]

Lower Riemann Stieltjes integrals of f over [a, b], respectively. If the upper and lower Riemann Stieltjes integrals are equal, we say that f is Riemann Stieltjes integrable on [a, b]

∫ f d (x)α or ∫ f(x) dα∫ f d

overαThis is the Rientatm-Stieltjes integral (or simply the Slielljes integral of f with respect to (x) = x we see that the Riemann integral is the special case of the Riemannα[a,b]. If we put (x) = x

we see that the Riemann integral is the special case of the Riemanns

If f: [a, b] be a monotonically increasing functionα R is bounded function

. Lemma If f: [a, b] on [a, b]. Let P be any Partition of [a, b] .Then the upper and lower Riemann-Stietjes integrals of be a monotonically increasing function on [a, b]. Let P be any Partition of [a, b] .Then the upper and lower Riemann-Stietjes integrals of are bounded.αf with respect to

Proof. Since f is bounded, there exist two numbers m and M such that m ≤ f(x) ≤ M (a ≤ x ≤ b). Hence, for every partition P of [a, b] we have M≤ Mi ≤ mi ≤m iαmiΔΣ ≤i αmΔΣ ⇒ iαMiΔΣ ≤ iαMΔΣ ≤ , i = 1, 2, 3, …, n. (a)],α(b) – α) ≤ M[α) ≤ U(P, f, α(a)] ≤ L(P, f, α(b) – α m[⇒) form a bounded set.

Therefore by the definition ofα) and U(P, f, αso that the numbers L(P, f, lower and upper Riemann-Stietjes integrals this shows that the upper and lower integrals are defined for every bounded function f are bounded also. 1.1.6.

Lemma. If P* is a refinement of the partition P of [a, b], then).α) ≤ U(P, f , α) and U(P*, f , α) ≤ L(P*, f , αL(P, f ,

Proof. Let P = {x0 , x1, …, xn-1 , xn } such that a = x0 ≤ x1≤…… ≤ xn-1 ≤ xn = b be any Partition of [a, b] and P* the refinement of P contains just one point X* more than P such that xi-1< x*

where x i-1 and xi are two consecutive points of P.

Let mi, mi , mi′ are the infimum of f(x) in′′ mi≤[xi-1 , xi], [xi-1 , x*] and [x* , xi] respectively then clearly mi mi ≤ and mi ′ . Therefore′′) = miα) - L(P,f, αL(P*,f , (xi-1)] + miα(x*) - α [′ (xi-1)]α(xi) - α(x*)] - mi[α(xi) - α [′′ = mi (xi-1)] + miα(x*) - α[′ (xi-1)]α(x*) -

α(x*) + α(xi) - α(x*)] - mi[α(xi) - α['' = (mi (xi-1)] + (miα(x*) - α- mi)[' 0.≥(x*)] α(xi) - α- mi)['').

If P* contains k points more than P then byb repeating theα L(P*, f , ≤) αHence L(P, f ,) is analogousα) ≤ U(P, f , α)same process we arrive at the same result.

Definition 7.1. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k$$

is called a Riemann–Stieltjes sum of f with respect to α. We say f is Riemann-integrable with respect to α on $[a, b]$, and we write "$f \in R(\alpha)$ on $[a, b]$" if there exists a number A having the following property: For every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that for every partition P finer than P_ε and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \varepsilon$.

Theorem 7.2. *If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

Proof. Let $h = c_1 f + c_2 g$. Given a partition P of $[a, b]$, we can write

$$S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\,\Delta\alpha_k = c_1 \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\,\Delta\alpha_k$$
$$= c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha).$$

Given $\varepsilon > 0$, choose P'_ε so that $P \supseteq P'_\varepsilon$ implies $|S(P, f, \alpha) - \int_a^b f\, d\alpha| < \varepsilon$, and choose P''_ε so that $P \supseteq P''_\varepsilon$ implies $|S(P, g, \alpha) - \int_a^b g\, d\alpha| < \varepsilon$. If we take $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$, then, for P finer than P_ε, we have

$$\left|S(P, h, \alpha) - c_1 \int_a^b f\, d\alpha - c_2 \int_a^b g\, d\alpha\right| \le |c_1|\varepsilon + |c_2|\varepsilon,$$

and this proves the theorem.

Theorem 7.3. *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.$$

The proof is similar to that of Theorem 7.2 and is left as an exercise.

A result somewhat analogous to the previous two theorems tells us that the integral is also additive with respect to the interval of integration.

Theorem 7.4. *Assume that $c \in (a, b)$. If two of the three integrals in* (1) *exist, then the third also exists and we have*

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha. \tag{1}$$

Proof. If P is a partition of $[a, b]$ such that $c \in P$, let

$$P' = P \cap [a, c] \qquad \text{and} \qquad P'' = P \cap [c, b],$$

denote the corresponding partitions of $[a, c]$ and $[c, b]$, respectively. The Riemann–Stieltjes sums for these partitions are connected by the equation

$$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha).$$

Assume that $\int_a^c f\, d\alpha$ and $\int_c^b f\, d\alpha$ exist. Then, given $\varepsilon > 0$, there is a partition P'_ε of $[a, c]$ such that

$$\left|S(P', f, \alpha) - \int_a^c f\, d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P' \text{ is finer than } P'_\varepsilon,$$

and a partition P''_ε of $[c, b]$ such that

$$\left|S(P'', f, \alpha) - \int_c^b f\, d\alpha\right| < \frac{\varepsilon}{2} \qquad \text{whenever } P'' \text{ is finer than } P''_\varepsilon.$$

Then $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ is a partition of $[a, b]$ such that P finer than P_ε implies $P' \supseteq P'_\varepsilon$ and $P'' \supseteq P''_\varepsilon$. Hence, if P is finer than P_ε, we can combine the foregoing results to obtain the inequality

$$\left|S(P, f, \alpha) - \int_a^c f\, d\alpha - \int_c^b f\, d\alpha\right| < \varepsilon.$$

Definition 7.5. *If $a < b$, we define $\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha$ whenever $\int_a^b f\,d\alpha$ exists. We also define $\int_a^a f\,d\alpha = 0$.*

The equation in Theorem 7.4 can now be written as follows:

$$\int_a^b f\,d\alpha + \int_b^c f\,d\alpha + \int_c^a f\,d\alpha = 0.$$

Theorem 7.6. *If $f \in R(\alpha)$ on $[a, b]$, then $\alpha \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

NOTE. This equation, which provides a kind of reciprocity law for the integral, is known as the *formula for integration by parts.*

Proof. Let $\varepsilon > 0$ be given. Since $\int_a^b f\,d\alpha$ exists, there is a partition P_ε of $[a, b]$ such that for every P' finer than P_ε, we have

$$\left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right| < \varepsilon. \tag{2}$$

Consider an arbitrary Riemann–Stieltjes sum for the integral $\int_a^b \alpha\,df$, say

$$S(P, \alpha, f) = \sum_{k=1}^{n} \alpha(t_k)\,\Delta f_k = \sum_{k=1}^{n} \alpha(t_k)f(x_k) - \sum_{k=1}^{n} \alpha(t_k)f(x_{k-1}),$$

where P is finer than P_ε. Writing $A = f(b)\alpha(b) - f(a)\alpha(a)$, we have the identity

$$A = \sum_{k=1}^{n} f(x_k)\alpha(x_k) - \sum_{k=1}^{n} f(x_{k-1})\alpha(x_{k-1}).$$

Subtracting the last two displayed equations, we find

$$A - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^{n} f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})].$$

The two sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of $[a, b]$ obtained by taking the points x_k and t_k together. Then P' is finer than P and hence finer than P_ε. Therefore the inequality (2) is valid and this means that we have

$$\left| A - S(P, \alpha, f) - \int_a^b f\, d\alpha \right| < \varepsilon,$$

whenever P is finer than P_ε. But this is exactly the statement that $\int_a^b \alpha\, df$ exists and equals $A - \int_a^b f\, d\alpha$.

$b = g(d)$. *Let h and β be the composite functions defined as follows:*

$$h(x) = f[g(x)], \qquad \beta(x) = \alpha[g(x)], \qquad \textit{if } x \in S.$$

Then $h \in R(\beta)$ on S and we have $\int_a^b f\, d\alpha = \int_c^d h\, d\beta$. That is,

$$\int_{g(c)}^{g(d)} f(t)\, d\alpha(t) = \int_c^d f[g(x)]\, d\{\alpha[g(x)]\}.$$

Proof. For definiteness, assume that g is strictly increasing on S. (This implies $c < d$.) Then g is one-to-one and has a strictly increasing, continuous inverse g^{-1} defined on $[a, b]$. Therefore, for every partition $P = \{y_0, \ldots, y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $x_k = g(y_k)$. In fact, we can write

$$P' = g(P) \qquad \text{and} \qquad P = g^{-1}(P').$$

Furthermore, a refinement of P produces a corresponding refinement of P', and the converse also holds.

If $\varepsilon > 0$ is given, there is a partition P'_ε of $[a, b]$ such that P' finer than P'_ε implies $|S(P', f, \alpha) - \int_a^b f\, d\alpha| < \varepsilon$. Let $P_\varepsilon = g^{-1}(P'_\varepsilon)$ be the corresponding partition of $[c, d]$, and let $P = \{y_0, \ldots, y_n\}$ be a partition of $[c, d]$ finer than P_ε. Form a Riemann–Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k)\, \Delta\beta_k,$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$. If we put $t_k = g(u_k)$ and $x_k = g(y_k)$, then $P' = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$ finer than P'_ε. Moreover, we then have

$$\begin{aligned} S(P, h, \beta) &= \sum_{k=1}^{n} f[g(u_k)]\{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\} \\ &= \sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} = S(P', f, \alpha), \end{aligned}$$

since $t_k \in [x_{k-1}, x_k]$. Therefore, $|S(P, h, \beta) - \int_a^b f\, d\alpha| < \varepsilon$ and the theorem is proved.

Theorem 7.8. *Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and we have*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemann sum

$$S(P, g) = \sum_{k=1}^{n} g(t_k)\,\Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\,\Delta x_k.$$

The same partition P and the same choice of the t_k can be used to form the Riemann–Stieltjes sum

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k.$$

Applying the Mean-Value Theorem, we can write

$$\Delta\alpha_k = \alpha'(v_k)\,\Delta x_k, \qquad \text{where } v_k \in (x_{k-1}, x_k),$$

and hence

$$S(P, f, \alpha) - S(P, g) = \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\,\Delta x_k.$$

Since f is bounded, we have $|f(x)| \le M$ for all x in $[a, b]$, where $M > 0$. Continuity of α' on $[a, b]$ implies uniform continuity on $[a, b]$. Hence, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ (depending only on ε) such that

$$0 \le |x - y| < \delta \qquad \text{implies} \qquad |\alpha'(x) - \alpha'(y)| < \frac{\varepsilon}{2M(b - a)}.$$

If we take a partition P'_ε with norm $\|P'_\varepsilon\| < \delta$, then for any finer partition P we will have $|\alpha'(v_k) - \alpha'(t_k)| < \varepsilon/[2M(b - a)]$ in the preceding equation. For such P we therefore have

$$|S(P, f, \alpha) - S(P, g)| < \frac{\varepsilon}{2}.$$

On the other hand, since $f \in R(\alpha)$ on $[a, b]$, there exists a partition P''_ε such that P finer than P''_ε implies

$$\left|S(P, f, \alpha) - \int_a^b f\, d\alpha\right| < \frac{\varepsilon}{2}.$$

Combining the last two inequalities, we see that when P is finer than $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$, we will have $|S(P, g) - \int_a^b f\, d\alpha| < \varepsilon$, and this proves the theorem.

Theorem 7.9. *Given $a < c < b$. Define α on $[a, b]$ as follows: The values $\alpha(a)$, $\alpha(c)$, $\alpha(b)$ are arbitrary;*

$$\alpha(x) = \alpha(a) \qquad \text{if } a \le x < c,$$

and

$$\alpha(x) = \alpha(b) \qquad \text{if } c < x \le b.$$

Let f be defined on $[a, b]$ in such a way that at least one of the functions f or α is continuous from the left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on $[a, b]$ and we have

$$\int_a^b f\, d\alpha = f(c)[\alpha(c+) - \alpha(c-)].$$

Proof. If $c \in P$, every term in the sum $S(P, f, \alpha)$ is zero except the two terms arising from the subinterval separated by c, say

$$S(P, f, \alpha) = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)],$$

where $t_{k-1} \le c \le t_k$. This equation can also be written as follows:

$$\Delta = [f(t_{k-1}) - f(c)][\alpha(c) - \alpha(c-)] + [f(t_k) - f(c)][\alpha(c+) - \alpha(c)],$$

where $\Delta = S(P, f, \alpha) - f(c)[\alpha(c+) - \alpha(c-)]$. Hence we have

$$|\Delta| \le |f(t_{k-1}) - f(c)|\, |\alpha(c) - \alpha(c-)| + |f(t_k) - f(c)|\, |\alpha(c+) - \alpha(c)|.$$

If f is continuous at c, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|P\| < \delta$ implies

$$|f(t_{k-1}) - f(c)| < \varepsilon \qquad \text{and} \qquad |f(t_k) - f(c)| < \varepsilon.$$

In this case, we obtain the inequality

$$|\Delta| \le \varepsilon|\alpha(c) - \alpha(c-)| + \varepsilon|\alpha(c+) - \alpha(c)|.$$

But this inequality holds whether or not f is continuous at c. For example, if f is discontinuous both from the right and from the left at c, then $\alpha(c) = \alpha(c-)$ and $\alpha(c) = \alpha(c+)$ and we get $\Delta = 0$. On the other hand, if f is continuous from the left and discontinuous from the right at c, we must have $\alpha(c) = \alpha(c+)$ and we get

$|\Delta| \le \varepsilon|\alpha(c) - \alpha(c-)|$. Similarly, if f is continuous from the right and discontinuous from the left at c, we have $\alpha(c) = \alpha(c-)$ and $|\Delta| \le \varepsilon|\alpha(c+) - \alpha(c)|$. Hence the last displayed inequality holds in every case. This proves the theorem.

Definition 7.10 (Step function). *A function α defined on $[a, b]$ is called a step function if there is a partition*

$$a = x_1 < x_2 < \cdots < x_n = b$$

such that α is constant on each open subinterval (x_{k-1}, x_k). The number $\alpha(x_k+) - \alpha(x_k-)$ is called the jump at x_k if $1 < k < n$. The jump at x_1 is $\alpha(x_1+) - \alpha(x_1)$, and the jump at x_n is $\alpha(x_n) - \alpha(x_n-)$.

Step functions provide the connecting link between Riemann–Stieltjes integrals and finite sums:

Theorem 7.11 (Reduction of a Riemann–Stieltjes integral to a finite sum). *Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $x_1, \ldots, x_n$ are as described in Definition 7.10. Let f be defined on $[a, b]$ in such a way that not both f and α are*

discontinuous from the right or from the left at each x_k. Then $\int_a^b f\, d\alpha$ exists and we have

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k.$$

Proof. By Theorem 7.4, $\int_a^b f\, d\alpha$ can be written as a sum of integrals of the type considered in Theorem 7.9.

Definition 7.14. *Let P be a partition of $[a, b]$ and let*

$$M_k(f) = \sup \{f(x) : x \in [x_{k-1}, x_k]\},$$
$$m_k(f) = \inf \{f(x) : x \in [x_{k-1}, x_k]\}.$$

The numbers

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f) \, \Delta\alpha_k \qquad \text{and} \qquad L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f) \, \Delta\alpha_k,$$

are called, respectively, the upper and lower Stieltjes sums of f with respect to α for the partition P.

NOTE. We always have $m_k(f) \le M_k(f)$. If $\alpha \nearrow$ on $[a, b]$, then $\Delta\alpha_k \ge 0$ and we can also write $m_k(f)\,\Delta\alpha_k \le M_k(f)\,\Delta\alpha_k$, from which it follows that the lower sums do not exceed the upper sums. Furthermore, if $t_k \in [x_{k-1}, x_k]$, then

$$m_k(f) \le f(t_k) \le M_k(f).$$

Theorem 7.15. *Assume that $\alpha \nearrow$ on $[a, b]$. Then:*

i) *If P' is finer than P, we have*

$$U(P', f, \alpha) \le U(P, f, \alpha) \qquad \text{and} \qquad L(P', f, \alpha) \ge L(P, f, \alpha).$$

ii) *For any two partitions P_1 and P_2, we have*

$$L(P_1, f, \alpha) \le U(P_2, f, \alpha).$$

Proof. It suffices to prove (i) when P' contains exactly one more point than P, say the point c. If c is in the ith subinterval of P, we can write

$$U(P', f, \alpha) = \sum_{\substack{k=1 \\ k \ne i}}^{n} M_k(f) \, \Delta\alpha_k + M'[\alpha(c) - \alpha(x_{i-1})] + M''[\alpha(x_i) - \alpha(c)],$$

where M' and M'' denote the sup of f in $[x_{i-1}, c]$ and $[c, x_i]$. But, since

$$M' \le M_i(f) \qquad \text{and} \qquad M'' \le M_i(f),$$

we have $U(P', f, \alpha) \le U(P, f, \alpha)$. (The inequality for lower sums is proved in a similar fashion.)

To prove (ii), let $P = P_1 \cup P_2$. Then we have

$$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha).$$

as $n \to \infty$, we have

$$\int_0^{10} f(x)\,d\alpha(x) = 50 + 55 = 105.$$

E.2. Properties

Theorem E.4. *Let c_1, c_2 be two constants in $\mathbb{R}$.*

(1) *If $f, g \in R(\alpha)$ on $[a,b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a,b]$, and*

$$\int_a^b (c_1 f + c_2 g)\,d\alpha = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b g\,d\alpha.$$

(2) *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a,b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a,b]$, and*

$$\int_a^b f\,d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\,d\alpha + c_2 \int_a^b f\,d\beta.$$

(3) *If $c \in [a,b]$, then*

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Definition E.5. If $a < b$, we define

$$\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha.$$

Theorem E.6. *If $f \in R(\alpha)$ and α has a continuous derivative on $[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)\,dx$ exists and*

$$\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx.$$

POSSIBLE QUESTIONS
PART - B
(5 × 8 = 40)

1. Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\ dx$ exists and $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$
2. State and prove formula for integration by parts of a Riemann-Stieltjes integral.

3. Assume that c ∈ (a,b) if two of the three integrals $\int_a^c f\ d\alpha$ +$\int_c^b f\ d\alpha =$ =$\int_a^b f\ d\alpha$ exists then the third also exists

4.. State and prove Reduction of a Riemann-Stieltjes integral to a finite sum.

5. If f ∈ R (α) on [a,b] then α ∈ R (f) on [a,b,] we have $\int_a^b f(x)d$ α (x) + $\int_a^b$ α $(x)d$ f (x) =f (b) α (b) - f (a) α (a).

6. Assume that c ∈ (a,b) if two of the three integrals $\int_a^c f\ d\alpha$ +$\int_c^b f\ d\alpha =$ =$\int_a^b f\ d\alpha$ exists then the third also exists

Reg. No ----------------

(15MMU601)

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE-21

DEPARTMENT OF MATHEMATICS

Sixth Semester

I INTERNAL TEST - Jan '18

REAL ANALYSIS II

Date : .01.18 ( ) Time:2 hours

Class : III B.Sc (MATHEMATICS) Maximum Marks: 50

PART – A (20 x 1 = 20 Marks)

ANSWER ALL THE QUESTIONS

1. The function f(x) = x, is continuous at ------------
   a. for some real x b. for finite number of real x
   c. for all real x d. for no real x

2. If x < y = > f (x) < f(y) then f is ----------
   a. constant b. continuous
   c. increasing d. strictly increasing

3. The function f itself is called ----------
   a. curve b. path
   c. closed interval d. open interval

4. If f : S → R is continuous and A= { x: f(x) < 0 } then A -----
   a. closed b. open
   c. both open and closed d. neither open nor closed

5. If x < y = > f(x) < f(y) then f is -------------
   a. constant b. decreasing
   c. increasing d. strictly increasing

6. If f is continuous on S and S is compact then $f^{-1}\ is$
   a. not continuous b. continuous
   c. uniformly continuous d. constant

7. If X is connected and f is continuous then f(X) is ----
   a. connected b. open
   c. both open and connected d. disconnected

8. If f : R→ R is continuous is continuous then f ([0,1]) is ------
   a.connected b. compact
   c.not bounded d. not compact

9. A contraction of any metric space --------
   a. continuous b. not continuous
   c. Uniformly continuous d. onto

10. If f' is 0 everywhere on (a ,b) then f is constant on --------
    a.(a , b) b. (a, b]
    c.[a ,b) d. [a , b]

11. A curve is a ----------- subset of R^n
    a. compact b. connected
    c. compact and connected d. not compact

12. The function f itself is called ----------
    a. curve b. path
    c. closed interval d. open interval

13. If f is strictly monotonic then f is -------------
    a.onto b. 1-1
    c. bijection d. not 1-1

14. If f is of bounded variation then 1/f is -----------
    a. bounded b. of bounded variation
    c. need not be bounded variation d. not exists

15. If f can be expressed as a difference of two increasing functions then f is ------- on [a,b]
    a. bounded b. of bounded variation
    c. need not be bounded variation d. not exists

16. If both f and f^{-1} are continuous then f is called -------------
a. Increasing b. continuous
c. continuous increasing d. decreasing

17. A Contraction of any metric space is -------------
a. continuous b. discontinuous
c. uniformly continuous d. constant

18. Graph of f is called as -------------
a. curve b. path
c. closed interval d. open interval

19. The contraction constant α *is* ------------
a. < 2 b. > 2 c. < 1 d. > 1

20. If f (x) = x on A,then f ‘ (x) = -------- on A
a. 1 b. 0 c.3 d. ∞

PART – B (3 x 10 = 30 Marks)

ANSWER ALL THE QUESTIONS

21. a) State and prove sign preserving property of continuous functions.

(OR)

b) Prove that continuous image of an open set is open.

22. a) State and prove fixed point theorem for contraction.

(OR)

b) Prove that continuous image of a connected set is connected .Then prove that X is compact.

23. a). State and prove Intermediate Valued Theorem for continuous functions

(OR)

b) State and prove Connectedness

Reg. No ----------------

(15MMU601)

KARPAGAM ACADEMY OF HIGHER EDUCATION

COIMBATORE-21

DEPARTMENT OF MATHEMATICS

II INTERNAL TEST 8

Sixth Semester

REAL ANALYSIS II

Date : 26 .02.18 (FN) Time:2 hours

Class : III B.Sc (MATHEMATICS) Maximum : 50 Marks

PART A (20 x 1 = 20 Marks)

ANSWER ALL THE QUESTIONS

1. If f is monotonic on [a,b] the set of discontinuous of f is ---
   a) uncountable b) finite
   c) infinite d) countable
2. If f is monotonic on [a,b] then f is bounded variation on -------
   a) (a,b) b. (a,b] c. [a,b) d. [a,b]
3. If f has a derivative of order n then f is approximately a polynomial of order------------
   a)n b) n-1 c) 1 d) 3
4. If f and g are of bounded variation on [a,b] then f+g is-----------
   a) bounded b) of bounded variation
   c)constant d) not of bounded variation
5. If f is of bounded variation on [a,b] then f can be expressed as sum of ------------
   a) decreasing function b) increasing function
   c) constant function d) continuous function
6. If f is decreasing then f is ------------
   a) n b) n-1 c) 1 d) 3
7. The function f itself is called ---------
   a)curve b) path c) closed interval d) open interval
8. If f is 0 everywhere is on (a, b) then f is constant on ------------
   a) (a,b) b) (a,b] c) [a,b) d) [a,b]
9. If f is of bounded variation then 1/f -----------------
   a)is bounded b. is of bounded variation
   c. is need not be bounded variation d. not exists
10. Graph of f is called as --------
    a)curve b. path c. closed interval d. open interval
11. If f and g are of bounded variation on [a,b] then f-g is of ----------
    a)bounded on [a,b] b. bounded variation on [a,b]
    c. uniformly continuous d. onstant
12. Partition P of [a,b] is set of -------------
    a)finite points b. infinite points
    c. infinite points d. uncountable points
13. If f is continuous at c the f is --------------
    a)differentiable at c b) need not be differentiable at c
    c) 0 d)1
14. If f and g are of bounded variation on [a,b] then fg is --------------
    a.) bounded b) constant
    c) strictly decreasing d) bounded variation
15. If $\propto$ (x) = x then S (P ,f, $\propto$) = ------
    a.)$S(\mathrm{P}, \mathrm{f}, \mathrm{x})$ b.) $\mathrm{S}(\mathrm{P}, \mathrm{f})$
    c.) $S(\mathrm{P}, \mathrm{f}, 1)$ d.) $S(\mathrm{P}, \ , \mathrm{a})$
16. A partition P' is said to be finer than P if -------------
    a) $P' \subset P$ b) $\mathrm{P}' \neq \mathrm{P}$ c.) $\mathrm{P} \subset \mathrm{P}'$ d.) $\mathrm{P} \cap \mathrm{P}' = \mathrm{P}$
17. The constant function f(x)=1/100, is continuous at----------
    a) for some complex numbers x b) for complex numbers some
    c) for all complex x d) for some real x

18. $\left||P'|\right| \leq ||P||$ if ---------

a) $P' \subset P$ b) $P' \neq P$ c) $P \subset P'$ d.) $P \cap P' = P$

19. If ∝ (x) = x then S (P ,f, ∝) = --------

a).f ∈ R b) f ∈ ∝

c.) f ∈ R and $f \in \alpha$ d) f ∈ R or f ∈ α

20. The refinement of a partition P is --------------- if its norm increases

a) increases b) decreases

c)strictly increases d) strictly increasing

PART B (3 x 10 = 30 Marks)

ANSWER ALL THE QUESTIONS

(OR)

b) State and prove algebra for derivatives

22. a) Explain about chain rule .

(OR)

b) State and prove Generalized Mean valued theorem.

s formula with remainder .

(OR)

b) State and prove additive property of total variations.

1. If f is monotonic on [a,b] the set of discontinuous of f is ---
a) uncountable b) finite
c) infinite d) countable

2. If f is monotonic on [a,b] then f is bounded variation on -------
a) (a,b) b. (a,b] c. [a,b) d. [a,b]

3. If f has a derivative of order n then f is approximately a polynomial of order------------
a)n b) n-1 c) 1 d) 3

4. If f and g are of bounded variation on [a,b] then f+g is-----------
a) bounded b) of bounded variation
c)constant d) not of bounded variation

5. If f is of bounded variation on [a,b] then f can be expressed as sum of ------------
a) decreasing function b) increasing function
c) constant function d) continuous function

6. If f is decreasing then f is ------------
a) n b) n-1 c) 1 d) 3

7. The function f itself is called ---------
a)curve b) path c) closed interval d) open interval

8.If f is 0 everywhere is on (a, b) then f is constant on ------------
a) (a,b) b) (a,b] c) [a,b) d) [a,b]

9. If f is of bounded variation then 1/f -----------------
a)is bounded b. is of bounded variation
c. is need not be bounded variation d. not exists

10. Graph of f is called as --------
a)curve b. path c. closed interval d. open interval

11. If f and g are of bounded variation on [a,b] then f-g is of ----------
a)bounded on [a,b] b. bounded variation on [a,b]
c. uniformly continuous d. onstant

12. Partition P of [a,b] is set of -------------
a)finite points b. infinite points
c. infinite points d. uncountable points

13. If f is continuous at c the f is --------------
a)differentiable at c b) need not be differentiable at c
c) 0 d)1

14. If f and g are of bounded variation on [a,b] then fg is --------------
a.) bounded b) constant
c) strictly decreasing d) bounded variation

15. If $\propto (x) = x$ then S (P ,f, $\propto$) = ------
a.)$S(P, f, x)$ b.) $S(P, f)$
c.) $S(P, f, 1)$ d.) $S(P, \ , a)$

16. A partition P' is said to be finer than P if -------------
a) $P' \subset P$ b) $P' \neq P$ c.) $P \subset P'$ d.) $P \cap P' = P$

17. The constant function f(x)=1/100, is continuous at----------
a) for some complex numbers x b) for complex numbers some
c) for all complex x d) for some real x

18. $\left|\left|P'\right|\right| \leq ||P||$ if ---------
a) $P' \subset P$ b) $P' \neq P$ c) $P \subset P'$ d.) $P \cap P' = P$

19. If $\propto (x) = x$ then S (P ,f, $\propto$) = --------
a).f $\in$ R b) f $\in \propto$
c.) f $\in$ R and $f \in \alpha$ d) f $\in$ R or f $\in \alpha$

20. The refinement of a partition P is --------------- if its norm increases
a) increases b) decreases
c)strictly increases d) strictly increasing

2) State and prove algebra for derivatives

3. Explain about chain rule .

4 State and prove Generalized Mean valued theorem.

5

6. State and prove additive property of total variations.

7. State and prove Tota l n\variations

8. If f is B.V on (a,b) then f is bounded on (a,b)

KARPAGAM ACADEMY OF HIGHER EDUCATION

(Deemed to be University Established Under Section 3 of UGC Act 1956)

Coimbatore – 641 021.

LECTURE PLAN
DEPARTMENT OF MATHEMATICS

STAFF NAME : S.KOHILA
SUBJECT NAME: REAL ANLAYSIS II SUB.CODE:15MMU601
SEMESTER: VI CLASS: III B.SC MATHEMATICS

S.No	Lecture Duration Period	Topics to be Covered	Support Material/Page Nos
		UNIT-I	
1	1	Introduction and examples of continuous functions	T: Chapter 4, 80-81
2	1	Theorems on Continuity and inverse images of a set	T: Chapter 4, 81-82
3	1	Theorems on Continuity and inverse images of open and closed sets	T: Chapter 4, 82
4	1	Theorems on Functions continuous on compact sets	T: Chapter 4, 82: R1: Ch 5, 134-135
5	1	Theorems on bounded functions	T: Chapter 4, 82-83
6	1	Theorems for f^{-1} to be continuous	83
7	1	Examples and problems	T: Chapter 4, 82-83
8	1	Definition and examples for topological mappings	T: Chapter 4, 83-84
9	1	Definition and examples for topological mappings	T: Chapter 4, 84
10	1	sign preserving property	T: Chapter 4, 84
11	1	Bolzano's theorem	T: Chapter 4, 84
12	1	Continuation of Bolzano's theorem	T: Chapter 4, 84
13	1	Intermediate value theorem	T: Chapter 4, 84-85
14	1	Problems on IVT	T: Chapter 4, 85

15	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit 1=15		
		UNIT-II	
1	1	Introduction to Connectedness	T: Chapter 4, 86
2	1	Examples for Connectedness	T: Chapter 4, 86
3	1	Thereom on two valued function	T: Chapter 4, 86
4	1	Thereom on two valued function and Connectedness	T: Chapter 4, 87
5	1	Introduction to Connectedness	T: Chapter 4, 86
6	1	Thereom on continuous image of a connected set	T: Chapter 4, 87
7	1	IVT for real valued function	T: Chapter 4, 88
8	1	connected sets	T: Chapter 4, 89
9	1	Theroem on arcwise connectedness	T: Chapter 4, 89
10	1	Continuation of Theroem on arcwise connectedness	T: Chapter 4, 89: R2: Ch 6, 143-145
11	1	Continuation of Theroem on arcwise connectedness	T: Chapter 4, 89
12	1	Theorem on uniform connectivity	T: Chapter 4, 89-90
13	1	Thereom on Uniform continuity and compact sets	T: Chapter 4, 90
14	1	Fixed point theorem	T: Chapter 4, 92
15	1	COntinuation of Fixed point theorem	T: Chapter 4, 92
16	1	Thereom on Monotonic functions	T: Chapter 4, 94
17	1	Continuation of Monotonic functions	T: Chapter 4, 95
18	1	Recapitulation and discussion of possible questions	
24	1	Recapitulation and discussion of possible questions	
	Total No of Hours Planned For Unit II=24		
		UNIT-III	
1	1	Introduction and Definition	T: Chapter 5, 104-

		of derivative	105
2	1	Theorems on Derivative and continuity	T: Chapter 5, 105
3	1	Continuation of Theorems on Derivative and continuity	T: Chapter 5, 105-106
4	1	Theorems on Algebra of derivatives	T: Chapter 5, 106
5	1	The chain rule	T: Chapter 5, 106-107
6	1	One sided derivatives and infinite derivatives	T: Chapter 5, 107-108
7	1	Theorems on Functions with non-zero derivatives	T: Chapter 5, 108-109
8	1	Theorems on Zero derivatives and local extrema	T: Chapter 5, 109-110
9	1	Rolle's theorem	T: Chapter 5, 110
10	1	The mean value theorem for derivatives	T: Chapter 5, 110
11	1	Generlized mean value theorem for derivatives	T: Chapter 5, 110-111
12	1	Corralory of Generlized mean value theorem	T: Chapter 5, 110-111
13	1	Corralory of mean value theorem	T: Chapter 5, 113
14	1	Taylor's formula with remainder	T: Chapter 5, 113-114
15 16	1 1	Corralory of Taylor's formula with remainder	T: Chapter 5, 113-114
17	1	Recapitulation and discussion of possible questions	
1	1	Properties of monotonic functions	T: Chapter 6, 127
2	1	Properties of monotonic functions	T: Chapter 6, 127-128
3	1	Theorems on bounded variation	T: Chapter 6, 128
4	1	Theorems on bounded variation	T: Chapter 6, 128-129
5	1	Examples for bounded variation	T: Chapter 6, 128-129

6	1	Theorems Total Variation	T: Chapter 6, 129
7	1	Continuation of theorems on Total Variation	T: Chapter 6, 129-130
8	1	Additive properties of total variation on (a, x) as a function of x	T: Chapter 6, 130
9	1	Continuation Additive properties of total variation on (a, x) as a function of x	T: Chapter 6, 130-131
10	1	Total variation on (a, x) as a function of x	T: Chapter 6, 131-132
11	1	Theorems on Continuous functions of bounded variation	T: Chapter 6, 132
12	1	Continuation of continuous functions of bounded variation.	T: Chapter 6, 132
13	1	Continuation of Continuous functions of bounded variation.	T: Chapter 6, 133
14	1	Continuation of continuous functions of bounded variation.	T: Chapter 6, 133
	Total No of Hours Planned for unit IV=14		
1	**1**	The Riemann - Stieltjes integral-Introduction	T: Chapter 7, 140
2	**1**	Notation of Riemann Stieltjes integral	T: Chapter 7, 141
3	**1**	Definition of Riemann Stieltjes integral	T: Chapter 7, 141
4	**1**	Theorems on linear properties	T: Chapter 7, 142
5	**1**	Continuation of Theorems on linear properties	T: Chapter 7, 142-143
6	**1**	Theorems on Integration by parts	T: Chapter 7, 144
7	**1**	Continuation of theorems on Integration by parts	T: Chapter 7, 144
8	**1**	Theorems on Change of variable	T: Chapter 7, 144-145
9	**1**	Continuation of Theorems on Change of variable in a Riemann Stieltjes integral	T: Chapter 7, 144-145
10	**1**	Theorems on Reduction to a Riemann integral.	T: Chapter 7, 145-146
11	**1**	Continuation of Theorems onReduction to a Riemann integral.	T: Chapter 7, 145-146
12	**1**	Continuation of Theorems on Reduction to a Riemann	T: Chapter 7, 145-

		integral.	146
Total Planned Hours	**120**		

TEXT BOOK

1. Apostol.T.M.,1990. Mathematical Analysis, Second edition, Narosa Publishing Company, Chennai.

REFERENCES

1. Balli. N.P, 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.
2. Gupta . S.L , and N.R. Gupta ., 2003.Principles of Real Analysis, Second edition, Pearson Education Pvt.Ltd,Singapore.
3. Royden .H.L , 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.
4. Rudin. W,1976 .Principles of Mathematical Analysis, Mcgraw hill, Newyork .
5. Sterling. K. Berberian , 2004.A First Course in Real Analysis, Springer Pvt Ltd,

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Reg. No ----------------

(15MMU601)

Karpagam Academy of Higher Education

COIMBATORE-21

MODEL EXAMINATION -MAR '18

MATHEMATICS

REAL ANALYSIS –II

Class : III B.Sc (MATHEMATICS) Time:3 hours

Date : .3.18 ( ) Maximum Marks: 60 Marks

PART – A (20 x 1 = 20 marks)

ANSWER ALL THE QUESTIONS

1. If $f: R \to R$ by $f(x) = c$ then image of an open set is -------------
   a. open b. closed c. not open d. both open and closed
2. A real function f defined on S is said to be bounded if $|f(x)| - - - - - -$ M for all x in S
   a. $\geq$ b. $\leq$ c. $<$ d. $>$
3. Inverse image of closed set is ----------
   a. closed b. open
   c. both open and closed d. neither open nor closed
4. If f is continuous on S and S is compact then f^{-1} is ----------
   a. not continuous b. continuous
   c. uniformly continuous d. constant
5. If $f(x) = f(y)$ for all x and y then f is
   a. constant b. decreasing
   c. increasing d. strictly increasing
6. If $f: R \to R$ is continuous then the image of $[a, b]$ is -------
   a. bounded b. unbounded
   c. closed d. compact.
7. A contraction of any metric space is ---------
   a. continuous b. discontinuous
   c. uniformly continuous d. constant
8. A real valued function f is said to two valued if the range of $f \subset$
   a. (0,1) b. [0,1] c. {0,1} d. (0 ,1]
9. If f has derivative at a and b and continous on (a, b) then $f'(c) = 0$ for $- - -$
   a. some c in (a,b) b. for all c in (a,b)
   c. for no c in (a,b) d. only one c in (a,b)
10. If $f'(x) \leq 0$ for all $x \in I$ then f is $- - - - -$ on I
    a. increasing b. strictly decreasing
    c. strictly decreasing d. decreasing
11. If f' is 0 everywhere on (a, b) the f is constant on ----------
    a. (a,b) b. (a,b] c. [a,b)
12. If f and g are continuous on $[a, b]$ and $f - g$ is $- -$ on $[a, b]$ ---------
    a. constant b. not constant c.1-1 d. onto
13. Graph of f is called as ------------
    a. curve b. path . closed interval d. open interval
14. $V_f(a, b) =$ ---------
    a. $V_f(a, c) - V_f(c, b)$ b. . $V_f(a, c) + V_f(c, b)$
    c. . $V_f(a, c) \times V_f(c, b)$ d. 0
15. If f is decreasing then –f is -----------
    a. decreasing b. increasing
    c. strictly decreasing d. strictly increasing
16. If f and g are of bounded variation on [a,b] then fg is ---------
    a. bounded b. constant c. continuous d. bounded variation.
17. A curve is a -----------subset of R^n
    a. compact b. connected c. compact and connected d. not compact
18. If $\alpha(x) = x$ then $- - - - -$
    a. $f \in R$ b. $f \in \alpha$ c. $f \in R$ and $f \in \alpha$ d. $f \in R$ or $f \in \alpha$
19. $\int_a^b f \, d(c\alpha)$ =-------
    a. $c\int_a^b f \, d\alpha$ b. 0 c. 1 d. 1
20. Let A be a compact subset of S and f is continuous on S -------------
    a. continuous on A b. uniformly continuou
    c. unifromly but not continuous on A d. continuous but not u

PART – B (5 x 8 = 40 Marks)
ANSWER ALL THE QUESTIONS

21. a) Prove that f is continuous iff inverse image of a closed set is closed. also prove that continuous image of a closed set is need not be closed.

(OR)

b) State and prove Connectedness.

22. a) Prove that a metric space S is connected if and only if every two valued function on S is constant.

(OR)

b) Prove that continuous image of a Connected set is Connected.

23. a)State and prove mean value Theorem (Derivatives)

(OR)

b) State and prove Taylor's theorem.

24. a) Prove that if f is monotonic on [a,b] then the set of discontinuous of f is countable

(OR)

b) Prove that a metric space S is connected if and only if every two valued function on S is constant.

25. a) State and prove formula for a Riemann- -Stieltjes integral

(OR)

b) State and prove formula for integration by parts of a Riemann-Stieltjes integral.

1. If $f: R \to R$ by $f(x) = c$ then image of an open set is -------------
   a. open b. closed c. **not open** d. both open and closed
2. A real function f defined on S is said to be bounded if $|f(x)| - - - - - - M$ for all x in S
   a. $\geq$ b. $\leq$ c. < d. >
3. Inverse image of closed set is ----------
   a. closed b. open c. **both open and closed** d. neither open nor closed
4. If f is continuous on S and S is compact then f^{-1} is ----------
   a. not continuous b. continuous c. **uniformly continuous** d. constant
5. If $f(x) = f(y)$ for all x and y then f is
   a. **constant** b. decreasing c. increasing d. strictly increasing
6. If $f: R \to R$ is continuous then the image of $[a, b]$ is -------
   a. **bounded** b. unbounded c. closed d. compact.
7. A contraction of any metric space is ---------
   a. continuous b. discontinuous c. **uniformly continuous** d. constant
8. A real valued function f is said to two valued if the range of $f \subset$
   a. (0,1) b. **[0,1]** c. {0,1} d. (0,1]
9. If f has derivative at a and b and continous on (a, b) then $f'(c) = 0$ for
   a. some c in (a,b) b. **for all c in (a,b)** c. for no c in (a,b) d. only one c in (a,b)
10. If $f'(x) \leq 0$ for all $x \in I$ then fis $- - - - -$ on I
   a. increasing b. strictly decreasing c. strictly decreasing d. **decreasing**
11. If f' is 0 everywhere on $(a, b)$the f is constant on
   a. **(a,b)** b. (a,b] c. [a,b) d. [a,b]
12. If f and g are continuous on $[a, b]$ and have equal finite derivatives the $f - g$ is $- -$on $[a, b]$
   a. constant b. not constant c. **1-1** d. onto
13. Graph of f is called as ------------
   a. **curve** b. path c. closed interval d. open interval
14. $V_f(a, b) =$ ---------
   a. $\mathbf{V_f(a, c) - V_f(c, b)}$ b. . $V_f(a, c) + V_f(c, b)$ c. . $V_f(a, c) \times V_f(c, b)$
15. If f is decreasing then –f is -----------
   a. decreasing b. **increasing** c. strictly decreasing d. strictly i
16. If f and g are of bounded variation on [a,b] then fg is ---------
   a. bounded b. constant c. continuous d. **bounded variation**.
17. If $\alpha(x) = x$ then $S(P, f, \alpha) =$ ---------
   a. $\mathbf{S(P, f, x)}$ b. $S(P, f)$ c. $S(P, f, 1)$ d. $S(P, \alpha, \alpha)$
18. If $\alpha(x) = x$ then $- - - - -$
   a. $f \in R$ b. $f \in \alpha$ c. $\mathbf{f \in R}$ **and** $\mathbf{f \in \alpha}$ d. $f \in R$ or $f \in \alpha$
19. A partition P' is said to be finer than P if -------
   a. $P' \subset P$ b. $P' \neq P$ c. $\mathbf{P \subset P'}$ d. $P \cap P' = P$
20. Let A be a compact subset of S and f is continuous on S -------------
   a. continuous on A b. **uniformly continuo**
   c. unifromly but not continuous on A d. continuous but not un

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SUB : REAL ANALYSIS II **Subject Code: 15MMU601**						
Class : III B.SC MATHEMATICS **Semester : VI**						
Unit I	OPTION 1	OPTION 2	OPTION 3	OPTION 4		ANSWERS
The constant function f(x)=c, is continuous at	for some real numbers x	for finite number of re	for all real x	for no real x		for all real x
The constant function f(x)=c, is continuous at	for some complex num	for finite number of co	for all complex x	for no complex x		for all complex x
The identity function f(x)=x, is continuous at	for some complex num	for finite number of co	for all complex x	for no complex x		for all complex x
The identity function f(x)=x, is continuous at	for some real numbers x	for finite number of re	for all real x	for no real x		for all real x
	1	0	-1	2		1
The image of an open set under continuous function is	open	need not be open	closed	need not be closed		need not be open
The image of a closed set under continuous function is	open	need not be open	closed	need not be closed		need not be cloded
The image of a compact set under continuous function is	compact	need not be compact	connected	need not be connected		compact
The image of a connected set under continuous function is	compact	need not be compact	connected	need not be connected		connected
The inverse image of an open set under continuous function is	open	need not be open	closed	need not be closed		open
The inverse image of a closed set under continuous function is	open	need not be open	closed	need not be closed		closed
Which of the following is not a bounded function?	sin x	cos x	tan x	sec x		tan x
If f is continuous on a compact subset S of X then f is	bounded	unbounded	constant	identity		bounded
The homeomorphic image of an open set is	open	need not be open	closed	need not be closed		open
The homeomorphic image of an open set is	open	need not be open	closed	need not be closed		closed
The topological image of an interval is	simple arc	circle	square	rectangle		simple arc
The topological image of an interval is	simple arc	circle	square	rectangle		simple arc
A simple closed curve is the topological image of	simple arc	circle	square	rectangle		circle
If f(a)f(b)<0, then there is -------------point c between a nd b such that f(c)=0	atmost one	atleast one	finite number of	infinite number of point		atleast one
The constant function f(x)=-1, is continuous at	for some real numbers x	for finite number of re	for all real x	for no real x		for all real x
The constant function f(x)=1/100, is continuous at	for some complex num	for finite number of co	for all complex x	for no complex x		for all complex x
The constant function f(x)=-100, is continuous at	for some real numbers x	for finite number of re	for all real x	for no real x		for all real x
The constant function f(x)=10000/100, is continuous at	for some complex num	for finite number of co	for all complex x	for no complex x		for all complex x

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Class : III B.SC MATHEMATICS **Semester : VI**

UNIT III	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWERS
$If\ \ f'\ is\ first\ derivative\ of\ f\ then\ \ f'\ is$	an one to one function	a function	an onto function	a bijection	a function
$If\ f\ is\ constant\ on\ A\ then\ on\ A, f'$	>0	<0			
$If\ f(x) = x\ on\ A\ then\ on\ A, f'$	>0	<0			
$If\ \ f(x) = x^4 - 4x^3 + 4x^2 - 1\ then\ the\ set\ of\ values\ at\ which\ \ f'\ is\ zero\ \ is$	{1,2}	{0,-1,-2}	{0}	{0,1,2}	
$If\ \ f'(x) > 0\ for\ all\ x\ in\ I\ then\ f\ is - - - - - - - on\ I$	increasing	decreasing	strictly increasing	strictly decreasing	increasing
$If\ \ f'(x) < 0\ for\ all\ x\ in\ I\ then\ f\ is - - - - - - - on\ I$	increasing	decreasing	strictly increasing	strictly decreasing	decreasing
$The\ function f(x) = 2x^3 - 15x^2 + 36x + 6\ is\ \ strictly\ increasing\ in\ the\ interval$	(2,3)	(3,4)	$(-\infty, 3) \cup (4, \infty)$	$(-\infty, 2) \cup (3, \infty)$	(2,3)
$The\ function f(x) = 2x^3 - 15x^2 + 36x + 6\ is\ \ strictly\ decreasing\ in\ the\ interval$	(2,3)	(3,4)	$(-\infty, 3) \cup (4, \infty)$	$(-\infty, 2) \cup (3, \infty)$	(3,4)
$If\ the\ function\ f\ defined\ by\ f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1\ then\ f'(0) =$	1	-1	100	-100	1
$Assume\ \ f\ and\ g\ are\ defined\ on\ (a,b) and\ differentiable\ at\ c.\ Then\ (f+g)'(c) =$	$f'(c)g'(c)$	$f'(c) - g'(c)$	$f'(c) + g'(c)$	$\frac{f'(c)}{g'(c)}$	$f'(c) + g'(c)$
$Assume\ \ f\ and\ g\ are\ defined\ on\ (a,b) and\ differentiable\ at\ c.\ Then\ (f-g)'(c)$	$f'(c)g'(c)$	$f'(c) - g'(c)$	$f'(c) + g'(c)$	$\frac{f'(c)}{g'(c)}$	$f'(c) - g'(c)$
$Assume\ \ f\ and\ g\ are\ defined\ on\ (a,b) and\ differentiable\ at\ c.\ Then\ (fg)'(c) =$	$f'(c)g'(c)$	$f'(c) - g'(c)$	$f'(c) + g'(c)$	$f'(c)g(c) + f(c)g'(c)$	$f'(c)g(c) + f(c)g'(c)$
$Assume\ \ f\ and\ g\ are\ defined\ on\ (a,b) and\ differentiable\ at\ c.\ \ Which\ of\ the\ following\ not\ exists?$	$f'(c) - g'(c)$	$f'(c) + g'(c)$	$f'(c)g(c) + f(c)g'(c)$	$\frac{f'(c)}{g'(c)}$	$\frac{f'(c)}{g'(c)}$
$To\ apply\ Rolle's\ theorem\ we\ \ must\ have$	$f(a) < f(b)$	$f(a) = f(b)$	$f(a) \neq f(b)$	$f(a) > f(b)$	$f(a) = f(b)$
$If\ f(a) = f(b)\ then\ by\ Rolle's\ \ theorem\ \ there\ is\ atleast\ one\ point\ c \in (a,b)\ \ \ at\ which$	$f'(c) = 0$	$f'(c) > 0$	$f'(c) < 0$	$f'(c) \neq 0$	$f'(c) = 0$
$If\ f(a) = f(b)\ then\ by\ Rolle's\ \ theorem\ \ there\ is - - - - -\ one\ point\ \ c \in (a,b)\ \ \ at\ which\ \ f'(c) = 0$	atleast	atmost	exactly	no	atleast
$If\ \ f\ satisfies\ \ all\ the\ conditions\ of\ mean\ value\ theorem\ then$	$f(b) - f(a) = f'(a)(b-a)$	$f(b) - f(a) = f'(b)(b-a)$	$f(b) - f(a) = f'(c)(b-a)$	$f(b) - f(a) = (b-a)$	$f(b) - f(a) = f'(c)(b-a)$
$If\ \ f'\ takes\ only\ positive\ values\ on\ (a,b)\ \ then\ \ f\ is - - - - - - - - - function\ on\ (a,b)$	increasing	striclty increasing	decreasing	striclty decreasing	striclty increasing
$If\ \ f'\ takes\ only\ non\ negative\ \ values\ on\ (a,b)\ \ then\ \ f\ is - - - - - - - - - function\ on\ (a,b)$	increasing	striclty increasing	decreasing	striclty decreasing	increasing
$If\ \ f'\ takes\ only\ \ negative\ \ values\ on\ (a,b)\ \ then\ \ f\ is - - - - - - - - - function\ on\ (a,b)$	increasing	striclty increasing	decreasing	striclty decreasing	striclty decreasing
$If\ \ f'\ takes\ only\ \ non\ positive\ \ values\ on\ (a,b)\ \ then\ \ f\ is - - - - - - - - - function\ on\ (a,b)$	increasing	striclty increasing	decreasing	striclty decreasing	decreasing
$If\ f\ \ \ and\ \ \ \ g\ are\ are\ continuous\ \ on\ (a,b)\ and\ have\ equal\ and\ finite\ derivatives\ the\ f-g\ is - - - - - - on\ (a,b)$	non constant	constant	striclty decreasing	striclty increasing	constant
$A\ sufficiently\ smooth\ curve\ joining\ two\ points\ \ A\ \ and\ B\ has - - - - - - - line\ with\ the\ same\ \ slope\ as\ the\ chord\ AB$	tangent	normal	no	both tangent and normal	tangent
$If\ \ f'(x) \neq 0\ for\ x \in (a,b) then\ f\ is - - - - - - - - function\ on\ (a,b)$	striclty increasing	striclty decreasing	monotonic	constant	monotonic
$f'\ is\ continuous\ if\ f'\ exists\ \ and\ f\ is - - - - -$	constant	monotnoic	striclty increasing	striclty decreasing	monotnoic

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Class : III B.SC MATHEMATICS Semester : VI

UNIT IV	OPTION 1	OPTION 1	OPTION 1	OPTION 1	ANSWERS
If f is monotmonic on $[a,b]$ then the set of discontinuities of f is	countable	atmost countable	finite	uncountable	countable
If f is ---------- $[a,b]$ then the set of discontinuities of f is countable	monotonic	constant	striclty increasing	strictly decreasing	monontonic
If $f(x) > f(y)$ for $x < y$ in $[a,b]$ then the set of discontinuities of f is	countable	atmost countable	finite	uncountable	countable
If $f(x) < f(y)$ for $x < y$ in $[a,b]$ then the set of discontinuities of f is	countable	atmost countable	finite	uncountable	countable
If $f(x) = f(y)$ for x,y in $[a,b]$ then the set of discontinuities of f is	countable	atmost countable	finite	uncountable	countable
If $[a,b]$ is a compact interval then the set of points ------is called a partition of $[a,b]$	$a = x_0 < \cdots < x_n = b$	$a = x_0 \leq \cdots \leq x_n = b$	$a = x_0 \geq \cdots \geq x_n = b$	$a = x_0 \neq \cdots \neq x_n = b$	$a = x_0 < \cdots < x_n = b$
Which of the following is a partition of $[0,1]$?	$\{0,\frac{1}{2},\frac{1}{4},1\}$	$\{0,\frac{1}{2},\frac{1}{2},1\}$	$\{0,\frac{1}{5},\frac{1}{2},1\}$	$\{0,\frac{13}{5},\frac{1}{2},1\}$	$\{0,\frac{1}{5},\frac{1}{2},1\}$
Number of partition of $[0,1]$ with each subinterval length $\frac{1}{2}$ is	1	2	3	4	1
Number of partition of $[0,1]$ with each subinterval length 0 is	0	1	2	3	0
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ then $\sum_k \Delta x_k =$	b-a	a-b	a	b	b-a
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[0,1]$ then $\sum_k \Delta x_k =$	1	0	2	3	1
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[0,1]$ then $x_0 =$	1	0	2	3	0
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[0,1]$ then $x_n =$	1	0	2	3	1
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ with $f(x) = x_0$ then $\Delta f_k =$	1	0	2	3	0
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ with $f(x) = x_0$ then $\Delta f_k = 0$ for	some k	all k	only one k	no k	all k
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ with $f(x) = x$ then $\Delta f_k =$	$x_k - x_{k-1}$	x_k	x_{k-1}	0	$x_k - x_{k-1}$
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ with $f(x) = x$ then $\sum \lvert\Delta f_k\rvert =$	$-\sum \Delta f_k$	$\sum \Delta f_k$	0	1	$\sum \Delta f_k$
If $P\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$ with $f(x) = 1$ then $\sum \Delta f_k =$	1	2	3	0	0
f is said to be bounded variation on $[a,b]$ if for all partitions $[a,b]$, $\sum \lvert\Delta f_k\rvert$	$\leq M$	$\geq M$	$= M$	$< M$	$\leq M$
f is said to be bounded variation on $[a,b]$ if for --- partitions $[a,b]$, $\sum \lvert\Delta f_k\rvert \leq M$	all	some	no	only one	all
If f is ----on $[a,b]$ the f is of bounded variation on $[a,b]$	monotonic	decreasing	increasing	constant	monotonic
Which of the following is true?	If f is of bounded variation on $[a,b]$ then f is bounded on $[a,b]$	If f is bounded on $[a,b]$ then f is of boundedvariation on $[a,b]$	both a and b	neither a nor b	If f is of bounded variation on $[a,b]$ then f is bounded on $[a,b]$
If $P_n = \left\{0,\frac{1}{2n},1\right\}, n = 1,2,3,\dots$ is a partition of $[0,1]$ then $\bigcup_{n\geq 1} P_n =$	$\{0,\frac{1}{2},1\}$	$\{0,\dots,\frac{1}{8},\frac{1}{2},1\}$	$\{0,\dots,\frac{1}{16},\frac{1}{8},\frac{1}{4},\frac{1}{2},1\}$	$\{0,\dots,\frac{1}{8},\frac{1}{6},\frac{1}{4},\frac{1}{2},1\}$	$\{0,\dots,\frac{1}{8},\frac{1}{6},\frac{1}{4},\frac{1}{2},1\}$
If $P_n = \left\{0,\frac{1}{2n},1\right\}, n = 1,2,3,\dots$ is a partition of $[0,1]$ then maximum subinterval length in $\bigcup_{n\geq 1} P_n =$	1	0.5	2	3	0.5
If $P_n = \left\{0,\frac{1}{2n},1\right\}, n = 1,2,3,\dots$ is a partition of $[0,1]$ then minimum subinterval length in $P_1 \cup P_2 \cup P_3 =$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{6}$
$f(x) = x^{\frac{1}{3}}$ is	monotonic	bounded variation	both a and b	neither a nor b	both a and b
If $f(x) = x^{\frac{1}{3}}$ then $f'(x) \to$ ---- as $x \to 0$	0	1	2	∞	∞
If f is of bounded variation on $[a,b]$ then $V_f(a,b)$ is	infinite	finite	not exists	1	finite
$V_f(a,b)$	≥ 0	≤ 0	0	1	≥ 0
If f is constant then $V_f(a,b)$	≥ 0	≤ 0	0	1	0
If $V_f(a,b) = 0$ the f is	strictly increasing	strictly decreasing	constant	monotonic	constant
$V_{f\pm g} \leq$	$V_f \pm V_g$	$V_f + V_g$	$V_f - V_g$	0	$V_f + V_g$
$V_f(a,b)$ ----$V_f(a,c) + V_g(c,b)$ for $c \in (a,b)$	$\leq$	$=$	$\geq$	$\neq$	$=$
If $V(x) = V_f(a,b), a < x \leq b$ and $V(a) = 0$ then V is ----function on $[a,b]$	striclty increasing	incresing	stricly decreasing	decreasing	increasing
If $V(x) = V_f(a,b), a < x \leq b$ and $V(a) =$ ------- then V is increasing function on $[a,b]$	1	2	3	0	0
If $V(x) = V_f(a,b), a < x \leq b$ and $V(a) =$ ------- then $V - f$ is increasing function on $[a,b]$	1	2	3	0	0
If $V(x) = V_f(a,b), a < x \leq b$ and $V(a) = 0$ then $V - f$ is --- function on $[a,b]$	striclty increasing	incresing	stricly decreasing	decreasing	increasing
If f is of bounded variation on $[a,b]$ then f is difference of two --- functions on $[a,b]$	striclty increasing	incresing	stricly decreasing	decreasing	increasing
If f is of bounded variation on $[a,b]$ then f is --- of two increasing functions on $[a,b]$	sum	difference	product	quotient	difference
If $f(x) = x$ for $x \in [2,3]$ then f is	striclty increasing	incresing	stricly decreasing	decreasing	strictly increasing
If $f(x) = \sqrt{x}$ for $x \in [2,3]$ then f is	stricIty increasing	incresing	stricly decreasing	decreasing	striclty decreasing
If $f(x) = \frac{1}{x}$ for $x \in [2,3]$ then f is	striclty increasing	incresing	stricly decreasing	decreasing	striclty decreasing

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UNIT III	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWERS
Suppose A and B are disjoint nonempty opensets in S then S is called disconnected if	$S = A \cup B$	$S = A \cap B$	$S \neq A \cap B$	$S \neq A \cup B$	$S = A \cup B$
Suppose A and B are disjoint nonempty opensets in S then S is called connected if	$S = A \cup B$	$S = A \cap B$	$S \neq A \cap B$	$S \neq A \cup B$	$S \neq A \cup B$
The metric space $S = R - \{0\}$ with Euclidean metric is	connected	disconnected	compact	closed	disconnected
Every open interval in R is	connected	disconnected	compact	closed	connnected
The set of all rational numbers Q with Euclidean metric is	connected	disconnected	compact	closed	disconnected
Every singleton set is	connected	disconnected	compact	open	connected
A real valued function f which is continuous on a metric space S is called two valued if	$f(S) \subset (0,1)$	$f(S) \subset [0,1)$	$f(S) \subset [0,1]$	$f(S) \subset \{0,1\}$	$f(S) \subset \{0,1\}$
A real valued function f is continuous on S and $f(S) \subset \{a, b\}$ is called two valued if	$a = b$	$a \neq b$	$a < b$ only	$a > b$ only	$a \neq b$
A metric space S is connected if every two valued function defined on S is	constant	not constant	not continuous	continuous	constant
If a metric space S is connected and $f: S \to \{0,1\}$ then	$f(S) \subset \{0\}$ only	$f(S) \subset \{1\}$ only	the $f(S) \subset \{1\}$ only or $f(S) \subset \{0\}$ only	both $f(S) \subset \{1\}$ and $f(S) \subset \{0\}$	either $f(S) \subset \{1\}$ only or $f(S) \subset \{0\}$ only
Continuous image of a connected set is	connected	disconnected	compact	closed	connected
If $f: R \to R$ is continuous then $f(R)$ is	connected	disconnected	compact	closed	connected
If $f: R \to R$ is continuous then $f((0,1))$ is	connected	disconnected	compact	closed	connected
If $f: R \to R$ is continuous then $f(\{a\})$ is	connected	disconnected	compact	closed	connected
If $f: R \to R$ is continuous then $f(R - \{a\})$ is	connected	disconnected	compact	closed	disconnected
If $f: R \to R$ is continuous then $f(R - \{0\})$ is	connected	disconnected	compact	closed	disconnected
If $f: R \to R$ is continuous then $f(Q)$, Q is the set of all rational numbers, is	connected	disconnected	compact	closed	disconnected
Every curve in R^n is	connected	disconnected	compact	closed	connected
If $f: R \to R$ is continuous function, then image of an interval is	connected	disconnected	compact	closed	connected
If $f: R \to R^n$ is continuous function, then image of an interval is	connected	disconnected	compact	closed	connected
If $f: R \to R^n$ is continuous function, then image of an interval is	curve	interval	open interval	closed interval	curve
Every nonempty set contains −−−−−−− connected subset	atleast one	atmost one	exactly one	no	atleast one
Suppose S is any metric space and $a \in S$. Which of the following is connected subset of S?	$\{a\}$	$\{\}$	$S - \{a\}$	S	$\{a\}$
Component of $S, U(x)$ is −−−−−−− connected subset of S which contains x	minimal	maximal	only	not	maximal
Every point x of a metric space S belongs to −−−−−−− connected subset of S	atleast one	atmost one	exactly one	no	atleast one
Union of connected sets is connected set if the intersection of connected sets is	non empty	empty	singleton set	fininte set	non empty
Suppose $A = \{(a,b): a, b \in R\}$. Then $\bigcup_{(a,b)\in A} (a,b)$ is connected provided $\bigcap (a,b)$ is	non empty	empty	singleton set	fininte set	non empty
Suppose S is any metric space and $S = \{1,2,3, \ldots\}$. Then $\bigcup_{x\in S} \{x\}$ is	connected	disconnected	compact	closed	diconnected
Let $f(x) = \frac{1}{x}$ for $x > 0$ and $A = (0,1]$. Then f is continuous at	every point of A	no point of A	finite number of points of A	1 only	every point of A
Let $f(x) = \frac{1}{x}$ for $x > 0$ and $A = (0,1]$. Then f is	continuous on A	uniformly continuous on A	continuous but not	unifromly but not	continuous but not uniformly on A
Let $f(x) = x^2$ for $x > 0$ and $A = (0,1]$. Then f is	continuous on A	uniformly continuous on A	continuous but not	unifromly but not	uniformly continuous on A
Let A be a compact subset of S and f is continuous on S, then f is	continuous on A	uniformly continuous on A	continuous but not	unifromly but not	uniformly continuous on A
Let A =[0,1] be a subset of S and f is continuous on S, then f is	continuous on A	uniformly continuous on A	continuous but not	unifromly but not	uniformly continuous on A
Let A =[a,b] be a subset of R and f is continuous on R, then f is	continuous on A	uniformly continuous on A	continuous but not	unifromly but not	uniformly continuous on A
The contraction constant α is	>1	<1	1	0	<1
Number of fixed point of a constant function is	0	1	2	3	1
If $f: Z \to Z$ be a identity function then number of fixed points of f is	countable	uncountable	finite	infinite	countable
If $f: R \to R$ be a identity function then number of fixed points of f is	countable	uncountable	finite	infinite	uncountable
If $f: Q \to Q$ be a identity function then number of fixed points of f is	countable	uncountable	finite	infinite	countable
Contraction of any metric space is	continuous only	uniformly continuos	uniformly continuous but not continuous	continuous but not uniformly continuous	uniformly continuos
A contraction f of a complete metric space has −−−−−−−−−− fixed point	a unique	a finite number of	countable	uncountable	a unique
Which of the following is a connected subset of R?	$\emptyset$	$\{1,2,3\}$	$\{1,2,3,4,5,6,7\}$	$\{1,2,3, \ldots\ldots\}$	$\emptyset$
Which of the following is a connected subset of R?	{1}	$\{1,2,3\}$	$\{1,2,3,4,5,6,7\}$	$\{1,2,3, \ldots\ldots\}$	{1}
If X is connected the closure of X is	disconnected	connected	open	both open and disconnected	connected
Let $U(x)$ be the cpomponent of S containing x is	closed	opne	neither open nor closed	both open and closed	closed
If $x < y \Rightarrow f(x) < f(y)$ then f is called	constant function	identity function	increasing function	strictly increasing	strictly increasing function
If $x \leq y \Rightarrow f(x) \leq f(y)$ then f is called	constant function	identity function	increasing function	strictly increasing	increasing function
If $x > y \Rightarrow f(x) < f(y)$ then f is called	constant function	identity function	decreasing function	strictly decreasing	strictly decreasing function
If $x \geq y \Rightarrow f(x) \leq f(y)$ then f is called	constant function	identity function	decreasing function	strictly decreasing	decreasing function
If f is strictly increasing then f is	one-to-one	onto	both one-to-one and onto	neither one-to-one nor onto	one-to-one
If f is strictly decreasing then f is	one-to-one	onto	both one-to-one and onto	neither one-to-one nor onto	one-to-one
If f is stricity monotonic then f is	one-to-one	onto	both one-to-one and onto	neither one-to-one nor onto	one-to-one

KARPAGAM ACADEMY OF HIGHER EDUCATION
(Deemed to be University Established Under Section 3 of UGC Act 1956)
Pollachi Main Road, Eachanari (Po),
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SUB : REAL ANALYSIS II **Subject Code: 15MMU601**

Class : III B.SC MATHEMATICS **Semester : VI**

UNIT III	OPTION 1	OPTION 2	OPTION 3	OPTION 4	ANSWERS	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}, n = 1,2,3, \dots is\ a\ partition\ of\ [0,1]\ then\ \bigcup_{n\geq 1} P_n =$	$\{0, \frac{1}{2}, 1\}$	$\{0, \dots, \frac{1}{8}, \frac{1}{2}, 1\}$	$\{0, \dots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\}$	$\{0, \dots, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1\}$	$\{0, \dots, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1\}$	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}, n = 1,2,3, \dots is\ a\ partition\ of\ [0,1]\ then\ maximum\ subinterval\ length\ in\ \bigcup_{n\geq 1} P_n =$	1	0.5	2	3	0.5	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}, n = 1,2,3, \dots is\ a\ partition\ of\ [0,1]\ then\ minimum\ subinterval\ length\ in\ P_1 \cup P_2 \cup P_3 =$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{6}$	
$If\ P\{x_0, x_1, \dots, x_n\} is\ a\ partition\ of\ [a,b]\ then\ \sum_k \Delta x_k =$	b-a	a-b	a	b	b-a	
$If\ P\{x_0, x_1, \dots, x_n\} is\ a\ partition\ of\ [0,1]\ then\ \sum_k \Delta x_k =$	1	0	2	3	1	
$If\ P\{x_0, x_1, \dots, x_n\} is\ a\ partition\ of\ [0,1]\ then\ x_0 =$	1	0	2	3	0	
$If\ P\{x_0, x_1, \dots, x_n\} is\ a\ partition\ of\ [0,1]\ then\ x_n =$	1	0	2	3	1	
$If\ P\{x_0, x_1, \dots, x_n\}\ is\ a\ partition\ of\ [a,b]\ with\ f(x) = x_0\ then\ \Delta f_k =$	1	0	2	3	0	
$If\ P\{x_0, x_1, \dots, x_n\}\ is\ a\ partition\ of\ [a,b]\ with\ f(x) = x_0\ then\ \Delta f_k = 0\ for$	some k	all k	only one k	no k	all k	
$If\ P\{x_0, x_1, \dots, x_n\}\ is\ a\ partition\ of\ [a,b]\ with\ f(x) = x\ \ then\ \Delta f_k =$	$x_k - x_{k-1}$	x_k	x_{k-1}	0	$x_k - x_{k-1}$	
$If\ P\{x_0, x_1, \dots, x_n\}\ is\ a\ partition\ of\ [a,b]\ with\ f(x) = x\ \ then\ \sum \lvert\Delta f_k\rvert =$	$-\sum \Delta f_k$	$\sum \Delta f_k$	0	1	$\sum \Delta f_k$	
$If\ P\{x_0, x_1, \dots, x_n\}\ is\ a\ partition\ of\ [a,b]\ with\ f(x) = 1\ \ then\ \sum \Delta f_k =$	1	2	3	0	0	
$If\ \ [a,b]\ is\ a\ compact\ interval\ \ then\ the\ set\ of\ points ------ is\ called\ a\ partition\ of\ [a,b]$	$a = x_0 < \cdots < x_n = b$	$a = x_0 \leq \cdots \leq x_n = b$	$a = x_0 \geq \cdots \geq x_n = b$	$a = x_0 \neq \cdots \neq x_n = b$	$a = x_0 < \cdots < x_n = b$	
$Which\ of\ the\ following\ is\ a\ partition\ of\ [0,1]?$	$\{0, \frac{1}{2}, \frac{1}{4}, 1\}$	$\{0, \frac{1}{2}, \frac{1}{2}, 1\}$	$\{0, \frac{1}{5}, \frac{1}{2}, 1\}$	$\{0, \frac{13}{5}, \frac{1}{2}, 1\}$	$\{0, \frac{1}{5}, \frac{1}{2}, 1\}$	
$Number\ of\ partition\ of\ [0,1]\ \ with\ each\ subinterval\ length\ \frac{1}{2}\ \ \ is$	1	2	3	4	1	
$Number\ of\ partition\ of\ [0,1]\ \ with\ each\ subinterval\ length\ 0\ \ \ is$	0	1	2	3	0	
$A\ partition\ P'\ of\ [a,b]\ is\ a\ refinement\ of\ P\ if$	$P' = P$	$P' \subset P$	$P \subset P'$	$P \neq P'$	$P \subset P'$	
$A\ partition\ P'\ of\ [a,b]\ is\ \ finer\ than\ \ P\ if$	$P' = P$	$P' \subset P$	$P \subset P'$	$P \neq P'$	$P \subset P'$	
$The\ norm\ of\ a\ partition\ P\ \ is$	the largest subinterval of P	the smallest subinterval of P	the sum all subintervals	the number of points in P of P	the largest subinterval of P	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}\ then\ \lVert P_1 \rVert =$	1	2	3	1.5	1.5	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}\ then\ \lVert P_2 \rVert =$	1	2	3	0.25	0.25	
$If\ P_n = \left\{0, \frac{1}{2n}, 1\right\}\ then\ \lVert P_3 \rVert =$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	0	$\frac{1}{6}$	
$If\ P' = P\ then\ \lVert P \rVert =$	0	1	2	$\lVert P' \rVert$	$\lVert P' \rVert$	
$If\ P' \subset P\ then$	$\lVert P \rVert = \lVert P' \rVert$	$\lVert P \rVert \leq \lVert P' \rVert$	$\lVert P' \rVert \leq \lVert P \rVert$	$\lVert P' \rVert = \lVert P \rVert = 0$	$\lVert P \rVert \leq \lVert P' \rVert$	
$If\ \lvert S(P,f,\alpha) - \int f\,d\alpha \rvert < \epsilon\ then$	$f \in R$	$f \in R(\alpha)$	$R \in f$	$R \in f(\alpha)$	$f \in R(\alpha)$	
$If\ \ f \in R(\alpha)\ and\ f \in R(\beta)\ then$	$f \in\ R(\alpha+\beta)$	$f \in\ R(\alpha\beta)$	$f \in\ R(\frac{\alpha}{\beta})$	$f \in\ R(\frac{\beta}{\alpha})$	$f \in\ R(\alpha+\beta)$	
$\int_a^b f\,d\alpha = 0\ \ \ \ if$	a=0	b=0	a=b	a=1	a=b	

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Coimbatore – 641 021.

SYLLABUS

Semester - VI

		L T P C
15MMU601	**REAL ANALYSIS II**	**5 0 0 4**

Scope : After the completion of this course, the learner get a clear knowledge in the concept of analysis which is the motivating tool in the study of applied Mathematics.

Objectives : To introduce the concepts which provide a strong base to understand and analysis
mathematics.

UNIT I

Examples of continuous functions –continuity and inverse images of open or closed sets – functions continuous on compact sets –Topological mappings –Bolzano‟s theorem.

UNIT II

Connectedness –components of a metric space – Uniform continuity :
Uniform continuity and compact sets –fixed point theorem for contractions – monotonic functions.

UNIT III

Definition of derivative –Derivative and continuity –Algebra of derivatives – the chain rule –one sided derivatives and infinite derivatives –functions with non-zero derivatives –zero derivatives and local extrema –Roll‟s theorem –The mean value theorem for derivatives.

UNIT IV

Properties of monotonic functions –functions of bounded variation –total Variation –additive properties of total variation on (a, x) as a function of x – functions of bounded variation expressed as the difference of increasing functions.

UNIT V

The Riemann - Stieltjes integral : Introduction –Notation –The definition of Riemann –Stieltjes integral –linear properties –Integration by parts –change of variable in a Riemann –stieltjes integral – Reduction to a Riemann integral.

TEXT BOOK

1.Apostol.T.M.,1990. Mathematical Analysis, Second edition, Narosa Publishing Company, Chennai.

1. Balli. N.P, 1981. Real Analysis, Laxmi Publication Pvt Ltd, New Delhi.

2. Gupta . S.L , and N.R. Gupta ., 2003.Principles of Real Analysis, Second

edition, Pearson Education Pvt.Ltd,Singapore.

3. Royden .H.L , 2002. Real Analysis, Third edition, Prentice hall of India,New Delhi.

4. Rudin. W,1976 .Principles of Mathematical Analysis, Mcgraw hill, Newyork .

UNIT-I
SYLLABUS

Examples of continuous functions –continuity and inverse images of open or closed sets –functions continuous on compact sets –Topological mappings –Bolzano‟s theorem.

Unit I

Example 1 *Consider a function $f(x) = c$, a constant.*
Clearly,the above function is continuous.
so,constant function function are continuous.

Example 2 *Consider a function $f(x) = x$.*

The above function is called identity function and the above function is continuous

Problem 1 *Consider a polynomial of degree $n \geq 0$, $f(x) = a_0 + a_1x + \cdots a_nx^n$ where $a_0 + a_1 + \cdots a_n$ are real numbers. Prove that the polynomial is continuous.*

Solution We have to prove the theorem by induction on n.

Consider, $n = 0$

$f(x) = a_0$ a_0 is constant. $f(x) = a_0$ is a continuous function. $\therefore$ The theorem is true for $n = 0$.

Now consider $n = 1$

$f(x) = a_0 + a_1 x$

We know that,

$f(x) - x$ is a continuous function

$a_1 f(x) = a_1 x$ is also a continuous function as for $n = 0$ and $n = 1$ the function is continuous.

$\therefore$ The polynomial function

$f(x) = a_0 + a_1 x + \cdots a_n x^n$ is a continuous
function. Hence the proof.

Remark 1 *The familiar real value functions of elementary calculus such as the exponentials, trigonometric and logarithmic functions are all continuous where ever they define.*

Continuity and inverse images of open (or) closed sets

Definition of inverse image:

Let f be a function from S to T ($f : S \to T$)be a function from a set S to a set T. If Y is a subset of T, the inverse image of Y under f, denoted by $f_{-}(Y)$, defined to be the largest subset of S which maps into Y.(i.e) $f^{-}(Y) = x \in S \mid f(x) \in Y$

Example 3 *Let* $S = \{1, 2, 3, 4\}$ *and* $T = \{a, b, c\}$ *and f be a function from S to T such that f* $(1) = a$

$f(2) = a$

$f(3) = b$

$f(4) = c$

let $Y = \{b, c\}$

Clearly $Y \subseteq T$

$f^{-}(y) = \{3, 4\}$

Remark 2 *If A is a subset of B then* $f^{-}(A)$ *subset of* $f^{-}(B)$

Solution

Suppose $A \subseteq B$

To prove: $f^{-}(A) \subseteq f^{-}(B)$

Let $x \in f^{-}(A)$ be arbitrary.

$\therefore$ There is a $x \in S / f(x) \in A$

$f(x) \in A \subseteq B$

$f(x) \in B$

(i.e) $x \in S$ and $f(x) \in B$

$\therefore x \in f^{-}(B)$

$\therefore f^{-}(A) \subseteq f^{-}(B)$

Theorem 1 *Let $f : S \to T$ be a function from $S \to T$. If $x \subseteq S$ and $Y \subseteq T$, Then we have*

a) $x = f^{-}(Y)$ implies $f(x) \subseteq Y$ b)Y
$= f(x)$ implies $X \subseteq f^{-}(Y)$

Theorem 2 *Let $f : S \to T$ be a function from one metric space (S , d_s) to another (T, d_T). Then f is continuous if and only if for every open set Y in T, the inverse image $f^{-}(Y)$ is open in S.*

Proof Let f be continuous on S

Let Y be open in T .
Suppose $f^{-}(Y) = \varphi$
Then, clearly $f^{-}(y)$ is open in S.

Suppose $f^{-}(Y)$ Ç φ

Then, there is a point $p \in f^{-}(Y)$

$\therefore$There is a point y such that $f(p) = y$

(i.e) $y \in Y$ such that $f(p) = y$

Since Y is open, y is an interior point of Y.

$\therefore$ There is an open ball$B_T(y, z)$

Since f is continuous at p, there is a $\delta > 0$ such that

$f(B_S(p, \delta)) \subseteq B_T(y, z)$

$\therefore B_S(p, \delta) \subseteq f^{-1} f(B_S(p, \delta))$

$\subseteq f^{-1}(B_T(y, z))$

$\subseteq f^{-1}(Y)$

$\therefore$ p is an interior point of $f^{-1}(Y)$

$\therefore f^{-1}(Y)$ is open.

Conversely,

Assume that $f^{-1}(Y)$ is open in S for every open subset Y in T. Let $p \in s$ be arbitrary

Then $f(p) \in f(s)$

$\therefore f(p) = y$ (say)

Claim: f is continuous at p

for every $z > 0$, the ball $(B_T(y,z))$ is open in T. By our assumption,

$f^{-1} B_T(y_T, z)$ is open in s.

$\therefore p \in f^{-1} B_T(y_T, z)$

Then p is an interior point of $f^{-1} B_T(y_T, z)$

$\therefore$There exist $\delta > 0$ such that

$B_s(p, \delta) \subseteq f^{-1} B_T(y, z)$

$\Rightarrow f(B_s(p, \delta)) \subseteq B_T(y, z)$

$\Rightarrow$ f is continuous at p.

Example 4 *The image of an open set under a continuous function is not necessarily open.*

Solution Let f be a continuous function defined on S to R (i.e) $f : R \to R$ such that

$f(x) = c$, a constant for all $x \in s$

Let x be open set in S.

Then $f(x) = c$ is closed in R.

Theorem 3 *Let $f : S \to T$ then f is continuous on S if and only if for every closed set Y in T, the inverse image $f^{-1}(Y)$ is closed in S.*

Proof Let y be a closed set in T Then Y^c is open in T

(i.e) $Y^c = T - Y$ is open in T.

Claim : Now, $f^{-1}(Y)(T-Y) = S - f^{-1}(Y)$

Let $x \in f^{-1}(Y)$ be arbitrary

Then $x \in S$ and $f(x) \in T - Y$

$\Rightarrow x \in S$ and $f(x) \in T$ and $f(x)$ g Y

$\Rightarrow x \in S$ and $f(x)$ g Y

$\Rightarrow x \in S$ and x g $f^{-1}(Y)$

$\Rightarrow x \in S - f^{-1}(Y)$

$\Rightarrow f^{-1}(T - Y) \subseteq S - f^{-1}(Y)$

Similarly we can prove

$S - f^{-1}(Y) \subseteq f^{-1}(T - Y)$

$\therefore f^{-1}(T - Y) = S - f^{-1}(T - Y)$

Suppose f is continuous Then $f^{-1}(T - Y)$ is open in S

(i.e)$S - f^{-1}(T - Y)$ is open in S

(i.e)$(f^{-1}(T - Y))^c$ is open in S

$\therefore f^{-1}(T - Y)$ is closed in S Conversely,

Assume that for every closed set Y in T,the inverse image $f^{-1}(Y)$ is closed in S.

$\therefore (f^{-1}(Y))^c$ is open in S.

$\therefore S - f^{-1}(Y)$ is open in S

(i.e) $f^{-1}(T - Y)$ is open in S

$\therefore$ we have T-Y is open in T

$\Rightarrow f^{-1}(T-Y)$ is open
in S By previous
theorem,

f is continuous
Hence the proof

Example 5 *The image of an closed set under a continuous function need not to be closed.*

Solution Let f be a function defined on R to the open interval $(\frac{-\pi}{2}, \frac{\pi}{2})$ such that $f(x) = tan^{-1}(x)$

We know that, R is closed and $(\frac{-\pi}{2}, \frac{\pi}{2})$ is open.

But $f(R) = (\frac{-\pi}{2}, \frac{\pi}{2})$ is open

Hence, continuous image of closed set need not be closed.

Continuous functions and Compact set:

Theorem 4 *Let $f : S \to T$ if f is continuous on a compact subset X of S then, the image $f(x)$ is a compact subset of T. In particular $f(X)$is closed and bounded in T*

Proof

Let X be a compact subset
of S Let A_α be an open
covering of X.

Then, $X \subseteq \bigcup_{i=1}^{n} A_i$

Let f be a open covering of $f(x)$

$\therefore f(x) \subseteq \bigcup_{A \in F} A$

where each A is open in T

Since f is continuous, inverse image of open set is open.

$\therefore$ Each $f^{-1}(A)$ is open in S

The sets $f^{-1}(A)$ form an open covering of X.

Since X is compact we have finite number of $f^{-1}(A)$ also covers X (i.e)
$X \subseteq f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots \cup f^{-1}(A_n)$
$\therefore f(X) \subseteq f[f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots \cup f^{-1}(A_n)]$

$\subseteq f[f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots \cup f^{-1}(A_n)]$

$\subseteq (A_1) \cup (A_2) \cup \cdots \cup (A_n)$

$\therefore f(X)$ has a finite sub-cover

$\therefore f(X)$ is compact

$\therefore f(X)$ is closed and bounded.

Definition 1 *A function $f : S \to R^k$ is called bounded on s if there is positive number M such that*

$||f(X)|| \leq M$ for all $x \in S$.

Theorem 5 *Let $f : S \to R^k$ if f is continuous on a compact subset X of S then f is bounded on S.*

Proof

Let X be a compact subset of S and f is continuous
function Then $f(x)$ is compact
Then, $f(x)$ is closed and bounded

Since $f(x)$ is bounded and we have $a \leq f(x) \leq b$

where a = greatest lower
bound b = least upper
bound
$\therefore$ f is bounded

Theorem 6 *Let $f : S \to T$. Assume that f is one-to -one on S so that the inverse function f^{-1} exists. If S is compact and if f is continuous on S, then f^{-1} is continuous on $f(S)$.*

Proof Given $f : S \to T$

Then $f^{-1} : f(S) \to S$

To prove:

f^{-1} is continuous

It is enough to prove for every closet set X in S the image (the inverse image) $f(X)$ is closed in T Since, X is closed and S is compact, We have X is compact

$\therefore f(X)$ is compact

$\therefore f(X)$ is closed and bounded (i.e) $f(X)$ is continuous.

Remark 3 *Compactness of domain set S is an essential for f^{-1} to be continuous.*

Example 6 *Let f be a function from R with discrete metric space to R with usual metric, defined by $f(x) = x$*

Proof Let X be an open subset of R

Then, $f^{-1}(X)$ is a subset of R with discrete metric space.

Since, every subset of discrete metric space is open, we have $f^{-1}(X)$ is open

$\therefore$ f is continuous

Let $\{x\} \subseteq R$ with discrete metric space

$\{x\}$ is open subset of R

Then, $(f^{-1})^{-1}(\{x\}) = f(x)$

$= \{x\}$

But x is not open in R with usual metric

$\therefore f^{-1}$ is not
continuous Note that
R is not compact

Topological Mappings:

Let $f : S \to T$. Assume that f is one -to- one on S. So, that the inverse image f^{-1} exists. If f is continuous on S and if f^{-1} is continuous on $f(S)$ then f is called a topological mapping or homomorphism and the metric space S, d_S and (T, d_T) are said to be homomorphic

Remark 4 · *If f is homomorphism then so is f^{-1}.*

- *A homomorphism maps open subsets of S onto open subset of $f(S)$.*
- *A homomorphism maps closed subsets of S onto closed subset of $f(S)$.*

$\Rightarrow$

Definition 2 *A function $f : S \to T$ is called isometry if f is one to one on S and preserves the metric. If there is an isometry from $S \to T$ then the two metric spaces are called isometric*

Bolzano's Theorem

Theorem 7 *sign preserving property*

Let f be defined on an open interval S in R. Assume f is continuous at a point c in S and that $f(c)$ Ç 0. Then there is a one ball $B(c, \delta) \bigcap S$.

Proof Let us assume that f
$(c) > 0$ Given that, f is
continuous at $c \in S$

$\therefore$ for every $z > 0$ there is a$\delta > 0$ such that $(f(x), f(c) < z$ if $d(x, c) > \delta$

$|f(x) - f(c)| < z$ if $x \in B(c, \delta) \bigcap S$

$\therefore -z < f(x) - f(c) < z$ if $x \in B(c, \delta) \bigcap S$

$f(c) - z < f(x) < z + f(c)$ if $x \in B(c, \delta) \bigcap S$

let $\frac{f(c)}{2} z$

$\therefore \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c)$ if $x \in B(c, \delta) \bigcap S$

$\therefore f(x) > 0$

$\therefore f(x)$ has the same sign as $f(c)$

The proof is similar for $f(c) < 0$ with $z = -\frac{f(c)}{2}$

Theorem 8 *Balzano Theorem*

Let f be a real valued and continuous on a compact interval [a,b] in R, and suppose that $f(a)$ and $f(b)$ have opposite signs (i.e), assume $f(a) f(b) < 0$. Then there is atleast one point c in the open interval (a,b) such that $f(c) = 0$

Proof Given that $f(a)$ and $f(b)$ have opposite signs
Suppose $f(a) > 0$ and $f(b) < 0$
Let $A = \{x : x \in [a, b] and\ f(x) \geq 0\}$

$\therefore$ A is non empty

Since, A is subset of[a,b] A is bounded above by b Let c = sup A
Claim: $f(c) = 0$
Suppose $f(c)$ Ç 0
By previous theorem, there is a one ball $B(c,\delta)$ in which f has the same sign as f(c) Suppose $f(c) > 0$
Then there are points $x > c$ at which $f(x) > 0$

POSSIBLE QUESTIONS
PART - B
(5 × 8 = 40)
UNIT I

1..State and prove Bolzano's theorem for continuous functions

2.Prove that a function f is continuous iff every inverse image of an open set is open.

3. Prove that continuous image of a compact is compact.

4. State and prove sign preserving property of continuous functions

5..Let f: S→ T ,If f is continuous on a compact subset X of S ,Then the image f(x) is a compact subset of T.In Particular f(x)is closed and bounded

6. Prove that a metric space S is connected if and only if every two valued function on S is constant.

7. Prove that f is continuous iff inverse image of a closed set is closed. Also prove that continuous image of a closed set is need not be closed.

UNIT-II
SYLLABUS

Connectedness –components of a metric space – Uniform continuity :

Uniform continuity and compact sets –fixed point theorem for

UNIT II

Definition 3 *The metric space S is called disconnected if* $S = A \cup B$ *where A and B are disjoint non-empty open sets in S.*

We call S connected if it is not disconnected.

Example 7 *Let* $S = R - \{0\}$

$\therefore S = (-\infty, 0) \cup (0, \infty)$

Clearly, $(-\infty, 0)$ *and* $(0, \infty)$ *are open sets and* $(-\infty, 0)$

$\cap (0, \infty)$ $(-\infty, 0) \cap (0, \infty) = \varphi$

$\therefore$ *S is disconnected*

Example 8 *Every open interval in R is*

connected R is connected

$\therefore$ *Every open interval is connected and* $R = (-\infty, \infty)$

$\therefore$ *R is connected*

Remark 5 • *For each p in S the set* $\{p\}$ *is connected*

· *Any discrete metric space S with more than one point is disconnected Let S be a discrete metric space with more than one point*

Let A be a proper non-empty subset of S

Since, S has more than one point such a

set exists Now, A^c is also non-empty

Since, S is a discrete metric space, we have A and A^c are open

Also, $S = A \cup A^c$

$\therefore$ *S is disconnected*

$\Rightarrow$

Two valued function

Definition 4 *Two valued function*

A real valued function f which is continuous on a metric space on S is said to be two valued on S if f(S) is a subset of $\{0, 1\}$

Remark 6 *A two valued function is a continuous function whose only possible values are 0 and 1*

This can be considered as a continuous function from S to the metric space $T = \{0, 1\}$*, where T is the discrete metric space.*

Theorem 10 *A metric space S is connected if and only if every two valued function on S is constant*

Proof Assume that S is connected

Let f be a continuous two valued

function on S To prove:

f is constant

Let $A = f^{-1}\{0\}$ and $B = \{1\}$ be the inverse image of the subsets $\{0\}$ and $\{1\}$ Since, $\{0\}$ and $\{1\}$ are open in discrete metric space $\{0, 1\}$ and f is continuous We have A and B are open in S

Also we have $A \cup B = S$

where A and b are disjoint open sets

Since, S is connected either A is empty or B is empty

$\therefore$ we must have S = A (or) S = B

Hence,
f is constant.

Conversely,

Assume that every two valued function on S is

constant To prove:

S is connected

Suppose S is

disconnected then $S = A$

$\cup$ B,where $A \cap B = \varphi$ A,

B Ç φ and A,B are open

sets

Let f be function from S to R such that

$f(x) = 0$ if $x \in A$

1 if $x \in B$

Since, A and B are non empty, f takes both values 1 and 0

$\therefore$.vf is not constants

Also, f is continuous on S, because the inverse image of every open subset of {0, 1} is open in S.

$\therefore$.The two valued function f is not constant.

Contradiction to every two valued function is

constant. Hence, S is connected

Continuous function and connected set

Theorem 11 *The continuous image of a connected set is connected.*

Proof Let f be a continuous function from

H to Y To prove:

f(X) is connected

Suppose f(x) is disconnected

Then, there exist a non-empty proper subset A of f(x) such that A is both open and closed.

Then, $f^{-1}(A)$ is a non-empty proper subset of X then X is

disconnected which is contradiction to X is connected

$\therefore$ f(X) is

connected

Hence the proof.

NOTE:

$\Rightarrow$ A metric space (X, d) is disconnected if and only if there exist a non empty

proper subset of X which is both open and closed.

$\Rightarrow$ In a metric space (X, d) is disconnected if there exist two non empty sets A and B such that $X = A \bigcup B, \bar{A} \cap \bar{B} = A \cap B = \varphi$

$\Rightarrow$ Let A and B be two connected subsets of X then, $A \cup B$ is also connected if $A \cap B$ Ç φ

Problem:

Let f be a continuous real valued function defined on a metric space S. Let $A = \{x$

$\in S \mid f(x) \geq 0\}$. Prove that A is closed.

Solution:

Given $f : S \to R$ is continuous

function Also,$A = \{x \in S \mid f(x) \geq$

$0\}$

$= \{x \in S \mid f(x) \in [0, \infty]\}$

$= \dot{} x \in S \,|(x) \in f^{-1}([0, \infty])^{\Sigma}$

Since,$[0, \infty] = ((-\infty, 0))^{c}$ is closed in R and f is continuous,we have $f^{-1}([0, \infty])$ is closed

$\therefore$ A is closed.

Theorem 12 *If A and B are connected subsets of S and if* $A \cap B$ Ç φ *then* $A \cup B$ *is connected.*

Proof Let $f : A \cup B \to \{0, 1\}$ be a continuous

function. Since, $A \cap B$ Ç φ $x_o \in A \cup B$ is

possible.

Let $f(x_o) = 0$

since, f is continuous $f|_A : A \to \{0, 1\}$ is also

continuous since, A is connected, we have $f|_A$

is also constant.

$\therefore f|_A$ is not onto

$\therefore$ f(x) = 0 or f(x) = 1 for all $x \in A$

Since, $f(x_0) = 0$ and $x_0 \in A$

$\therefore f(x) = 0$ for all $x \in A$

Similarly,

We can prove f(x) = 0 for all $x \in B$

f(x)= 0 for all $x \in A \cup B$

$\therefore$ f is constant

function Hence $A \cup B$

is connected. Hence

the proof

Remark 7 *Every interval in R is*

connected. Every curve in R^n *is*

connected.

Every subset S of R is connected if and only if S is an open interval.

<u>Intermediate value theorem</u>

Theorem 13 *Intermediate valued theorem for real continuous functions:*

Proof :

Let f be a real valued and continuous on a connected subsets S of R^1. If f takes an two different values in S say a and b then for each real c between a and b there exist a point x in S such that,

$f(x) = c$. Even $f : S \to R$ is continuous Let a,b belongs to S and $f(a)$ Ç $f(b)$ Suppose $f(a) < f(b)$

Let c be such that $f(a) < c < f(b)$

Since, S is connected and f is continuous, we have $f(s)$ is connected and is subset of R.

$\therefore f(s)$ is an interval.

Also. $f(a), f(b)$

$\therefore [f(a), f(b)] \subseteq f(s)$

$Since, f(a) < c < f(b), c \in f(s)$

$\therefore c = f(x)$ for some x in S.

Remark 8 *Let* $A = \{(x, y) : x^2 + y^2 = 1\}$ *is a connected subset of* R^2

Every point x in a metric space S belongs to atleast one connected subset of S, namely $\{x\}$

The union of all connected subsets which contain x is also connected we call this union a component of s and is denoted by $U(x)$

$U(x)$ *is maximal connected subset of S which contains x.*

Theorem 14 *Every point or a metric space S belongs to a uniquely determined component of S. In other words, the components of S form a collection of disjoint sets whose union is S.*

Proof

Let $x \in S$ be

arbitrary To prove:

$\{U(x)\}$ form a disjoint components of S and whose union is S.

(i.e) $S = \bigcup_{x \in S} U(x)$ and $\cap U(x) = \varphi$

and $U(x) \cap U(y) = \varphi$ for all $x \neq y \in S$

Suppose $x \in U(x)$ and $U(y)$

$\Rightarrow x \in U(x) \cup U(y)$

$\Rightarrow U(x) \cap U(y) \neq \varphi$ U)(x) and U(y)

connected sets Clearly,

$U(x) \subseteq U(x) \cup U(y)$

and $U(y) \subseteq U(x) \cup U(y)$

$\Rightarrow \Leftrightarrow$ to U(x) and U(y) are components

$\therefore U(x) \cap U(y) = \varphi$

$\therefore$ Two distinct components cannot contain a point x.

Fixed point theorem for contractions

Definition 5 *Let $f : S \to S$ be a function on a metric space (S , d). A point $p \in S$ is called a fixed point of f if $f(p) = p$.*

Definition 6 *The function $f : S \to S$ is called a contraction of S if there is a constant $\alpha < 1$ such that*

$$d(f(x), f(y)) \leq \alpha\, d(x, y) \text{ for all } x, y \in S .$$

Remark 9 *A contraction f of any metric space S is uniformly continuous on S .*

Theorem 15 *A contraction f on a complete metric space has a unique fixed point p.*

Proof Let $x \in S$ be arbitrary and f be a contraction of S .

Consider a sequence $\{p_n\}$ such that

$$p_0 = x$$

$$p_1 = f(p_0) = f(x)$$

$$p_2 = f(p_1) = f(f(x))$$

Now

$$\begin{aligned} d(p_{n+1}, p_n) &= d(f(p_n), f(p_{n-1})) \\ &< \alpha\, d(p_n, p_{n-1}) \\ &= \alpha\, d(f(p_{n-1}), f(p_{n-2})) \\ &\leq \alpha\, \alpha\, d(f(p_{n-2}), f(p_{n-2})) \\ &= \alpha^2 d(p_{n-2}, p_{n-2}) \end{aligned}$$

$$\vdots$$

$$\leq \alpha^n \; d(p_1, p_0)$$

$$= \alpha^n c$$

$$\text{where } c = d(p_1, p_0)$$

Suppose $m > n$.

Then $n < n + 1 < \cdots < m - 1 < m$

Now

$$d(p_m, p_n) \leq d(p_m, p_{m-1}) + d(p_{m-1}, p_n)$$

$$\leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-1}) + d(p_{m-2}, p_n)$$

$$\leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-1}) + d(p_{m-2}, p_n) + \cdots + d(p_{n+1}, p_n)$$

$m-$
1

$$(p_{k+1}, p_k)$$

As $n \to \infty$, we have $d(p_m, p_n) \to 0$. Hence $\{p_n\}$ is a cauchy sequence in S.

$$\leq c\alpha^n \sum_{k=0}^{\infty} \alpha^k$$

$$= c\alpha^n \frac{1}{1-\alpha}$$

Since S is complete metric space, we have $\{p_n\}$ converges. That is $p_n \to p$.

Now

$$f(p) = f(\lim_{n\to\infty} p_n)$$

$$= \lim_{n\to\infty} f(p_n) \text{ since } f \text{ is continuous}$$

$$= \lim_{n\to\infty} p_{n+1}$$

$$= p$$

Hence f has a fixed point p.

Uniqueness Suppose f has two fixed points p and q. Then $f(p) = p$ and $f(q) = q$.

Since f is contraction of S, we have

$$d(f(p), f(q)) \leq$$

$$\alpha d(p,q) \; d(p,$$

$$q) \leq \alpha d(p,q)$$

Since $\alpha < 1$, we must have $d(p, q) = 0$.

Hence $p = q$. ❑

Monotonic functions

Definition 7 *Let f be a real valued function defined on a subset S of* R. *Then f is said to be increasing function on f if for every pair of points $x, y \in S$,*

$$x < y \quad \Rightarrow \; f(x) \leq f(y)$$

If $x < y \Rightarrow f(x) < f(y)$, then f is said to be strictly increasing function.

Remark 10 • *Similarly we can define decreasing function and strictly decreasing function.*

· *A function f is said to be monotonic if it is either increasing or decreasing.*

· *If f is an incresing then $-f$ is decreasing function. Hence it is sufficient to consider increasing function in situations involving monotonic functions.*

Theorem 16 *Let f be a strictly increasing function on $S \subset$ R. Then f^{-1} exists and is striclty in- creasing on $f(S)$.*

Proof Let f be a strictly increasing function on $S \subset$ R.

Then

$$x < y \quad \Rightarrow \; f(x) < f(y)$$

$$\text{i.e. } x \text{ Ç } y \quad \Rightarrow f(x) \text{ Ç } f(y)$$

Hence f is one-to-one on S.

Therefore f^{-1} exits. i.e. f^{-1} is a function on f (S). Claim: f^{-1} is strictly increasing.

Suppose $y_1 < y_2$.

Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.

Then we have one of the possiblity $x_1 = x_2$ or $x_1 > x_2$ or $x_1 < x_2$.

Supoose $x_1 = x_2$.

$$\Rightarrow \quad f^{-1}(y_1) = f^{-1}(y_2)$$
$$\Rightarrow \quad f(x_1) = f(x_2)$$
$$\Rightarrow \quad y_1 = y_2$$
$$\Rightarrow\Leftarrow \quad \text{to } y_1 < y_2$$

Supoose $x_1 > x_2$.

$$\Rightarrow \quad f^{-1}(y_1) > f^{-1}(y_2)$$
$$\Rightarrow \quad f(x_1) > f(x_2)$$
$$\Rightarrow \quad y_1 > y_2$$
$$\Rightarrow\Leftarrow \quad \text{to } y_1 < y_2$$

Hence we must have $x_1 < x_2$.

Therefore f^{-1} is striclty increasing function.

POSSIBLE QUESTIONS
PART - B
(5 × 8 = 40)

1. State and prove Connectedness.
2. Let f be strictly increasing on a set S in R ,then f^{-1} exists and its strictly increasing on f(s).
3. State and prove intermediate value theorem for continuous functions
4. Prove that continuous image of a connected set is connected .Then prove that X is compact.
5. Prove that a metric space S is connected if and only if every two valued function on S is constant.
6. State and prove fixed point theorem for contraction.

UNIT-III

SYLLABUS

Definition of derivative –Derivative and continuity –Algebra of derivatives – the chain rule –one sided derivatives and infinite derivatives –functions with non-zero derivatives –zero derivatives and local extrema –Roll"s theorem –The mean value theorem for derivatives.

UNIT III

Definition 8 *Let f be defined on an interval* (a, b) *and* $c \in (a, b)$*. Then f is said to be differentiable at c whwnever*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. The limit, denoted by $f'(c)$ *and is called the derivative of f at c.*

Remark 11 • *The limit process defines a new function* f' *whose domain consists of those points in* (a, b) *at which f is differentiable.*

- *The function* f' *is called the first derivative of f .*
- *The process of finding* f' *from f is called differentiation.*

Theorem 17 *If f is defined on* (a, b) *and differentiable at a point c in* (a, b)*, then there is function* f^* *which is continuous at c and which satisfies the equation*

$$f(x) - f(c) = (x - c) f^*(x)$$

for all $x \in (a, b)$ *with* $f^{y}(c) = f^{j}(c)$. *Conversely, if there is function* f^{y}*, continous at c, which satisfies the above equation , then f is differentiable at c and* $f^{y}(c) = f^{j}(c)$.

Proof

Suppose f is differentiable at a point c in (a, b).

Then we have

$$f^{j}(c) = \lim \frac{f(x) - f(c)}{}$$

Let f be defined on (a, b) as

$$f(x) - f(c) = (x - c) f^{y}(x), \text{ if } x \text{ Ç } c$$

and $f^{y}(c) = f^{j}(c)$.

Then

$$f^{y}(x) = \frac{f(x) - f(c)}{(x - c)}$$

$$\lim_{x \to c} f^{y}(x) = \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)}$$

$$\lim_{x \to c} f^{y}(x) = f^{j}(c)$$

$$\lim_{x \to c} f^{y}(x) = f^{y}(c)$$

Hence f^{y} is continuous at c.

Conversely, suppose f^{y} is continuous at c with

$$f(x) - f(c) = (x-c) f^{y}(x), \text{ if } x \text{ Ç } c$$

and $f^{y}(c) = f^{j}(c)$.

Then

$$\frac{f(x)-f(c)}{x-c} = f^{y}(x)$$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f^{y}(x)$$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f^{y}(c)$$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f^{j}(c)$$

Therefore the limit exists and is equal to $f^{j}(c)$. Hence f is differentiable at c.

Theorem 18 *If f is differentiable at c, then f is continous at c.*

Proof Suppose f is differentiable at c. and $f^{y}(c) = f^{j}(c)$.

Then by previous theorem, there is a function f^{y} continuous at c such that

$$f(x) - f(c) = (x-c) f^{y}(x), \text{ if } x \text{-}\!\rightarrow c$$

and $f^{*}(c) = f'(c)$.

Therefore

$$\frac{f(x)-f(c)}{x-c} = f^{*}(x)$$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f^{*}(x)$$

$$f'(c) = f^{*}(c)$$

Algebra of derivatives

Theorem 19 *Assume f and g are defined on* (a, b) *and differentiable at c. Then* $f + g$, $f - g$ *and* $f \cdot g$ *are also differentiable at c. This is also true of* $\frac{f}{g}$ *if* $g(c) \neq 0$*. The derivatives at c are given by the following formulas*

a $(f \pm g)'(c) = f'(c) \pm g'(c)$

b $(f \cdot g)'(c) = f(c)g'(c) + g(c)f'(c)$

b $\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}$. *provided* $g(c) \neq 0$

Proof Suppose f and g are defined on (a, b) and differentiable at c.

By previous theorem, we have

$$f(x) - f(c) = (x - c) f^{*}(x), \text{ if } x \neq c$$

$$g(x) - g(c) = (x - c)g^{*}(x), \text{ if } x \neq c$$

Now

$$f(x) \pm g(x) = f(c) + (x-c) f^{y}(x) \pm g(c) + (x-c) g^{y}(x)$$

$$= f(c) \pm g(c) + (x-c) [f^{y}(x) \pm g^{y}(x)]$$

$$[f(x) \pm g(x)] - [f(c) \pm g(c)] = (x-c) [f^{y}(x) \pm g^{y}(x)]$$

$$\frac{[f(x) \pm g(x)] - [f(c) \pm g(c)]}{x-c} = [f^{y}(x) \pm g^{y}(x)]$$

$$\lim_{x \to c} \frac{[f(x) \pm g(x)] - [f(c) \pm g(c)]}{x-c} = \lim_{x \to c} [f^{y}(x) \pm g^{y}(x)]$$

$$(f \pm g)^{j}(c) = [f^{y}(c) \pm g^{y}(c)]$$

$$= [f^{j}(c) \pm g^{j}(c)]$$

$$f(x)g(x) = f(c)g(c) + (x-c) f(c) g^{y}(x) + (x-c) g(c) f^{y}(x) + (x-c)^{2} f^{y}(x) g^{y}(x)$$

$$f(x)g(x) - f(c)g(c) = (x-c) [f(c) g^{y}(x) + g(c) f^{y}(x)] + (x-c)^{2} f^{y}(x) g^{y}(x)$$

$$\frac{f(x)g(x) - f(c)g(c)}{x-c} = [f(c) g^{y}(x) + g(c) f^{y}(x)] + (x-c) f^{y}(x) g^{y}(x)$$

$$\lim \frac{f(x)g(x) - f(c)g(c)}{} = \lim [f(c) g^{y}(x) + g(c) f^{y}(x)] + \lim (x-c) f^{y}(x) g^{y}(x)$$

The chain rule

Theorem 20 *Let f be defined on an open interval S, let g be defined on $f(S)$, and consider the composite function $g \circ f$ defined on S by the equation*

$$(g \circ f)(x) = g[f(x)]$$

Assume there is a point c in S such that $f(c)$ is an interior point of $f(S)$. If f is differentiable at c and if g is differentiable at $f(c)$ then $g \circ f$ is differentiable at c and we have

$$(g \circ f)'(c) = g'[f(c)]f'(c)$$

Proof Given that f is differentiable at c and g is differentiable at c.

By prevoius theorem, there is a function f^* continuous at c such that

$$f(x) - f(c) = (x - c)f^*(x)$$

for all $x \in S$ with $f^*(c) = f'(c)$ and there is a function g^* continuous at $f(c)$ such that

$$g(y) - g[f(c)] = (y - f(c))g^*(y)$$

for all y in some open interval T of $f(S)$ with $g^*[f(c)]$ $= g'[f(c)]$. Let $x \in S$ such that $y = f(x) \in T$.

Then we have

$$g[f(x)] - g[f(c)] = (f(x) - f(c))g^*[f(x)]$$

$$= (x - c) f'(x) g'[f(x)]$$

Since f' continuous at c and g' continuous at $f(c)$, we have

$$\lim_{x\to c} g'[f(x)] = g'[f(c)]$$

$$= g'[f(c)]$$

Hence $\dfrac{g[f(x)] - g[f(c)]}{(x-c)} = g' \cdot f(c) \cdot f'(c)$

Functions with nonzero derivative

Theorem 21 *Let f be defined on an open interval (a,b) and assume that for some $c \in (a,b)$ we have $f'(c) > 0$ or $f'(c) = +\infty$. Then there is a 1-ball $B(c) \subset (a,b)$ in which*

$$f(x) > f(c) \text{ if } x > c \text{ and } f(x) < f(c) \text{ if } x < c$$

Proof Suppose $f'(c) > 0$.

i.e. f is differentiable at c and $f'(c)$ is finite and positive.

By prevoius theorem, there is a function f^* continuous at c such that

$$f(x) - f(c) = (x - c) f^*(x)$$

for all $x \in S$ with $f^*(c) = f'(c)$.

By sign preserving property of continuous functionsnthere is a 1-ball $B(c) \subset (a, b)$ in which $f^*(x)$ has the same sign as $f^*(c)$.

Since $f'(c) > 0$, we have $f^*(c) > 0$. Therefore $f^*(x) > 0$.

Suppose $x - c < 0$.

Then

$$\begin{aligned} f(x) - f(c) &= (x-c) f^*(x) \\ &< 0 \end{aligned}$$

Suppose $x - c > 0$.

Then

$$\begin{aligned} f(x) - f(c) &= (x-c) f^*(x) \\ &> 0 \end{aligned}$$

Hence $f(x) - f(c)$ has the same sign as $x - c$.

Rolle's theorem

Theorem 22 *Assume f has a derivative(finite or infinite) at each point of an open interval* (a, b)*, and assume that f is continuous at both endpoints a and b. If* $f(a) = f(b)$ *there is at least one interior point c at which* $f'(c) = 0$.

Proof Given that f is differential

on (a,b). Then f is continuous on

(a, b).

Also given that f is continuous at both end points a

and b. Therefore f is continous on $[a, b]$.

Since $[a, b]$ is compact, $f([a, b])$ is compact.

i.e. $f([a, b])$ is closed and

bounded. Then $m \leq f(x) \leq M$

for all $x \in [a, b]$.

To prove: There is at least one interior point c at which $f'(c) = 0$. Suppose there is no interior point c at which $f'(c) = 0$. $f'(c) \neq 0$ for all $c \in (a, b)$.

The Mean value theorem for derivatives

Theorem 23 Generalized mean value theorem *Let f and g be two functions, each having a derivative at each point of an open interval* (a, b) *and each* *continuous at the end points a and b. Assume also that there is no interior point x at which both* $f'(x)$ *and* $g'(x)$ *are finite. Then for some interior point c we have*

$$f'(c)\,[g(b)-g(a)] = g'(c)\,[f(b)-f(a)]$$

Proof Let $h(x) = f(x)\,[g(b)-g(a)] - g(x)\,[f(b)-f(a)]$. Suppose both $f'(x)$ and $g'(x)$ are finite.

Then $h'(x)$ is also finite.

Suppose either $f'(x)$ or $g'(x)$ is infinite. Then $h'(x)$ is also infinite.

Since f is continous at the end points a and b, $f(x)\,[g(b)-g(a)]$ is continuous at the end points a and b.

Similarly, $g(x)\,[f(b)-f(a)]$ is continuous at the end points a and b. Hence $h(x)$ is continuous at the end points a and b.

Also

$$\begin{aligned} h(a) &= f(a)\,[g(b)-g(a)] - g(a)\,[f(b)-f(a)] \\ &= f(a)g(b) - g(a)f(b) \\ h(b) &= f(b)\,[g(b)-g(a)] - g(b)\,[f(b)-f(a)] \\ &= f(a)g(b) - g(a)f(b) \end{aligned}$$

By Rolle's theorem, for some interior point c we have $h'(c) = 0$.

$$f'(c) \,.\, g(b) - g(a) \,.\, = g'(c) \,.\, f(b$$

Theorem 24 Mean value theorem *Assume that f has a derivative (finite or infinite) at each point o f an open interval* (a, b) *and also assume that f is continuous at both end points a and b.*

Then for

some interior point c we have

Proof Let $g(x) = x$ on a,

$$f^{j}(c)\,[b-a] = \dot{}\, f(b) - f(a)\dot{}$$

Theorem 25 *Assume f has a derivative (finite or infinite) at each point od an open interval* (a, b) *and that f is continous at the end points a and b.*

a) *If* f^{j} *takes only positive values (finite or infinite) in* (a,b), *then* f *is strictly increasing on* $[a,b]$.
b) *If* f^{j} *takes only negative values (finite or infinite) in* (a, b), *then* f *is strictly decreasing on* $[a, b]$.
c) *If* f^{j} *is zero in* (a, b), *then f is constant on* $[a, b]$.

Proof Let $x < y$ and $[x, y] \subset [a, b]$.

By mean value theorem, $f^{j}(c)(y-x) = f(y) - f(x)$ where $c \in (x, y)$

a) Suppose f^{j} takes only positive values. Then $f^{j}(c) > 0$.

Since $(y-x) > 0$, $f^{j}(c)(y-x) > 0$.

i.e. $f(y) - f(x) > 0$.

f is strictly increasing on $[x, y]$.

b) Suppose f^{j} takes only negative values. Then $f^{j}(c) < 0$.

Since $(y-x) > 0$, $f^{j}(c)(y-x) < 0$.

i.e. $f(y) - f(x) < 0$.

f is strictly decreasing on $[x, y]$.

c) Suppose f^{j} is zero in (x, y). Then $f^{j}(c) = 0$.

Hence $f(y) - f(x) = 0$.

i.e. $f(y) = f(x)$.

f is constant on $[x, y$.

Theorem 26 *If f and g are continuous on $[a,b]$ and have equal finite derivatives in (a,b), then $f-g$ is constant on $[a,b]$.*

Proof Given that f and g are continuous on $[a, b]$. Then $f-g$ is continous on $[a, b]$.

Also given that f and g have finite derivatives in (a, b) Then $f-g$ has a finite derivative in $[a, b]$.

Now

$$(f-g)^{j}(x) \quad = f^{j}(x) - g^{j}(x)$$

$$= \quad 0$$

By previous theorem, $f - g$ is constant on $[a, b]$.

Intermediate value theorem for derivatives

Theorem 27 *Assume f has a derivative (finite or infinite) at each point od an open interval* (a, b) *and that f is continous at the end points a and b. If* $f^{j}(x)$ Ç 0 *for all x in* (a, b) *then f is strictly monotonic.*

Proof Suppose $f^{j}(x)$ = Ç 0 for all x in (a, b). Then either $f^{j}(x) > 0$ or $f^{j}(x) <$ By previous theorem, we have f is strictly increasing or strictly decreasing on $[a, b]$.

POSSIBLE QUESTIONS
PART - B
(5 × 8 = 40)
UNIT III

1. State and prove Rolle's theorem.
2. Let f & g be two functions, each having a derivative at each point of (a, b). At the end points assume that the limits $f(a+)$, $g(a+), f(b-)$ and $g(b-)$ exists at finite values. Assume further that there is no interior point x at which both $f'(x)$ and $g'(x)$ are infinite. Then for some interior point c we have to prove that $f'(c)[g(b-) - g(a+)] = g'(c)[f(b-) - f(a+)]$
3. State and prove intermediate value theorem for derivatives.
4. State and prove generalized mean value theorem for derivatives.
5. State and prove function of function for derivatives.
6. Prove that f is monotonic on [a,b] then the set of discontinuous of f is countable
7. Prove that a metric space S is connected if and only if every two valued function on S is constant.

UNIT-IV

SYLLABUS

Properties of monotonic functions –functions of bounded variation –total Variation –additive properties of total variation on (a, x) as a function of x – functions of bounded variation expressed as the difference of increasing functions.

Definition 9 *Let $F : \mathbb{R} \to \mathbb{R}$ be a function. We say that F has bounded variation and write $B \in BV(\mathbb{R})$*

If

$$\sup\left\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : -\infty < x_0 < \dots < x_n < +\infty\right\} < +\infty.$$

Suppose that F is a function of bounded variation. We define the variation function of F by

$$T_F(x) = \sup\left\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : -\infty < x_0 < \dots < x_n = x\right\}$$

Clearly, T_F is a non-decreasing function and $T_F(+\infty) = \lim_{x\to+\infty} T_F(x) < +\infty$.

Example 9 *1. A constant function has bounded variation.*

2. *each monotone bounded function has bounded variation.*

3. *If $F, G \in BV(\mathbb{R})$ then $aF + bG \in BV(\mathbb{R})$ for any $a, b \in \mathbb{R}$.*

4. *$F(x) = \sin x$ has unbounded variation on $(-\infty, +\infty)$ but bounded variation on any finite inter- val.*

5. *$F(x) = \sin(1/x)$ has unbounded variation on $(0, 1)$.*

6. *$F(x) = x\sin(1/x) \in C([0, 1])$ and has unbounded variation on $[0, 1]$.*

Properties of monotonic functions

Theorem 28 *Let $F \in BV(\mathbb{R})$, then $T_F - F$ is non-decreasing.*

Proof Suppose that $x < y$ we want to show that $T_F(x) - F(x) \le T_F(y) - F(y)$. For any z > 0 we can find $-\infty < x_0 < ... < x_n = x$ such that $T_F(x) < \sum |F(x_{j-1} - x_j)| + z$. Then we have

$$T_F(y) - F(y) \ge \sum |F(x_{j-1} - x_j)| + |F(y) - F(x)| - F(y) >$$

$$T_F(x) - z + F(y) - F(x) - F(y) = T_F(x) - F(x) - z.$$

Since it is true for any z > 0 we get the required inequality.

Functions of bounded variation

Definition 10 *If* $[a, b]$ *is a compact interval, a set of points* $P = \{x_0, x_1, \cdots, x_n\}$ *satisfying the in- equalities* $a = x_0 < x_1 < \cdots < x_n = b$, *called a partition of* $[a, b]$. *The interval* $[x_{k-1}, x_k]$ *is called the kth subinterval of P and we write* $\Delta x_k = x_k - x_{k-1}$, *so that* $\sum_{k=1}^{n} \Delta x_k = b - a$. *The collection of all partitions of* $[a, b]$ *will be denoted by* $P[a, b]$

Definition 11 *Let f be defined on* $[a, b]$. *If* $P = \{x_0, x_1, \cdots, x_n\}$ *is a partition of* $[a, b]$, *write* $\Delta f_k = f(x_k) - f(x_{k-1})$, *for* $k = 1, 2, , n$. *If there exists a positive number M such that* $\sum_{k=1}^{n} |\Delta f_k| \le M$ *for all partitions of* $[a, b]$, *then f is said to be of bounded varition on* $[a, b]$.

Theorem 29 *If f is monotonic on* $[a, b]$, *then f is of bounded variation on* $[a, b]$.

Proof Let f be increasing function. Then $x_{k-1} < x_k$ implies $f(x_{k-1}) \le f(x_k)$. Therefore for every partition of $[a, b]$,

$$\Delta f_k = f(x_k) - f(x_{k-1}) \ge 0$$

Now

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} \Delta f_k$$

$$= \sum_{k=1}^{n} f(x_k) - f(x_{k-1})$$

$$= \quad f(b) - f(a)$$

Since $f(b) - f(a) > 0$, there is a positive mumber M such that $f(b) - f(a) \leq M$.

Hence $\sum_{k=1}^{n} |\Delta f_k| \leq M$.

$\therefore$ f is of bounded variation on $[a, b]$.

Theorem 30 *If f is continuous on $[a,b]$ and if f' exists and is bounded in the interior, say $|f'(x)| \leq A$ for all $x \in (a, b)$, then f is of bounded varition on $[a,b]$.*

Proof Applying mean value theorem, we have

Now

$$f_k = f(x_k) - f(x_{k-1})$$

$$= f'(t_k)(x_k - x_{k-1})$$

$$\leq A \sum_{k=1} \Delta x_k$$

$$= A(b - a)$$

Hence f is of bounded variation on $[a, b]$.

Theorem 31 *If f is of bounded variation on $[a, b]$, say $\sum_{k=1}^{n} |\Delta f_k| \leq M$ for all partitions of $[a, b]$, then f is bounded on $[a, b]$. In fact, $|f(x)| \leq |f(a)| + M$ for all $x \in a, b$.*

Let $x \in (a, b)$.

Then $P = \{a, x, b\}$ is a partition of $[a, b]$. Since f is of bounded variation on $[a, b]$,

$$\sum_{k=1}^{n} |\Delta f_k| \leq M$$

$$|f(x)-f(a)|+|f(b)-f(x)| \leq M$$

$$|f(x)-f(a)| \leq M-|f(b)-f(x)| \leq M$$

WKT

$$|f(x)|-|f(a)| \leq |f(x)-f(a)|$$

$$\leq M$$

$$|f(x)| \leq M+|f(a)|$$

Hence *f* is bounded on [a,

on [0, 1].

Clearly f *is continuous on*

[0, 1]. *Let* $P = \{0, \frac{1}{2n-1}, \frac{1}{2}, \cdots,$

$\frac{1}{}, 1\}$.

Then P *is a partition of* [0, 1].

$\therefore$ f *is not of bounded variation on* $[a, b]$.

Total variation

Definition 12 *Let* f *be of bounded variation on* $[a, b]$, *and let* $\sum(P)$ *denote the sum* $\sum_{k=1}^{n} |\Delta f_k|$ *corresponding to the partition* $P = \{x_0, x_1, \cdots, x_n\}$ *of* $[a, b]$. *The number*

$$V_f(a,b) = \sup\left\{\sum(P) : P \in \mathcal{P}([a, b])\right\},$$

is called the total variation of f *on the interval* $[a, b]$.

Remark 12 • *We will write* V_f *instead of* $V_f(a,b)$.

- *Since* f *is of bounded variation on* $[a, b]$, V_f *is finite*
- *Since each sum* $\sum(P) \geq 0$, $V_f \geq 0$
- *Suppose* f *is constant. i.e.* $f(x) = c$, *for all* $x \in [a, b]$

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

$$= \sum_{k=1}^{n} |c - c|$$

$$= \quad 0$$

Hence $\bar{}(P) = 0$ *for all partitions of*

$[a, b]$. *Therefore,* $V_f = 0$.

Converse of the above is alos true.

Theorem 32 *Assume f and g are of bounded variation on* $[a, b]$. *Then so are their sum,difference and product. Also, we have* $V_{f\pm g} \le V_f + V_g$ *and* $V_{f\cdot g} \le A\, V_f + B\, V_g$ *where*

$$A \;=\; \sup\,\{|g(x)| : x \in [a, b]\}\;,$$

$$B \;=\; \sup\,\{|\,f\,(x)| : x \in [a, b]\}\;,$$

Additive property of total variation

Total variation on $[a, x]$ **as a function of** x

Functions of bounded variation expressed as the difference of increasing functions

Now we may give a different characterization of functions of bounded variation.

Theorem 33 *The function* $F : \mathrm{R} \to \mathrm{R}$ *has bounded variation if and only if F is the difference of two bounded non-decreasing functions.*

Proof Suppose that $F \in BV(\mathrm{R})$; then F is bounded (Q1:check it!). We can write $F(x) = T_F\,(x) - (T_f\,(x) - F(x))$. Both functions T_F and $T_F - F$ are non-decreasing; T_F is bounded by the definition of $BV(\mathrm{R})$. Further, $T_F - F$ is also bounded since F is bounded.

POSSIBLE QUESTIONS
PART - B
(5 × 8 = 40)

1.Prove that f is monotonic on [a,b] then the set of discontinuous of f is countable

2. Prove that a metric space S is connected if and only if every two valued function on S is constant.

3.Assume $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\ dx$ exists and $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$

4.State and prove formula for integration by parts of a Riemann-Stieltjes integral.

5. Additive properties of total variation?

6. Continuous function on bounded variation ?

7. Reduction and concept of Riemann integral